3. Bayesian Analysis

$P(X=H)=\theta$, outcomes: HHH

$D=\{X_1,\dots,X_n\}$, $X_i = 1$ if heads, $X_i=0$ if tails

$X_i \sim \text{Bernoulli}(\theta)$

let $p=\theta$

$$L(p) = \prod_{i=1}^{n} p(x_i|p) = \prod p^{x_i}(1-p)^{1-x_i}$$

$$\ln(L(p)) = \sum_{i=1}^{n} \ln\left[p^{x_i}(1-p)^{1-x_i}\right] = \sum_{i=1}^{n} x_i \ln(p) + (1-x_i)\ln(1-p)$$

$$\frac{d}{dp}\ln[L(p)] = \sum \frac{x_i}{p} + \frac{x_i-1}{1-p}$$

$$= \frac{\sum x_i}{p} + \frac{\sum x_i}{1-p} - \frac{\sum(1)}{1-p}$$, $\sum_{i=1}^{n} x_i$ counts number of heads

$\therefore$ let n_1 = number of heads

$$0 = \frac{n_1}{p} + \frac{n_1}{1-p} - \frac{n}{1-p}$$

$$= n_1\left(\frac{1}{p}+\frac{1}{1-p}\right) - \frac{n}{1-p}$$

$$= n_1\left(\frac{1}{p(1-p)}\right) - \frac{pn}{p(1-p)} = 0$$

$$n_1 = pn$$

$$\hat{p} = \frac{n_1}{n} \equiv \hat{\theta}_{MLE} = \frac{n_1}{n}$$

MLE would make a poor estimator for this coin flipping scenario:

observations: HHH

$\therefore n = 3, \quad n_1 = 3$

$$\hat{\theta} = \frac{n_1}{n} = \frac{3}{3} = 1.$$

Thus, MLE would be poor for predictions, as no prior convictions is considered.

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)} = \frac{p(x|\theta)p(\theta)}{\int_{\theta' \in \Theta} p(x|\theta')p(\theta')d\theta}$$

$$p(x|\theta) = \theta^h (1-\theta)^{n-h}$$ n = tosses, h = # heads

$$p(\theta) = \frac{1}{B(\alpha;\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$ $\sim Beta(\theta;\alpha,\beta)$ prior

where $$B(\alpha;\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt.$$

Altogether:

$$p(\theta|x) = \frac{\theta^h(1-\theta)^{n-h} \cdot \frac{1}{B(\alpha;\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\int_0^1 p(x|\theta')p(\theta')d\theta}$$

since $\theta \in [0,1]$

posterior:

$$p(\theta|x) = \frac{\theta^{h+\alpha-1}(1-\theta)^{(n-h+\beta-1)}}{\int_0^1 \theta'^{\,h+\alpha-1}(1-\theta')^{n-h+\beta-1}\,d\theta'}$$

$$\sim \text{Beta}(\theta;\ h+\alpha,\ n-h+\beta)$$

Mean of Beta: Mean of θ

$$\frac{\alpha}{\alpha+\beta} \implies \frac{h+\alpha}{h+\alpha+n-h+\beta}$$

In our example: $n=3$, $h=3$, $\alpha=50$, $\beta=50$

$$\frac{3+50}{3+50+3-3+50} = \frac{53}{103} \approx 0.51$$

Mode of Beta:

$$\frac{\alpha-1}{\alpha+\beta-2} \longrightarrow \frac{(h+\alpha)-1}{(h+\alpha)+(n-h+\beta)-2}$$

In our example

$$\frac{3+50-1}{(3+50)+(3-3+50)-2} = \frac{52}{101} \approx 0.51$$

Plot: See below

No, $\hat{\theta}_{MLE}$ predicts $\frac{3}{3} = 1.00$

With $n=3$, $h=3$, $\alpha=\beta=50$, $p(0.99 \le \theta \le 1.00|x) \approx 0$

This is not a good summary of distribution

Q3plot

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1 Comp 652 A1Q3

```
In [2]: import numpy as np
        from scipy.stats import beta
        from matplotlib import pyplot as plt

In [3]: a = 3 + 50 # posterior alpha -> 3 heads + 50
        b = 3 - 3 + 50 # posterior beta -> 3 tosses - 3 heads + 50

In [4]: dist = beta(a,b)
        x = np.arange(0,1,0.001)

In [5]: y = [dist.pdf(val) for val in x]

In [8]: plt.plot(x,y)
        plt.show()
```