习题四.

5. $E(X(t)) = E[\cos(w_0 t+\xi)] = E[\cos(w_0 t)\cos(\xi) - \sin(w_0 t)\sin(\xi)]$

$= \cos(w_0 t) E[\cos(\xi)] - \sin(w_0 t) E[\sin(\xi)]$. ①

$R_X(t,t+\tau) = E[\cos(w_0 t+\xi)\cos(w_0(t+\tau)+\xi)]$

$= E\{\frac{1}{2}[\cos(w_0\tau) + \cos(w_0\tau + 2w_0 t + 2\xi)]\}$

$= \frac{1}{2}\cos(w_0\tau) + \frac{1}{2}E[\cos(w_0\tau + 2w_0 t + 2\xi)]$

$= \frac{1}{2}\cos(w_0\tau) + \frac{1}{2}\cos(w_0\tau + 2w_0 t)E[\cos(2\xi)] - \frac{1}{2}\sin(w_0\tau + 2w_0 t)E[\sin(2\xi)]$ ②

$X(t)$为平稳过程 $\Leftrightarrow \begin{cases} E[X(t)] = C & (与t无关) \\ R_X(t,t+\tau) = R_X(\tau). & (与t无关). \end{cases}$

注意.①和②中 $E[\cos(\xi)]$. $E[\sin(\xi)]$. $E[\cos(2\xi)]$. $E[\sin(2\xi)]$ 均为常数.

①式与t无关 $\Leftrightarrow E[\cos(\xi)] = E[\sin(\xi)] = 0$,

②式与t无关 $\Leftrightarrow E[\cos(2\xi)] = E[\sin(2\xi)] = 0$, 故.

$X(t)$为平稳过程 $\Leftrightarrow E[\cos(\xi)] = E[\sin(\xi)] = E[\cos(2\xi)] = E[\sin(2\xi)] = 0$.

$\Leftrightarrow \begin{cases} \varphi(1) = E(e^{j\cdot 1\cdot \xi}) = E[\cos(\xi)] + jE[\sin(\xi)] = 0. \\ \varphi(2) = E(e^{j2\xi}) = E[\cos(2\xi)] + jE[\sin(2\xi)] = 0. \end{cases}$ 证毕.

8. 证: ① $E[X^2(t)] = R_X(t,t) = R_X(t-t) = R_X(0)$ 为常数.

② $R_{X^2}(t,t+\tau) = E[X^2(t)X^2(t+\tau)] \xlongequal[\text{由例 例3.13.(2)知}]{X(t)为正态过程}$

$= E[X^2(t)]E[X^2(t+\tau)] + 2[E(X(t)X(t+\tau))]^2$

$= R_X^2(0) + 2R_X^2(\tau)$ 故只与τ有关.

$\therefore \{X^2(t), t\in R\}$ 为平稳过程.

11. 解：$E(\xi)=\int_0^1 E[X(t)]dt=\int_0^1 1\,dt=1$.

$E[\xi^2]=E\left[\int_0^1 X(t)dt\right]^2=\int_0^1\int_0^1 R_X(t-s)ds\,dt=2\int_0^1(1-\tau)R(\tau)d\tau$

$=2\int_0^1(1-\tau)(1+e^{-2|\tau|})d\tau=2\int_0^1(1-\tau)(1+e^{-2\tau})d\tau$

$=\frac{3}{2}+\frac{1}{2}e^{-2}$.

$D(\xi)=E(\xi^2)-E^2(\xi)=\frac{1}{2}+\frac{1}{2}e^{-2}$.

题目中 Y 应为 $\frac{1}{2T}\int_{-T}^{T}X(t)dt$

12. (1) $m_Y=E[Y]=\frac{1}{2T}\int_{-T}^{T}E(X(t))dt=0$.

(2) $E(Y^2)=\frac{1}{4T^2}E\left[\int_{-T}^{T}X(t)dt\right]^2=\frac{1}{4T^2}\times 2\int_0^{2T}$ ~~$(2T-\theta)R_X(\theta)d\theta$~~ $(2T-\tau)R_X(\tau)d\tau$

~~$=\frac{1}{2\tau^2}\int_0^{2\tau}(2\tau-\theta)e^{-2\lambda\theta}d\theta$~~

$=\frac{1}{2T^2}\int_0^{2T}(2T-\tau)e^{-2\lambda\tau}d\tau$.

$=\frac{1}{2\lambda T}-\frac{1-4\lambda T}{8\lambda^2T^2}$ （积分过程类似11.）

13. 因 $R(\tau)=e^{-|\tau|}\cos\tau$ 在 $\tau=0$ 处连续，故均方连续，且均方可积，又

$R'(0+)=\lim_{\tau\to 0^+}\frac{R(\tau)-R(0)}{\tau}=\lim_{\tau\to 0^+}\frac{e^{-\tau}\cos\tau}{\tau}=\lim_{\tau\to 0^+}\frac{(1-\tau+\frac{\tau^2}{2!}+o(\tau^2))(1-\frac{\tau^2}{2!}+o(\tau^4))}{\tau}$

$=+\infty$

$R'(0-)=\lim_{\tau\to 0^-}\frac{R(\tau)-R(0)}{\tau}=\lim_{\tau\to 0^-}\frac{e^{\tau}\cos\tau}{\tau}=\lim_{\tau\to 0^-}\frac{[1+\tau+o(\tau^2)][1-\frac{\tau^2}{2!}+o(\tau^2)]}{\tau}=-\infty$.

故 $R(\tau)$ 在 $\tau=0$ 处一阶导数不存在，二阶导数亦不存在，故 $X(t)$ 不均方可导。

若 $E[X(t)]=0$，(题目中缺条件)

$\lim_{\tau\to+\infty}R_X(\tau)=\lim_{\tau\to+\infty}e^{-\tau}\cos\tau=0=E[X(t)]$ 故满足遍历性（均值）

14. 见复习材料第五题.

17. 证 ① ∵ ξ、η 相互独立

$$E[X(t)]=E(\xi)E[\cos(\beta t+\eta)]=0.$$

$$R_X(t,t+\tau)=E\left[\xi^2\cdot\cos(\beta t+\eta)\cos(\beta(t+\tau)+\eta)\right]$$

$$=E[\xi^2]\cdot E\left[\frac{1}{2}\cos\beta\tau+\frac{1}{2}\cos(2\beta t+\beta\tau+2\eta)\right]=\frac{1}{2}\cos\beta\tau+\frac{1}{2}E\left[\cos(2\beta t+\beta\tau+2\eta)\right]$$

$$=\frac{1}{2}\cos\beta\tau+\frac{1}{2}\int_0^{2\pi}\cos(2\beta t+\beta\tau+2\theta)\cdot\frac{1}{2\pi}d\theta=\frac{1}{2}\cos\beta\tau$$

故 $X(t)$ 为平稳过程.

又 $\lim\limits_{T\to\infty}\frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)C_X(\tau)\,d\tau$

$$=\lim_{T\to\infty}\frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)R_X(\tau)d\tau=\lim_{T\to\infty}\frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)\frac{1}{2}\cos\beta\tau\,d\tau$$

$$=\underbrace{\lim_{T\to\infty}\frac{1}{T}\frac{1}{2\beta}\underset{\text{sin 有界}}{\underbrace{\sin\beta\tau}}\Big|_0^{2T}}_{=0}-\lim_{T\to\infty}\frac{1}{4T^2}\int_0^{2T}\tau\cos\beta\tau\,d\tau=-\lim_{T\to\infty}\frac{1}{4T^2}\int_0^{2T}\tau\cos\beta\tau\,d\tau$$

$$=-\lim_{T\to\infty}\frac{1}{4T^2}\int_0^{2T}\tau\,d\left(\frac{\sin\beta\tau}{\beta}\right)=-\frac{1}{4T^2}\tau\frac{\sin\beta\tau}{\beta}\Big|_0^{2T}+\lim_{T\to\infty}\frac{1}{4T^2}\int_0^{2T}\underbrace{\frac{\sin\beta\tau}{\beta}}_{\text{有界}}d\tau.$$

$=0.$

故均值满足均方遍历性.

注：此题用充分条件做有问题.

20. 此题不作要求. 有两种方法. ① 用定义积分需用留数的知识 ② 用傅立叶变换及性质.
仅提供第1种做法：(不做要求).

$$R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S(\omega)e^{j\omega\tau}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{(1+\omega^2)^2}e^{j\omega\tau}d\omega\quad \text{令 } f(\omega)=\frac{1}{(1+\omega^2)^2}e^{j\omega\tau}$$

$=\frac{1}{2\pi}\cdot 2\pi j\times \mathrm{Res}(f(\omega),j)$ (*) 因 $f(\omega)=\dfrac{e^{j\omega\tau}}{(\omega+j)^2(\omega-j)^2}$ 在 $\omega=j$ 有二阶极点

$$\mathrm{Res}(f(\omega),j)=\lim_{\omega\to j}\frac{d}{d\omega}\left[(\omega-j)^2f(\omega)\right]=\lim_{\omega\to j}\frac{d}{d\omega}\left(\frac{e^{j\omega\tau}}{(\omega+j)^2}\right)=\frac{j\tau e^{j\omega\tau}(\omega+j)^2-e^{j\omega\tau}2(\omega+j)}{(\omega+j)^4}\Bigg|_{\omega=j}$$

$=-\frac{1}{4}(j\tau e^{-\tau}+je^{-\tau})$ 代入(*)式有 $R(\tau)=\frac{1}{4}(\tau e^{-\tau}+e^{-\tau})$.

21. $R(\tau)=\frac{1}{2\pi}\int_{-1}^{1}(1-|w|)e^{jw\tau}dw=\frac{1}{\pi}\int_{0}^{1}(1-w)\cos w\tau dw=\frac{1}{\pi}\int_{0}^{1}\cos w\tau d\tau-\frac{1}{\pi}\int_{0}^{1}w\cos w\tau dw$

$$=\frac{1}{\pi\tau^2}(1-\cos\tau).$$

23. $S(w)=\int_{-\infty}^{+\infty}R(\tau)e^{-jw\tau}d\tau$

$$=\int_{-1}^{0}(1+\tau)e^{-jw\tau}d\tau+\int_{0}^{1}(1-\tau)e^{-jw\tau}d\tau$$

$$=-\frac{1}{jw}e^{-jw\tau}\Big|_{-1}^{0}+\frac{\tau}{jw}e^{-jw\tau}\Big|_{-1}^{0}-\frac{e^{-jw\tau}}{(jw)^2}\Big|_{-1}^{0}+(-\frac{1}{jw}e^{-jw\tau}\Big|_{0}^{1}$$

$$-\frac{\tau}{jw}e^{-jw\tau}\Big|_{0}^{1}+\frac{e^{-jw\tau}}{(jw)^2}\Big|_{0}^{1}$$

$$=\frac{2-2\cos w}{w^2}.$$