On the Complexity of Approximating Weighted Satisfiability Problems*

(Extended Abstract)

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Abstract

The maximum weighted satisfiability (MAX WEIGHTED SAT) problem and the minimum weighted satisfiability (MIN WEIGHTED SAT) problem are optimization variants of the standard maximum satisfiability (MAX SAT) problem, in which every variable is assigned a non-negative weight and one tries to find a truth assignment satisfying the formula, that maximize or minimize the sum of the weights of the variables set to true. In this work a complete classification of the problems obtained by making restrictions on the size of variables per clause and on the number of occurrences of variables in the clauses is given. Moreover, the approximation properties of special cases of weighted satisfiability, that contain classical NPO problems, are analyzed.

Keywords: Computational Complexity, Approximation, Approximation Classes, Weighted Satisfiability Problems.

1 Introduction

The maximum weighted satisfiability (MAX WEIGHTED SAT) problem and the minimum weighted satisfiability (MIN WEIGHTED SAT) problem are optimization variants of the standard maximum satisfiability (MAX SAT) problem, in which every variable is assigned a non-negative weight and one tries to find a truth assignment satisfying the formula that maximize or minimize the sum of the weights of the variables set to true.

Among NP-hard optimization problems, MAX WEIGHTED SAT and MIN WEIGHTED SAT play an important role. They indeed are the first problems that have been proved to be complete for the class NPO of the optimization problems whose decision version belong to NP. In particular, in [3, 23], completeness results have been exhibited in the class of the maximization NPO problems (MAX NPO), and in the class of the NPO minimization problems (MIN NPO). Then, it has been show that any MAX NPO-complete (MIN NPO-complete) problem is indeed complete for the whole class NPO [8]. Besides, the first examples (MAX BOUNDED WEIGHTED SAT and MAX POLYNOMIALLY BOUNDED WEIGHTED SAT [9]) of APX-complete problems are variants of the maximization problem, based on constraints on the sum of the weights of the variables and on a slight modification of the objective function.

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Variants of MAX WEIGHTED SAT and MIN WEIGHTED SAT have been considered in the literature, defined by making restrictions on the size of the clauses in the formulae and on the values of the variable weights. In this context, the NPO-completeness of the restriction of MIN WEIGHTED SAT to 3-CFN formulae and the NPO PB-completeness of the restrictions of both MAX WEIGHTED SAT and MIN WEIGHTED SAT to 3-CFN formulae (also called MAX DISTINGUISHED ONES and MIN DISTINGUISHED ONES) in which variables are assigned unitary weight have been proved [18, 19, 23]. Better approximations results have been instead achieved for MIN WEIGHTED SAT restricted to 2-CFN formulae that has been proved to be in APX [13, 16].

In this work the approximability of weighted satisfiability problems is further investigated, and a complete classification of the problems obtained by making restrictions on the size of variables per clause and on the number of occurrences of variables in the clauses is given, both if variables are assigned arbitrary weight and if variables are assigned unitary weight.

In analyzing the approximability of an optimization problem, we are interested in defining a precise boundary between cases that fall into different approximation classes. In this paper we show that, for weighted satisfiability problems, the dividing line between NPO-complete (NPO PB-complete) cases, in terms both of size of the clauses and of occurrences of variables, is exactly between three and two. Namely, we show that, for both maximization and minimization problems, neither the bound k on the size of the clauses, nor the bound B on the number of variable occurrences is helpful in approximation if k and B are both greater than 2.

A precise characterization of the approximability of the restrictions of the problems to 2-CFN formulae is then provided. We show that, although the polynomial solvability of 2-SAT guarantees that it is possible to decide whether the set of feasible solutions is empty in polynomial time, the maximization problem is not in APX, even if the number of variable occurrences is bounded. On the contrary the minimization version belong to APX [13, 16].

Then, we show that, in terms of occurrences of variables, the dividing line between two and three define the precise boundary between the polynomial and the NP-hard cases, by showing that, in this case, the problems can be reduced to maximum weighted matching problems.

Finally, we focus on special cases of weighted satisfiability, that are classical NPO problems such as the maximum weighted independent set (MAX WEIGHTED IND SET) problem, the minimum weighted vertex cover (MIN WEIGHTED VERTEX COVER) problem and the minimum weighted hitting set (MIN WEIGHTED HITTING SET) problem. Namely, the variant of the maximization problems in which all literals in the clauses are negative and the variant of the minimization problems in which all literals in the clauses are positive.

Tables 1 and 1 summarize all the discussed results.

The remainder of the paper is organized as follows. In Section 2, we state the basic definitions. In Section 3 NPO-complete and NPO PB-complete problems are considered. Sections 4 and 5 are focussed on the restrictions to MAX WEIGHTED 2-SAT and MAX WEIGHTED SAT-2, respectively. In Section 6 we analyze the approximability of the restrictions of MAX WEIGHTED SAT and of MIN WEIGHTED 2-SAT to formulae in which all variables are negative and positive, respectively.

2 Basic Definitions and Preliminaries

In the following we use the standard definitions of NP optimization (NPO) problems and approximation classes given in [1]. In particular, NPO is the class of all NPO problems, NPO PB is the class of all polynomially bounded NPO problems, and APX is the class of all NPO problems that can be approximated within some constant ratio.

The problems that are studied in the paper are defined in the appendix.

Several different kinds of approximation preserving reducibility have been defined and thus various concepts of completeness have been introduced [2, 7, 9, 10, 20, 23, 25].

Among them, the PTAS-reducibility introduced in [10] is perhaps the most general one appearing in the literature.

MAXIMUM	Arbitrary weight	Unitary weight	Arbitrary weight negative variables	Unitary weight negative variables
MAX WEIGHTED SAT MAX WEIGHTED k-SAT $k \geq 3$	NPO-complete	NPO PB-complete	not in APX in exp-APX	not in APX in poly-APX
MAX WEIGHTED SAT$-B$ MAX WEIGHTED k-SAT$-B$ $k, B \geq 3$	NPO-complete	NPO PB-complete	APX-complete	
MAX WEIGHTED 2-SAT	not in APX in exp-APX	not in APX in poly-APX	not in APX in exp-APX	not in APX in poly-APX
MAX WEIGHTED 2-SAT$-B$ $B \geq 3$	not in APX in exp-APX	not in APX in poly-APX	APX-complete	
MAX WEIGHTED SAT-2	polynomial			

Table 1: Complexity of maximum weighted satisfiability problems.

Definition 2.1 *Let $A = (I_A, SOL_A, m_A, opt_A)$ and $B = (I_B, SOL_B, m_B, opt_B)$ be two NPO problems. A is said to be PTAS-reducible to B, in symbols $A \leq_{PTAS} B$, if three computable function f, g and c exist such that:*

1. *For any $x \in I_A$ and for any $r > 1$, $f(x,r) \in I_B$ is computable in time polynomial with respect to $|x|$.*
2. *For any $x \in I_A$ for any $r > 1$ and for any $y \in SOL_B(f(x,r))$, $g(x,y,r) \in SOL_A(x)$ is computable in time polynomial with respect to both $|x|$ and $|y|$.*
3. *$c : (1,\infty) \to (1,\infty)$ is a computable function.*
4. *For any $x \in I_A$, for any $r > 1$ and for any $y \in SOL_B(f(x,r))$,*

$$R_B(f(x,r),y) \leq c(r) \quad \textit{implies} \quad R_A(x, g(x,y,r)) \leq r$$

The triple (f,g,c) is said to be a PTAS-reduction from A to B.

It is possible to show that the PTAS-reducibility preserves membership in PTAS. Furthermore, if function c is invertible, membership in APX is also preserved. Therefore, if $A \leq_{\mathrm{PTAS}} B$ and $B \in$ PTAS (respectively, $B \in$ APX) , then $A \in$ PTAS (respectively, $A \in$ APX). Using PTAS-reducibility one can thus prove NPO-completeness, APX-hardness and APX-completeness results.

Let $X = \{x_1, \ldots, x_n\}$ be a set of boolean variables. A *truth assignment* for X is a function $\tau : X \to \{true, false\}$. If x is a variable in X, then x and $\overline{x}$ are literals over X. A CFN formula is a conjunctive normal form formula and a k-CFN formula is a CFN formula whose clauses have size at most k.

MINIMUM	Arbitrary weight	Unitary weight	Arbitrary weight positive variables	Unitary weight positive variables
MIN WEIGHTED SAT	NPO-complete	NPO PB-complete	not in APX in log-APX	
MIN WEIGHTED k-SAT MIN WEIGHTED SAT$-B$ MIN WEIGHTED k-SAT$-B$ $k, B \geq 3$	NPO-complete	NPO PB-complete	APX-complete	
MIN WEIGHTED 2-SAT MIN WEIGHTED 2-SAT$-B$ $B \geq 3$	APX-complete			
MIN WEIGHTED SAT-2	polynomial			

Table 2: Complexity of minimum weighted satisfiability problems.

3 NPO-Completeness and NPO PB-Completeness Results

In this section we focus on restrictions of MAX WEIGHTED SAT and MIN WEIGHTED SAT that remain NPO-complete (NPO PB-complete if variables are assigned unitary weight).

We show that, for both problems, neither the bound k on the size of the clauses, nor the bound B on the variable occurrences is helpful in approximation if k and B are both greater than 2.

Theorem 3.1 MAX WEIGHTED SAT *is NPO-complete even if restricted to 3-CFN formulae in which the number of occurrences of each variable is bounded by three.*

Proof. Since MAX WEIGHTED SAT is NPO-complete [3, 8], we first give a PTAS-reduction from MAX WEIGHTED SAT to MAX WEIGHTED 3-SAT and then a PTAS-reduction from MAX WEIGHTED 3-SAT to MAX WEIGHTED 3-SAT-3.

MAX WEIGHTED SAT $\leq_{PTAS}$ MAX WEIGHTED 3-SAT.
This reduction is analogous to the one from MIN WEIGHTED SAT to MIN WEIGHTED k-SAT [23]. The standard reduction used in the proof for the NP-hardness of 3-SAT [12] is used to transform a formula Φ of MAX WEIGHTED SAT into a formula Φ_3 such that Φ_3 is satisfiable if and only if Φ is satisfiable. The auxiliary variables introduced in the reduction are assigned zero weight.

MAX WEIGHTED 3-SAT $\leq_{PTAS}$ MAX WEIGHTED 3-SAT-3.
This reduction is similar to the standard reduction used to show the NP-hardness of SAT-3. Let x be a variable of weight w that occurs h times in the formula. Split x into h variables $x_1, \ldots x_h$ of weight w/h. In order to guarantee $x_1, \ldots x_h$ to have the same value in every satisfying truth assignment, add h clauses $(x_i \vee \overline{x}_{i+1})$, $1 \leq i \leq h-1$, and $(x_h \vee \overline{x}_1)$. □

Theorem 3.2 *The variation of* MAX WEIGHTED SAT *in which variables are assigned unitary weight is NPO PB-complete even if restricted to 3-CFN formulae in which the number of occurrences of each variable is bounded by three.*

Proof. Since MAX WEIGHTED 3-SAT with unitary weights is NPO PB-complete [18], we PTAS-reduce this problem to MAX WEIGHTED 3-SAT-3.

MAX WEIGHTED 3-SAT $\leq_{PTAS}$ MAX WEIGHTED 3-SAT-3.
The reduction is similar to the reduction used in Theorem 3.1. Nevertheless, in this case, we are interested to obtain a formula in which all variables are assigned unitary weight, or, equivalently, in which all variables are assigned the same weight. Therefore, each variable x is now split into the same number H of variables, where H is the maximum value among the variable occurrences in the formula. □

Theorem 3.3 MIN WEIGHTED SAT *is NPO-complete even if restricted to 3-CFN formulae in which the number of occurrences of each variable is bounded by three.*

Theorem 3.4 *The variation of* MIN WEIGHTED SAT *in which variables are assigned unitary weight is NPO PB-complete even if restricted to 3-CFN formulae in which the number of occurrences of each variable is bounded by three.*

The proof of the above theorems are analogous to the proofs of Theorems 3.1 and 3.2, and therefore are omitted.

As a consequence of these results, the maximization and the minimization versions of the problems are NPO-complete (NPO PB-complete if variables are assigned unitary weight) even if the size of the formula is bounded by k and the number of variable occurrences is bounded by B, and both k and B are greater than 2.

4 Approximability of Weighted 2-Satisfiability

It is well know that, for decision problems, the dividing line in terms of size of the clauses, between NP-complete cases and polynomial cases is exactly between two and three. Therefore, for weighted satisfiability problems restricted to 2-CNF formula, the problems of deciding if feasible solutions exist and, in that case, of determining one of such solutions can be solved in polynomial time.

It is then interesting to restrict ourselves to study these problems, in order to provide a precise characterization of their approximability properties. In this context, MIN WEIGHTED 2-SAT has been proved to be approximable to within approximation ratio 2 [13, 16].

In this section we provide a precise characterization of the approximability of MAX WEIGHTED 2-SAT, MIN WEIGHTED 2-SAT and of their restrictions to formulae in which the number of variable occurrences is bounded. We first show that, MAX WEIGHTED 2-SAT is hard to approximate to within any constant ratio and belongs to exp-APX (poly-APX for variables of unitary weight)[1]. Then we prove that, in both the maximization version and the minimization version, the corresponding approximation properties remain almost the same even if the number of variable occurrences is bounded by a constant greater 2.

Before analyzing the approximation properties of MAX WEIGHTED 2-SAT, let us consider that this problem contains as special case a simple encoding of MAX WEIGHTED IND SET. Namely, given a graph $G = (V, E)$, for each vertex $v \in V$, create a variable x_v with the same weight, and for each edge $(u, v) \in E$ construct a clause $(\overline{x}_u \vee \overline{x}_v)$. Then, let Φ be the conjunction of all such clauses. It is easy to see that any satisfying truth assignment τ for Φ corresponds to an independent set for G, where a vertex v is included in the independent set if and only if $\tau(x_v) = true$, and therefore that instances of MAX WEIGHTED IND SET can be naturally expressed as instances of MAX WEIGHTED 2-SAT in which all variables are negative.

Theorem 4.1 MAX WEIGHTED 2-SAT *and* MAX WEIGHTED 2-SAT$-B$ *($B \geq 3$);*
1. are not in APX, both for variables of arbitrary weight and for variables of unitary weight;
2. belong to exp-APX (poly-APX for variables of unitary weight).

[1]Notice that, the symmetry existing for maximization version and minimization version of the problems studied in the previous section disappears in the case of 2-CNF formulae.

Proof.

1. For MAX WEIGHTED 2-SAT, we observe that MAX IND SET $\subseteq$ MAX WEIGHTED 2-SAT, and MAX IND SET is not in APX [4].
   For MAX WEIGHTED 2-SAT$-B$, we give a PTAS-reduction from MAX WEIGHTED 2-SAT to MAX WEIGHTED 2-SAT$-B$, analogous to the reduction from MIN WEIGHTED SAT to MIN WEIGHTED SAT$-B$.
2. It follows immediately from the fact that 2-SAT is polynomially solvable.

□

It is easy ti see that, similarly to the encoding of instances of MAX WEIGHTED IND SET as instances of MAX WEIGHTED 2-SAT, it is possible to express instances of MIN WEIGHTED VERTEX COVER as instances MIN WEIGHTED 2-SAT. Therefore we have the following theorems.

Theorem 4.2 MIN WEIGHTED 2-SAT *and* MIN WEIGHTED 2-SAT$-B$ *are APX-complete, both for variables of arbitrary weight and for variables of unitary weight.*

Proof. MIN WEIGHTED 2-SAT and MIN WEIGHTED 2-SAT$-B$ are approximable within 2 and thus belong to APX [13, 16].

MIN WEIGHTED 2-SAT and MIN WEIGHTED 2-SAT$-B$ contain MIN VERTEX COVER$-B$, that is APX-complete [25]. Therefore, both of them are APX-hard. □

5 Complexity of Weighted Satisfiability-2

In this section we show that in terms of occurrences of variables the dividing line between two and three define the precise boundary between the polynomial and the NP-hard cases.

In particular the following theorem shows that if variable occurrences are bounded by two, problems can be reduced to MAX WEIGHTED MATCHING [11, 21, 24].

Theorem 5.1 MAX WEIGHTED SAT-2 *and* MIN WEIGHTED SAT-2 *belong to P.*

Proof. We prove the theorem for MAX WEIGHTED SAT-2. The proof for MIN WEIGHTED SAT-2 is similar, and therefore is omitted for sake of brevity.

We give a polynomial reduction from MAX WEIGHTED SAT-2 to MAX WEIGHTED MATCHING. Suppose we are given an instance of MAX WEIGHTED SAT-2 with n variables $x_1, \ldots x_n$ and m clauses $c_1, \ldots c_m$. Let w_i be the weight of variable x_i, $1 \leq i \leq n$. Without lose of generality we can assume that there are no variable that occurs (once or twice) positive.

Construct a graph G with a vertex set consisting of two vertices x_i^1 and x_i^2 (the variable vertices) for each variable and one vertex c_j (the clause vertices) for each clause. For each variable x_i we construct the following edges:

- if x_i occurs once (negative) in clause c_j, one edge (x_i^1, c_j) with weight $\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i} + 1$;
- if x_i occurs once positive in clause c_j and once negative in clause c_h, two edges (x_i^1, c_j) with weight $w_i + \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i} + 1$ and (x_i^1, c_h) with weight $\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i} + 1$;
- if x_i occurs twice negative in clauses c_j and c_h, two edges (x_i^1, c_j) and (x_i^2, c_h) both with weight $\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i} + 1$.

For every variable x_i, we also include one edge (x_i^1, x_i^2) with weight w_i. An edge between a variable vertex x_i and a clause vertex c_j is denoted by positive (negative) edge if x_i occurs positive (negative) in c_j.

Now we have a correspondence between assignments that satisfy the formula and maximal matchings such that all clause vertices are matched (i.e., incident to edges in the matching).

Let M be a maximal matching that satisfies the above condition. Then, for each variable x_i, if there is (at least) one negative edge in M incident to one of the variable vertices corresponding to x_i than $\tau(x_i) = false$, otherwise $\tau(x_i) = true$.

The weight of M is

$$W_{MATCH}(M) = m \cdot \left(\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i} + 1 \right) + \sum_{\tau(x_i)=true} w_i = m \cdot \left(\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i} + 1 \right) + W_{WSAT}(\tau).$$

Moreover, since the weight of any matching such that at least one clause vertex is unmatched is at most $(m-1)\cdot(\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i}+1)+\sum_{i=1}^{n} w_i = m\cdot\sum_{i=1}^{n} w_i + m - 1$, and the weight of any matching without unmatched clause vertices is at least $m \cdot (\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i} + 1) = m \cdot \sum_{i=1}^{n} w_i + m$, all clause vertices must be matched by any maximum matching.

Therefore:

$$W^*_{MATCH}(M) = m \cdot \left(\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i} + 1 \right) + W^*_{WSAT}(\tau).$$

□

6 Approximability of All Negative Maximum Weighted Satisfiability and All Positive Minimum Weighted Satisfiability

We first focus on the restriction of MAX WEIGHTED SAT in which all variables are negative, and analyze its approximability. Analogously to general case, restrictions of this problems in term of size of the clauses and occurrences of the variables are considered, both for variables of arbitrary weight and for variables of unitary weight.

The following theorem shows that, if all variables are negative, the bound k on the size of the clauses is not helpful in approximation if $k \geq 2$.

Theorem 6.1 MAX WEIGHTED SAT *and* MAX WEIGHTED k-SAT *($k \geq 2$):*
1. *are not in APX, both for variables of arbitrary weight and for variables of unitary weight;*
2. *belong to exp-APX (poly-APX for variables of unitary weight).*

Proof.
1. MAX IND SET $\subseteq$ MAX WEIGHTED k-SAT (if $k = 2$ it is MAX IND SET) and MAX IND SET is not in APX [4].
2. A feasible solution can be found in polynomial time.

□

Now, we prove that the problem becomes APX-complete, if we restrict to formulae in which variables occur a number $B \geq 3$ of times.

Namely, we show that the problem belongs to APX since they are particular cases of MAX WEIGHTED GSAT(B) for $B \geq 4$, that is in the class MAX NP [25].

We give a PTAS-reduction from MAX WEIGHTED SAT$-B$ to WEIGHTED GSAT($B+1$) Suppose we are given an instance of MAX WEIGHTED SAT$-B$, that is a formula Φ over the set $X = \{x_1, \ldots x_n\}$ of variables, where each variable occurs at most B times. Let w_i be the weight of variable x_i, $1 \leq i \leq n$.

Construct an instance Φ' of WEIGHTED GSAT(B+1) of n clauses $c_1, \ldots c_n$ over the set X of variables with in the following way. For each variable x_i construct a clause c_i of Φ' that is satisfied if and only if $\tau(x_1) = true$ and all the clauses in Φ that contains x_i are satisfied. Namely, the clause c_i is a disjunction of conjunctions such that if $\tau(x_1) = true$ and all the clauses in Φ that contains x_i are satisfied, then at least one disjunct in c_i is true.

Let us consider the following example for $k = 3$ and $B = 3$.

Example: Let $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$, $(\overline{x_1} \vee \overline{x_4} \vee \overline{x_5})$ and $(\overline{x_1} \vee \overline{x_6} \vee \overline{x_7})$ be the three clauses in Φ that contain x_1.

We construct the clause c_1 as follows.

$$(x_1 \wedge \overline{x_2} \wedge \overline{x_4} \wedge \overline{x_6}) \vee (x_1 \wedge \overline{x_2} \wedge \overline{x_4} \wedge \overline{x_7}) \vee (x_1 \wedge \overline{x_2} \wedge \overline{x_5} \wedge \overline{x_6}) \vee (x_1 \wedge \overline{x_2} \wedge \overline{x_5} \wedge \overline{x_7})$$
$$\vee (x_1 \wedge \overline{x_3} \wedge \overline{x_4} \wedge \overline{x_6}) \vee (x_1 \wedge \overline{x_3} \wedge \overline{x_4} \wedge \overline{x_7}) \vee (x_1 \wedge \overline{x_3} \wedge \overline{x_5} \wedge \overline{x_6}) \vee (x_1 \wedge \overline{x_3} \wedge \overline{x_5} \wedge \overline{x_7})$$

Notice that the size of the formula of MAX WEIGHTED GSAT$(B+1)$ is polynomial in the size of the corresponding MAX WEIGHTED SAT$-B$ formula. Indeed, the number of clauses in MAX WEIGHTED GSAT$(B+1)$ is equal to the number n of variables of MAX WEIGHTED SAT$-B$, and the number of literals in any clause is bounded by $(n-1)^B \cdot (B+1)$.

Then we are able to state the following theorem.

Theorem 6.2 MAX WEIGHTED SAT$-B$ *and* MAX WEIGHTED k-SAT$-B$ *($B \geq 3$, $k \geq 2$) are APX-complete, both for variables of arbitrary weight and for variables of unitary weight.*

Proof. Since MAX WEIGHTED SAT$-B$ $\leq_{PTAS}$ MAX WEIGHTED GSAT$(B+1)$ that belongs to MAX NP, MAX WEIGHTED SAT$-B$ and MAX WEIGHTED k-SAT$-B$ belong to APX.

The APX-hardness follows from MAX IND SET$-B$ $\subseteq$ MAX WEIGHTED k-SAT$-B$[2] [25]. □

Consider now the restriction of MIN WEIGHTED SAT in which all variables are positive. Since, in this case, MIN WEIGHTED SAT (MIN WEIGHTED 2-SAT) is an encoding of MIN WEIGHTED HITTING SET(MIN WEIGHTED VERTEX COVER), the following results hold. MIN WEIGHTED SAT is APX-hard and belongs to log-APX, and MIN WEIGHTED k-SAT, MIN WEIGHTED SAT$-B$ and MIN WEIGHTED k-SAT$-B$ are APX-complete [2, 5, 6, 14, 15, 17, 22, 25].

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Appendix

In the following we give the definition of the NPO problems that have been considered throughout the work.

MAX WEIGHTED SAT and MIN WEIGHTED SAT

INSTANCE: Set X of variables, CNF formula Φ over X, weight function $w : X \rightarrow R^+$.
SOLUTION: Truth assignment τ for X that satisfies Φ.
MEASURE: $W(\tau) = max\{1, \sum_{i=1}^{n} w(x_i)\tau(x_i)\}$.

Variation in which:
- Φ is a k-CFN formula,
- each variable occurs at most B times,
- Φ is a k-CFN formula and each variable occurs at most B times

are denoted by MAX WEIGHTED k-SAT (MIN WEIGHTED k-SAT), MAX WEIGHTED SAT$-B$ (MIN WEIGHTED SAT$-B$) and MAX WEIGHTED k-SAT$-B$ (MIN WEIGHTED k-SAT$-B$), respectively.

MAX WEIGHTED IND SET

INSTANCE: Graph $G = (V, E)$, , weight function $w : V \rightarrow R^+$.
SOLUTION: An independent set for G, i.e., a subset $V' \subseteq V$ such that no two vertices in V' are joined by an edge in E.
MEASURE: Sum of the weights of the independent set, i.e. $\sum_{v_i \in V'} w(v_i)$.

Variation in which the graph G has degree bounded by B is indicated by MAX WEIGHTED IND SET$-B$.

Unweighted versions are denoted by MAX IND SET and MAX IND SET$-B$, respectively.

MIN WEIGHTED HITTING SET

INSTANCE: Collection C of subset of a finite set S, weight function $w : S \rightarrow R^+$.
SOLUTION: A hitting set for C , i.e., a subset $S' \subseteq S$ such that S' contains at least one element from each subset in C.
MEASURE: Sum of the weights of the hitting set, i.e. $\sum_{s_i \in S'} w(s_i)$.

Variation in which:
- the size of C is at most k,
- the number of occurrences of any element in C is bounded by B,
- the size of C is at most k and the number of occurrences of any element in C is bounded by B

are denoted by MIN WEIGHTED k-HITTING SET, MIN WEIGHTED HITTING SET$-B$ and MIN WEIGHTED k-HITTING SET$-B$, respectively.

Unweighted versions are denoted by MIN k-HITTING SET, MIN HITTING SET$-B$ and MIN k-HITTING SET$-B$, respectively.

MIN WEIGHTED VERTEX COVER

INSTANCE: Graph $G = (V, E)$, weight function $w : V \rightarrow R^+$.
SOLUTION: A vertex cover for G, i.e., a subset $V' \subseteq V$ such that for all $(u, v) \in E$ at least one of u and v is included in V'.
MEASURE: Sum of the weights of the vertex cover , i.e. $\sum_{v_i \in V'} w(v_i)$.

Variation in which the graph G has degree bounded by B is indicated by MIN WEIGHTED VERTEX COVER$-B$.

Unweighted versions are denoted by MIN VERTEX COVER and MIN VERTEX COVER$-B$, respectively.

MAX WEIGHTED GSAT(B)

INSTANCE: Set X of variables, set $\Phi = \{c_1, \ldots, c_m)\}$, of disjunctive form clauses[3] over X, all of whose disjuncts are conjunctions containing up to B literals. weight function $w : \Phi \to R^+$.
SOLUTION: Truth assignment for X.
MEASURE: Weight of the set of clauses from Φ that are satisfied by the truth assignment.

[3]Notice that, in this definition, we make use of a generalized notion of clause. Namely, in this context, clauses are disjunctions of conjunctions, such that each conjunction has at most B literals. In other words, clauses are DNF formulas, such that each term in a DNF has at most B literals.