Grover's Search Algorithm

- Large database : N elements.

$a_1, \ldots \to a_N$, Does $\exists x$ $a_x = y$.

↳ $O(N)$ time. Best, classically.

Q. Can we do better.
A: Yes, (in quantum) $\longrightarrow$ Grover's Search.

An oracle, $f : [N] \longrightarrow \{0, 1\}$.

Goal : Find ...

Find x s.t. $f(x) = 1$.

$$x \to \boxed{O_f} \to f(x)$$

Quantum oracle:

$$O_f(|x\rangle|y\rangle) = |x\rangle|y \oplus f(x)\rangle$$

input reg. $\lceil \log N \rceil$ qubits

o/p register (1 qubit).

$$O_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle$$

$$O_f|x\rangle|1\rangle = |x\rangle|1 \oplus f(x)\rangle$$

$$= |x\rangle |f(x)\rangle .$$

$$O_f |x\rangle |-\rangle = O_f \left(\frac{|x\rangle|0\rangle - |x\rangle|1\rangle}{\sqrt{2}} \right)$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

$$= \frac{|x\rangle|f(x)\rangle - |x\rangle|\overline{f(x)}\rangle}{\sqrt{2}}.$$

Case I: $f(x) = 0$.

$$O_f|x\rangle|-\rangle = \frac{|x\rangle|0\rangle - |x\rangle|1\rangle}{\sqrt{2}}$$

$= |x\rangle|-\rangle$

Case II : $f(x) = 1$

$$O_f|x\rangle|-\rangle = \frac{|x\rangle|1\rangle - |x\rangle|0\rangle}{\sqrt{2}}$$

$$= -|x\rangle|-\rangle$$

$$\boxed{O_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle}$$

phase oracle

- Query superposition ...

[illegible]

- start w/. $|\psi_0\rangle = |U\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{x} |x\rangle$

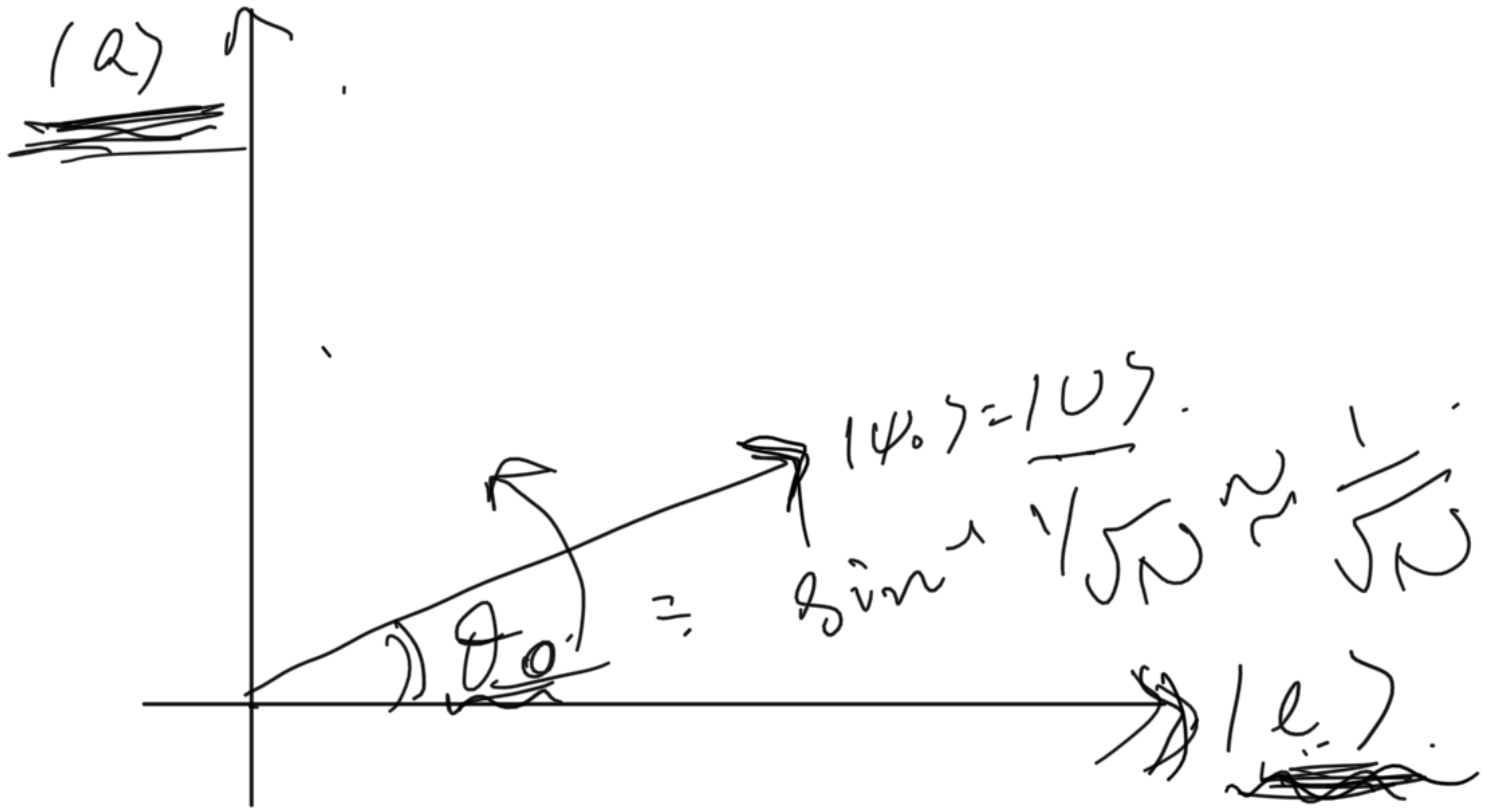
Assume $\boxed{f(a) = 1}$, $f(x) = 0$
$\forall x \neq a$.

$$|\psi_0\rangle = \frac{1}{\sqrt{N}}\left(|a\rangle + \sum_{x \neq a} |x\rangle\right).$$

$$= \frac{1}{\sqrt{N}} |a\rangle + \frac{\sqrt{N-1}}{\sqrt{N}} \frac{\sum_{x \neq a} |x\rangle}{\sqrt{N-1}}$$

$$= \frac{1}{\sqrt{N}}|a\rangle + \sqrt{1-\frac{1}{N}}\,|e\rangle$$

Nay



$$\cos\theta_0 = \frac{|\langle e|U\rangle|}{\quad}$$

$$= \sqrt{\quad}$$

$$= \sqrt{1 - \frac{1}{N}}$$

$$\Rightarrow \sin\theta_0 = \underline{\frac{1}{\sqrt{N}}}$$

Oracle.

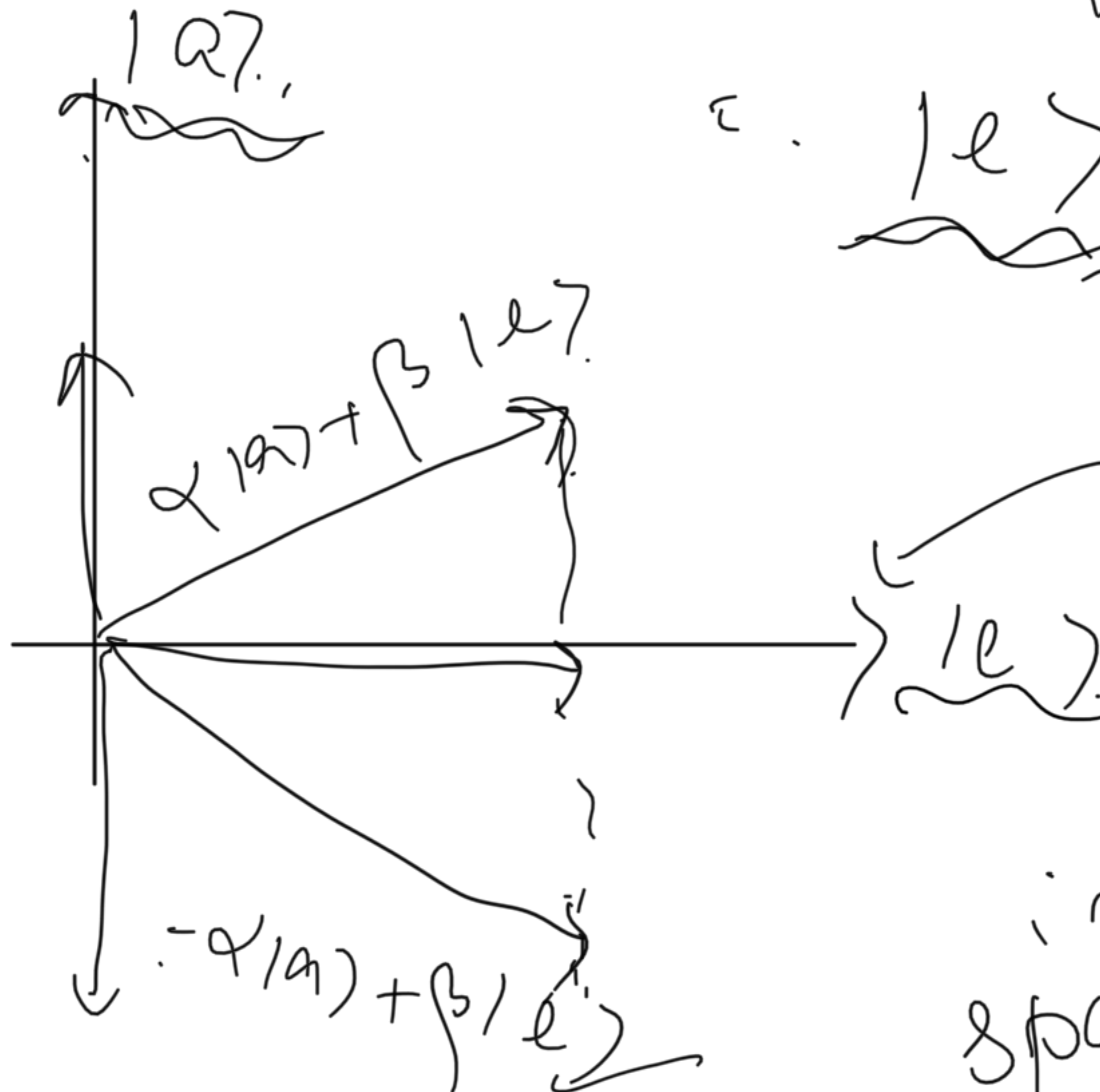
$$O_f |a\rangle \boxed{|-\rangle} = (-1)^{f(a)} |a\rangle |-\rangle$$

$$= -|a\rangle |-\rangle$$

$$O_f |e\rangle |-\rangle = O_f \left(\frac{\sum_{x \neq a} |x\rangle}{\sqrt{N-1}} |-\rangle \right)$$

$$= (-1) \sum_{x \neq a} \frac{|x\rangle}{\sqrt{N-1}} \, |-\rangle$$

$$= |e\rangle \boxed{|-\rangle}$$

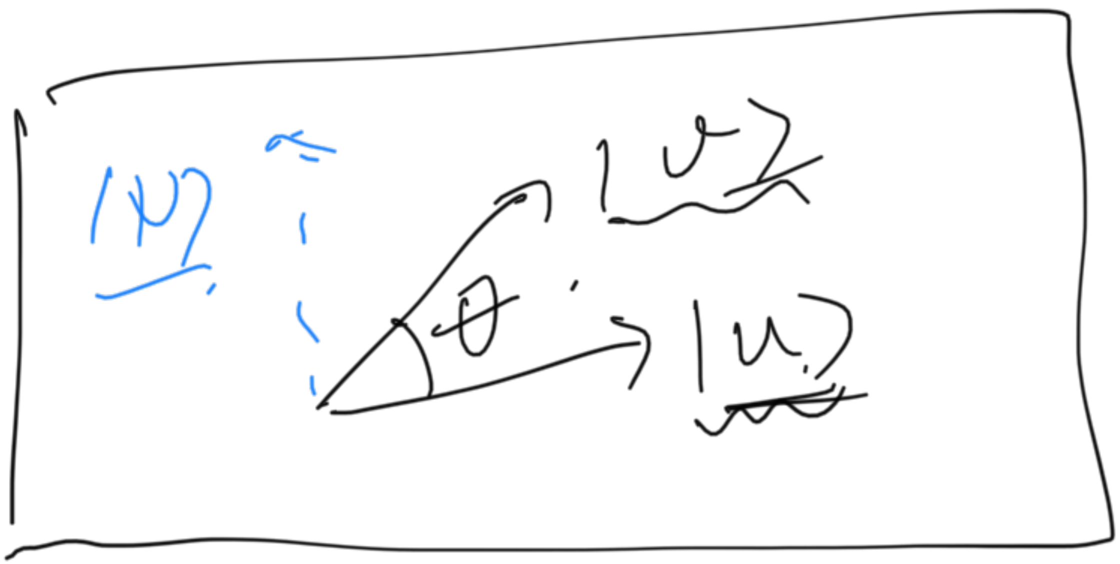


Reflection about $|e\rangle$ in span$\{|e\rangle, |a\rangle\}$

Q. Can we achieve rotation through reflections?

reflection?

- Can achieve this from 2 consecutive reflections!

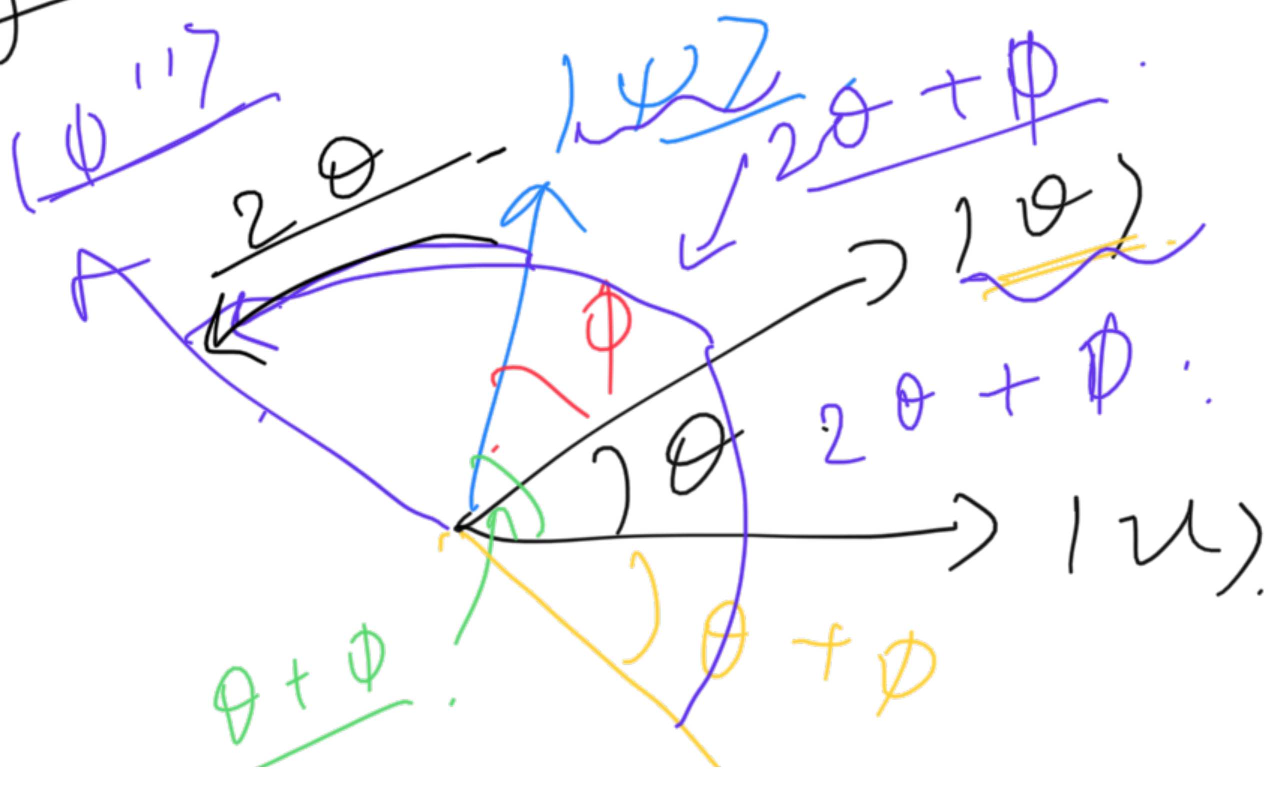


$$R_{|v\rangle}\left(R_{|u\rangle}|\psi\rangle\right)$$

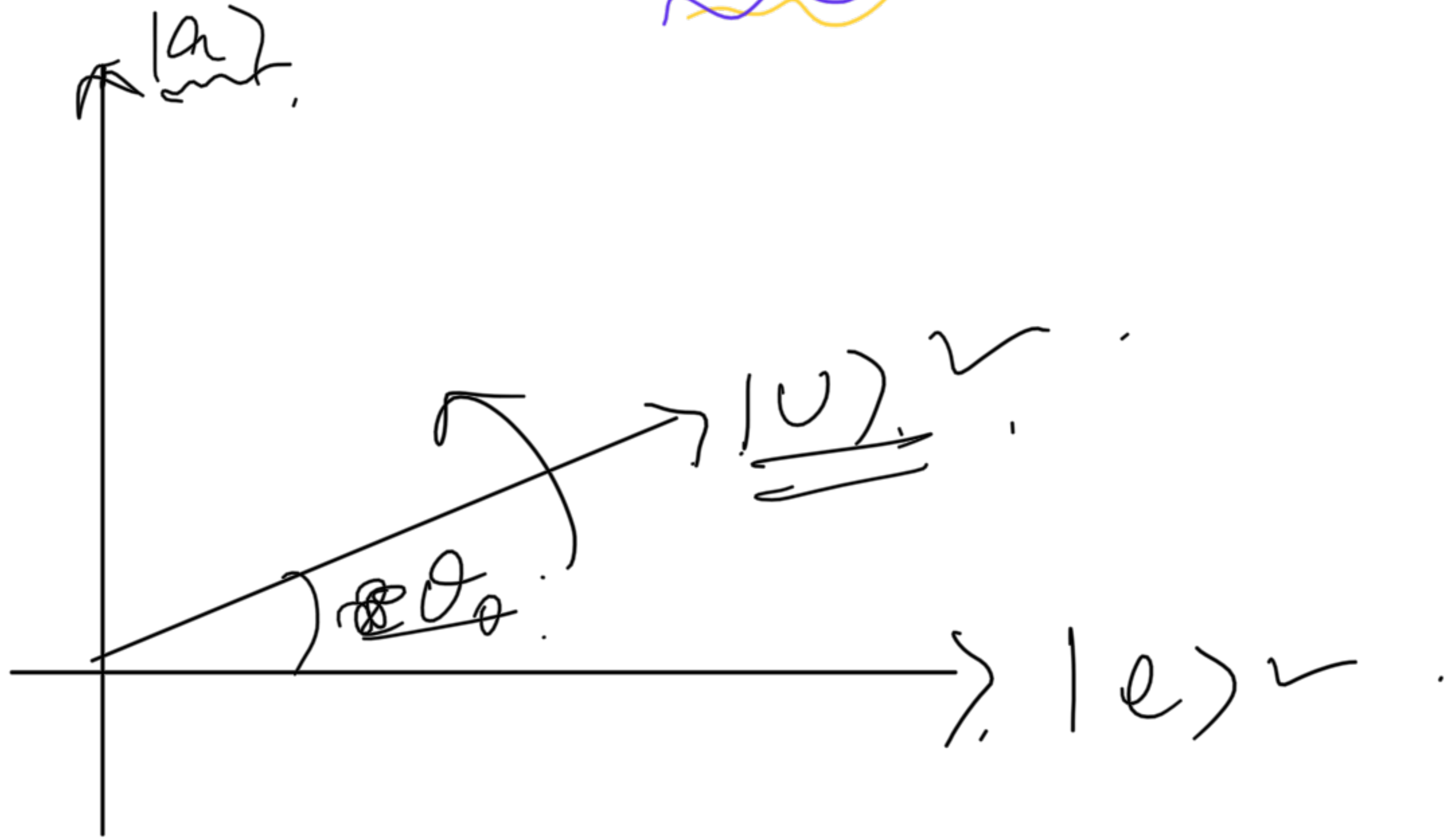
Lemma: • 2 successive reflections

in a plane lead to a rotation by 2θ, where θ = angle b/w the consecutive reflecting lines.

Pf:



$|\psi'\rangle$



- Consider a pair of reflⁿ. abt $|e\rangle$ i.e. $R_{|e\rangle}$ and abt $|v\rangle$ i.e. $R_{|v\rangle}$

$\rightsquigarrow$ leads to rotn by $2\theta_0$

$\approx \frac{2}{\sqrt{N}}$.

Q. Can we reflect abt. $|u\rangle$?

$N = 2^n$

$|u\rangle = H^{\otimes n}|0\cdots 0\rangle$

$= \frac{1}{\sqrt{2^n}} \sum |x\rangle$

$V2^n \quad x$

$\Rightarrow |v\rangle = H^{\otimes n} |0 \cdots 0\rangle$.

& $|0 \cdots 0\rangle = (H^{\otimes n})^{+} |v\rangle$.

$= H^{\otimes n} |v\rangle$.

$R_{|0 \cdots 0\rangle} |0 \cdots 0\rangle$

$= |0 \cdots 0\rangle$.

$|b_1 \cdots b_n\rangle$

$R_{|0 \cdots 0\rangle} = -|b_1 \cdots b_n\rangle$

$\ldots b_n \;\ldots\; b_n + 0 \ldots 0$

$$R_{|0\cdots0\rangle} = (-1)^{b_1 \vee \cdots \vee b_n}$$

$$R_{|N\rangle} = H^{\otimes n} R_{|0\ldots0\rangle} H^{\otimes n}$$

Grover's Algorithm:

- (1) Create $|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle$

  (for $N = 2^n$, $H^{\otimes n}|0\ldots0\rangle$)

② For $t = 1 \ldots T$ do {

(a) Reflect $|\psi_{t-1}\rangle$ abt $|e\rangle$ using O_f to set $|\tilde{\psi}_t\rangle$

(b) Reflect $|\tilde{\psi}_t\rangle$ abt $|v\rangle$ to create $|\psi_t\rangle$ }

Analysis:

- Each step (a) & (b) combined rotate by $2\theta_0$.

- Initial angle: θ_0.
- After T iterations,
  angle $= (2T+1)\,\theta_0$.

$$\frac{\pi}{4} \le (2T+1)\,\theta_0 \le 3\pi/4.$$

$$\Rightarrow \underline{2T+1} \geq \frac{\pi}{4\theta_0}.$$

$$\Rightarrow \underset{\sim}{T} \geq \frac{\frac{\pi}{4\theta_0} - 1}{2}.$$

choose $T = \left\lceil \frac{\pi}{8\theta_0} - 1/2 \right\rceil.$

$$\theta_0 = \sin^{-1} 1/\sqrt{N} \geq \frac{1}{\sqrt{N}}.$$

$$T \leq O(\sqrt{N}).$$

- Perform measurement in

Std. basis $\longrightarrow |a\rangle$

w/p $\geq \sin^2 \pi/4 = 1/2$

Randomized algo:

Running time: $O(\sqrt{N})$

$P_{success} \geq 1/2$