A Formal Proofs

Note for all proofs in this section, we represent the forward and reverse mappings in AP-reduction, as $\pi(\cdot)$ and $\sigma(\cdot,\cdot)$ instead of $\pi(\cdot,\cdot)$ and $\sigma(\cdot,\cdot,\cdot)$, since the mappings used here do not depend on the rational r in Definition 2.8. Also, we assign integer values to Boolean functions: 0 for False and 1 for True.

A.1 General Case

Theorem 3.1. QPBO is exp-APX-complete.

Proof. We reduce from the following problem.

Problem 2 ([4], Section 8.3.2). **W3SAT-triv**
INSTANCE: Boolean CNF formula F with variables $x_1, \cdots, x_n$ and each clause assuming exactly 3 variables; non-negative integer weights $w_1, \cdots, w_n$.
SOLUTION: Truth assignment τ to the variables that either satisfies F or assigns the trivial, all-true assignment.
MEASURE: $\min \sum_{i=1}^{n} w_i \tau(x_i)$.

W3SAT-triv is known to be exp-APX-complete [4]. We use an AP-reduction from W3SAT-triv to prove the same completeness result for QPBO. The optimal value of W3SAT-triv is upper bounded by $M := \sum_i w_i$ because the all-true assignment is feasible. The objective weight is represented in QPBO as unary terms $f_i(x_i) = w_i x_i$. For every Boolean clause $C(x_i, x_j, x_k) \in F$ we construct a triple-wise term

$$\delta_{ijk}(x_i, x_j, x_k) = M(1 - C(x_i, x_j, x_k)). \tag{9}$$

This term takes the large value M iff C is not satisfied and 0 otherwise. Further, the Boolean clause $C(x_i, x_j, x_k)$ can be represented uniquely as a multi-linear cubic polynomial. For example, a clause $x_1 \vee \bar{x}_2 \vee \bar{x}_3$ can be represented as

$$1 - (1 - x_1)x_2x_3 = x_1x_2x_3 - x_2x_3 + 1. \tag{10}$$

Then we obtain similar representation with a single third order term and a second order multi-linear polynomial for δ_{ijk}:

$$\delta_{ijk} = M(ax_ix_jx_k + \sum_J b_J \prod_{l \in J} x_l), \tag{11}$$

where $J \subseteq \{i, j, k\}, |J| \leq 2$, $\prod_{l \in J} x_l$ is set to 1 if J is empty, $a \in \{-1, 1\}$, and $b_J \in \{-1, 0, 1\}$. We now apply the quadratization techniques [23] to δ_{ijk}. After introducing an auxiliary variable x_w with $w > n$, we observe the following identities:

$$-x_ix_jx_k = \min_{x_w \in \{0,1\}} -x_w(x_i + x_j + x_k - 2) \tag{12}$$

$$x_ix_jx_k = \min_{x_w \in \{0,1\}} \Big((x_w - 1)(x_i + x_j + x_k - 1) + (x_ix_j + x_ix_k + x_jx_k)\Big) \tag{13}$$

In either case, substituting the cubic term $ax_ix_jx_k$ in δ_{ijk} with the expression inside the min operator, we can have a unified quadratic form

$$\psi_{ijk} := M \sum_{J_w} b_{J_w} \prod_{l \in J_w} x_l, \tag{14}$$

where $J_w \subseteq \{i, j, k, w\}, |J_w| \leq 2$ and $\prod_{i \in J_w} x_i$ is set to 1 if J_w is empty. In both cases, the quadratic form takes the same optimal values as its cubic counterpart given the optimal assignment, i.e.,

$$\min_{x_i, x_j, x_k, x_w} \psi_{ijk} = \min_{x_i, x_j, x_k} \delta_{ijk}, \tag{15}$$

but the transformation expands the original range of the cubic term from $\{-1, 0\}$ to $\{-1, 0, 1, 2\}$ and from $\{0, 1\}$ to $\{0, 1, 3\}$ respectively for $a = -1$ and $a = 1$. Therefore, the cost of the constructed instance of QPBO is bounded in the absolute value by $3M$ and the number of added variables is exactly the number of clauses in F. Clearly, this construction can be computed in polynomial time. *Note that when approximation is used, this transformation is no longer exact ($\psi_{ijk} \neq \delta_{ijk}$), as the optimality of the auxiliary variable x_w cannot be guaranteed. However, it can be verified that under all possible assignments (ignoring the min operator) in either case, $\psi_{ijk} \geq 0$, which is the key for the reduction to be an approximation preserving (AP) one.*

The construction above defines a mapping π from any instance of W3SAT-triv ($p_1 \in I_1$) to an instance of QPBO ($p_2 \in I_2$). The mapping σ from feasible solutions of p_2 ($x \in S_2(p_2)$) to that of p_1 is defined as follows: if $f(x) \geq M$, then let the mapped solution $\sigma(p_1, x)$ be the all true assignment, otherwise let the mapped solution $\sigma(p_1, x)$ be $x_i, i \in \{1, ..., n\}$.

Now, we need to show that (π, σ) together with a constant α is an AP-reduction. Let m_1, m_2, m_1^* and m_2^* to be short for $m_1(p_1, \sigma(p_1, x)), m_2(p_2), m_1^*(p_1)$, and $m_2^*(\pi(p_2))$ respectively, where $*$ indicates the optimal solution. First, note that $\sigma(p_1, x)$ is always feasible for *W3SAT*-triv: either it satisfies F or $f(x) \geq M$ and therefore $\sigma(p_1, x)$ is the all-true assignment. In the first case, since every quadratic term is non-negative, we have

$$m_1 = \sum_{i=1}^{n} x_i w_i \tag{16}$$

$$\leq \sum_{i=1}^{n} x_i w_i + \sum_{C_{ijk} \in F} \psi_{ijk}(x_i, x_j, x_k) = f(x) = m_2. \tag{17}$$

In the second case, by construction

$$m_1 = M \leq f(x) = m_2. \tag{18}$$

Therefore, no matter which case $m_1 \leq m_2$.

Now for the optimal solution, if F is satisfiable, then by construction $m_1^* = m_2^*$. Recall from Definition 2.5, $R = m/m^*$. For any instance $p_1 \in I_1$, for any rational $r > 1$, and for any $x \in S_2(p_2)$, if

$$R_2(p_2, x) \leq r, \tag{19}$$

then

$$m_1 \leq m_2 \leq rm_2^* = rm_1^* \tag{20}$$

$$R_1(p_1, \sigma(p_1, x)) = \frac{m_1}{m_1^*} \leq r \tag{21}$$

If F is not satisfiable, $m_1^* = M \leq m_2^*$ and $m_2 \geq m_2* \geq M$. Thus, for any instance $p_1 \in I_1$, for any rational $r > 1$, and for any $x \in S_2(p_2)$,

$$R_1(p_1, \sigma(p_1, x)) = \frac{m_1}{m_1^*} = \frac{M}{M} = 1 \leq r \tag{22}$$

Therefore $(\pi, \sigma, 1)$ is an AP-reduction. Since W3SAT-triv is exp-APX-complete and QPBO is in exp-APX, we prove that QPBO is exp-APX-complete. □

Corollary 3.2. k-label energy minimization is exp-APX-complete for $k \geq 2$.

Proof. We create an AP-reduction from QPBO to k-label energy minimization by setting up the unary and pairwise terms to discourage a labeling with the additional $k-2$ labels.

Denote QPBO as $\mathcal{P}_1 = (\mathcal{I}_1, \mathcal{S}_1, m_1, \text{MIN})$ and k-label energy minimization as $\mathcal{P}_2 = (\mathcal{I}_2, \mathcal{S}_2, m_2, \text{MIN})$. Given an instance $p_1 = (\mathcal{G} = (\mathcal{V}, \mathcal{E}), \mathcal{L}_1, f) \in \mathcal{I}_1$, let $M(p_1)$ be a large number such that all for all $y_1 \in \mathcal{S}_1$, $m_1 < M$. For example, we can let

$$M = \sum_{u \in \mathcal{V}} \sum_{x_u \in \mathcal{L}_1} |f_u(x_u)| + \sum_{(u,v) \in \mathcal{E}} \sum_{x_u \in \mathcal{L}_1} \sum_{x_v \in \mathcal{L}_1} |f_{uv}(x_u, x_v)| + 1. \tag{23}$$

We define the forward mapping π from any $p_1 \in I_1$ to $p_2 = (\mathcal{G} = (\mathcal{V}, \mathcal{E}), \mathcal{L}_2, g) \in I_2$ as follows:

- $g_u(a) = f_u(a)$, for $\forall a \in \mathcal{L}_1$, and $\forall u \in \mathcal{V}$;
- $g_u(a) = M$, for $\forall a \notin \mathcal{L}_1$, and $\forall u \in \mathcal{V}$;
- $g_{uv}(a, b) = f_{uv}(a, b)$, for $\forall a, b \in \mathcal{L}_1$, and $\forall (u, v) \in \mathcal{E}$;
- $g_{uv}(a, b) = M$ if either a or $b \notin \mathcal{L}_1$ for $\forall (u, v) \in \mathcal{E}$.

This setup has two properties:

- $m_2 \geq M$ if and only if the labeling $\mathbf{x}_2 \in \mathcal{S}_2$ includes labels that are not in $\mathcal{L}_1$;
- $m_1^* = m_2^*$, for any p_1 and $p_2 = \pi(p_1)$.

Then we define the reverse mapping σ from any $(p_2, \mathbf{x}_2)$ to $\mathbf{x}_1 \in \mathcal{S}_1$ to be

- $\mathbf{x}_1 = \mathbf{x}_2$, if $m_2 < M$;
- $\mathbf{x}_1$ be any fixed feasible solution (e.g., all nodes are labeled as the first label), if $m_2 \geq M$.

Observe that in both cases, $m_1 \leq m_2$. For any instance $p_1 \in I_1$, for any rational $r > 1$, and for any $\mathbf{x}_2 \in S_2$, if

$$R_2(p_2, \mathbf{x}_2) = \frac{m_2}{m_2^*} \leq r, \tag{24}$$

then

$$m_1 \leq m_2 \leq rm_2^* = rm_1^* \tag{25}$$

$$R_1(p_1, \mathbf{x}_1) = \frac{m_1}{m_1^*} \leq r \tag{26}$$

Therefore $(\pi, \sigma, 1)$ is an AP-reduction. As QPBO is exp-APX-complete and all energy minimization problems are in exp-APX, we conclude that k-label energy minimization is exp-APX-complete for $k \geq 2$. $\square$

The above construction also formally shows that the energy minimization problem can only become harder when having a larger labeling space, irrespective of the graph structure and the interaction type.

The next result is used in Section 5.2.

Corollary A.2. An $O(\log k)$-approximation implies an $O(\log |x|)$-approximation for k-label energy minimization problems.

Proof. Observe that an instance of the energy minimization problem (1) is completely specified by a set of all unary terms f_u and pairwise terms f_{uv}. This defines a natural encoding scheme to describe an instance of an energy minimization problem with binary alphabet $\{0, 1\}$. Therefore, the input size

$$|x| = O(k^2|V|^2). \tag{27}$$

Assume we have an r-approximation algorithm, and $r = O(\log k)$, then

$$r = O(\log k + \log |V|) = O(\log k|V|) = O(log|x|), \tag{28}$$

which implies an $O(\log |x|)$-approximation algorithm.

$\square$

Relation to Bayesian Networks There are substantial differences between results for Bayesian networks [2] and our result. Bayesian networks have a probability density function $p(x)$ that factors according to a directed acyclic graph, e.g., as $p(x_1, x_2, x_3) = p(x_1|x_2, x_3)p(x_2)p(x_3)$. Finding the MAP assignment (same as the most probable estimate (MPE)) in a Bayesian network is related to energy minimization (1) by letting $f(x) = -\log(p(x))$. The product is transformed into the sum and so, e.g., factor $p(x_1|x_2, x_3)$ corresponds to term $f_{1,2,3}(x_1, x_2, x_3)$.

The inapproximability result of Abdelbar and Hedetniemi [2] holds even when restricting to binary variables and factors of order three. However, [2, Section 6.1] count incoming edges of the network. For a factor $p(x_1|x_2, x_3)$, there are two, but the total number of variables it couples is three and therefore such a network does not correspond to QPBO. If one restricts to factors of at most two variables, e.g., $p(x_1|x_2)$, in a Bayesian network, then only tree-structured models can be represented, which are easily solvable.

In the other direction, representing pairwise energy (1) as a Bayesian network may require to use factors of order up to $|V|$ composed of conditional probabilities of the form $p(x_i \,|\, x_j, x_k, \cdots)$ with the number of variables depending on the vertex degrees. It is seen that while the general problems are convertible, fixed-parameter classes (such as order and graph restrictions) differ significantly. In addition, the approximation ratio for probabilities translates to an absolute approximation (an additive bound) for energies. The next corollary of our main result illustrates this point.

Corollary A.3. It is NP-hard to approximate MAP in the value of probability (2) with any exponential ratio $\exp(r(n))$, where r is polynomial.

Recall that the probability $p(x)$ is given by the exponential map of the energy: $p(x) = \exp(-f(x))$. Assume for contradiction that there is a polynomial time algorithm $\mathcal{A}$ that finds solution x and a polynomial $r(n) \geq 0$ for $n > 0$ such that

$$\frac{p(x^*)}{p(x)} \leq e^{r(n)} \tag{29}$$

for all instances of the problem. Taking the logarithm,

$$-f(x^*) + f(x) \leq r(n). \tag{30}$$

or,

$$f(x) \leq r(n) + f(x^*). \tag{31}$$

Divide by $f(x^*)$, which, by definition of NPO is positive, we obtain

$$\frac{f(x)}{f(x^*)} \leq 1 + \frac{1}{f(x^*)} r(n) \leq 1 + r(n). \tag{32}$$

where we have used that $f(x^*)$ is integer and positive and hence it is greater or equal to 1. Inequality (32) provides a polynomial ratio approximation for energy minimization. Since the latter is exp-APX-complete (Corollary 3.2), this contradicts existence of the polynomial algorithm $\mathcal{A}$, unless P = NP. □

Note, this corollary provides a stronger inapproximiability result for probabilities than was proven in [2].

Remark A.4. Abdelbar and Hedetniemi [2] have shown also the following interesting facts. For Bayesian networks, the following problems are also APX-hard (in the value of probability):

- Given the optimal solution, approximate the second best solution;
- Given the optimal solution, approximate the optimal solution conditioned on changing the assignment of one variable.

A.2 Planar Case

Theorem 4.1. Planar 3-label energy minimization is exp-APX-complete.

Proof. We create an AP-reduction from 3-label energy minimization to planar 3-label energy minimization by introducing polynomially many auxiliary nodes and edges.

Denote 3-label energy minimization as $\mathcal{P}_1 = (\mathcal{I}_1, \mathcal{S}_1, m_1, \mathrm{MIN})$ and planar 3-label energy minimization as $\mathcal{P}_2 = (\mathcal{I}_2, \mathcal{S}_2, m_2, \mathrm{MIN})$. Given an instance $p_1 \in \mathcal{I}_1$, we compute a large number $M(p_1)$ as in Equation (23) in the proof for Corollary 3.2.

The gadget-based reduction presented in Section 4, defines a forward mapping π from any $p_1 = (\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1), \mathcal{L}, f) \in I_1$ to $p_2 = (\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2), \mathcal{L}, g) \in I_2$. Let $\mathcal{V}_3$ be the nodes added during the reduction, then $\mathcal{V}_2 = \mathcal{V}_1 \cup \mathcal{V}_3$. The two gadgets

SPLIT and UNCROSSCOPY are used 4 times each to replace an edge crossing (point of intersection not at end points) with a planar representation (Figure 4), introducing 22 auxiliary nodes. Since the gadgets can be drawn arbitrarily small so that they are not intersecting with any other edges, we can repeatedly replace all edge crossings in $\mathcal{G}_1$ with this representation. There can be up to $O(|\mathcal{E}_1|^2)$ edge crossings, and we have $|V_3|$ = $O(|\mathcal{E}_1|^2)$. Given that the reduction adds only a polynomial number of auxiliary nodes, the forward mapping π can be computed by a polynomial time algorithm.

This setup has two properties:

- $m_2 \leq M$ if and only if the labeling $\mathbf{x}_1$ is the same as the partial labeling in $\mathbf{x}_2$ restricting to nodes in $\mathcal{V}_1$ in $\mathcal{G}_2$.
- $m_1^* = m_2^*$, for any p_1 and $p_2 = \pi(p_1)$.

Then we define the reverse mapping σ from any $(p_2, \mathbf{x}_2)$ to $\mathbf{x}_1 \in \mathcal{S}_1$ to be

- $\mathbf{x}_1 = \mathbf{x}_2$, if $m_2 < M$;
- $\mathbf{x}_1$ be any fixed feasible solution (e.g., all nodes are labeled as the first label), if $m_2 \geq M$.

Observe that in both cases, $m_1 \leq m_2$. For any instance $p_1 \in I_1$, for any rational $r > 1$, and for any $\mathbf{x}_2 \in S_2$, if

$$R_2(p_2, \mathbf{x}_2) = \frac{m_2}{m_2^*} \leq r, \tag{33}$$

then

$$m_1 \leq m_2 \leq r m_2^* = r m_1^* \tag{34}$$

$$R_1(p_1, \mathbf{x}_1) = \frac{m_1}{m_1^*} \leq r \tag{35}$$

Therefore $(\pi, \sigma, 1)$ is an AP-reduction. As 3-label energy minimization is exp-APX-complete (Corollary 3.2) and all energy minimization problems are in exp-APX, we conclude that planar 3-label energy minimization is exp-APX-complete.

□

Corollary 4.2. Planar k-label energy minimization is exp-APX-complete, for $k \geq 3$.

Proof. The proof of Corollary 3.2 is graph structure independent. Therefore, the same proof applies here. □