Finsler notes

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1 Basic Background

1.1 Definition of Finsler Manifolds

Let V be a real finite dimensional vector space, and let $\{e_i\}$ be a basis for V. Furthermore, let $\frac{\partial}{\partial y^i}$ be partial differentiation in the e_i direction. Here we use *Einstein summing convention* throughout. For example, for $v \in V$, $v = v^i e_i$ [3].

Definition 1 ([3]) *A function $f : V \to \mathbb{R}$ is (positively)* ***Homogeneous of degree*** *$s \in \mathbb{R}$ (or* ***s-homogeneous****) if $f(\lambda v) = \lambda^s f(v)$ for all $v \in V$, $\lambda > 0$.*

Proposition 1 ([3]) *Suppose f is smooth and s-homogeneous. Then ∂f is $(s-1)$-homogeneous, and*

$$\frac{\partial f}{\partial y^i}(v)v^i = sf(v), \ v \in V \quad \textit{(Euler's theorem).}$$

Definition 2 ([1]) *A (global defined)* ***Finsler structure*** *of M is a function $F : TM \to [0, \infty)$ with the following properties:*

(i) ***Regularity*** *F is C^∞ on $TM \setminus \{0\}$,*

(ii) ***Positive homogeneity*** *$F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$, i.e. F is 1-homogeneous of y.*

(iii) ***strong convexity*** *The $n \times n$ Hessian matrix*

$$(g_{ij}) := ([\frac{1}{2}F^2]_{y^i y^j})$$

is positive-definite at every point of $TM \setminus \{0\}$.

The pair (M, F) is a **Finsler manifold**.

Theorem 1 ([1]) *Let F be a non-negative real-valued function on $\mathbb{R}^n$ with the properties:*

- *F is C^∞ on the punctured space $\mathbb{R}^n \setminus 0$.*
- *$F(\lambda y) = \lambda F(y)$ for all $\lambda > 0$*
- *The $n \times n$ matrix (g_{ij}), where $g_{ij}(y) := [\frac{1}{2}F^2]_{y^i y^j}(y)$, is positive definite at all $y \neq 0$.*

Then we have the following conclusions:

- *(Positivity)* $F(y) > 0$ *where* $y \neq 0$.
- *(Triangle inequality)* $F(y_1 + y_2) \leq F(y_1) + F(y_2)$, *where equality holds if and only if* $y_2 = \alpha y_1$ *or* $y_1 = \alpha y_2$ *for some* $\alpha \geq 0$.
- *(Fundamental inequality)* $w^i F_{y^i}(y) \leq F(w)$ *at all* $y \neq 0$, *and equality holds if and only if* $w = \alpha y$ *for some* $\alpha \geq 0$.

Remark 1

(i) [1] The hypotheses of the above theorem define a ***Minkowski norm*** *on* $\mathbf{R}^n$. *According to this theorem, there is no need to hypothesize that* F *be positive at* $y \neq 0$; *it is necessary so.*

(ii) [1] If the Minskowski norm satisfies $F(-y) = F(y)$, *then one has the absolute homogeneity* $F(\lambda y) = |\lambda| F(y)$.

(iii) [1] In view of the first two conclusions of the above theorem, every absolutely homogeneous Minkowski norm is a norm in the sense of functional analysis.

(iv) [3] For $u, v \in T_xM$ *and* $y \in T_xM \setminus \{0\}$, *we have* $g_y(u,v) = g_{ij}(y)u^i v^j$ *where*

$$g_{ij}(y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(y).$$

Hence g_y *is bilinear.*

(v) [6] for all $u, v \in T_xM$ *and* $y \in T_xM \setminus \{0\}$, *we have* $g_y(u,v) = \frac{1}{2}\frac{\partial^2 F^2(y+su+tv)}{\partial s \partial t}\Big|_{t=s=0}$

(vi) [3] Suppose $u, v \in T_xM$, $y \in T_xM \setminus \{0\}$. *Then*

$$\begin{aligned} g_{\lambda y}(u,v) &= g_y(u,v),\ \lambda > 0 \\ g_y(y,u) &= \frac{1}{2}\frac{\partial F^2}{\partial y^i}(y)u^i = \frac{1}{2}\frac{\partial F^2(y+tu)}{\partial t}\Big|_{t=0}, \\ g_y(y,y) &= F^2(y). \end{aligned}$$

(vii) [3] $F(x,y) = 0$ *if and only if* $y = 0$.

1.2 Another definition of Finsler Manifolds

Definition 3 ([5]) *A* C_p ***Banach Manifold*** $\mathcal{M}$ *for* $p \in \mathbb{N} \cup \{\infty\}$ *is an Hausdorff topological space together with a covering by open sets* $(U_i)_{i\in I}$, *a family of Banach vector spaces* $(E_i)_{i\in I}$ *and a family of continuous mappings* $(\varphi_i)_{i\in I}$ *from* U_i *inton* E_i *such that*
(i) for every $i \in I$, φ_i:

$$U_i \longrightarrow \varphi_i(U_i) \text{ is an homeomorphism.}$$

(ii) for every pair of indices $i \neq j$ *in* I,

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \subset E_i \longrightarrow \varphi_j(U_i \cap U_j) \subset E_j$$

is a C^p *diffeomorphism.*

Definition 4 ([5]) *A Banach manifold $\mathcal{V}$ is called C^p-**Banach Space Bundle** over another Banach manifold $\mathcal{M}$ if there exists a Banach space E, a submersion π from $\mathcal{V}$ to $\mathcal{M}$, a covering $(U_i)_{i\in I}$ of $\mathcal{M}$ and a family of homeomorphism from $\pi^{-1}U_i$ into $U_i \times E$ such that the following diagram commutes*

$$\begin{tikzcd} \pi^{-1}U_i \arrow[r, "\tau_i"] \arrow[dr, "\pi"'] & U_i \times E \arrow[d, "\rho"] \\ & U_i \end{tikzcd}$$

where ρ is the canonical projections from $U_i \times E$ onto U_i.
The restriction of τ on each fiber $\mathcal{V}_x := \pi^{-1}(\{x\})$ for $x \in U_i$ realizes a continuous isomorphism onto E_i. Moreover, the map $x \in U_i \cup U_j \to \tau_i \circ \tau_j^{-1}|_{\pi^{-1}(x)} \in \mathcal{L}(E, E)$ is C^p.

A submersion is a differentiable map between differentiable manifolds whose differential is everywhere surjective.

$\mathcal{L}(X, Y)$ is the space of all continuous linear maps from a topological vector X to another topological space Y.

Definition 5 ([5]) *Let $\mathcal{M}$ be a normal Banach manifold and let $\mathcal{V}$ be a Banach Space Bundle over M. A **Finsler structure** on $\mathcal{V}$ is a continuous function $\|\cdot\|_x := \|\cdot\||_{\pi^{-1}(x)}$ is a norm on $\mathcal{V}_x$.*

Definition 6 ([5]) *Let $\mathcal{M}$ be a normal C^p Banach manifold. $T\mathcal{M}$ equipped with a Finsler structure is called a **Finsler Manifold**.*

Remark: A Finsler structure on $T\mathcal{M}$ defines in a canonical way a dual Finsler structure on $T^*\mathcal{M}$.

1.3 Relative definitions

Example 1 ([3]) *Let (M, g) be a Riemannian manifold. Set $F|_{T_xM}(y) = \sqrt{g_x(y, y)}$. Then F is a Finsler norm on M.*

Suppose M is a manifold. Then a **curve** is a smooth mapping $c : (a, b) \to M$ such that $(Dc)_t \neq 0$ for all t [3]. Such a curve has a **canonical lift** $\hat{c} : (a, b) \to TM \setminus \{0\}$ defined as $\hat{c}(t) = (Dc)(t)$, where Dc is the tangent of c. If (M, F) is a Finsler manifold, we define the **length** of c as

$$L(c) = \int_a^b F(\hat{c}(t))dt.$$

Definition 7 ([6]) *Let U be an open subset of a Finsler manifold (M, F). Let νU be the space of smooth vector fields on U and $\nu U^+ \subset \nu U$ be the subset of nowhere vanishing vector fields. For $V \in \nu U^+$ and for all $X, Y, Z \in \nu U$ define a trilinear form $< \cdot, \cdot, \cdot >_V$ by*

$$< X, Y, Z >_V = \frac{1}{4} \frac{\partial^3}{\partial r \partial s \partial t} F^2(V + rX + sY + tZ)|_{r=s=t=0},$$

*called the **Cartan tensor**.*

The Cartan tensor is a non-Riemannian quantity. A Finsler metric reduces to a Riemannian metric if and only if its Cartan tensor vanishes [6].

Definition 8 ([6]) *An* ***affine connection*** ∇^V *is a map*

$$\begin{aligned} \nabla^V : \nu U \times \nu U &\to \nu U \\ (X, Y) &\to \nabla^V_X Y \end{aligned}$$

(i) linear in Y *(not necessary linear in* X*)*

(ii) $\nabla^V_X(fY) = f\nabla^V_X Y + X(f)U$

(iii) $\nabla^V_{fX} Y = f\nabla^V_X Y$

for all $f \in C^\infty(U)$ *and* $X, Y \in \nu U$*.*

Definition 9 ([3]) *suppose* $c : (a, b) \to M$ *is a curve. Then a* ***variation*** *of* c *is a continuous mapping* $H : [a, b] \times (-\epsilon, \epsilon) \to M$ *for some* ϵ *such that*

(i) H *is smooth on* $(-\epsilon, \epsilon) \times (a, b)$ *and with notation* $c_s(\cdot) = H(\cdot, s)$

(ii) $c_0(t) = c(t)$*, for all* $t \in [a, b]$

(iii) $c_s(a), c_s(b) \in M$ *are constants not depending on* $s \in (-\epsilon, \epsilon)$*.*

Definition 10 ([3]) *A curve* c *in a Finsler manifold is a* ***geodesic*** *if* L *is stationary at* x*, that is, for any variation* (c_s) *of* c*,*

$$\frac{d}{ds} L(c_s)\bigg|_{s=0} = 0.$$

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