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Completeness in approximation classes beyond APX

Bruno Escoffier, Vangelis Th. Paschos*

LAMSADE, Université Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France

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Abstract

We present a reduction that allows us to establish completeness results for several approximation classes mainly beyond **APX**. Using it, we extend one of the basic results of S. Khanna, R. Motwani, M. Sudan, and U. Vazirani (On syntactic versus computational views of approximability, SIAM J. Comput. 28 (1998) 164–191) by proving sufficient conditions for getting complete problems for the whole **Log-APX**, the class of problems approximable within ratios that are logarithms of the size of the instance, as well as for any approximability class beyond **APX**. We also introduce a new approximability class, called **Poly-APX(Δ)**, dealing with graph-problems approximable with ratios functions of the maximum degree Δ of the input-graph. For this class also, using the proposed reduction, we establish complete problems.

Keywords: Complexity; **NP**-optimization problems; Polynomial approximation; Reductions; Completeness

1. Introduction and preliminaries

Consider an **NPO** problem[1] $\Pi = (\mathcal{I}_\Pi, \mathrm{Sol}_\Pi, m_\Pi, \mathrm{goal}_\Pi)$, where: $\mathcal{I}_\Pi$ denotes the set of instances of Π; for any instance $x \in \mathcal{I}_\Pi$, $\mathrm{Sol}_\Pi(x)$ is the set of feasible solutions of x; for any $x \in \mathcal{I}_\Pi$ and any $y \in \mathrm{Sol}(x)$, $m_\Pi(x, y)$ denotes the value of y; finally, goal_Π is max, or min. For any $x \in \mathcal{I}_\Pi$, let $\mathrm{opt}_\Pi(x)$ be the value of an optimal solution for x. Then, the approximation ratio of an algorithm $\mathtt{A}$ computing a feasible solution $\mathtt{A}(x) = y \in \mathrm{Sol}(x)$ is defined by $\rho_\Pi(x, y) = m_\Pi(x, y)/\mathrm{opt}_\Pi(x)$. The objective of the polynomial approximation theory is double: on the one hand, it aims at devising polynomial algorithms achieving good approximation ratios for **NP**-hard problems; on the other hand, it aims at building a hierarchy of these problems, elements of which correspond to strata of problems sharing common approximability properties (they notably are approximable within comparable—in some predefined sense—approximation ratios) and at investigating relations between problems in the same stratum (notably to exhibit problems that are harder than others). This second objective is called in short structure in approximability classes.

Study of structure in approximability classes is at the heart of the research in polynomial approximation since the seminal papers [11,12,7]. By using suitable approximation-preserving reductions, the existence of natural complete problems for almost all the known approximation classes has been established. For instance, MAX WSAT for **NPO** under AP-reduction [4,6], or PTAS-reduction [8], MAX WSAT-B for **APX**, the class of the problems approximable within

* Corresponding author.
E-mail addresses: escoffier@lamsade.dauphine.fr (B. Escoffier), paschos@lamsade.dauphine.fr (V.Th. Paschos).

[1] An **NPO** problem is an optimization problem, the decision version of which is in **NP**.

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(fixed) constant ratios, under P-reduction [7], MAX 3-SAT for **APX** under AP-reduction [4,6], or PTAS-reduction [8], MAX PLANAR INDEPENDENT SET for **PTAS**, the class of the problems solvable by polynomial time approximation schemata, under FT-reduction [5], MAX INDEPENDENT SET for **Poly-APX** under PTAS-reduction [5]. Also, under E-reduction, completeness in **Log-APX-PB**, the subclass of **Log-APX** dealing with polynomially bounded problems (the class of problems whose values are bounded by a polynomial in the size of the instance), has also been established. One problem that, to our knowledge, remains open from this ambitious but so successful research program, is to establish the completeness for the whole class **Log-APX**.

As one can see by the unravelling of the fascinating history of the approximation-preserving reductions, the one that allows achievement of most of the completeness results is the PTAS-reduction, originally introduced in [8]. Let Π and Π' be two maximization **NPO** problems (the case of minimization is completely analogous). We say that Π *PTAS-reduces to* Π' if and only if there exist three functions f, g and c such that:

- for any $x \in \mathcal{I}_\Pi$ and any $\varepsilon \in]0,1[$, $f(x,\varepsilon) \in \mathcal{I}_{\Pi'}$; f is computable in time polynomial with $|x|$;
- for any $x \in \mathcal{I}_\Pi$, any $\varepsilon \in]0,1[$ and any $y \in \mathrm{Sol}(f(x,\varepsilon))$, $g(x,y,\varepsilon) \in \mathrm{Sol}(x)$; g is computable in time polynomial with $|x|$ and $|y|$;
- $c:]0,1[\to]0,1[$;
- for any $x \in \mathcal{I}_\Pi$, any $\varepsilon \in]0,1[$ and any $y \in \mathrm{Sol}(f(x,\varepsilon))$, $\rho_{\Pi'}(f(x,\varepsilon),y) \geqslant 1-c(\varepsilon)$ implies $\rho_\Pi(x,g(x,y,\varepsilon)) \geqslant 1-\varepsilon$.

This is the descendant of a number of powerful reductions as the L-reduction [12] or the E-reduction [10] that, even if they allowed achievement of completeness results in natural approximability sub-classes (e.g., **Max-SNP** $\subset$ **APX**, for the former, or **Log-APX-PB** $\subset$ **Log-APX**, for the latter) were not able to extend them to the whole of the classes dealt. In fact, as shown in [8], these reductions suffered from the fact that they map optimal solutions to optimal solutions and, in this sense, it is very unlikely that they can allow completeness of a polynomially bounded problem in the classes dealt (unless $\mathbf{P}^{\mathrm{SAT}} = \mathbf{P}^{\mathrm{SAT}[\mathrm{O}(\log(n))]}$, where $\mathbf{P}^{\mathrm{SAT}}$ and $\mathbf{P}^{\mathrm{SAT}[\mathrm{O}(\log(n))]}$ are the classes of decision problems solvable by using, respectively, a polynomial and a logarithmic number of calls to an oracle solving SAT). On the contrary, PTAS-reductions, by allowing functions f and g to depend on ε, do not necessarily map optimal solutions between them and consequently, they allow that general problems are PTAS-reducible to polynomially bounded ones. Indeed, by means of a PTAS-reduction it is proved in [8] that MAX WSAT-B reduces to MAX WSAT-B with weights bounded by a polynomial in the size of the instance.

In [5] the power of PTAS-reduction has been further confirmed since it has allowed the achievement of **Poly-APX**-complete problems. This was not possible with the E-reduction under which only **Poly-APX-PB** completeness have been obtained in [10]. Given a family **F** of functions, denote by **F-APX** the subclass of **NPO** whose problems are approximable in polynomial time within ratio $g(n)$ (in the case of minimization), or $1/g(n)$ (for maximization), for a $g \in \mathbf{F}$. Here, we further confirm its scope by generalizing the result of [5], providing a way to handle complete problems under PTAS-reducibility, for any approximation class **F-APX**, where **F** is a class of polynomially bounded functions. This is of particular interest for the class **Log-APX**, since in [10] only **Log-APX-PB** completeness conditions have been established under the E-reduction.

We now recall some key notions that will be used in what follows. They are about additivity and canonical hardness of a maximization **NPO** problem and have been originally introduced in [10].

A problem $\Pi \in$ **NPO** is said *additive* if and only if there exist an operator $\oplus$ and a function f, both computable in polynomial time, such that:

- $\oplus$ associates with any pair $(x_1,x_2) \in \mathcal{I}_\Pi \times \mathcal{I}_\Pi$ an instance $x_1 \oplus x_2 \in \mathcal{I}_\Pi$ with $\mathrm{opt}(x_1 \oplus x_2) = \mathrm{opt}(x_1) + \mathrm{opt}(x_2)$;
- with any solution $y \in \mathrm{Sol}_\Pi(x_1 \oplus x_2)$, f associates two solutions $y_1 \in \mathrm{Sol}_\Pi(x_1)$ and $y_2 \in \mathrm{Sol}_\Pi(x_2)$ such that $m_\Pi(x_1 \oplus x_2, y) = m_\Pi(x_1,y_1) + m_\Pi(x_2,y_2)$.

A set **F** of functions from $\mathbb{N}$ to $\mathbb{N}$ will be called *downward close* if, for any function $g \in \mathbf{F}$ and any constant c, if $h(n) = \mathrm{O}(g(n^c))$, then $h \in \mathbf{F}$. A function $g : \mathbb{N} \to \mathbb{N}$ is *hard* for **F** if and only if, for any $h \in \mathbf{F}$, there exists a constant c such that $h(n) = \mathrm{O}(g(n^c))$; if, in addition, $g \in \mathbf{F}$, then g is said *complete* for **F**.

A maximization (resp., minimization) problem $\Pi \in$ **NPO** is *canonically hard for **F-APX***, for a downward close family **F** of functions, if and only if there exist a polynomially computable function f, two constants n_0 and c and a function F, hard for **F**, such that:

- for any instance φ of 3SAT on $n \geqslant n_0$ variables and for any $N \geqslant n^c$, $f(\varphi, N)$ belongs to $\mathcal{I}_\Pi$;
- if φ is satisfiable, then $\mathrm{opt}(f(\varphi,N)) = N$;
- if x is not satisfiable, then $\mathrm{opt}(f(\varphi,N)) = N/F(N)$ (resp., $NF(N)$);

- for any $y \in \mathrm{Sol}_{\Pi}(f(\varphi, N))$ such that $m(f(\varphi, N), y) > N/F(N)$ (resp., $m(f(\varphi, N), y) < NF(N)$), one can polynomially determine a truth assignment satisfying φ.

More generally, since 3SAT is **NP**-complete, a maximization (resp., minimization) problem Π is canonically hard for **F-APX** if and only if, for any decision problem $\Pi' \in$ **NP**, there exist a polynomially computable function f and two constants n_0 and c such that, given an instance x' of Π' of size greater than n_0, one can construct for any $N \geqslant |x'|^c$ an instance $x = f(x', N)$ of Π such that:

- if x' is a positive instance, then $\mathrm{opt}(x) = N$;
- if x' is a negative instance, then $\mathrm{opt}(x) = N/F(N)$ (resp. $NF(N)$);
- given a solution $y \in \mathrm{Sol}_{\Pi}(x)$ such that $m(x, y) > N/F(N)$ (resp., $m(x, y) < NF(N)$), one can polynomially determine a certificate proving that x' is a positive instance.

In [10], the following major theorem is proved, that constitutes the starting point for our work.

Theorem 1 (*Khanna et al. [10]*). *Let **F** be any class of downward close polynomially bounded functions and Ω be an additive maximization problem canonically hard for **F**. Then, any maximization problem $\Pi \in$ **F-APX-PB** (the class of problems in **F-APX** whose values are bounded by a polynomial with the size of the instance) E-reduces to Ω.*

We have already mentioned that it seems very unlikely that use of E-reducibility is able to establish completeness for the whole **F-APX** classes (at least for polynomially bounded problems). Thus, in order to extend Theorem 1 in order to capture them, we introduce in what follows a modification of PTAS-reducibility, called MPTAS-reducibility, M standing for multivalued, where function f is allowed to be multivalued. Formally, an MPTAS-reduction can be defined as follows.

Definition 1. Let Π and Π' be two maximization **NPO** problems (the case of minimization problems is completely analogous). Then, Π *MPTAS-reduces to* Π', if and only if there exist two functions f and g, computable in polynomial time, and a function c such that:

- for any $x \in \mathcal{I}_{\Pi}$ and any $\varepsilon \in]0, 1[$, $f(x, \varepsilon) = (x'_1, x'_2, \ldots, x'_M)$ is a family of instances of Π' (where M is polynomially bounded in $|x|$);
- for any $x \in \mathcal{I}_{\Pi}$, any $\varepsilon \in]0, 1[$ and any family of feasible solutions $y = (y_1, y_2, \ldots, y_M)$, where y_i is a feasible solution of x'_i, $g(x, y, \varepsilon) \in \mathrm{Sol}(x)$;
- $c :]0, 1[\rightarrow]0, 1[$;
- there exists an index j such that, for any $x \in \mathcal{I}_{\Pi}$ and any $\varepsilon \in]0, 1[$, $\rho_{\Pi'}(x'_j, y_j) \geqslant 1 - c(\varepsilon)$ implies $\rho_{\Pi}(x, g(x, y, \varepsilon)) \geqslant 1 - \varepsilon$.

It is easy to see that an MPTAS-reduction preserves membership in **PTAS**.

We show that using MPTAS-reducibility, **F-APX**-completeness can be extended for any class **F** of functions even if they are not polynomially bounded. Furthermore, the fact that function f in Definition 1 is allowed to be multivalued, relaxes the restriction to additive problems.

The approximation classes beyond **APX** dealt until now in the literature are defined with respect ratios depending on the size of the instances and not on other parameters even natural. This can be considered as somewhat restrictive given that a lot of approximation (and of inapproximability results) are established with respect to other parameters of the instances. Such parameters can be, for example, the maximum or the average degree of the input-graph, when dealing with graph-problems, the maximum set-cardinality, when dealing with problems on set-systems as the MIN SET COVER, or the MAX SET PACKING, etc. So, the third and last issue of this paper is to consider the class **Poly-APX(Δ)**, of maximization (resp., minimization) graph-problems approximable within ratios which are inverse polynomials (resp., polynomials) of the maximum degree Δ of the input-graph. We prove that MAX INDEPENDENT SET is complete for this class, under MPTAS.

2. F-APX-completeness for any downward close class of functions

We show in this section that, using MPTAS-reduction (Definition 1), we can apprehend completeness for any class **F-APX** even containing exponential functions. In a first time we shall deal with maximization problems. Extension to minimization problems will be performed after the proof of Theorem 2 that follows.

Theorem 2. *Let **F** be a class of downward close functions, and $\Omega \in$ **NPO** be a maximization problem canonically hard for **F-APX**. Then, any maximization **NPO** problem in **F-APX** MPTAS-reduces to Ω.*

Proof. Let Π be a maximization problem of **F-APX** and let $\mathtt{A}$ be an algorithm achieving approximation ratio $1/r(\cdot)$, $r \in \mathbf{F}$. Since $\Pi \in$ **NPO**, the value of the solutions for any instance $x \in \mathcal{I}_\Pi$ is bounded above by $2^{p(|x|)}$ for some polynomial p. Let Ω be as assumed in theorem's statement. Let F be a function hard for **F**, and k, n_0 and c' be constants such that, for $n \geqslant n_0$, $r(n) - 1 \leqslant k(F(n^{c'}) - 1)$. Finally, consider $x \in \mathcal{I}_\Pi$ and $\varepsilon \in (0, 1)$.

In order to build function $f(x, \varepsilon)$, claimed by Definition 1, we first partition the interval $[1, 2^{p(|x|)}]$ of the possible values for $\mathrm{opt}(x)$ into a polynomial number of subintervals:

$$\left[\left(\frac{1}{1-\varepsilon}\right)^{i-1}, \left(\frac{1}{1-\varepsilon}\right)^{i}\right], \quad i = 1, \ldots, M = \left\lceil p(|x|)\frac{\ln 2}{\ln\left(\dfrac{1}{1-\varepsilon}\right)}\right\rceil$$

(i.e., $(1/(1-\varepsilon))^M \geqslant 2^{p(|x|)}$). Consider then, for $i = 1, \ldots, M$, the languages $L_i = \{x \in \mathcal{I}_\Pi : \mathrm{opt}_\Pi(x) \geqslant (1/(1-\varepsilon))^{i-1}\}$. Set $N = |x|^{c'}$. Since Ω is canonically hard for **F-APX**, we can build, for any $i = 1, \ldots, M$, an instance $\omega_i \in \mathcal{I}_\Omega$ such that, if $x \in L_i$, then $\mathrm{opt}_\Omega(\omega_i) = N$, otherwise, $\mathrm{opt}_\Omega(\omega_i) = N/F(N)$. We set $f(x, \varepsilon) = \Upsilon = (\omega_i, 1 \leqslant i \leqslant M)$.

We now show how $g(x, y, \varepsilon)$ can be built. Consider a solution $y = (y_1, \ldots, y_M)$ in $\mathrm{Sol}(\Upsilon)$. Then, for any $i \in \{1, \ldots, M\}$, if $m_\Omega(\omega_i, y_i) > N/F(N)$, one can find a polynomial certificate proving that $x \in L_i$, i.e., a solution $y_i' \in \mathrm{Sol}(x)$ such that

$$m_\Pi(x, y_i') \geqslant \left(\frac{1}{1-\varepsilon}\right)^{i-1} \tag{1}$$

otherwise (i.e., if $m_\Omega(\omega_i, y_i) \leqslant N/F(N)$), we consider solution $\mathtt{A}(x)$. Finally, $g(x, y, \varepsilon)$ is the best among the solutions so produced.

Let us now prove that we really deal with an MPTAS-reduction. Let j be an index verifying:

$$\left(\frac{1}{1-\varepsilon}\right)^{j-1} \leqslant \mathrm{opt}_\Pi(x) \leqslant \left(\frac{1}{1-\varepsilon}\right)^{j} \tag{2}$$

and set $c(\varepsilon) = \varepsilon/(\varepsilon + k(1-\varepsilon))$.

Assume first that $m_\Omega(\omega_j, y_j) > N/F(N)$. Then, using (1) and (2), we get:

$$\frac{m_\Pi(x, g(x, y, \varepsilon))}{\mathrm{opt}_\Pi(x)} \geqslant 1 - \varepsilon.$$

On the other hand, if $m_\Omega(\omega_j, y_j) \leqslant N/F(N)$, then $\rho_\Omega(\omega_j, y_j) \geqslant 1 - c(\varepsilon)$ implies:

$$\frac{1}{F(N)} \geqslant \frac{k(1-\varepsilon)}{\varepsilon + k(1-\varepsilon)}. \tag{3}$$

By the assumptions made above, we have:

$$F(N) \geqslant 1 + \frac{r(|x|) - 1}{k}, \tag{4}$$

$$\rho_\Pi(x, g(x, y, \varepsilon)) \geqslant \frac{1}{r(|x|)}. \tag{5}$$

Using (3)–(5), we get the following implications:

$$\rho_\Omega(\omega_j, y_j) \geqslant 1 - c(\varepsilon) \Longrightarrow 1 + \frac{r(|x|) - 1}{k} \leqslant \frac{\varepsilon + k(1-\varepsilon)}{k(1-\varepsilon)} \Longrightarrow r(|x|) \leqslant 1 + k\left(\frac{\varepsilon}{k(1-\varepsilon)}\right) = \frac{1}{1-\varepsilon}$$

i.e., $\rho_{\Pi}(x, g(x, y, \varepsilon)) \geqslant 1 - \varepsilon$. So, the reduction exhibited is indeed a MPTAS-reduction and the proof of the theorem is complete. □

In order to extend the result of Theorem 2 to minimization problems, we are based upon an analogous result in [10]. Let Π be a maximisation problem in **F-APX** approximable within ratio $1/f(\cdot)$. One can define a minimization problem Π', identical to Π up to its objective function defined as: $m_{\Pi'}(x, y) = \lfloor 2M^2(x)/m_{\Pi}(x, y) \rfloor$, where $M(x)$ is an upper bound for the values in $\mathrm{Sol}(x)$. Then, Π E-reduces (hence, MPTAS-reduces also) to Π' and, furthermore, Π' is approximable within ratio $f(\cdot)(1+2/M(x))$. In completely analogous way, one can MPTAS-reduce a minimization problem, approximable within ratio $f(\cdot)$, to a maximization problem approximable within ratio $1/(f(\cdot)(1+2/M(x)))$. This allows the existence of additive canonically hard *minimization* problems also to be **F-APX**-complete, since the above transformation derives, from such a problem Π, an additive canonically hard maximization problem Π' (notice that $f(\cdot)$ and $f(\cdot)(1 + 2/M(x))$ have the same hardness status with respect to **F**). Then, it suffices that one applies Theorem 2 as stated. The remark above applies for any family **F**. As a consequence, for the rest of the paper, we restrict ourselves to maximization problems.

Theorem 3. *Let **F** be a class of downward close functions, and $\Omega \in$ **NPO** be a problem canonically hard for **F**. Then, any **NPO** problem in **F-APX** MPTAS-reduces to Ω.*

There exist two main differences between Theorems 1 and 3:
1. Theorem 3 applies to the whole of the problems of any class **F-APX** and not only to the bounded ones;
2. the use of a multivalued version of PTAS-reduction allows us to relax additivity from the conditions of Theorem 1 enlarging so the scope of the applicability of Theorem 3.

Another advantage of using MPTAS-reductions is that we can relax the first two items of the definition of canonical hardness. Indeed, dealing with maximization problems, we only need the following conditions:
- if x' is positive, then $\mathrm{opt}(x) \geqslant N$;
- if x' is negative, then $\mathrm{opt}(x) \leqslant N/F(N)$.

Consider now **Log-APX**, the class of **NPO** problems approximable with logarithmic (resp., inversely logarithmic when goal = max) ratios, i.e., within ratios $\mathrm{O}(\log(\cdot))$ (resp., $\mathrm{O}(1/\log(\cdot))$). MIN SET COVER, an instance of which is defined by a set-system $\mathcal{S}$ over a ground set C is approximable within ratio $\mathrm{O}(\log|C|)$ [15]; hence, it belongs to **Log-APX**. Furthermore, it is claimed in [3,13] that it is **NP**-hard to approximate MIN SET COVER within a factor $c \log|C|$, for some constant c.

In these reductions (not formally written yet but very widely considered as true and largely used in the literature), $|C|$, the number of the ground elements in the derived MIN SET COVER-instance, is, very likely, polynomially related to the size $|x|$ of the instance x of the "left-hand side problem" of the reduction, i.e., $|x|^d \leqslant |C| \leqslant |x|^{d'}$, for some suitable constants d and d' [16]. In other words, given an instance x of an **NP** problem, one can polynomially build an instance of MIN SET COVER, i.e., a set-system over a ground set C, introducing a gap of $c \log|C| \geqslant cd \log|x|$.

Consider some function $r \in$ **Log-APX**. For sufficiently large $|x|$, $r(|x|) \leqslant kcd \log|x| \leqslant kc \log|C|$ for a suitable constant k (or, equivalently, $r(|x|) - 1 \leqslant k(c \log|C| - 1)$). Using this inequality, one can apply the proof of Theorem 2 in order to derive the following result about the status of MIN SET COVER.

Theorem 4. MIN SET COVER *is **Log-APX**-complete under MPTAS-reducibility.*

Now, consider the class **Exp-APX**. The inapproximability result for MIN TSP in [14] (see also [9]) can immediately be extended for showing that MIN TSP is canonically hard for **Exp-APX**. Consequently, the **Exp-APX**-completeness of MIN TSP can be derived as an immediate corollary of Theorem 3.

Corollary 1. MIN TSP *is **Exp-APX**-complete under MPTAS-reducibility.*

3. Completeness under PTAS

In the previous section, we dealt with MPTAS reductions. In what follows, we will show that, if we deal with functions that are polynomially bounded, we can prove an equivalent of Theorem 3 but with respect to PTAS-reducibility. As noticed above, we can restrict ourselves to maximization problems.

Theorem 5. *Let* $\mathbf{F}$ *be any class of downward close polynomially bounded functions and* Ω *be an additive maximization problem canonically hard for* ***F-APX***. *Then, any maximization problem* $\Pi \in$ ***F-APX*** *PTAS-reduces to* Ω.

Proof. The proof goes along the lines of the corresponding proof of [5] for **Poly-APX**. Let Π be a maximization problem of **F-APX** and let $\mathtt{A}$ be an algorithm achieving approximation ratio $1/r(\cdot)$, $r \in \mathbf{F}$. Let Ω be as assumed in theorem's statement. Let F be a function hard for $\mathbf{F}$, and k, n_0 and c' be constants such that, for $n \geqslant n_0$, $r(n) \leqslant k(F(n^{c'}) - 1)$. Finally, consider $x \in \mathcal{I}_\Pi$, $\varepsilon \in (0, 1)$ and set $|x| = n$.

We first construct function $f(x, \varepsilon)$. Set $m(x, \mathtt{A}(x)) = m \geqslant \mathrm{opt}_\Pi(x)/r(n)$. Partition the interval $[0, mr(n)]$ of the possible values for $\mathrm{opt}_\Pi(x)$ into $q(n)$ regular subintervals, where $q(n)$ is bounded by polynomial of n. More precisely, we set $q(n) = r(n)/\varepsilon$. Consider now, for $i \in \{1, \ldots, q(n)\}$, the sets of languages $L_i = \{x \in \mathcal{I}_\Pi : \mathrm{opt}_\Pi(x) \geqslant imr(n)/q(n)\}$ and set $N = n^{c'}$. Since Ω is canonically hard for **F-APX**, one can build, for any i, an instance $\omega_i \in \mathcal{I}_\Omega$ such that, if $x \in L_i$, then $\mathrm{opt}_\Omega(\omega_i) = N$ and if $x \notin L_i$, then $\mathrm{opt}_\Omega(\omega_i) = N/F(N)$. We define $f(x, \varepsilon) = \Upsilon = \bigoplus_{1 \leqslant i \leqslant q(n)} \omega_i$. Observing that $r(n)/q(n) = \varepsilon$, we get:

$$\mathrm{opt}_\Omega(\Upsilon) = N|\{i \leqslant q(n) : \mathrm{opt}_\Pi(x) \geqslant im\varepsilon\}| + \frac{N}{F(N)}(q(n) - |\{i \leqslant q(n) : \mathrm{opt}_\Pi(x) \geqslant im\varepsilon\}|). \tag{6}$$

We now construct function $g(x, y, \varepsilon)$. Consider a solution $y \in \mathrm{Sol}(\Upsilon)$. By the additivity of Ω, one can compute, for any ω_i, a solution y_i in such a way that $m_\Omega(\Upsilon, y) = \sum_i m_\Omega(\omega_i, y_i)$. Let j be the largest among the indices i such that $m_\Omega(\omega_i, y_i) > N/F(N)$ ($j = 0$, if no i verifies the inequality). Then, one can find a polynomial certificate proving that $x \in L_j$, i.e., a solution $y_1' \in \mathrm{Sol}(x)$ verifying: $m_\Pi(x, y_1') \geqslant jm\varepsilon$. Furthermore, $m_\Omega(\Upsilon, y) \leqslant Nj + (q(n) - j)N/F(N)$.

We then define $g(x, y, \varepsilon) = y'$ as the best (the largest value one) among y_1' and $y_2' = \mathtt{A}(x)$. Obviously, $m_\Pi(x, y') \geqslant \max\{m, jm\varepsilon\}$.

Let us now prove that the reduction described is indeed a PTAS-reduction. We first notice that, using expressions for m_Π and m_Ω, the following inequality is derived (after some calculation) for the approximation ratio $\rho_\Omega(\Upsilon, y)$, using that $q(n) = r(n)/\varepsilon \leqslant k(F(N) - 1)/\varepsilon$:

$$\rho_\Omega(\Upsilon, y) \leqslant \frac{j + k/\varepsilon}{\mathrm{opt}_\Pi(x)/\varepsilon m - 1 + k/\varepsilon}$$

We now consider two cases, namely, $j \leqslant 1/\varepsilon$ and $j \geqslant 1/\varepsilon$. For the first one, $j \leqslant 1/\varepsilon$, we get from the expression for ρ_Ω (after some algebra and using also that $\rho_\Pi(x, y') \geqslant m/\mathrm{opt}_\Pi(x)$):

$$\rho_\Omega(\Upsilon, y) \leqslant \frac{\rho_\Pi(x, y')(1 + k)}{1 + \rho_\Pi(x, y')(k - \varepsilon)}.$$

On the other hand, if $j \geqslant 1/\varepsilon$, noticing that, using expression for m_Π, $\rho_\Pi(x, y') \geqslant jm\varepsilon/\mathrm{opt}_\Pi(x)$, we also get:

$$\rho_\Omega(\Upsilon, y) \leqslant \frac{\rho_\Pi(x, y')(1 + k)}{1 + \rho_\Pi(x, y')(k - \varepsilon)}.$$

Assume $c(\varepsilon) = \varepsilon^2/(1 + (1 - \varepsilon)(k - \varepsilon))$. Then, after some algebra one gets: $\rho_\Pi(x, y') \geqslant 1 - \varepsilon$, proving so that the reduction just devised is indeed a PTAS-reduction and completing the proof of the theorem. □

4. Completeness in F-APX(Δ)

We now deal with a new approximation class, **F-APX(Δ)**, namely, the class of minimization (resp. maximization) graph-problems approximable with ratio $f(\Delta)$ (resp., $1/f(\Delta)$), where Δ is the degree of the input-graph, i.e., the maximum degree of its vertices, $\mathbf{F}$ a downward close class of functions and $f \in \mathbf{F}$.

Definition 2. Let $\mathbf{F}$ be a downward close class of functions and $\Pi \in$ **NPO** be a maximization (resp., minimization) graph-problem. Then, Π is said *canonically hard* for **F-APX(Δ)** if, for any problem $\Pi' \in$ **NP** there exist three functions

f, α and β, computable in polynomial time, a constant Δ_0 and a function F, hard for **F** such that:
- for any instance x of Π' and any $\Delta \geqslant \Delta_0$, $G_x = f(x, \Delta)$ is a graph (instance of Π) of maximum degree Δ;
- if x is a positive instance, then $\text{opt}_\Pi(G_x) \geqslant \alpha(G_x)$ (resp., $\text{opt}_\Pi(G_x) \leqslant \alpha(G_x)$);
- if x is a negative instance, then $\text{opt}_\Pi(G_x) \leqslant \beta(G_x)$ (resp., $\text{opt}_\Pi(G_x) \geqslant \beta(G_x)$);
- $\alpha(G_x)/\beta(G_x) \geqslant F(\Delta)$ (resp., $\beta(G_x)/\alpha(G_x) \geqslant F(\Delta)$);
- given a solution $y \in \text{Sol}(G_x)$ of value greater (resp., smaller) than $\beta(G_x)$, one can determine in polynomial time a witness showing that x is a positive instance.

For the same reasons as previously, we will restrict ourselves to maximization problems and are going to prove the following theorem.

Theorem 6. *If **F** is a downward close family of functions and Ω is a maximization problem canonically hard for **F-APX**($\boldsymbol{\Delta}$), then any problem in **F-APX**($\boldsymbol{\Delta}$) MPTAS-reduces to Ω.*

Proof. Consider a maximization problem $\Pi \in$ **F-APX**($\boldsymbol{\Delta}$), a graph $G \in \mathcal{I}_\Pi$ of order n and an algorithm $\mathtt{A}$ for Π achieving approximation ratio $1/r(\Delta(G))$, where $r \in$ **F**. The proof goes along the lines of Theorem 2.

In order to construct function $f(G, \varepsilon)$ (Definition 1), we partition the interval $[1, 2^{p(n)}]$ in the same way as in the proof of Theorem 2, and we consider, for any i, the analogous sets L_i of languages. We set $\Delta = \Delta^{c'}(G)$ and we build, for any i, an instance H_i of Ω, of maximum degree Δ such that:
- if $G \in L_i$, then $\text{opt}_\Omega(H_i) \geqslant \alpha(H_i)$,
- otherwise, $\text{opt}_\Omega(H_i) \leqslant \beta(H_i)$

with $\alpha(H_i)/\beta(H_i) \geqslant F(\Delta)$. Finally, we set $f(G, \varepsilon) = H = (H_i, 1 \leqslant i \leqslant M)$.

Function $g(G, y, \varepsilon)$ is defined as in the proof of Theorem 2, setting $\beta(H_i)$ instead of $N/F(N)$. The proof for the transfer of the approximation ratios is also done in the same way as in Theorem 2 (with $\beta(H_i)$ instead of $N/F(N)$). □

Denote by **Poly-APX**($\boldsymbol{\Delta}$) the subclass of the graph-problems in **NPO** which are approximable within polynomials of Δ^{-1} (of Δ when dealing with minimization problems). We are going now to establish an interesting completeness result for this class by showing that one of the most paradigmatic problems for the polynomial approximation theory and the combinatorial optimization, the MAX INDEPENDENT SET, is complete for **Poly-APX**($\boldsymbol{\Delta}$). For this we will use the following theorem [1].

Theorem 7 (*Alon et al. [1]*). *Let $(a, b) \in [0, 1]^2$, $a > b$. There exists $\varepsilon_0 > 0$ such that, for any $\Delta \geqslant 3$, there exists a function f that transforms an instance G of* MAX INDEPENDENT SET *into an instance of* MAX INDEPENDENT SET-Δ, *i.e., into a graph with degree bounded by Δ, and two positive functions α and β such that*:
- *if* $\text{opt}(G) \geqslant an$, *then* $\text{opt}(f(G)) \geqslant \alpha(f(G))$;
- *if* $\text{opt}(G) \leqslant bn$, *then* $\text{opt}(f(G)) \leqslant \beta(f(G))$;
- $\alpha(f(G))/\beta(f(G)) \geqslant \Delta^{\varepsilon_0}$.

Thanks to the **PCP** theorem, stating that there exist a and b such that it is **NP**-complete to distinguish graphs with maximum independent set with size at least a times their order, from graphs having maximum independent set at most b times their order, the following corollary can be derived [1].

Corollary 2 (*Alon et al. [1]*). *There exists an ε_0 such that, for any $\Delta \geqslant 3$,* MAX INDEPENDENT SET-Δ *is not approximable within ratio $\Delta^{-\varepsilon_0}$.*

In order to get the canonical hardness of MAX INDEPENDENT SET-Δ, we have to strengthen slightly Theorem 7 as follows.

Lemma 1. *Let $(a, b) \in [0, 1]^2$, $a > b$. There exists $\varepsilon_0 > 0$ such that, for any $\Delta \geqslant 3$, there exists a function f that transforms an instance G of* MAX INDEPENDENT SET *into an instance of* MAX INDEPENDENT SET-Δ, *i.e., into a graph*

with degree bounded by Δ, and two positive functions α and β such that:
1. *if* $\mathrm{opt}(G) \geqslant an$, *then* $\mathrm{opt}(f(G)) \geqslant \alpha(f(G))$;
2. *if* $\mathrm{opt}(G) \leqslant bn$, *then* $\mathrm{opt}(f(G)) \leqslant \beta(f(G))$;
3. $\alpha(f(G))/\beta(f(G)) \geqslant \Delta^{\varepsilon_0}$;
4. *given an independent set of $f(G)$ of size greater than $\beta(G)$, one can determine, in polynomial time, an independent set of G of size greater than bn*;
5. *the complexity of f is polynomial in Δ.*

Proof. For legibility, we first sketch the proof of Theorem 7 in [1]. Given a graph $G(V, E)$ of order n, consider a subset V' of V^k of size nc^k, for some constant c, where $k = \mathrm{O}(\log \Delta)$. Set V' is the set of vertices of the *derandomized graph product* $f(G)$ of G. Then, two vertices $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ are linked in $f(G)$ if and only if the set $(u_1, \ldots, u_k, v_1, \ldots, v_k)$ does not form an independent set in G. Hence, in particular, if a set of k-tuples forms an independent set in $f(G)$, then the union of this k-tuples forms an independent set in G. The construction of V' is based on walks on a particular expander graph, chosen (together with k) in such a way that $f(G)$ is of degree Δ, and that $\mathrm{opt}(f(G))$ and $\mathrm{opt}(G)$ are strongly related.

More precisely, if S is an independent set in V, then the number t of elements in V' that are in S^k verifies $h(|S|) \leqslant t \leqslant l(|S|)$ for some particular functions h and l. Since these t elements form an independent set in $f(G)$: if $\mathrm{opt}(G) \leqslant bn$, then $\mathrm{opt}(f(G)) \leqslant l(bn)$; on the other hand, if $\mathrm{opt}(G) \geqslant an$, then $\mathrm{opt}(f(G)) \geqslant h(an)$. Moreover, h and l are such that $h(an)/l(bn) \geqslant \Delta^{\varepsilon}$, for some constant ε.

Item 4 follows from the correspondence between independent sets in G and in $f(G)$. Given an independent set S' of $f(G)$ (of size greater than $l(bn)$), one can find an independent set S of G (of size greater than bn) by simply considering the union of the k-tuples of S'.

Finally, f is clearly polynomial in Δ since the number of vertices is nc^k (where c is a constant) which is polynomial even for $\Delta = n$ (since $k = \mathrm{O}(\log \Delta)$). □

We will also use the following proposition, derived from an immediate application of the **PCP** theorem.

Proposition 1. *There exists $(a, b) \in [0, 1]^2$, $a > b$, such that, for any $\Pi \in$ **NP** there exists a function f which transforms an instance x of Π into a graph $G = f(x)$, instance of* MAX INDEPENDENT SET, *that verifies*:
1. *if x is positive*, *then* $\mathrm{opt}(G) \geqslant an$;
2. *if x is negative*, *then* $\mathrm{opt}(G) \leqslant bn$;
3. *given an independent set S in G with size $|S| > bn$, one can determine, in polynomial time, a witness proving that x is positive.*

Proposition 1, is an approximation preserving transfer, by say an L-reduction [12], of an analogous result dealing with MAX 3-SAT (appearing in [2]) to MAX INDEPENDENT SET.

Combination of Lemma 1 and Proposition 1, leads immediately to the following result.

Theorem 8. *There exists an $\varepsilon_0 > 0$ such that, for any $\Delta \geqslant 3$, given $\Pi \in$ **NP**, there exists a function f computable in polynomial time, that transforms an instance x of Π into a graph $G = f(x)$, instance of* MAX INDEPENDENT SET, *and two positive functions α and β verifying*:
1. *if x is positive*, *then* $\mathrm{opt}(G) \geqslant \alpha(G)$;
2. *if x is negative*, *then* $\mathrm{opt}(G) \leqslant \beta(G)$;
3. *the maximum degree of G is Δ*;
4. $\alpha(G)/\beta(G) \geqslant \Delta^{\varepsilon_0}$;
5. *given an independent set S of G such that $|S| > \beta(G)$, one can build, in polynomial time, a certificate proving that x is positive.*

Consider now **Poly-APX(Δ)** and MAX INDEPENDENT SET-Δ. Using item 5 of Lemma 1, we can apply Theorem 6. So, the following theorem concludes the paper.

Theorem 9. MAX INDEPENDENT SET-Δ *is **Poly-APX(Δ)**-complete under MPTAS-reducibility.*

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