ECO316: Applied game theory

Lecture 2

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  - General model
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- Comparison of Bertrand's and Cournot's models
- Finding Nash equilibrium using best response functions

Example: Investing in a joint project

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- Noninvestors unaffected by outcome of project
- So for every person,

$$\text{successful project} \succ \text{not investing} \succ \text{failed project}$$

Investing in a joint project

Strategic game

- Players:

Investing in a joint project

Strategic game

- Players: n people

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  if player chooses *Don't invest*, 0 regardless of others' actions

Investing in a joint project

Nash equilibrium

- k people invest?

Investing in a joint project

Nash equilibrium

- k people invest?
- n people invest?

Investing in a joint project

Nash equilibrium

- k people invest?
- n people invest?
- no one invests?

Investing in a joint project

Nash equilibrium

- k people invest?
- n people invest?
- no one invests?
- some other number of people invest?

Investing in a joint project

Reminder of payoffs:

- *Invest* $\implies$ 100 if $\geq k$ investors, -10 if $< k$ investors
- *Don't invest* $\implies$ 0

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- no one invests: Nash equilibrium because player deviates $\implies$ gets -10 rather than 0
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- between k and $n-1$ people invest: not Nash equilibrium because noninvestor deviates $\implies$ gets 100 rather than 0

deviation include both in and out, consider player who are not in the game currently

Investing in a joint project

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  so no players will deviate
- no one invests: Nash equilibrium because player deviates $\implies$ gets -10 rather than 0
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  some of the players will deviate
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Summary

Exactly two Nash equilibria:

- everyone invests
- no one invests

Traveler's Dilemma

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where a_i is i's action and a_j is the other player's action

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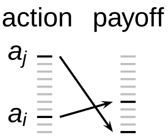

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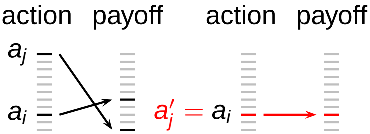

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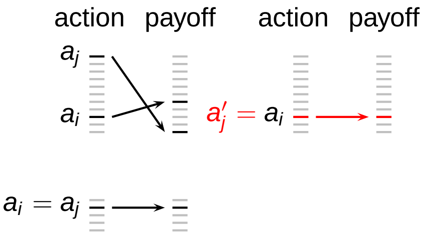

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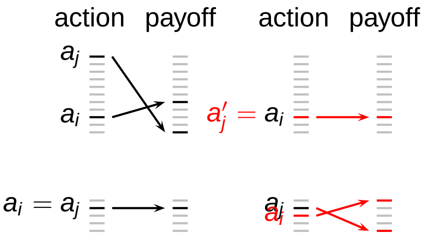

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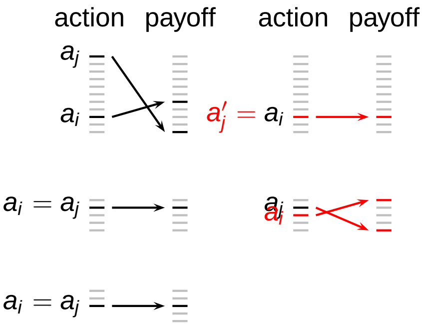
  - If $a_i = a_j = 2$, NE! If either player increases amount, payoff $= 0$



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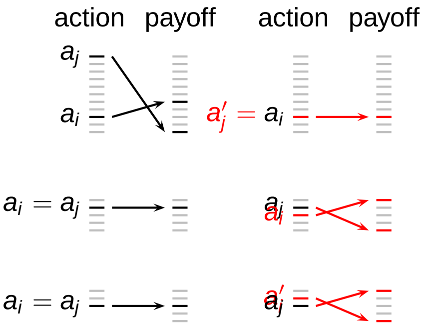

Traveler's Dilemma

Summary

Unique Nash equilibrium: both travelers name the lowest possible valuation, $2

Competition between firms

- Topic at heart of classical economic theory

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- Need model in which each firm takes others into account
- Can study impact on number of firms on the outcome

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- Firms producing same good compete for customers
- Each firm's profit depends on behavior of all firms
- Model interaction between firms as strategic game
- What are properties of Nash equilibrium?
- How is Nash equilibrium related to “competitive” outcome?
  How does it depend on number of firms?

Bertrand's model

- Each firm chooses a unit price

Joseph Louis François Bertrand
1822–1900

Bertrand's model

- Each firm chooses a unit price
- Each firm produces enough output to satisfy the demand it faces, given the prices charged by all firms

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Strategic game

- Players:

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- Players: firms

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Strategic game

- Players: firms
- For each firm
  - possible actions:

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- For each firm
  - possible actions: prices
  - payoff:

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Strategic game

- Players: firms
- For each firm
  - possible actions: prices
  - payoff: profit

Example of Bertrand's game

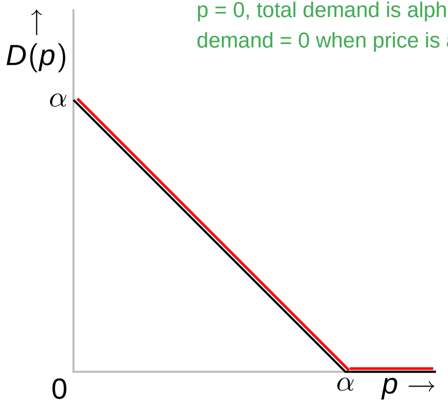
- Two firms

Example of Bertrand's game

- Two firms
- Constant unit cost, same for both firms:
  $C_i(q_i) = cq_i$ where $c > 0$ and q_i is output of firm i

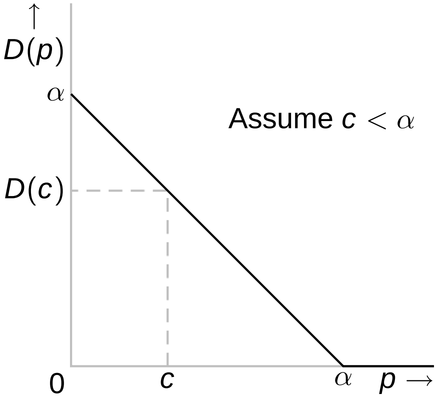
Example of Bertrand's game

- Two firms
- Constant unit cost, same for both firms:
  $C_i(q_i) = cq_i$ where $c > 0$ and q_i is output of firm i
  cost based on quantity produced
- Linear demand function: $D(p) = \alpha - p$ for $p \leq \alpha$
  demand based on price
  p = 0, total demand is alpha
  demand = 0 when price is alpha



Example of Bertrand's game

- Two firms
- Constant unit cost, same for both firms:
  $C_i(q_i) = cq_i$ where $c > 0$ and q_i is output of firm i
- Linear demand function: $D(p) = \alpha - p$ for $p \leq \alpha$



Example of Bertrand's game

Strategic game

- Players:

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- Players: two firms

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- Players: two firms
- For each firm i,
  - possible actions:

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Strategic game

- Players: two firms
- For each firm i,
  - possible actions: prices

Example of Bertrand's game

Strategic game

- Players: two firms
- For each firm i,
  - possible actions: prices (nonnegative numbers)

Example of Bertrand's game

Strategic game

Price can be any number—not restricted to multiples of discrete unit (e.g. multiples of a cent)

- Players: two firms
- For each firm i,
  - possible actions: prices (nonnegative numbers)

Example of Bertrand's game

Strategic game

- Players: two firms
- For each firm i,
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  - payoff:

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$$\pi_i(p_1, p_2) = \begin{cases} & \text{if } p_i < p_j \\ & \\ & \end{cases}$$

where j is the other firm ($j = 2$ if $i = 1$, and $j = 1$ if $i = 2$).

Example of Bertrand's game

Strategic game

- Players: two firms
- For each firm i,
  - possibl Revenue from selling ative numbers)
  - payoff: total demand at price p_i

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Profit from selling total demand at price p_i

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Strategic game

- Players: two firms
- For each firm i,
  - possible actions: prices (nonnegative numbers)
  - payoff: pro

Simplify expression

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)D(p_i) & \text{if } p_i < p_j \\ & \\ & \end{cases}$$

where j is the other firm ($j = 2$ if $i = 1$, and $j = 1$ if $i = 2$).

Example of Bertrand's game

Strategic game

- Players: two firms
- For each firm i,
  - possible actions: prices (nonnegative numbers)
  - payoff: Substitute $\alpha - p_i$ for $D(p_i)$ (for $p_i \leq \alpha$)

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ & \\ & \end{cases}$$

(assuming $p_i \leq \alpha$)

where j is the other firm ($j = 2$ if $i = 1$, and $j = 1$ if $i = 2$).

Example of Bertrand's game

Strategic game

- Players: two firms
- For each firm i,
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$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \\ & \text{if } p_i > p_j, \end{cases} \quad \text{(assuming } p_i \leq \alpha)$$

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Example of Bertrand's game

Strategic game

- Players: two firms
- For each firm i,
  - possible actions: prices (nonnegative numbers)
  - payoff: profit, which is

High price $\Rightarrow$ no customers

$$\pi_i(p_1, p_2) = \begin{cases} \quad \alpha - p_i) & \text{if } p_i < p_j \\ 0 & \text{if } p_i > p_j, \end{cases}$$

(assuming $p_i \leq \alpha$)

where j is the other firm ($j = 2$ if $i = 1$, and $j = 1$ if $i = 2$).

Example of Bertrand's game

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Example of Bertrand's game

Strategic game

- Players: two firms
- For each firm i,
  - possible actions: prices (nonnegative numbers)
  - payoff: profit, which is

Equal prices $\Rightarrow$
demand split equally

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j, \end{cases}$$

(assuming $p_i \leq \alpha$)

where j is the other firm ($j = 2$ if $i = 1$, and $j = 1$ if $i = 2$).

Example of Bertrand's game

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Example of Bertrand's game

Exploration of payoffs:

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases} \quad (\text{assuming } p_i \leq \alpha)$$

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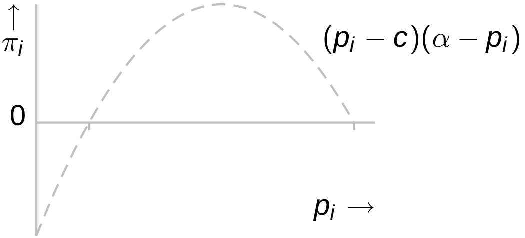

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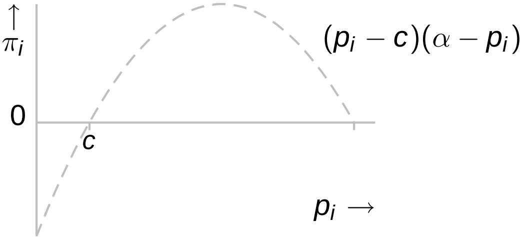

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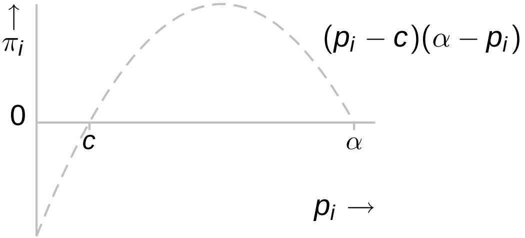

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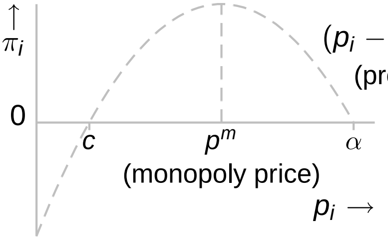

i seems like the only company in the market

Example of Bertrand's game

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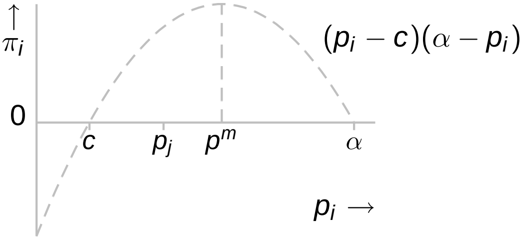

$c < p_j \leq p^m$

Example of Bertrand's game

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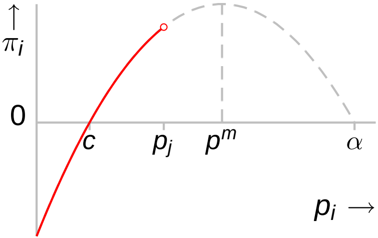

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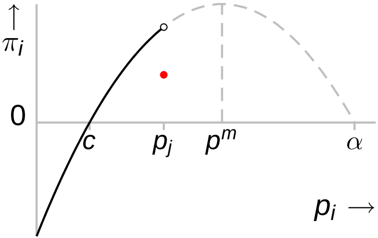

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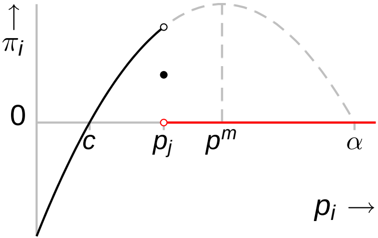

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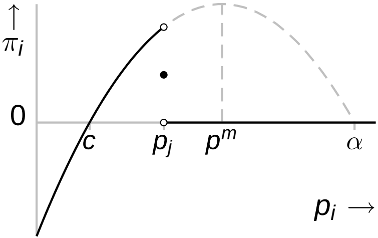

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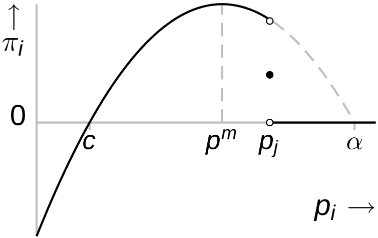
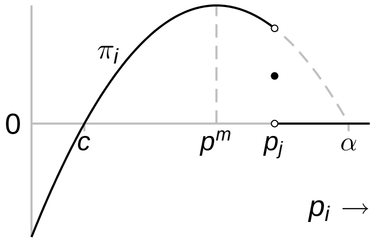


$$p^m < p_j < \alpha$$

Example of Bertrand's game

Best value of p_i given p_j?



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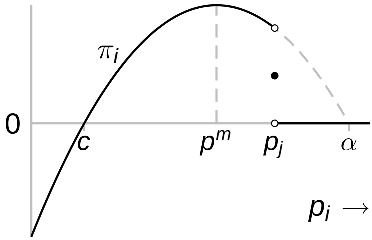

$$p^m < p_j \leq \alpha$$

- If $p_j > p^m$, firm i's best price is p^m

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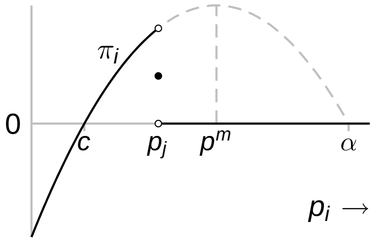

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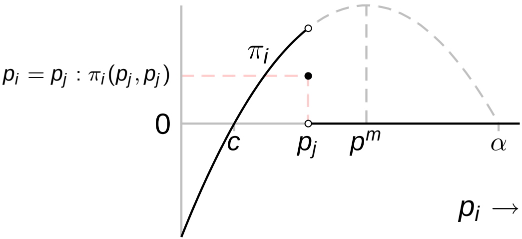

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Example of Bertrand's game

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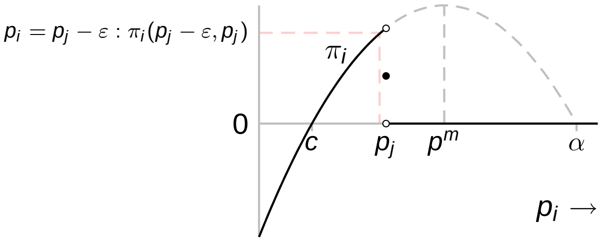
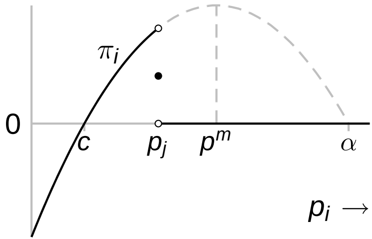


$c < p_j \leq p^m$

- If $p_j > p^m$, firm i's best price is p^m
- If $p_j < p^m$, firm i can almost double its profit by charging a bit less than p_j

Example of Bertrand's game

Best value of p_i given p_j?



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- $\Rightarrow$ incentive to "undercut" other firm's price

Example of Bertrand's game

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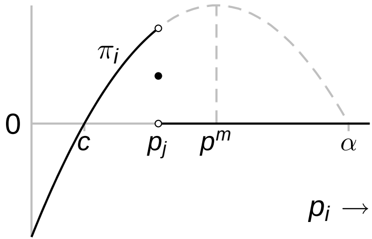

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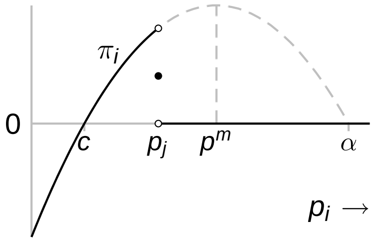
- Prices less than c yield losses

Example of Bertrand's game

Best value of p_i given p_j?



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- If $p_j < p^m$, firm i can almost double its profit by charging a bit less than p_j
- $\Rightarrow$ incentive to "undercut" other firm's price
- Prices less than c yield losses
- So perhaps (c, c) is only equilibrium?

Example of Bertrand's game

Proof that (c, c) is a Nash equilibrium

- $u_1(c, c) =$

Example of Bertrand's game

Proof that (c, c) is a Nash equilibrium

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- If $p_1 < c$, then $u_1(p_1, c)$

Example of Bertrand's game

Proof that (c, c) is a Nash equilibrium

- $u_1(c, c) = 0$
- If $p_1 < c$, then $u_1(p_1, c) < 0$ (given $\alpha > c$)

Example of Bertrand's game

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Example of Bertrand's game

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Thus

$$u_1(c, c) \geq u_1(p_1, c) \text{ for all } p_1$$

and similarly for firm 2.

Example of Bertrand's game

Proof that (c, c) is a Nash equilibrium

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Thus

$$u_1(c, c) \geq u_1(p_1, c) \text{ for all } p_1$$

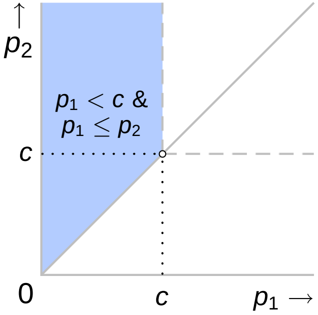
and similarly for firm 2.

Hence (c, c) is a Nash equilibrium.

Example of Bertrand's game

Proof that no pair $(p_1, p_2) \neq (c, c)$ is Nash equilibrium

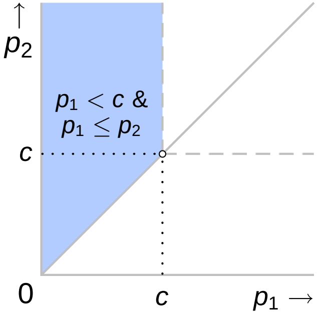
- $p_1 < c$ and $p_1 \leq p_2$?



Example of Bertrand's game

Proof that no pair $(p_1, p_2) \neq (c, c)$ is Nash equilibrium

- $p_1 < c$ and $p_1 \leq p_2$? No: $u_1(p_1, p_2) < 0$ and $u_1(c, p_2) = 0$, so firm 1 can increase its payoff by deviating to c



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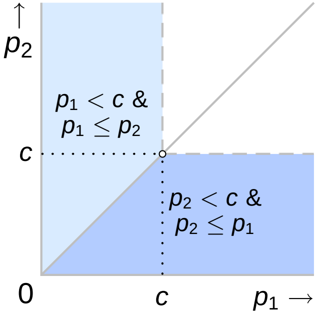
- $p_2 < c$ and $p_2 \leq p_1$?



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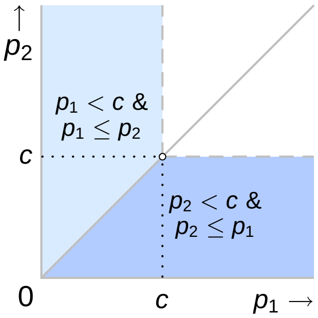
- $p_2 < c$ and $p_2 \leq p_1$? No: firm 2 can profitably deviate to c



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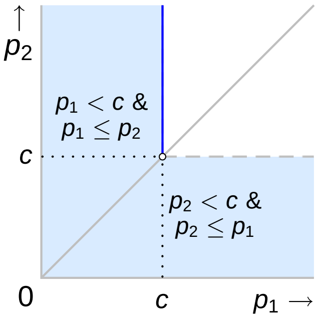
- $p_2 < c$ and $p_2 \leq p_1$? No: firm 2 can profitably deviate to c
- $p_1 = c$ and $p_2 > c$?



Example of Bertrand's game

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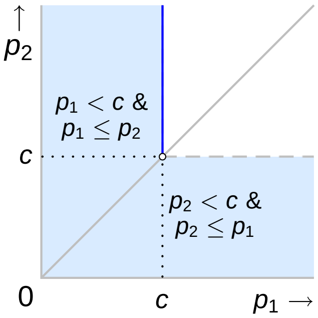
- $p_1 < c$ and $p_1 \leq p_2$? No: $u_1(p_1, p_2) < 0$ and $u_1(c, p_2) = 0$, so firm 1 can better increase its payoff by deviating to c
- $p_2 < c$ and $p_2 \leq p_1$? No: firm 2 can profitably deviate to c
- $p_1 = c$ and $p_2 > c$? No: firm 1 can profitably *raise* its price: $u_1(c, p_2) = 0$ and $u_1(p_1, p_2) > 0$ for $c < p_1 < p_2$ and $p_1 < \alpha$



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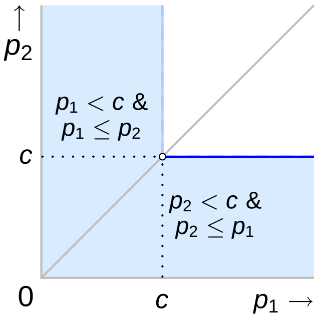

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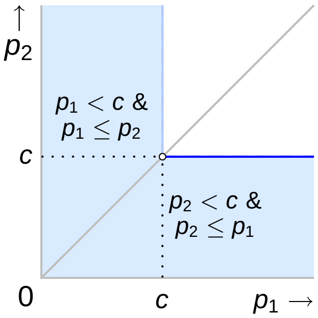
- $p_2 = c$ and $p_1 > c$? No: similar reason



Example of Bertrand's game

Proof that no pair $(p_1, p_2) \neq (c, c)$ is Nash equilibrium

- $p_1 < c$ and $p_1 \leq p_2$? No: $u_1(p_1, p_2) < 0$ and $u_1(c, p_2) = 0$, so firm 1 can increase its payoff by deviating to c
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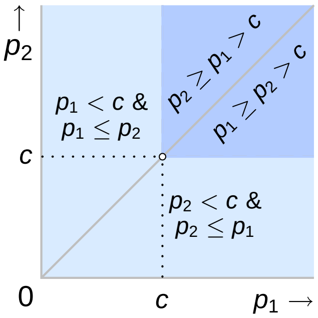

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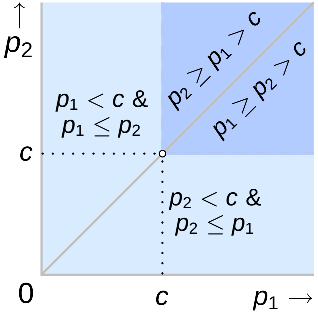

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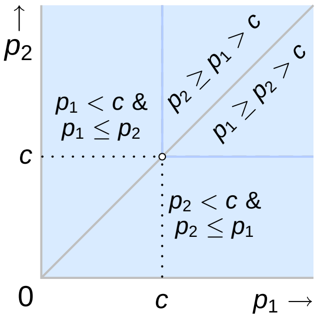

Bertrand's model: summary

- Unique Nash equilibrium, in which price $=$ unit cost for both firms

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Questions

- What about other demand functions?
- What about other cost functions?
- What happens with more than two firms?
- Is there a way for the firms to collude?

Cournot's model

- Each firm chooses an output

Antoine Augustin Cournot
1801–1877

Cournot's model

- Each firm chooses an output
- Price is determined by demand function, given firms' total output

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Strategic game

- Players:

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Strategic game

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Strategic game

- Players: firms
- For each firm
  - possible actions: outputs
  - payoff: profit

Example of Cournot's game

- Two firms

Example of Cournot's game

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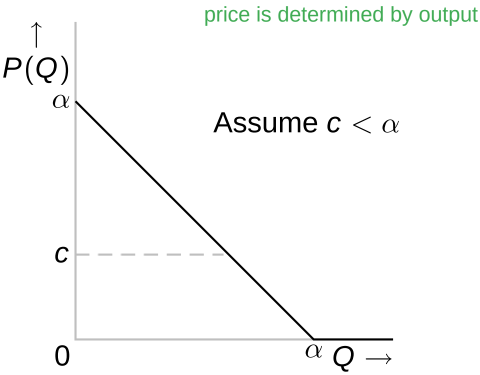
- Constant unit cost, same for both firms: $C_i(q_i) = cq_i$ where $c > 0$ and q_i is output of firm i

Example of Cournot's game

- Two firms
- Constant unit cost, same for both firms: $C_i(q_i) = cq_i$ where $c > 0$ and q_i is output of firm i
- Linear inverse demand function: $P(Q) = \alpha - Q$ for $Q \leq \alpha$



Example of Cournot's game

Strategic game

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- Players: two firms
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Example of Cournot's game

Strategic game

- Players: two firms
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Example of Cournot's game

Strategic game

- Players: two firms
- For each firm
    - possible actions: outputs (nonnegative numbers)

Example of Cournot's game

Strategic game

- Players: two firms
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  - possible actions: outputs (nonnegative numbers)

Output can be any number—not restricted to multiples of discrete unit

Example of Cournot's game

Strategic game

- Players: two firms
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$$\pi_i(q_1, q_2)$$

Example of Cournot's game

Strategic game

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$$\pi_i(q_1, q_2) = \text{revenue} - \text{cost}$$

Example of Cournot's game

Strategic game

- Players: two firms
- For each firm
  - possible actions: outputs (nonnegative numbers)
  - payoff: profit of firm i

$$\pi_i(q_1, q_2) = q_i P(q_1 + q_2) - \text{cost}$$

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Example of Cournot's game

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$$\begin{aligned}\pi_i(q_1, q_2) &= q_i P(q_1 + q_2) - cq_i \\ &= q_i(\alpha - q_1 - q_2) - cq_i \qquad \text{(if } q_1 + q_2 \leq \alpha\text{)}\end{aligned}$$

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for $i = 1, 2$

by changing the number, the same function can be used to represent both companies' profit

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Nash equilibrium

- We want to find a pair (q_1^*, q_2^*) of outputs such that

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$$\text{solution of } \max_{q_1} \pi_1(q_1, q_2)$$

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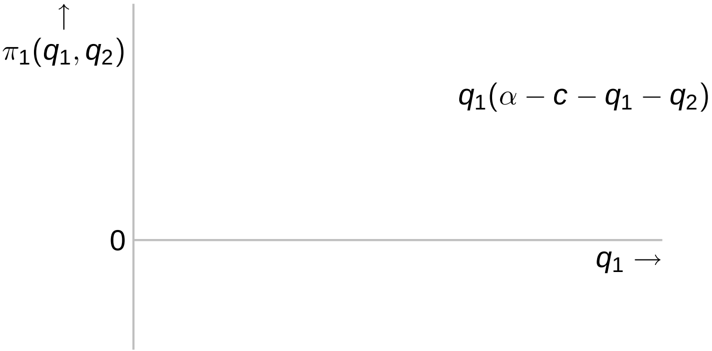
$$\implies \text{solution of } \max_{q_1} q_1(\alpha - c - q_1 - q_2)$$

Example of Cournot's game

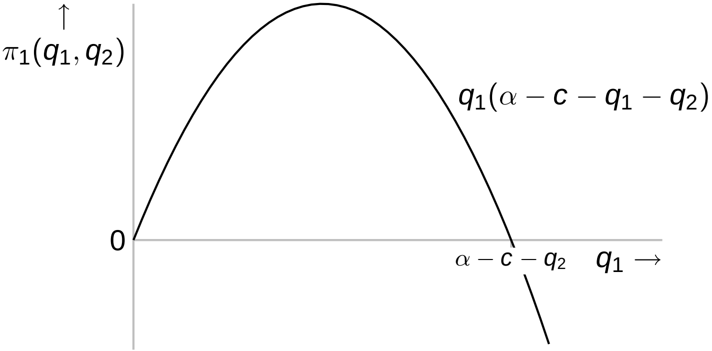
Nash equilibrium



Firm 1's payoff as a function of q_1, given q_2

Example of Cournot's game

Nash equilibrium



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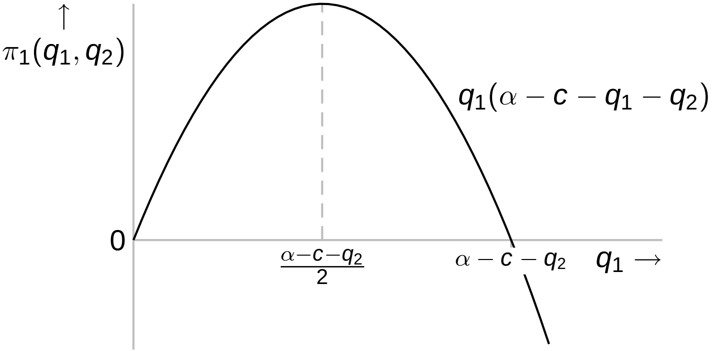

Firm 1's payoff as a function of q_1, given q_2

$\Rightarrow$ optimal q_1 given q_2 is $\frac{1}{2}(\alpha - c - q_2)$

Example of Cournot's game

Nash equilibrium

- Can write

$$b_1(q_2) = \frac{1}{2}(\alpha - c - q_2)$$

$\rightarrow$ *best response* of firm 1 to firm 2's output

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- Similarly
$$b_2(q_1) = \tfrac{1}{2}(\alpha - c - q_1)$$
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- Nash equilibrium:
$$q_1^* = b_1(q_2^*)$$
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or
$$q_1^* = \tfrac{1}{2}(\alpha - c - q_2^*)$$
$$q_2^* = \tfrac{1}{2}(\alpha - c - q_1^*)$$

Example of Cournot's game

Nash equilibrium

$$q_1^* = \tfrac{1}{2}(\alpha - c - q_2^*) \; (= b_1(q_2^*))$$
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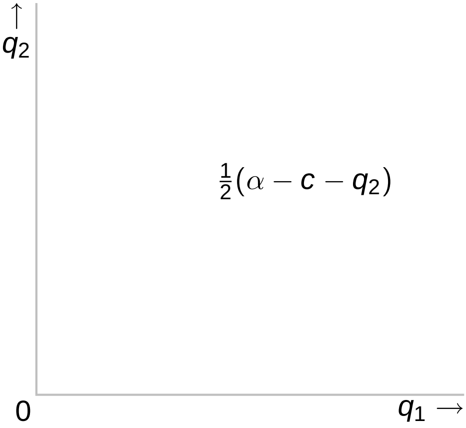

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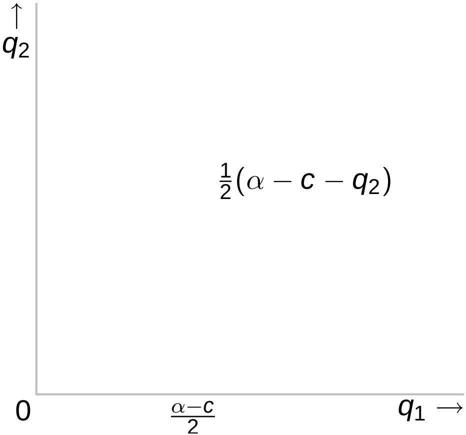

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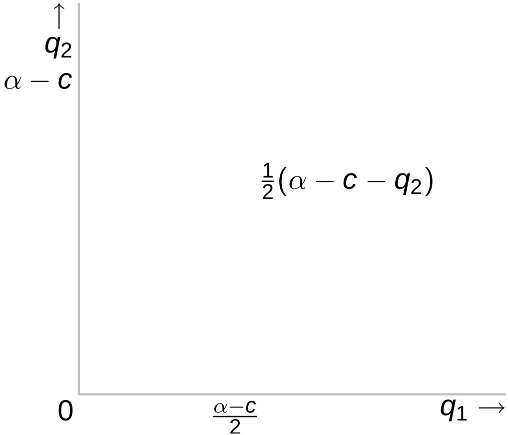

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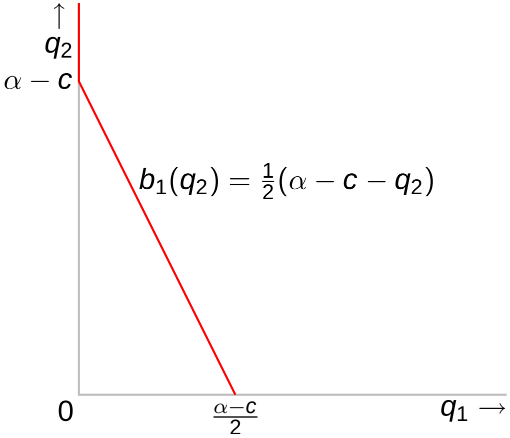

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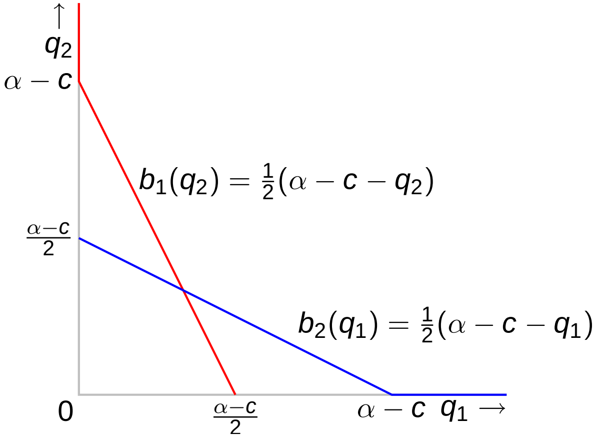

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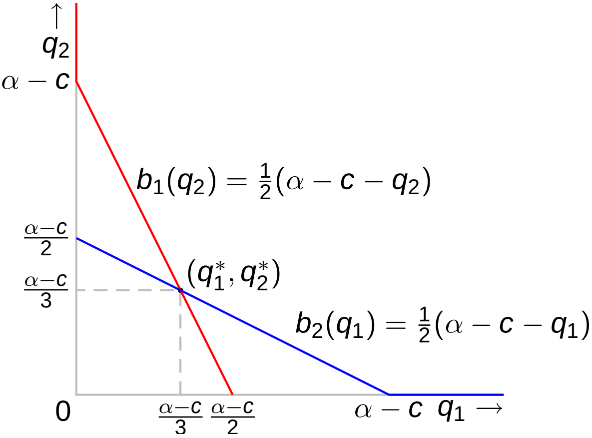

Example of Cournot's game

Nash equilibrium

$$q_1^* = \frac{1}{2}(\alpha - c - q_2^*)$$
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Example of Cournot's game

Nash equilibrium

$$q_1^* = \frac{1}{2}(\alpha - c - q_2^*)$$
$$q_2^* = \frac{1}{2}(\alpha - c - q_1^*)$$

$\Longrightarrow$

$$q_1^* = \frac{1}{2}(\alpha - c - \frac{1}{2}(\alpha - c - q_1^*))$$

Example of Cournot's game

Nash equilibrium

$$q_1^* = \frac{1}{2}(\alpha - c - q_2^*)$$
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$\Longrightarrow$

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$$q_1^* = \frac{1}{4}(\alpha - c) + \frac{1}{4}q_1^*$$

Example of Cournot's game

Nash equilibrium

$$q_1^* = \frac{1}{2}(\alpha - c - q_2^*)$$
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$\Longrightarrow$

$$q_1^* = \frac{1}{2}(\alpha - c - \frac{1}{2}(\alpha - c - q_1^*))$$
$$q_1^* = \frac{1}{4}(\alpha - c) + \frac{1}{4}q_1^*$$
$$\frac{3}{4}q_1^* = \frac{1}{4}(\alpha - c)$$

Example of Cournot's game

Nash equilibrium

$$q_1^* = \frac{1}{2}(\alpha - c - q_2^*)$$
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$$\frac{3}{4}q_1^* = \frac{1}{4}(\alpha - c)$$
$$q_1^* = \frac{1}{3}(\alpha - c)$$

Example of Cournot's game

Nash equilibrium

$$q_1^* = \tfrac{1}{2}(\alpha - c - q_2^*)$$
$$q_2^* = \tfrac{1}{2}(\alpha - c - q_1^*)$$

$\Longrightarrow$

$$q_1^* = \tfrac{1}{2}(\alpha - c - \tfrac{1}{2}(\alpha - c - q_1^*))$$
$$q_1^* = \tfrac{1}{4}(\alpha - c) + \tfrac{1}{4}q_1^*$$
$$\tfrac{3}{4}q_1^* = \tfrac{1}{4}(\alpha - c)$$
$$q_1^* = \tfrac{1}{3}(\alpha - c)$$

Substitute back to get $q_2^* = \frac{1}{2}(\alpha - c - \frac{1}{3}(\alpha - c)) = \frac{1}{3}(\alpha - c)$

Example of Cournot's game

Nash equilibrium

Conclusion: Unique Nash equilibrium,

$$(q_1^*, q_2^*) = (\tfrac{1}{3}(\alpha - c), \tfrac{1}{3}(\alpha - c))$$

Example of Cournot's game

Nash equilibrium

Conclusion: Unique Nash equilibrium,

$$(q_1^*, q_2^*) = (\tfrac{1}{3}(\alpha - c), \tfrac{1}{3}(\alpha - c))$$

Properties of Nash equilibrium

How is the price in the equilibrium related to unit cost, c?

Total output $=$

Example of Cournot's game

Nash equilibrium

Conclusion: Unique Nash equilibrium,

$$(q_1^*, q_2^*) = (\tfrac{1}{3}(\alpha - c), \tfrac{1}{3}(\alpha - c))$$

Properties of Nash equilibrium

How is the price in the equilibrium related to unit cost, c?

Total output $= q_1^* + q_2^*$

Example of Cournot's game

Nash equilibrium

Conclusion: Unique Nash equilibrium,

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Properties of Nash equilibrium

How is the price in the equilibrium related to unit cost, c?

$$\text{Total output } = q_1^* + q_2^* = \tfrac{1}{3}(\alpha - c) + \tfrac{1}{3}(\alpha - c)$$

Example of Cournot's game

Nash equilibrium

Conclusion: Unique Nash equilibrium,

$$(q_1^*, q_2^*) = (\tfrac{1}{3}(\alpha - c), \tfrac{1}{3}(\alpha - c))$$

Properties of Nash equilibrium

How is the price in the equilibrium related to unit cost, c?

$$\text{Total output } = q_1^* + q_2^* = \tfrac{1}{3}(\alpha - c) + \tfrac{1}{3}(\alpha - c) = \tfrac{2}{3}(\alpha - c)$$

Example of Cournot's game

Nash equilibrium

Conclusion: Unique Nash equilibrium,

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Properties of Nash equilibrium

How is the price in the equilibrium related to unit cost, c?

$$\text{Total output } = q_1^* + q_2^* = \tfrac{1}{3}(\alpha - c) + \tfrac{1}{3}(\alpha - c) = \tfrac{2}{3}(\alpha - c)$$

$$\implies \text{price } =$$

Example of Cournot's game

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Properties of Nash equilibrium

How is the price in the equilibrium related to unit cost, c?

$$\begin{aligned} \text{Total output } &= q_1^* + q_2^* = \tfrac{1}{3}(\alpha - c) + \tfrac{1}{3}(\alpha - c) = \tfrac{2}{3}(\alpha - c) \\ \implies \text{price } &= P(\tfrac{2}{3}(\alpha - c)) \end{aligned}$$

Example of Cournot's game

Nash equilibrium

Conclusion: Unique Nash equilibrium,

$$(q_1^*, q_2^*) = (\tfrac{1}{3}(\alpha - c), \tfrac{1}{3}(\alpha - c))$$

Properties of Nash equilibrium

How is the price in the equilibrium related to unit cost, c?

$$\begin{aligned} \text{Total output } &= q_1^* + q_2^* = \tfrac{1}{3}(\alpha - c) + \tfrac{1}{3}(\alpha - c) = \tfrac{2}{3}(\alpha - c) \\ \implies \text{ price } &= P(\tfrac{2}{3}(\alpha - c)) = \alpha - \tfrac{2}{3}(\alpha - c) \end{aligned}$$

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We have $\alpha > c$, so price $> c$

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$$q^m \text{ solves } \max_q q(\alpha - q) - cq$$

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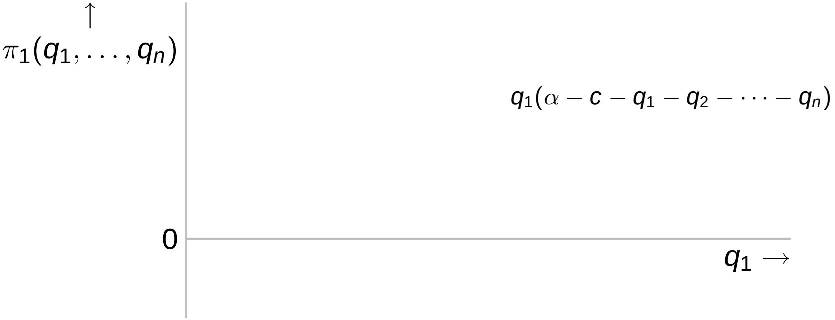

Firm 1's payoff as a function of q_1, given $q_2, \ldots, q_n$

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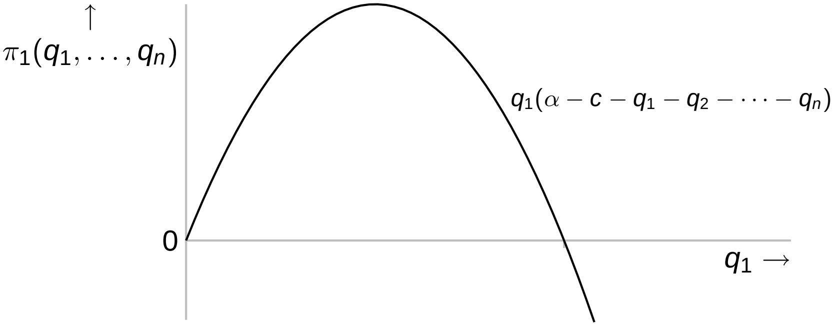

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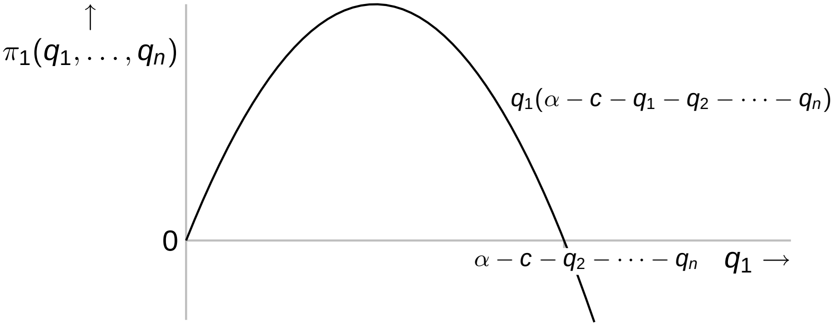

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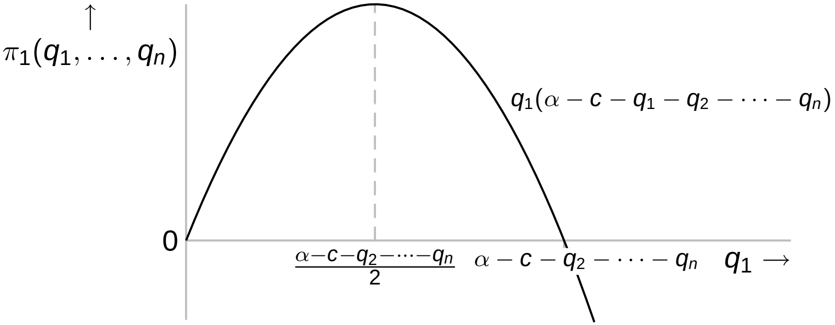

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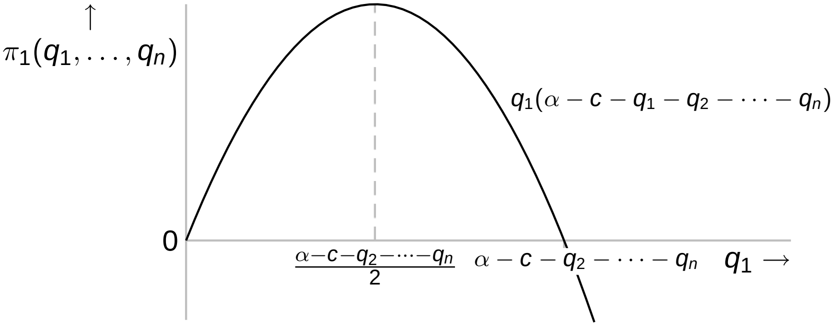

Firm 1's payoff as a function of q_1, given $q_2, \ldots, q_n$

$\Rightarrow$ optimal q_1 given $q_2, \ldots, q_n$ is $\frac{1}{2}(\alpha - c - q_2 - \cdots - q_n)$

Example of Cournot's game: many firms

- Best response function of firm 1 is

$$b_1(q_{-1}) = \frac{1}{2}(\alpha - c - q_2 - \cdots - q_n) \quad (\text{if } q_2 + \cdots + q_n \leq \alpha - c)$$

where q_{-1} stands for $(q_2, \ldots, q_n)$

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Example of Cournot's game: many firms

- So for Nash equilibrium

$$
\begin{aligned}
q_1^* &= \tfrac{1}{2}(\alpha - c \qquad\quad - q_2^* - q_3^* - \cdots - q_n^*) \\
q_2^* &= \tfrac{1}{2}(\alpha - c - q_1^* \qquad\quad - q_3^* - \cdots - q_n^*) \\
&\vdots \\
q_n^* &= \tfrac{1}{2}(\alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^* \qquad)
\end{aligned}
$$

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\end{aligned}
$$

- Multiply each equation by 2:

$$
\begin{aligned}
2q_1^* &= \alpha - c \phantom{{}- q_1^*} - q_2^* - q_3^* - \cdots - q_n^* \\
2q_2^* &= \alpha - c - q_1^* \phantom{{}- q_2^*} - q_3^* - \cdots - q_n^* \\
&\vdots \\
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Example of Cournot's game: many firms

- From previous slide:

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- Subtract q_i^* from both sides of each equation i:

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$$

- Right-hand side of every equation is the same! So

$$q_1^* = q_2^* = \cdots = q_n^*$$

Example of Cournot's game: many firms

- Let $q_1^* = q_2^* = q_3^* = q_4^* = \cdots = q_n^* = q^*$

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- So

$$q^* = \frac{\alpha - c}{n+1}$$

Example of Cournot's game: many firms

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- As n increases, this price decreases to c
- As number of firms increases, equilibrium outcome approaches competitive outcome

Comparison of Bertrand's and Cournot's games

Bertrand

- strategic variable is price

Cournot

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- Outcome $\rightarrow$ competitive as number of firms increases

Finding Nash equilibrium using best response functions

Technique used to find Nash equilibrium in example of Cournot's game can be used in other games

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1. Find the best response function b_i of each player i (optimization problem)
2. Find profiles a^* of actions for which

$$a_1^* = b_1(a_{-1}^*)$$
$$\vdots$$
$$a_n^* = b_n(a_{-n}^*)$$

where a_{-i}^* is the list of actions of the players other than i (typically n equations in n unknowns)

Finding Nash equilibrium using best responses

Example

Finding Nash equilibrium using best responses

Example

Players Two people

Finding Nash equilibrium using best responses

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Actions Each player can choose any nonnegative number

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Finding Nash equilibrium using best responses

Example

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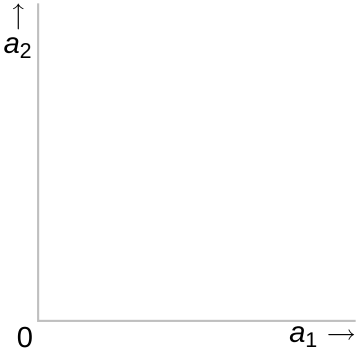

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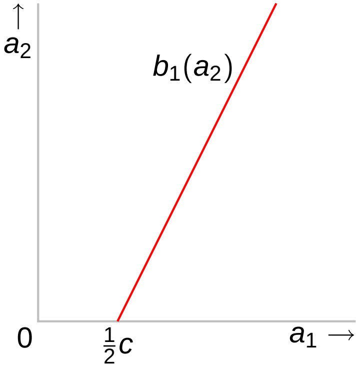

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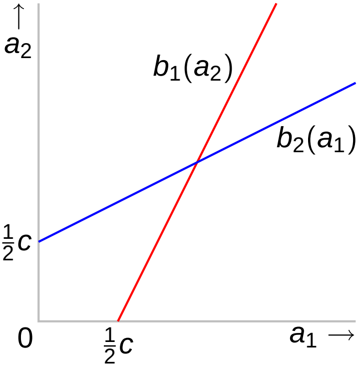

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Example

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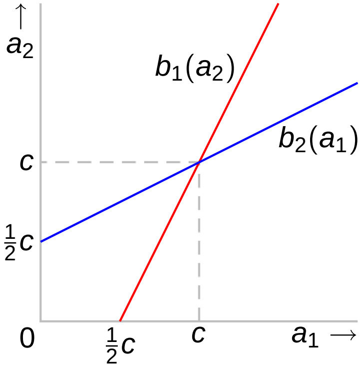
$$a_2^* = b_2(a_1^*) = \tfrac{1}{2}(c + a_1)$$



Unique Nash equilibrium: $(a_1^*, a_2^*) = (c, c)$

Finding Nash equilibrium using best responses

Example

Algebraically:

$$a_1^* = \frac{1}{2}(c + a_2^*)$$

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$$
\begin{aligned}
a_1^* &= \tfrac{1}{2}(c + a_2^*) \\
&= \tfrac{1}{2}(c + \tfrac{1}{2}(c + a_1^*))
\end{aligned}
$$

Finding Nash equilibrium using best responses

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Algebraically:

$$\begin{aligned} a_1^* &= \tfrac{1}{2}(c + a_2^*) \\ &= \tfrac{1}{2}(c + \tfrac{1}{2}(c + a_1^*)) \\ &= \tfrac{3}{4}c + \tfrac{1}{4}a_1^* \end{aligned}$$

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so

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   - Find profiles a^* of actions for which

     $$a_i^* = b_i(a_{-i}^*) \text{ for every player } i$$

     where a_{-i}^* is list of actions of other players (typically n equations in n unknowns)

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   - Can in principle be used for any game, but computing complete best response functions may be difficult and unnecessary (as for Bertrand's game)