ENPM 667 Control of Robotics Systems

Review Report On

A Unified SO(3) Approach to the Attitude Control Design for Quadrotors

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Abstract

This report aims to review and replicate the results of the selected research paper, Jen-Te Yu, "A Unified SO(3) Approach to the Attitude Control Design for Quadrotors," IEEE Access, vol. 9, 20 April 2021. The paper proposes a two phased problem reformulation approach which resolves the quadrotor Unmanned Aerial Vehicle attitude control problem. The aim is to design a control to move the UAV to a desired location, given an initial attitude. The first part declares three types of errors concerning attitude discrepancy and is followed by using virtual control which is then merges the angular velocity and the attitude dynamics under the SO(3) framework. Cancellation of unwanted terms and entry wise treatment of the remining dynamics gives a three-dimensional linear time invariant system, which can be stabilized using standard feedback designs. To account for the parametric uncertainties, external disturbances, and sensor errors and noises, another controller is used based on H_∞ control. A numerical example and computer simulations are reported to assess and compare the effectivity of the proposed design.

Index Terms: Define Quadrotor UAV, attitude, rotation matrix, Special Orthogonal group SO(3), virtual control, linear quadratic regulator, disturbance attenuation, H_∞ control, Special Orthogonal group SO(3), Linear Time Invariant(LTI) system and state feedback form

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I. INTRODUCTION

The concept of Unmanned Aerial vehicles (UAVs) or Drones is not unheard of. In recent times, there has been quite a lot of interest in this specific field. They are so popular because of its significant advantages over a regular aerial vehicle. Advantages such as vertical take-off and landing, and rapid manoeuvring. Some applications include photography/videography, patrolling, firefighting, detecting incidents, aerial mappings, delivering goods surveillance, spraying pesticides and fertilizers, monitoring, defence, etc. With such a great potential in so many fields, its operation and maintenance costs are comparatively low.

However, the dynamics of a quadrotor can get very complex and hence it is divided into two parts – translation and rotational motion. Translation can also be called as the outer loop, which consists of altitude, linear speed, and acceleration. Rotation, also called as the inner loop consists of the angular velocities and the attitude angles. Figure 1 shows the schematic for the rotation motion of a quadrotor.

Translational motion of the quadrotor depends on its rotational motion, but the inverse is not true. So, attitude control can also be designed independently. There is also research that deals with both types of motion simultaneously. The paper we chose, concentrates on the former. Therefore, the goal of the paper is to design a controller that makes the quadrotor to track a desired and predetermined attitude, through attitude control. This report reviews every step in this research and provides detailed explanation.

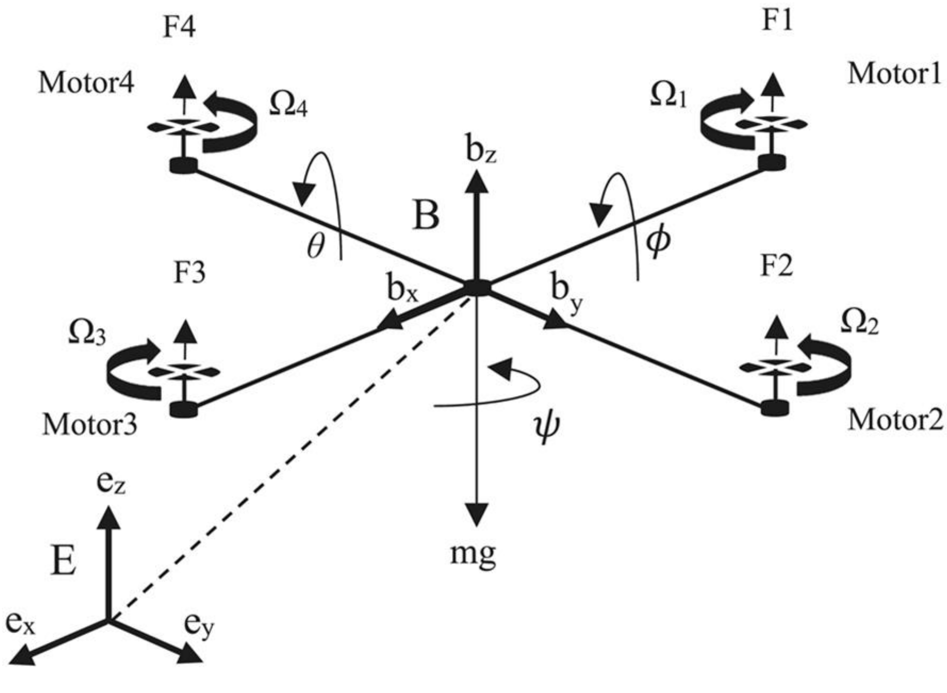


Figure 1. Schematic of rotational motion of quadrotor with body and earth reference frame (source: https://doi.org/10.1002/asjc.1758)

The paper proposes a two phased transformation which unifies the attitude dynamics of a quadrotor under the SO(3) framework and in turn linearizes the attitude control. By using this method, the design problem becomes simpler, the controller design becomes easy and reduces the dimension of the problem such that many classical control schemes are applicable.

This approach specifically uses SO(3) to parameterize the attitude. Other classical methods include Euler angle, Rodriguez vector and unit-quaternion. Some of them are Euclidean and some are not. Except the rotation matrix, all the attitude representations are not simultaneously globally defined and unique. For example, the Euler angle can represent any attitude but it is not unique at certain angle. Although unit quaternion can represent any angle and does not have the singularity problem, it redundantly covers the set of rotation matrix. In other words, the $\pm$ Q $\in \mathbb{S}^3$ mappings to rotation matrix are in fact identical. This defect leads to unwinding or renders only local attitude manoeuvres.

This paper chooses a rotation matrix to parameterize the attitude. Besides being global and unique, there are some appealing features exhibited by the rotation matrix. The set of the rotation matrix belongs to a special orthogonal group which is denoted by SO(3) which comprises the rotation space. Uniqueness is essential when another attitude representation is mapped to SO(3).

In this paper, first, three types of errors pertaining to attitude discrepancy are defined. Virtual control is employed to drive the attitude error to zero, considering the error dynamics. Second, the dynamics of quadrotor's attitude and angular velocities are unified under the framework of SO(3), using three algebraic facts. The first component of control is chosen naturally as it readily cancels the unwanted terms in the error dynamics. The remaining matrix error dynamics is treated entry-wise, which then becomes a three-dimensional linear time-invariant system (LTI). Stabilization of such a system using static (constant) feedback gain can easily achieved by employing classical control schemes, provided there are no uncertainties or disturbances. Finally, another static design called the H infinity control theory is proposed, to deal with the parametric uncertainties, external disturbances and sensor measurement error and noise. This makes the proposed scheme more robust and practical.

The report is organized as follows. First section starts with the brief about the problem statement and introduction of the paper. The second section starts with the dynamics model, control objective, and some basic assumptions. There is also a brief description about rotation matrices and SO(3) group. In section III, we define the three types of error related to attitude discrepancy. A virtual control is used to begin the design. Further in section IV, using the three algebraic facts, the dynamics of the quadrotor attitude and the angular velocity are unified under the SO(3) framework leading to the choice of the first component of control. The choice of the first component of control emerges in a natural manner at this point, as it can be used to cancel some unwanted terms appearing in the error dynamics. An important result out of this cancelation is that the remaining matrix error dynamics, when treated entry-wise, becomes a three-dimensional linear time-invariant (LTI) system. Stabilization of such low dimensional LTI systems using static (constant) feedback gain can be easily achieved by classical control schemes, provided no uncertainties and/or disturbances exist. Then in section V, we assume zero disturbance or uncertainties. We then, address the choice of feedback gain for the LTI system and propose to two designs for the nominal system, Linear Quadratic Regulator (LQR) and a specialized one. Section VI, provides a static attitude control design, using the H_∞ control theory from robust controls, which equips the system with disturbance attenuation capabilities. Section VII, solely talks about the stability of this closed loop system while section VIII, validates the entire approach with a numerical example and computer simulation. Finally in section IX, we describe the entire result of the project, the outcomes of it and possible future works in the field and the challenges that still need to be addressed.

Some notations used throughout the paper: -

- Vector "ω" refers to the angular velocity of the quadrotor
- R refers to the rotation matrix associated with a set of Euler angles that represent an attitude
- E refers to the attitude error
- I_R refer to integral error associated with the rotation matrix
- Δ and *d* refers to lumped disturbances and/ or uncertainties
- γ refers to disturbance attenuation level
- I_3 is an identity matrix of order 3
- $\hat{A}$ or $(A)^{\Lambda}$ refers to skew symmetric matrix associated to A
- λ refers to eigenvalue
- Cofactor(J) or Cof(J) refers to the cofactor matrix of J
- |J| or det(J) refers to the determinant of J
- ||.|| refers to matrix norm or vector length
- A < 0 means that matrix A is negative definite

II. THE SYSTEM MODEL AND CONTROL OBJECTIVES:

Quadrotor UAV: The term quadrotor is used to define an aerial vehicle which is controlled with four rotors. Each of these rotors consists of a motor and a propeller. So, a quadrotor UAV is a remotely controlled or automated aerial vehicle which has four rotors. The quadrotor is a Vertical Take-Off and Landing(VTOL) aircraft.

Attitude: Attitude is the set of values that describes how an object is placed in the space. So, attitude control is the process of controlling the orientation of the concerned body with respect to an inertial frame or reference frame.

Rotation Matrix: Rotational Matrix is a transformation matrix used to describe rotation in the Euclidean space. In our case, we deal with 3 D rotations. When pre-multiply, the following three matrices rotate vectors by $\theta\ about\ x, y, and\ z$ axis, using the right-hand rules.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & Cos\theta & -Sin\theta \\ 0 & Sin\theta & Cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} Cos\theta & 0 & Sin\theta \\ 0 & 1 & 0 \\ -Sin\theta & 0 & Cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} Cos\theta & -Sin\theta & 0 \\ Sin\theta & Cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Roll: ϕ→X axis : It is defined as an angle of rotation about X axis.

Pitch: θ→Y axis : It is defined as an angle of rotation about Y axis.

Yaw: ψ→Z axis : It is defined as an angle of rotation about Z axis.

Special Orthogonal Group SO(3): Special Orthogonal Group SO(3) is a Lie Group that has an algebraic structure in the operation of matrix multiplication and a differential geometric structure. Moreover, it features global attitude definition and one-to-one mapping. In simple words, it is a three-dimensional rotational group which consists of all rotations about the origin of three-dimensional Euclidean space. Rotations are linear transformations of $\Re^3$. In our case SO (3), an orthonormal basis is chosen, so every rotation is described by an orthonormal 3 × 3 matrix. Therefore, the set of 3 × 3 matrices, coupled with matrix multiplication and follows the below mentioned properties forms a group called SO(3).

Properties of SO(3): -

R = $[a \quad b \quad c]$, a,b,c ∈ $\mathfrak{R}^3$ is a rotation matrix for $\mathfrak{R}^3$ if

→ Det(R) = (a × b) . c = 1

→ Identity : RI = IR

$R_1, R_2, R_3 \in$ SO(3)

→ Closure :

$$(R_1,R_2)^T(R_1R_2) = R_2^T R_1^T R_1 R_2$$
$$= R_2^T I R_2$$
$$= R_2^T R_2$$
$$= I$$

→ Associativity : $R_1(R_2R_3) = (R_1R_2R_3)$

→ $R^TR = I$

- **Attitude Dynamics:**

Figure 1 shows the schematic of the quadrotor. In this figure, coordinate frame B denoted by unit vectors [b_x by b_z] is attached to the body fixed frame and placed at the center of mass of the quadrotor and coordinate frame E denoted by unit vector [e_x e_y e_z] is attached to the reference frame or inertial frame.

Unit vector [b_x b_y b_z] are along the **principal axis** of the quadrotor. Under normal circumstances, quadrotors are symmetric and hence the off-diagonal term of the **inertial matrix** of quadrotor is zero. There for the inertia matrix of quadrotor in the frame B is given by following expression.

$$J = \begin{bmatrix} J1 & 0 & 0 \\ 0 & J2 & 0 \\ 0 & 0 & J3 \end{bmatrix}$$

Where J_1, J_2 and J_3 are the moment of inertia of quadrotor along the b_x, b_y and b_z axis.

Angular velocity of the quadrotor ω_{EB} can be represented in the frame B with the following expression.

$$\omega_{EB} = \omega_1 b_1 + \omega_2 b_2 + \omega_3 b_3$$

Where ω_1, ω_2 and ω_3 are the component of the ω_{EB} along b_1, b_2 and b_3

Now, angular momentum of quadrotor in frame B can be expressed as below:

$$H_C = J\,\omega_{EB}$$

$$= \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$= \begin{bmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{bmatrix}$$

In order to find the Euler's equation of motion, we need to differentiate this angular momentum term in an inertial frame E. The angular momentum can be represented in the frame E by following expression

$$H_C = R_{EB} \begin{bmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{bmatrix}$$

Where $R_{EB} \in R^{3X3}$ is the rotation matrix representing transformation of frame B into frame E. Henceforth R_{EB} will be written as R.

Differentiating above expression

$$\dot{H}_C = R \begin{bmatrix} J_1\dot{\omega}_1 \\ J_2\dot{\omega}_2 \\ J_3\dot{\omega}_3 \end{bmatrix} + \dot{R} \begin{bmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{bmatrix}$$

To transform back the above derivative to frame B, we need to multiply the above expression with R_{EB}^T

$$\dot{H}_C = R^T R \begin{bmatrix} J_1\dot{\omega}_1 \\ J_2\dot{\omega}_2 \\ J_3\dot{\omega}_3 \end{bmatrix} + R^T \dot{R} \begin{bmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{bmatrix}$$

Since R_{EB} is the rotational matrix, $R_{EB}^T R_{EB} = I$, where I is a 3 X 3 identity matrix.

$R_{EB}^T \dot{R}_{EB} = \widehat{\omega}$ where $\widehat{\omega}$ can be expressed as below skew schematic matrix

$$\widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\therefore \dot{H}_C = \begin{bmatrix} J_1\dot{\omega}_1 \\ J_2\dot{\omega}_2 \\ J_3\dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{bmatrix}$$

$$\therefore \dot{H}_c = J\dot{\omega} + \omega \times (J\omega)$$

Where J = $\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$ is the inertia matrix of the quadrotor represented in the body fixed frame,

$\dot{\omega} = \begin{bmatrix} \dot{\omega_1} \\ \dot{\omega_2} \\ \dot{\omega_3} \end{bmatrix}$ is the angular acceleration in frame B

$$\therefore\ \dot{H}_C = J\dot{\omega} - (J\omega) \times \omega$$

$$\therefore\ J\dot{\omega} = (J\omega) \times \omega + \dot{H}_C$$

Now rate of change of angular momentum $\dot{H}_C$ is equivalent to the moment applied by the rotor. Hence it is the input to the system and can be represented as u + Δ where u is the designed input and Δ is the lumped uncertainty and disturbance

$$\therefore\ J\dot{\omega} = (J\omega) \times \omega + u + \Delta \qquad \text{.....................................(1)}$$

$\omega \in \Re^3$, $u \in \Re^3$, $\Delta \in \Re^3$

$$\dot{R} = R\widehat{\omega}\ ,\ R \in \Re^{3X3}\ ,\ \widehat{\omega} \in \Re^{3X3} \qquad \text{.....................................(2)}$$

Note that matrix $\widehat{\omega}$ can be viewed as the skew-symmetrified counterpart of the velocity vector ω defined as

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix},\ \widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad \text{.....................................(3)}$$

$$R^T R = R R^T = I_3\ ,\ \widehat{\omega}^T = -\widehat{\omega} \qquad \text{.....................................(4)}$$

- Control Objective:

Let us suppose that R_d is the desired rotation matrix. This rotation matrix is orthonormal meaning that its rows and columns form an orthonormal vector, and its determinant is equal to 1. It also satisfies the below property.

$$R_d^T R_d = R_d R_d^T = I_3$$

Without loss of generality (meaning that we are narrowing the premises to a particular case, but it does not affect the proof in general), this paper focuses on the case where the desired rotation matrix R_d is constant. Equivalently, given an admissible rotation matrix R, the goal is to approach R_d using the control u. Rotation matrix comes directly from the roll, pitch and yaw angle that define the attitude of a quadrotor. We will derive the rotation matrix from the given roll, pitch, and yaw in the next the section.

III.THREE TYPES OF ERRORS AND VIRTUAL CONTROL:

The attitude of a quadrotor is the orientation of the body's fixed frame in the inertial frame or reference frame. The attitude of the quadrotor, in general, is defined by a set of 3D Euler angles – the so-called roll, pitch and yaw angles (denoted as $\phi, \ \theta, and\ \psi$). Associated with this set of Euler angles is a rotation matrix resulting from a body-fixed axis rotation sequence that corresponds to the given set of Euler angles.

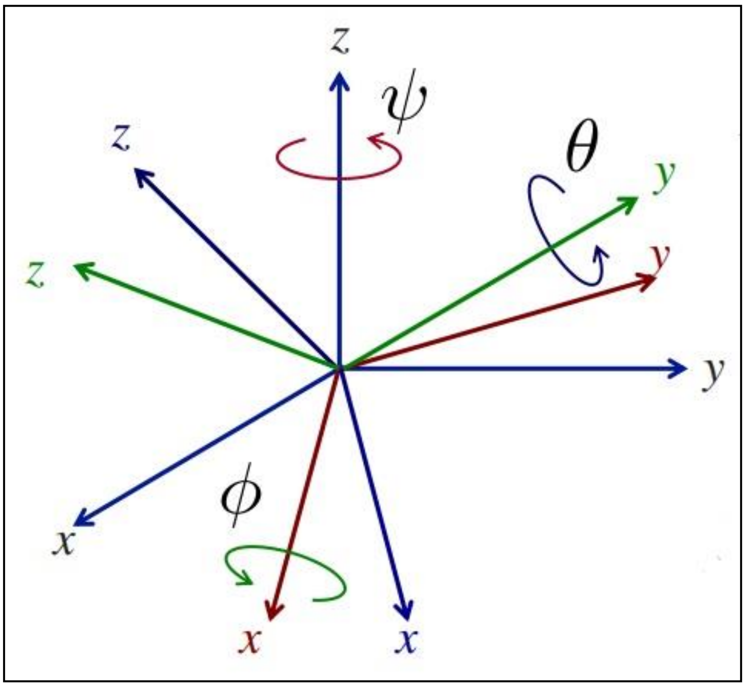


Figure: 2
(Source: Edx.org)

Let us suppose that the sequence of rotation of the quadrotor about the body-frame is as shown in the figure 2.

Rot (z, ψ) --- Yaw
Rot (x, θ) --- Roll
Rot (y, ϕ) --- Pitch

Here $\phi, \ \theta, and\ \psi$ are the 3D Euler angle.

Therefore, the rotation matrix can be given by the following formula:

R = Rot(z, ψ)Rot(x, θ)Rot(y, ϕ)

$$\therefore \text{R} = \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\phi & 0 & sin\phi \\ 0 & 1 & 0 \\ -sin\phi & 0 & cos\phi \end{bmatrix}$$

$$\therefore \text{R} \ = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & c\phi s\psi & c\psi s\theta + c\theta s s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\theta s c\psi \\ c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$

Since the rotation matrix is corresponds to the Euler angle, the attitude error in the form of Euler angle is equivalent to the attitude error in the form of rotation matrix.

Here, R is the initial admissible rotation matrix and R_d is the desired admissible rotation matrix. The objective is to design the control input u to drive the discrepancy between the two to zero. The design is corresponding to the two phases and corresponding to which there will be two different sets of gains to be determined, which will be detailed shortly.

In the first phase of design, we will consider the error in the rotation matrix. The error in the rotation matrix can be given by following expression.

$$E_R = R - R_d \qquad \text{..................................(6)}$$

Which is to be driven to zero. Differentiating the above error with respect to time

$$\dot{E}_R = \dot{R} - \dot{R}_d$$

Since R_d is constant, $\dot{R}_d = 0$ and from equation (2), $\dot{R} = R\widehat{\omega}$

$$\therefore \dot{E}_R = R\widehat{\omega} \qquad \text{...................................... (7)}$$

Now, we would like to define Lyapunov function by vectorizing attitude error matrix. It is shown as follows,

$$\overrightarrow{E_R} = e_R = \vec{R} - \overrightarrow{R_d}$$

Lyapunov function is defined as,

$$V = e_R^T e_R$$

Because of its quadratic form, it will always be positive unless the error equals to zero. To ensure the stability, we let the derivative of Lyapunov function to be negative by input. As a result, we can get a virtual control of the angular velocity in derivatives of Lyapunov function. It could drive the function to be zero.

$$\dot{V} = 2e_R^T(\overrightarrow{R\widehat{w}})$$

If matrix $R\widehat{\omega}$ were the control, it may be chosen to drive the attitude error to zero. In other words, let us assume that $R\widehat{\omega}$ is our control input. So, it will drive the attitude error to zero. Without loss of generality,

$$(R\widehat{\omega})_{vir} = -g_p E_R, \quad g_p > 0 \qquad \text{.................................... (8)}$$

Where the subscript "vir" stands for virtual and g_p is a positive scalar proportional gain at our disposal. To reduce steady-state error, an integral control may be added, which is defined as

Now,

$$I_R = \int_0^t E_R(\tau)d\,\tau \qquad \text{.....................................} \ (9)$$

With this integration term, the

$$(R\widehat{\omega})_{vir} = g_i I_R - g_p E_R, \qquad 0<|g_i| << 1 \qquad \text{.....................................} \ (10)$$

Where g_i is a scalar integral gain at our disposal. Under normal circumstances, small g_i suffices. Larger g_i may result into oscillatory response and sometime instability.

Apparently, discrepancy exist between the virtual control and its actual value hence another error may be defined as

$$E_\omega = R\widehat{\omega} - (R\widehat{\omega})_{vir} = R\widehat{\omega} - g_i I_R + g_p E_R, \qquad \text{.....................................} \ (11)$$

IV. Problem reformulation under a unified SO(3) framework and entry-wise stabilization

This section is divided into 2 parts.

1. To facilitate the skew symmetrification operation, we use three algebraic facts in this section, stated and defined below.

   - Fact 1: (Da) × (Db) = Cof(D)(a × b) ,
     Where a and b are any 3-dimensional vectors, and D is any 3 × 3 matrix
     Proof of Fact 1: -
     $A^{-1} = \frac{1}{|A|}$ Adj[A] ; Adj{A] = $Cof(A)^T$
     $A^{-1} = \frac{1}{|A|}$ $Cof(A)^T$
     $A^{-1}|A| = Cof(A)^T$
     Taking transpose both sides,
     $|A|(A^{-1})^T = Cof(A)$
     $(Ma) \times (Mb) = M(a \times b) = |M|(M^{-1})^T (a \times b) = cof(M)\ (a \times b)$

   - Fact 2: $(a \times b)\hat{} = ba^T - ab^T$,

     Where (a × b)^ is a skew symmetric matrix corresponding to (a × b) vector

     Proof of Fact 2: -

     Let c = $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ and d = $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

     $$c \times d = \begin{bmatrix} c_2 d_3 - C_3 d_2 \\ c_3 d_1 - c_1 d_3 \\ c_1 d_2 - c_2 d_1 \end{bmatrix}$$

     $$(c \times d)^{\wedge} = \begin{bmatrix} 0 & c_2 d_1 - c_1 d_2 & c_3 d_1 - c_1 d_3 \\ c_1 d_2 - c_2 d_1 & 0 & C_3 d_2 - c_2 d_3 \\ c_1 d_3 - c_3 d_1 & c_2 d_3 - C_3 d_2 & 0 \end{bmatrix}$$

     $c^T = [c_1 \quad c_2 \quad C_3]$; $d^T = [d_1 \quad d_2 \quad d_3]$

     $$cd^T = \begin{bmatrix} C_1 d_1 & C_2 d_1 & C_3 d_1 \\ C_1 d_2 & c_2 d_2 & c_3 d_2 \\ C_1 d_3 & C_2 d_3 & C_3 d_3 \end{bmatrix}$$

     $$dc^T = \begin{bmatrix} c_1 d_1 & C_2 d_1 & C_3 d_1 \\ C_1 d_2 & c_2 d_2 & c_3 d_2 \\ c_1 d_3 & C_2 d_3 & C_3 d_3 \end{bmatrix}$$

$$cd^T - dc^T = \begin{bmatrix} 0 & C_2d_1 - C_1d_2 & C_3d_1 - C_1d_3 \\ C_1d_2 - C_2d_1 & 0 & C_3d_2 - C_2d_3 \\ C_1d_3 - C_3d_1 & C_2d_3 - C_3d_2 & 0 \end{bmatrix}$$

Therefore ,

$(c \times d)^{\wedge} = cd^T - dc^T$

- Fact 3: $J^{-1} = |J^{-1}|\, Cof(J)$,
  Where Cof stands for cofactor and $|J|$ is the determinant of J
  Proof of Fact 3: -
  $A^{-1} = \frac{1}{|A|} Adj[A]$; $Adj\{A] = Cof(A)^T$
  $A^{-1} = \frac{1}{|A|} Cof(A)^T$
  $A^{-1} = |A|^{-1} Cof(A)^T$
  Taking Transpose both sides
  $(A^{-1})^T = |A|^{-1} Cof(A)^T$
  If A is a diagonal matrix, A^{-1} is also a diagonal matrix. So, $(A^{-1}) = A^{-1}$
  $A^{-1} = |A|^{-1} Cof(A)$

2. Entry wise treatment of remaining dynamics and followed by cancelation of unwanted terms which leads to dimension reduction and linearization, allowing to use the classical control designs.

Now we already know that,

$$J\dot{\omega} = (J\omega) \times \omega + u + \Delta \quad \text{[From (1)]}$$

Pre multiplying by $|J^{-1}|$,

$$\dot{\omega} = J^{-1} [(J\omega) \times \omega + u + \Delta\,] \quad \text{……………………………………………………………………….(12)}$$

Note : The two dynamic equations (1) – (2) are coupled, so for consistency, the design will be performed on the dynamics of $\widehat{\omega}$ instead of ω.

Now using the three facts,

$$\dot{\omega} = |J^{-1}| Cof(J)[\,(J\omega) \times \omega] + J^{-1}u + J^{-1}\Delta \quad \text{(Using Fact 3)}$$

$$\therefore \dot{\omega} = |J^{-1}|\,[Cof(J)\,((J\omega) \times \omega)] + J^{-1}u + J^{-1}\Delta \quad \text{(Using Fact 1)}$$

$$\therefore \dot{\omega} = |J^{-1}|\,[(J^2\omega) \times (J\omega)] + J^{-1}u + J^{-1}\Delta$$

$$\therefore \dot{\omega} = |J^{-1}|\,[(J^2\omega) \times (J\omega)] + J^{-1}u + J^{-1}\Delta$$

And,

$$(J^{-1}u) = \begin{bmatrix} J_1^{-1} \\ J_2^{-1} \\ J_3^{-1} \end{bmatrix} \quad \text{....................................(14)}$$

Now, using the skew symmetrification operator throughout the equation,

$$\dot{\widehat{\omega}} = |\ J^{-1}|[J\omega\omega^T J^2 - J^2\omega\omega^T J] + (J^{-1}u)^\wedge + (J^{-1}\Delta)^\wedge \ \text{..................................(16)} \ \text{(Using Fact 3)}$$

Now the control becomes,

$$(J^{-1}u)^\wedge = \begin{bmatrix} J_1^{-1}\, u_1 \\ J_2^{-1}\, u_2 \\ J_3^{-1}\, u_3 \end{bmatrix} = \begin{bmatrix} 0 & -J_3^{-1}\, u_3 & J_2^{-1}\, u_2 \\ J_3^{-1}\, u_3 & 0 & -J_1^{-1}\, u_1 \\ -J_2^{-1}\, u_2 & J_1^{-1}\, u_1 & 0 \end{bmatrix} \ \text{..(17)}$$

We know that,

$$E_\omega = R\widehat{\omega} - g_i I_R + g_p E_R$$

$$\therefore \dot{E}_\omega = \dot{R}\widehat{\omega} + R\dot{\widehat{\omega}} - g_i E_R + g_p \dot{E}_R$$

$$\therefore \dot{E}_\omega = R\widehat{\omega}^2 + |\ J^{-1}|R[J\omega\omega^T J^2 - J^2\omega\omega^T J] - g_i E_R + g_p \dot{E}_R + R(J^{-1}u)^\wedge + R(J^{-1}\Delta)^\wedge \ \text{(Using (2) and (16))}$$

$$(J^{-1}u)^\wedge = \widehat{\omega}^2 + |\ J^{-1}|[J\omega\omega^T J^2 - J^2\omega\omega^T J] - \frac{g_i E_R}{R} + \frac{g_p \dot{E}_R}{R} + (J^{-1}\Delta)^\wedge - \frac{\dot{E}_\omega}{R} \ \text{.........................(18)}$$

The control is then divided into 2 parts,

$$(J^{-1}u)^\wedge = (J^{-1}u)_1^\wedge + (J^{-1}u)_2^\wedge$$

Let us choose $(J^{-1}u)_1^\wedge$ in such as way that it will cancel the unwanted non-linear term in equation 18

$$(J^{-1}u)_1^\wedge = |\ J^{-1}|[J^2\omega\omega^T J - J\omega\omega^T J^2] - \widehat{\omega}^2 \ \text{...(19)}$$

For convenience, define F as

$$F = R\,(J^{-1}u)_2^\wedge \ \text{..(20)}$$

For simplicity,

$$F = -(k_i I_R + k_p E_R + k_\omega E_\omega) \ \text{...(21)}$$

Where k_i,k_p and k_w belong to the second set of gains for the design.

$$(J^{-1}u)_2^\wedge = R^T F = -R^T(k_i I_R + k_p E_R + k_\omega E_\omega) \ \text{..(22)}$$

$$(J^{-1}u)^\wedge = |\ J^{-1}|[J^2\omega\omega^T J - J\omega\omega^T J^2] - \widehat{\omega}^2 - R^T(k_i I_R + k_p E_R + k_\omega E_\omega) \ \text{..................................(23)}$$

Now, that the control relies on values of attitude and angular velocity that are measurement based which leaves room for sensor error and noises. All of this can be incorporated into *Δ* which stands for disturbances and lumped uncertainties.

Now, the error dynamics looks like,

$$\dot{I}_R = E_R$$

$$\dot{E}_R = g_i I_R - g_p E_R + E_\omega$$

$$\dot{E}_\omega = (g_i g_p - k_i)\, I_R - (g_i + g_p^2 + k_p)\, E_R + (g_p - k_\omega)\, E_\omega + R(J^{-1}\Delta)^\wedge$$

All these three equations are a system of first order linear differential matrix equation with constant coefficients. So, if we treat them entry-wise, we get a 3-dimensional time-invariant(LTI) system

$$\dot{e} = Ae + Bf + Bd$$

$$e = [\, e_I \;\; e_R \; e_\omega]^T \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

$\in \mathrm{R}^3$, $d \in \mathrm{R}$, $f \in \mathrm{R}$

where e_I is the corresponding entry of the error matrix I_R

e_R is the corresponding entry of the error matrix E_R

e_ω is the corresponding entry of the error matrix E_ω

Scalar control f is the corresponding entry of F to stabilize the LTI system

d is the corresponding entry of the limped disturbances and/ or uncertainties.

Now, the open loop matrices are,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ g_i & -\, g_p & 1 \\ g_i g_p & -\, g_i -\; g_p^2 & g_p \end{bmatrix} \quad , B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

$$F = -Ke$$

$$K = [k_i \; k_p \; k_\omega] \in \mathrm{R}^{1\times 3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

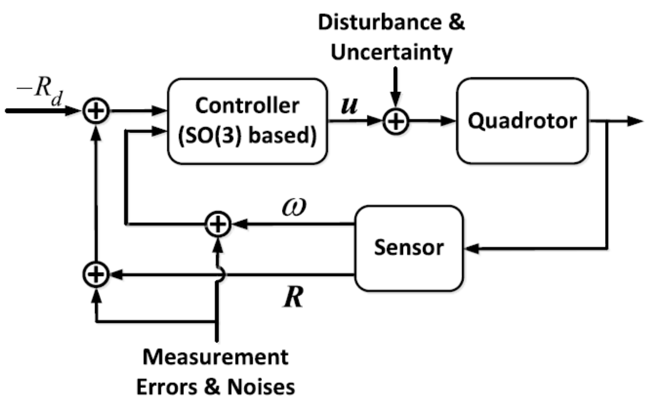


Figure 3. A schematic diagram of the proposed control system

(source: 10.1109/ACCESS.2021.3074381)

V. THE GAIN DESIGN OF THE NOMINAL SYSTEM:

Up to this point, the choice of the stabilizing state feedback gain K has not been addressed yet. The details are provided in the present section and next sections.

There are plentiful standard designs that are readily applicable to determine the gain of the nominal system such as classical LQR and pole placement methods. For easy exposition, only the classical LQR design and a specialized one are presented.

A. STANDARD LQR DESIGN

- Linear Quadratic Regulator (LQR):

For the continuous-time linear system

$$\dot{X}(t) = A(t)X(t) + B(t)U(t)$$

If the pair (A, B) is stabilizable, then we can look for the state feedback gain K that will minimize the following cost function:

$$J(K, X(0)) = \int_0^\infty (X^T(t)QX(t) + U^T(t)RU(t))dt$$

Q and R are the symmetric and positive definite matrices. Q is the error weighted matrix and R is the control weighted matrix. These matrices are guaranteed to stabilize the close loop system and provide a way to trade-off speed of response with control error.

The feedback control laws that minimize the value of the cost is:

$$U(t) = -K X(t)$$

The optimal feedback gain can be given by the following formula:

$$K = R^{-1} B^T P$$

Where P is the symmetric positive definite solution of the following continuous-time algebraic Riccati equation:

$$A^T P + P A - P B R^{-1} B^T P = -Q$$

Now the cost function

$$J(K, X(0)) = \int_0^{\infty} X^T(t)\ Q\ X(t) + U(t)\ R\ U(t)$$

Substitute U(t) = – K X(t) in the above equation

$$\therefore \quad J(K, X(0)) = \int_0^{\infty} (X^T(t) Q X(t) + (-\ K\,X(t))^T R\ (-K\,X(t)))\ dt$$

$$= \int_0^{\infty} (X^T(t) Q X(t) + X^T(t) K^T R\ K\ X(t))\ dt$$

$$= \int_0^{\infty} (X^T(t)\ (Q + K^T R\ K)\ X(t))\ dt$$

Now,

$$\dot{X}(t) = A(t)X(t) + B(t)U(t)$$

$$\therefore \dot{X}(t) = A(t)X(t) + B(t)\ (-K\,X(t))$$

$$\therefore \dot{X}(t) = (A(t) - B(t)\ K)\ X(t)$$

$$\therefore X(t) = e^{(A-BK)} X(0)$$

Substitute the above value into the J (K, X (0))

$$\therefore \quad J(K, X(0)) = \int_0^{\infty} (X^T(0) e^{(A-BK)^T t}\ (Q + K^T R\ K)\ e^{(A-BK)} X(0))\ dt$$

$$= \int_0^{\infty} X^T(0) (e^{(A-BK)^T t}\ (Q + K^T R\ K)\ e^{(A-BK)})\ dt\ X(0)$$

$$= X^T(0)\ P\ X(0)$$

Where P is the symmetric positive definite solution of

$$(A - BK)^T\ P + P\ (A - BK) = -\ (Q + K^T R K)$$

Using completion of the square techniques, we will re-write the equation as follow:

$$A^T P - K^T B^T P + PA - P\ B\ K = -\ Q - K^T\ R\ K$$

$$\therefore A^T P + PA = -\ Q - K^T\ R\ K + K^T B^T P + P\ B\ K$$

Put K = K = $R^{-1} B^T P$ into above equation

$$\therefore A^T P + PA = -\ K^T\ R\ K + K^T B^T P - Q + P\ B\ \ R^{-1} B^T P$$

Note that, $P^T B R^{-1} R\ K - \ P^T B {R^{-1}}^T R R^{-1} B^T P = 0$. So, we can subtract this term from the RHS of the above equation

$$\therefore A^T P + PA = -\ K^T\ R\ K + K^T R\ R^{-1} B^T P - P^T B R^{-1} R\ K + P^T B {R^{-1}}^T R R^{-1} B^T P - Q + P\ B\ \ R^{-1} B^T P$$

$$\therefore A^T P + PA = -\ (K^T) R (K - R^{-1} B^T P) - (P^T B {R^{-1}}^T) R (K - R^{-1} B^T P) - Q + P\ B\ \ R^{-1} B^T P$$

$$\therefore A^T P + PA = [K - R^{-1} B^T P]^T R [K - R^{-1} B^T P] - Q + P\ B\ \ R^{-1} B^T P$$

Since K = $R^{-1}B^T$P above equation become Riccati equation

$$\therefore A^T\text{P} + \text{PA} = [0]^T R[0] - \text{Q} + \text{P B} \;\; \text{R}^{-1} B^T\text{P}$$

$$\therefore A^T\text{P} + \text{PA} - \text{P B} \;\; \text{R}^{-1} B^T\text{P} = -\text{Q}$$

In this case for simplicity, we will take error weight matrix Q and control weight matrix R to be identity matrix

So, Linear Quadratic Regulator (LQR) design takes the form

$$\text{K} = [\, k_i \; k_p \; k_\omega \,] = B^T \text{ P} \qquad \text{.. (30)}$$

$$A^T\text{P} + \text{PA} - \text{P B} \;\; \text{R}^{-1} B^T\text{P} + \text{I} = 0, \qquad \text{P} = \text{P}^\text{T} > 0 \qquad \text{......................................(31)}$$

Here, the matrix P is symmetric and positive definite.

From equation (25), (26), (27) and (28), we can say that closed matrix A_C of the state space parameter can be given by below formula.

$$\text{A}_\text{C} = \begin{bmatrix} 0 & 1 & 0 \\ g_i & -g_p & 1 \\ g_i g_p - k_i & -g_i - g_p^2 - k_p & g_p - k_\omega \end{bmatrix} \qquad \text{..................................... (32)}$$

Remarks: The convergence speed of the proposed control algorithm can be controlled by the selection of the feedback gain K in terms of its numeric values. For example, smaller control weighting in the LQR design in general will give rise to a larger gain, which in turn will lead to a higher conversion speed. In other word, if the controlled weighted matrix R assumed small value meaning that penalty associated with control input is small. Hence, system will have a more control energy which will lead to higher conversion speed. However, since more control energy is put into the system, it inevitably costs more in some sense. As such, there is a trade-off between the two, apparently. LQR controller is a technique that will explicitly define a cost function (a quadratic sum of the state and the control) that is used as a measure to evaluate the overall performance (hence the so-called state cost and control cost)

B. A SPECIALIZED DESIGN:

Besides the standard LQR and pole placement schemes, an interesting gain design is presented below that feeds back E_ω only. Given below are two facts about the closed loop eigenvalues

$$\text{Trace}(A_C) = \sum_{i=1}^{3} \lambda_i (A_C) = -k_\omega \qquad \text{.............................. (33)}$$

$$\text{Det}(A_C) = \prod_{i=1}^{3} \lambda_i (A_C) = -g_i k_\omega - k_i \qquad \text{............................... (34)}$$

Apparently, the following expression must hold

$$0 < k_\omega, \quad g_i k_\omega < k_\omega \qquad \text{..................................... (35)}$$

If we choose $k_i = 0$, $k_p = 0$ and $g_i < 0$, we obtain the following

$$K = [0 \; 0 \; k_\omega], \qquad \text{..................................... (36)}$$

$$\widehat{V_2} = R^T F = -k_\omega \; R^T E_\omega \qquad \text{..................................... (37)}$$

The number of gains used by $\widehat{V_2}$ in this specialized scheme is reduced from three to one. The same can be verified by substituting $k_i = 0$ and $k_p = 0$ in the equation (32)

$$A_C = \begin{bmatrix} 0 & 1 & 0 \\ g_i & -g_p & 1 \\ g_i g_p & -g_i - g_p^2 & g_p - k_\omega \end{bmatrix} \qquad \text{..................................... (38)}$$

An illustrative example will be given in the later section that employs this specialized design.

VI. Classical Robust Control Design

The controller designed in the section V works well for the nominal system meaning that by choosing the appropriate feedback gain K, controller will ensure stability of the system in absence of external disturbance or noise. However, in the presence of the external disturbance and noise, stability of the system may not be guaranteed. In this section we will discuss about the H_∞ control which is robust control technique. This controller will ensure the stability of the system in presence of noise and external disturbance as well as the uncertainties.

- H_∞ norms for the system:

$$\|G(s)\|_\infty = \sup_\omega \bar{\sigma}[G(j\omega)]$$

Interpretation:
$\|G(s)\|_\infty$ is the "energy gain" from the input u to output y

$$\|G(s)\|_\infty = \max_{u(t)\neq 0} \frac{\int_0^\infty y^T(t)y(t)dt}{\int_0^\infty u^T(t)u(t)dt}$$

Achieve this maximum gain using a worst-case input signal that is essentially a sinusoid at frequency ω^* with input direction that yields $\bar{\sigma}[G(j\omega^*)]$ as the amplification.

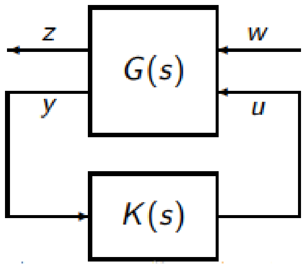
H_∞ is concerned primarily with the peaks in the frequency response.

Reference to H_∞ control is that we would like to design a stabilizing controller that ensures that the peaks in the transfer function matrix of interest are knocked down.

e.g., we want $\max_\omega \bar{\sigma}[G(j\omega)] \equiv \|G(s)\|_\infty < 0.70$

- **H_∞ control:**

Consider the system describe by:



$$G(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}$$

Optimal H_∞ control: Find all the admissible controller K(s) such that $\|T_{ZW}\|_\infty$ is minimized.
Suboptimal H_∞ control: Given γ > 0, find all admissible controller K(s), if there are any such that $\|T_{ZW}\|_\infty$ < γ.

- ## $\boldsymbol{H_\infty}$ controller:

From equation (27), error dynamics can be represented as below 3-dimensional linear time-invariant (LTI) system

$$\dot{e} = Ae + Bf + Bd,$$

Feedback law with H_∞ controller can be given by below formula.

$$F = -Ke, \quad K = B^TP, \ \eta = \begin{bmatrix} e \\ f \end{bmatrix} \qquad \text{.........................(39)}$$

Where P is the solution of the below Riccati equation.

$$A^TP + PA - (1 - \gamma^{-2})\, PBB^TP + I = 0 \qquad \text{.....................(40)}$$

$$A_C = A - BK, \quad A_{C\gamma} = A - (1 - \gamma^{-2})\, BK \qquad \text{.....................(41)}$$

Here η stands for performance output at our disposal, and γ refers to prescribed disturbance attenuation level, and both A_C and $A_{C\gamma}$ are asymptotically stable

Here, if feedback gain K in (39) is employed then the transfer function from disturbance to the performance output η is guaranteed to fall below the prescribed value γ. In other words, following holds.

$$\|G_{\eta d}(s)\|_\infty < \gamma \qquad \text{.............................(42)}$$

To hold the above condition, the disturbance attenuations level must be chosen in such a way that the Hamiltonian matrix H has no imaginary axis eigen-value. In this case the Hamiltonian matrix takes the below form:

$$\begin{bmatrix} A - BK & \gamma^{-2}BB^T \\ \begin{bmatrix} I \\ -K \end{bmatrix} & -(A - BK)^T \end{bmatrix} \qquad \text{...........................(43)}$$

If γ is too small, the Hamilton matrix has a eigenvalue on the imaginary axis and hence the condition mentioned in the equation (42) is not satisfied. In this case, the system cannot be stabilised.

Equation (40) can be rewritten as below:

$$A_{c\gamma}^T P + PA_\gamma + \gamma^{-2}\, PBB^TP + I - PBB^TP = 0 \qquad \text{.......................(44)}$$

To find the value of P, the above Riccati equation can be solved using the heuristic iteration algorithm as follow. Here j subscript for "j" stands for index of iteration

$$A_{C\gamma j}^T P_{j+1} + P_{j+1}A_{C\gamma j} + \gamma^{-2}P_{j+1}BB^TP_{j+1} + I - P_jBB^TP_j = 0 \qquad \text{....................(45)}$$

Remarks: There are many other schemes like sliding-mode controls, disturbance-tailored super twisting algorithm, multivariable super-twisting sliding mode approach etc... Ideally, these control algorithms should perform better than the H_∞ design presented above, but the trade-off apparently is that they are computationally more demanding/costly due to their structural complexities. Unlike the proposed static H_∞ design, those sliding-mode controller and the-like are dynamics, hence are more involved from the implementation point of view.

VII. STABILITY OF CLOSED LOOP SYSTEM

We will analyze the stability of the system in this section. After entry wise treatment of the error dynamics, we were able to obtain a 3-dimensional LTI system in which matrix K may be any stabilizing state feedback gain. Error dynamics can be rewritten by substituting the value of f=−Ke from the equation (29) into equation (27)

$$\dot{e} = A_c e + Bd \qquad \text{Where } A_c = A - BK \quad \ldots\ldots (46)$$

A. NOMINAL SYSTEM

Nominal system: It is the type of system in which uncertainty, disturbance and noise are not present. Hence lumped disturbance d is assumed to be zero

With the proper choice of stabilizing gain K, real part of all the eigen value of A_c will become negative and residing in the open left complex plane. Hence, the system is asymptomatically stable and the state of such a nominal system will decay to zero exponentially. To see that construct a quadratic Lyapunov function for equation (46) as follow

$$V = e^T Y e, \quad Y = Y^T > 0$$

Where, Y is the constant symmetric positive definite matrix and it satisfied the following Lyapunov equation (Take Q as a identity matrix

$$A_C^T Y + Y A_C + Q = 0$$

$$A_C^T Y + Y A_C + I = 0 \qquad \ldots\ldots (47)$$

Note that the since the Y is a positive definite matrix, eigenvalues of Y are real and positive (if eigenvalue of Y is negative or zero than eTYe is either less than zero or equal to zero which contradict the definition of positive definite matrix), hence V is always positive when the error is non-zero. Consider the time derivative of the Lyapunov function

Now,

$$V = e^T Y e$$

Take the derivative with respect to time,

$$\dot{V} = \dot{e}^T Y e + e^T Y \dot{e}$$

Substitute the value of $\dot{e}$ from equation (46) taking d = 0

$$\therefore \dot{V} = (A_c e)^T Y e + e^T Y (A_c e)$$

$$\therefore \dot{V} = e^T A_C^T Y e + e^T Y A_c e$$

$$\therefore \dot{V} = e^T (A_C^T Y + Y A_c) e = -\|e\|^2 < 0, \ \ \forall\, e \neq 0$$

$$(as\ A_C^T Y + Y A_c = -I)$$

$$\dot{V} = e^T (A_C^T Y + Y A_C) e = -\|e\|^2 < 0, \qquad \forall\, e \neq 0 \qquad \ldots\ldots (48)$$

The Lyapunov function satisfies the following inequalities

$$\|e\|^2\underline{\lambda}\,(Y) \leq V \;\leq\; \bar{\lambda}(Y)\|e\|^2 \qquad \text{.................................}(49)$$

Where $\underline{\lambda}\,(Y)$ and $\bar{\lambda}(Y)$ denote the minimum eigenvalue and maximum eigen value of Y, respectively.

Now,

$$V \;\leq\; \bar{\lambda}(Y)\|e\|^2$$

$$\therefore -\mathrm{V} \geq -\bar{\lambda}(Y)\|e\|^2$$

$$\therefore -\;\|e\|^2 \leq\; -\frac{1}{\bar{\lambda}(Y)}\mathrm{V}$$ ($\bar{\lambda}(Y)$ is the eigen value of Y and it is always positive)

From equation (48)

$$\dot{V} \leq -\frac{1}{\bar{\lambda}(Y)}\mathrm{V} \qquad \text{....................................}(50)$$

Hence, as the derivative of Lyapunov function is less zero (as in equation (50), V and $\bar{\lambda}(Y)$ is always positive), it can be concluded that the error dynamics of the nominal system is exponentially stable when the lumped disturbance and uncertainty d is assumed to be negligible.

B. B. NON-NOMINAL SYSTEM

Non- Nominal system: It is the type of system in which uncertainty, disturbance and noise are present. Hence lumped disturbance d is not equal to zero.

In practise, external disturbance, system uncertainty, and sensor measurement errors and/or noise may not be negligible. Assume the lumped disturbance/uncertainty is bounded for all the time and satisfies

$$\|d\| \leq\; \bar{d}, \qquad \text{.................................}(51)$$

Where $\bar{d}$ is a constant representing the upper bound of the lumped disturbance/uncertainty. Error dynamics can be rewritten by substituting the value of f=−Ke from the equation (29) into equation (27)

$$e =\; \dot{A_c} + Bd \qquad \text{Where } A_c = A - BK$$

With the proper choice of stabilizing gain K, real part of all the eigen value of A_c will become negative and residing in the open left complex plane. Hence, the system is asymptomatically stable and the state of such a nominal system will decay to zero exponentially. To see that construct a quadratic Lyapunov function for equation (46) as follow

$$\mathrm{V} = \mathrm{e^T Y e}, \quad \mathrm{Y} = \mathrm{Y^T} > 0$$

Where, Y is the constant symmetric positive definite matrix, and it satisfied the following Lyapunov equation (Take Q as a identity matrix)

$$A_C^T Y + Y A_C + Q = 0$$

$$A_C^T Y + Y A_C + I = 0$$

From the above derivation (when d = 0) of Lyapunov derivative, we can write the derivative of Lyapunov function when d ≠ 0 as below:

$$\dot{V} = e^T (A_C^T Y + Y A_c) e + 2 e^T Y B d$$

$$\leq -\|e\|(\|e\| - 2\bar{d}\|Y\|) \qquad \text{..................................(52)}$$

So, if $\|e\| \leq 2\bar{d}\|Y\|$, then $\dot{V} \leq 0$.

It can be concluded that if after certain period, the magnitude of the error $\|e\|$ remains inside a ball with its radius equal to $2\bar{d}\|Y\|$ the stability can be guaranteed.

Apparently, the closed loop stability no longer belongs to the exponential type when the lumped disturbance/uncertainty is not negligible.

Hence, unlike section VI where induced norms of the transfer function from the disturbance to the performance output is predetermined, the degree of robustness of the two-controller presented in the section V for the nominal system cannot be specified in advanced and cannot be guaranteed.

VIII. Validation and Performance Assessment

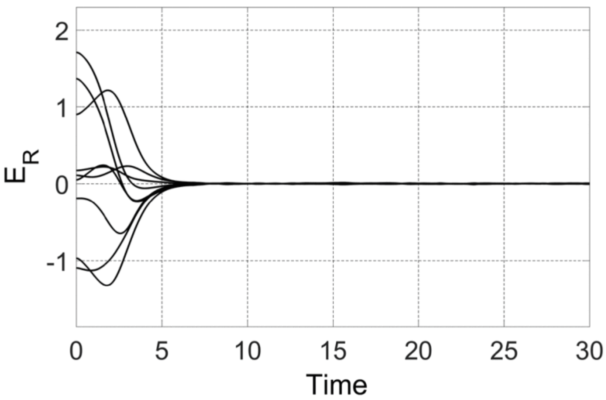


Figure 4. The trajectory of E_R with E_ω feedback only
(source: 10.1109/ACCESS.2021.3074381)

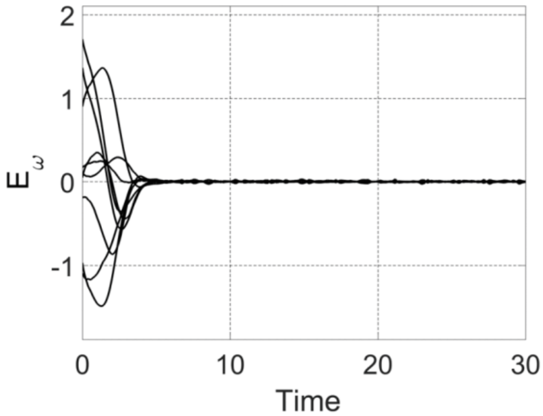


Figure 6. The trajectory of E_ω with E_ω feedback only
(source: 10.1109/ACCESS.2021.3074381)

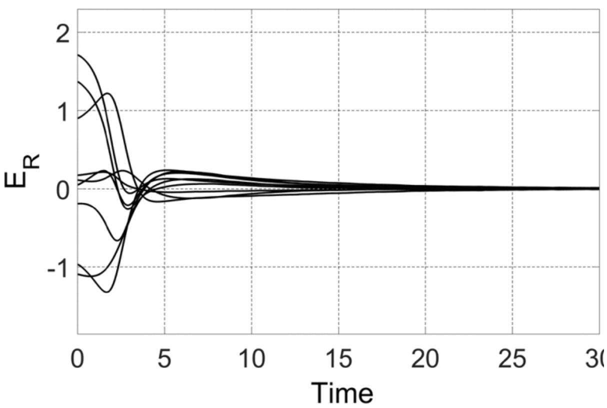


Figure 5. The trajectory of E_R with H_∞ design
(source: 10.1109/ACCESS.2021.3074381)

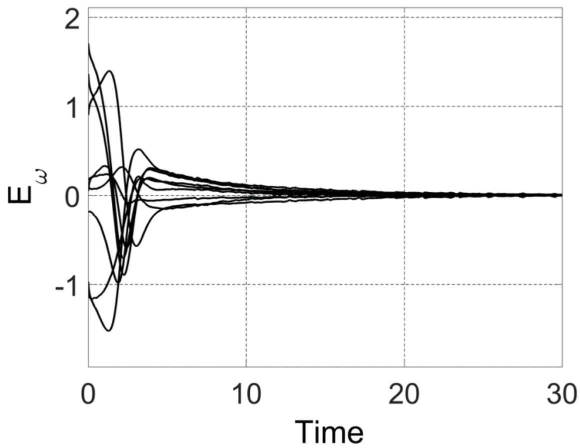


Figure 7. The trajectory of E_ω with H_∞ design
(source: 10.1109/ACCESS.2021.3074381)

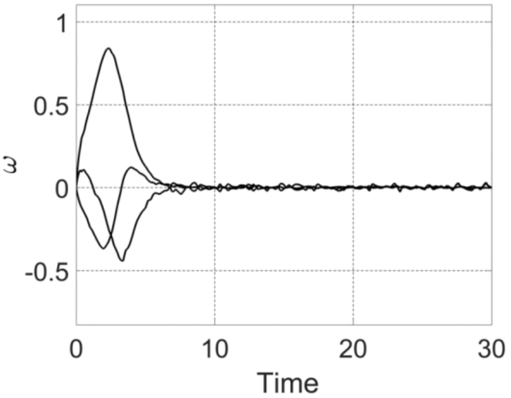


Figure 8. The trajectory of ω with E_ω feedback only

(source: 10.1109/ACCESS.2021.3074381)

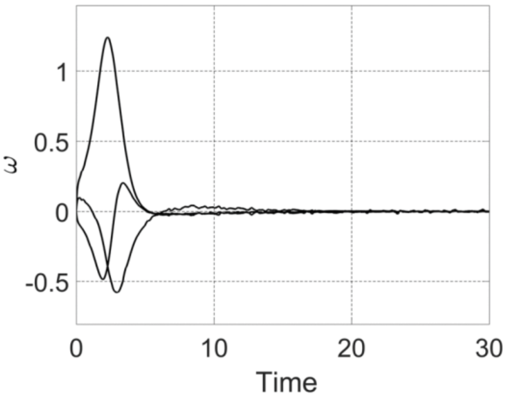


Figure 9. The trajectory of ω with H∞ design

(source: 10.1109/ACCESS.2021.3074381)

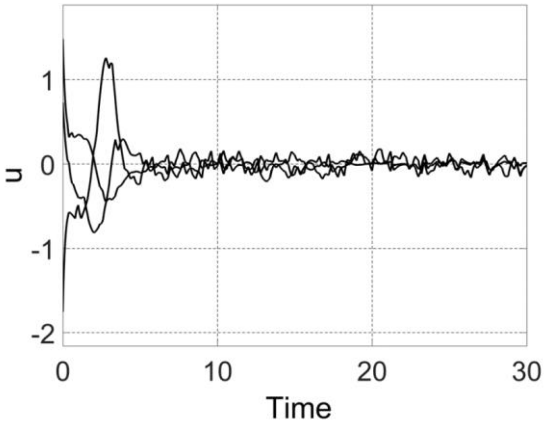


Figure 10. The trajectory of u with E_ω feedback only

(source: 10.1109/ACCESS.2021.3074381)

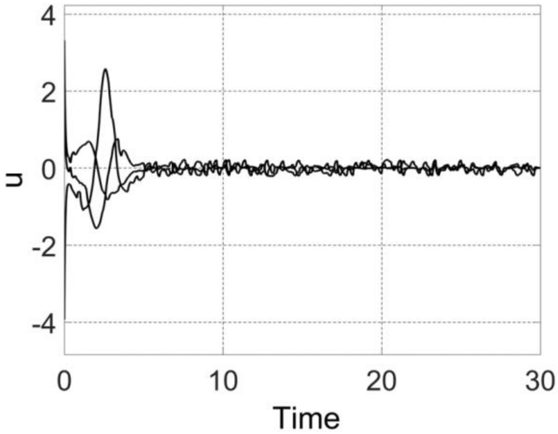


Figure 11. The trajectory of u with H∞ design

(source: 10.1109/ACCESS.2021.3074381)

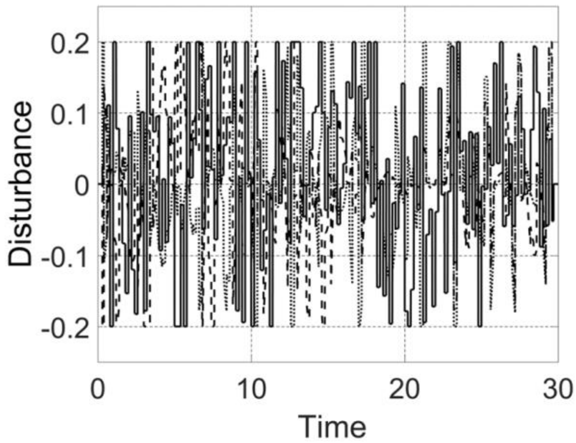


Figure 12. The trajectory of disturbances put into the system

(source: 10.1109/ACCESS.2021.3074381

Example:

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} \psi(0) \\ \theta(0) \\ \psi(0) \end{bmatrix} = \begin{bmatrix} -80^o \\ -50^o \\ 20^o \end{bmatrix}, \begin{bmatrix} \psi_d \\ \theta_d \\ \psi_d \end{bmatrix} = \begin{bmatrix} 100^o \\ 70^o \\ 150^o \end{bmatrix}$$

$$R(0) = \begin{bmatrix} 0.1116 & 0.8799 & -0.4618 \\ -0.6330 & 0.4212 & 0.6495 \\ 0.7660 & 0.2198 & 0.6040 \end{bmatrix}$$

$$R(0) = \begin{bmatrix} -0.0594 & 0.7713 & 0.6337 \\ 0.3668 & 0.6131 & -0.7146 \\ -0.9397 & 0.1710 & -0.2692 \end{bmatrix}$$

g_i = -0.0004, g_p =1,

Nominal design: k_ω= 5

Robust design : k_i = 3.2733, k_p= -4.6626, k_ω = 15.8696

We conducted the simulations with the above-mentioned data to generate trajectories of errors, angular velocities, controls, and the disturbance put into the system. However, the plots we received from our simulations were slightly different from the plots provided in the paper. Therefore, to provide accurate results, we have considered the plots provided in the paper.

Since the governing differential matrix error is treated entry-wise, we get 9 plots for E_R and E_ω for both the specialised designed gain feedback (Section V) and robust control (Section VI). Meanwhile, ω and u are vectors so there are 3 plots each for the specialised designed gain feedback (Section V) and the robust control(Section VI).

The major difference between both the proposed schemes lie in the second control component f. The specialized non robust scheme only uses E_ω, whereas the robust scheme uses all the error information available to it in the H_∞ sense. The trajectories for E_R in both the cases narrow down to 0 rapidly, but the scheme with H_∞takes a little longer. Similarly, for E_ω, the trajectories narrow down immediately, but takes longer for H_∞ design. Additionally, for ω, the robust scheme takes longer time to narrow down it to zero. Even though the response time for the H_∞ design is larger than the specialized feedback control, the robust control posses the ability to deal with the uncertainties or disturbances that may occur over time. This can be viewed in the plots for the control input. The trajectory for the control input in the graph for the robust design shows that it can handle the uncertainties or disturbances that occur over time better as compared to the other design.

The simulations clearly portray that both the proposed schemes work well when put under noticeable disturbances and initial attitude errors. All the undesired errors dropped to small values promptly. However, the non-robust design cannot guarantee stability as it does not have disturbance attenuation whereas the robust design does, which leads to the conclusion that robust design is a better approach.

IX. Conclusion

The attitude control problem of a quadrotor was studied and replicated. The speciality of this specific paper was that it used a unified SO(3) framework, the advantages of which were explicitly stated in the report. Virtual control is implemented, and the remaining error dynamics is transformed into a 3-dimensional LTI system by treating it entry-wise. This was much easier to solve as the classical methods to solves them are widely available. A feedback controller is then designed. Alongside, we also design a robust control based on H_∞ theory with capability of disturbance attenuation, to deal with the uncertainties and disturbances that may occur during the use of the system.

We have also used an example to validate the proposed designs. Simulations were conducted and plots for the trajectories of each type of errors, the control input and the disturbances are produced. The results of this proves the legitimacy of the proposed designs.

However, there is still a lot of work to be done behind this scheme to make it work in real life. There needs to be a testbed setup to experimentally validate the approach. The proposed schemes need to be further extended to enhance the disturbance rejection capabilities and to achieve finite-time convergence.

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