Universality of the Local Marginal Polytope

Daniel Průša and Tomáš Werner

Center for Machine Perception
Czech Technical University
Prague, Czech Republic

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LP relaxation of min-sum problem

Pairwise min-sum problem with graph (V, E) and label set K:

$$\min_{\mathbf{k}\in K^V} \Big[\sum_{u\in V} f_u(k_u) + \sum_{\{u,v\}\in E} f_{uv}(k_u, k_v) \Big].$$

All weights $f_u(k), f_{uv}(k,\ell) \in \mathbb{R} \cup \{\infty\}$ form a vector $\mathbf{f}$.
Problem instance is defined by $(V, E, K, \mathbf{f})$.

LP relaxation = linear optimization over local marginal polytope:

$$\begin{aligned} \langle \mathbf{f}, \boldsymbol{\mu} \rangle &\to \min \\ \sum_{\ell\in K} \mu_{uv}(k,\ell) = \mu_u(k), &\quad \{u,v\} \in E,\ k \in K \\ \sum_{k\in K} \mu_u(k) = 1, &\quad u \in V \\ \boldsymbol{\mu} &\geq \mathbf{0} \end{aligned}$$

where in scalar product $\langle \mathbf{f}, \boldsymbol{\mu} \rangle$ we define $\infty \cdot 0 = 0$.
Components $\mu_u(k)$ and $\mu_{uv}(k,\ell)$ of $\boldsymbol{\mu}$ are pseudomarginals.

Main result

Theorem (Průša-Werner-CVPR2013)

Any linear program can be reduced in linear time to the LP relaxation of a pairwise min-sum problem with 3 labels.

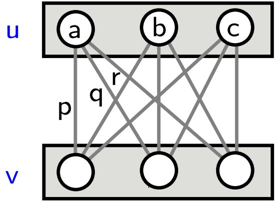
Some consequences:

- Finding an efficient algorithm to solve LP relaxation of min-sum problem might be as hard as improving the complexity of the best known algorithm for LP.
- LP relaxation of min-sum problem with 3+ labels is inherently more complex than for 2 labels
  (because for 2 labels, it reduces in linear time to max-flow).
- When solving LP relaxation of min-sum problem by the simplex method, finding a pivot rule that prevents stalling would mean that the rule applies to any LP.

Elementary min-sum problems

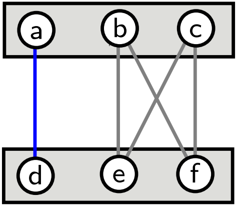
The reduction is done by combining elementary min-sum problems.

- They perform simple operations on unary pseudomarginals.
- Depicting a pair $\{u, v\} \in E$ with $|K| = 3$ labels:



$$p + q + r = a$$
$$a + b + c = 1$$

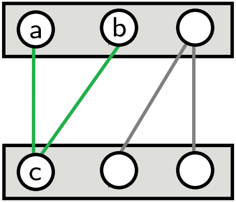
- Visible edges have weights $f_{uv}(k, \ell) = 0$.
  Invisible edge have weights $f_{uv}(k, \ell) = \infty$, implying $\mu_{uv}(k, \ell) = 0$.



Enforces $a = d$.

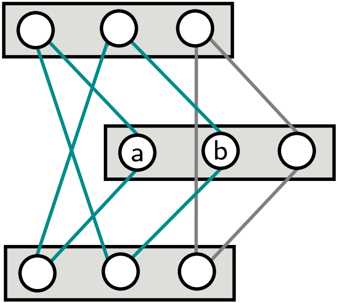
Precisely:
Given any feasible unary pseudomarginals a, b, c, d, e, f,
feasible pairwise pseudomarginals exist if and only if $a = d$.

Elementary min-sum problem ADDITION

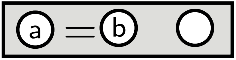


Enforces $c = a + b$.

Elementary min-sum problem EQUALITY

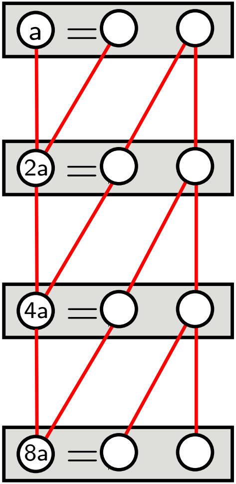


Enforces $a = b$.



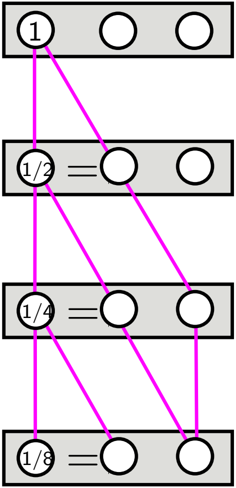
shorthand

Elementary min-sum problem POWERS



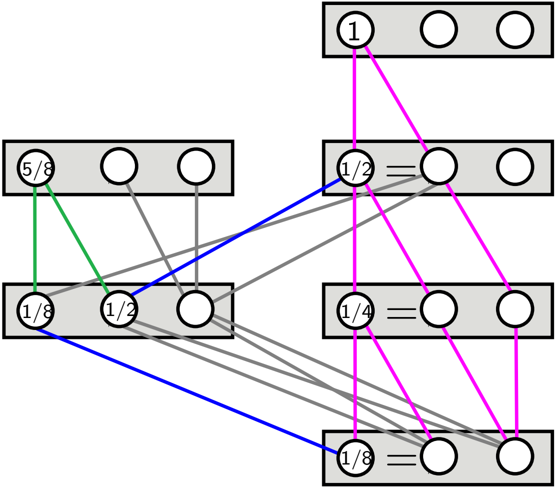
Constructs unary pseudomarginals with values $2^i a$ for $i = 0, \ldots, d$, where d is the depth of the problem.

Elementary min-sum problem NegPowers



Constructs unary pseudomarginals with values 2^{-i} for $i = 0, \ldots, d$.

Example of combining elementary min-sum problems



Constructs a unary pseudomarginal with value $5/8 = 5 \cdot 2^{-d}$.
Similarly, we can construct any multiple of 2^{-d} (not greater than 1).

The input LP

The input of the reduction is the LP

$$\min\{\, \langle \mathbf{c}, \mathbf{x} \rangle \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \,\}$$

where $\mathbf{A} \in \mathbb{Z}^{m \times n}$, $\mathbf{b} \in \mathbb{Z}^m$, $\mathbf{c} \in \mathbb{Z}^n$, $m \leq n$.

Before reduction, the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is rewritten as

$$\mathbf{A}^{+}\mathbf{x} = \mathbf{A}^{-}\mathbf{x} + \mathbf{b}$$

where all entries of $\mathbf{A}^{+}, \mathbf{A}^{-}, \mathbf{b}$ are non-negative and $\mathbf{A} = \mathbf{A}^{+} - \mathbf{A}^{-}$.

Bounding the variable ranges

Lemma

Let $\mathbf{x}$ *be a vertex of the polyhedron* $\{\, \mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \,\}$. *Then each component* x_j *of* $\mathbf{x}$ *satisfies either* $x_j = 0$ *or* $M^{-1} \leq x_j \leq M$, *where*

$$
\begin{aligned}
M &= m^{m/2}(B_1 \times \cdots \times B_{n+1}) \\
B_j &= \max\{1, |a_{1j}|, \ldots, |a_{mj}|\}, \quad j = 1, \ldots, n \\
B_{n+1} &= \max\{1, |b_1|, \ldots |b_m|\}.
\end{aligned}
$$

Lemma

Let the polyhedron $\{\, \mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \,\}$ *be bounded. Then for any* $\mathbf{x}$ *from the polyhedron, each component of* $\mathbf{A}^{+}\mathbf{x}$ *and* $\mathbf{A}^{-}\mathbf{x} + \mathbf{b}$ *is not greater than* $N = M(B_1 + \cdots + B_{n+1})$.

Initializing the reduction

The reduction algorithm:

- Its input is $(\mathbf{A}, \mathbf{b}, \mathbf{c})$, assuming w.l.o.g. that the polyhedron $\{\, \mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \,\}$ is bounded.
- Its output will be a min-sum problem $(V, E, K, \mathbf{f})$ with $V = \{1, \ldots, |V|\}$ and $K = \{1, 2, 3\}$.

The algorithm is initialized as follows:

1. For each variable x_j in the input LP, introduce a new object j into V and set $f_j(1) = c_j$.
   (Pseudomarginal $\mu_j(1)$ will represent variable x_j.)
2. For each such object $j \in V$, build $\textsc{Powers}$ with the depth $d_j = \lfloor \log_2 B_j \rfloor$ based on label 1.
3. Build $\textsc{NegPowers}$ with the depth $d = \lceil \log_2 N \rceil$.

Encoding the equality constraints

Each equation

$$a_{i1}^{+}x_1 + \cdots + a_{in}^{+}x_n = a_{i1}^{-}x_1 + \cdots + a_{in}^{-}x_n + b_i$$

of the system $\mathbf{A}^{+}\mathbf{x} = \mathbf{A}^{-}\mathbf{x} + \mathbf{b}$ is encoded as follows:

1. Construct pseudomarginals with values $a_{ij}^{+}x_j$ and $a_{ij}^{-}x_j$ by summing selected values from the POWERS.
2. Construct a pseudomarginal with value $2^{-d}b_i$ by summing selected values from the NEGPOWERS. (The number 2^{-d} plays the rôle of the unit.)
3. Sum the terms on each side of the equation by repetitively applying ADDITION and COPY.
4. Enforce equality of the two sides of the equation by COPY.

Finally, set $f_i(k) = 0$ for all $i > n$ or $k > 1$.

$\min\{\, 2x - 5y + z \mid x + 2y + 2z = 3;\ x = 3y + 1;\ x, y, z \geq 0 \,\}$

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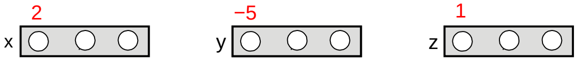

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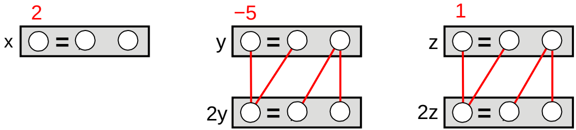

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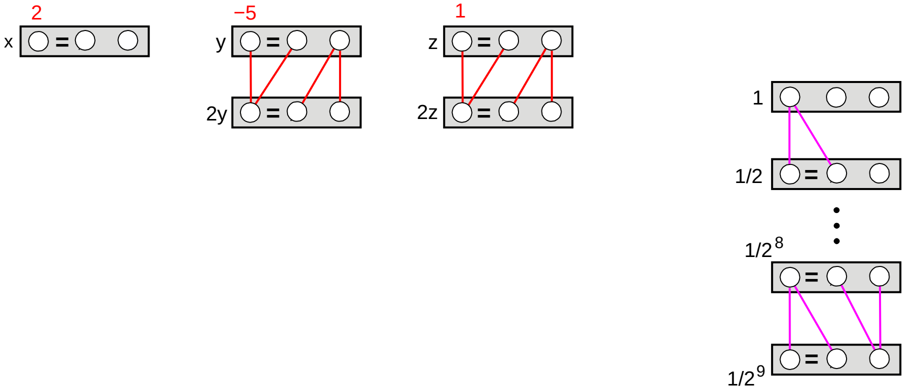

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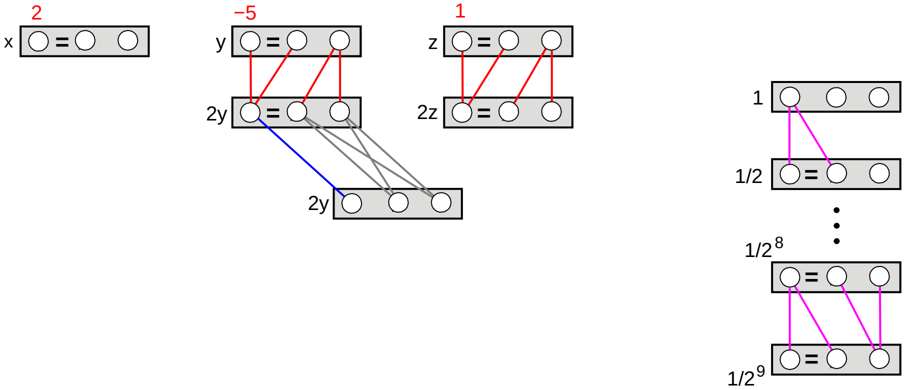

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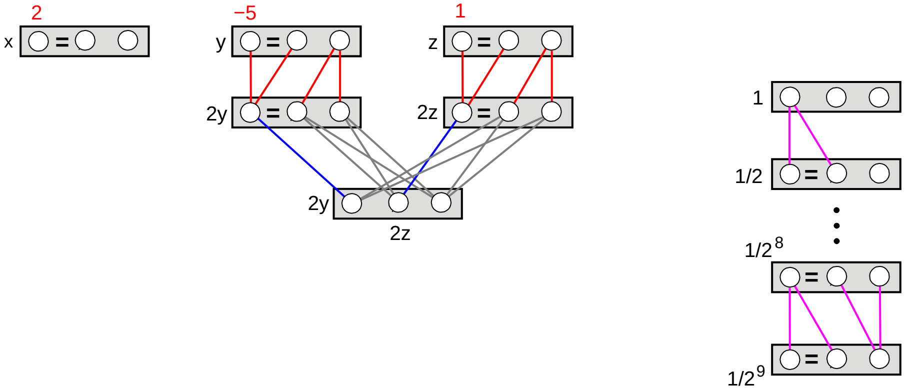

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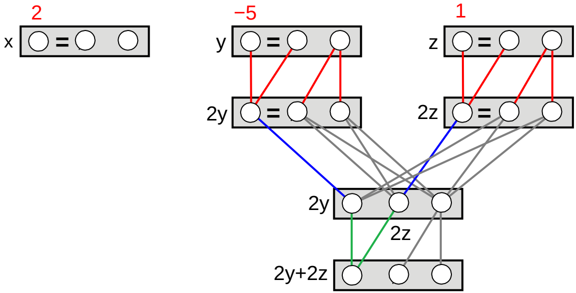

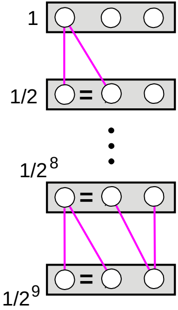

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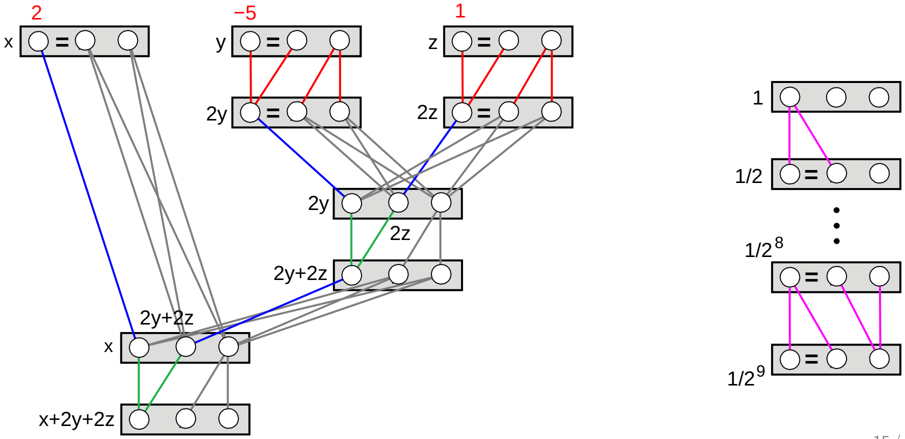

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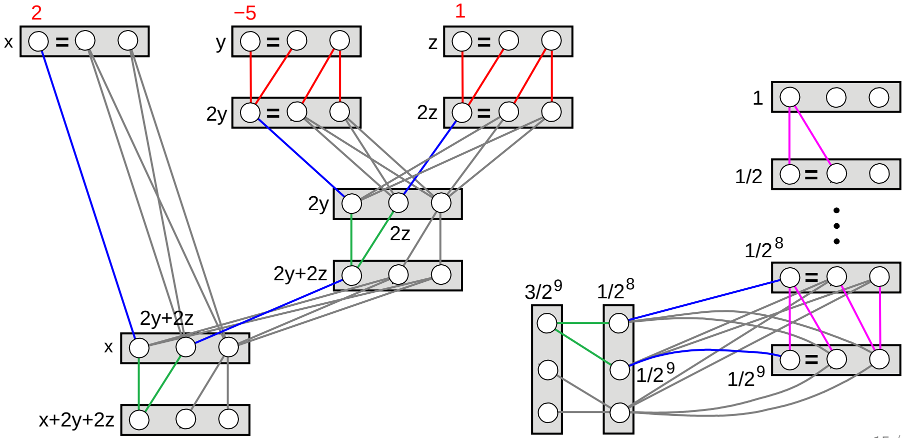

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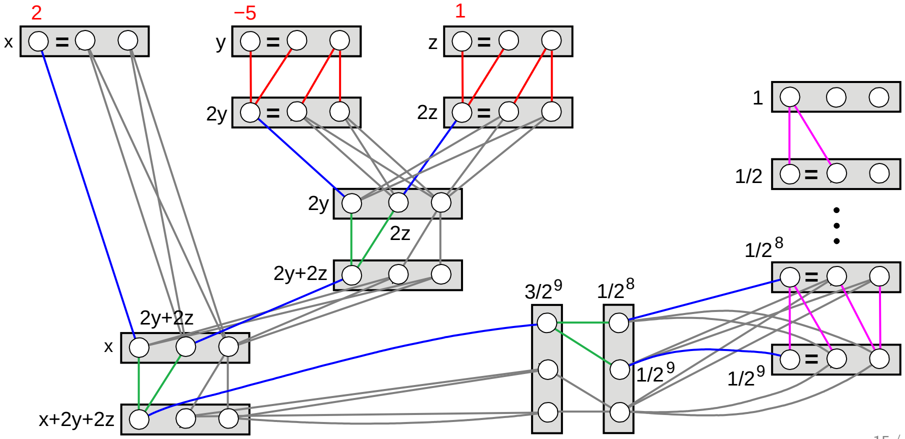

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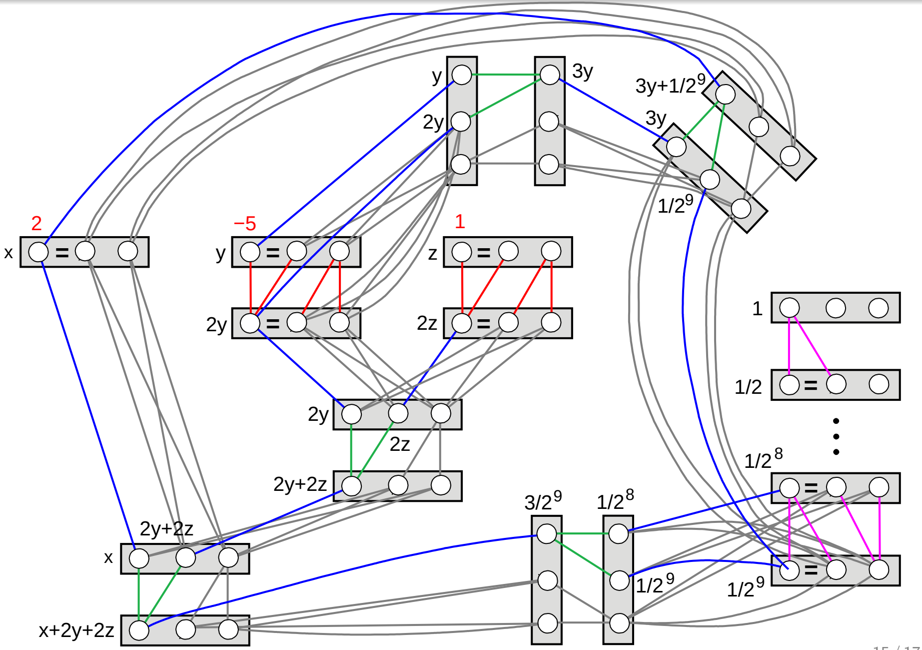

Complexity of the reduction

Let L be the number of bits of the binary representation of $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. Want to prove that the reduction time is $\mathcal{O}(L)$.

This is easy:

- Let the output of the reduction be $(V, E, K, \mathbf{f})$.
- Clearly, the reduction time is $\mathcal{O}(|E|)$.
- Clearly, $|E| = \mathcal{O}(|V|)$.
- Thus we need to prove $|V| = \mathcal{O}(L)$.
- For that, it suffices to prove that the numbers $d_j = \lceil \log_2 B_j \rceil$ and $d = \lceil \log_2 N \rceil$ are $\mathcal{O}(L)$.

Other results

Corollary

Every polytope is (up to scale) a coordinate-erasing projection of a face of a local marginal polytope with 3 labels, whose description can be computed from the description of the original polytope in linear time.

If only finite weights are allowed ($f_u(k), f_{uv}(k, \ell) \in \mathbb{R}$) then:

Theorem

Any linear program can be reduced in time and space $\mathcal{O}(L(L + L'))$ to a linear optimization over a local marginal polytope with 3 labels, where L' is the length of the binary representation of $\mathbf{c}$.