



Multilevel Modeling

Day 2

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Standard regression equation:

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

```
> grade.mod <- lm(grade ~ skipped_class, data = grade_data)
```

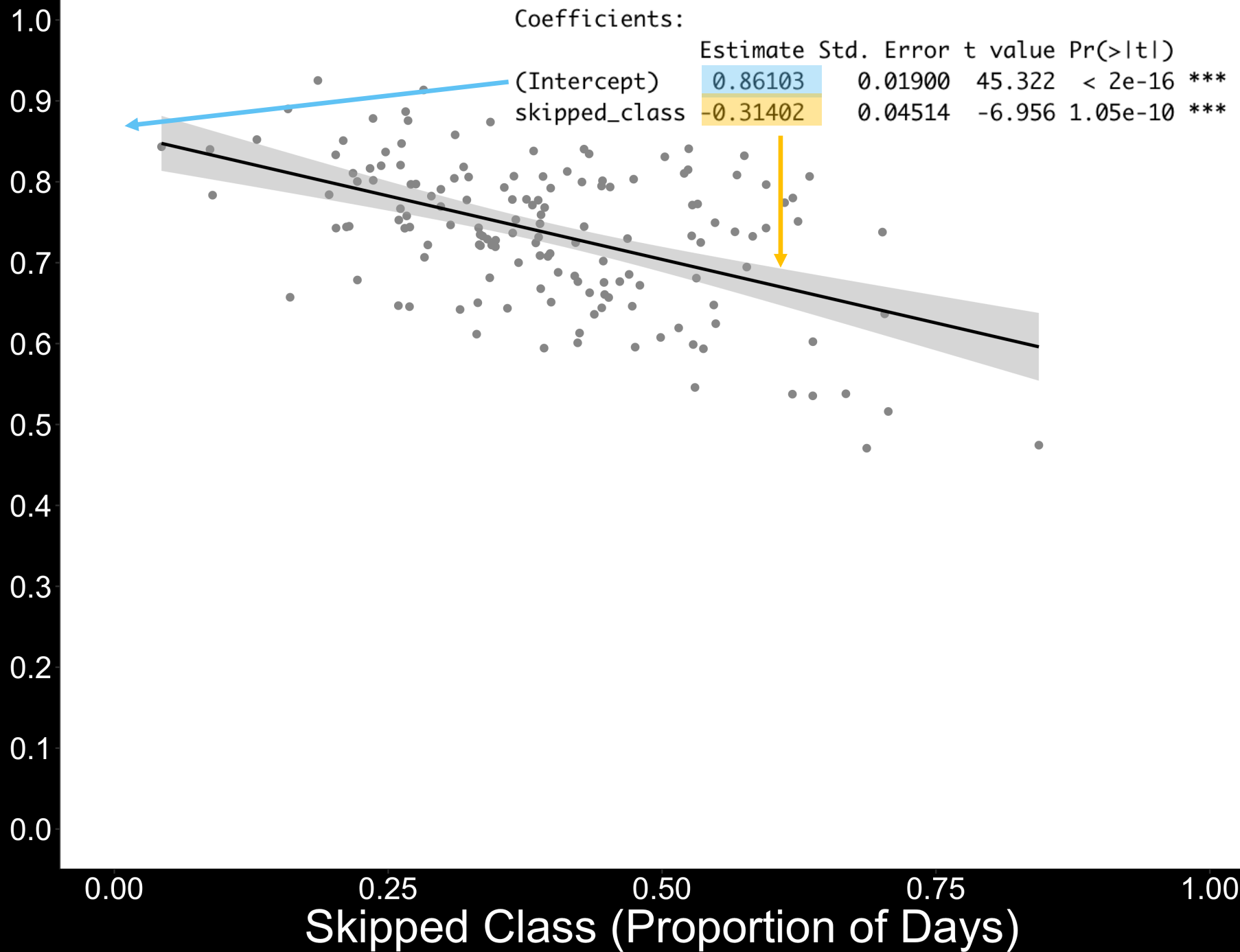
```
> summary(grade.mod)
```

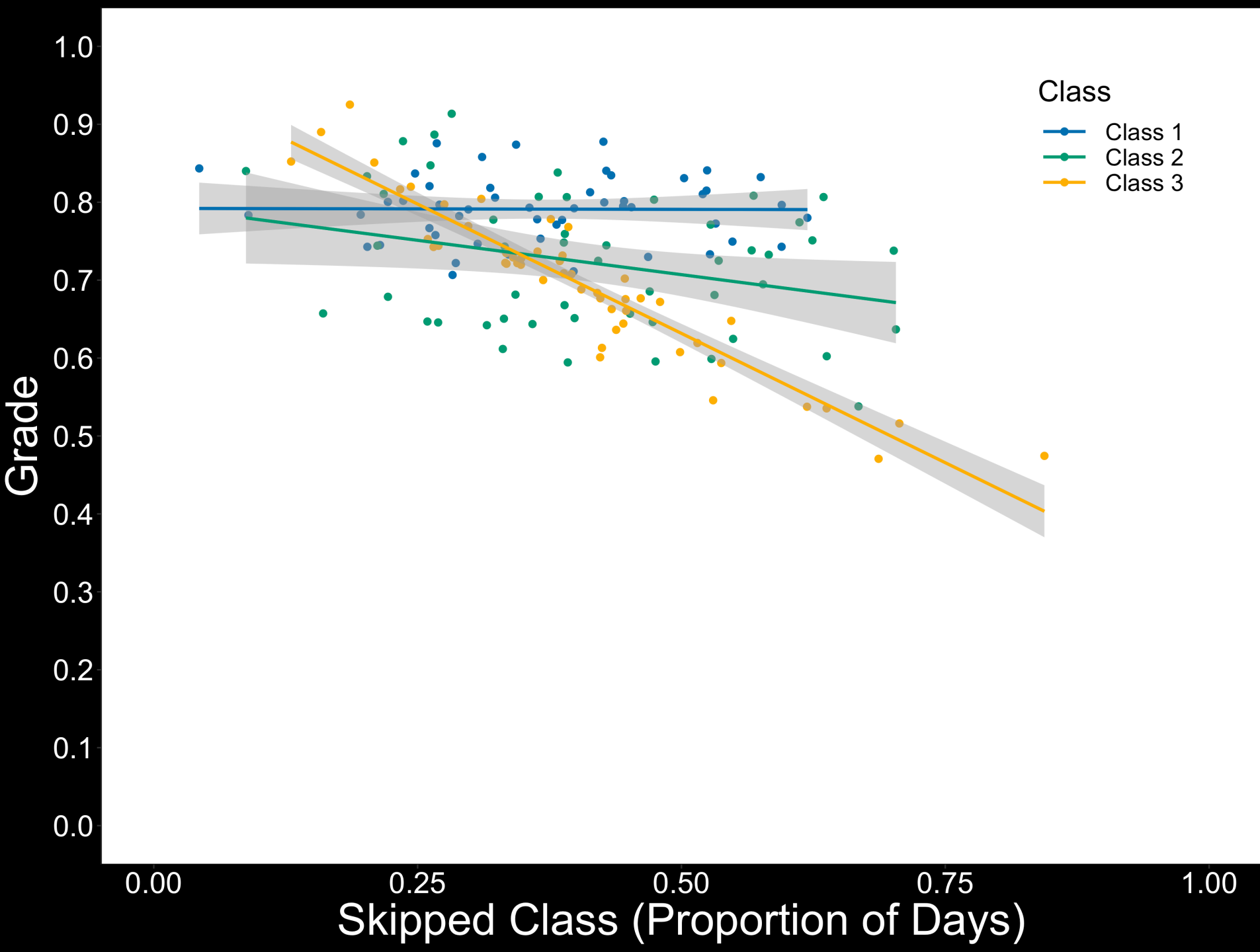
Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	0.86103	0.01900	45.322	< 2e-16	***
skipped_class	-0.31402	0.04514	-6.956	1.05e-10	***

$$\hat{grade}_i = 0.861 - 0.314 X_i$$

Grade



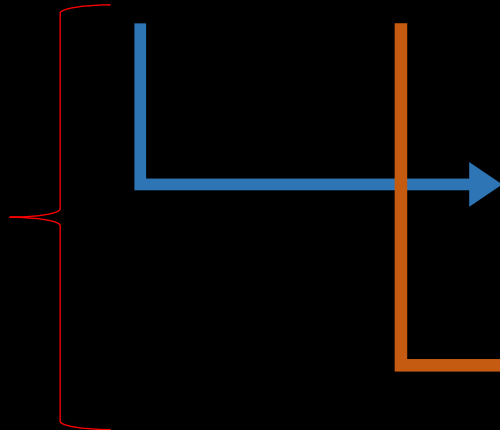


Multilevel model

Level 1

$$\hat{grade}_i = 0.861 - 0.314X_i$$

Level 2


$$\begin{aligned} \text{intercept}_j &= 0.861 + u_{0j} \\ \text{slope}_j &= -0.314 + u_{1j} \end{aligned}$$

Multilevel model

$$grade_{i,j} = \underbrace{(0.861 + u_{0j})}_{\text{Intercept}} + \underbrace{(-0.314 + u_{1j})}_{\text{Slope}} X_{ij} + \epsilon_{ij}$$

$\hat{\beta}_{0j}$ $\hat{\beta}_{1j}$

Multilevel model

$$grade_{i,j} = \underbrace{(0.861 + u_{0j})}_{\text{Intercept}} + \underbrace{(-0.314 + u_{1j})}_{\text{Slope}} X_{ij} + \epsilon_{ij}$$

$\hat{\beta}_{0j}$ $\hat{\beta}_{1j}$

Multilevel model

Fixed effects

Random effects

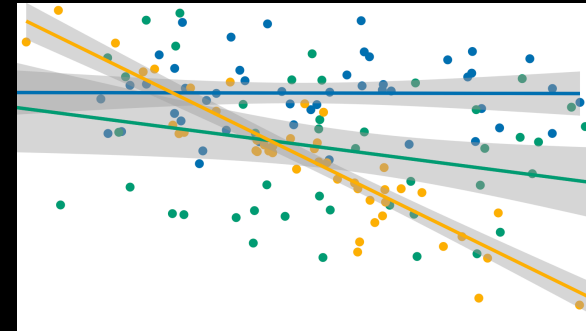
$$grade_{i,j} = (0.861 + u_{0j}) + (-0.314 + u_{1j})X_{ij} + \epsilon_{ij}$$

Intercept

Slope

$$\hat{\beta}_{0j}$$

$$\hat{\beta}_{1j}$$



Multilevel model

Class 1 $\hat{grade}_{i,1} = 0.792 - 0.002X_i$

Class 3 $\hat{grade}_{i,3} = 0.964 - 0.664X_i$

$$\hat{grade}_i = 0.861 - 0.314X_i$$

Multilevel model

Class 1 $\hat{grade}_{i,1} = 0.792 - 0.002X_i$

Class 3 $\hat{grade}_{i,3} = 0.964 - 0.664X_i$

	Δ Intercept	Δ Slope
Class 1	-0.069	0.312
Class 3	0.103	-0.350

$$\hat{grade}_i = 0.861 - 0.314X_i$$

Multilevel model

$$grade_{i,j} = (0.861 + u_{0j}) + (-0.314 + u_{1j})X_{ij} + \epsilon_{ij}$$

Intercept

$$\hat{\beta}_{0j}$$

Slope

$$\hat{\beta}_{1j}$$

Multilevel model

$$grade_{i,j} = (0.861 + u_{0j}) + (-0.314 + u_{1j})X_{ij} + \epsilon_{ij}$$

$$grade_{i,1} = (0.861 + -0.069) + (-0.314 + 0.312)X_i + \epsilon_i$$

$$grade_{i,1} = (0.792) + (-0.002)X_i + \epsilon_i$$

	Δ Intercept	Δ Slope
Class 1	-0.069	0.312
Class 3	0.103	-0.350

Multilevel model

$$grade_{i,j} = (0.861 + u_{0j}) + (-0.314 + u_{1j})X_{ij} + \epsilon_{ij}$$

$$grade_{i,1} = (0.861 + -0.069) + (-0.314 + 0.312)X_i + \epsilon_i$$

$$grade_{i,1} = (0.792) + (-0.002)X_i + \epsilon_i$$

$$grade_{i,3} = (0.861 + 0.103) + (-0.314 + -0.350)X_i + \epsilon_i$$

$$grade_{i,3} = (0.964) + (-0.664)X_i + \epsilon_i$$

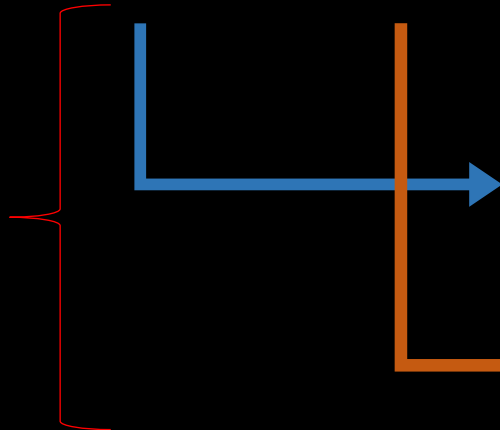
	Δ Intercept	Δ Slope
Class 1	-0.069	0.312
Class 3	0.103	-0.350

Multilevel model

Level 1

$$\hat{grade}_i = 0.861 - 0.314X_i$$

Level 2



The diagram illustrates the relationship between the Level 1 equation and the Level 2 components. A red bracket on the left groups the Level 2 equations. A blue arrow points from the intercept term (0.861) of the Level 1 equation to the intercept equation in Level 2. An orange arrow points from the slope term (-0.314) of the Level 1 equation to the slope equation in Level 2.

$$\begin{aligned} \text{intercept}_j &= 0.861 + u_{0j} \\ \text{slope}_j &= -0.314 + u_{1j} \end{aligned}$$

Multilevel model

Level 1

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + \epsilon_{ij}$$

Level 2

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + u_{1j}$$

$$Y_{ij} = (\gamma_{00} + u_{0j}) + (\gamma_{10} + u_{1j})X_{ij} + \epsilon_{ij}$$

Multilevel model

Level 1

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + \epsilon_{ij}$$

Level 2


$$\beta_{0j} = \gamma_{00} + u_{0j}$$

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Multilevel model

Level 1

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Multilevel model

Level 1

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + \epsilon_{ij}$$

Random effects

Fixed effects

Level 2

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + u_{1j}$$

$$Y_{ij} = (\gamma_{00} + u_{0j}) + (\gamma_{10} + u_{1j})X_{ij} + \epsilon_{ij}$$

Some notes

- Level 1 random effects (residuals), ϵ_{ij} , vary *within* level 2 units

$$\epsilon_{ij} \sim N(0, \sigma^2)$$

- Constant *across* level 2 units
- Level 1 coefficients (β_{0j} β_{1j}) are constant *within* level 2 units but vary *across* level 2 units

Some notes

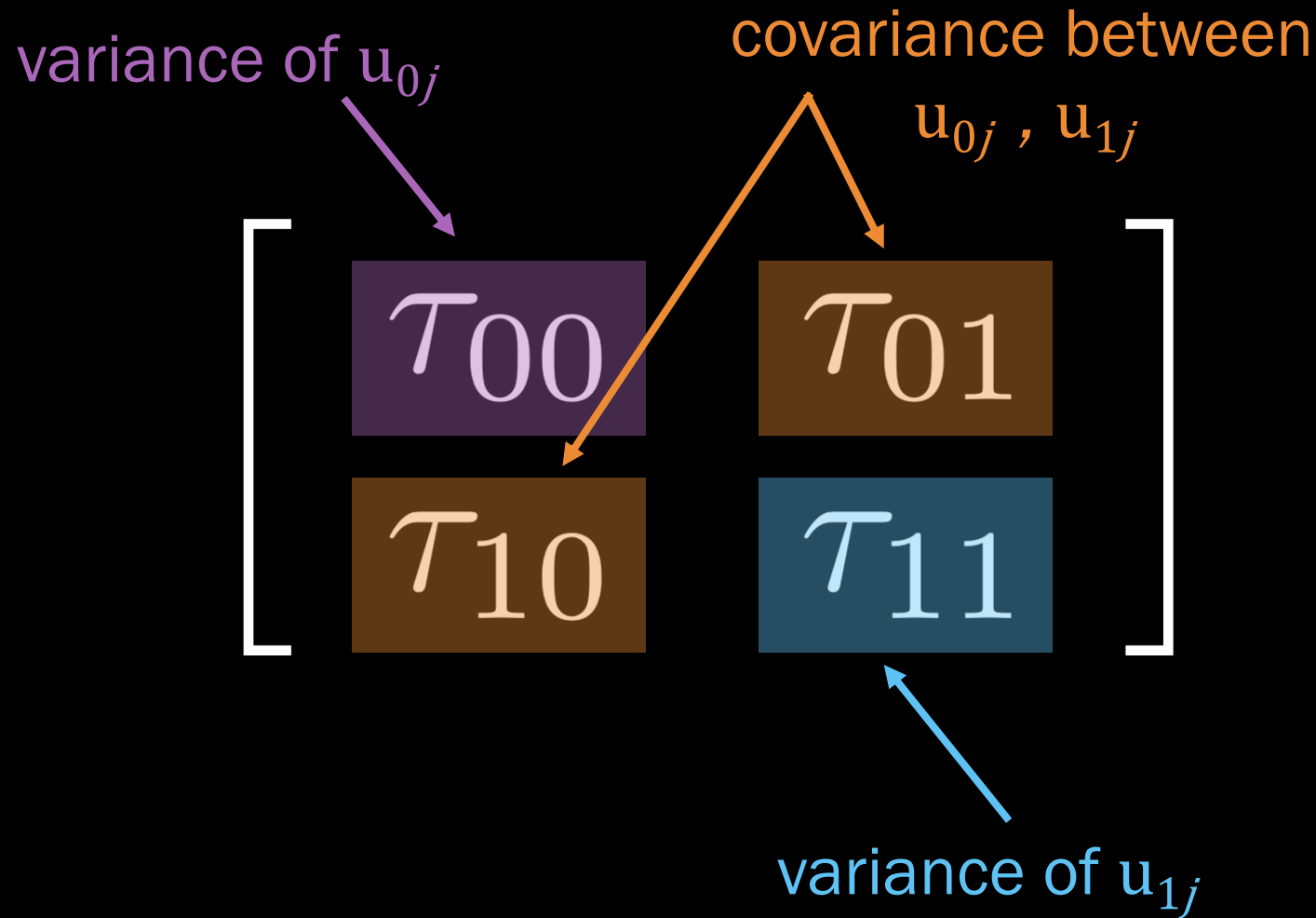
- Level 2 random effects/residuals (u_{0j} , u_{1j}) vary across level 2 units

$$u_{0j} \sim N(0, \tau_{00})$$

$$u_{1j} \sim N(0, \tau_{11})$$

- u_{0j} , u_{1j} can be correlated! But are uncorrelated with ϵ_{ij}
 - Tau matrix, T, contains variance and covariance for random effects

Tau (T) matrix



Some notes

- Level 2 coefficients (γ_{00} , γ_{10}) are constant across level 2 units
 - Fixed effects!

Implementation in R

```
> install.packages("lme4")
```

```
> library(lme4)
```

- **Fixed effect:** proportion of classes skipped
- **Random effects:**
 - by-class random intercepts
 - by-class random slopes for proportion of classes skipped

Implementation in R

```
> lmer(grade ~ 1 + skipped_class +  
        (1 + skipped_class | Class),  
        data = grade_data)
```


Implementation in R

```
> lmer(grade ~ 1 + skipped_class +  
      (1 + skipped_class|Class),  
      data = grade_data)
```

skipped_class	grade	Class
0.645541354	0.4810248	Class 3
0.158089206	0.5184626	Class 2
0.558733023	0.5245797	Class 3
0.392118288	0.5292537	Class 2
0.544739291	0.5686268	Class 3
0.574664954	0.5766249	Class 3

Implementation in R

Fixed effects:

	Estimate	Std. Error	t value
(Intercept)	0.84229	0.05101	16.512
skipped_class	-0.25669	0.17380	-1.477

Implementation in R

Fixed effects:

	Estimate	Std. Error	t value
(Intercept)	0.84229	0.05101	16.512
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Random effects:

Groups	Name	Variance	Std.Dev.	Corr
Class	(Intercept)	0.007011	0.08373	
	skipped_class	0.086015	0.29328	-0.91
Residual		0.004751	0.06893	

```
lmer(grade ~ skipped_class +  
      (1 + skipped_class|Class) +  
      data = grade_data)
```

Implementation in R

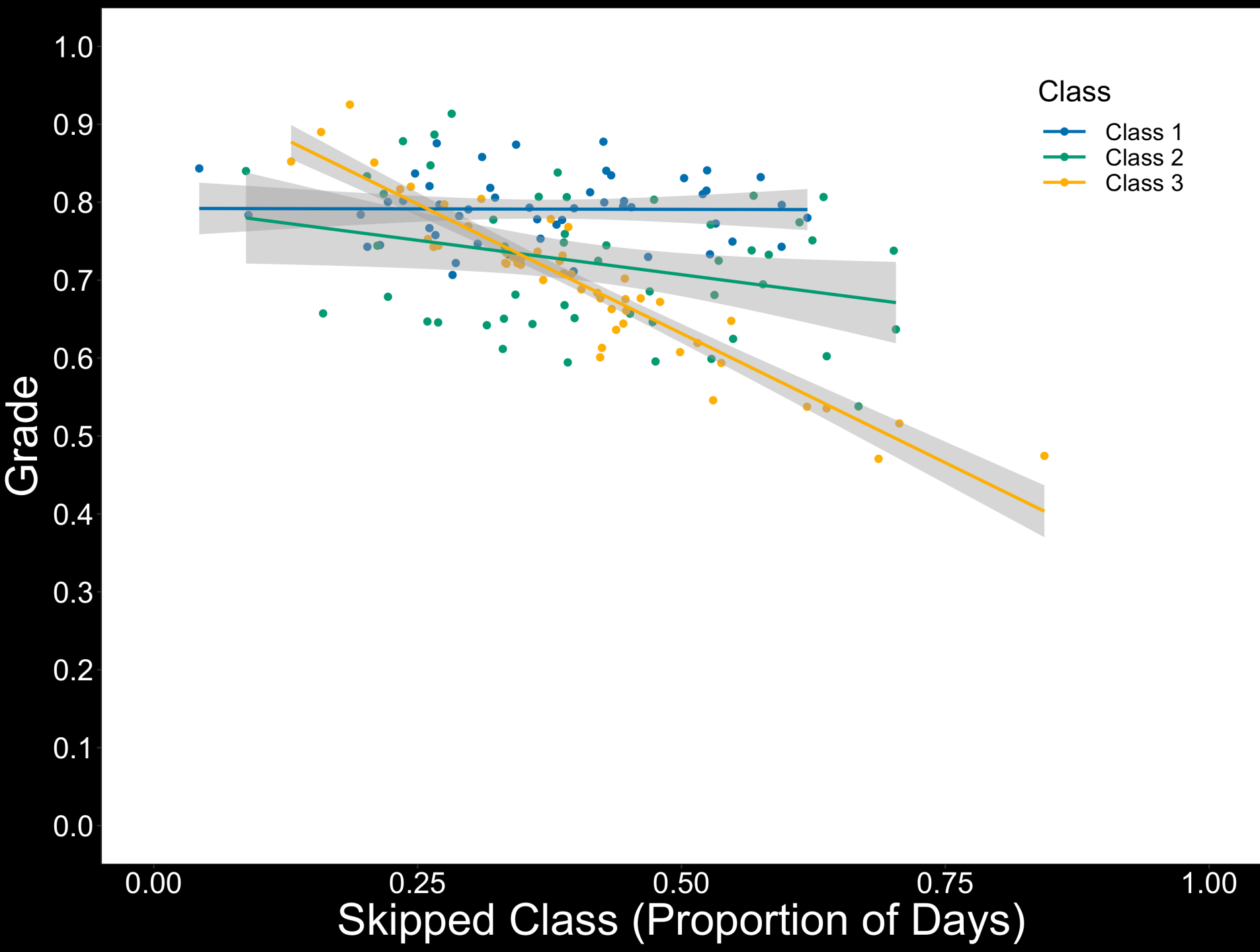
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Implementation in R

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$$\begin{bmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{bmatrix}$$

Whew!