1. a)

Linear Stochastic Bandits:

$X \subseteq \mathbb{R}^n$

$\theta_* \in \mathbb{R}^n$

$f(x) = \langle x, \theta_* \rangle$

$r_t = \langle x_t, \theta_* \rangle + \varepsilon$ R-subgaussian

Gaussian environment:

$$\mathcal{E}_N^k : \left\{ N(\mu_i, \sigma_i^2) \mid \mu \in \mathbb{R}^k, \sigma^2 \in [0, \infty)^k \right\}$$

Stochastic:

$k = \{1, \dots, k\}$ actions

$v = (P_1, \dots, P_k)$

$\mu_k(v) \rightarrow$ expectation of P_k in config v.

$\hat{\mu}_k(t)$

For each round t:

Select $k_t \in k$

play k_t

observe $r_t \sim N(\mu_k, \sigma_k^2)$

Now, $r_t = \langle x_t, \theta_* \rangle + \varepsilon$, ε is R-subgaussian

$N(0, \sigma^2)$ is σ-subgaussian

Proof) $X \sim (0,1)$

$$E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} = e^{t^2/2} \leq e^{\frac{\sigma^2 t^2}{2}}$$

$\Rightarrow$ σ-subgaussian, $\sigma = 1$.

$$Y \sim \mathcal{N}(0, \sigma^2) \Rightarrow Y/\sigma \sim \mathcal{N}(0,1)$$

$$\Rightarrow E\left[e^{tY}\right] = E\left[e^{t\sigma X}\right] \quad \text{let } t' = \sigma t$$

$$\Rightarrow E\left[e^{t'X}\right] = e^{t'^2/2} = e^{\sigma^2 t^2/2} \leq e^{\frac{\sigma^2 t^2}{2}}$$

$\Rightarrow \sigma$-subgaussian. ■

Thus, $r_t - \mu_n \sim \mathcal{N}(0, \sigma^2)$

$$\Rightarrow r_t = \mu_n + \varepsilon^{\sim \sigma\text{-subgaussian}}$$

Altogether:

Linear Stochastic Bandit

$\mathcal{X} \subseteq \mathbb{R}^k$, $X = \{\vec{I}_1, \ldots, \vec{I}_k\}$ where

$\vec{I}_i$ is a vector of length K, with 0's for all entries except for a 1 at entry i

At each round t:

1. Select action $X_t \in X$
2. Play action X_t.
3. Observe reward:

$$r_t = \langle X_t, \theta_* \rangle + \varepsilon_t^{\sim \mathcal{N}(0, \sigma_{X_t}^2)}$$

where $\theta_* = (\mu_1, \ldots, \mu_k) \in \mathbb{R}^k$

$= \mu$

b) Regret:

$$R_T = \sum_{t=1}^{T} \langle x_*, \theta_* \rangle - \sum_{t=1}^{T} \langle x_t, \theta_* \rangle$$

Suppose we have $K = \{1, 2, 3, 4, 5\}$, where arm 5 has the largest mean.

$$\therefore R_T = \sum_{t=1}^{T} \langle [0,0,0,0,1], [\mu_1, \mu_2, \mu_3, \mu_4, \mu_5] \rangle$$

$$- \sum_{t=1}^{T} \langle x_t, \theta_* \rangle$$

$$= R_T = \sum_{t=1}^{T} \mu_5 - \sum_{t=1}^{T} \langle x_t, \theta_* \rangle$$

$$= T\mu_5 - \sum_{t=1}^{T} \langle x_t, \theta_* \rangle$$

Now, let's look at $\sum_{t=1}^{T} \langle x_t, \theta_* \rangle$

at time t, we chose action $x_j = \vec{I}_j$

$\Rightarrow \langle x_j, \theta_* \rangle = \langle \vec{I}_j, \theta_* \rangle = \mu_j$.

This can be reformulated as, at time t:

let k_t = index j of $x_j \equiv$ we pulled arm j

$\Rightarrow \langle x_j, \theta_* \rangle = \mu_{k_t}$

Thus, $R_T = \sum_{t=1}^{T} \langle x_*, \theta_* \rangle - \sum_{t=1}^{T} \langle x_t, \theta_* \rangle$ (linear setting regret)

$$= T\mu_* - \sum_{t=1}^{T} \mu_{k_t}$$

where μ_* is the optimal μ_i for the configuration and

μ_{u_t} is the μ_j for arm $j = u_t$ in the given configuration.

=> Linear setting regret is equivalent to finitely-armed regret, given the gaussian reward configuration in the stochastic bandit setting

c) OFUL:

On round t:

1. choose optimistic $\tilde{\theta}_t = \arg\max_{\theta \in C_{t-1}} (\max_{x \in X} \langle x, \theta \rangle)$
2. $x_t = \arg\max_{x \in X} \langle x, \tilde{\theta}_t \rangle$
3. play x_t
4. $r_t = \langle x_t, \theta_* \rangle + \varepsilon_t$

UCB:

Actions k, confidence level δ

Play each action once,

$\forall t > k$:

Select $k_t = \arg\max_{k \in K} UCB_k(t-1, \delta)$

For UCB, we maintain upper confidence bounds $\forall k \in K = \{1, \dots, K\}$, and we select the greediest action to maximize the upper confidence bound

For OFUL, we maintain a confidence set $C_{t-1} \subseteq \mathbb{R}^k$ for θ_*, and we select the greediest action x_t to maximize $\langle x, \tilde{\theta}_t \rangle$, where $\tilde{\theta}_t = \arg\max_{\theta \in C_{t-1}} (\max_{x \in X} \langle x, \theta \rangle)$

Equivalently $$(x_t, \tilde{\theta}_t) = \arg\max_{(x, \theta) \in X \times C_{t-1}} \langle x, \theta \rangle$$

i.e. $x_t = \vec{1}_i$, so we are selecting greediest action $k_t = i^{th}$ action to maximize confidence set $C_{t-1, i} \equiv UCB_i(t-1, \delta)$

C_{t-1} is an ellipsoid around $\mu = \theta_*$, with confidence level δ or equivalently

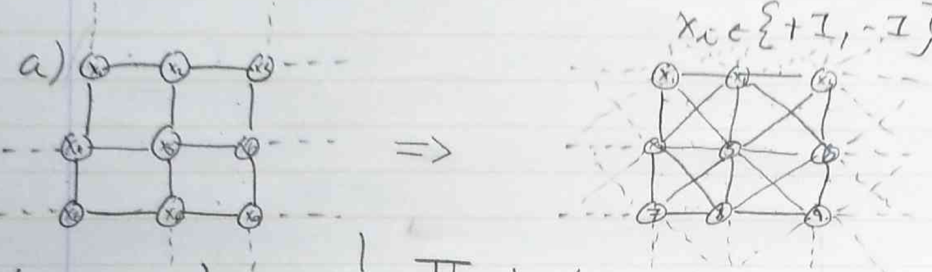
$$C_{t-1} = \left(UCB_1(t-1, \delta), \dots, UCB_k(t-1, \delta)\right)$$

Thus, UCB in the finitely-armed setting maintains k confidence intervals for $\mu_1, \dots, \mu_k$, whereas OFUL maintains C_t confidence set, $C_t \subseteq \mathbb{R}^k$, which can be viewed as k confidence intervals, for $\mu_1, \dots, \mu_k$.

Thus, both seek to maximize the upper confidence bound, but the notation (k confidence intervals vs. k-element confidence set) is 'slightly' different.

Hilroy

2. Markov Random Field

a) 

$x_i \in \{+1, -1\}$

$$p(x_1, \dots, x_9) = \frac{1}{Z} \prod_{i \sim j} \phi_{ij}(x_i, x_j) \quad \text{(ising model)}$$

We can represent the joint distribution as a product of clique potentials

$$\Rightarrow p(x_1, \dots, x_9) = \frac{1}{Z} \prod_{\text{cliques } c} \psi_c(x_c)$$

where x_c are the values for the variables that participate in clique c.

4 neighbours (left model): consider $\{x_1, \dots, x_9\}$.

Cliques:
- size 1: $\{\{x_1\}, \dots, \{x_9\}\}$ 9 parameters.
- size 2: $\{\{x_1, x_2\}, \{x_1, x_4\}, \dots \{x_8, x_9\}\}$

$\Rightarrow$ 12 parameters.

size 3: 0

$\Rightarrow$ 21 parameters / energies.

8 neighbours (right model): consider $\{x_1, \dots, x_9\}$.
- size 1: 9 paramters
- size 2: 20 paramters (i.e count edges)

size 3:
Count triangles: 4 per square · 4 squares
= 16 parameters.

size 4:
count squares with diagonals
=> 4 parameters.

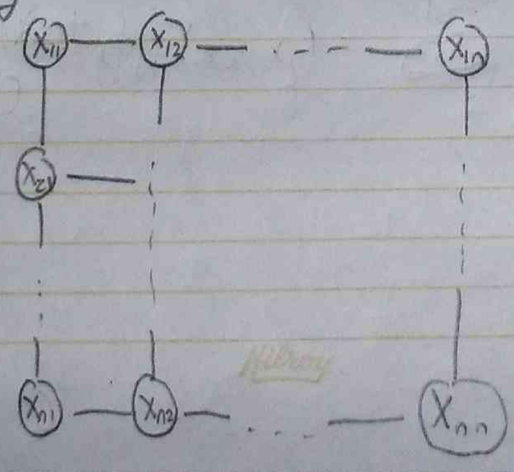
Thus, there's an increase in parameters, as we:
a) have more 2-cliques.
b) have 3-cliques
c) have 4-cliques.

The parameters, thus, can be expressed as 1, 2, 3, 4 - cliques.

b) Disadvantages:
More complex model means
a) higher variance b) increased computational cost since we have many more parameters to learn.

Advantages:
- More complex patterns (non-linear) can be captured.

c) Ising Model



X_{ij} is node : i^{th} row, j^{th} column
$i = 1, \ldots, n \quad j = 1, \ldots, n.$

let e_i, $i = 1, \ldots, n$ be the evidence injected for nodes $X_{i,1}$

$$E(\sigma) = -\sum_{i \sim j} \sigma_i \sigma_j,$$

σ_i: configuration $= \pm 1$ for node i.

↳ sum over all pairs of sites i, j which are neighbors

$$\pi_\sigma = \frac{e^{-\beta E(\sigma)}}{\sum_\tau e^{-\beta E(\tau)}}$$

prob dist on set of configurations

$\beta > 0$ favors neighbors of similar spin

$\beta < 0$ favors high-energy config.

let $\sigma_{-h} = (\sigma_1, \ldots, \sigma_{h-1}, \sigma_{h+1}, \ldots, \sigma_{n^2})$

$\sigma_h^+ = (\sigma_1, \ldots, \sigma_{h-1}, +1, \sigma_{h+1}, \ldots, \sigma_{n^2})$

$\sigma_h^- = (\sigma_1, \ldots, \sigma_{h-1}, -1, \sigma_{h+1}, \ldots, \sigma_{n^2})$

$$\Rightarrow p(\sigma_h = +1 \mid \sigma_{-h}) = \frac{P(\sigma_h^+)}{P(\sigma_{-h})} = \frac{P(\sigma_h^+)}{P(\sigma_h^+) + P(\sigma_h^-)}$$

$$= \frac{e^{-\beta E(\sigma_h^+)}}{e^{-\beta E(\sigma_h^+)} + e^{-\beta E(\sigma_h^-)}}$$

$$= \frac{1}{1 + e^{\beta(E(\sigma_k^+) - E(\sigma_k^-))}}$$

now, $E(\sigma_k^+) = -\sum_{\substack{i\sim j \\ k=1}} \sigma_i \sigma_j$

$$= -\left(\sum_{\substack{i\sim j \\ i,j \neq k}} \sigma_i \sigma_j + \sum_{i\sim k} \sigma_i \right)$$

$$E(\sigma_k^-) = -\left(\sum_{\substack{i\sim j \\ i,j \neq k}} \sigma_i \sigma_j - \sum_{i\sim k} \sigma_i \right)$$

$$\Rightarrow E(\sigma_k^+) - E(\sigma_k^-) = -2 \sum_{i\sim k} \sigma_i$$

$$\Rightarrow p(\sigma_k = +1 \mid \sigma_{-k}) = \frac{1}{1 + e^{-2\beta \sum_{i\sim k} \sigma_i}}$$

and $p(\sigma_k = -1 \mid \sigma_{-k}) = 1 - p(\sigma_k = +1 \mid \sigma_{-k})$

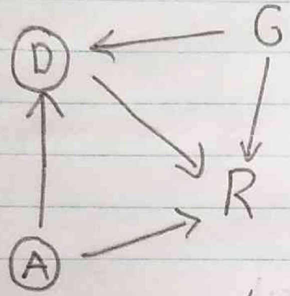
Algorithm: (num is some number of iterations ≥ 0)

```
X_ij = random ∀i,j (let X̃ be lattice)
for num:
    if num = 0
        ∀j = 1,...,n  i = 1,...,n:
            j = 1:
                x_ij = e_i
            else:
                x_ij = sample p(σ_{i+j} | σ_{-(i+j)})
    else:
        ∀j = 1,...,n  i = 1,...,n
            X_ij = sample p(σ_{i+j} | σ_{-(i+j)})
return X̃
```

3. $D \mid A, G$

$R \mid D, A, G$

A indp of G



$$\Rightarrow P(A, D, G, R)$$
$$= P(A)P(G)P(D \mid A, G)P(R \mid D, A, G)$$

$$p(R=r \mid D=d) = \frac{p(R=r, D=d)}{p(D=d)}$$

$r :=$ recover
$d :=$ drug

$$P(R=r, D=d) = \sum_{A \in \{o, y\}} \sum_{G \in \{m, f\}} P(A, G, D=d, R=r)$$

$$= \sum_A \sum_G P(A)P(G)P(D=d \mid A, G)P(R=r \mid D=d, A, G)$$

$$= \sum_A P(A) \left[\sum_G P(G)P(D=d \mid A, G)P(R=r \mid D=d, A, G) \right]$$

$$P(D=d) = \sum_A \sum_G \sum_R P(A)P(G)P(D=d \mid A, G)P(R \mid D=d, A, G)$$

$$= \sum_A P(A) \sum_G P(G)P(D=d \mid A, G) \underbrace{\sum_R P(R \mid D=d, A, G)}_{=1}$$

$$= \sum_A P(A) \sum_G P(G)P(D=d \mid A, G)$$

$$\Rightarrow P(R=r \mid D=d) = \frac{\sum_A P(A) \sum_G P(G)P(D=d \mid A, G)P(R=r \mid D=d, A, G)}{\sum_A P(A) \sum_G P(G)P(D=d \mid A, G)}$$

Thus, to calculate $p(r|d)$,

i) $P(\text{old}) \cdot \Big(P(\text{male}) P(\text{drug}|\text{male}, \text{old}) P(\text{recover}|\text{drug}, \text{old}, \text{male})$

$\quad + P(\text{Female}) P(\text{drug}|\text{Female}, \text{old}) P(\text{recover}|\text{drug}, \text{old}, \text{Female}) \Big)$

$+ P(\text{young}) \Big(P(\text{male}) P(\text{drug}|\text{male}, \text{young}) P(\text{recover}|\text{drug}, \text{young}, \text{male})$

$\quad + P(\text{female}) P(\text{drug}|\text{Female}, \text{young}) P(\text{recover}|\text{drug}, \text{young}, \text{Female}) \Big)$

ii) $P(\text{old}) \Big(P(\text{male}) P(\text{drug}|\text{male}, \text{old})$

$\quad + P(\text{female}) P(\text{drug}|\text{Female}, \text{old}) \Big)$

$+ P(\text{young}) \Big[P(\text{male}) P(\text{drug}|\text{male}, \text{young}) + P(\text{Female}) P(\text{drug}|\text{female}, \text{young}) \Big]$

$$p(r|d) = \frac{(i)}{(ii)}$$

Where probabilitres can be ascertained by the doctor

$p(A)$: $\dfrac{\text{\# people in age group a}}{\text{Total people.}}$

$p(G)$: $\dfrac{\text{\# people of gender g}}{\text{Total people.}}$

$p(D{=}d|A,G)$: $\dfrac{\text{\# of people of age group a, gender g given drug d}}{\text{\# of people of age group a, gender g.}}$

$p(R{=}r|D,A,G)$: $\dfrac{\text{\# of people of a, g, given (or not) drug d, who did recover}}{\text{\# of people of a, g, d.}}$

$$P(R=r \mid D=d, A=y) = \frac{P(R=r, D=d, A=y)}{P(D=d, A=y)}$$

$$P(R=r, D=d, A=y) = \sum_G P(A=y)\,P(G)\,P(D=d \mid A=y, G)\,P(R=r \mid D=d, A=y, G)$$

$$= P(A=y) \sum_G P(G)\,P(D=d \mid A=y, G)\,P(R=r \mid D=d, A=y, G)$$

$$P(D=d, A=y) = \sum_G \sum_R P(A=y)\,P(G)\,P(D=d \mid A=y, G)\,P(R \mid D=d, A=y, G)$$

$$= P(A=y) \sum_G P(G)\,P(D=d \mid A=y, G) \underbrace{\sum_R P(R \mid D=d, A=y, G)}_{=1}$$

$$\Rightarrow P(R=r \mid D=d, A=y) = \frac{\sum_G P(G)\,P(D=d \mid A=y, G)\,P(R=r \mid D=d, A=y, G)}{\sum_G P(G)\,P(D=d \mid A=y, G)}$$

Thus, to calculate $p(r \mid d, y)$

i) $P(\text{male})\,P(\text{drug} \mid \text{young}, \text{male})\,P(\text{recover} \mid \text{drug}, \text{young}, \text{male})$
$+ P(\text{female})\,P(\text{drug} \mid \text{young}, \text{female})\,P(\text{recover} \mid \text{drug}, \text{young}, \text{female})$

ii) $P(\text{male})\,P(\text{drug} \mid \text{young}, \text{male}) + P(\text{female})\,P(\text{drug} \mid \text{young}, \text{female})$

$$p(r \mid d, y) = \frac{(i)}{(ii)}$$

Probabilities ascertained similarly to previous page.

3b) $p(a,b,c) = p(a|b)p(b|c)p(c)$

$$c \longrightarrow b \longrightarrow a$$

a, b, c are binary

$p(a|b)p(b|c)p(c)$

$p(c) \rightarrow \psi \;;\; p(\neg c) = 1-\psi$

$$p(b|c) \rightarrow \begin{array}{l} p(b|c=0) = \psi, \; p(\neg b|c=0) = 1-\psi \\ p(b|c=1) = \alpha, \; p(\neg b|c=1) = 1-\alpha \end{array}$$

$$p(a|b) \Rightarrow \begin{array}{ll} 1 \text{ parameters} & b=0 \\ 1 \text{ param} & b=1 \end{array}$$

$\Rightarrow 1 + 2 + 2$

$= 5$ parameters $\left(\frac{10}{2} \right.$ since knowing true for conditionals and probabilities gives false $\left. (1 - P(\text{true})) \right)$

$p(a,b,c)$: $2^3 = 8$ parameters

(but, one is for free: 1 − summation of the 7 other parameters)

$\Rightarrow 7$ parameters.

4a) 1000 training
$d = 6$ dimensions
binary output
100 test. set.

A big concern is the distribution of the classes in the training set. If the training set is unbalanced, we may need to oversample the minority class (ex. SMOTE [Synthetic Minority Over-sampling Technique] Boosting). Moreover, using a feed forward neural network may be too complex => add dropout to the network. Since we don't know if the function $f: X \longrightarrow Y \in \{0,1\}$ is linear or not, a neural network w/ non-linear activation is the most appropriate, since it is able to learn basis functions to project the data into a space where it may be linearly separable (if $f: X \rightarrow Y$ is non-linear). If $f: X \rightarrow Y$ is linear, then the neural network will be able to find a function to linearly separate the data, without the need to project the data.

Thus, use a feed-forward neural network (w/ SMOTE boosting if unbalanced dataset).

b) We know:

10,000 dimensions

binary output

input $\xrightarrow{\text{linear}}$ output

We don't want to use too complex of a model, since overfitting is a big issue.

Use <u>Linear SVM</u>: these are perfect for classification (as it is a non-probabilistic binary linear classifier that will find a hyperplane to separate the data. Since the output is a linear function of the input, a linear kernel is utilized. Moreover, we can solve the Dual problem for SVM, which is more efficient (since $d = 10{,}000$ and only 1,000 training samples).

4 c)

Stock prices are an example of a time-series: data points indexed in time order. In analyzing time-series, trends and seasonality can be noticed and described, since the data can correlate with itself. Thus, we need a model that considers the past and the sequential relationship w/ previous points

↳ <u>LSTM</u> should be used.

We can use the previous week's data to predict the next day's price (t+1). Then, using the data + our prediction, we can predict t+2 days and, similarly, t+3.

LSTM's utilize the data and the order of the data to make predictions, whereas feed-forward neural networks wouldn't learn any information from the ordering of the data ⇒ LSTM's are more appropriate.

Hilroy

d) Logistic regression will give us the probability of failure we are interested.

Moreover, if $Y \in \{\overset{1}{\text{Failure}}, \overset{0}{\text{No-failure}}\}$, X is predictor w/ density f.

$$\frac{P(Y=1 \mid X=x)}{P(Y=0 \mid X=x)} = \frac{P(Y=1)\, f_{X|Y=1}(x)}{P(Y=0)\, f_{X|Y=0}(x)}$$

$$\log\left(\frac{P(Y=1|X)}{P(Y=0|X)}\right) = \log\left(\frac{P(Y=1)}{P(Y=0)}\right) + \log\left(\frac{f_{X|Y=1}(x)}{f_{X|Y=0}(x)}\right)$$

here, we can encode our prior information (i.e leverage the fact that we know the likelihood of a random failure.

=> Use logistic regression.