Verifying Equivalence of Spark Programs

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──── Abstract ────

We present a novel approach for verifying the equivalence of Spark programs. Spark is a popular framework for writing large scale data processing applications. Such frameworks, intended for data-intensive operations, share many similarities with traditional database systems, but do not enjoy a similar support of optimization tools. Our goal is to enable such optimizations by first providing the necessary theoretical setting for verifying the equivalence of Spark programs. This is challenging because such programs combine relational algebraic operations with *User Defined Functions* (*UDF*s) and aggregate operations. We define a model of Spark as a programming language which imitates Relational Algebra queries in the bag semantics and allows for user defined functions expressible in Presburger Arithmetics as well as aggregations. We present a technique for verifying the equivalence of an interesting class of Spark programs.

1   Introduction

Unlike traditional relational databases, which are accessed using a standard query language, NoSQL databases are often accessed via an entire program. As NoSQL databases are typically huge, optimizing such programs is an important problem. Unfortunately, although query optimization has been studied extensively over the last few decades, the techniques presented in the past no longer carry over when access to the data is via a rich programming language.

Standard query optimization techniques are often based on the notion of query equivalence, i.e., the ability to determine if different queries are guaranteed to return the same results over all database inputs. If one can decide equivalence between queries, it may be possible to devise a procedure that simplifies a given query to derive a cheaper, yet equivalent, form. In addition, the ability to decide query equivalence is a necessary component to allow rewriting of queries with views or previously computed queries. This, again, is an important optimization technique.

This paper studies the equivalence problem for fragments of the Spark programming language. Spark is a popular framework for writing large scale data processing applications. It was developed in reaction to the data flow limitations of the Map-Reduce paradigm. Importantly, Spark programs are centered on the resilient distributed dataset (RDD) structure, which contains a bag of (distributed) items. An RDD r can be accessed using operations such as *map*, which applies a function to all items in r, *filter*, which filters items in r using a given Boolean function, and *fold* which aggregates items together, again using a user defined function (UDF). Intuitively, map, filter and fold can be seen as extensions to project, select and aggregation, respectively, with arbitrary UDFs applied. Besides accessing RDDs, Spark programs can also perform arithmetic and Boolean computations.

Unsurprisingly, in the general case, the problem of determining equivalence between Spark programs is undecidable. Thus, this paper focuses on fragments of Spark programs, and gives a sound condition for equivalence. For special cases, we also provide conditions that are complete for determining equivalence. In addition, we show that, even within our limited fragments, there are still undecidable cases.

The techniques used in this paper for determining equivalence of Spark programs are rather different than traditional query equivalence characterizations. Query equivalence is usually determined by either *(1)* showing that equivalence boils down to the existence of some special mapping (e.g., isomorphism, homomorphism) between the queries *or (2)* proving that equivalence over arbitrary databases can be determined by checking for equivalence over some finite set of canonical databases. (Actually, often, both such characterizations are shown.) In this paper we use a different approach. Intuitively, we present a method to translate Spark programs into an augmented version of Presburger arithmetic. Since equivalence of Presburger arithmetic is decidable, we derive decidability of Spark programs. We note that some of the intricacies arise from the fact RDDs are bags (and not individual items), and Spark programs can contain aggregation (using the fold operation).

2 Overview

We begin with a few example programs representative of the primary results of this paper. Below we show two equivalent programs in a simple language we define, called SparkLite. SparkLite is a general purpose programming language with a special data type called RDD and operations on its. Programs have straight-line-code with conditions, but no loops.

	P1$(R \colon RDD_{\mathtt{Int}})$**:**	**P2**$(R \colon RDD_{\mathtt{Int}})$**:**
1	$R' = \mathtt{map}(\lambda x.2 * x)(R)$	$R' = \mathtt{filter}(\lambda x.x \geq 50)(R)$
2	$\mathtt{return}\ \mathtt{filter}(\lambda x.x \geq 100)(R')$	$\mathtt{return}\ \mathtt{map}(\lambda x.2 * x)(R')$

The variables of RDD type represent our dataset, and $\mathtt{map}$, $\mathtt{filter}$ are operations that apply a function on each element in the dataset. The functions given to the operations are expressible in Presburger Arithmetic (theory of integers with addition, without multiplication). To verify the equivalence, we encode this program symbolically over First-Order Logic, such that the question of equivalence boils down to proving the validity of a formula. In this example, we encode $P1$ as: $ite(2 * x < 100, 2 * x, \bot)$, and $P2$ as: $(\lambda x.2 * x)(ite(x < 50, x, \bot))$. The variable symbol x denotes an arbitrary element in the dataset R. The constant symbol $\bot$ encodes an element that was deleted from the dataset by a $\mathtt{filter}$ operation.

Next, we consider programs with aggregate operations, which go over all elements in the dataset and return a single result. For example, the following equivalent programs check if the sum of all elements in the dataset is divisible by 5:

	P1$(R \colon RDD_{\mathtt{Int}})$**:**	**P2**$(R \colon RDD_{\mathtt{Int}})$**:**
1	$s = \mathtt{fold}(0, \lambda A, x.A + x)(R)$	$R' = \mathtt{map}(\lambda x.3 * x)(R)$
2	$\mathtt{return}\ s\%5 = 0$	$s' = \mathtt{fold}(0, \lambda A, x.A + x)(R')$
3		$\mathtt{return}\ s'\%5 = 0$

We encode the summation of the elements as $[x]_{0, \lambda A, x.A + x}$. The $[\cdot]$ operator indicates we apply a 'big' operator, so in more conventional mathematical notation we would write it as: $\Sigma_{x \in X} x$. In our example, we have the following encoding for $P1$: $[x]_{0, \lambda A, x.A + x}\%5 = 0$ and similarly for $P2$. For equivalence, we require:

$$(*)[x]_{0, \lambda A, x.A + x}\%5 = 0 \iff [3x]_{0, \lambda A, x.A + x}\%5 = 0$$

In general cases with aggregate operations, we encode the equivalence condition as an inductive claim — the programs should be equal in the initial state, and in every step of the implicit loop in `fold` (which goes over all elements and performs the aggregation). However, there are cases in which the fold functions and the term for the elements in both programs behave as distributive functions. For these cases, it is enough to show the equality on the initial step, and the 1st step of the aggregation. For the initial state in $(*)$, both programs return True, as $0\%5 = 0$. Otherwise, as every pair of applications of the sum function can be written as a single summation $((A + x) + y = A + (x + y), (A + 3x) + 3y = A + 3(x + y))$, we can prove the induction step for the first step only, instead for the n th. Therefore, it is enough to show $\forall x.x\%5 = 0 \iff 3x\%5 = 0$ to prove $(*)$.

We can find a wide range of example programs for which equivalence can be proven using direct induction. For example, suppose we maintain a database of products and their prices, and we write a program that verifies all products' prices are not below some threshold, say 100\$. Then, we wish to make a discount, while maintaining the threshold. So we give a 20% discount for all products whose prices are above 125\$, and give no discount otherwise: $discount((prod, p)) = \textbf{if } p \geq 125 \textbf{ then } (prod, 0.8p) \textbf{ else } (prod, p)$. To verify that our discount program preserves the threshold, we verify if the below programs are equivalent:

Let: $min2 = \lambda A, (x, y).min(A, y)$

	P1$(R\colon RDD_{\texttt{Prod}\times\texttt{Int}})$**:**	**P2**$(R\colon RDD_{\texttt{Prod}\times\texttt{Int}})$**:**
1	$minP = \texttt{fold}(+\infty, min2)(R)$	$minDiscountP = \texttt{fold}(+\infty, min2)(R)$
2	$\texttt{return } minP \geq 100$	$\texttt{return } minDiscountP \geq 100$

This can be written in our encoding of programs as formulas as follows:

$$[(prod, p)]_{+\infty, \lambda A, (x,y).min(A,y)} \geq 100 \iff [discount((prod, p))]_{+\infty, \lambda A, (x,y).min(A,y)} \geq 100$$

In the induction base, we check the empty database, for which both programs return True. Otherwise, we assume that after n prices checked, the minimum M in the 1st program, and the minimum M' in the 2nd program, are both either no less than 100 in both programs, or not. The programs will be equivalent, if this invariant is kept after checking the next price. Formally, we prove:

$$\forall (prod, p), M, M'.(M \geq 100 \iff M' \geq 100) \implies$$
$$(min(M, p) \geq 100 \iff min(M', p_2(discount((prod, p)))) \geq 100)$$

3 Main Results

The main contributions of this paper are as follows:

- We present a simplified model of Spark programs over RDDs by defining a programming language called *SparkLite*, in which UDFs are expressed as simply typed λ-calculus restricted to Presburger arithmetics (see Section 6).
- We show that the equivalence of arbitrary SparkLite programs is undecidable (Section 8.2.1).
- We define a sound method for checking equivalence of a broad class of SparkLite programs (Sections 8.2 and 8.2.4).
- We introduce an interesting and nontrivial subclass of SparkLite in which checking program equivalence is decidable, called $AggPair^1_{sync}$. Interestingly, programs in $AggPair^1_{sync}$ allow both UDFs and aggregations. The decidability of the equivalence is proven by observing that common SparkLite aggregates are *closed* in the sense that operations composed one on another can be simulated by a single operation (Section 8.2.3).

4   Outline

In Section 5 we present the preliminaries for this paper . Section 6 introduces the syntax and operational semantics of the SparkLite language, which is modeling conjunctive queries with aggregations and UDFs. In Section 7, we present alternative syntax with equivalent semantics for SparkLite, and a compilation procedure from the natural syntax of SparkLite to a symbolic one. This novel presentation of SparkLite allows to verify various properties based on the programs' representation as logical terms, and program equivalence in particular, as is discussed in Section 8. There, we first show that the general program equivalence problem is undecidable for SparkLite. Following it is a discussion of various classes of SparkLite, focusing on the decidability of the program equivalence problem. We present a decision procedure for conjunctive queries without aggregations (which is unlike known existing algorithms for the problem), and more interestingly, for a subclass of SparkLite programs with aggregations. For other classes, we present sound algorithms for verifying the equivalence. We conclude and present future work in Section 9.

5   Preliminaries

In this section, we describe a simple extension of Presburger arithmetic [10], which is the first-order theory of the natural numbers with addition, to tuples of integers, and state its decidability.

Notations.   We denote the sets of natural numbers, positive natural numbers, and integers by $\mathbb{N}$, $\mathbb{N}^+$, and $\mathbb{Z}$, respectively. We denote a (possibly empty) sequence of elements coming from a set X by $\overline{X}$. We write $ite(p, e, e')$[1] to denote an expression which evaluates to e if p holds and to e' otherwise. We use $\bot$ to denote the *undefined* value. A *bag m* over a domain X is a multiset, i.e., a set which allows for repetitions, with elements taken from X. We denote the *multiplicity* of an element x in bag m by $m(x)$, where for any x, either $0 < m(x)$ or $m(x)$ is undefined. We write $x \in m$ as a shorthand for $0 < m(x)$. We write $\{\!\{x; n(x) \mid x \in X \wedge \phi(x)\}\!\}$ to denote a bag with elements from X satisfying some property ϕ with multiplicity $n(x)$, and omit the conjunct $x \in X$ if X is clear from context. We denote the *size* (number of elements) of a set X by $|X|$ and that of a bag m of elements from X by $|m|$, i.e., $|m| = \Sigma_{x \in X}\, ite(x \in m, m(x), 0)$.

Presburger Arithmetic.   We consider a fragment of first-order logic (FOL) with equality over the integers, where expressions are written in the rather standard syntax specified in Figure 1.[2] Disregarding the tuple expressions $((pe, \overline{pe})$ and $\boldsymbol{p}_i(e))$ and ite, the resulting first-order theory with the usual $\forall$ and $\exists$ quantifiers is called the *Presburger Arithmetic*. The problem of checking whether a sentence in Presburger arithmetic is valid has long been known to be decidable [5,10], even when combined with Boolean logic [2,7],[3] and infinities [8].[4] For example, *Cooper's Algorithm* [4] is a standard decision procedure for Presburger Arithmetic[5].

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[1]   ite is shorthand for if-then-else
[2]   We assume the reader is familiar with FOL, and omit a more formal description for brevity.
[3]   Originally, Presburger Arithmetic was defined as a theory over natural numbers. However, its extension to integers and booleans is also decidable. (See, e.g., [2].)
[4]   We denote inifinities as $+\infty, -\infty \in \mathbb{Z}$.
[5]   The complexity of Cooper's algorithm is $O(2^{2^{2^{pn}}})$ for some $p > 0$ and where n is the number of symbols in the formula [9]. However, in practice, our experiments show that Cooper's algorithm on most

Arithmetic Expression	ae	$::=$	$c \mid v \mid ae + ae \mid -ae \mid c * ae \mid ae \mathbin{/} c \mid ae \mathbin{\%} c$
Boolean Expression	be	$::=$	$\mathtt{true} \mid \mathtt{false} \mid b \mid e = e \mid ae < ae \mid \neg be \mid be \wedge be \mid be \vee be$
Primitive Expression	pe	$::=$	$ae \mid be$
Basic Expression	e	$::=$	$pe \mid \boldsymbol{v} \mid (pe, \overline{pe}) \mid \boldsymbol{p}_i(e) \mid \mathtt{ite}(be, e, e)$

c, v, and b denote integer numerals, integer variables, and boolean variables, respectively. %
denotes the modulo operator.

■ **Figure 1** Terms of the Augmented Presburger Arithmetic

First-Order Functions	$Fdef$	$::=$	$\mathtt{def}\ \boldsymbol{f} = \lambda \overline{\boldsymbol{y} \colon \boldsymbol{\tau}}. e \colon \boldsymbol{\tau}$
Second-Order Functions	$PFdef$	$::=$	$\mathtt{def}\ \boldsymbol{F} = \lambda \overline{\boldsymbol{x} \colon \boldsymbol{\tau}}. \lambda \overline{\boldsymbol{y} \colon \boldsymbol{\tau}}. e \colon \boldsymbol{\tau}$
Function Expressions	f	$::=$	$\boldsymbol{f} \mid \boldsymbol{F}(\overline{e})$
RDD Expressions	re	$::=$	$\mathtt{cartesian}(\boldsymbol{r}, \boldsymbol{r}) \mid \mathtt{map}(f)(\boldsymbol{r}) \mid \mathtt{filter}(f)(\boldsymbol{r})$
Aggregation Exp.	ge	$::=$	$\mathtt{fold}(e, f)(\boldsymbol{r})$
General Expressions	η	$::=$	$e \mid re \mid ge$
Let expressions	E	$::=$	$Let\ \boldsymbol{x} = \eta\ in\ E \mid \eta$
Programs	$Prog$	$::=$	$\boldsymbol{P}(\overline{\boldsymbol{r} \colon RDD_{\boldsymbol{\tau}}}, \overline{\boldsymbol{v} \colon \boldsymbol{\tau}}) = \overline{Fdef}\ \ \overline{PFdef}\ \ E$

The syntax of basic expressions e is defined in Figure 1

■ **Figure 2** Syntax for SparkLite

In this paper, we consider a simple extension to this language by adding a *tuple constructor* $(pe, \overline{pe})$, which allows us to create k-tuples, for some $k \geq 1$, of primitive expressions, and a projection operator $\boldsymbol{p}_i(e)$, which returns the i-th component of a given tuple expression e. We extend the equality predicate to tuples in a pointwise manner, and call the extended logical language *Augmented Presburger Arithmetic* (APA). The decidability of Presburger Arithmetic, as well as Cooper's Algorithm, can be naturally extended to APA. Intuitively, verifying the equivalence of tuple expressions can be done by verifying the equivalence of their corresponding constituents.

▸ **Proposition 1.** *The theory of formulas over $\mathbb{Z}^n$ with terms in the Augmented Presburger Arithmetic is decidable.*

It will be useful in the next sections to discuss an extension of APA in which terms are allowed to contain two additional constructs: *ite* expressions, and $\perp$ values. We denote this extension APA⁺, and we later show how formulas in APA⁺ can be converted to APA formulas.

6 The SparkLite language

In this section, we define the syntax of SparkLite, a simple functional programming language which allows to use Spark's *resilient distributed datasets* (*RDDs*) [11].

6.1 Syntax

The syntax of SparkLite is defined in Figure 2. SparkLite supports two primitive types: *integers* (Int) and *booleans* (Boolean). On top of this, the user can define *record types* $\boldsymbol{\tau}$,

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non-trivial formulas in this paper returns almost instantly, even on commodity hardware.

which are Cartesian products of primitive types, and RDDs: RDD_τ is (the type of) bags containing elements of type τ. We refer to primitive types and tuples of primitive types as *basic types*, and, by abuse of notation, range over them using τ. We denote the set of (syntactic) variable names by $\mathtt{Vars}$, and range over it using v, b, and r, for variables of type integer and records, boolean, and RDD, respectively.

A program $P(\overline{r : RDD_\tau}, \overline{v : \tau}) = \overline{Fdef}\,\overline{PFdef}\,E$ is comprised of a *header* and a *body*, which are separated by the = sign. The header contains the name of the program (P) and the name and types of its input parameters, which may be $RDDs$ ($\overline{r}$) or integers ($\overline{v}$). The body of the program is comprised of two sequences of function declarations ($\overline{Fdef}$ and $\overline{PFdef}$) and the program's *main expression* (E). $\overline{Fdef}$ binds function names f with first-order lambda expressions, i.e., to a function which takes as input a sequence of arguments of basic types and return a value of a basic type. For example,

$$\mathtt{def\ isOdd} = \lambda y\colon \mathtt{Int}.\ \neg(y\,\%\,2 = 0)\colon \mathtt{Boolean}$$

defines $\mathtt{isOdd}$ to be a function which determines whether its integer argument $\mathtt{y}$ is odd or not. $\overline{PFdef}$ associates function names F with a restricted form of second-order lambda expressions, which we refer to as *parametric functions*.[6] A parametric function F receives a sequence of basic expressions and returns a first order function. Parametric functions can be instantiated to form an unbounded number of functions from a single pattern. For example,

$$\mathtt{def\ addC} = \lambda x\colon \mathtt{Int}.\ \lambda y\colon \mathtt{Int}.\ x + y\colon \mathtt{Int}$$

allows to create any first order function which adds a constant to its argument, e.g., $\mathtt{addC(1)}$ is the function $\lambda x\colon \mathtt{Int}.\ 1 + x\colon \mathtt{Int}$ which returns the value of its input argument incremented by one.

The program's main expression is comprised of a sequence of *let* expression which bind general expressions to variables. A general expression is either a basic expression (e), an RDD *expression* (re) or an *aggregate expression* (ge). A basic expression is a term in augmented Presburger arithmetics (see Section 5). The RDD expression $\mathtt{cartesian}(r, r')$ returns the cartesian product, under bag semantics, of $RDDs$ r and r'. The RDD expressions $\mathtt{map}$ and $\mathtt{filter}$ generalize the *project* and *select* operators in *Relational Algebra* (RA) [1, 3], with *user-defined functions* ($UDFs$): $\mathtt{map}(f)(r)$ evaluates to an RDD obtained by applying the UDF f to every element x of RDD r, with the same multiplicity x had in r. $\mathtt{filter}(f)(r)$ evaluates to a copy of r, except that all elements in r which do not satisfy f are removed. The *aggregation expression* $\mathtt{fold}$ is a generalization of aggregate operations in SQL, e.g., $\mathtt{SUM}$ or $\mathtt{AVERAGE}$, with *UDFs*: $\mathtt{fold}(e, f)(r)$ accumulate the results obtained by iteratively applying f to every element x in r, starting from the *initial element e* and applying f a total of $r(x)$ times for every $x \in r$.

▸ Remark. To ensure that the meaning of $\mathtt{fold}(e, f)(r)$ is well defined, we require that f be a commutative function.[7] Also, as is common in functional languages, we assume that variables are never reassigned.

▸ Remark. The signature of UDFs given to either *map*, *filter*, or *fold* should match to the type of the RDD on which they are applied. See Figure 7.

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[6] Parametric functions were inspired by the *Kappa Calculus* [6], which contains only first-order functions, but allows lifting them to larger product types, which is exactly the purpose of parametric functions in SparkLite.

[7] Formally, we refer to the following notion of commutativity, unlike the traditional definition: $f : X \times Y$ is commutative if $\forall x, y_1, y_2. f(f(x, y_1), y_2) = f(f(x, y_2), y_1)$.

$$isOdd = \lambda x\colon \mathtt{Int}.\ \neg(x\,\%\,2 = 0)$$

Let: $\quad doubleAndAdd = \lambda c\colon \mathtt{Int}.\lambda x\colon \mathtt{Int}.\ 2 * x + c$

$\qquad\quad sumFlatPair = \lambda A\colon \mathtt{Int}, (x, y)\colon \mathtt{Int} \times \mathtt{Int}.\ A + x + y$

$\mathbf{P1}(R_0\colon RDD_{\mathtt{Int}}, R_1\colon RDD_{\mathtt{Int}})$:

```
1   A = filter(isOdd)(R_0)
2   B = map(doubleAndAdd(1))(A)
3   C = cartesian(B, R_1)
4   v = fold(0, sumFlatPair)(C)
5   return v
```

Figure 3 Example SparkLite Program

▷ **Example 1.** Consider the example SparkLite program in Figure 3. From the example program we can see the general structure of SparkLite programs: First, the functions that are used as UDFs in the program are declared and defined: $isOdd, sumFlatPair$ defined as $Fdef$, and $doubleAndAdd$ defined as a $PFdef$. The name of the program ($\mathbf{P} = P1$) is declared with a list of input RDDs (R_0, R_1) (*Prog* rule). Instead of writing *let* l_1 *in let* l_2 *in* ..., we use syntactic sugar, where each line of code contains a single l_i, and the last line denotes the return value using the `return` keyword. Here, 3 variables of RDD type (A, B, C) and one integer variable (v) are bound by *let*s. We can see in the definition of A an application of the *filter* operation, accepting the RDD R_0 and the function *isOdd*. For B's definition we apply the *map* operation with a parametric function *doubleAndAdd* with the parameter 1, which is interpreted as $\lambda x.\ 2 * x + 1$. C is the cartesian product of B and input RDD R_1. We apply an aggregation using *fold* on the RDD C, with an initial value 0 and the function *sumFlatPair*, which 'flattens' elements of tuples in C, taking their sum. The sum total of all these elements is stored in the variable v. The returned value is the integer variable v. The program's signature is $P1(RDD_{\mathtt{Int}}, RDD_{\mathtt{Int}})\colon \mathtt{Int}$.

6.2 Operational Semantics

Program Environment.   We define a unified semantic domain $\mathcal{D}$ for all types in SparkLite. The *program environment* type: $\mathcal{E} = \mathtt{Vars} \to \mathcal{D}$ is a mapping from each variable in $\mathtt{Vars}$ to its value, according to type. A variable's type does not change during the program's run, nor does its value.

Data flow.   We start with an initial environment function ρ_0 that maps all input variables and function definitions. We define the *semantic interpretation* of expressions based on an environment $\rho \in \mathcal{E}$, and specifically for $x \in \overline{r} \cup \overline{v}$, $[\![x]\!](\rho) = \rho(x)$. The semantics of composite expressions are straight-forward using the semantics of their components. The semantics of *let* is to create a new environment by binding the variable name. In Section 6.2 we specify the behavior of $[\![\cdot]\!](\cdot)$ for all expressions and statements.

Example.   In Figure 3, suppose we were given the following input: $R_0 = \{\!\{(1; 7), (2; 1)\}\!\}, R_1 = \{\!\{(3; 4), (5; 2)\}\!\}$. Then: $\rho(A) = \{\!\{(1; 7)\}\!\}$, $\rho(B) = \{\!\{(3; 7)\}\!\}$, $\rho(C) = \{\!\{((3, 3); 28), ((3, 5); 14)\}\!\}$, $\rho(v) = 28 * (3 + 3) + 14 * (3 + 5)$ and the program returns $\rho(v) = 280$.

$$
\begin{array}{lcl}
[\![c]\!](\rho) & = & c \\
[\![\boldsymbol{v}]\!](\rho) & = & \rho(v) \\
[\![\texttt{unOp}\,e]\!](\rho) & = & unOp\,[\![e]\!](\rho) \\
[\![e_1\,\texttt{binOp}\,e_2]\!](\rho) & = & [\![e_1]\!](\rho)\,binOp\,[\![e_2]\!](\rho) \\
[\![(e_1,\cdots,e_n)]\!](\rho) & = & ([\![e_1]\!](\rho),\cdots,[\![e_n]\!](\rho)) \\
[\![\texttt{if}\,e_1\,\texttt{then}\,e_2\,\texttt{else}\,e_3]\!](\rho) & = & ite\big([\![e_1]\!](\rho),[\![e_2]\!](\rho),[\![e_3]\!](\rho)\big) \\
[\![\texttt{map}(f)(r)]\!](\rho) & = & \{\!\{\rho(f)(x) \mid x \in \rho(r)\}\!\} \\
[\![\texttt{filter}(b)(r)]\!](\rho) & = & \{\!\{x \mid x \in \rho(r) \wedge \rho(b)(x)\}\!\} \\
[\![\texttt{cartesian}(r_1,r_2)]\!](\rho) & = & \{\!\{(x_1,x_2) \mid x_1 \in \rho(r_1) \wedge x_2 \in \rho(r_2)\}\!\} \\
[\![\texttt{fold}(a_0,f)(r)]\!](\rho) & = & q\big([\![a_0]\!](\rho),\rho(r)\big), \;\; where
\end{array}
$$

$$
q(v_0,s) = \begin{cases} a_0 & s = \{\} \\ \rho(f)\big(x,q(v_0,s')\big) & s = \{\!\{x;1\}\!\} \cup s' \end{cases}
$$

$$
\begin{array}{lcl}
[\![Let\;\boldsymbol{x}=\eta\;in\;E]\!](\rho) & = & [\![E]\!](\rho[\boldsymbol{x}\mapsto[\![\eta]\!](\rho)]) \\
[\![\boldsymbol{P}(\cdots)=\cdots E]\!](\rho_0) & = & [\![E]\!](\rho_0)
\end{array}
$$

$Prog = \boldsymbol{P}(\overline{\boldsymbol{r}:RDD_\tau},\overline{\boldsymbol{v}:\tau}) = \overline{Fdef}\;\;\overline{PFdef}\;\;E$. unOp and binOp are taken from Figure 1:
unOp $\in \{-,\neg,\boldsymbol{p}_i\}$, binOp $\in \{+,*,/,\%,=,<,\wedge,\vee,(,)\}$

🟨 **Figure 4** Semantics of SparkLite

$$
\begin{array}{lcl}
\phi_P(Let\;x=\eta\;in\;E) & = & \phi_P(E')[\phi_P(\eta)/x] \\
\phi_P(e) & = & e \\
\phi_P(\texttt{map}(f)(r)) & = & f(\phi_P(r)) \\
\phi_P(\texttt{filter}(f)(r)) & = & ite(f(\phi_P(r))=tt,\phi_P(r),\bot) \\
\phi_P(\texttt{cartesian}(r_1,r_2)) & = & (\phi_P(r_1),\phi_P(r_2)) \\
\phi_P(\texttt{fold}(f,e)(r)) & = & [\phi_P(r)]_{e,f} \\
\phi_P(r) & = & \begin{cases} \mathbf{x}_r & r \in \overline{r} \\ r & otherwise \end{cases}
\end{array}
$$

$$
Let\;P:\boldsymbol{P}(\overline{\boldsymbol{r}},\overline{\boldsymbol{v}}) = \overline{\boldsymbol{F}}\,\overline{\boldsymbol{f}}\,E
$$
$$
\Phi(P) = \phi_P(E)
$$

🟨 **Figure 5** Compiling SparkLite to Logical Terms

$$
\begin{array}{lcl}
[\![\mathbf{x}_{r_i}]\!](\overline{v},\overline{r}) & = & r_i \\
[\![f(t)]\!](\overline{v},\overline{r}) & = & \{\!\{[\![f]\!](z) \mid z \in [\![t]\!](\overline{v},\overline{r})\}\!\} \\
[\![ite(f(t),t,\bot)]\!](\overline{v},\overline{r}) & = & \{\!\{z \mid z \in [\![t]\!](\overline{v},\overline{r}) \wedge [\![f]\!](z)\}\!\} \\
[\![(t,t')]\!](\overline{v},\overline{r}) & = & \{\!\{(z,z') \mid z \in [\![t]\!](\overline{v},\overline{r}) \wedge z' \in [\![t]\!](\overline{v},\overline{r})\}\!\} \\
[\![[t]_{e,f}]\!](\overline{v},\overline{r}) & = & [\![\texttt{fold}(e,f)(r')]\!][r' \mapsto [\![t]\!](\overline{v},\overline{r})]
\end{array}
$$

🟨 **Figure 6** Semantics of Terms

7 Term Semantics for SparkLite

In this section, we define an alternative, equivalent semantics for SparkLite where the program is interpreted as a term in APA$^+$.[8] This term is called the *program term* and denoted $\Phi(P)$ for program P, specified in Figure 5. These terms are assigned their own semantics such that the semantics of $\Phi(P)$ is identical to the semantics of P as defined earlier. Special variables are used to refer to elements of input RDDs,[9] and a new language construct is added to denote the fold operation. As an example, we take the program $P1$ from Figure 3, and show how to construct $\Phi(P1)$ by recursively applying ϕ_P and simplifying the formula.

$$
\begin{aligned}
\phi_{P1}(A) \quad &= \phi_{P1}(\texttt{filter}(isOdd)(R_0)) = ite(isOdd(\phi_{P1}(R_0)), \phi_{P1}(R_0), \bot) \\
&= ite(isOdd(\mathbf{x}_{R_0}), \mathbf{x}_{R_0}, \bot) \\
\phi_{P1}(B) \quad &= \phi_{P1}(doubleAndAdd(1)(A)) = doubleAndAdd(1)(\phi_{P1}(A)) \\
&= doubleAndAdd(1)(ite(isOdd(\mathbf{x}_{R_0}), \mathbf{x}_{R_0}, \bot)) \\
\phi_{P1}(C) \quad &= \phi_{P1}(\texttt{cartesian}(B, R_1)) = (\phi_{P1}(B), \phi_{P1}(R_1)) \\
\Phi(P1) \quad &= \phi_{P1}(\texttt{fold}(0, sumFlatPair)(C)) = [\phi_{P1}(C)]_{0, sumFlatPair} \\
& [(\phi_{P1}(B), \phi_{P1}(R_1))]_{0, sumFlatPair} \\
&= [(doubleAndAdd(1)(ite(isOdd(\mathbf{x}_{R_0}), \mathbf{x}_{R_0}, \bot)), \mathbf{x}_{R_1})]_{0, sumFlatPair}
\end{aligned}
$$

Representative elements of RDDs. The variables assigned by ϕ for input RDDs are called *representative elements*. In a program that receives an input RDD r, we denote the representative element of r as: $\mathbf{x}_r$. The set of possible valuations to that variable is equal to the bag defined by r, and an additional 'undefined' value ($\bot$), for the empty RDD. Therefore $\mathbf{x}_r$ ranges over $\text{dom}(r) \cup \{\bot\}$. By abuse of notations, the term for a non-input RDD, computed in a SparkLite program, is also called a representative element.

Formalization of the Term Semantics for SparkLite. Let P be a SparkLite program. We use standard notations $\overline{r}$ for the inputs of P, and r^{out} for the output of P. The term $\Phi(P)$ is called the *program term of P* as before. The *Term Semantics* (TS) of a program that returns an RDD-type output is the bag that is obtained from all possible valuations to the free variables:

$$TS(P)(\overline{v}, \overline{r}) = [\![\Phi(P)]\!](\overline{v}, \overline{r})$$

Where the meaning of $[\![\Phi(P)]\!]$ is determined according to Figure 6. Assigning a concrete valuation to the free variables of $\Phi(P)$ returns an element in the output RDD r^{out}. By taking all possible valuations to the term with elements from $\overline{r}$, we get the bag equal to r^{out}.

▶ **Proposition 2.** *Let $P : \boldsymbol{P}(\overline{\boldsymbol{r}}) = \overline{\boldsymbol{F}}\,\overline{\boldsymbol{f}}\,E$ be a* SparkLite *program. Let $[\![P]\!]$ be the interpretation of its output according to the operational semantics, and $TS(P)$ according to the term semantics of P. Then, for any input $\overline{v}, \overline{r}$, we have:*

$$TS(P)(\overline{v}, \overline{r}) = [\![P]\!]([\![\overline{v}]\!], [\![\overline{r}]\!])$$

---

[8] Reminder: APA$^+$ is equal to APA with additional *ite* and $\bot$ expressions,
[9] To avoid overhead of notations, we assume the programs do not contain self-products (for every cartesian product in the program, the sets of variables appearing in each component must be disjoint).

8 Verifying Equivalence of SparkLite Programs

The Program Equivalence (*PE*) problem. Let P_1 and P_2 be SparkLite programs, with signature $P_i(\overline{T}, \overline{RDD_T}){:}\tau$ for $i \in \{1,2\}$. We use $[\![P_i]\!]([\![\overline{v}]\!],[\![\overline{r}]\!])$ to denote the result of P_i. We say that P_1 and P_2 are *equivalent*, if for all input values $\overline{v}$ and RDDs $\overline{r}$, it holds that $[\![P_1]\!]([\![\overline{v}]\!],[\![\overline{r}]\!]) = [\![P_2]\!]([\![\overline{v}]\!],[\![\overline{r}]\!])$.

8.1 Verifying Equivalence of SparkLite Programs without Aggregations

We are given two programs P_1, P_2 receiving as input a series of RDDs $\overline{r} = (r_1, \ldots, r_n)$. We assume w.l.o.g. the programs do not receive non-RDD arguments $\overline{v}$.[10] We let $\overline{x} = (x_1, \ldots, x_n)$ be a concrete valuation for all input RDDs representative elements: $\mathbf{x}_{r_i}$ will map to x_i. We denote the substitution of the concrete valuation in a term t over $\overline{\mathbf{x}_r}$ as: $t(\overline{x}) = t[x_1/\mathbf{x}_{r_1}, \ldots, x_n/\mathbf{x}_{r_n}]$.

We present a class of programs which do not contain any aggregate expressions in the program term. In other words, the returned value of the program is not affected by *fold* operations.

▸ **Definition 1** (The *NoAgg* class). *A program P satisfies $P \in NoAgg$ if $\Phi(P)$ does not contain any aggregate terms (i.e. terms of the form: $[\,]_{i,f}$).*

Comparing representative elements. For two program terms of *NoAgg* programs to be comparable, they must depend on the same input RDDs. For example, let $P1(R_0, R_1) = \mathtt{map}(\lambda x.1)(R_0)$ and $P2(R_0, R_1) = \mathtt{map}(\lambda x.1)(R_1)$. $P1$ and $P2$ have the same program term (the constant 1),[11] but the multiplicity of that element in the output bag is different and depends on the source input RDD. In $P1$, its multiplicity is the same as the size of R_0, and in $P2$ it is the same as the size of R_1. $P1$ and $P2$ are therefore not equivalent, for inputs R_0, R_1 of different sizes. Therefore, for each program term $\Phi(P)$ we consider the *set of free variables*, $FV(\Phi(P))$. Each free variable has some source input RDD. In the example, $FV(\Phi(P1)) = \{\mathbf{x}_{R_0}\}$, and $FV(\Phi(P2)) = \{\mathbf{x}_{R_1}\}$.

We show that in two programs without aggregations, different sets of free variables of the programs terms imply the existence of input RDDs for which the two program terms evaluate to different bags, thus they are semantically inequivalent:

▸ **Proposition 3.** *Let there be two programs $P_1, P_2 \in NoAgg$, over input RDDs $\overline{r}$ and program terms $\Phi(P_i) = t_i$. such that $FV(t_1) \neq FV(t_2)$, and $t_1 \neq \bot \lor t_2 \neq \bot$. Then $\exists \overline{r}.[\![t_1]\!](\overline{r}) \neq [\![t_2]\!](\overline{r})$.*

The program terms are written in APA$^+$, and may contain *ite* and $\bot$ expressions, therefore we need to encode the formulas in APA by eliminating *ite* and $\bot$. We present a translation procedure for converting APA$^+$ formulas to APA.

▸ **Proposition 4** (Translation of APA$^+$ to APA). *Let φ be a formula in APA^+. We denote by $\Diamond$ the set of binary equality predicates without inequality $(=,<,\leq,>,\geq)$, and by $\Box$ the full set of equality predicates. We denote variables by x. Functions are denoted f.*

---

[10] The extension to equivalence of terms which is based also on non-RDD inputs is immediate by quantification on the non-RDD variables in the term.

[11] To be more precise, the term is an application of the function $\lambda x.1$ on a free variable, $\mathbf{x}_{R_0}$ or $\mathbf{x}_{R_1}$. $\phi()$ does not apply beta-reduction on the UDFs.

1. *We define a transformation $\mathcal{N}_F{:}APA^+{\to}APA^+$ that extends all function definitions to ite expressions and to $\bot$, without changing the value. The definition of $\mathcal{N}_F$ presented here is insensitive to the location of the arguments handled.*

    a. $\bigvee_i \varphi_i = \bot \implies \mathcal{N}_F(f(\varphi_1, \ldots, \varphi_n)) = \bot$

    b. $\mathcal{N}_F(f(ite(c, \varphi, \varphi'))) = ite(c, \mathcal{N}_F(f(\varphi)), \mathcal{N}_F(f(\varphi')))$

    c. $\mathcal{N}_F(f(ite(c, \varphi_1, \varphi_2), \varphi')) = ite(c, \mathcal{N}_F(f(\varphi_1, \varphi')), \mathcal{N}_F(f(\varphi_2, \varphi')))$

    d. $\mathcal{N}_F(f(ite(c, \varphi_1, \varphi_2), ite(c', \varphi'_1, \varphi'_2))) =$
        $ite(c \wedge c', \mathcal{N}_F(f(\varphi_1, \varphi'_1)), ite(c \wedge \neg c', \mathcal{N}_F(f(\varphi_1, \varphi'_2)), ite(\neg c \wedge c', \mathcal{N}_F(f(\varphi_2, \varphi'_1)), \mathcal{N}_F(f(\varphi_2, \varphi'_2)))))$

    e. *By induction, a function f that receives n ite expressions as arguments, will be transformed to 2^n nested ite expressions, before the recursive application of $\mathcal{N}_F$ on the $\varphi_i s$.*

2. *We define a transformation $\mathcal{N}{:}APA^+{\to}APA$, such that $\mathcal{N}(\varphi)$ is equivalent to φ.*[12]

    a. $\mathcal{N}(x) = x$

    b. $\mathcal{N}(f(x)) = f(\mathcal{N}(x))$

    c. $\mathcal{N}(\varphi \wedge \varphi') = \mathcal{N}(\varphi) \wedge \mathcal{N}(\varphi')$

    d. $\mathcal{N}(\forall x.\varphi) = \forall x.\mathcal{N}(\varphi)$, $\mathcal{N}(\exists x.\varphi) = \exists x.\mathcal{N}(\varphi)$.

    e. $\mathcal{N}(\bot = \bot) = tt$

    f. *If φ does not contain either ite or $\bot$, then $\mathcal{N}(\varphi \diamond \bot) = f\!f$, $\mathcal{N}(\varphi \neq \bot) = tt$.*

    g. *If φ, φ' do not contain either ite or $\bot$, $\mathcal{N}(\varphi \square \varphi') = \varphi \square \varphi'$*

    h. $\mathcal{N}(ite(c, \varphi, \varphi') \square \bot) = (\mathcal{N}(c = tt) \implies \mathcal{N}(\varphi \diamond \bot)) \wedge (\mathcal{N}(c = f\!f) \implies \mathcal{N}(\varphi' \square \bot))$

    i. $\mathcal{N}(ite(c, \varphi_1, \varphi_2) \square ite(c', \varphi'_1, \varphi'_2)) = (\mathcal{N}(c \wedge c' = tt) \implies \mathcal{N}(\varphi_1 \square \varphi'_1)) \wedge (\mathcal{N}(c \wedge \neg c' = tt) \implies \mathcal{N}(\varphi_1 \square \varphi'_2)) \wedge (\mathcal{N}(\neg c \wedge c' = tt) \implies \mathcal{N}(\varphi_2 \square \varphi'_1)) \wedge (\mathcal{N}(\neg c \wedge \neg c' = tt) \implies \mathcal{N}(\varphi_2 \square \varphi'_2))$

    j. $\mathcal{N}(f(\varphi) \square \varphi') = \mathcal{N}(\mathcal{N}_F(f(\varphi)) \square \varphi')$

    k. $\mathcal{N}(f(\varphi) \square f'(\varphi')) = \mathcal{N}(\mathcal{N}_F(f(\varphi)) \square \mathcal{N}_F(f'(\varphi')))$

▸ **Theorem 2** (Decidability of the *NoAgg* class). *Given two* SparkLite *programs $P_1, P_2 \in$ NoAgg,* PE *is decidable.*

Proof. For non-RDD return types, the absence of aggregate operators implies we can use Proposition 1, as the returned expression is expressible in APA. For RDD return type, we provide the following algorithm, which is a decision procedure:

- If both program terms evaluate to empty bag for any choice of input RDDs, the algorithm detects it in step (1) and outputs the programs are equivalent.
- Otherwise, the algorithm checks syntactically in step 2 that $FV(\Phi(P_1)) = FV(\Phi(P_2))$. If that is not the case, then by Proposition 3 we can conclude the programs are not equivalent.
- The correctness of the algorithm in step 3 follows from the equivalence of the TS semantics and the operational semantics defined in 6.2 (Proposition 2). If such an $\bar{x}$ is found, we take input RDDs $\bar{R}$, where $R_i = \{\!\{x_i; 1\}\!\}$, for which $[\![\Phi(P_i)]\!](\bar{R} = \Phi(P_i)[\bar{x}/FV(\Phi(P_i))]$, thus $[\![\Phi(P_1)]\!](\bar{R}) \neq [\![\Phi(P_2)]\!](\bar{R})$. Otherwise, the formula is unsatisfiable. As $FV(\Phi(P_1)) = FV(\Phi(P_2))$ from the previous step, we know that the additional multiplicity donated by any valuation $\bar{x} \in \bar{R}$ is equal to $\Pi_i R_i(x_i)$ in both programs. We conclude that for all

---

[12] While elimination of *ite* and $\bot$ completely may lead to a blow-up in formula size, it is possible to write more concise versions, tailored for the the filter operation, in which the term always takes the form $ite(c, \varphi, \bot)$. For example, when φ, φ' do not contain $\bot$, $\mathcal{N}(ite(c, \varphi, \bot) = ite(c', \varphi', \bot)) = (c \iff c') \wedge (c \implies \mathcal{N}(\varphi = \varphi'))$

if $\mathcal{N}(\Phi(P_1) = \bot \wedge \Phi(P_2) = \bot) = tt$ **then**
 | **return** *equivalent*
else
 if $FV(\Phi(P_1)) \neq FV(\Phi(P_2))$ **then**
  | **return** *not equivalent*
 else
  if $\mathcal{N}(\exists \overline{x}.\Phi(P_1)[\overline{x}/FV(\Phi(P_1))] \neq \Phi(P_2)[\overline{x}/FV(\Phi(P_2))]) = tt$ **then**
   | **return** *not equivalent*
  else
   | **return** *equivalent*
  end
 end
end

possible choices of input RDDs, the resulting bags have the same elements, and those elements have the same multiplicities in each bag, as required.

                                  ◄

▸ Remark. The equivalence formula generated does not contain aggregate terms, and applications of *ite* are normalized using the transformation $\mathcal{N}$ as defined in Proposition 4, resulting in a formula definable in APA. The algorithm generates formulas in APA several times: once in Step 1, to verify whether the programs do not return the empty bag for all inputs, and second in Step 3, to test for equivalence.[13] From Proposition 1 and Proposition 4, all the formulas checked by the algorithm can be decided using Cooper's Algorithm.

▸ Remark. All examples use syntactic sugar for '*let*' expressions. For brevity, instead of applying ϕ on the underlying '*let*' expressions, we apply it line-by-line from the top-down. In addition, we assume that in programs returning an RDD-type, the RDD is named r^{out}, and the programs always end with `return` r^{out}. Thus, $\Phi(P) = \phi_P(r^{out})$.

▸ Example 1 (Basic optimization — operator pushback). This example shows a common optimization of pushing the filter/selection operator backward, to decrease the size of the dataset.

P1(R: $RDD_{\mathtt{Int}}$)**:**	**P2**(R: $RDD_{\mathtt{Int}}$)**:**
1 $R' = \mathtt{map}(\lambda x.4 * x/5)(R)$	$R' = \mathtt{filter}(\lambda x.x \geq 125)(R)$
2 `return` $\mathtt{filter}(\lambda x.x \geq 100)(R')$	`return` $\mathtt{map}(\lambda x.4 * x/5)(R')$

Both programs may return an non-empty RDD of integers, and the sets of free variables are equal: $FV(\Phi(P1)) = \{\mathbf{x}_R\} = FV(\Phi(P2))$. We analyze the representative elements:

$$\phi_{P1}(R') = 4 * \mathbf{x}_R/5$$
$$\phi_{P1}(r^{out}) = ite(\phi_{P1}(R') \geq 100, \phi_{P1}(R'), \bot) = ite(4 * \mathbf{x}_R/5 \geq 100, 4 * \mathbf{x}_R/5, \bot)$$
$$\phi_{P2}(R') = ite(\mathbf{x}_R \geq 125, \mathbf{x}_R, \bot)$$
$$\phi_{P2}(r^{out}) = (\lambda x.4 * x/5)(\phi_{P1}(R')) = (\lambda x.4 * x/5)(ite(\mathbf{x}_R \geq 125, \mathbf{x}_R, \bot))$$

We need to verify that:

$$\forall \mathbf{x}_R.ite(4 * \mathbf{x}_R/5 \geq 100, 4 * \mathbf{x}_R/5, \bot) = (\lambda x.4 * x/5)(ite(\mathbf{x}_R \geq 125, \mathbf{x}_R, \bot))$$

---

[13] In the next section, program terms may contain aggregate expressions. In that case, there may be more formulas generated, and subsequently more calls to Cooper's Algorithm.

Using Proposition 4:

$$\mathcal{N}(ite(4*\mathbf{x}_R/5 \geq 100, 4*\mathbf{x}_R/5, \bot) = (\lambda x.4*x/5)(ite(\mathbf{x}_R \geq 125, \mathbf{x}_R, \bot))) =$$
$$\mathcal{N}(ite(4*\mathbf{x}_R/5 \geq 100, 4*\mathbf{x}_R/5, \bot) = ite(\mathbf{x}_R \geq 125, 4*\mathbf{x}_R/5, \bot)) = \ldots$$
$$= ((4*\mathbf{x}_R/5 \geq 100 \wedge \mathbf{x}_R \geq 125) \implies \mathcal{N}(4*\mathbf{x}_R/5 = 4*\mathbf{x}_R/5))$$
$$\wedge((4*\mathbf{x}_R/5 \geq 100 \wedge \neg\mathbf{x}_R \geq 125) \implies \mathcal{N}(4*\mathbf{x}_R/5 = \bot))$$
$$\wedge((\neg 4*\mathbf{x}_R/5 \geq 100 \wedge \mathbf{x}_R \geq 125) \implies \mathcal{N}(\bot = 4*\mathbf{x}_R/5))$$
$$\wedge((\neg 4*\mathbf{x}_R/5 \geq 100 \wedge \neg\mathbf{x}_R \geq 125) \implies \mathcal{N}(\bot = \bot)) =$$
$$((4*\mathbf{x}_R/5 \geq 100 \wedge \mathbf{x}_R \geq 125) \implies 4*\mathbf{x}_R/5 = 4*\mathbf{x}_R/5)$$
$$\wedge((4*\mathbf{x}_R/5 \geq 100 \wedge \neg\mathbf{x}_R \geq 125) \implies \mathit{ff})$$
$$\wedge((\neg 4*\mathbf{x}_R/5 \geq 100 \wedge \mathbf{x}_R \geq 125) \implies \mathit{ff})$$
$$\wedge((\neg 4*\mathbf{x}_R/5 \geq 100 \wedge \neg\mathbf{x}_R \geq 125) \implies \mathit{tt})$$

8.2 Verifying Equivalence of a Class of SparkLite **Programs with Aggregation**

In the following section we discuss how the existing framework can be extended to prove equivalence of SparkLite programs containing aggregate expressions. The terms for aggregate operations are given using a special operator - $[t]_{i,f}$ - where t is the term being folded, i is the initial value, and f is the fold function. The operator *binds* all free variables in the term t, thus the free variables of t are not contained in the free variables set of the term that includes the aggregate term.

We begin by showing there is no algorithm to decide *PE* for general SparkLite programs with aggregation.

8.2.1 Proof of Undecidability of *PE*

▸ **Theorem 3.** PE *for general* SparkLite *programs is undecidable.*

Proof. We show a reduction of Hilbert's 10th problem to *PE*. Hilbert's 10th problem is the problem of finding a general algorithm that given a polynomial with integer coefficients, decides whether it has integer roots. We assume towards a contradiction that *PE* is decidable. Let there be a polynomial p over k variables $x_1, \ldots, x_k$, and coefficients $a_1, \ldots, a_m$. We use SparkLite operations and the input RDDs R_i to represent the value of the polynomial p for some valuation of the x_i. We define a translation φ from polynomials to SparkLite expressions:

- $\varphi(x_i) = [\texttt{map}(\lambda x.1)(R_i)]_{0, \lambda A, x.A+x}$
- $\varphi(x_i x_j) = [\texttt{cartesian}(\texttt{map}(\lambda x.1)(R_i), \texttt{map}(\lambda x.1)(R_j))]_{0, \lambda A, (x,y).A+x}$
- $\varphi(x_i^2) = [\texttt{cartesian}(\texttt{map}(\lambda x.1)(R_i), \texttt{map}(\lambda x.1)(R_i))]_{0, \lambda A, (x,y).A+x}$. This rule as well as the rest of the powers follows directly from the previous rule. For a degree k monom, we apply the *cartesian* operation k times, and fold it with $\lambda A, (x)_{i=1}^k.A+x_1$.
- $\varphi(x_i^0) = 1$, trivially.
- $\varphi(a m(x_{i_1}, \ldots, x_{i_j})) = a\varphi(m(x_{i_1}, \ldots, x_{i_j}))$ where $m(x_{i_1}, \ldots, x_{i_j}) = x_{i_1}^{l_1} \cdots x_{i_j}^{l_j}$, for which φ is defined by induction according to the previous rules.
- $\varphi(p) = \sum_{i=1}^k \varphi(a_i m_i(x_{i_1}, \ldots, x_{i_k})))$, where $p = \sum_{i=1}^k a_i m_i(x_{i_1}, \ldots, x_{i_k})$.

Given a polynomial $p(a_1 \ldots, a_m; x_1, \ldots, x_k)$, we generate the following instance of the *PE* problem:

```
      P1(R₁,...,Rₖ:RDD_Int):   P2(R₁,...,Rₖ:RDD_Int):
  1   return φ(p) ≠ 0          return tt
```

By choosing input RDDs such that the size of R_i is equal to the matching variable x_i, we can simulate any valuation to the polynomial p. If $P1$ returns true, then the valuation is not a root of the polynomial p. Thus, if it is equivalent to the 'true program' $P2$, then the polynomial p has no roots. Therefore, if the algorithm solving PE outputs 'equivalent' then the polynomial p has no roots, and if it outputs 'not equivalent' then the polynomial p has some root, where $x_i = |[\![R_i]\!]|$ for the R_i's which are the witness for nonequivalence. Thus we have a reduction of Hilbert's 10th problem, proving PE is undecidable.   ◀

8.2.2   Single Aggregate

The simplest class of programs in which an aggregation operator appears, is the class of programs whose program terms are a function of an aggregate term, that is have the form $g([t]_{i,f})$.

▸ **Definition 2** (The Agg^1 class). *Let there be a program P with $\Phi(P) = g([t]_{i,f})$. $P \in Agg_R^1$ if t does not contain aggregate terms.*

▸ **Lemma 4** (Sound method for verifying equivalence of Agg^1 programs). *Let P_1, P_2 be Agg^1 programs, with $\Phi(P_i) = g_i([\varphi_i]_{init_i, f_i})$. The terms φ_1, φ_2 are representative elements of two RDDs R_1, R_2 of types σ_1, σ_2, respectively. Let $f_1 : \xi_1 \times \sigma_1 \to \xi_1, f_2 : \xi_2 \times \sigma_2 \to \xi_2$ be two fold functions, $init_1 : \xi_1, init_2 : \xi_2$ be initial values, and $g_1 : \xi_1 \to \xi, g_2 : \xi_2 \to \xi$ be functions. We have $g_1([\varphi_1]_{init_1, f_1}) = g_2([\varphi_2]_{init_2, f_2})$ if:*

$$FV(\varphi_1) = FV(\varphi_2) \tag{1}$$

$$g_1(init_1) = g_2(init_2) \tag{2}$$

$$\forall \overline{v}, A_{\varphi_1} : \xi_1, A_{\varphi_2} : \xi_2. g_1(A_{\varphi_1}) = g_2(A_{\varphi_2}) \implies \tag{3}$$
$$g_1(f_1(A_{\varphi_1}, \varphi_1(\overline{v}))) = g_2(f_2(A_{\varphi_2}, \varphi_2(\overline{v})))$$

The application of Lemma 4 to the algorithm presented in Theorem 2 involves one syntactic check (Equation (1)), and two calls to Cooper's algorithm (Equations (2) and (3)). Lemma 4 shows that an inductive proof of the equality of folded values is *sound*. Therefore, given two folded expressions which are not equivalent, the lemma is guaranteed to report so.

▸ Example 2 (Maximum and minimum). Below is an example of two equivalent programs belonging to Agg^1:

$$\text{Let:} \quad \begin{array}{l} max = \lambda M, x.ite(x > M, x, M) \\ min = \lambda M, x.ite(x < M, x, M) \end{array}$$

P1($R : RDD_{\text{Int}}$):	**P2**($R : RDD_{\text{Int}}$):
1 `return fold(-∞, max)(R)`	$R' = \mathtt{map}(\lambda x. - x)(R)$
2	`return - fold(+∞, min)(R')`

The programs compute the maximum element of a numeric RDD in two different methods: in the first program by getting the maximum directly, and in the second by getting the additive inverse of the minimum of the additive inverses of the elements. The equivalence formula is:

$$[\mathbf{x}_R]_{-\infty, max} = -[-\mathbf{x}_R]_{+\infty, min}$$

We apply Lemma 4: The two program apply a fold operation on a term of the same RDD R. $init_0 = -\infty, init_1 = +\infty$ and $g_1 = \lambda x.x., g_2 = \lambda x. - x$, therefore $g_1(-\infty) = g_2(+\infty)$ as required. We check the inductive claim:

$$\forall x, A, A'. A = -A' \implies max(A, \mathbf{x}_R(x)) = -min(A', -\mathbf{x}_R(x))$$

We assume $A = -A'$ and attempt to prove $max(A, x) = -min(A', -x)$, after normalizing to APA:

$$max(A, x) \qquad =^? -min(A', -x)$$
$$ite(x > A, x, A) \quad =^? -ite(-x < A', -x, A') = ite(-x < A', x, -A')$$

And we verify the following APA formula:

$$\forall x, A, A'. A = -A' \implies ((x > A \land -x < A' \implies x = x) \land (x > A \land -x \geq A' \implies x = -A')$$
$$\land (x \leq A \land -x < A' \implies A = x) \land (x \leq A \land -x \geq A' \implies A = -A'))$$

which is true. ∎

8.2.3   A Complete Subclass

There are several cases in which one or more of the requirements of Lemma 4 are not satisfied, yet the aggregate expressions are equal. The first requirement, $FV(\varphi_0) = FV(\varphi_1)$, is not necessary when the fold functions applied on the terms in both programs are both *trivial*.

▸ **Definition 3** (Trivial fold). $[\varphi]_{init, f}$ *is a trivial fold if:*

$$\forall \overline{v}. f(init, \varphi(\overline{v})) = init$$

If two instances of Agg^1 have trivial folds, then Equation (2) in Lemma 4 is a sufficient condition for the equivalence:

$$g([\varphi_0]_{init_0, f_0}) = g(init_0) \land g'([\varphi_1]_{init_1, f_1}) = g'(init_1) \land g(init_0) = g'(init_1) \implies$$
$$g([\varphi_0]_{init_0, f_0}) = g'([\varphi_1]_{init_1, f_1})$$

Conversely, when the fold is not trivial, the proof of Lemma 4 requires the sets of free variables to be equal. Otherwise, the induction termination is not well defined. One possibility is that the size of participating RDDs may not be equal. Assuming one set of free variables is contained in the other, any non-constant result of the fold function on these additional elements will lead to inequal results. To avoid such peculiarities, we require equal sets of free variables in aggregate terms. We proceed with an example showing a case which Lemma 4 does not cover due to Equation (3).

▸ **Example 3** (Non-injective modification of folded expressions). Non-injective transformations can weaken the inductive claim, resulting in failure to prove it. As a result, Lemma 4 fails to prove the equivalence of the following two Agg_1 programs.

P1($R: RDD_\texttt{Int}$)**:**	**P2**($R: RDD_\texttt{Int}$)**:**
1 $R' = \texttt{map}(\lambda x. x\%3)(R)$	$v = \texttt{fold}(0, \lambda A, x. A + x)(R)$
2 `return` $\texttt{fold}(0, \lambda A, x. (A + x)\%3)(R') = 0$	`return` $v\%3 = 0$

To prove the equivalence, we should check by induction the equality of both boolean results. Taking $g_1(x) = \lambda x. x = 0$, $g_2(x) = \lambda x. (x\%3) = 0$, the attempt to use Lemma 4 fails with a counterexample to Equation (3):

$$[x\%3]_{0, +\%3} = 0 \Leftrightarrow [x]_{0, +}\%3 = 0$$
$$\forall x, A, A'. A = 0 \iff A'\%3 = 0 \implies (A + x\%3)\%3 = 0 \iff (A' + x)\%3 = 0$$

The counter-example is: $A = 1, A' = 2, x = 1$. The hypothesis $A = 0 \iff A'\%3 = 0$ is satisfied, but $(A + x\%3)\%3 = 2 \neq 0$, and $(A' + x)\%3 = 0$.

Despite the fact that Lemma 4 did not cover Example 3, this example belongs to a subclass of Agg^1 for which a sound and complete equivalence test method exists. This class is characterized by a verifiable semantic property of the fold functions and the initial values. First, for both programs, it is possible to shrink a sequence of iterated applications of the fold function starting from the initial value, to a sequence of size 1. Secondly, given a sequence of elements on which the fold functions are applied, the same element is used to shrink both sequences of applications.

▸ **Definition 4** (The $AggPair^1_{sync}$ class). *Let there be two Agg^1 programs P_1, P_2 with equal signature, whose program terms are $g_i([\varphi_i]_{init_i, f_i})$ for $i = 1, 2$. We say that $\langle P_1, P_2 \rangle \in AggPair^1_{sync}$ if:*

$$FV(\varphi_1) = FV(\varphi_2) \tag{4}$$

$$\forall \overline{v_1}, \overline{v_2}. \exists \overline{v}'. f_1(f_1(init_1, \varphi_1(\overline{v_1})), \varphi_1(\overline{v_2})) = f_1(init_1, \varphi_1(\overline{v}')) \tag{5}$$
$$\wedge f_2(f_2(init_2, \varphi_2(\overline{v_1})), \varphi_2(\overline{v_2})) = f_2(init_2, \varphi_2(\overline{v}'))$$

The $AggPair^1_{sync}$ class contains pairs of programs in which repeated application of Equation (5) can shrink multiple applications of the *fold* function starting from the same initial value and on the same sequence of valuations, to a single application of the *fold* function, and it can be done using the same valuation for both programs.

▸ **Theorem 5** (Equivalence in $AggPair^1_{sync}$ is *decidable*). *Let P_1, P_2 such that $\langle P_1, P_2 \rangle \in AggPair^1_{sync}$, with input RDDs $\overline{r}$. We denote $\Phi(P_i) = g_i([\varphi_i]_{init_i, f_i})$. Then, $\Phi(P_1) = \Phi(P_2)$ if and only if:*

$$g_1(init_1) = g_2(init_2) \tag{6}$$

$$\forall \overline{v}, \overline{y}, A_{\varphi_1}, A_{\varphi_2}.(A_{\varphi_1} = f_1(init_1, \varphi_1(\overline{y})) \wedge A_{\varphi_2} = f_2(init_2, \varphi_2(\overline{y})) \wedge g_1(A_{\varphi_1}) = g_2(A_{\varphi_2})) \tag{7}$$
$$\implies g_1(f_1(A_{\varphi_1}, \varphi_1(\overline{v}))) = g_2(f_2(A_{\varphi_2}, \varphi_2(\overline{v})))$$

▸ Example 4 (Completing Example 3). We have:

$$f_0(f_0(0, x\%3), y\%3) = x\%3\%3 + y\%3\%3 = f_0(0, (x + y)\%3) = (x + y)\%3\%3$$
$$f_1(f_1(0, x), y) = x + y = f_1(0, x + y)$$

So Equation (5) is true (for arbitrary x, y, $x + y$ can reduce the two fold applications), and the programs belong to $AggPair^1_{sync}$. We are left with proving:

$$\forall x, y.((0 + y\%3)\%3 = 0 \iff (0 + y)\%3 = 0) \implies ((y + x\%3)\%3 = 0 \iff (y + x)\%3 = 0)$$

which is correct — so we were able to prove the equivalence with the stronger lemma.

Note that checking if two programs P_1, P_2 belong to $AggPair^1_{sync}$ involves a syntactic check of the free variables, and verification of an additional APA formula (Equation (5)).

8.2.4   Sound Methods for Additional Classes

A natural extension of the Agg^1 class is to programs that use the aggregated expression to perform an operation on RDDs. For example, a program that returns an RDD where all elements are stricty larger than all elements in another RDD:

```
        P1(R_0 : RDD_Int, R_1 : RDD_Int):
1   filter((λx.λy.y > x)(fold(-∞, max)(R_1)))(R_0)
```

. The program term is $ite(\mathbf{x}_{R_0} > [\mathbf{x}_{R_1}]_{-\infty,max}, \mathbf{x}_{R_0}, \perp)$.

▶ **Definition 5** (The Agg_R^1 class)**.** *Let there be a program P with $\Phi(P) = \psi$. We say that $P \in Agg_R^1$ if ψ contains a single instance of an aggregate term $\gamma = [\varphi]_{init,f}$, and φ has no aggregate terms. We write $\Phi(P) = \psi(\gamma)$.*

▶ **Lemma 6** (Lifting Lemma 4 to Agg_R^1)**.** *Let there be two* SparkLite *programs $P_1, P_2 \in Agg_R^1$ with terms ψ_i and aggregate expressions $\gamma_i = [\varphi_i]_{init_i, f_i}$, $i \in \{1, 2\}$. P_1 is equivalent to P_2 if:*

$$FV(\varphi_1) = FV(\varphi_2) \tag{8}$$

$$FV(\psi_1) = FV(\psi_2) \tag{9}$$

$$\forall \overline{x}.\psi_1(init_1)(\overline{x}) = \psi_2(init_2)(\overline{x}) \tag{10}$$

$$\forall \overline{x}, \overline{v}, A_1, A_2.(\psi_1(A_1)(\overline{x}) = \psi_2(A_2)(\overline{x})) \implies$$
$$(\psi_1(f_1(A_1, \varphi_1(\overline{v})))(\overline{x}) = \psi_2(f_2(A_2, \varphi_2(\overline{v})))(\overline{x})) \tag{11}$$

Lemmas 4,6 show that Agg^1, Agg_R^1 have a sound equivalence verification method, and Theorem 5 shows that $AggPair_{sync}^1$ has a sound and complete equivalence verification method.[14] The sound technique can be further generalized to programs with multiple aggregate terms, where the aggregated terms are not nested in one another. Each aggregate term does not contain an aggregate term in its definition. We denote this class Agg^n.

▶ **Definition 6** (The Agg^n class)**.** *Let there be a program P with $\Phi(P) = g([t_1]_{i_1, f_1}, \ldots, [t_n]_{i_n, f_n})$. $P \in Agg^n$ if $t_1, \ldots, t_n$ do not contain aggregate terms.*

▶ **Lemma 7.** *Let P_1, P_2 be two programs in Agg^n, such that $\Phi(P_i) = g_i(\overline{[\varphi_i]_{init_i, f_i}})$. We have $g_1(\overline{[\varphi_1]_{init_1, f_1}}) = g_2(\overline{[\varphi_2]_{init_2, f_2}})$ if:*

$$\bigcup FV(\overline{\varphi_1}) = \bigcup FV(\overline{\varphi_2}) \tag{12}$$

$$g_1(\overline{init_1}) = g_2(\overline{init_2}) \tag{13}$$

$$\forall \overline{v}, \overline{A_{\varphi_1}}, \overline{A_{\varphi_2}}.g_1(\overline{A_{\varphi_1}}) = g_2(\overline{A_{\varphi_2}}) \implies g_1(\overline{f_1(A_{\varphi_1}, \varphi_1(\overline{v}))}) = g_2(\overline{f_2(A_{\varphi_2}, \varphi_2(\overline{v}))}) \tag{14}$$

▶ **Example 5** (Independent fold)**.** Below are 2 programs which return a tuple containing the sum of positive elements in its first element, and the sum of negative elements in the second element. We show that by applying lemma 7, we are able to show the equivalence.

$$\text{Let:} \quad h : (\lambda(P, N), x.ite(x \geq 0, (P + x, N), (P, N - x))$$

	P1(R: RDD_{Int})**:**	**P2**(R: RDD_{Int})**:**
1	`return fold((0,0), h)(R)`	$R_P = $ `filter`$(\lambda x.x \geq 0)(R)$
2		$R_N = $ `map`$(\lambda x. - x)($`filter`$(\lambda x.x < 0)(R))$
3		$p = $ `fold`$(0, \lambda A, x.A + x)(R_P)$
4		$n = -$`fold`$(0, \lambda A, x.A + x)(R_N)$
5		`return` (p, n)

$$\Phi(P1) = [\mathbf{x}_R]_{(0,0),h}; \quad \Phi(P2) = ([\phi_{P2}(R_P)]_{0,+}, -[\phi_{P2}(R_N)]_{0,+})$$
$$\phi_{P2}(R_P) = ite(\mathbf{x}_R \geq 0, \mathbf{x}_R, \perp); \phi_{P2}(R_N) = ite(\mathbf{x}_R < 0, -\mathbf{x}_R, \perp)$$

---

[14] Even when the completeness criterion for $AggPair_{sync}^1$ is not met, we may be successful in proving the equivalence using Lemma 4. For example, $[((\lambda x.1)(\mathbf{x}_{r_0}), (\lambda x.1)(\mathbf{x}_{r_1}))]_{0, \lambda A, (x,y).A+x+y} = [((\lambda x.1)(\mathbf{x}_{r_1}), (\lambda x.1)(\mathbf{x}_{r_0}))]_{0, \lambda A, (x,y).A+x+y}$ can be proved by induction, but for $f = \lambda A, (x, y).A + x + y$, $f(f(A, (1, 1)), (1, 1)) = A + 4 \neq f(A, (1, 1)) = A + 2$ (the choice of valuation does not change the result). Thus, it does not satisfy the completeness criterion.

We let $g_1 = g_2 = \lambda(x,y).(x,y)$. We need to prove:

$$[\mathbf{x}_R]_{(0,0),h} = ([ite(\mathbf{x}_R \geq 0, \mathbf{x}_R, \bot)]_{0,+}, -[ite(\mathbf{x}_R < 0, -\mathbf{x}_R, \bot)]_{0,+})$$

Induction base case is trivial. Induction step:

$$\forall x, A, B, C. p_1(A) = B \wedge p_2(A) = C \implies$$
$$p_1(h(A,x)) = B + ite(x \geq 0, x, 0) \wedge p_2(h(A,x)) = C + ite(x < 0, -x, 0)$$

Substituting for h, we get a formula in APA. ∎

9   Conclusion and Future Work

To conclude, we saw that the problem of checking query equivalence (where queries were written as programs in the SparkLite language), can be modeled with logical formulas. Specifically, we looked at formulas over a decidable extension of the Presburger arithmetic, and added a special operation for representing fold operations. We also showed that in the presentation of a query equivalence instance as a logical formula, the solver for the formula is capable of proving equivalence of conjunctive queries (Theorem 2). Furthermore, we provided a classification of programs with the fold operations (aggregates), and presented a sound method for equivalence testing for each (Lemmas 4, 6, 7), and a sound and complete method for one particular non-trivial class of programs (Theorem 5). We hope the foundations laid in this paper will open numerous new possibilities: First, it allows writing tools that handle formal verification and optimization of clients written in Spark and similar frameworks, by building upon the concepts presented here to more elaborate structures involving queries with nested aggregation, unions, and multiple step-inductions for self joins. In addition, other decidable theories are applicable to programs used in practice, replacing Presburger arithmetics.

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A   Extending Cooper's Algorithm to the Augmented Presburger Arithmetic

Proof. Let φ be a quantified formula over $\bigcup_n \mathbb{Z}^n$ with terms in the Augmented Presburger Arithmetic. We shall translate φ to a formula in the Presburger Arithmetic. For any atom $A := a = b$, and $a, b \in \mathbb{Z}^k$ for some $k > 0$, we build the following formula: $\bigwedge_{i=1}^{k} p_i(a) = p_i(b)$ and replace it in place of A. In the resulting formula, we assign new variable names, replacing the projected tuple variables: For $a \in \mathbb{Z}^k$ we define $x_{a,i} = p_i(a)$ for $i \in \{1, \ldots, k\}$. Variable quantificaion extends naturally, i.e. $\forall a$ becomes $\forall x_{a,1}, \ldots, x_{a,k}$, and similarly for $\exists$.   ◀

B   Typing rules for SparkLite

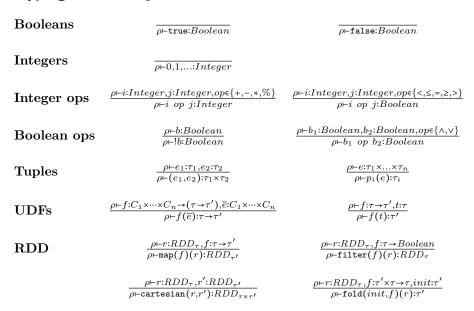|		
Booleans	$\overline{\rho \vdash \texttt{true}: Boolean}$	$\overline{\rho \vdash \texttt{false}: Boolean}$
Integers	$\overline{\rho \vdash 0,1,\ldots: Integer}$	
Integer ops	$\dfrac{\rho \vdash i: Integer, j: Integer, op \in \{+,-,*,\%\}}{\rho \vdash i \ op \ j: Integer}$	$\dfrac{\rho \vdash i: Integer, j: Integer, op \in \{<,\leq,=,\geq,>\}}{\rho \vdash i \ op \ j: Boolean}$
Boolean ops	$\dfrac{\rho \vdash b: Boolean}{\rho \vdash !b: Boolean}$	$\dfrac{\rho \vdash b_1: Boolean, b_2: Boolean, op \in \{\wedge,\vee\}}{\rho \vdash b_1 \ op \ b_2: Boolean}$
Tuples	$\dfrac{\rho \vdash e_1: \tau_1, e_2: \tau_2}{\rho \vdash (e_1,e_2): \tau_1 \times \tau_2}$	$\dfrac{\rho \vdash e: \tau_1 \times \ldots \times \tau_n}{\rho \vdash p_i(e): \tau_i}$
UDFs	$\dfrac{\rho \vdash f: C_1 \times \cdots \times C_n \to (\tau \to \tau'), \overline{e}: C_1 \times \cdots \times C_n}{\rho \vdash f(\overline{e}): \tau \to \tau'}$	$\dfrac{\rho \vdash f: \tau \to \tau', t: \tau}{\rho \vdash f(t): \tau'}$
RDD	$\dfrac{\rho \vdash r: RDD_\tau, f: \tau \to \tau'}{\rho \vdash \texttt{map}(f)(r): RDD_{\tau'}}$	$\dfrac{\rho \vdash r: RDD_\tau, f: \tau \to Boolean}{\rho \vdash \texttt{filter}(f)(r): RDD_\tau}$
	$\dfrac{\rho \vdash r: RDD_\tau, r': RDD_{\tau'}}{\rho \vdash \texttt{cartesian}(r,r'): RDD_{\tau \times \tau'}}$	$\dfrac{\rho \vdash r: RDD_\tau, f: \tau' \times \tau \to \tau, init: \tau'}{\rho \vdash \texttt{fold}(init,f)(r): \tau'}$

Figure 7 Typing Rules for SparkLite

C   Proof of Proposition 3

Proof. By symmetry, we assume w.l.o.g. $t_1 \neq \bot$. Therefore, there is an element in the RDD defined by t_1: $\exists \overline{x}, y. y = t_1(\overline{x}) \wedge y \neq \bot$. Denoting $\overline{x} = (x_1, \ldots, x_l)$, we choose input RDDs $\overline{r}$ such that each input RDD has a single element x_i of multiplicity n_i: $r_i = \{\!\{x_i; n_i\}\!\}$, for $i = 1, \ldots, l$. If $t_2(\overline{x}) \neq y$ then $[\![t_1]\!](\overline{r}) \neq [\![t_2]\!](\overline{r})$, as required. Otherwise, the multiplicity of y in $[\![t_1]\!]$ is $\Pi_{\mathbf{x}_{r_i} \in FV(t_1)} n_i$, and in $[\![t_2]\!]$ it is $\Pi_{\mathbf{x}_{r_i} \in FV(t_2)} n_i$. As $FV(t_1) \neq FV(t_2)$ and $\forall i = 1, \ldots, l. n_i > 0$, the multiplicities are different, thus $[\![t_1]\!](\overline{r}) \neq [\![t_2]\!](\overline{r})$, as required.   ◀

D   Proof of Lemma 4

Proof. First we recall the semantics of the `fold` operation on some RDD R, which is a bag. We choose an arbitrary element $a \in R$ and apply the fold function recursively on a and on R with a single instance of a removed. We then write a sequence of elements in the order they are chosen by `fold`: $\langle a_1, \ldots, a_n \rangle$, where n is size of the bag R. We also know that a requirement of aggregating operations' UDFs is that they are *commutative*, so the order

of elements chosen does not change the final result. We also extend f_i to $\xi_i \times (\sigma_i \cup \{\bot\})$ by setting $f_i(A, \bot) = A$ ($\bot$ is defined to behave as the neutral element for f_i). We denote $[\![\varphi_1]\!] = R_1, [\![\varphi_2]\!] = R_2$. To prove $g_1([\varphi_1]_{init_1, f_1}) = g_2([\varphi_2]_{init_2, f_2})$, it is necessary to prove that

$$g_1([\![\texttt{fold}]\!](f_1, init_1)(R_1)) = g_2([\![\texttt{fold}]\!](f_2, init_2)(R_2))$$

We set $A_{\varphi_j, 0} = init_j$ for $j = 1, 2$. Each element of R_1, R_2 is expressible by providing a concrete valuation to the free variables of φ_1, φ_2, namely the vector $\overline{v}$. We prove the equality by induction on the *size* of the RDDs R_1, R_2, denoted n. We choose an arbitrary sequence of n valuations $\langle \overline{a}_1, \ldots, \overline{a}_n \rangle$, and plug them into the *fold* operation for both R_1, R_2. The result is 2 sequences of *intermediate values* $\langle A_{\varphi_1, 1}, \ldots, A_{\varphi_1, n} \rangle$ and $\langle A_{\varphi_2, 1}, \ldots, A_{\varphi_2, n} \rangle$. From the semantics of $\texttt{fold}$, we have that $A_{\varphi_j, i} = f_j(A_{\varphi_j, i-1}, \varphi_j(\overline{a}_i))$ for $j = 1, 2$. Our goal is to show $g(A_{\varphi_1, n}) = g'(A_{\varphi_2, n})$ for all n.

Case $n = 0$: $R_1 = R_2 = \varnothing$, so $[\![\texttt{fold}]\!](f_1, init_1)(R_1) = init_1$ and $[\![\texttt{fold}]\!](f_2, init_2)(R_2) = init_2$. From Equation (2), $g_1(init_1) = g_2(init_2)$, as required.

Case $n = i$, assuming correct for $n \leq i - 1$: By assumption, we know that the sequence of intermediate values up to $i - 1$ satisfies: $g_1(A_{\varphi_1, i-1}) = g_2(A_{\varphi_2, i-1})$. We are given the i'th valuation, denoted $\overline{a}_i$. We need to show $A_{\varphi_1, i} = A_{\varphi_2, i}$, so we use the formula for calculating the next intermediate value:

$$
\begin{aligned}
A_{\varphi_1, i} &= f_1(A_{\varphi_1, i-1}, \varphi_1(\overline{a}_i)) \\
A_{\varphi_2, i} &= f_2(A_{\varphi_2, i-1}, \varphi_2(\overline{a}_i))
\end{aligned}
$$

We use Equation (3), plugging in $\overline{v} = \overline{a}_i$, $A_{\varphi_1} = A_{\varphi_1, i-1}$, and $A_{\varphi_2} = A_{\varphi_2, i-1}$. By the induction assumption, $g_1(A_{\varphi_1, i-1}) = g_2(A_{\varphi_2, i-1})$, therefore $g_1(A_{\varphi_1}) = g_2(A_{\varphi_2})$, so Equation (3) yields $g_1(f_1(A_{\varphi_1}, \varphi_1(\overline{a}_i))) = g_2(f_2(A_{\varphi_2}, \varphi_2(\overline{a}_i)))$. By substituting back A_{φ_j} and the formula for the next intermediate value, we get: $g_1(A_{\varphi_1, i}) = g_2(A_{\varphi_2, i})$ as required. ◄

E   Proof Theorem 5

Proof. Sound (if): We prove the equality $g_1([\varphi_1]_{init_1, f_1}) = g_2([\varphi_2]_{init_2, f_2})$ by induction on the size of the RDDs $[\![\varphi_1]\!], [\![\varphi_2]\!]$, denoted n. For $n = 0$, $[\![\varphi_1]\!](\overline{r}) = [\![\varphi_2]\!](\overline{r}) = \varnothing$, thus $[\varphi_i]_{init_i, f_i} = init_i$ $(i = 1, 2)$, and the equality follows from Equation (6). Assuming for n and proving for $n+1$: We let a sequence of intermediate values $A_{\varphi_i, k}$, $(i = 1, 2; k = 1, \ldots, n+1)$, for which we know in particular that $g_1(A_{\varphi_1, n}) = g_2(A_{\varphi_2, n})$, and we need to prove $g_1(A_{\varphi_1, n+1}) = g_2(A_{\varphi_2, n+1})$. We denote $A_{\varphi_i, 0} = init_i$, and then we have $A_{\varphi_i, k} = f_i(A_{\varphi_i, k-1}, \varphi_i(\overline{a_k}))$ $(k = 1, \ldots, n+1)$ for some $\overline{a_k}$. According to Equation (5), $A_{\varphi_i, 2} = f_i(A_{\varphi_i, 1}, \varphi_i(\overline{a_2})) = f_i(f_i(init_i, \varphi_i(\overline{a_1})), \varphi_i(\overline{a_2}))$ yields $\exists \overline{a_2'}. \bigwedge_{i=1,2} A_{\varphi_i, 2} = f_i(init_i, \varphi_i(\overline{a_2'}))$. We can thus use Equation (5) to prove by induction that $\exists \overline{a_k'}. \bigwedge_{i=1,2} A_{\varphi_i, k} = f_i(init_i, \varphi_i(\overline{a_k'}))$, and in particular $\exists \overline{a_n'}. \bigwedge_{i=1,2} A_{\varphi_i, n} = f_i(init_i, \varphi_i(\overline{a_n'}))$. By applying Equation (7) for $\overline{v} = \overline{a_{n+1}}, \overline{y} = \overline{a_n'}$, we get:

$$
\begin{aligned}
g_1(f_1(f_1(init_1, \varphi_1(\overline{y}), \varphi_1(\overline{v}))) &= g_2(f_2(f_2(init_2, \varphi_2(\overline{y}), \varphi_2(\overline{v}))) &&\Longrightarrow \\
g_1(f_1(f_1(init_1, \varphi_1(\overline{a_n'}), \varphi_1(\overline{a_{n+1}}))) &= g_2(f_2(f_2(init_2, \varphi_2(\overline{a_n'}), \varphi_2(\overline{a_{n+1}}))) &&\Longrightarrow \\
g_1(f_1(A_{\varphi_1, n}, \varphi_1(\overline{a_{n+1}}))) &= g_2(f_2(A_{\varphi_2, n}, \varphi_2(\overline{a_{n+1}}))) &&\Longrightarrow \\
g_1(A_{\varphi_1, n+1}) &= g_2(A_{\varphi_2, n+1})
\end{aligned}
$$

as required.

Complete (only if): Assume towards a contradiction that either Equation (6) or Equation (7) are false. If the requirement of Equation (6) is not satisfied, yet the aggregates are equivalent, i.e.

$$g_1([\varphi_1]_{init_1, f_1}) = g_2([\varphi_2]_{init_2, f_2}) \wedge g_1(init_1) \neq g_2(init_2)$$

then we can get a contradiction by choosing all input RDDs to be empty. Thus, for $R = \varnothing$, $[\![[\varphi_1]_{init_1,f_1}]\!](R) = init_1 \wedge [\![[\varphi_2]_{init_2,f_2}]\!](R) = init_2 \implies g_1(init_1) = g_2(init_2)$, contradiction. The conclusion is that Equation (6) is a necessary condition for equivalence. Therefore, we assume just Equation (7) is false. Let there be counter-examples $\overline{v}, \overline{y}$ to Equation (7) (The A_{φ_i} are determined immediately), and let:

$$F_i = f_i(f_i(init_i, \varphi_i(\overline{y}), \varphi_i(\overline{v}))$$

Then $g_1(F_1) \neq g_2(F_2)$. By Equation (5) we can write F_i as: $F_i = f_i(init_i, \varphi_i(\overline{w}))$ for some $\overline{w}$. We take an RDD $R = \{\!\{\overline{w}; 1\}\!\}$. Then $[\![\varphi_j]\!](R) = \{\!\{\varphi_j(\overline{w}); 1\}\!\}$, for which: $[\![[\varphi_j]_{init_j,f_j}]\!](R) = F_i$. By the assumption, $[\![g_1([\varphi_1]_{init_1,f_1})]\!](R) = [\![g_2([\varphi_2]_{init_2,f_2})]\!](R)$, but then $g_1(F_1) = g_2(F_2)$. Contradiction. ◄

F   Proof of Lemma 6

Proof. The proof follows along the lines of the proof of Lemma 4. We need to prove $\Phi(P_1) = \Phi(P_2)$, or $\forall \overline{x}.\psi_1(\gamma_1)(\overline{x}) = \psi_2(\gamma_2)(\overline{x})$, where $\gamma_i = [\varphi_i]_{init_i,f_i}$ and $\overline{x}$ is a vector of valuations to $FV(\psi_1), FV(\psi_2)$ which are equal sets (Equation (9)). We shall prove it by induction on the size of the RDDs R_1, R_2, generating the underlying terms of γ_1, γ_2.

For size 0, we have $\gamma_i = init_i$, and from Equation (10) we have $\Phi(P_1)(R_1) = \Phi(P_2)(R_2)$ as required.

Assuming for size n and proving for $n + 1$: The RDDs R_1, R_2 are now generated using $a_1, \ldots, a_{n+1}$, with intermediate values $A_{\varphi_i,1}, \ldots, A_{\varphi_i,n+1}$ for $i = 1, 2$. By assumption, $\forall x.\psi_1(A_{\varphi_1,n}) = \psi_2(A_{\varphi_2,n})$, and we need to prove $\forall \overline{x}.\psi_1(A_{\varphi_1,n+1})(\overline{x}) = \psi_2(A_{\varphi_2,n+1})(\overline{x})$. In addition, $A_{\varphi_i,n+1} = f_i(A_{\varphi_i,n}, a_{n+1})$ for $i = 1, 2$. We let some $\overline{x}$ and we need to prove for it $\psi_1(A_{\varphi_1,n+1})(\overline{x}) = \psi_2(A_{\varphi_2,n+1})(\overline{x})$. We apply Equation (11) with $\overline{x}$, $\overline{v} = a_{n+1}$, and $A_{\varphi_1,n}, A_{\varphi_2,n}$, concluding that: $\psi_1(f_1(A_{\varphi_1,n}, a_{n+1}))(\overline{x}) = \psi_2(f_2(A_{\varphi_2,n}, a_{n+1}))(\overline{x})$. Replacing for $A_{\varphi_i,n+1}$, we get what had to be proven. ◄