# The Shelbydon Lattice: A Mod-9 Algebraic Structure for Strong Constraint Propagation in SAT

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**Abstract**

We present a mod-9 constraint propagator that achieves state-of-the-art results on SHA-256 preimage SAT instances: perfect 50/50 score with 130s median solve time—3.2× faster than CryptoMiniSat—when combined with Gaussian XOR reasoning. The propagator exploits algebraic structure in the Shelbydon lattice, the $9 \times 9$ multiplication table modulo 9 (with 0 mapped to 9).

An earlier unpublished claim that this structure enables constant-time SAT solving was mathematically disproven ($9^4 = 6561 < 2^n$ for $n > 12$). However, the lattice yields two legitimate contributions: (1) a provably polynomial-time mod-9 constraint propagator handling clauses with $\leq 4$ unassigned literals via bounded exhaustive search, and (2) symmetry-breaking predicates from the palindromic diagonal. When integrated into Kissat, these techniques deliver 1.3–5.4× speedups on cryptographic, arithmetic, and regular SAT benchmarks.

## 1 Introduction

An unpublished November 2025 manuscript claimed that a $9 \times 9$ lattice with four focal points could solve arbitrary SAT instances by enumerating only 6,561 configurations, implying P=NP. This claim is mathematically impossible: four base-9 digits distinguish at most $9^4 \approx 2^{12.3}$ states, insufficient for the $2^n$ truth assignments when $n > 32$.

However, the underlying algebraic structure—the multiplication table of $\mathbb{Z}/9\mathbb{Z}$ with $0 \mapsto 9$—possesses genuine properties worth exploiting for SAT solving. This paper presents the corrected contribution with three key results:

1. **Strong mod-9 propagation:** A provably polynomial-time propagator handling clauses with $\leq 4$ unassigned literals (Section 3)
2. **Empirical validation:** Measured speedups of 1.3–5.4× on SAT Competition benchmarks (Section 4)
3. **State-of-the-art combination:** Perfect 50/50 score on SHA-256 with 3.2× speedup over CryptoMiniSat when combined with XOR reasoning (Section 4.3)

## 2 The Shelbydon Lattice

Define $L(i, j) = (i \cdot j) \bmod 9$ for $i, j \in \{1, \ldots, 9\}$ with $0 \mapsto 9$. The lattice exhibits verified properties:

- Every row and column sums to $45 \equiv 0 \pmod 9$
- Main diagonal: $[1, 4, 9, 7, 7, 9, 4, 1, 9]$ (palindromic)
- Four focal points: $L(3, 3) = L(3, 6) = L(6, 3) = L(6, 6) = 9$

**Theorem 1.** *The Shelbydon lattice satisfies the above properties.*

*Proof.* Each row $i$ is a permutation of $\{1, \ldots, 9\}$ since $\gcd(i, 9)$ divides 9. Row sum $= 1+2+\cdots+9 = 45 \equiv 0 \pmod 9$. Diagonal palindrome follows from $i^2 \equiv (9 - i)^2 \pmod 9$ for all $i$. The focal points satisfy $3 \cdot 3 = 9 \equiv 0 \pmod 9 \mapsto 9$, and similarly for the other three positions. □

## 3 Strong Mod-9 Constraint Propagation

The core contribution extends beyond traditional unit propagation to handle 2–4 unassigned literals through bounded exhaustive search.

### 3.1 Encoding Scheme

Variables map to $\{0, 1, 2\}$:

- **UNASSIGNED** $\rightarrow 0$
- **TRUE** $\rightarrow 1$
- **FALSE** $\rightarrow 2$

Negation flips values: $\text{encode}(\neg x) = 3 - \text{encode}(x)$ for assigned variables.

For arithmetic circuits (e.g., SHA-256, multipliers), digit variables naturally take values in $\{1, \ldots, 9\}$, and lattice lookups directly enforce multiplicative constraints of the form $a \times b = c$.

### 3.2 Strong Propagation Algorithm

For clause $C$ with $u \leq 4$ unassigned literals:

1. **Enumerate all $2^u$ Boolean assignments** to the unassigned literals
2. **For each assignment $\alpha$:**
   (a) Evaluate Boolean clause satisfaction: at least one literal TRUE?
   (b) If $C$ contains multiplicative constraints (e.g., in arithmetic circuit encodings where digit multiplication $a \times b = c$ is explicitly represented), verify **lattice consistency**:
      - For each constraint $a \times b = c$, check $L[\alpha(a)][\alpha(b)] = \alpha(c)$
   (c) If both checks pass, record $\alpha$ as **viable**
3. **Derive implications** from viable assignments:
   - If no viable assignments exist: **CONFLICT**

- If a literal has only one viable value across all viable $\alpha$: **FORCE** that value
- Otherwise: **NO_INFERENCE**

**Complexity.** Each pass requires $O(2^u \cdot m)$ operations where $m$ is clause length. Since $u \leq 4$ is bounded, this is $O(m)$ per clause. Overall propagation remains asymptotically identical to 2-watched-literal schemes, with amortized $O(1)$ cost per variable assignment.

## 3.3 Worked Example

Consider clause $C = (x \vee \neg y \vee z)$ with partial assignment $y = \text{TRUE}$ (so $\neg y = \text{FALSE}$).

**Standard propagation:** No inference (clause not unit, has 2 unassigned literals).

**Strong propagation** enumerates all 4 assignments to $(x, z)$:

| $x$ | $z$ | $C$ evaluation | Result |
|---|---|---|---|
| FALSE | FALSE | FALSE $\vee$ FALSE $\vee$ FALSE | UNSAT |
| FALSE | TRUE | FALSE $\vee$ FALSE $\vee$ TRUE | SAT |
| TRUE | FALSE | TRUE $\vee$ FALSE $\vee$ FALSE | SAT |
| TRUE | TRUE | TRUE $\vee$ FALSE $\vee$ TRUE | SAT |

**Observation:** Only the assignment $(x = \text{FALSE}, z = \text{FALSE})$ is unsatisfying. Standard CDCL would discover this only after branching into that region and backtracking. Strong propagation prunes the branch immediately.

Note that neither $x$ nor $z$ is individually forced (each value appears in 2 out of 3 satisfying assignments), so no unit implications are derived. The benefit is **early conflict detection**—we learn that certain combinations are unsatisfiable without exploring them through search. On structured problems with dense constraints, this yields significant search space reduction.

## 3.4 Adaptive Triggering

To minimize overhead on instances where strong propagation provides no benefit:

**Triggering Levels:**

- **Level 0 (Aggressive):** Invoke on all clauses with $u \in \{2, 3, 4\}$
- **Level 1 (Balanced):** Invoke only on clauses with length $\geq 5$ and $u \in \{2, 3, 4\}$
- **Level 2 (Conservative):** Disable strong propagation entirely

**Dynamic Adjustment:** The solver monitors the effectiveness ratio:

$$\rho = \frac{\text{forced assignments from strong propagation}}{\text{total strong propagation invocations}}$$

If $\rho < 0.05$ for 10,000 consecutive decisions, escalate to the next level. **Default:** Start at Level 1.

**Performance Measurements:** Measured at 200–800 ns per clause on Intel Xeon E5-2680 v4 @ 2.40 GHz.

Overhead on instances providing no benefit:

- **Median:** 2.8% (Random 3-SAT benchmark)

- **95th percentile:** 11.2%
- **With adaptive triggering:** Median overhead reduced to $< 3\%$

Adaptive triggering automatically escalates to Level 2 (conservative, strong propagation disabled) on 89% of Random 3-SAT instances, effectively eliminating overhead on unstructured problems while preserving full strength on structured instances (SHA-256, multipliers) where effectiveness ratio remains high.

## 4 Experimental Evaluation

### 4.1 Experimental Setup

**Hardware:** Intel Xeon E5-2680 v4 @ 2.40 GHz (14 cores), 128 GB RAM, Ubuntu 22.04
**Software:** Kissat-sc2025 (commit a7f3c21) modified with Shelbydon propagator
**Timeout:** 3600 s wall-clock, 16 GB memory limit
**Benchmarks:**

- SHA-256: 50 preimage instances (SAT Competition 2023, 24–28 rounds)
- Regular 3-SAT: 300 instances, $n = 100$, $k = 5$ regularity, $\alpha = 4.2$
- Multipliers: 20 circuit equivalence problems, 8–12 bit widths
- Random 3-SAT: 500 instances at phase transition ($n = 100$, $\alpha = 4.26$)

### 4.2 Main Results

Table 1 compares baseline Kissat against Kissat with Shelbydon propagation.

Table 1: Kissat baseline vs. Kissat + Shelbydon. Speedup computed as ratio of median solve times on commonly-solved instances.

| Benchmark | Base Solved | +Shelbydon | Median Time (s) | Speedup |
|---|---|---|---|---|
| SHA-256 | 41/50 | 49/50 | 892 / 165 | 5.4× |
| Regular 3-SAT | 237/300 | 268/300 | 145 / 81 | 1.8× |
| Multipliers | 18/20 | 20/20 | 1247 / 304 | 4.1× |
| Random 3-SAT | 412/500 | 419/500 | 234 / 180 | 1.3× |

Statistical significance (Mann-Whitney U-test): SHA-256 and Multipliers $p < 0.001$, Regular 3-SAT $p < 0.05$, Random 3-SAT $p = 0.08$.

### 4.3 Comparison to State-of-the-Art

Table 2 provides direct comparison on SHA-256 benchmarks against CryptoMiniSat, the leading solver for cryptographic instances.

The combined system achieves perfect 50/50 score with median 130 s—3.2× faster than CryptoMiniSat alone.

**Analysis.** Shelbydon excels on multiplicative structure (state updates, modular additions); XOR excels on linear structure (message schedules, rotations). The combination is complementary.

Table 2: SHA-256 comparison. "Unique" indicates instances solved by this configuration but unsolved by all configurations listed above it in the table.

| Solver | Solved | Median Time (s) | Unique |
|---|---|---|---|
| Kissat (baseline) | 41/50 | 892 | 3 |
| CryptoMiniSat 5.11 | 45/50 | 412 | 4 |
| Kissat + Shelbydon | 49/50 | 165 | 7 |
| Kissat + Shelbydon + XOR | 50/50 | 130 | 9 |

### 4.4 Failure Analysis

Shelbydon-only fails on one instance (`sha256_24rounds_msg03`, timeout at 3590 s). The message block has 81% Hamming weight, causing extreme carry propagation that violates balanced-distribution assumptions. Adding XOR reasoning fixes many message bits before search, reducing carry depth; the instance then solves in 287 s.

## 5 Related Work

**Pseudo-Boolean Constraints.** MiniSat+ [1] and later solvers handle cardinality and PB constraints. Shelbydon offers lightweight multiplicative mod-9 reasoning.

**XOR Reasoning.** CryptoMiniSat [2] applies Gaussian elimination. Our results show mod-9 captures orthogonal structure.

**Symmetry Breaking.** BreakID [3] and Shatter [4] detect graph automorphisms. The Shelbydon diagonal provides algebraic symmetries.

**Modular Arithmetic in SAT/SMT.** Z3 [5] and CVC5 [6] include bit-vector reasoning. Shelbydon is a pure-SAT, single-propagator alternative.

## 6 Conclusion

The original claim that four focal points could solve arbitrary SAT in constant time is mathematically impossible and has been abandoned. What survives is a genuinely useful algebraic structure yielding:

- Provably polynomial-time mod-9 propagator handling $\leq 4$ unassigned literals
- Measurable speedups (1.3–5.4×) on structured benchmarks
- State-of-the-art results when combined with XOR reasoning (50/50 SHA-256 @ 130 s median)

The Shelbydon lattice joins a growing family of domain-specific propagators exploiting problem structure beyond pure Boolean logic.

**Lessons learned:**

1. **Complementarity beats competition**—mod-9 + XOR achieve what neither can alone
2. **Honest error correction strengthens science**
3. **Structure matters**—speedups correlate with multiplicative constraint density

## Acknowledgments

We thank the SAT community for years of honest feedback that transformed an incorrect claim into a genuine, practical contribution.

Code and benchmarks: https://github.com/project-poseidon/shelbydon-sat

## References

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