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# AN INTRODUCTION TO FINSLER GEOMETRY

Xiaohuan Mo

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# FINSLER GEOMETRY

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Vol. 1: An Introduction to Finsler Geometry
*by Xiaohuan Mo (Peking University, China)*

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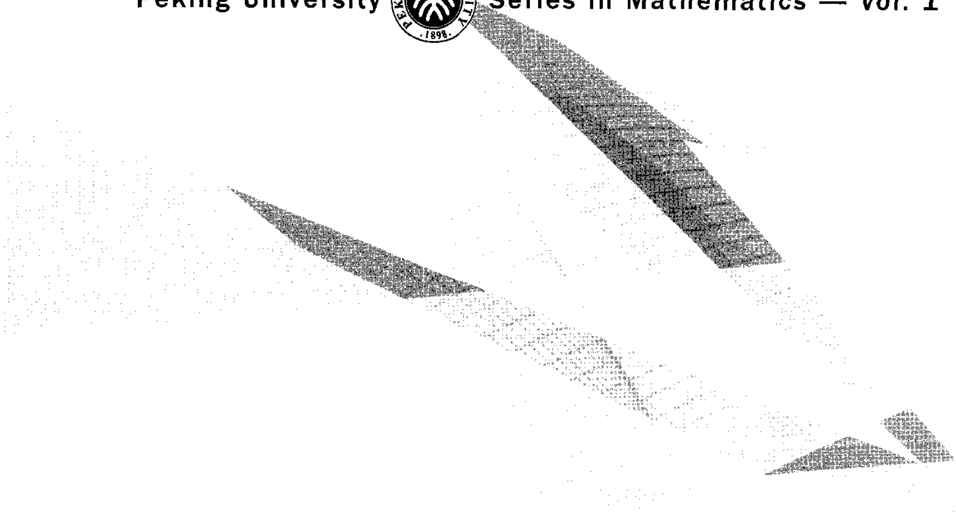

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# Preface

This book uses the moving frame as a tool and develops Finsler geometry on the basis of the discovery of Chern connection and projective sphere bundle. It systematically introduces three classes of geometrical invariants on Finsler manifolds and their intrinsic relation, analyses local and global results from classic and modern Finsler geometry, and gives non-trivial examples of Finsler manifolds satisfying kinds of curvature conditions.

Most of the geometric invariants and fundamental equations in this book lie on the projective sphere bundle, that is, they are positively homogenous degree zero with respect to the tangent vectors. The content of this book is brief and to the point. We reveal the studying method of modern Fisher geometry by introducing the theory of harmonic maps of a Fisher space and the geometry of a projective sphere bundle. This book is easy to understand.

The author has given a one-semester course Finsler Geometry three times to graduate students of mathematics of Peking University. The book is based on the teaching of these courses. S. S. Chern, a renowned leader in differential geometry, provided me with generous support and guidance for my development of this manuscript and preparation for my teaching. I would like to take this opportunity to express my sincere gratitude to him. I am also grateful to C. Yang and L. Huang for generous help. I would like to express my deep gratitude to Z. Shen numerous suggestions which led to significant improvements of both the style and content of the book.

*X. H. Mo*

# Contents

# Chapter 1

# Finsler Manifolds

Finsler geometry is just Riemannian geometry without the quadratic restriction. It also asserts itself in applications, most notably in theorey of relativity, control theory and mathematical biology. In this chapter, we will introduce Finsler manifolds and discuss the relation between Finsler manifolds and Riemannian manifolds. Also, we will give other useful examples of Finsler manifolds[AIM, 1993].

## 1.1 Historical remarks

The geometry based on the element of arc

$$ds = F(x^1, \cdots, x^n; dx^1, \cdots, dx^n) \tag{1.1}$$

where $F$ is positively homogeneous of degree 1 in $dx^i$ is now called Riemannian-Finsler geometry (Finsler geometry for short). Roughly speaking, $F$ is a collection of Minkowskian norms $F_x$ in the tangent space at $x$ such that $F_x$ varies smoothly in $x$. In fact, metric (1.1) was introduced by Riemann in his famous 1854 Habilitationsvortag "Uber die Hypotheser welche der geometric zugrund liegen". He paid particular attention to a metric defined by the positive square root of a positive definite quadratic differential form, i.e.

$$F^2(x, dx) = g_{ij}(x)dx^i dx^j$$

In the famous Paris address of 1900, Hillbert devoted the last problem to variational calculous of $\int ds$ and its geometrical overtone. A few years later, the general development took a curious turn away from the basic aspects and methods of the theory as developed by P. Finsler. The latter did not make use of the tensor calculous, being guided in principle by the notions of the calculous of variations. Finsler's thesis, which treats curves and surfaces of (1.1), must be regarded as the first step in this direction. The name "Finsler geometry" came from his thesis in 1918.

## 1.2 Finsler manifolds

Let $M$ be an $n$-dimensional smooth manifold. Denote by $T_xM$ the tangent space at $x \in M$, and by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of $M$. Each element of $TM$ has the form $(x, y)$, where $x \in M$ and $y \in T_xM$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$.

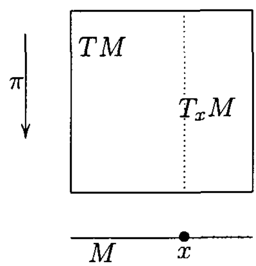


Fig. 1.1

Let $(\varphi, x^i)$ be a local coordinate system on an subset $U \subset M$. i.e.

$$\forall x \in U, \quad \varphi(x) = (x^1, \cdots, x^n).$$

Taking a curve on $U$

$$\gamma_i(t) = \varphi^{-1}(x^1, \cdots, x^{i-1}, x^i + t, x^{i+1}, \cdots, x^n),$$

then $\frac{\partial}{\partial x^i}|_x = \gamma_i'(0)$ are the induced coordinate bases for $T_xM$. We said $x^i$ gives rise to local coordinates $(x^i, y^i)$ on $\pi^{-1}U \subset TM$ through the mechanism

$$y = y^i \frac{\partial}{\partial x^i}. \tag{1.2}$$

The $y^i$ are fibrewise global. Whenever possible, let us make no distinction between $(x, y)$ and its coordinate representation $(x^i, y^i)$. Function $H$ that defined on $TM$ can be locally expressed as

$$H(x^1, \cdots, x^n; y^1, \cdots, y^n).$$

We employed the following convention, namely, denoted by $H_{y^i}, H_{y^iy^j}, \cdots$, etc, the partial derivative(s) of $H$ with respect to the coordinate $y^i$. Adopt a similar notation for the partial derivative with respect to the coordinate $x^i$.

**Lemma 1.2.1** *Let $V$ be an open subset on $M$, such that $V \cap U \neq O$, Let $(\tilde{x}^i, \tilde{y}^i)$ be the local coordinate on $\pi^{-1}V \subset TM$. Then*

$$(i) \quad H_{\tilde{y}^j} = \frac{\partial x^i}{\partial \tilde{x}^j} H_{y^i}. \tag{1.3}$$

$$(ii) \quad H_{\tilde{x}^j} = \frac{\partial x^i}{\partial \tilde{x}^j} H_{x^i} + \tilde{y}^k \frac{\partial^2 x^i}{\partial \tilde{x}^j \partial \tilde{x}^k} H_{y^i}. \tag{1.4}$$

***Proof.*** By (1.2) we have

$$y^i = \frac{\partial x^i}{\partial \tilde{x}^j} \tilde{y}^j. \tag{1.5}$$

This implies that

$$\frac{\partial y^i}{\partial \tilde{y}^j} = \frac{\partial x^i}{\partial \tilde{x}^j}. \tag{1.6}$$

From (1.5) and (1.6) and the chain rule, we have (1.3) and (1.4). □

**Definition 1.2.2** A function $F : TM \to [0, \infty)$ is called a *Finsler structure*, if, in a local coordinate systerm $(x^i, y^i)$,

$$F(x, y) = F\left(y^i \frac{\partial}{\partial x^i}|_x\right)$$

satisfies

(i) $F(x, \lambda y) = \lambda F(x, y); \quad \lambda > 0.$
(ii) $F(x, y)$ is $C^\infty$ for $y \neq 0$.
(iii) $\frac{1}{2}(F^2)_{y^iy^j} \quad (y \neq 0)$ is positive definite.

A $C^\infty$ manifold $M$ with its Finsler structure $F$ is said a *Finsler manifold* or *Finsler space*.

## 1.3 Basic examples

(I) Riemannian manifold

A Riemannian metric $g$ on a manifold $M$ is a family of inner products $\{g_x\}_{x\in M}$ such that the quantities

$$g_{ij}(x) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

are smooth in local coordinates.

The Finsler structure $F(x,y)$ of a Riemannian manifold satisfies

$$F(x,y) = \sqrt{g_{ij}(x)y^i y^j}.$$

In that case, $\frac{1}{2}(F^2)_{y^i y^j}$ is simply $g_{ij}(x)$, which is independent of $y$.

Let $|\cdot|$ and $\langle,\rangle$ be the standard Euclidean norm and inner product in $R^n$. We have the following interesting Finsler metrics defined on $B^n$

$$F = \sqrt{\frac{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}{1-|x|^2}}, \tag{1.6}$$

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2}. \tag{1.7}$$

For metric (1.6), we have

$$\begin{aligned} g_{ij} =& \left(\frac{F^2}{2}\right)_{y^i y^j} \\ =& \frac{1}{2}\left[\frac{y^k y^l \delta_{kl}(1-|x|^2) + (\sum_k x^k y^k)^2}{1-|x|^2}\right]_{y^i y^j} \\ =& \frac{\delta_{ij}(1-|x|^2) + x^i x^j}{1-|x|^2} = g_{ij}(x). \end{aligned}$$

So metric (1.6) is a Riemannian metric. Similarly we can show that the metric defined in (1.7) is also Riemannian.

These Riemannian metrics have special properties. Riemannian metric in (1.6) has constant curvature 1, and this in (1.7) has constant curvature -1. The notion of curvature will be defined later. The metric in (1.7) is called the *Klein metric.*

(II) Minkowski manifold

**Definition 1.3.1** Let $(M, F)$ be a Finsler manifold, we say that $(M, F)$ is a *Locally Minkowski manifold* if for arbitrary chart $(U, x^i)$ of $M$, the fundamental tensor satisfies $g_{ij}(x, y) = g_{ij}(y)$. Furthermore, we call $(M, F)$ a *(globally) Minkowski manifold* if $M$ is a vector space.

**Example 1.3.2** Consider the case $n = 2$, so $y = (y^1, y^2)$. In order to avoid the excessive use of parenthese, we shall abbreviate $y^1, y^2$ as $p, q$ respectively. Define

$$F_{\lambda,k} = \sqrt{p^2 + q^2 + \lambda(p^{2k} + q^{2k})^{\frac{1}{k}}},$$

where $\lambda \in [0, \infty), k \in \{1, 2, \cdots\}$. The case $\lambda = 0$ is Riemannian (in fact, Euclidean)[Bao and Shen, 2002]. It can be checked that this $F$ is actually $C^\infty$ on $TM\backslash\{0\}$, calculations give

$$\frac{1}{2}\frac{\partial^2 F^2_{\lambda,k}}{\partial p^2} = 1 + \lambda\omega p^{2(k-1)}[p^{2k} + (2k-1)q^{2k}] > 0,$$

$$\frac{1}{2}\frac{\partial^2 F^2_{\lambda,k}}{\partial p \partial q} = 2\lambda(1-k)\omega(pq)^{2k-1},$$

where

$$\omega = (p^{2k} + q^{2k})^{\frac{1}{k}-2}.$$

Hence

$$\det\left(\frac{\partial^2 F^2_{\lambda,k}}{\partial y^i \partial y^j}\right) > 0. \tag{1.8}$$

One obtains a Minkowski structure.

(III) Randers manifold

**Definition 1.3.3** Let $\alpha = \sqrt{a_{ij}(x)}$ be a Riemannian metric on a differential manifold $M$ and $\beta = b_i(x)y^i$ be a 1-form on $M$. Suppose that

$$\|\beta\|_\alpha = \sqrt{a^{ij}b_i b_j} < 1,$$

where $(a^{ij}) = (a_{ij})^{-1}$. Define

$$F = \alpha + \beta$$

$F$ is a (positively definite) Finsler metric. $F$ is called the *Randers metric.* It was first studied by physicist G. Randers, in 1941 from the standard point of general relatively [Randers, 1941].

**Example 1.3.4** The following Randers metrics are very special. It is the deformation of Klein metric in (1.7).

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1-|x|^2} \pm \frac{\langle x, y\rangle}{1-|x|^2} \tag{1.9}$$

$$x \in B^n, \quad y \in T_x B^n = T_x R^n.$$

It is easy to show

$$(i) \quad \beta = \frac{\langle x, y\rangle}{1-|x|^2}$$

is an exact form. In particular, $\beta$ is closed;

$$(ii) \quad \|\beta\|_\alpha < 1.$$

The pair of metrics in (1.9) are called the *Funk metrics* on the unit ball.

**Example 1.3.5** Consider

$$\alpha = \frac{(1-\epsilon^2)\langle x, y\rangle^2 + \epsilon|y|^2(1+\epsilon|x|^2)}{1+\epsilon|x|^2},$$

$$\beta = \frac{\sqrt{1-\epsilon^2}\langle x, y\rangle}{1+\epsilon|x|^2},$$

where $x \in R^2$, $y \in T_x R^2$, $\epsilon \in (0, 1)$, then

$$\|\beta\|_\alpha = \sqrt{1-\epsilon^2}\sqrt{\frac{|x|^2}{\epsilon + |x|^2}} < 1.$$

Hence $F = \alpha + \beta$ is a Randers metric.

## 1.4 Fundamental invariants

(I) Fundamental tensor

**Lemma 1.4.1** *Let $V$ be a vector space and $H : V \to R$ be positively homogeneous of degree $r$. That is,*

$$H(\lambda y) = \lambda^r H(y) \quad for \quad all \quad \lambda > 0. \tag{1.10}$$

*Then*

$$y^i H_{y^i}(y) = rH(y), \tag{1.11}$$

*where $(y^1, \cdots, y^n)$ is the coordinate of $y \in V$, and*

$$H_{y^i} := \frac{\partial H}{\partial y^i}.$$

***Proof.*** Suppose $H$ satisfies (1.10). Fix $y$. Differentiating this equation with respect to $\lambda$ gives

$$y^i H_{y^i}(\lambda y) = r\lambda^{r-1} H(y).$$

Setting $\lambda$ equal to 1 gives (1.11). □

**Corollary 1.4.2** *Let $(M, F)$ be a Finsler manifold. Then*

$$y^i F_{y^i} = F, \tag{1.12}$$

$$y^i F_{y^i y^j} = 0, \tag{1.13}$$

$$y^i F_{y^i y^j y^k} = -F_{y^i y^j}. \tag{1.14}$$

***Proof.*** Succesively substituting $F, F_{y^i}, F_{y^i y^j}$ for $H$. Note that $F, F_{y^i}, F_{y^i y^j}$ is positively homogeneous of degree $1, 0$, and $-1$, respectively. □

Let $(M, F)$ be a Finsler manifold. Setting

$$g := g_{ij}(x, y) dx^i \otimes dx^j,$$

where

$$g_{ij} := \frac{1}{2}(F^2)_{y^i y^j} = FF_{y^i y^j} + F_{y^i} F_{y^j}. \tag{1.15}$$

It is easy to show that the symmetric covariant 2-tensor $g$ is intrinsically defined on $TM$ by using Lemma 1.2.1. We call $g$ the *fundamental tensor* of $F$.

**Lemma 1.4.3** *The components of the fundamental tensor satisfy the following properties:*

$$(i) \quad y^i g_{ij} = FF_{y^j}, \tag{1.16}$$

$$(ii) \quad y^i y^j g_{ij} = F^2, \tag{1.17}$$

$$(iii) \quad y^i \frac{\partial g_{ij}}{\partial y^k} = y^j \frac{\partial g_{ij}}{\partial y^k} = y^k \frac{\partial g_{ij}}{\partial y^k} = 0. \tag{1.18}$$

***Proof.*** (i) and (ii) follows from (1.12), (1.13) and (1.15).

(iii) Substituting $g_{ij}$ for $H$ in Lemma 1.4.1. Note that $g_{ij}$ is positively homogeneous of degree 0. □

(II) Hilbert form

Let $M$ be a smooth manifold and $SM$ the projective sphere bundle of $M$, with canonical projection map $p : SM \to M$ given by $(x, [y]) \to x$ and $S_xM := p^{-1}(x) \approx S^{n-1}$, the projective sphere at $x$.

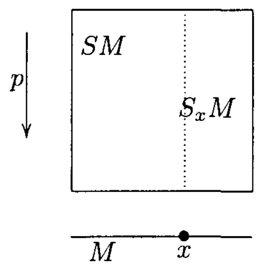


Fig. 1.2

Let $(M, F)$ be a Finsler space and $(x^i, y^i)$ the local coordinate on $TM$. Setting $\omega := F_{y^j} dx^j$, then we have the following

**Lemma 1.4.4** *$\omega$ is a globally defined differential form on $SM$.*

***Proof.*** Taking another local coordinate on $TM$ (cf. Lemma 1.2.1). From (1.2) we have

$$F_{\tilde{y}^j} d\tilde{x}^j = F_{y^j} dx^j.$$

Thus $\omega$ is globally defined. On the other hand, it is easy to see

$$F_{y^j}(x, \lambda y) = F_{y^j}(x, y), \quad \forall \lambda \in R^+.$$

This concludes that $\omega$ is defined on $SM$. $\square$

**Definition 1.4.5** Let $(M,\ F)$ be a Finsler manifold . We call $\omega$ the *Hilbert form* of $(M,\ F)$.

Let $(M, F)$ be an $n$-dimensional Finsler manifold and $p : SM \to M$ the canonical projection map. We pull back $TM$ and $T^*M$, and denote them by $p^*TM$ and $p^*T^*M$ respectively, making them vector bundles (with $n$-dimensional fibres ) over the $(2n-1)$-dimensional base $SM$. We call $p^*TM$ (resp. $p^*T^*M$) the *Finsler bundle* (resp. the *dual Finsler bundle*)[Chern, 1992].

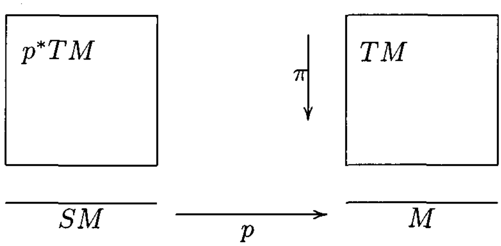


Fig. 1.3

It is easy to see that the Hilbert form is a section of $p^*T^*M$.

## 1.5 Reversible Finsler structures

The Finsler structure in Example 1.3.2 has the following important property

$$F_{\lambda,k}(y) = F_{\lambda,k}(-y).$$

In general, a Finsler structure $F : TM \to R$ is said to be *reversible* if

$$F(x, y) = F(x, -y), \tag{1.19}$$

for all $(x, y) \in TM$. It is easy to show that a Finsler structure is reversible if and only if

$$F(x, \lambda y) = |\lambda| F(x, y), \tag{1.20}$$

for all $\lambda \neq 0$. Of importance to our treatment with respect to reversible Finsler structure is the projectivized tangent bundle $PTM$ obtained from $TM$ by identifying its non-zero vectors which differ from each other by a multiplicative factor. Geometrically $PTM$ is the manifold of the line elements of $M$.

Let $(M, F)$ be a reversible Finsler manifold. Since $F_{y^i}$ is homogeneous of degree zero in $y$, the Hilbert form lives in $PTM$. In classical calculus of variations it is called Hilbert's invariant integral. Similarly, the fundermental tensor in Section 1.4 also lives in $PTM$.

**Lemma 1.5.1** *Let $(M, F)$ be a Randers space. If $F$ is reversible, then $(M, F)$ is a Riemann manifold .*

***Proof.*** Consider a Randers metric $F = \alpha + \beta$, where

$$\alpha = \sqrt{a_{ij}(x) y^i y^j}; \quad \beta = b_i(x) y^i$$

Thus

$$F(x, -y) = \sqrt{a_{ij}(x)(-y^i)(-y^j)} + b_i(x)(-y^i) = \sqrt{a_{ij}(x) y^i y^j} - b_i(x) y^i.$$

Since $F$ is reversible, $F(x, -y) = F(x, y)$. This implies that $b_i(x) y^i = \beta = 0$. We obtain $F = \alpha$ is Riemannian. $\square$

# Chapter 2

# Geometric Quantities on a Minkowski Space

There are several geometric invariants on a Finsler manifold. Some of them vanish on a Riemannian manifold. So we call them non-Riemannian quantities. In this chapter, we introduce the first class of geometric invariants. These invariants describes the non-Euclidean properties of Minkowski tangent spaces over a Finsler manifold.

## 2.1 The Cartan tensor

Let $(M, F)$ be a Finsler manifold with its fundermental tensor $g_{ij}$ (cf. Section 1.4). Put

$$A_{ijk} := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{F}{4}(F^2)_{y^i y^j y^k}, \tag{2.1}$$

$$A := A_{ijk} dx^i \otimes dx^j \otimes dx^k.$$

It is easy to see that $A$ is well defined on $SM$. We call $A$ the *Cartan tensor* of $(M,\ F)$.

**Proposition 2.1.1** *Let $(M, F)$ be a Finsler manifold. Then*
*(i)* $A_{ijk}$ *is totally symmetric, i.e.* $A_{ijk} = A_{jik} = A_{ikj}$*;*
*(ii)* $y^i A_{ijk} = 0$*;*
*(iii)* $(M, F)$ *is Riemannian if and only if* $A = 0$.

***Proof.*** (i) and (iii) follow from (2.1).
(ii) Follows (1.18). □

**Remark** The Cartan tensor gives a measure of the failure of $(M, F)$ to be a Riemannian manifold. Put $\varrho := (g_{ij})$ and $(g^{ij}) := (g_{ij})^{-1}$. It is easy to see that

$$F\frac{\partial}{\partial y^k}\log\sqrt{\det\varrho} = \frac{F}{2}\frac{\partial}{\partial y^k}\log\det\varrho = \frac{F}{2}\frac{1}{\det\varrho}\frac{\partial g_{ij}}{\partial y^k}G_{ij}, \tag{2.2}$$

where $G_{ij}$ is the algebraic complement of $g_{ij}$. It follows that

$$G_{ij} = (\det\varrho)g^{ji} = (\det\varrho)g^{ij}. \tag{2.3}$$

Plugging it into (2.2) and using (2.1) yields

$$F\frac{\partial}{\partial y^k}\log\sqrt{\det\varrho} = \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k}g^{ij} = A_{ijk}g^{ij}.$$

Then we have following

**Lemma 2.1.2** *Let $(M, F)$ be a Finsler space. Then* $\det\varrho$ *is independent of $y$ if and only if $A_i \equiv 0$ where*

$$A_i := g^{jk}A_{ijk}. \tag{2.4}$$

## 2.2 The Cartan form and Deicke's Theorem

Setting $\eta := A_i dx^i$ where $A_i$ is defined in (2.4). Then $\eta$ is a well-defined 1-form on $SM$, and $\eta$ is called *Cartan form.*

Let $F : R^n \to [0, \infty)$ be a Minkowski norm on $R^n$. Thus (i) $F$ is positively $y$-homogeneous of degree one; (ii) $F$ is $C^\infty$ on $R^n\backslash\{0\}$ and the following symmetric matrix

$$g_{ij} := \left(\frac{F^2}{2}\right)_{y^i y^j}$$

is positive definite. Put $\varrho := (g_{ij})$ and $(g^{ij}(x)) = (g_{ij}(x))^{-1}$.

**Lemma 2.2.1** *For any $x \in R^n\backslash\{0\}$, define $\phi_x : R^n\backslash\{0\} \to R$ by $\phi_x(y) = trace[\varrho^{-1}(x)\varrho(y)]$. If* $\det\varrho = const$, *then*

$$\min_{y\in R^n\backslash\{0\}}\phi_x(y) = \phi_x(x).$$

***Proof.*** Denote the eigenvalues of $\varrho^{-1}(x)\varrho(y)$ by $\lambda_1, \cdots, \lambda_n$. Then

$$\begin{aligned}\phi_x(y) &= \lambda_1 + \cdots + \lambda_n \\ &\geq n(\lambda_1\lambda_2\cdots\lambda_n)^{\frac{1}{n}} \\ &= n\{det[\varrho^{-1}(x)\varrho(y)]\}^{\frac{1}{n}} \\ &= n\{[\det\varrho(y)]^{-1}[\det\varrho(y)]\}^{\frac{1}{n}} \\ &= n \quad (\text{since } \det\varrho(y) = const) \\ &= \operatorname{tr}[\varrho^{-1}(x)\varrho(x)] = \phi_x(x).\end{aligned}$$

□

**Lemma 2.2.2** *Consider the following operator $\triangle = g^{ij}(x)\frac{\partial^2}{\partial y^i \partial y^j}$. Then we have*

$$\triangle\varrho = \left(\frac{\partial^2\phi_x}{\partial y^h \partial y^k}\right).$$

***Proof.***

$$\begin{aligned}\triangle g_{hk} &= g^{ij}(x)\frac{\partial^2 g_{hk}}{\partial y^i \partial y^j} \\ &= g^{ij}(x)\frac{\partial^4(\frac{F^2}{2})}{\partial y^h \partial y^k \partial y^i \partial y^j} \\ &= g^{ij}(x)\frac{\partial^4(\frac{F^2}{2})}{\partial y^i \partial y^j \partial y^h \partial y^k} \\ &= g^{ij}(x)\frac{\partial^2 g_{ij}}{\partial y^h \partial y^k} \\ &= \frac{\partial^2[g^{ij}(x)g_{ij}(y)]}{\partial y^h \partial y^k} = \frac{\partial^2\phi_x}{\partial y^h \partial y^k}.\end{aligned}$$

□

**Lemma 2.2.3** *Let $F : R^n \to [0, \infty)$ be a Minkowski norm. Assume that $\det\varrho = constant$ then for all $i, j$, $g_{ij} = constant$.*

***Proof.*** Setting $S := \{x \in R^n\backslash\{0\} \mid F(x) = 1\}$. Then $S$ is compact. It follows that for any fixed $h \in \{1, 2, \cdots, n\}$, there exist $x_0 \in S$, such that

$$g_{hh}(x_0) = \max_{x\in S} g_{hh}(x) = \max_{x\in R^n\backslash\{0\}} g_{hh}(x) \tag{2.5}$$

here we have used the fact that $g_{hh}$ is positively $y$-homogeneous of degree zero. By using Lemma 2.2.1 and the principle of calculus, we have $\left(\frac{\partial^2\phi_x}{\partial y^i \partial y^j}\right)(x)$ is semi-positive. Together with Lemma 2.2.2 we obtain $\triangle g_{hh} \geq 0$. Notice that (2.5) and Hopf's maximal principle we have

$g_{hh} = constant$. Thus $\triangle g_{hh} = 0$. Since $\triangle g$ is semi-positive, $\triangle g_{hk} = 0$ for all $h, k$. A similar discussion implies that $g_{hk} = constant$ for $h \neq k$. $\square$

**Theorem 2.2.4**([Deicke,1953]) *Let $(M, F)$ be a Finsler manifold. Then it is Riemannian if and only if its Cartan form vanishs identically.*

***Proof.*** From Lemma 2.1.2 and Lemma 2.2.3. $\square$

**Remark** The Cartan form characterizes the Riemannian manifolds among Finsler manifolds.

## 2.3 Distortion

Let $B^n$ denote the unit ball in $R^n$. Let $(M, F)$ be a Finsler manifold. Let

$$\tau = \ln \frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma_F(x)},$$

where $\sigma_F(x) := \frac{Vol(B^n)}{\{(y^i)\in R^n | F(x, y^i \frac{\partial}{\partial x^i}) < 1\}}$. $\tau$ is called the *distortion* [Shen, 2001; Shen, 1997]. It is easy to see $\tau$ is positively y-homogeneous of degree zero. Observe that

$$\begin{aligned} \frac{\partial \tau}{\partial y^k} &= \frac{\partial}{\partial y^k} \ln \left[ \frac{\sqrt{det(g_{ij}(x,y))}}{\sigma_F(x)} \right] \\ &= \frac{\partial}{\partial y^k} \ln \sqrt{det(g_{ij}(x,y))} - \frac{\partial \ln \sigma_F(x)}{\partial y^k} \\ &= \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{F} A_{ijk} g^{ij} = \frac{1}{F} A_k. \end{aligned} \tag{2.6}$$

(2.6) gives a relation between distortion and Cartan form. By Theorem 2.2.4 , we have

**Proposition 2.3.1** *Let $(M, F)$ be a Finsler manifold. Then $(M, F)$ is a Riemannian manifold if and only if its distortion is defined on $M$.*

## 2.4 Finsler submanifolds

Let $(\tilde{M}, \tilde{F})$ be a Finsler manifold and $M$ a smooth manifold. Let $f : M \to \tilde{M}$ be an immersion. Define $f_* : TM \to T\tilde{M}$ by

$$f_*(x, y) = (f(x), (df)_x(y)), \tag{2.7}$$

where $(df)_x : T_xM \to T_{f(x)}\tilde{M}$ is the differentiation of $f$ at $x \in M$. Put $F := \tilde{F} \circ f_*$. It is easy to show that $F$ is a Finsler structure on $M$. $F$ is called the *Finsler structure* induced by $\tilde{F}$.

**Lemma 2.4.1** *Let $f : M \to (\tilde{M}, \tilde{F})$ be an immersion into a Finsler manifold $(\tilde{M}, \tilde{F})$ and $F$ the Finsler structure induced by $\tilde{F}$. Then the fundamental tensor $g$ and Cartan tensor $A$ of $F$ satisfies*

$$g_{ab}(x, y) = \tilde{g}_{ij}(\tilde{x}, \tilde{y}) \frac{\partial f^i}{\partial x^a} \frac{\partial f^j}{\partial x^b}, \tag{2.8}$$

$$A_{abc}(x, y) = \tilde{A}_{ijk}(\tilde{x}, \tilde{y}) \frac{\partial f^i}{\partial x^a} \frac{\partial f^j}{\partial x^b} \frac{\partial f^k}{\partial x^c}, \tag{2.9}$$

*where $f^i$ is the component function of $f$ with respect to $\tilde{x}^i$, $x^a$ is the local coordinate on $M$, and*

$$\tilde{x} := f(x), \quad \tilde{y} = (df)_x(y). \tag{2.10}$$

***Proof.*** By (2.7),(2.10) and $F = \tilde{F} \circ f_*$, we have

$$F(x, y) = \tilde{F}(\tilde{x}, \tilde{y}). \tag{2.11}$$

Thus

$$F_{y^a}(x, y) = \tilde{F}_{\tilde{x}^i}(\tilde{x}, \tilde{y}) \frac{\partial \tilde{x}^i}{\partial y^a} + \tilde{F}_{\tilde{y}^i}(\tilde{x}, \tilde{y}) \frac{\partial \tilde{y}^i}{\partial y^a} = \tilde{F}_{\tilde{y}^i}(\tilde{x}, \tilde{y}) \frac{\partial f^i}{\partial x^a}(x), \tag{2.12}$$

where we have used (2.10) and

$$\frac{\partial \tilde{y}^i}{\partial y^a} = \frac{\partial f^i}{\partial x^a}.$$

Similarly,

$$F_{y^a y^b}(x, y) = \tilde{F}_{\tilde{y}^i \tilde{y}^j}(\tilde{x}, \tilde{y}) \frac{\partial f^i}{\partial x^a}(x) \frac{\partial f^j}{\partial x^b}(x). \tag{2.13}$$

Now (2.8) is an immediate conclusion of (2.11)–(2.13). From (2.1) (2.8) and (2.11), we have

$$\begin{aligned}A_{abc}(x,y) &= \frac{1}{2}F(x,y)\frac{\partial}{\partial y^c}g_{ab}(x,y)\\ &= \frac{\tilde{F}(\tilde{x},\tilde{y})}{2}\frac{\partial}{\partial \tilde{y}^k}\left[\tilde{g}_{ij}(x,y)\frac{\partial f^i}{\partial x^a}(x)\frac{\partial f^j}{\partial x^b}(x)\right]\frac{\partial \tilde{y}^k}{\partial y^c}\\ &= \frac{\tilde{F}(\tilde{x},\tilde{y})}{2}\left[\frac{\partial}{\partial \tilde{y}^k}\tilde{g}_{ij}(x,y)\right]\frac{\partial f^i}{\partial x^a}(x)\frac{\partial f^j}{\partial x^b}(x)\frac{\partial f^k}{\partial x^c}(x).\end{aligned}$$

Hence we get (2.9). □

**Corollary 2.4.5** *Let $(\tilde{M},\tilde{F})$ be a Riemannian manifold. Then its induced Finsler structure is Riemannian.*

Let $V$ be an $n$-dimensional vector space and $f$ a Minkowski norm on $V$. Fix a basis $\{e_i\}$ on $V$. Then $y = y^i e_i$ for $y \in V$. Thus $(y,u)$ has its local coordinate $(y^1,\cdots,y^n;\, u^1,\cdots,u^n)$, where $u^i = dy^i(u)$, and $(y,u) \in TV$. Define $F : TV \to [0,+\infty)$ by

$$F(y,u) = f(u^i e_i). \tag{2.14}$$

Then it is easy to show that
(i) $F$ is well defined, i.e. $F$ is independent of the choose of $\{e_i\}$ ;
(ii) $F$ is a Finsler structure on the linear manifold $V$;
(iii) $F(y,u) = F(u)$.

Put $S := \{y \in V | F(y) = 1\}$. Denote the natural imbedding from $S$ into $V$ by $h$. By (2.11) and (2.14), the induced Finsler structure by $F$ satisfies that

$$\begin{aligned}\tilde{F}(u,\omega^a\tfrac{\partial}{\partial u^a}) &= F(h(u), h_*(\omega^a\tfrac{\partial}{\partial u^a}))\\ &= F(h(u),\omega^a\tfrac{\partial h^i}{\partial u^a}\tfrac{\partial}{\partial y^i})\\ &= f(\omega^a\tfrac{\partial h^i}{\partial u^a}e_i).\end{aligned} \tag{2.15}$$

For instance, consider Example 1.3.2 and $e_1 = (1,0)$, $e_2 = (0,1)$. Then from (2.15)

$$\tilde{F}_{\lambda,k}(u,\omega) = \sqrt{\sum_{i=1}^{2}(\omega^a\frac{\partial h^i}{\partial u^a})^2 + \lambda\left\{\sum_{i=1}^{2}(\omega^a\frac{\partial h^i}{\partial u^a})^{2k}\right\}^{\frac{1}{k}}}.$$

## 2.5 Imbedding problem of submanifolds

Recall well-known Nash Theorem: Let $M$ be an $n$-dimensional Riemannian manifold. Then there exist an isometric imbedding $M \hookrightarrow R^m$ where

$$m = \begin{cases} \frac{n(3n+11)}{2}, & \text{if} \quad M \quad \text{is} \quad \text{compact;} \\ 2(2n+1)(3n+8), & \text{if} \quad M \quad \text{is} \quad \text{noncompact,} \end{cases}$$

Consider a Finsler manifold $(M, F)$, its tangent space on each point is Minkowski. It is natural to ask the following:

**Problem** If any $n$-dimensional Finsler space can be isometrically imbedded into a Minkowski space ?

Let $(M, F)$ be a Finsler manifold and $A$ the Cartan tensor of $F$. Write

$$I_x := \{y \in T_xM | F(y) = 1\}$$

and

$$\|A\|_x := \sup_{y \in I_x} \sup_{u \in I_x} \frac{|A_{x,y}(u,u,u)|}{|g_{x,y}(u,u)|^{\frac{3}{2}}}.$$

We call $\|A\|_x$ the *norm* of $A$ at $x$. In fact

$$\|A\|_x := \sup_{y \in T_xM \setminus \{0\}} \sup_{u \in T_xM \setminus \{0\}} \frac{|A_{x,y}(u,u,u)|}{|g_{x,y}(u,u)|^{\frac{3}{2}}}.$$

**Proposition 2.5.1** *Let $f : (M, F) \hookrightarrow (\tilde{M}, \tilde{F})$ be an isometric immersion. Then for any $x \in M$, we have*

$$\|A\|_x \leq \|\tilde{A}\|_{f(x)}, \tag{2.16}$$

*where $A$ and $\tilde{A}$ are the Cartan tensor of $F$ and $\tilde{F}$ respectively.*

***Proof.***

$$\begin{aligned}\|A\|_x :&= \sup_{y\in I_x}\sup_{u\in I_x}\frac{|A_{x,y}(u,u,u)|}{|g_{x,y}(u,u)|^{\frac{3}{2}}}\\ &= \sup_{y\in I_x}\sup_{u\in I_x}\frac{|\tilde{A}_{(f(x),f_*(y))}(f_*u,f_*u,f_*u)|}{|\tilde{g}_{(f(x),f_*(y))}(f_*u,f_*u)|^{\frac{3}{2}}}\\ &\le \sup_{\tilde{y}\in I_{f(x)}}\sup_{\tilde{u}\in I_{f(x)}}\frac{|\tilde{A}_{(f(x),f_*(y))}(\tilde{u},\tilde{u},\tilde{u})|}{|g_{(f(x),\tilde{y})}(\tilde{u},\tilde{u})|^{\frac{3}{2}}}\\ &= \|A\|_{f(x)}\end{aligned}$$

□

**Corolary 2.5.2** *If a Finsler manifold can be embedded into a finite-dimensional Minkowski manifold, then the Cartan tensor must be bounded.*

***Proof.*** Let $V$ be a finite dimentional vector space and $\tilde{F}_0$ a Minko wski norm on $V$. By Section 2.4, we define a Finsler structure $\tilde{F}$ from $\tilde{F}_0$ such that

$$\tilde{F}(\tilde{x},\tilde{y}) = \tilde{F}_0(\tilde{y}). \tag{2.17}$$

Let $f : (M,F) \rightarrowtail (V,\tilde{F})$ be an isometric immersion. From (2.17) we have

$$\|\tilde{A}\|_x = constant.$$

Together with Proposition 2.5.1,

$$\sup_{x\in M}\|A\|_x \le \sup_{x\in M}\|\tilde{A}\|_{f(x)} = constant < +\infty.$$

□

Consider the following Minkowski manifold $(R^n, F)$ where

$$F = F_\lambda(y) = \sqrt{\sum |y^i|^2 + \lambda r^2} \qquad (\lambda \ge 0), \tag{2.18}$$

$$r = (\sum |y^i|^4)^{\frac{1}{4}}. \tag{2.19}$$

Notice that when $n = 2$, then $F_\lambda = F_{\lambda,2}$ (see Example 1.3.2). Now

$$F^2 = \sum |y^i|^2 + \lambda r^2, \tag{2.20}$$

$$r^4 = \sum |y^i|^4. \tag{2.21}$$

By (2.21) we get

$$rr_{y^i} = r^{-2}(y^i)^3. \tag{2.22}$$

Thus

$$(r^{-2})_{y^i} = -2r^{-6}(y^i)^3, \tag{2.23}$$

$$(r^{-6})_{y^i} = -6r^{-10}(y^i)^3, \tag{2.24}$$

From (2.20) and (2.22), we have

$$\left(\frac{F^2}{2}\right)_{y^i} = y^i + \lambda r^{-2}(y^i)^3. \tag{2.25}$$

Together with (2.22) and (2.23), we get

$$g_{ij}(y) = \left(\frac{F^2}{2}\right)_{y^i y^j} = \delta_{ij} + 3\lambda r^{-2} y^i y^i \delta_{ij} - 2\lambda r^{-6}(y^i y^j)^3. \tag{2.26}$$

A simple calculation give the following formula

$$(y^i y^i \delta_{ij})_{y^k} = 2y^i \delta_{ijk}, \tag{2.27}$$

where

$$\delta_{ijk} = \begin{cases} 1, & \text{if} \quad i = j = k; \\ 0, & \text{other.} \end{cases} \tag{2.28}$$

Similarly

$$(y^i y^j)^3_{y^k} = 3[y^i y^i (y^j)^3 \delta_{ik} + y^j y^j (y^i)^3 \delta_{jk}]. \tag{2.29}$$

Thus, from (2.23),(2.24),(2.26),(2.27) and (2.29)

$$\begin{aligned} A_{ijk}(y) &= \tfrac{F}{2}\tfrac{\partial g_{ij}}{\partial y^k} \\ &= \tfrac{\lambda F}{2}[3r^{-2}y^i y^i \delta_{ij} - 2r^{-6}(y^i y^j)^3]_{y^k} \\ &= 3\lambda F[r^{-2}y^i \delta_{ijk} + 2r^{-10}(y^i y^j y^k)^3 \\ &\quad -r^{-6}(y^i y^i (y^j)^3 \delta_{ik} + y^j y^j (y^i)^3 \delta_{jk} + y^i y^i (y^k)^3 \delta_{ij})]. \end{aligned} \tag{2.30}$$

In particular, from (2.26) and (2.30), we get

$$g_{11}(y) = 1 + 3\lambda r^{-2}|y^1|^2 - 2\lambda r^{-6}|y^1|^6, \tag{2.31}$$

$$A_{111}(y) = 3\lambda F[r^{-2}y^1 + 2r^{-10}(y^1)^9 - 3r^{-6}(y^1)^5]. \tag{2.32}$$

Taking

$$y_0 = (1, \lambda^{\frac{1}{2}}, 0, \cdots, 0), \tag{2.33}$$

then

$$r(y_0) = (1+\lambda^2)^{\frac{1}{4}}. \tag{2.34}$$

Together with (2.31) and (2.33),

$$g_{11}(y_0) = 1 + \lambda \frac{1+3\lambda^2}{(1+\lambda^2)^{\frac{3}{2}}}. \tag{2.35}$$

Consider the following function

$$f(x) := \frac{x+3x^2}{(1+x^2)^{\frac{3}{2}}}. \tag{2.36}$$

Then

$$f'(x) := \frac{x+7x^2}{(1+x^2)^{\frac{5}{2}}} > 0 \qquad \forall x. \tag{2.37}$$

Thus

$$\sup f(x) = f(+\infty) = 3.$$

It follows that

$$g_{11}(y_0) \leq 4. \tag{2.38}$$

On the other hand, from (2.20) and (2.34), we have

$$F_\lambda^2(y_0) = 1 + \lambda + \lambda(1+\lambda^2)^{\frac{1}{2}}. \tag{2.39}$$

Together with (2.32) and (2.34) we get

$$\begin{aligned} A_{111}(y_0) &= \frac{3\lambda^3(\lambda^2-1)[1+\lambda+\lambda(1+\lambda^2)^{\frac{1}{2}}]}{(1+\lambda^2)^{\frac{5}{2}}} \\ &\geq \frac{3\lambda^4(\lambda^2-1)}{(1+\lambda^2)^2} \\ &\geq \frac{3\lambda^4(\lambda-1)}{(1+\lambda)^3}. \end{aligned} \tag{2.40}$$

Put

$$O = (0, \cdots, 0), \qquad e_1 = (1, 0, \cdots, 0),$$

then

$$\|A\|_o \geq \frac{|A_{(o,y_0)}(e_1,e_1,e_1)|}{|g_{(o,y_0)}(e_1,e_1)|^{\frac{3}{2}}} = \frac{|A_{111}(y_0)|}{|g_{11}(y_0)|^{\frac{3}{2}}} \geq \frac{3}{8}\frac{\lambda^4(\lambda-1)}{(1+\lambda)^3} \tag{2.41}$$

from (2.38) and (2.40).

**Proposition 2.5.3**([Shen, 1998]) *On $R^n$ we define $F : TR^n \to [0,\infty)$*

$$F : (x,y) \mapsto \sqrt{\sum |y^i|^2 + \|x\|(\sum |y^i|^4)^{\frac{1}{4}}}.$$

*Then $(R^n, F)$ cannot be isometrically into any embedded finite-dimen sional Minkowski manifold.*

***Proof.*** By (2.41)

$$\|A\|_x \geq \frac{3}{8}\frac{\|x\|^4(\|x\|-1)}{(1+\|x\|)^3} \to +\infty \quad (\|x\| \to +\infty).$$

Hence the Cartan tensor of $(R^n, F)$ is not bounded. Now our conclusion is an immediate consequence of Corollary 2.5.2. □

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# Chapter 3

# Chern Connection

Besides the geometric invariants of Minkowski tangent spaces, there are several geometric quantities on a Finsler manifold. We will describe these invariants by introducing the connection on a Finsler manifold. From the point of view of differential geometry the most notable connection was Chern's connection defined in 1948, which is derived by using exterior differentiation on the Hilbert form [Bao and Shen, 1994; Chern, 1948], Chern connection is torsion-free and 'almost' compatible with the fundamental tensor. In this chapter, we will give the explicit construction of Chern connection and analysis the main properties of this connection.

## 3.1 The adapted frame on a Finsler bundle

Let $(M, F)$ be a Finsler manifold and $SM$ its projective sphere bundle. Consider the Finsler bundle $p^*TM$ of $(M, F)$ (see Section 1.4). There is a global section $l$ of $p^*TM$. It is defined by

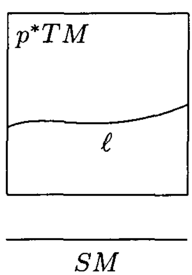


Fig. 3.1

$$l = l_{(x,[y])} = (x, [y], \frac{y}{F(x,y)}) \approx \frac{y}{F(x,y)}.$$

We call $l$ the *distinguished section.* Let $g := g_{ij}dx^i \otimes dx^j$ denote the fundamental tensor of $(M, F)$. Then $g$ is positively homogeneous of degree zero with respect to $y$. Hence $g \in \Gamma(\odot^2 p^*T^*M)$ where $p^*T^*M$ is the dual Finsler bundle.

**Lemma 3.1.1** *For any $(x,[y]) \in SM$, there exists an open subset $V$ and $e_1, \cdots, e_n \in \Gamma_V(p^*T^*M)$, such that $(x,[y]) \in V \subset SM$, $g(e_i, e_j) = \delta_{ij}$ and $e_n = l|_V$.*

**Proof.** For each $x \in M$, there exists a local coordinate chart $(U; x^1, \cdots, x^n)$ such that $x \in U$. Thus

$$\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, l \in \Gamma_{p^{-1}(U)}(p^*TM).$$

Since $l$ is nowhere zero and

$$l_{(x,[y])} = \frac{y}{F(x,y)} = \frac{y^i}{F(x,y)} \frac{\partial}{\partial x^i}, \tag{3.1}$$

we assume that

$$y^n > 0. \tag{3.2}$$

Thus there exists open subset $V \subset p^{-1}(U)$ of $(x,[y])$ such that $y^n|_V > 0$. It follows $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{n-1}}, l$ are linear independent on whole $V$. Setting $e_n = l|_V$. By using the Schmidt orthogonalization, we get $e_1, \cdots, e_n \in \Gamma_V(p^*TM)$ satisfying $g(e_i, e_j) = \delta_{ij}$. For instance,

$$\xi_{n-1} := l - \frac{1}{g(\frac{\partial}{\partial x^n}, l)} \frac{\partial}{\partial x^{n-1}},$$

$$e_{n-1} := \xi_{n-1}/\sqrt{g(\xi_{n-1}, \xi_{n-1})}.$$

We call the frame field in Lemma 3.1.1 is *adapted.*

**Lemma 3.1.2** *If $\omega^1, \cdots, \omega^n \in \Gamma_V(p^*T^*M)$ is the dual frame field of a adapted frame, then $\omega^n$ is the Hilbert form.*

**Proof.** From (1.12),(1.13),(1.15) and (3.1), we have

$$F_{y^i}(x,y) = g_{ij}(x,y)\frac{y^j}{F(x,y)} = g_{ij}(x,y)dx^j(e_n).$$

Thus, for $k = 1, \cdots, n$, we get

$$\begin{aligned}\omega(e_k) &= [F_{y^i}(x,y)dx^i](e_k) \\ &= g_{ij}(x,y)dx^i \otimes dx^j(e_k, e_n) \\ &= g(e_k, e_n) = \delta_{kn} = \omega^n(e_k).\end{aligned}$$

Notice that $\omega^j_{(x,[y])} \in T^*_x M$, for any $(x,[y]) \in V$ and $j \in \{1, \cdots, n\}$. The frame $\omega^j$ can be expressed in terms of $dx^j$. That is

$$\omega^j := v^j_k(x,[y])dx^k. \tag{3.3}$$

Hence

$$\omega^j \in \Gamma_V(T^*SM). \tag{3.4}$$

It is easy to verify the following lemma using (3.1),(3.3), Lemma 3.1.2 and $g^{kl} = \sum_i dx^k(e_i)dx^l(e_i)$.

**Lemma 3.1.3** *Let $\{e_i\}$ denote the adapted frame field of $(M,F)$ and $v^j_i$ the functions defined in (3.3). Then*

$$(i) \quad \frac{\partial}{\partial x^j} = v_j{}^l e_l, \tag{3.5}$$

$$(ii) \quad v_k{}^n = F_{y^k}, \tag{3.6}$$

$$(iii) \quad v_k{}^i g^{kl} v_l{}^j = \delta^{ij}, \tag{3.7}$$

$$(iv) \quad v_i{}^k \delta_{kl} v_j{}^l = g_{ij}, \tag{3.8}$$

$$(v) \quad v_k{}^\alpha y^k = 0, \tag{3.9}$$

$$(vi) \quad v_j{}^\alpha v_k{}^\beta \delta_{\alpha\beta} = FF_{y^j y^k}, \tag{3.10}$$

*where*

$$1 \le i,\ j,\ k,\ \cdots \le n,$$

$$1 \le \alpha,\ \beta,\ \gamma,\ \cdots \le n-1.$$

Similarly, setting

$$e_j := u_j{}^l(x, [y])\frac{\partial}{\partial x^l},$$

and $\{\omega^i\}$ is the dual frame of $\{e_i\}$, then we have

**Lemma 3.1.4**

$$(i) \quad dx^i = u_k{}^i \omega^k, \tag{3.11}$$

$$(ii) \quad u_n{}^l = y^l/F, \tag{3.12}$$

$$(iii) \quad u_k{}^i g_{ij} u_l{}^j = \delta_{kl}, \tag{3.13}$$

$$(iv) \quad u_k{}^i \delta^{kl} v_l{}^j = g^{ij}, \tag{3.14}$$

$$(v) \quad F_{y^k} u_\alpha{}^k = 0, \tag{3.15}$$

$$(vi) \quad u_\alpha{}^j u_\beta{}^k F F_{y^j y^k} = \delta_{\alpha\beta}. \tag{3.16}$$

**Corollary 3.1.5**

$$v_k{}^i u_j{}^k = \delta_j{}^i; \quad u_k{}^i v_j{}^k = \delta_j{}^i, \tag{3.17}$$

$$u_j{}^l = \delta_{ji} v_k{}^i g^{kl}; \quad v_k{}^i = \delta^{ij} u_j{}^l g_{lk}. \tag{3.18}$$

## 3.2 Construction of Chern connection

**Lemma 3.2.1** *Let $\omega$ denote the Hilbert form of $(M, F)$. Then*

$$d\omega \equiv 0 \quad mod \quad \omega^i \wedge \omega^j; \quad \omega^\alpha \wedge dy^j. \tag{3.19}$$

***Proof.***

$$\begin{aligned} d\omega &= d[F_{y^i}(x,y)dx^i] \\ &= dF_{y^i} \wedge dx^i \\ &= (F_{y^i x^j} dx^j + F_{y^i y^j} dy^j) \wedge u_k{}^i \omega^k \\ &= F_{y^i x^j} u_l{}^j u_k{}^i \omega^l \wedge \omega^k + F_{y^i y^j} u_\alpha{}^i dy^j \wedge \omega^\alpha + F_{y^i y^j} \tfrac{y^i}{F} dy^j \wedge \omega \\ &= F_{y^i x^j} u_l{}^j u_k{}^i \omega^l \wedge \omega^k + F_{y^i y^j} u_\alpha{}^i dy^j \wedge \omega^\alpha \end{aligned} \tag{3.20}$$

from (1.13),(3.11) and (3.12). □

**Lemma 3.2.2** *There exist $\omega_j{}^n \in \Gamma(T^*SM)$, such that*

$$d\omega = \omega^j \wedge \omega_j{}^n. \tag{3.21}$$

*Moreover $\omega_j{}^n$ can be taken as*

$$\omega_n{}^n = 0, \tag{3.22}$$

$$\omega_\alpha{}^n = (u_\alpha{}^j u_\beta{}^k F_{y^k x^j} + \lambda_{\alpha\beta})\omega^\beta + \frac{u_\alpha{}^j}{F}(F_{x^j} - y^k F_{y^j x^k})\omega - u_\alpha{}^k F_{y^j y^k} dy^j, \tag{3.23}$$

*where $\lambda_{\alpha\beta} : V(\subset SM) \to R$ satisfy that $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$.*

***Proof.*** By using (1.12) and (3.12), we have

$$\begin{aligned} F_{y^i x^j} u_l{}^j u_k{}^i \omega^l \wedge \omega^k &= F_{y^i x^j}(\tfrac{y^j}{F} u_\alpha{}^i \omega \wedge \omega^\alpha + \tfrac{y^i}{F} u_\alpha{}^j \omega^\alpha \wedge \omega \\ &\quad + u_\alpha{}^j u_\beta{}^i \omega^\alpha \wedge \omega^\beta) \\ &= (F_{x^j} u_\alpha{}^j / F)\omega^\alpha \wedge \omega - F_{y^i x^j} \tfrac{y^j}{F} u_\beta{}^i \omega^\beta \wedge \omega \\ &\quad + F_{y^i x^j} u_\alpha{}^j u_\beta{}^i \omega^\alpha \wedge \omega^\beta. \end{aligned}$$

Substituting it into (3.20) yields (3.21)–(3.23). □

**Lemma 3.2.3**

$$d\omega^\alpha = \omega^i \wedge \omega_i{}^\alpha \tag{3.24}$$

*where*

$$\omega_\beta{}^\alpha = v_k{}^\alpha du_\beta{}^k + \xi_\beta{}^\alpha \omega + \mu^\alpha{}_{\beta\gamma}\omega^\gamma \tag{3.25}$$

$$\omega_n{}^\alpha = \frac{1}{F} v_k{}^\alpha dy^k + \xi_i{}^\alpha \omega^i \tag{3.26}$$

*and* $\mu^\alpha{}_{\beta\gamma} : V \to R$ *satisfying* $\mu^\alpha{}_{\beta\dot{\gamma}} = \mu^\alpha{}_{\gamma\beta}$, $\xi_i{}^\alpha : V \to R$.

***Proof.*** From (3.3),(3.9),(3.11),(3.12) and (3.17), we have

$$\begin{aligned} d\omega^\alpha &= d(v_k{}^\alpha dx^k) \\ &= dv_k{}^\alpha \wedge dx^k \\ &= u_i{}^k dv_k{}^\alpha \wedge \omega^i \\ &= -v_k{}^\alpha du_i{}^k \wedge \omega^i \\ &= \omega^\beta \wedge v_k{}^\alpha du_\beta{}^k + \omega \wedge v_k{}^\alpha d(\tfrac{y^k}{F}) \\ &= \omega^\beta \wedge v_k{}^\alpha du_\beta{}^k + \omega \wedge (v_k{}^\alpha / F) dy^k. \end{aligned} \tag{3.27}$$

On the other hand, since $\mu^\alpha{}_{\beta\gamma} = \mu^\alpha{}_{\gamma\beta}$, we get

$$\omega^\beta \wedge [\xi_\beta{}^\alpha \omega + \mu^\alpha{}_{\gamma\beta}\omega^\gamma] + \omega \wedge \xi_i{}^\alpha \omega^i. \tag{3.28}$$

It is easy to show (3.24),(3.25) and (3.26) from (3.27) and (3.28). □

**Lemma 3.2.4** *Taking*

$$\xi_n{}^\alpha = -\delta^{\alpha\sigma}\frac{u_\sigma{}^j}{F}(F_{x^j} - y^k F_{y^j x^k}), \tag{3.29}$$

$$\xi_\beta{}^\alpha = -\delta^{\alpha\sigma}(u_\sigma{}^j u_\beta{}^k F_{y^k x^j} + \lambda_{\sigma\beta}), \tag{3.30}$$

*then*

$$\omega_\alpha{}^n + \delta_{\alpha\beta}\omega_n{}^\beta = 0, \tag{3.31}$$

$$\omega_\beta{}^\alpha = v_k{}^\alpha du_\beta{}^k - \delta^{\alpha\sigma}(u_\sigma{}^j u_\beta{}^k F_{y^k x^j} + \lambda_{\sigma\beta})\omega + \mu^\alpha{}_{\beta\gamma}\omega^\gamma. \tag{3.32}$$

***Proof.*** By using (3.10) and (3.17), we have

$$u_\alpha{}^k F_{y^j y^k} = \frac{v_j{}^\alpha}{F}. \tag{3.33}$$

Now it is easy to obtain (3.31) and (3.32) from (3.23),(3.25),(3.26),(3.29), (3.30) and (3.33). □

**Lemma 3.2.5**

$$\begin{aligned}\omega_{\rho\sigma}+\omega_{\sigma\rho} = &- u_\sigma{}^j u_\rho{}^i[d(FF_{y^iy^j})+(F_{y^jx^i}+F_{y^ix^j})\omega]\\ &-2\lambda_{\rho\sigma}\omega+(\delta_{\alpha\sigma}\mu^\alpha{}_{\rho\gamma}+\delta_{\alpha\rho}\mu^\alpha{}_{\sigma\gamma})\omega^\gamma\end{aligned} \tag{3.34}$$

***Proof.*** From (3.17) and (3.10), we have

$$\delta_{\alpha\sigma}v_i{}^\alpha du_\rho{}^i+\delta_{\alpha\rho}v_i{}^\alpha du_\sigma{}^i=-u_\sigma{}^j u_\rho{}^i d(FF_{y^iy^j}). \tag{3.35}$$

On the other hand

$$\omega_{\rho\sigma}+\omega_{\sigma\rho}:=\omega_\rho{}^\alpha\delta_{\alpha\sigma}+\omega_\sigma{}^\alpha\delta_{\alpha\rho}. \tag{3.36}$$

Plugging (3.32) into (3.36) and using (3.35) yields (3.34). ☐

**Lemma 3.2.6** *Let $V$ be an $n$-dimensional vector space, and assume that $F : V \to [0,+\infty)$ is positively homogeneous of degree one and $(\frac{F^2}{2})_{y^iy^j}$ is positive definite. Then*

$$(i)\quad F(y)>0 \quad for \quad y\neq 0, \tag{3.37}$$

$$(ii)\quad F_{y^iy^j}\xi^i\xi^j\geq 0,\quad \forall\xi\in V, \tag{3.38}$$

*and " $=0$" only if $\xi$ and $y$ are collinear.*

***Proof.*** Setting

$$g_{ij}:=\left(\frac{F^2}{2}\right)_{y^iy^j}, \tag{3.39}$$

then

$$g_{ij}:=FF_{y^iy^j}+F_{y^i}F_{y^j}, \tag{3.40}$$

(1.12) and (1.13) imply that

$$g_{ij}(y)y^iy^j=F^2(y). \tag{3.41}$$

Notice that $g_{ij}$ is positive definite, we have (3.37). Also,

$$\langle\xi,\eta\rangle_y:=g_{ij}(y)\xi^i\xi^j$$

is an inner product on $V$ for any $y \in V\backslash\{0\}$ . By the Cauchy-Schwarz inequality

$$[g_{ij}(y)\xi^i\eta^j]^2 \leq [g_{ij}(y)\xi^i\xi^j][g_{kl}(y)\eta^k\eta^l], \tag{3.42}$$

where equality holds iff $\xi$ and $\eta$ are collinear. In particular, we take $\eta = y$ and use (3.41) and (3.42)

$$[g_{ij}(y)\xi^i y^j]^2 \leq F^2(y)g_{ij}(y)\xi^i\xi^j, \tag{3.43}$$

where equality holds iff $\xi$ and $y$ are collinear. Thus, from (3.40) and (3.43)

$$\begin{aligned} F_{y^iy^j}(y)\xi^i\xi^j &= F^{-1}(g_{ij} - F_{y^i}F_{y^j})\xi^i\xi^j \\ &= F^{-1}g_{ij}\xi^i\xi^j - F^{-1}(F_{y^j}\xi^j)^2 \\ &= F^{-1}g_{ij}\xi^i\xi^j - F^{-1}(F^{-1}(g_{ij}y^i\xi^j)^2 \\ &= F^{-3}[F^2 g_{ij}\xi^i\xi^j - (g_{ij}y^i\xi^j)^2] \geq 0 \end{aligned}$$

where equality holds iff $\xi$ and $y$ are collinear. □

**Lemma 3.2.7** *Let $M$ be a Finsler manifold. Put*

$$\theta_k = F_{y^jy^k}dy^j \in \Gamma(T^*SM),$$

*then we have at least $n-1$ linear independent forms among $\theta_1, \cdots, \theta_n$.*

***Proof.*** Denote the complementary minor of $F_{y^iy^j}$ in $det(F_{y^ky^l})$ by $f^{ij}$. It is easy to see that

$$\theta_1 \wedge \cdots \wedge \hat{\theta_j} \wedge \cdots \wedge \theta_n = f^{ij}dy^1 \wedge \cdots \wedge \hat{dy^i} \wedge \cdots \wedge dy^n. \tag{3.44}$$

Normalizing the homogeneous coordinate $(y^1, \cdots, y^n)$ we have

$$y^1dy^1 + \cdots + y^ndy^n = 0. \tag{3.45}$$

Consider a $(x, [y]) \in SM$ such that $y^n \neq 0$, we get

$$dy^n = -\frac{\sum y^\alpha dy^\alpha}{y^n}. \tag{3.46}$$

By using (3.44), (3.45) and (3.46), we get

$$
\begin{aligned}
\theta_1 &\wedge \cdots \wedge \hat{\theta_j} \wedge \cdots \wedge \theta_n \\
&= f^{nj} dy^1 \wedge \cdots \wedge dy^{n-1} \\
&\quad + f^{\alpha j} dy^1 \wedge \cdots \wedge \hat{dy^\alpha} \wedge \cdots \wedge dy^{n-1} \wedge (-\tfrac{y^\beta}{y^n} dy^\beta) \\
&= \tfrac{1}{y^n} \textstyle\sum_i (-1)^{n-i} f^{ij} y^i dy^1 \wedge \cdots \wedge dy^{n-1} \\
&= \frac{(-1)^{n-j}}{y^n} \begin{vmatrix} F_{y^1y^1} & \cdots & F_{y^1y^{j-1}} & y^1 & F_{y^1y^{j+1}} & \cdots & F_{y^1y^n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ F_{y^ny^1} & \cdots & F_{y^ny^{j-1}} & y^n & F_{y^ny^{j+1}} & \cdots & F_{y^ny^n} \end{vmatrix}.
\end{aligned}
\tag{3.47}
$$

Note by Lemma 3.2.6 we have $rank(F_{y^iy^j}) = n-1$. Since $F_{y^iy^j}y^i = 0$ (cf. Lemma 1.4.1),

$$y(\neq 0) \perp (F_{y^1y^l}, \cdots, F_{y^ny^l}).$$

Hence there exist $j \in \{1, \cdots, n\}$, such that $\theta_1 \wedge \cdots \wedge \hat{\theta_j} \wedge \cdots \wedge \theta_n \neq 0$. □

**Proposition 3.2.8** *$\omega^i$, ${\omega_\alpha}^n$ are the linear independent Pfaff forms on $SM$.*

***Proof.*** Assume that

$$\mu^\alpha {\omega_\alpha}^n + v_i \omega^i = 0. \tag{3.48}$$

Substituting (3.23) into (3.48) yields

$$\mu^\alpha {u_\alpha}^k \theta_k = 0, \tag{3.49}$$

where $\theta_k = F_{y^jy^k} dy^j$ (cf. Lemma 3.2.7). By (1.13) we have

$$y^1\theta_1 + \cdots + y^{n-1}\theta_{n-1} + y^n\theta_n = 0. \tag{3.50}$$

From Lemma 3.2.7, we can assume that $\theta_1, \cdots, \theta_{n-1}$ are linear independent. If $y^n = 0$ then (3.50) implies that $y = 0$. This is a contradiction. Setting

$$\mu^\alpha {u_\alpha}^n = \lambda y^n. \tag{3.51}$$

Using (3.51) we can remove $\theta_n$ from (3.49) and (3.50) and get

$$(\mu^\alpha u_\alpha^\beta - \lambda y^\beta)\theta_\beta = 0.$$

Since $\theta_1, \cdots, \theta_{n-1}$ are linear independent, we get, together with (3.51)

$$\mu^\alpha {u_\alpha}^k = \lambda y^k. \tag{3.52}$$

Thus

$$0 = \mu^{\alpha} u_{\alpha}{}^{k} v_{k}{}^{n} = \lambda y^{k} v_{k}{}^{n} = \lambda y^{k} F_{y^{k}} = \lambda F(x, y)$$

from (3.6) and (3.52). Note that $(x, [y]) \in SM$, Lemma 3.2.6 implies that $\lambda = 0$. Substituting it into (3.52) yields

$$\mu^{\alpha} u_{\alpha}{}^{k} = 0.$$

On the other hand, since $rank(u_i{}^j) = n$, we get $rank(u_{\alpha}{}^k) = n-1$. Without lose of generality, we assume that $\det(u_{\alpha}{}^{\beta}) \neq 0$. Thus $\mu^{\alpha} u_{\alpha}{}^{\beta} = 0$ implies taht $\mu^{\alpha} = 0$. Plugging it into (3.48) yields $\sum_i v_i \omega^i = 0$. Hence $v_i = 0$. □

**Remark** From Proposition 3.2.8, $\omega^1 \wedge \cdots \wedge \omega^n \wedge \omega_1{}^n \wedge \cdots \wedge \omega_{n-1}{}^n$ can be considered as a volume form on $SM$. Setting

$$\Pi := \delta_{ij} \omega^i \otimes \omega^j + \delta_{\alpha\beta} \omega_n{}^{\alpha} \otimes \omega_n{}^{\beta}.$$

Then $SM$ is a $(2n$-$1)$-dimensional Riemannian manifold with Sasaki-type Riemannian metric $G$[Bao and Shen, 1994].

## 3.3 Properties of Chern connection

First, Chern connection is torsion-free from (3.21) and (3.24). Note Proposition 3.2.8, we put

$$d(FF_{y^i y^j}) = K_{ij}{}^{\alpha} \omega_{\alpha}{}^{n} + G_{ijk} \omega^k, \tag{3.53}$$

where $K_{ij}{}^{\alpha}$ and $G_{ijk}$ are symmetric with respect to $i$ and $j$.

**Proposition 3.3.1** *In (3.25) and (3.30), we choose respectively*

$$\lambda_{\rho\sigma} = -\frac{1}{2} u_{\rho}{}^{i} u_{\sigma}{}^{j} (G_{ijn} + F_{y^j x^i} + F_{y^i x^j}), \tag{3.54}$$

$$\mu_{\rho\sigma}{}^{\alpha} = \frac{1}{2} \delta^{\alpha\beta} (u_{\beta}{}^{i} u_{\rho}{}^{j} G_{ij\sigma} - u_{\rho}{}^{i} u_{\sigma}{}^{j} G_{ij\beta} + u_{\sigma}{}^{i} u_{\beta}{}^{j} G_{ijl}). \tag{3.55}$$

*Then*

$$\omega_{\rho\sigma} + \omega_{\sigma\rho} = -2 H_{\rho\sigma\alpha} \omega_n{}^{\alpha}, \tag{3.56}$$

*where*

$$H_{abc} = A_{ijk} u_a{}^i u_b{}^j u_c{}^k \tag{3.57}$$

*and*

$$A_{ijk} = A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right).$$

***Proof.*** From (3.53)

$$\frac{\partial}{\partial y^k}(FF_{y^iy^j})dy^k \equiv K_{ij}{}^{\alpha}\omega^n \quad mod \quad dx^i. \tag{3.58}$$

Plugging (3.23) into (3.58) and comparing the coefficients of $dy^k$ yield

$$-K_{ij}{}^{\alpha}u_{\alpha}{}^{l}F_{y^ky^l} = F_{y^k}F_{y^iy^j} + FF_{y^iy^jy^k}. \tag{3.59}$$

Combine with (3.15) and (3.16), we have

$$-K_{ij}{}^{\alpha}\delta_{\alpha\beta} = F^2F_{y^iy^jy^k}u_{\beta}{}^{k}. \tag{3.60}$$

Substituting (3.15), (3.53)–(3.55) and (3.57) into (3.34) and using (3.60) yield

$$\omega_{\rho\sigma} + \omega_{\sigma\rho} = -F^2F_{y^iy^jy^k}u_{\beta}{}^{k}u_{\sigma}{}^{j}u_{\rho}{}^{i}\omega_n{}^{\beta} = -H_{\rho\sigma\alpha}\omega_n{}^{\alpha}.$$

□

**Remark** (1) $H_{abc}$ is zero whenever any of its three indices has the value $n$.

(2) (3.31) and (3.56) combine to give

$$\omega_{ik} + \omega_{ki} = -H_{ikj}\omega_n{}^{j}. \tag{3.61}$$

It means that Chern connection is almost compatible with metric $g$.

**Lemma 3.3.2** *Having spectifield $\lambda_{\rho\sigma}$ in (3.54) , we have the following formula for $\omega_{\alpha}{}^{n}$, namely,*

$$\begin{aligned}\omega_{\alpha}{}^{n} = -u_{\alpha}{}^{s}F_{y^sy^k}dy^k + u_{\alpha}{}^{s}[\frac{1}{2}\frac{y^r}{F}F_{y^k}(G_{rsn} + F_{y^rx^s} - F_{y^sx^r}) \\ - \frac{1}{2}(G_{skn} + F_{y^sx^k} - F_{y^kx^s})]dx^k.\end{aligned} \tag{3.62}$$

***Proof.*** Direct calculations give that

$$y^k F_{y^k x^j} = (y^k F_{y^k})_{x^j} = F_{x^j} \tag{3.63}$$

and

$$u_\beta{}^k v_i{}^\beta = u_l{}^k v_i{}^l - u_n{}^k v_i{}^n = \delta_i{}^k - \frac{y^k}{F} F_{y^i}. \tag{3.64}$$

Plugging (3.54) into (3.23) and using (3.63) and (3.64) yields (3.62). □

**Lemma 3.3.3** *Having specified $K_{ij}{}^\alpha$ in (3.53), we obtain the following*

$$K_{ij}^\alpha u_\alpha{}^s = -y^s F_{y^i y^j} - F^2 F_{y^i y^j y^k} g^{ks}. \tag{3.65}$$

***Proof.*** From (3.18) and (3.60)

$$K_{ij}^\alpha = -v_l{}^\alpha F^2 F_{y^i y^j y^k} g^{kl}. \tag{3.66}$$

Together with (3.64) we have

$$K_{ij}^\alpha u_\alpha{}^s = y^s y^t g_{tl} F_{y^i y^j y^k} g^{kl} - F^2 F_{y^i y^j y^k} g^{ks} = -y^s F_{y^i y^j} - F^2 F_{y^i y^j y^k} g^{ks}.$$

□

**Corollary 3.3.4** *We can rewrite the formula for $K_{ij}^\alpha u_\alpha{}^s$ as*

$$K_{ij}^\alpha u_\alpha{}^s = -g^{ks} F \frac{\partial (F F_{y^i y^j})}{\partial y^k}. \tag{3.67}$$

***Proof.*** By using (3.6),(3.12) and (3.18) we have

$$g^{ij} F_j = \frac{y^i}{F}. \tag{3.68}$$

Together with (3.65) we get (3.67). □

**Lemma 3.3.5** *Having specified $G_{ijl}$ in (3.53), we obtain the following*

$$\begin{aligned} G_{ijl} =& (F_{x^k} F_{y^i y^j} + F F_{y^i y^j x^k}) u_l{}^k + (y^s F_{y^i y^j} + F^2 F_{y^i y^j y^r} g^{rs}) \\ & \times [\frac{1}{2}\frac{y^k}{F}\delta_{nl}(G_{ksn} - F_{y^s x^k} + F_{y^k x^s}) - \frac{1}{2} u_l{}^k (G_{skn} - F_{y^k x^s} + F_{y^s x^k})]. \end{aligned} \tag{3.69}$$

***Proof.*** *Substitute (3.62) and (3.3) into (3.53) and compare the coeffieients of $dx^k$. Then we contract it with $u_l{}^k$. Finally we plug (3.65) into this formula.* □

**Lemma 3.3.6** With the same notation, we have

$$G_{abn} = FF_{y^a y^b y^c} g^{cs}(F_{x^s} - y^l F_{y^s x^l}) + \frac{F_{y^a x^b}}{F} y^l F_{x^l} + y^l F_{y^a y^b x^l}. \qquad (3.70)$$

***Proof.*** *Using $u_n{}^k = \frac{y^k}{F}$ and (3.69).* □

**Lemma 3.3.7**

$$g^{ij} F_{y^i}(y^k F_{y^j x^k} - F_{x^j}) = 0. \qquad (3.71)$$

***Proof.*** By using (3.63) and (3.68). □

Put

$$G = \frac{1}{2} F^2, \qquad (3.72)$$

$$G_l = \frac{1}{2}(y^s G_{y^l x^s} - G_{x^l}), \qquad (3.73)$$

$$G^i = g^{il} G_l. \qquad (3.74)$$

**Definition 3.3.8** The local functions $G^i$ defined in (3.74) are called the *geodesic coefficient* ([Shen, 2001]).

**Remark** The projection of an integral curve of $G := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is called the *geodesic* of $(M, F)$.

**Lemma 3.3.9**

$$(i) \quad G_l = \frac{1}{2}(y^s F_{x^s} F_{y^l} + y^s F F_{y^l x^s} - F F_{x^l}). \qquad (3.75)$$

$$\begin{aligned}(ii)\quad -\frac{g_{st}}{F}\frac{\partial G^t}{\partial y^k} =& -\frac{1}{F}\frac{\partial G_s}{\partial y^k} + \frac{g^{tl}}{2}(y^a F_{y^l x^a} - F_{x^l}) \\ &\times (F_{y^k} F_{y^s y^t} + F F_{y^s y^t y^k} + F_{y^s} F_{y^t y^k}).\end{aligned} \tag{3.76}$$

***Proof.*** (i) A direct calculation.

(ii) By using (3.68), (3.71) and (3.75), we have

$$G_l \frac{g^{tl}}{F}\frac{\partial g_{st}}{\partial y^k} = \frac{g^{tl}}{2}(y^a F_{y^l x^a} - F_{x^l}) \times (F_{y^k} F_{y^s y^t} + F F_{y^s y^t y^k} + F_{y^s} F_{y^t y^k}). \tag{3.77}$$

Now (3.76) is an immedinat consequence of (3.74) and (3.77). □

**Lemma 3.3.10**

$$\begin{aligned}&\frac{1}{2}\frac{y^r}{F}F_{y^k}(G_{rsn} + F_{y^r x^s} + F_{y^s x^r}) - \frac{1}{2}(G_{skn} + F_{y^s x^k} + F_{y^k x^s}) \\ &= -\frac{g_{st}}{F}\frac{\partial G^t}{\partial y^k} + \frac{1}{2}\frac{F_{y^s}}{F}\{F F_{y^k y^t} g^{tl}(F_{x^l} - y^r F_{y^l x^r}) + F_{x^k} + y^r F_{y^k x^r}\}\end{aligned} \tag{3.78}$$

***Proof.*** It is easy to see

$$y^i F_{y^i y^j} = 0, \quad y^i F_{y^i y^j x^k} = 0; \quad y^r F_{y^r x^s} = F_{x^s}.$$

Together with (3.70), we obtain (3.78). □

**Theorem 3.3.11** (i) *The one forms ${\omega_\alpha}^n$ on $SM$ can be more compactly re-expressed as*

$${\omega_\alpha}^n = -{u_\alpha}^s \left[\frac{g_{st}}{F}\frac{\partial G^t}{\partial y^k}dx^k + F_{y^s y^k}dy^k\right]; \tag{3.79}$$

(ii) *Let $\nabla : \Gamma(p^*TM) \to \Gamma(p^*T^*M \otimes p^*TM)$ denote covariant differentiation, on $p^*TM$, relative to the Chern connection $({\omega_i}^j)$, i.e.*

$$\nabla e_k = {\omega_k}^i \otimes e_i,$$

*then $\nabla$ is metric compatible if and only if $(M, F)$ is Riemannian, hence $\nabla$ reduces to usual Levi-Civita connection.*

***Proof.*** (i) Plugging (3.78) into (3.62) and using (3.15) we obtain (3.79).

(ii) $\bigtriangledown$ preserves inner product if and only if $\omega_{ij} = \omega_{ji} = 0$. By using (3.61), it equivalents to $H_{ijk} = 0$. □

## 3.4 Horizontal and vertical subbundles of $SM$

Denote the fibres of $SM$ by $S_xM := p^{-1}(x)$. Thus

$$TS_xM = \{\omega^j = 0\},$$

where $\omega^1, \cdots, \omega^n$ is the dual frame field of a adapted frame. Hence

$$H := \{b \in TSM | \omega_n{}^\alpha(b) = 0\}$$

is orthogonal to $TS_xM$ with respect to Riemannian metric II. We call $H$ the *horizontal subbundle* of $SM$. Write $\bigcup\limits_{x \in M} TS_xM$ by $V$, we call it the *vertical subbundle* of $SM$[Mo, 1998; Mo, 2000].

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# Chapter 4

# Covariant Differentiation and Second Class of Geometric Invariants

In this chapter we will introduce the second class of geometric invariants on a Finsler manifold. These quantities describes the changing rates of the first class of geometric invariants along geodesics, or Hilbert form. First, we define the following.

## 4.1 Horizontal and vertical covariant derivatives

Let $(M, F)$ be a Finsler manifold and $\omega^1, \cdots, \omega^n$ the adapted frame fields on the dual Finsler bundle (cf. Lemma 3.1.2). Let $\omega_i{}^j$ be Chern connection 1-form. Thus

$$\omega^1 \wedge \cdots \wedge \omega^n \wedge \omega_n{}^1 \wedge \cdots \wedge \omega_n{}^{n-1} \neq 0, \tag{4.1}$$

$$d\omega^j = \omega^k \wedge \omega_k{}^j, \tag{4.2}$$

$$\delta_{ik}\omega_j{}^k + \delta_{jk}\omega_i{}^k = -2H_{ij\alpha}\omega_n{}^\alpha. \tag{4.3}$$

By using $\omega_i{}^j$, we can define the covariant differentiation of the tensors on $P^*TM$. For instance, let $f : SM \to R$ be a smooth function. Write

$$Df := df := f_{|i}\omega^i + f_{;\alpha}\omega_n{}^\alpha, \tag{4.4}$$

we call $|i$ (resp. $;\alpha$) the horizontal (resp.vertical) covariant derivative. Recall that

$$H = \{\omega_n{}^\alpha = 0\}, \quad V = \{\omega^i = 0\}.$$

Consider the Cartan tensor $H_{ij\alpha}$ on $(M, F)$ and define

$$\begin{aligned} DH_{ij\alpha} &:= dH_{ij\alpha} - \sum H_{kj\alpha}\omega_i{}^k - \sum H_{ik\alpha}\omega_j{}^k - \sum H_{ijk}\omega_\alpha{}^k \\ &:= \sum H_{ij\alpha|k}\omega^k + \sum H_{ij\alpha;\beta}\omega_n{}^\beta. \end{aligned} \tag{4.5}$$

Usually, we call $|n$ the covariant derivative along the Hilbert form. Similarly, we can define the covariant derivative for other type tensors on $p^*TM$.

## 4.2 The covariant derivative along geodesic

Let $\Pi$ be the Sasaki-type Riemannian metric on $SM$ and $\flat : \Gamma(SM) \to T^*(SM)$ the musical isomorphism induced by $G$, i.e.

$$\Pi(X, Y) = X^\flat(Y). \tag{4.6}$$

Denote the horizontal left of $y$ by $\hat{y}$. Then

$$\left(\frac{\hat{y}}{F}\right)^\flat = \omega. \tag{4.7}$$

In fact

$$\hat{y} = y^i\left(\frac{\partial}{\partial x^i} - \frac{\partial G^j}{\partial y^i}\frac{\partial}{\partial y^j}\right) \tag{4.8}$$

and

$$G^j = \frac{1}{4}g^{jl}\left[2\frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l}\right]y^iy^k. \tag{4.9}$$

Note that $G^j$ are the geodesic coefficients[Shen, 2001], i.e., we have the following

**Proposition 4.2.1**

$$\begin{aligned} &\frac{1}{2}g^{il}(y)\left[(\frac{F^2}{2})_{y^lx^k}(y)y^k - (\frac{F^2}{2})_{x^l}(y)\right] \\ &= \frac{1}{4}g^{il}(y)\left[2\frac{\partial g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l}(y)\right]y^jy^k. \end{aligned} \tag{4.10}$$

***Proof.***

$$\begin{aligned}\sum \frac{\partial g_{jl}}{\partial x^k} y^j &= \sum \frac{\partial}{\partial x^k}\left[\left(\frac{F^2}{2}\right)_{y^j y^l} y^j\right] \\ &= \sum \frac{\partial}{\partial x^k}\left[\left(\frac{F^2}{2}\right)_{y^l y^j} y^j\right] \\ &= \left(\frac{F^2}{2}\right)_{y^l x^k},\end{aligned}$$

$$\frac{\partial g_{jk}}{\partial x^l} y^j y^k = \frac{\partial}{\partial x^l}(\sum g_{jk}(y) y^j y^k) = \frac{\partial}{\partial x^l} F^2.$$

These formulas imply (4.10). □

A $C^\infty$-curve $c(t)$, $t \in I$, is called a *geodesic* if the canonical lift $\dot{c}(t)$ in $TM$ is an integral curve of the vector field

$$y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}. \tag{4.11}$$

**Lemma 4.2.2** *Let $c(t)$ be a geodesic on $(M, F)$, and $f : SM \to R$ a smooth function. Then, on $(x, y)$, we have*

$$Fdf \equiv \frac{d}{dt}[f(\dot{c}(t))]_{t=0}\omega \quad mod \quad \omega_\alpha, \quad {\omega_n}^\alpha$$

*where $\dot{c}(0) = y$.*

***Proof.*** It is sufficient to show that, from (4.7)

$$\hat{y}(f) = \frac{d}{dt}[f(\dot{c}(t))]_{t=0}.$$

In fact,

$$\begin{aligned}\frac{d}{dt}[f(\dot{c}(t))] &= \frac{\partial f}{\partial x^i}\dot{c}^i + \frac{\partial f}{\partial y^i}\ddot{c}^i \\ &= \frac{\partial f}{\partial x^i}\dot{c}^i - 2\frac{\partial f}{\partial y^i}G^i(\dot{c}) \\ &= \dot{c}^i(\frac{\partial f}{\partial x^i} - \frac{\partial G^j}{\partial y^i}(\dot{c})\frac{\partial f}{\partial y^j}) \\ &= \hat{\dot{c}}(f)\end{aligned}$$

from (4.8),(4.11) and the following

$$y^i \frac{\partial G^j}{\partial y^i} = 2G^j.$$

□

**Remark** Sometimes we denote the covariant derivative along any geodesic, i.e. the Hilbert form, by " $\cdot$ ". For example, we have $\dot{f} := f_{|n}$ and $\dot{H}_{ij\alpha} := H_{ij\alpha|n}$, ect. Note that $\dot{f} : SM \to R$ is a function and $\dot{H}_{ij\alpha}$ is a three order tensor.

## 4.3 Landsberg curvature

Let $H_{ij\alpha}$ be the Cartan tensor of $(M, F)$(cf. (3.57)) . Put

$$L_{\alpha\beta\gamma} := -\dot{H}_{\alpha\beta\gamma}$$

and

$$L := \sum L_{\alpha\beta\gamma}\omega^\alpha \otimes \omega^\beta \otimes \omega^\gamma$$

we call $L$ the *Landsberg curvature.*

**Definition 4.3.1** Let $(M, F)$ be a Finsler manifold. If it has vanishing Landsberg curvature, then we call it a *Landsberg manifold.*

Let $(N, h)$ be a Riemannian manifold, and $\phi : M \looparrowright (N, h)$ an isometric immersion. Let $\tilde{\nabla}$ denote the Levi-Civita connection for $h$ and $\nabla$ the Levi-Civita connection for induced Riemannian metric $\phi^*h$. Then

$$\tilde{\nabla}_{\phi_* X}\phi_* Y = \phi_*(\nabla_X Y) + B(X, Y)$$

for $X, Y \in \Gamma(TM)$, where

$$\phi_*(\nabla_X Y) \in \Gamma(\phi_* TM) \quad \text{and} \quad B(X, Y) \in \Gamma(T^\perp M).$$

We call $B$ the *second fundamental form* for $\phi$ and $trace_{\phi^* h} B$ the *mean curvature* for $\phi$, denote by $H$. $\phi$ is called to be *minimal* (resp. *totally geodesic*), if its mean curvature (resp. its second fundamental form) vanishes. It is easy to see that $\phi : M \looparrowright (N, h)$ is totally geodesic if and only if $\phi$ carries any geodesic on $M$ into the geodesic on $N$. The Landsberg spaces have the following interesting geometric characterization.

**Theoem 4.3.2**([Ai, 1993]) *Let $(M, F)$ be a Finsler manifold and $SM$ its projective sphere bundle. Then $(M, F)$ is of Landsberg type if and only if all projective spheres in $SM$ are totally geodesic.*

Similarly, set

$$J_\alpha := -\eta_{\alpha|n} = -\dot{\eta}_\alpha$$

where $\eta = \sum_\alpha \eta_\alpha e_\alpha$ denote the Cartan form. We call $J_\alpha \omega^\alpha$ the *mean Landsberg curvature* and denote it by $J$.

**Definition 4.3.3** A Finsler manifold is said to be of *weak Landsberg type* if $J = 0$.

Z.Shen gives following geometric characterization of weak Landsberg manifold

**Theoem 4.3.4**([Shen, 1994]) *A Finsler manifold is of weak Landsberg type if and only if all projective spheres in the projective sphere bundle are minimal.*

Let $Vol(x)$ denote the volume of the unit Finsler sphere ([Bao and Chern, 1996])

$$\{y \in T_x M | F(x, y) = 1\}.$$

For functions $a^i = a^i(x)$, we have

$$a^i \frac{\partial Vol(x)}{\partial x^i} = \int_{F(x,y)=1} g\left(\sum J_\alpha e_\alpha, a^i \frac{\partial}{\partial x^i}\right) dV, \tag{4.12}$$

where

$$dV := \sqrt{det(g_{ij})} \sum_{k=1}^{n} (-1)^{k-1} y^k dy^1 \wedge \cdots \wedge \hat{dy^k} \wedge \cdots \wedge dy^n.$$

It follows that

**Theoem 4.3.5**([Bao and Chern, 1996]) *Let $(M, F)$ be a weak Landsberg manifold. Then Vol(x)=constant.*

**Example** Define $F : TR^2 \to [0,\infty)$ by

$$F(x,y) := \sqrt{p^2 + q^2 + \lambda(x)(p^4 + q^4)}$$

where $y = (p,q) \in T_xR^2$ and $\lambda$ is a smoothly non-negative function. It's easy to see

$$(g_{ij}) = \begin{pmatrix} 1 + \frac{\lambda p^2(p^4+3q^4)}{(p^4+q^4)^{\frac{3}{2}}} & \frac{-2\lambda p^3 q^3}{(p^4+q^4)^{\frac{3}{2}}} \\ \frac{-2\lambda p^3 q^3}{(p^4+q^4)^{\frac{3}{2}}} & 1 + \frac{\lambda q^2(3p^4+q^4)}{(p^4+q^4)^{\frac{3}{2}}} \end{pmatrix}$$

(cf. Example 1.3.2). Hence

$$\sqrt{\det(g_{ij})} = \sqrt{1 + \lambda \frac{(p^2+q^2)^3}{(p^4+q^4)^{\frac{3}{2}}} + \lambda^2 \frac{3p^2q^2}{p^4+q^4}}.$$

By using (4.12), one finds that as the value of $\lambda(x)$ is increased from 0 to $\infty$, the value of $Vol(x)$ decreases monotonically from the expected $2\pi$ to 5.4414 ([Bao and Shen, 2002]).

D.Bao and S.S.Chern give the Gauss-Bonnet theorem for Finsler spaces whose $Vol(x)$ are constant. Together with Theorem 4.3.5 we have the following

**Theoem 4.3.6** *Let $(M,F)$ be an oriented compact 2k-dimentional weak Landsberg manifolds. Then*

$$\int_M [X]^*\{Pf(\Omega) + \mathcal{F}\} = \frac{Vol(Finsler S^{2k-1})}{Vol(S^{2k-1})}\chi(M)$$

*where $[X] : M \to SM$ is an arbitrary section with isolated singularities and $Pf(\Omega)$ is the Pfaffian of the curvature for the Chern connection and $\mathcal{F}$ the additional term in [Bao and Chern, 1996].*

## 4.4 $S$-curvature

Let $(M,F)$ be a Finsler manifold and $\tau$ its distortion. Put

$$S = \dot{\tau}F,$$

we call $S$ the *$S$-curvature* of $(M,F)$. Clearly

$$S(x,\lambda y) = \lambda S(x,y), \quad \lambda > 0. \tag{4.13}$$

In a standard local coordinate systerm $(x^i, y^i)$, let $G^i = G^i(x,y)$ denote the geodesic coefficients of $F$.

**Lenmma 4.4.1** *The $S$-curvature of $(M,F)$ can be expressed as*

$$S = \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial x^k}y^k - 2F^{-1}A_k G^k - \frac{y^j}{\sigma(x)}\frac{\partial \sigma}{\partial x^j}. \tag{4.14}$$

***Proof.*** From Lemma 4.2.2

$$S(\dot{c}) = \frac{d}{dt}[\tau(\dot{c}(t))],$$

where $c(t)$ is a geodesic of $(M,F)$. By the definition of distortion, we have

$$S(\dot{c}) = \frac{d}{dt}\left[ln\frac{\sqrt{det(g_{ij}(\dot{c}))}}{\sigma(c)}\right] = \frac{d}{dt}\left(ln\sqrt{det(g_{ij}(\dot{c}))}\right) - \frac{d}{dt}\left(ln\sigma(c)\right).$$

A similar calculation of (2.2) gives

$$S(\dot{c}) = \frac{1}{2}g^{ij}(\dot{c})\frac{\partial g_{ij}}{\partial x^k}(\dot{c})\dot{c}^k + g^{ij}(\dot{c})F^{-1}A_{ijk}\ddot{c}^k - \frac{\dot{c}^j}{\sigma(c)}\frac{\partial \sigma}{\partial x^j}(c).$$

By geodesic equation,

$$\ddot{c}^i + 2G^i(\dot{c}) = 0,$$

hence

$$S(\dot{c}) = \frac{1}{2}g^{ij}(\dot{c})\frac{\partial g_{ij}}{\partial x^k}(\dot{c})\dot{c}^k - 2F^{-1}A_k G^k(\dot{c}) - \frac{\dot{c}^j}{\sigma(c)}\frac{\partial \sigma}{\partial x^j}(c),$$

where $A_k$ is the Cartan form. $\square$

**Definition 4.4.2** Let $(M,F)$ be a Finsler manifold. $F$ is said to have *isotropic $S$-curvature* if

$$S = (n+1)cF,$$

where $c = c(x)$ is a scalar function on $M$. In particular, it is said to have *constant $S$-curvature* if $S = (n+1)cF$ for some constant $c$.

**Example 4.4.3** On the unit ball $B^n$ the following Randers metrics

$$F_a(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} \pm \frac{\langle x,y\rangle}{1-|x|^2} \pm \frac{\langle a,y\rangle}{1+\langle a,x\rangle}, \; y \in T_xR^n$$

have constant $S$-curvature $\frac{1}{2}(n+1)$, where $a \in R^n$ is a constant vector with $|a| < 1$. Take $a = 0$. The resulting Randers metrics are the well-known Funk metrics on the unit ball $B^n$[Mo, 2005].

**Example 4.4.4** Let $\zeta$ be an arbitrary constant and

$$\Omega = \begin{cases} R^n, & \text{if } \zeta \geq 0; \\ B\left(\sqrt{-\frac{1}{\zeta}}\right), & \text{if } \zeta < 0. \end{cases} \tag{4.15}$$

Define $F : T\Omega \to [0,\infty)$ by

$$\alpha(x,y) := \frac{\sqrt{\kappa^2\langle x,y\rangle^2 + \epsilon|y|^2(1+\zeta|x|^2)}}{1+\zeta|x|^2} \tag{4.16}$$

and

$$\beta(x,y) := \frac{\kappa\langle x,y\rangle}{1+\zeta|x|^2}, \tag{4.17}$$

where $\epsilon$ is an arbitrary positive constant, and $\kappa$ is an arbitrary constant. Thus $F$ has isotropic $S$-curvature (cf. Proposition 4.4.6), i.e. $S = (n+1)cF$ where

$$c = \frac{\kappa}{2[\epsilon + (\epsilon\zeta + \kappa^2)|x|^2]}. \tag{4.18}$$

Notice that when $\zeta = -1$, $\kappa = \pm 1$ and $\epsilon = 1$ our metrics reduce the famous Funk metrics.

The following result gives an equivalent condition for a Randers metric to be of isotropic $S$-curvature.

**Proposition 4.4.5**([Chen and Shen, 2003]) *Let $F = \alpha + \beta$ be a Randers metric on an n-dimentional manifold $M$ where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$. Then $F$ has isotropic $S$-curvature, i.e.*

$$S = (n+1)cF$$

*if and only if*

$$r_{ij} = 2c(x)(a_{ij} - b_ib_j) - b_is_j - b_js_i, \tag{4.19}$$

*where*

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}); \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$$

$$s_i := b_k a^{kj} s_{ji}, \quad (a^{kj}) = (a_{kj})^{-1}$$

$$b_{i|j} := \frac{\partial b_i}{\partial x^j} - b_k \gamma_{ij}^k$$

*and* $\gamma_{ij}^k$ *is the Christoffel symbol with respect to the Riemannian metric* $\alpha$. *In particular, if* $\beta$ *is closed, then* $S = (n+1)c(x)F$ *holds if and only if*

$$b_{i|j} = 2c(x)(a_{ij} - b_i b_j). \tag{4.20}$$

***Proof.*** For the sake of simplicity, we choose an orthonormal basis for $T_x M$ such that $a_{ij} = \delta_{ij}$. Let

$$\rho := ln\sqrt{1 - \|\beta\|_\alpha^2} \tag{4.21}$$

and $d\rho = \rho_i dx^i$, i.e.

$$\rho_i = -\frac{\sum b_j b_{j|i}}{1 - \|\beta\|_\alpha^2},$$

where $\|\beta\|_\alpha^2$ is defined in Section 1.3. According to (20) in [Shen, 2003], we have the following formula for $S$-curvature,

$$S = (n+1)(P - d\rho) \tag{4.22}$$

where

$$P := \frac{1}{2F}(r_{ij}y^i y^j - 2\alpha s_i y^i).$$

It follows that $S = (n+1)cF$ if and only if

$$r_{ij}y^i y^j - 2\alpha s_i y^i = 2(\alpha + \beta)\rho_i y^i + 2c(\alpha + \beta)^2. \tag{4.23}$$

(4.23) is equivalent to the following two equations

$$r_{ij} = b_j \rho_i + b_i \rho_j + 2c(a_{ij} - b_i b_j), \tag{4.24}$$

$$-s_i = \rho_i + 2cb_i. \tag{4.25}$$

First we assume that $S=(n+1)cF$. Then (4.24) and (4.25) hold. Plugging (4.25) into (4.24) gives (4.19).

Now we assume that (4.19) holds. Note that

$$\sum_j s_j b_j = \sum_{i,j} b_i s_{ij} b_j = 0.$$

Contracting (4.19) with $b_j$ yield

$$\sum b_j r_{ij} = 2c(1-\|\beta\|^2_\alpha)b_i - \|\beta\|^2_\alpha s_i \tag{4.26}$$

that is,

$$\sum b_j b_{i|j} + \sum b_j b_{j|i} = 4c(1-\|\beta\|^2_\alpha)b_i - \|\beta\|^2_\alpha \sum (b_j b_{i|j} + b_j b_{j|i}).$$

We obtain

$$\sum b_j b_{i|j} = 4cb_i - \frac{1+\|\beta\|^2_\alpha}{1-\|\beta\|^2_\alpha}\sum b_j b_{j|i}. \tag{4.27}$$

It follows from (4.22) and (4.27)

$$s_i = -2cb_i + \frac{1}{2}\left(\sum b_j b_{j|i} + \frac{1+\|\beta\|^2_\alpha}{1-\|\beta\|^2_\alpha}\sum b_j b_{j|i}\right) = -2cb_i - \rho_i. \tag{4.28}$$

Thus

$$r_{ij}y^i y^j - 2\alpha s_i y^i = 2F\rho_i y^i + 2cF^2, \tag{4.29}$$

we obtain

$$S = (n+1)(cF^2 + \rho_i y^i - \rho_i y^i) = (n+1)cF.$$

□

**Proposition 4.4.6** *Let $F=\alpha+\beta : T\Omega \to [0,\infty)$ be any function given in (4.16) and (4.17). Then $F$ has the following properties:*

(i) *$F$ is a Randers metric and $\beta$ an exact form;*

(ii) *$F$ has isotropic S-curvature, i.e. $S=(n+1)cF$, where $c$ is given in (4.18).*

***Proof.*** Set

$$\omega := 1+\zeta|x|^2, \quad \varrho^2 := \epsilon\zeta + \kappa^2 \tag{4.30}$$

$$\alpha^2 = a_{ij}y^i y^j, \quad \beta = b_i y^i. \tag{4.31}$$

Then

$$a_{ij} = \frac{\epsilon\delta_{ij}}{\omega} + \frac{\kappa^2 x^i x^j}{\omega^2}, \quad b_i = \frac{\kappa x^i}{\omega}. \tag{4.32}$$

Since $1 + \zeta|x|^2 > 0$, it is easy to see that if $y \neq 0$ then

$$\alpha^2 = \frac{\kappa^2\langle x, y\rangle^2 + \epsilon|y|^2 + \epsilon\zeta|x|^2|y|^2}{\omega^2} \geq \frac{\epsilon(1 + \zeta|x|^2)|y|^2}{\omega^2} > 0.$$

Moreover $\alpha(x, y) = 0$ if and only if $y = 0$. It implies that $\alpha$ is a Riemannian metric. Write

$$(a^{ij}) = (a_{ij})^{-1}. \tag{4.33}$$

By using (4.32), we have

$$\delta_i^j = a_{ik}a^{kj} = \sum_i \left(\frac{\epsilon\delta_{ik}}{\omega} + \frac{\kappa^2 x^i x^k}{\omega^2}\right)a^{kj} = \frac{\epsilon a^{ij}}{\omega} + \frac{\kappa^2 x^i t^j}{\omega^2}, \tag{4.34}$$

where

$$t^j = \sum_i x^i a^{ij}. \tag{4.35}$$

(4.34) is equivalent to

$$\epsilon\omega a^{ij} = \omega^2\delta_i^j - \kappa^2 x^i t^j. \tag{4.36}$$

From (4.35) and (4.36) we have

$$\epsilon\omega t^j = \epsilon\omega\sum_i x^i a^{ij} = \omega^2 x^j - \kappa^2|x|^2 t^j. \tag{4.37}$$

It implies that

$$t^j = \frac{\omega^2 x^j}{\epsilon + \varrho^2|x|^2} = \frac{2c\omega^2 x^j}{\kappa}, \tag{4.38}$$

where $c := c(x)$ is defined in (4.18). Substituting (4.38) into (4.36), we get

$$a^{ij} = \frac{1}{\epsilon\omega}(\omega^2\delta_i^j - \kappa^2 x^i t^j) = \frac{\omega}{\epsilon}\left(\delta_i^j - \frac{\kappa^2 x^i x^j}{\epsilon + \varrho^2|x|^2}\right) = \frac{\omega}{\epsilon}(\delta^{ij} - 2c\kappa x^i x^j). \tag{4.39}$$

Using (4.32) and (4.39), we have

$$\|\beta\|_\alpha^2 = a^{ij}b_i b_j = \frac{\omega}{\epsilon}(\delta^{ij} - 2c\kappa x^i x^j)\frac{\kappa x^i}{\omega}\frac{\kappa x^j}{\omega} = \frac{\kappa^2|x|^2}{\epsilon + \zeta^2|x|^2} < 1 \tag{4.40}$$

by $1 + \zeta|x|^2 > 0$. It follows that $F := \alpha + \beta$ is a Randers metric.

Consider $\beta$ as a 1-form, then

$$\beta = \frac{\kappa \sum_i x^i dx^i}{1+\zeta^2|x|^2} = d\left[\frac{\kappa}{2\zeta} ln(1+\zeta^2|x|^2)\right].$$

Thus $\beta$ is exact. In particular, we have $d\beta = 0$. From (4.30), we get

$$\frac{\partial \omega}{\partial x^i} = 2\zeta x^i. \tag{4.41}$$

Together with (4.32) we have

$$\frac{\partial a_{ij}}{\partial x^k} = \frac{\kappa^2}{\omega^2}(x^i\delta_{jk} + x^j\delta_{ik}) - \frac{2\epsilon\zeta}{\omega^2}\delta_{ij}x^k - \frac{4\zeta\kappa^2}{\omega^3}x^i x^j x^k. \tag{4.42}$$

This implies that

$$\begin{aligned}\gamma_{ijk} := &\tfrac{1}{2}\left(\frac{\partial a_{ij}}{\partial x^k} + \frac{\partial a_{ik}}{\partial x^j} - \frac{\partial a_{jk}}{\partial x^i}\right)\\ = &\frac{\varrho^2}{\omega^2}\delta_{jk}x^i - \frac{\epsilon\zeta}{\omega^2}(\delta_{ij}x^k + \delta_{ik}x^j) - \frac{2\zeta\kappa^2}{\omega^3}x^i x^j x^k.\end{aligned} \tag{4.43}$$

By using (4.39) and (4.43), we get

$$\gamma^i_{jk} = a^{il}\gamma_{ljk} = \frac{\varrho^2\delta_{jk}}{\epsilon+\varrho^2|x|^2}x^i - \frac{\zeta}{\omega}(\delta^i_j x^k + \delta^i_k x^j). \tag{4.44}$$

Note that $b_i$ satisfies (4.32), we have

$$\gamma^i_{jk}b_k = \frac{\kappa}{\omega}\left[\frac{\varrho^2|x|^2\delta_{ij}}{\epsilon+\varrho^2|x|^2} - \frac{2\zeta x^i x^j}{\omega}\right] \tag{4.45}$$

and

$$\frac{\partial b_i}{\partial x^j} = \kappa\frac{\omega\delta_{ij} - 2\zeta x^i x^j}{\omega^2}. \tag{4.46}$$

Here we have used (4.41). By the definition of $b_{i|j}$, we get

$$b_{i|j} = \frac{\epsilon\kappa\delta_{ij}}{\omega(\epsilon+\varrho^2|x|^2)} = \frac{2\epsilon c}{\omega}\delta_{ij}. \tag{4.47}$$

On the other hand, from (4.32), we have

$$a_{ij} - b_i b_j = \frac{\epsilon}{\omega}\delta_{ij}.$$

This implies that

$$b_{i|j} = 2c(x)(a_{ij} - b_i b_j),$$

where $c(x)$ is defined in (4.18). Note that $\beta$ is closed. From (4.20) we get F has isotropic S-curvature. $\square$

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# Chapter 5

# Riemann Invariants and Variations of Arc Length

Besides the non-Riemannian quantities, there are several important geometric quantities in a Finsler manifold. These invariants are the natural extension of geometric quantities in a Riemannian manifold. We call them the Riemannian invariants.

The flag curvature is the most important Riemannian quantity in Finsler geometry because it lies in second variation formula of arc length and takes the place of the sectional curvature in the Riemannian case.

In this chapter we will introduce Riemannian invariants in a Finsler manifold by discussing the Chern curvature 2-form. Then we will give the first and second variation formulas of arc length.

## 5.1 Curvatures of Chern connection

Differential (4.2) and using (4.2) one deduces

$$\begin{aligned} 0 &= d^2\omega^j \\ &= d(\omega^k \wedge \omega_k{}^j) \\ &= (d\omega^k) \wedge \omega_k{}^j - \omega^k \wedge d\omega_k{}^j \\ &= -\omega^i \wedge [d\omega_i{}^j - \omega_i{}^k \wedge \omega_k{}^j]. \end{aligned} \tag{5.1}$$

Put

$$\Omega_i{}^j := d\omega_i{}^j - \omega_i{}^k \wedge \omega_k{}^j \in \Gamma(\wedge^2 SM). \tag{5.2}$$

We call $\Omega_i{}^j$ the *curvature 2-form of Chern connection.* By (4.1), $\Omega_i{}^j$ can be expressed by

$$\Omega_i{}^j := \frac{1}{2} R_i{}^j{}_{kl}\omega^k \wedge \omega^l + P_i{}^j{}_{k\alpha}\omega^k \wedge \omega_n{}^\alpha + Q_i{}^j{}_{\alpha\beta}\omega_n{}^\alpha \wedge \omega_n{}^\beta. \tag{5.3}$$

From (5.1), (5.2) and (5.3) we have

$$\begin{aligned} 0 &= -\omega^i \wedge \Omega_i{}^j \\ &= -\omega^i \wedge Q_i{}^j{}_{\alpha\beta}\omega_n{}^\alpha \wedge \omega_n{}^\beta \quad \mathrm{mod}\ \omega^i \wedge \omega^k \wedge \omega^l, \quad \omega^i \wedge \omega^k \wedge \omega_n{}^\alpha. \end{aligned}$$

It follows that

$$Q_i{}^j{}_{\alpha\beta}\omega^i \wedge \omega_n{}^\alpha \wedge \omega_n{}^\beta = 0.$$

Thus we obtain $Q_i{}^j{}_{\alpha\beta} = Q_i{}^j{}_{\beta\alpha}$, i.e.

$$Q_i{}^j{}_{\alpha\beta}\omega_n{}^\alpha \wedge \omega_n{}^\beta = 0.$$

Substituting this into (5.3) yields

$$\Omega_i{}^j = \frac{1}{2}R_i{}^j{}_{kl}\omega^k \wedge \omega^l + P_i{}^j{}_{k\alpha}\omega^k \wedge \omega_n{}^\alpha \tag{5.4}$$

we agree on

$$R_i{}^j{}_{kl} = -R_i{}^j{}_{lk}. \tag{5.5}$$

**Definition 5.1.1** ([Chern, 1992]) Let $(M, F)$ be a Finsler manifold. We call $R_i{}^j{}_{kl}$ (resp. $P_i{}^j{}_{k\alpha}$) the *Riemannian* (resp. *Minkowskian) curvature.*

Plugging (5.4) into (5.1) yields

$$\frac{1}{2}\sum R_i{}^j{}_{kl}\omega^i \wedge \omega^k \wedge \omega^l \equiv 0 \quad \mathrm{mod}\ \omega^i \wedge \omega^k \wedge \omega_n{}^\alpha.$$

Thus one gets the First Bianchi Identity

$$R_i{}^j{}_{kl} + R_k{}^j{}_{li} + R_l{}^j{}_{ik} = 0. \tag{5.6}$$

Similarly, from (5.1),(5.2) and (5.4), we have

$$\sum P_i{}^j{}_{k\alpha}\omega^i \wedge \omega^k \wedge \omega_n{}^\alpha \equiv 0 \quad \mathrm{mod}\ \omega^i \wedge \omega^k \wedge \omega^l.$$

Hence the Minkowskian curvature satisfies that

$$P_i{}^j{}_{k\alpha} = P_k{}^j{}_{i\alpha}. \tag{5.7}$$

Setting $\omega_{ij} = \sum \delta_{jk}\omega_i{}^k$. Substituting this into (4.3) yields

$$\omega_{ij} + \omega_{ji} = -2\sum_\alpha H_{ij\alpha}\omega_n{}^\alpha, \tag{5.8}$$

where $\{H_{ij\alpha}\}$ is the Cartan tensor of $(M, F)$. Differentiating (5.8), we have

$$d\omega_{ij} + d\omega_{ji} = -2\sum_{\alpha}(dH_{ij\alpha}) \wedge \omega_n^\alpha - 2\sum H_{ij\alpha} d\omega_n^\alpha. \tag{5.9}$$

Put $\Omega_{ij} = \sum \delta_{jk}\Omega_i{}^k$. From (5.2), we obtain

$$d\omega_{ij} = \Omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj},$$

$$d\omega_n{}^\alpha = \Omega_n{}^\alpha + \sum_\beta \omega_n{}^\beta \wedge \omega_\beta{}^\alpha \; (\because \omega_n{}^n = 0).$$

Plugging them into (5.9) yields

$$\Omega_{ij} + \Omega_{ji} + 2H_{ijk}\Omega_n{}^k + (I) = 0, \tag{5.10}$$

where

$$(I) = \underbrace{\sum_k \omega_{ik} \wedge \omega_{kj} + \sum_n \omega_{jk} \wedge \omega_{ki}}_{\text{(II)}} + 2\sum (dH_{ij\alpha}) \wedge \omega_n{}^\alpha$$
$$+ 2\sum H_{ij\alpha}\omega_n{}^\beta \wedge \omega_\beta{}^\alpha.$$

By using (5.8) we get

$$\begin{aligned}\text{(II)} &= \textstyle\sum_k \omega_{ik} \wedge (-\omega_{jk} - 2\sum H_{kj\alpha}\omega_n{}^\alpha) \\ &\quad + \textstyle\sum_k \omega_{jk} \wedge (-\omega_{ik} - 2\sum H_{ki\alpha}\omega_n{}^\alpha) \\ &= -2\textstyle\sum_k (H_{kj\alpha}\omega_{ik} + H_{ki\alpha}\omega_{jk}) \wedge \omega_n{}^\alpha.\end{aligned}$$

Thus

$$(I) = 2\sum_\alpha (DH_{ij\alpha}) \wedge \omega_n \alpha,$$

where $\{DH_{ij\alpha}\}$ is the covariant derivative of Cartan tensor defined in (4.5). Plugging this into (5.10) yields

$$\Omega_{ij} + \Omega_{ji} + 2\sum_k H_{ijk}\Omega_n{}^k + 2\sum_\alpha (DH_{ij\alpha}) \wedge \omega_n{}^\alpha = 0. \tag{5.11}$$

Together with (5.4) and (4.5) we have

$$
\begin{aligned}
0 = &\frac{1}{2}R_{ijkl}\omega^k \wedge \omega^l + P_{ijk\alpha}\omega^k \wedge \omega_n{}^\alpha + \frac{1}{2}R_{jikl}\omega^k \wedge \omega^l + P_{jik\alpha}\omega^k \wedge \omega_n{}^\alpha \\
&+ 2H_{ij\alpha}\left(\frac{1}{2}R_n{}^\alpha{}_{kl}\omega^k \wedge \omega^l + P_n{}^\alpha{}_{k\beta}\omega^k \wedge \omega_n{}^\beta\right) \\
&+ 2\left(H_{ij\alpha|k}\omega^k + H_{ij\alpha;k}\omega_n{}^\beta\right) \wedge \omega_n{}^\alpha
\end{aligned}
\tag{5.12}
$$

It follows that

$$(R_{ijkl} + R_{jikl} + 2H_{ij\alpha}R_n{}^\alpha{}_{kl})\omega^k \wedge \omega^l \equiv 0 \mod \omega^k \wedge \omega_n{}^\alpha,\ \omega_n{}^\alpha \wedge \omega_n{}^\beta.$$

By using (5.5), we have

$$R_{ijkl} + R_{jikl} + 2H_{ij\alpha}R_n{}^\alpha{}_{kl} = 0. \tag{5.13}$$

We can re-express (5.12) as

$$\left(P_{ijk\alpha} + P_{jik\alpha} + 2H_{ij\beta}P_{n}{}^{\beta}_{k\alpha} + 2H_{ij\alpha|k}\right)\omega^k \wedge \omega_n^\alpha \equiv 0 \mod \omega^i \wedge \omega^j,\ \omega_n^\alpha \wedge \omega_n^\beta.$$

Hence

$$P_{ijk\alpha} + P_{jik\alpha} + 2\Sigma H_{ij\beta}P_n{}^\beta{}_{k\alpha} + 2H_{ij\alpha|k} = 0. \tag{5.14}$$

Finally, we take the components of $\omega_n{}^\alpha \wedge \omega_n{}^\beta$ in (5.12) and obtain

$$H_{ij\alpha;\beta} = H_{ij\alpha;\beta}.$$

The following Proposition tells us the Minkowski curvature is a non-Riemannian quantity:

**Proposition 5.1.2** *The Minkowski curvature $P_{ijk\alpha}$ can be represent as follows*

$$P_{ijk\alpha} = -H_{ij\beta}L^\beta{}_{k\alpha} + H_{ki\beta}L^\beta{}_{j\alpha} - H_{ik\beta}L^\beta{}_{i\alpha} - H_{ij\alpha|k} + H_{ki\alpha|j} - H_{jk\alpha|i}.$$

***Proof.*** Putting

$$E_{ijk\alpha} := \frac{P_{ijk\alpha} + P_{jik\alpha}}{2}$$

and using (5.7) we have

$$\begin{aligned} P_{ijk\alpha} &= \frac{P_{ijk\alpha}+P_{jik\alpha}}{2}+\frac{P_{jki\alpha}+P_{kji\alpha}}{2}-\frac{P_{kij\alpha}+P_{ikj\alpha}}{2} \\ &= E_{ijk\alpha}+E_{jki\alpha}-E_{kij\alpha}. \end{aligned} \tag{5.15}$$

From (5.14) we have

$$E_{ijk\alpha} = -H_{ij\beta}P_n{}^{\beta}{}_{k\alpha} - H_{ij\alpha|k}.$$

Plugging it into (5.15) yields

$$\begin{aligned} P_{ijk\alpha} = &-H_{ij\beta}P_n{}^{\beta}{}_{k\alpha} - H_{ij\alpha|k} - H_{jk\beta}P_n{}^{\beta}{}_{i\alpha} - H_{jk\alpha|i} \\ &+H_{ki\beta}P_n{}^{\beta}{}_{j\alpha} + H_{ki\alpha|j}\,. \end{aligned} \tag{5.16}$$

On the other hand, from (4.5) we have

$$\begin{aligned} H_{ni\alpha|j}\omega^j &\equiv dH_{ni\alpha} - H_{ki\alpha}\omega_n{}^k - H_{nk\alpha}\omega_i{}^k - H_{nik}\omega_\alpha{}^k \mod \omega_n{}^\beta \\ &= -H_{\beta i\alpha}\omega_n{}^\beta \equiv 0 \mod \omega_n{}^\beta \end{aligned}$$

It follows that

$$H_{ni\alpha|j} = 0. \tag{5.17}$$

Combining with (5.16) one yields that

$$\begin{aligned} P_{n\beta n\alpha} &= -H_{n\beta\gamma}P_n{}^{\gamma}{}_{n\alpha} - H_{n\beta\alpha|n} - H_{\beta n\gamma}P_n{}^{\gamma}{}_{n\alpha} - H_{\beta n\alpha|n} \\ &\quad + H_{nn\gamma}P_n{}^{\gamma}{}_{\beta\alpha} + H_{nn\alpha|\beta} \\ &= 0 \end{aligned} \tag{5.18}$$

Together with (5.16) and (5.17) we obtain

$$\begin{aligned} P_{njk\alpha} &= -H_{nj\beta}P_n{}^{\beta}{}_{k\alpha} - H_{nj\alpha|k} - H_{jk\beta}P_n{}^{\beta}{}_{n\alpha} - H_{jk\alpha|n} \\ &\quad +H_{kn\beta}P_n{}^{\beta}{}_{j\alpha} + H_{kn\alpha|j} \\ &= -\dot{H}_{jk\alpha} = L_{jk\alpha}. \end{aligned}$$

Substituting it into (5.16), we have Proposition 5.1.2. □

**Definition 5.1.3** Let $(M,F)$ be a Finsler manifold. If its Minko wskian curvature vanishes, we call $(M,F)$ a *Berwald manifold.*

It is easy to see that all Berwald manifolds are of Landsberg type. Z.Szabo showed the following:

**Theorem 5.1.4** ([Szabó, 1981]) *Any Berwald surface is either Riemannian or locally Minkowskian.*

The following question is still open: is there any Landsberg manifold which is not of Berwald type?

## 5.2 Flag curvature

Let $(M, F)$ be a Finsler manifold, and $R_i{}^j{}_{kl}$ its Riemannian curvature. The *flag curvature tensor* is a two order tensor defined by

$$R_{\alpha\beta} := \delta_{\alpha\gamma} R_n{}^\gamma{}_{\beta n}.$$

It is easy to see $R_{\alpha\beta}$ is symmetric from (5.5) and the First Bianchi Identity.

**Definition 5.2.1** The eigenvalues of the flag curvature tensor, denoted by

$$\kappa_1 \leq \cdots \leq \kappa_{n-1}$$

are the most important intrinsic invariants of the Finsler metric. We call $\kappa_\alpha$ the *$\alpha$-th principal curvature.*

**Definition 5.2.2** The trace of the flag curvature tensor, i.e. $\sum_{\alpha=1}^{n-1} \kappa_\alpha$, is called *Ricci scalar.* Usually we denote Ricci scalar by Ric. Ricci scalar is a function on $SM$.

**Definition 5.2.3** Let $y \in T_xM \backslash \{0\}$ and $P$ a 2-dimensional linear subspace of $T_xM$ which includes $y$. Taking $b \in P$ such that

$$g_{(x,[y])}(b, y) = 0, \quad g_{(x,[y])}(b, b) = 1.$$

Then $b = b^\alpha e_\alpha$. The quantity $R_{\alpha\beta} b^\alpha b^\beta$ is a function of the flag $\{x, y, P\}$, denoted by $\kappa(P, y)$ which is called the *flag curvature* at $\{x, [y], P\}$.

**Definition 5.2.4** We say that a Finsler manifold $(M, F)$ is of *scalar curvature* if there is a scalar function $\kappa$ on SM such that for any $(x, [y]) \in SM$ the principal curvatures $\kappa_\alpha = \kappa$, $\alpha = 1, \cdots, n-1$. In particular, we say

$(M, F)$ is of *constant (flag) curvature* $\kappa$ if it is of constant scalar curvature $\kappa$.

It is easy to show the following:

**Proposition 5.2.5** *If* $(M, F)$ *has scalar curvature* $\kappa = \kappa(x)$ *and* $\dim M \geq 3$, *then* $(M, F)$ *is of constant (flag) curvature.*

## 5.3 The first variation of arc length

In this section, we use the method of differential forms to describe the first variation. Consider the rectangle

$$\triangle := [t_0, t_1] \times [-1, 1].$$

Let $(M, F)$ be a Finsler manifold. We map it into $M$ using $\sigma : \triangle \to M$, $(t, u) \mapsto \sigma(t, u)$ such that the $t$-curves ($u =$ *constant*) are smooth. One obtains two vector fields defined over the square:

$$T := \sigma_* \left( \frac{\partial}{\partial t} \right), \quad U := \sigma_* \left( \frac{\partial}{\partial u} \right). \tag{5.19}$$

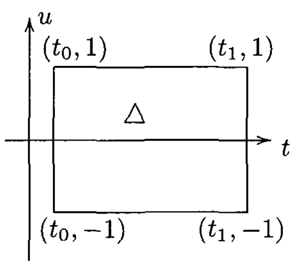


Fig. 5.1

$T$ gives the velocity vectors to the $t$-curves. Strictly speaking, T and U are maps from $\triangle$ into TM; but we will occasionally find it convenient to regard them, by a slight abuse of notation, as maps from $\triangle$ into $p^*TM$. The map $\sigma$ admits a lift $\tilde{\sigma} : \triangle \to SM$, defined by

$$\tilde{\sigma}(t, u) := (\sigma(t, u), T(t, u)). \tag{5.20}$$

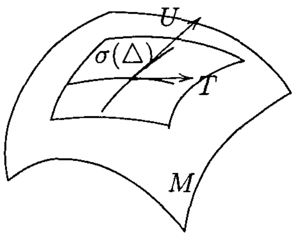


Fig. 5.2

$$\begin{tikzcd} & SM \arrow[d, "p"] \\ \triangle \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & M \end{tikzcd}$$

Correspondingly, one gets the following vector fields over $\tilde{\sigma}(\triangle)$:

$$\tilde{T} := \tilde{\sigma}_* \frac{\partial}{\partial t}, \quad \tilde{U} := \tilde{\sigma}_* \frac{\partial}{\partial u}. \tag{5.21}$$

For any $(t,u) \in \triangle$

$$T(t,u) \approx (\tilde{\sigma}(t,u), T(t,u)) \in p^*TM.$$

Put

$$\|T(t,u)\| := \sqrt{g(T,T)} \tag{5.22}$$

$$T(t,u) = T^i \frac{\partial}{\partial x^i}$$

$$\sigma(t,u) \to (\sigma^1, \cdots, \sigma^n).$$

Then, from (5.22), we have

$$\|T(t,u)\| = \sqrt{T^i T^j g_{ij}} = F(\sigma^1, \cdots, \sigma^n; T^1, \cdots, T^n).$$

Hence

$$T = \|T\| e_n = F(\sigma, T) e_n, \tag{5.23}$$

where $\{e_i\}$ is the adapted frame field. With respect to $\{e_i\}$, we obtain the dual co-frame field $\{\omega^i\}$ and the connection 1-forms $\{\omega_i{}^j\}$. These 1-forms can be pulled back to $\triangle$ by $\tilde{\sigma}^*$, i.e.

$$\tilde{\sigma}^* : T^*SM \longrightarrow T^* \triangle .$$

We write

$$\tilde{\sigma}^*(\omega^i) = a^i dt + b^i du, \tag{5.24}$$

$$\tilde{\sigma}^*(\omega_i^j) = a_i^j dt + b_i^j du. \tag{5.25}$$

Denote $\sigma$ by $(\sigma^1, \cdots, \sigma^n)$. Thus

$$T = \frac{\partial \sigma^i}{\partial t} \frac{\partial}{\partial x^i}.$$

From (5.20)

$$\tilde{\sigma} = (\sigma^1, \cdots, \sigma^n; \frac{\partial \sigma^1}{\partial t}, \cdots, \frac{\partial \sigma^n}{\partial t}).$$

Togother with (5.21) we get

$$\tilde{T} = \frac{\partial \sigma^i}{\partial t} \frac{\partial}{\partial x^i} + \frac{\partial^2 \sigma^i}{\partial t^2} \frac{\partial}{\partial y^i} \equiv T \qquad \mathrm{mod} \quad \frac{\partial}{\partial y^i}.$$

Since $\omega^i \in \Gamma(p^* T^* M)$, one has

$$\omega^i(\tilde{T}) = \omega^i(T).$$

Similarly, we have

$$\omega^i(\tilde{U}) = \omega^i(U).$$

Together with (5.21) and (5.24), we have

$$\begin{aligned} a^i &= a^i dt(\tfrac{\partial}{\partial t}) \\ &= (\tilde{\sigma}^* \omega^i - b^i du)(\tfrac{\partial}{\partial t}) \\ &= \omega^i(\tilde{\sigma}_*(\tfrac{\partial}{\partial t})) \\ &= \omega^i(\tilde{T}) = \omega^i(T) = \|T\| \omega^i(e_n) \end{aligned}$$

and

$$b^i = \tilde{\sigma}^* \omega^i(\frac{\partial}{\partial u}) = \omega^i(\tilde{U}) = \omega^i(U).$$

Hence we get

$$a^\alpha = 0, \quad a^n = \|T\|; \quad b^i = U^i. \tag{5.26}$$

Since $\omega_n{}^n = 0$, (5.21) and (5.24),

$$a_i{}^j = \omega_i{}^j(\tilde{T}), \quad b_i{}^j = \omega_i{}^j(\tilde{U}), \quad a_n{}^n = 0, \quad b_n{}^n = 0 \tag{5.27}$$

and, from (3.31)

$$a_\alpha{}^n = -\delta_{\alpha\beta}a_n{}^\beta, \quad b_\alpha{}^n = -\delta_{\alpha\beta}b_n{}^\beta. \tag{5.28}$$

Using $\tilde{\sigma}^*$ to pull back the torsion-free condition (4.2), we have

$$d(\tilde{\sigma}^*\omega^i) = \tilde{\sigma}^*\omega^j \wedge \tilde{\sigma}^*\omega_j{}^i.$$

From (5.24) and (5.25)

$$d(\tilde{\sigma}^*\omega^i) = d(a^i dt + b^i du) = (-\frac{\partial a^i}{\partial u} + \frac{\partial b^i}{\partial t})dt \wedge du$$

and

$$\tilde{\sigma}^*\omega^j \wedge \tilde{\sigma}^*(\omega_j{}^i) = (a^j b_j{}^i - b^j a_j{}^i)dt \wedge du.$$

Thus one gets

$$-\frac{\partial a^i}{\partial u} + \frac{\partial b^i}{\partial t} = a^j b_j{}^i - b^j a_j{}^i. \tag{5.29}$$

Together with (5.26) and (5.27)

$$\frac{\partial b^\alpha}{\partial t} = a^n b_n{}^\alpha - b^j a_j{}^\alpha, \tag{5.30}$$

$$\frac{\partial a^n}{\partial u} = \frac{\partial b^n}{\partial t} + b^\alpha a_\alpha{}^n. \tag{5.31}$$

From (5.2) and (5.25)

$$\tilde{\sigma}^*(\Omega_k{}^i) = \tilde{\sigma}^*(d\omega_k{}^i - \omega_k{}^j \wedge \omega_j{}^i) = (-\frac{\partial a_k{}^i}{\partial u} + \frac{\partial b_k{}^i}{\partial t} - a_k{}^j b_j{}^i + b_k{}^j a_j{}^i)dt \wedge du.$$

On the other hand, by using (5.4),(5.5),(5.24),(5.25) and (5.26),

$$\begin{aligned}\tilde{\sigma}^*(\Omega_k{}^i) &= \tilde{\sigma}^*(\tfrac{1}{2}R_k{}^i{}_{jl}\omega^j \wedge \omega^l + P_k{}^i{}_{j\beta}\omega^j \wedge \omega_n{}^\beta) \\ &= (\tfrac{1}{2}R_k{}^i{}_{jl}a^j b^l - \tfrac{1}{2}R_k{}^i{}_{jl}a^l b^j \\ &\quad + P_k{}^i{}_{j\beta}a^j b_n{}^\beta - P_k{}^i{}_{j\beta}b^j a_n{}^\beta)dt \wedge du \\ &= (R_k{}^i{}_{nl}a^n b^l + P_k{}^i{}_{n\beta}a^n b_n{}^\beta - P_k{}^i{}_{j\beta}b^j a_n{}^\beta)dt \wedge du.\end{aligned}$$

Thus we obtain

$$\frac{\partial b_k{}^i}{\partial t} - \frac{\partial a_k{}^i}{\partial u} = a_k{}^j b_j{}^i - b_k{}^j a_j{}^i + R_k{}^i{}_{nl}a^n b^l + P_k{}^i{}_{n\beta}a^n b_n{}^\beta - P_k{}^i{}_{j\beta}b^j a_n{}^\beta, \tag{5.32}$$

in particular, specializing $k$ to $\alpha$, $i$ to $n$, and using (5.18) one finds that $P_\alpha{}^n{}_{n\beta} = 0$ and

$$\frac{\partial a_\alpha{}^n}{\partial u} = \frac{\partial b_\alpha{}^n}{\partial t} - a_\alpha{}^j b_j{}^n + b_\alpha{}^j a_j{}^n - R_\alpha{}^n{}_{nl} a^n b^l + P_\alpha{}^n{}_{j\beta} b^j a_n{}^\beta. \tag{5.33}$$

Let $\sigma_u : [t_0, t_1] \to M$ be a $t$-curve and $L(u)$ its length. From (5.19) and (5.26), we have

$$\begin{aligned} L(u) &= L(\sigma_u) \\ &= \int_{t_0}^{t_1} \|\sigma_* \tfrac{\partial}{\partial t}\| dt \\ &= \int_{t_0}^{t_1} \|T\| dt \\ &= \int_{t_0}^{t_1} a^n dt. \end{aligned} \tag{5.34}$$

The formula for the first variation of arc length now follows from (5.31) and (5.34)

$$\begin{aligned} L'(u) &= \tfrac{d}{du} \int_{t_0}^{t_1} a^n dt \\ &= \int_{t_0}^{t_1} \tfrac{\partial a^n}{\partial u} dt \\ &= \int_{t_0}^{t_1} (\tfrac{\partial b^n}{\partial t} + b^\alpha a_\alpha{}^n) dt \\ &= b^n \big|_{t_0}^{t_1} + \int_{t_0}^{t_1} b^\alpha a_\alpha{}^n dt. \end{aligned} \tag{5.35}$$

Suppose that all the $t$-curves have same end-points, i.e.

$$Im\sigma_{t_0} = x, \quad Im\sigma_{t_1} = y. \tag{5.36}$$

Then

$$U(t_0, u) = (\sigma_{t_0})_* \left( \frac{\partial}{\partial u} \right) = 0$$

from (5.19). Similarly

$$U(t_1, u) = 0.$$

Thus we get , from (5.26)

$$b^n \big|_{t_0}^{t_1} = U^n \big|_{t_0}^{t_1} = \omega^n(U) \big|_{t_0}^{t_1} = \omega^n(U \big|_{t_0}^{t_1}) = 0.$$

Substituting into (5.36) yields

$$L'(u) = \int_{t_0}^{t_1} b^\alpha a_\alpha{}^n dt. \tag{5.37}$$

**Lemma 5.3.1** *Let $(M, F)$ be a Finsler manifold and $c : [t_0, t_1] \to M$ a curve on $M$. Then $c$ is a geodesic if and only if for any $\sigma : \triangle \to M$ satisfying (5.36) and $\sigma(t, 0) = c(t)$, we have $L'(0) = 0$.*

**Proposition 5.3.2** *Let $\tau : [t_0, t_1]$ be a curve on a Finsler manifold. Then $\tau$ is a geodesic on $M$ if and only if ${a_n}^\alpha = 0$ where*

$$\tilde{\tau}^* {\omega_n}^\alpha = {a_n}^\alpha dt$$

*$\{{\omega_i}^\alpha\}$ is the Chern connection 1-form and $\tilde{\tau}$ satisfies $\tau = p \circ \tilde{\tau}$.*

***Proof.*** Taking a variation of $\tau$ with fixed end-points

$$\sigma(t, u) = p[exp_{\tau(t)}(\sum \epsilon(t) {a_\alpha}^n u e_\alpha)],$$

where $\epsilon : [t_0, t_1] \to R$ is differentiable and $\epsilon \mid_{(t_0, t_1)} > 0$, $\epsilon(t_0) = \epsilon(t_1) = 0$. Thus

$$\sigma(t, 0) = \tau(t),$$
$$\tilde{U} = \tilde{\sigma}^* \frac{\partial}{\partial u} = \sum_\alpha \epsilon(t) {a_\alpha}^n e_\alpha.$$

From (5.26)we get

$$0 = L'(0) = \int_{t_0}^{t_1} \sum_\alpha ({a_\alpha}^n)^2 \epsilon(t) dt.$$

It follows that ${a_\alpha}^n = 0$. Conversely, if ${a_\alpha}^n = 0$, then $\tau$ is a geodesic form of (5.37). □

## 5.4 The second variation of arc length

The setup is just like we had in Section 5.3, except now the base curve $\sigma(t) = \sigma(t, 0)$ in our variation $\sigma(t, u)$, with $(t, u) \in \triangle := [t_0, t_1] \times [-1, 1]$.

From (5.33)

$$\begin{aligned}
&\frac{\partial}{\partial u}(b^\alpha {a_\alpha}^n) \\
&= \frac{\partial b^\alpha}{\partial u} b^\alpha u {a_\alpha}^n + b^\alpha \frac{\partial {a_\alpha}^n}{\partial u} \\
&= \frac{\partial b^\alpha}{\partial u} {a_\alpha}^n + b^\alpha \left[ \frac{\partial {b_\alpha}^n}{\partial t} - {a_\alpha}^j {b_j}^n + {b_\alpha}^j {a_j}^n - {R_\alpha}^n{}_{nl} a^n b^l + {P_\alpha}^n{}_{j\beta} b^j {a_n}^\beta \right] \\
&= \frac{\partial}{\partial u}(b^\alpha {b_\alpha}^n) + {a_\alpha}^n [\frac{\partial}{\partial u} - {P_\beta}^n{}_j{}^\alpha b^\beta b^j + b^\beta {b_\beta}^\alpha] - a^n b^\alpha b^l {R_\alpha}^n{}_{nl} + (I),
\end{aligned}$$

where

$$(I) := -\frac{\partial b^\alpha}{\partial t} b_\alpha{}^n - b^\alpha a_\alpha{}^j b_j{}^n = b^n a_n{}^\alpha b_\alpha{}^n - a^n b_n{}^\alpha b_\alpha{}^n$$

from (5.27) and (5.30). Therefore

$$\begin{aligned}\tfrac{\partial^2 a^n}{\partial u^2} &= \tfrac{\partial}{\partial u}[\tfrac{\partial b^n}{\partial t} + b^\alpha a_\alpha{}^n] \\ &= \tfrac{\partial}{\partial t}\tfrac{\partial b^n}{\partial u} + \tfrac{\partial}{\partial t}(b^\alpha b_\alpha{}^n) + a_\alpha{}^n[\tfrac{\partial b^\alpha}{\partial u} - P_\beta{}^n{}_j{}^\alpha b^\beta b^j + b^\beta b_\beta{}^\alpha \\ &\quad -b^n \delta^{\alpha\beta} b_\beta{}^n] - a^n(b_n{}^\alpha b_\alpha{}^n + b^\alpha b^l R_\alpha{}^n{}_{nl}).\end{aligned} \tag{5.38}$$

Note that $\sigma(t) = \sigma(t,0)$ is geodesic, $a_n{}^\alpha = 0$ from Proposition 5.3.2. Thus we obtain the formula for the second variation of arc length along a geodesic

$$\begin{aligned}L''(0) &= \tfrac{\partial^2}{\partial u^2} L(u) \\ &= \int_{t_0}^{t_1} \tfrac{\partial^2 a^n}{\partial u^2} dt \\ &= \left[\tfrac{\partial b^n}{\partial u} + \delta_{ij} b^i b_n{}^j\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} a^n[\delta_{ij} b_n{}^i b_n{}^j + R_{ninj} b^i b^j] dt.\end{aligned} \tag{5.39}$$

For the remainder of this section, we shall discuss the covariant differentiation along curves and re-express variation formulas mentioned above.

Let $\gamma(s)$ be a curve on $M$ with velocity $S(s)$; let $W(s)$ be a vector field in $M$ defined along $\gamma$. Corresponding to any lift $\tilde{\gamma}(s) := (\gamma(s), y(s))$ of $\gamma$ from $M$ into $SM$, we set $\tilde{S} := \tilde{\gamma}_* \frac{d}{ds} = \frac{d}{ds}(\gamma(s), y(s))$ and define the following

**Definition 5.4.1** Denote Chern connection 1-form by $\{\omega_k{}^i\}$, and $\omega$ by $\omega^k e_k$. We call

$$D_S W := \left[\frac{dW^i}{ds} + W^k \omega_k{}^i(\tilde{S})\right] e_i$$

the *covariant derivative* of $W$ along $\gamma$ (with lift $\tilde{\gamma}$).

From (4.3), it is easy to show

$$(D_s g)(V, W) = V^j W^i H_{ji\beta} \omega_n{}^\beta(\tilde{S}),$$

where $D_s g$ is defined by

$$(D_s g)(V, W) = \frac{d}{ds}[g(V, W)] - g(D_s V, W) - g(V, D_s W)$$

for $\forall V, W \in \Gamma(P^*TM)$. We now mention two special cases in which this term drops out, i.e. $(D_s g)(V, W) = 0$:

1. if $V$ or $W$ is proportional to y;
2. if $(\omega_n{}^\beta)_{|\tilde{\gamma}}(\tilde{S}) = 0$.

In this sense, one can say Chern connection has just the right amount of metric-compatibility.

Return now to the rectangle $\sigma(tu)$, $\sigma : \triangle \to M$. We lift it into $SM$ using the velocity $T$ of the $t$-curves, namely through $\tilde{\sigma} := (\sigma, T)$. One finds that

$$D_T U = D_U T, \tag{5.40}$$

where $U = \sigma_*(\frac{\partial}{\partial u})$. By the definition of the curvature form, we have

$$\Omega_k{}^i(\tilde{U}, \tilde{T}) = \frac{\partial}{\partial u}\omega_k{}^i(\tilde{T}) - \frac{\partial}{\partial t}\omega_k{}^i(\tilde{U}) - \omega_k{}^j(\tilde{U})\omega_j{}^i(\tilde{T}) + \omega_k{}^j(\tilde{T})\omega_j{}^i(\tilde{U}).$$

Hence

$$D_U D_T Z = D_T D_U Z + Z^k \Omega_k{}^i(\tilde{U}, \tilde{T}) e_i \tag{5.41}$$

for any vector field $Z(t, u)$ in $\sigma(\triangle)$. On the other hand

$$\Omega_k{}^i(\tilde{U}, \tilde{T}) = \left(\frac{1}{2} R_k{}^i{}_{jl}\omega^j \wedge \omega^l + P_k{}^i{}_{j\alpha}\omega^j \wedge \omega_n{}^\alpha\right)(\tilde{U}, \tilde{T}).$$

Thus we obtain

$$\begin{aligned} D_U D_T Z = D_T D_U Z + Z^k [&R_k{}^i{}_{jn} U^j \|T\| + P_k{}^i{}_{j\alpha} U^j \omega_n{}^\alpha(\tilde{T}) \\ &- P_k{}^i{}_{n\alpha}\|T\|\omega_n{}^\alpha(\tilde{U})] e_i \end{aligned} \tag{5.42}$$

By using (5.40) and metric-compatibility we have the first variational formulas in terms of covariant derivatives

$$\begin{aligned} L'(u) &= \tfrac{d}{du}\int_{t_0}^{t_1} \sqrt{g(T,T)}dt \\ &= \int_{t_0}^{t_1} \{\tfrac{\partial}{\partial t}[g(U, \tfrac{T}{\|T\|})] - g(U, D_T(\tfrac{T}{\|T\|}))\}dt. \end{aligned} \tag{5.43}$$

The equation for a geodesic is thus

$$D_T\left(\frac{T}{\|T\|}\right) = 0. \tag{5.44}$$

If the geodesic is given a constant speed parametrization, then (5.44) reduces to the following form

$$D_T T = 0.$$

A direct calculation gives, from (5.43)

$$L''(0) = \left[g\left(D_U U, \frac{T}{\|T\|}\right)\right]_{t=0}^{t=t_1} + \int_{t_0}^{t_1} \left\{g\left(D_T U, D_U\left(\frac{T}{\|T\|}\right)\right) + g\left(U, [D_T D_U - D_U D_T]\left(\frac{T}{\|T\|}\right)\right)\right\}dt \tag{5.45}$$

By using (5.23) and (5.42)

$$\left(\frac{T}{\|T\|}\right) = [-R_n{}^i{}_{jn}U^j\|T\| - P_n{}^i{}_{j\alpha}U^j\omega_n{}^\alpha(\tilde{T}) + P_n{}^i{}_{n\alpha}\|T\|\omega_n{}^\alpha(\tilde{U})]e_n.$$

From (5.40), we have

$$\begin{aligned} g(D_T U, D_U(\frac{T}{\|T\|})) &= g\left(D_U T, \frac{-D_U(\|T\|)T + (D_U T)\|T\|}{\|T\|^2}\right) \\ &= \frac{1}{\|T\|}\left[g(D_U T, D_U T) - \left(\frac{\partial\|T\|}{\partial u}\right)^2\right]. \end{aligned}$$

Substituting them into (5.45) yields

$$\begin{aligned} L''(0) = &\left[g(D_U U, \frac{T}{\|T\|})\right]_{t=0}^{t=t_1} \\ &+ \int_{t_0}^{t_1} \frac{1}{\|T\|}\left[g(D_U T, D_U T) - \left(\frac{\partial\|T\|}{\partial u}\right)^2 + \|T\|^2 R_{ninj}U^i U^j\right] dt. \end{aligned} \tag{5.46}$$

We write the $R$ term as

$$\|T\|^2 R_{ninj}U^i U^j = g(R(T,U)T,U). \tag{5.47}$$

Plugging (5.40) and (5.47) into (5.46) yields

$$\begin{aligned} L''(0) = &\left[g(D_U U, \frac{T}{\|T\|})\right]_{t=0}^{t=t_1} + \int_{t_0}^{t_1} \frac{1}{\|T\|}\left[g(D_T U, D_T U) + g(R(T,U)T,U)\right] dt \\ &- \int_{t_0}^{t_1} \frac{1}{\|T\|}\left(\frac{\partial\|T\|}{\partial u}\right)^2 dt. \end{aligned} \tag{5.48}$$

**Definition 5.4.2** Let $V$ and $W$ be two arbitrary vector fields along geodesic $\sigma(t)$, $a \leq t \leq b$, with velocity $T(t)$. The following quadratic form

$$I(V,W) := \int_a^b \frac{1}{\|T\|}\left[g(D_T V, D_T W) + g(R(T,V)T,W)\right] dt \tag{5.49}$$

is called the *index form* along $\sigma$.

Note that in (5.49) $g$ is evaluated at the point $(\sigma(t), [T(t)])$ of $SM$. By using (5.47) and the property of Riemannian curvature, we see $I(V,W)$ is indeed symmetric. If the geodesic is given a constant speed parametrization, then we define $I(V,W)$ omitting the factor $\frac{1}{\|T\|}$.

# Chapter 6

# Geometry of Projective Sphere Bundle

The key idea in Finsler geometry is to consider the projective sphere bundle $SM$ on the $n$-dimensional Finsler manifold. The main reason is that all geometric quantities constructed from Finsler structure are homogeneous of degree zero in $y$ and thus naturally live on $SM$, even though Finsler structure itself does not. $SM$ is a $(2n-1)$-dimensional Riemannian manifold with its Sasaki type metric induced by the fundamental tensor.

In this chapter, we will give intrinsic relation between several important subbundle, such as Finsler bundle of $SM$ and Finsler manifold itself. In particular, we show that Finsler bundle is integrable if and only if manifold itself has zero flag curvature, and Finsler bundle is minimal if and only if manifold itself is Riemannian.

## 6.1 Riemannian connection and curvature of projective sphere bundle

Let $(M, F)$ be an $n$-dimensional Finsler manifold and $SM$ its projective sphere bundle with the Sasaki type metric $G$. Then $(SM, G)$ is a $(2n$-$1)$-dimensional Riemannian manifold where

$$G := \delta_{ij}\omega^i \otimes \omega^j + \delta_{\alpha\beta}\omega_n{}^\alpha \otimes \omega_n{}^\beta. \tag{6.1}$$

Throughout this chapter, our index conventions are as follows:

$$1 \leq i, j, k, \cdots \leq n,$$

$$1 \leq \alpha, \beta, \gamma, \cdots \leq n-1,$$

$$1 \leq a, b, c, \cdots \leq 2n-1.$$

We introduce the abbreviations

$$\psi_i = \delta_{ij}\omega^j; \quad \psi_{\bar{\alpha}} = \delta_{\alpha\beta}\omega_n{}^\beta, \quad \bar{\alpha} = n + \alpha. \tag{6.2}$$

Then from (6.1) the metric $\Pi$ can be rewritten as

$$\Pi = \sum_a \psi_a \otimes \psi_a = \sum_\alpha \psi_\alpha \otimes \psi_\alpha + \psi_n \otimes \psi_n + \sum_\alpha \psi_{\bar{\alpha}} \otimes \psi_{\bar{\alpha}}. \tag{6.3}$$

From (4.1),(4.2),(4.3) and (6.2), we have

$$\begin{aligned} d\psi_\alpha &= d\omega^\alpha \\ &= \omega^k \wedge \omega_k{}^\alpha \\ &= \sum \omega^\beta \wedge \omega_\beta{}^\alpha + \omega^n \wedge \omega_n{}^\alpha \\ &= \textstyle\sum_\beta \psi_\beta \wedge \omega_{\beta\alpha} + \psi_n \wedge \psi_{\bar{\alpha}}, \end{aligned} \tag{6.4}$$

$$d\psi_n = d\omega^n = \omega^k \wedge \omega_k{}^n = \omega^\beta \wedge \omega_\beta{}^n = -\sum_\alpha \psi_\alpha \wedge \psi_{\bar{\alpha}}, \tag{6.5}$$

$$\begin{aligned} d\psi_{\bar{\alpha}} &= d\omega_n{}^\alpha \\ &= \Omega_n{}^\alpha + \omega_n{}^k \wedge \omega_k{}^\alpha \\ &= \tfrac{1}{2}\textstyle\sum_{i,j} R_n{}^\alpha{}_{ij}\omega_i \wedge \omega_j + \sum_i P_n{}^\alpha{}_{i\beta}\omega_i \wedge \omega_n{}^\beta + \omega_n{}^\beta \wedge \omega_\beta^\alpha \\ &= \tfrac{1}{2}R_n{}^\alpha{}_{\beta\gamma}\omega_\beta \wedge \omega_\gamma + \textstyle\sum_\beta R_n{}^\alpha{}_{\beta n}\omega_\beta \wedge \omega_n \\ &\quad + \textstyle\sum_\gamma P_n{}^\alpha{}_{\gamma\beta}\omega_\gamma \wedge \omega_n{}^\beta + \sum_\beta \theta_{\bar{\beta}} \wedge \omega_{\beta n} \\ &= \tfrac{1}{2}\textstyle\sum_{\beta,\gamma} R_{\alpha\beta\gamma}\theta_\beta \wedge \theta_\gamma + \sum_\beta R_{\alpha\beta}\omega_\beta \wedge \omega_\alpha \\ &\quad + \textstyle\sum_{\beta,\gamma} L_{\alpha\beta\gamma}\theta_\beta \wedge \theta_{\bar{\gamma}} + \sum_\beta \theta_{\bar{\beta}} \wedge \omega_{\beta\alpha}, \end{aligned} \tag{6.6}$$

where $R_{\alpha\beta\gamma} := \delta_{\alpha\epsilon}R_n{}^\epsilon{}_{\beta\gamma}$ and $L_{\alpha\beta\gamma}$ is the Landsberg curvature. Note that the length square of $R_{\alpha\beta}$, denoted by $\|S\|^2$, is a scalar function on $SM$.

**Lemma 6.1.1** *$(M, F)$ has scalar curvature $\kappa$ if and only if*

$$R_{\alpha\beta} = \kappa\delta_{\alpha\beta}. \tag{6.7}$$

*In particular, $(M, F)$ has zero flag curvature if and only if $\|S\|^2$ vanishes identically.*

***Proof.*** Obviously. □

Let $\psi_{ab}$ be the Levi-Civita connection 1-form satisfying

$$d\psi_a = -\sum_b \psi_{ab} \wedge \psi_b; \quad \psi_{ab} + \psi_{ba} = 0. \tag{6.8}$$

From (6.4) and (6.8) we have

$$\sum_\beta \psi_\beta \wedge (\psi_{\alpha\beta} - \omega_{\beta\alpha}) + \psi_n \wedge (\psi_{\alpha n} - \psi_{\bar\alpha}) + \sum_\beta \psi_{\bar\beta} \wedge \psi_{\alpha\bar\beta} = 0. \tag{6.9}$$

From (6.5) and (6.8) we have

$$\sum_\beta \psi_\beta \wedge (\psi_{n\beta} + \psi_{\bar\beta}) + \sum_\beta \psi_{\bar\beta} \wedge \psi_{n\bar\beta} = 0. \tag{6.10}$$

From (6.6) and (6.8) we get

$$\begin{aligned}\sum_\beta \psi_\beta \wedge (\psi_{\bar\alpha\beta} - \frac{1}{2}\sum_\gamma R_{\alpha\beta\gamma}\psi_\gamma - \sum_\gamma L_{\alpha\beta\gamma}\psi_{\bar\gamma})\\ + \psi_n \wedge (\psi_{\bar\alpha n} + \sum_\beta R_{\alpha\beta}\psi_\beta) + \sum_\beta \psi_{\bar\beta} \wedge (\psi_{\bar\alpha\bar\beta} - \omega_{\beta\alpha}) = 0.\end{aligned} \tag{6.11}$$

By (6.9)–(6.11), we have $\sum_b \theta_b \wedge \vartheta_{ab} = 0$, where

$$(\vartheta_{ab}) = \begin{bmatrix} \psi_{\alpha\beta} - \omega_{\beta\alpha} & \psi_{\alpha n} - \psi_{\bar\alpha} & \psi_{\alpha\bar\beta} \\ \psi_{n\beta} + \psi_{\bar\beta} & 0 & \psi_{n\bar\beta} \\ \begin{array}{c}\psi_{\bar\alpha\beta} - \frac{1}{2}\sum_\gamma R_{\alpha\beta\gamma}\psi_\gamma \\ -\sum_\gamma L_{\alpha\beta\gamma}\psi_{\bar\gamma}\end{array} & \psi_{\bar\alpha n} + \sum_\beta R_{\alpha\beta}\psi_\beta & \psi_{\bar\alpha\bar\beta} - \omega_{\beta\alpha} \end{bmatrix}. \tag{6.12}$$

By Cartan's lemma, we have

$$\vartheta_{ab} = \sum_c a_{abc}\psi_c, \quad a_{abc} = a_{acb}. \tag{6.13}$$

Substituting it into (6.12) yields

$$(\psi_{ab}) = \begin{bmatrix} \omega_{\beta\alpha} + \sum a_{\alpha\beta c}\psi_c & \psi_{\bar{\alpha}} + \sum a_{\alpha nc}\psi_c & \sum a_{\alpha\bar{\beta}c}\psi_c \\ -\psi_{\bar{\beta}} + \sum a_{n\beta c}\psi_c & 0 & \sum a_{n\bar{\beta}c}\psi_c \\ \begin{matrix} \frac{1}{2}\sum R_{\alpha\beta\gamma}\psi_\gamma \\ +\sum L_{\alpha\beta\gamma}\psi_{\bar{\gamma}} \\ +\sum a_{\bar{\alpha}\beta c}\psi_c \end{matrix} & \begin{matrix} \sum a_{\bar{\alpha}nc}\psi_c \\ -\sum R_{\alpha\gamma}\psi_\gamma \end{matrix} & \begin{matrix} \omega_{\beta\alpha} \\ +\sum a_{\bar{\alpha}\bar{\beta}c}\psi_c \end{matrix} \end{bmatrix}. \tag{6.14}$$

Put

$$b_{abc} = \frac{a_{abc} + a_{bac}}{2}. \tag{6.15}$$

Then

$$\sum_c b_{\alpha\beta c}\psi_c = \frac{\vartheta_{\alpha\beta} + \vartheta_{\beta\alpha}}{2} = -\frac{1}{2}(\omega_{\alpha\beta} + \omega_{\beta\alpha}) = \sum_\gamma H_{\alpha\beta\gamma}\psi_{\bar{\gamma}},$$

from (5.8), (6.12), (6.13) and (6.15).

Similarly, we have

$$\sum b_{\alpha nc}\psi_c = \frac{1}{2}(\vartheta_{\alpha n} + \vartheta_{n\alpha}) = 0,$$

and

$$\sum b_{\alpha\bar{\beta}c}\psi_c = \frac{1}{2}(\vartheta_{\alpha\bar{\beta}} + \vartheta_{\bar{\beta}\alpha}) = -\frac{1}{4}(\sum_\gamma R_{\beta\alpha\gamma}\psi_\gamma + 2\sum_\gamma L_{\alpha\beta\gamma}\psi_{\bar{\gamma}}).$$

Thus

$$(b_{\alpha\beta c}) = \begin{bmatrix} 0 \\ 0 \\ H_{\alpha\beta\gamma} \end{bmatrix}, \quad (b_{\alpha nc}) = 0, \quad (b_{\alpha\bar{\beta}c}) = -\frac{1}{4}\begin{bmatrix} R_{\beta\alpha\gamma} \\ 0 \\ 2L_{\alpha\beta\gamma} \end{bmatrix}. \tag{6.16}$$

Similarly, we get

$$(b_{n\bar{\alpha}c}) = \frac{1}{2}\begin{bmatrix} R_{\alpha\gamma} \\ 0 \\ 0 \end{bmatrix}, \quad (b_{nnc}) = 0, \quad (b_{\bar{\alpha}\bar{\beta}c}) = \begin{bmatrix} 0 \\ 0 \\ H_{\alpha\beta\gamma} \end{bmatrix}. \tag{6.17}$$

(6.15) together with $a_{abc} = a_{acb}$, suggests that the quantity $a_{abc}$ is algebraically similar to the Christoffel symbols of the first kind in Riemannian geometry, namely $\{ijk\} = \frac{1}{2}(g_{ij,k} - g_{jk,i} + g_{ki,j})$. Imitating the usual derivation of the formula for $\{ijk\}$, that is, applying (6.16) and (6.17) to the combination $b_{abc} - b_{bca} + b_{cab}$ leads to intermediate formulas for $a_{abc}$ which, in conjuction with (6.15), imply that

$$(\psi_{ab}) = \begin{bmatrix} \omega_{\beta\alpha} + \sum_\gamma (H_{\alpha\beta\gamma} + \frac{1}{2}R_{\gamma\beta\alpha})\psi_{\bar\gamma} & \psi_{\alpha n} & \psi_{\alpha\bar\beta} \\ \sum_\beta (\frac{1}{2}R_{\alpha\beta} - \delta_{\alpha\beta})\psi_{\bar\beta} & 0 & \frac{1}{2}\sum_\gamma R_{\beta\gamma}\psi_\gamma \\ \begin{matrix} -\sum_\gamma (H_{\alpha\beta\gamma} + \frac{1}{2}R_{\alpha\gamma\beta})\psi_\gamma \\ +\frac{1}{2}R_{\alpha\beta}\psi_n + \sum L_{\alpha\beta\gamma}\psi_{\bar\gamma} \end{matrix} & \psi_{\bar\alpha n} & \omega_{\beta\alpha} + \sum_\gamma H_{\alpha\beta\gamma}\psi_\gamma \end{bmatrix}. \tag{6.18}$$

Let $\Psi_{ab}$ denote the curvature form of $(SM, \Pi)$ and let $K_{abcd}$ be its components. Then

$$d\psi_{ab} = -\sum_c \psi_{ac} \wedge \psi_{cb} + \Psi_{ab}; \quad \Psi_{ab} = \frac{1}{2}\sum_{c,d} K_{abcd}\psi_c \wedge \psi_d. \tag{6.19}$$

Substituting (6.18) into (6.19) yields

$$\begin{aligned} \Psi_{\alpha n} &= d\psi_{\alpha n} + \sum_c \psi_{\alpha c} \wedge \psi_{cn} \\ &\equiv \sum_{\beta,\gamma} R_{\beta\gamma}(\delta_{\alpha\beta} - \frac{3}{4}R_{\alpha\beta})\psi_\gamma \wedge \psi_n \quad mod \quad \psi_\alpha \wedge \psi_\beta, \quad \psi_\alpha \wedge \psi_{\bar\alpha}, \end{aligned} \tag{6.20}$$

$$\Psi_{n\bar\alpha} \equiv \sum_{\beta,\gamma} R_{\alpha\beta}R_{\beta\gamma}\psi_n \wedge \psi_{\bar\beta}. \tag{6.21}$$

Comparing (6.19) with (6.20), we have

$$K_{\alpha n\alpha n} = \sum_{\alpha,\beta} R_{\alpha\beta}(\delta_{\alpha\beta} - \frac{3}{4}R_{\alpha\beta}). \tag{6.22}$$

Comparing (6.19) with (6.21) we get

$$K_{n\bar\alpha n\bar\alpha} = \frac{1}{4}\sum_\beta R^2_{\alpha\beta}. \tag{6.23}$$

Denote by $\hat{l}$ the dual vector of Hilbert form with respect to $\Pi$ (cf.(4.7)).

Then the Ricci curvature in the direction of $\hat{l}$ can be written as

$$\begin{aligned} Ric(\hat{l}) &= \sum_a K_{anan} \\ &= \sum_\alpha (K_{\alpha n\alpha n} + K_{n\bar{\alpha}n\bar{\alpha}}) \\ &= \sum_{\alpha,\beta} R_{\alpha\beta}(\delta_{\alpha\beta} - \tfrac{3}{4}R_{\alpha\beta}) + \tfrac{1}{4}\sum_{\alpha,\beta} R^2_{\alpha\beta} \\ &= Ric - \tfrac{1}{2}\|S\|^2. \end{aligned} \tag{6.24}$$

**Theorem 6.1.2** ([Mo, 1998]) *Let $(M, F)$ be an $n$-dimensional Finsler manifold. Then*

$$Ric(\hat{l}) \leq \min\{\frac{n-1}{2}, \quad Ricci \quad scalar\}$$

*and*

$$Ric(\hat{l}) \equiv \frac{n-1}{2} \quad (resp. \quad Ricci \quad scalar)$$

*if and only if $(M, F)$ has constant flag curvature 1 (resp.0 ).*

***Proof.*** Notice that

$$\begin{aligned} \|S\|^2 :&= \sum_{\alpha,\beta} R^2_{\alpha\beta} \\ &\geq \sum_\alpha R^2_{\alpha\alpha} \\ &\geq \tfrac{1}{n-1}(\sum R_{\alpha\alpha})^2 \\ &= \tfrac{1}{n-1} Ric^2. \end{aligned} \tag{6.25}$$

Plugging it into (6.24) yields

$$\begin{aligned} Ric(\hat{l}) &\leq Ric - \frac{1}{2(n-1)} Ric^2 \\ &\leq \frac{n-1}{2} - \left[\frac{Ric}{\sqrt{2(n-1)}} - \sqrt{\frac{n-1}{2}}\right]^2 \leq \frac{n-1}{2}. \end{aligned}$$

If $Ric(\hat{l}) = \frac{n-1}{2}$, then (6.25) implies that

$$R_{\alpha\beta} = 0 \quad if \quad \alpha \neq \beta$$

$$R_{11} = \cdots = R_{n-1,n-1}.$$

Hence

$$R_{\alpha\alpha} = \frac{1}{n-1} Ric = 1, \quad \forall \alpha.$$

By using (6.24) we can obtain another conclusion immediately. □

## 6.2 Integrable condition of Finsler bundle

Let $(M, F)$ be a Finsler manifold and $SM$ its projective sphere bundle. We study the horizontal subbundle $\mathcal{H}$ of $SM$. By (3.4)

$$\mathcal{H} := \{b \in TSM | \theta_{\bar{\alpha}}(b) = 0\}.$$

Because $\mathcal{H}$ is isomorphic to Finsler bundle $p^*TM$ (cf. Section 1.4), we call $\mathcal{H}$ the *Finsler bundle* of $(M, F)$. In this section we will investigate the integrability of Finlser bundle $\mathcal{H}$. Using Frobenius' theorem, $\mathcal{H}$ is integrable if and only if

$$d\psi_{\bar{\alpha}} \equiv 0 \qquad \text{mod} \quad \psi_{\bar{\alpha}}. \tag{6.26}$$

By using (6.6)

$$d\psi_{\bar{\alpha}} \equiv \frac{1}{2}\sum_{\beta,\gamma} R_{\alpha\beta\gamma}\theta_\beta \wedge \psi_\gamma + \sum_\beta R_{\alpha\beta}\psi_\beta \wedge \psi_n \qquad \text{mod} \quad \psi_{\bar{\alpha}},$$

and

$$R_{\alpha\beta\gamma} = -R_{\alpha\gamma\beta} := \delta_{\alpha\epsilon} R_n{}^\epsilon{}_{\beta\gamma}.$$

Hence (6.26) holds if and only if

$$R_{\alpha\beta} = 0, \quad R_{\alpha\beta\gamma} = 0. \tag{6.27}$$

**Lemma 6.2.1**

$$R_{\alpha\beta\gamma} = \frac{1}{3}(R_\beta{}^\alpha{}_{;\gamma} - R_\gamma{}^\alpha{}_{;\beta}), \tag{6.28}$$

*where* $R_\beta{}^\alpha{}_{;\gamma}$ *is defined by*

$$R_\beta{}^\alpha{}_{;\gamma}\omega_n{}^\gamma \equiv dR_\beta{}^\alpha + R_\beta{}^\gamma\omega_\gamma{}^\alpha - R_\gamma{}^\alpha\omega_\beta{}^\gamma \qquad \text{mod} \quad \omega^i.$$

***Proof.*** Review the curvature 2-form $\Omega_i{}^j$ is defined by

$$\Omega_i{}^j = d\omega_i{}^j - \omega_i{}^k \wedge \omega_k{}^j. \tag{6.29}$$

Differentiating it, one deduces the following second Bianchi identity

$$d\Omega_i{}^j = -\Omega_i{}^k \wedge \omega_k{}^j + \omega_i{}^k \wedge \Omega_k{}^j.$$

In particular, taking $i = n$, $j = \alpha$, we get

$$d\Omega_n{}^\alpha + \Omega_n{}^\beta \wedge \omega_\beta{}^\alpha - \Omega_\beta{}^\alpha \wedge \omega_n{}^\beta = 0. \tag{6.30}$$

Plugging (5.4) ($i = n$, $j = \alpha$) and its differential into (6.30), yields

$$R^\alpha{}_{\beta;\gamma} = R^\alpha{}_{\beta\gamma} + R_\gamma{}^\alpha{}_{\beta n} + L^\alpha{}_{\beta\gamma|n}. \tag{6.31}$$

On the other hand, using the First Bianchi Identity (5.6) we have

$$R_{\alpha\beta\gamma n} + R_{\gamma\beta n\alpha} + R_{n\beta\alpha\gamma} = 0, \tag{6.32}$$

From (6.31) we get

$$\begin{aligned} R_{\alpha\beta\gamma n} : &= \delta_{\beta\epsilon} R_\alpha{}^\epsilon{}_{\gamma n} \\ &= \delta_{\beta\epsilon}[R_\gamma{}^\epsilon{}_{;\alpha} - R^\epsilon{}_{\gamma\alpha} - L^\epsilon{}_{\gamma\alpha|n}] \\ &= R_\gamma{}^\beta{}_{;\alpha} - R_{\beta\gamma\alpha} - L_{\alpha\beta\gamma|n}. \end{aligned} \tag{6.33}$$

Similarly,

$$\begin{aligned} R_{\gamma\beta n\alpha} &= -R_{\gamma\beta\alpha n} \\ &= -(R_\alpha{}^\beta{}_{;\gamma} - R_{\beta\alpha\gamma} - L_{\gamma\beta\alpha|n}) \\ &= -R_\alpha{}^\beta{}_{;\gamma} - R_{\beta\gamma\alpha} + L_{\alpha\beta\gamma|n}. \end{aligned} \tag{6.34}$$

Plugging (6.33) and (6.34) into (6.32) yields

$$\begin{aligned} 0 &= R_\gamma{}^\beta{}_{;\alpha} - R_{\beta\gamma\alpha} - L_{\alpha\beta\gamma|n} + (-R_\alpha{}^\beta{}_{;\gamma} - R_{\beta\gamma\alpha} + L_{\alpha\beta\gamma|n}) + R_{n\beta\alpha\gamma} \\ &= R_\gamma{}^\beta{}_{;\alpha} - R_\alpha{}^\beta{}_{;\gamma} - 3R_{\beta\gamma\alpha}. \end{aligned}$$

□

**Theorem 6.2.2** ([Mo, 1998]) *Let $(M, F)$ be a Finsler manifold. Then its Finsler bundle $H$ is integrable if and only if $(M, F)$ has zero flag curvature. In particular, both Riemannian manifold and locally Minkowskian manifold have integrable Finsler bundle.*

***Proof.*** Suppose that the Finsler manifold $(M, F)$ has vanished flag curvature. Then $R_{\gamma\beta\alpha} \equiv 0$ from Lemma 6.2.1. Thus (6.27) holds and the Finsler bundle $\mathcal{H}$ is integrable. The necessity of Theorem 6.2.2 can be obtained from (6.26) and (6.27) immediately. □

Now we consider the vertical subbundle $\mathcal{V}$ of $SM$, i.e. the projective spheres $\{S_xM\}$ (cf. Section 3.4). $\mathcal{V}$ is determined by $\theta_i = 0$. Restricting (6.18) to $\mathcal{V}$, we have

$$\psi_{\alpha\bar{\beta}} = -\sum_{\gamma} L_{\alpha\beta\gamma}\psi_{\bar{\gamma}},$$

$$\psi_{n\bar{\alpha}} = 0.$$

Hence the components of the second fundamental forms of the immersion $S_xM \hookrightarrow SM$ are

$$h^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = -L_{\alpha\beta\gamma}; \quad h^{n}_{\bar{\beta}\bar{\gamma}} = 0. \tag{6.35}$$

Using Sasaki-type metric $G$, one decomposes the Finsler bundle $\mathcal{H}$ into direct sum as follows

$$H = H_1 \oplus \mathrm{span}\{\hat{l}\}.$$

From (6.35) it is easy to show the following

**Proposition 6.2.3** *The first normal spaces of projective spheres lie in $H_1$.*

**Remark** $(M, F)$ has flat horizontal foliations if and only if it has zero Riemannian curvature.

## 6.3 Minimal condition of Finsler bundle

Let $V$ now denote an arbitrary smooth distribution on a smooth Riemannian manifold $(N, h)$, which we call a vertical distribution. Let $H$ be the distribution orthogonal to $V$, we call this a horizonal distribution. Then $TN = H \oplus V$.

**Definition 6.3.1** $V$ is said to be *Riemannian* if $\mathcal{L}_U h(X, Y) = 0$ for all $x \in N$, $U \in V_x$ and $X, Y \in H_x$, where $\mathcal{L}_U$ denotes the Lie derivation with respect to $U$.

**Remark** If $V$ is integrable then $V$ is Riemannian if and only if its leaves are locally the fibres of a Riemannian submersion. In this case, we say that the foliation determined by $V$ is Riemannian.

Let $\nabla$ denotes the Levi-Civita connection for $h$ and let $\mathcal{V}$ denote the orthogonal projection onto $V$. By the *second fundamental form* of $H$ at $x \in N$ we mean the symmetric bilinear form

$$\zeta = \zeta_x : H_x \times H_x \to V_x,$$

$$(X, Y) \to \frac{1}{2}\mathcal{V}(\nabla_X Y + \nabla_Y X)$$

(where, to evaluate the covariant derivative, $X, Y$ are extended to local vector fields ). When $\|X\| = 1$, $\zeta(X, X)$ is called the *normal curvature* of $H$ in the direction $X$. The vector $\frac{1}{\dim V}\mathrm{Trace}\zeta_x$ is called *mean curvature* of $H$ at $x$. The Finsler bundle $H$ is called *minimal* (resp. *totally geodesic*) if its mean curvature (resp. its second funamental form) vanishes.

Now we characterize Riemannian manifolds in terms of their Finsler bundles. Using the metric $\Pi$ and its orthogonal coframe $\{\psi_a\}$, we define an isomorphism $\mu : H_1^* \to V$ by

$$\Pi(\mu(\theta), X) = [\lambda(\theta)](X).$$

where $\lambda : H_1^* \to V^*$ is given by

$$\lambda(\xi^\alpha \theta_\alpha) = \xi^\alpha \theta_{\bar{\alpha}}.$$

**Lemma 6.3.2** *Indentifying $H_1^*$ with $V$ via $\mu$, the second fundamental form of the Finsler bundle is just the same as the Cartan tensor of $(M, F)$. In particular, the mean curvature of the Finsler bundle is equal to the Cartan form.*

***Proof.*** From (6.18), we have

$$\psi_{\beta\bar{\alpha}} \equiv \sum (A_{\alpha\beta\gamma})\psi_\gamma - \frac{1}{2}R_{\alpha\beta}\psi_n \qquad \mathrm{mod}\ \psi_{\bar{\alpha}}, \tag{6.36}$$

$$\psi_{n\bar{\alpha}} \equiv \frac{1}{2}\sum_\beta R_{\alpha\beta}\psi_\beta. \tag{6.37}$$

Denote the second fundamental form of the Finsler bundle $H$ by $\zeta$. Then its components $\zeta_{\bar{\alpha}}$ satisfy that

$$\begin{aligned}\zeta_{\bar{\alpha}} := {} & \psi_{\bar{\alpha}}(\zeta) \\ \equiv {} & \frac{1}{2}\sum_i [\psi_{i\bar{\alpha}} \otimes \psi_{\bar{\alpha}} + \psi_i \otimes \psi_{i\bar{\alpha}}] \qquad \bmod \psi_{\bar{\alpha}} \\ = {} & (I) + (\mathrm{II}),\end{aligned} \tag{6.38}$$

where

$$\begin{aligned}2(I) : \equiv {} & \Sigma_\beta[\psi_{\beta\bar{\alpha}} \otimes \psi_\beta + \psi_\beta \otimes \psi_{\beta\bar{\alpha}}] \qquad \bmod \psi_{\bar{\alpha}} \\ = {} & \Sigma_\beta[\Sigma_\gamma(H_{\alpha\beta\gamma} + \frac{1}{2}R_{\alpha\gamma\beta})\psi_\gamma - \frac{1}{2}R_{\alpha\beta}\psi_n] \otimes \psi_\beta \\ & + \Sigma_\beta\psi_\beta \otimes [\Sigma_\gamma(H_{\alpha\beta\gamma} + \frac{1}{2}R_{\alpha\gamma\beta})\psi_\gamma - \frac{1}{2}R_{\alpha\beta}\psi_n] \\ = {} & \Sigma_{\beta,\gamma}(H_{\alpha\beta\gamma} + H_{\alpha\gamma\beta})\psi_\beta \otimes \psi_\gamma \\ & + \tfrac{1}{2}\Sigma_{\beta,\gamma}(R_{\alpha\beta\gamma} + R_{\alpha\gamma\beta})\psi_\beta \otimes \psi_\gamma \\ & - \frac{1}{2}\Sigma_\beta R_{\alpha\beta}(\psi_n \otimes \psi_\beta + \psi_\beta \otimes \psi_n) \\ = {} & 2\Sigma_{\beta,\gamma}H_{\alpha\beta\gamma}\psi_\beta \otimes \psi_\gamma - (\psi_{n\bar{\alpha}} \otimes \psi_n + \psi_n \otimes \psi_{n\bar{\alpha}}) \\ = {} & 2\Sigma_{\beta,\gamma}H_{\alpha\beta\gamma}\psi_\beta \otimes \psi_\gamma - 2(\mathrm{II}).\end{aligned} \tag{6.39}$$

□

With the identification $H_1^* \simeq V$, we get

$$\zeta \simeq \sum_\alpha \zeta_{\bar{\alpha}}\psi_\alpha = \sum_{\alpha,\beta,\gamma} \psi_\alpha \otimes \psi_\beta \otimes \psi_\gamma = A.$$

In particular,

$$\eta := \tfrac{1}{n}\mathrm{tr}\zeta \simeq \tfrac{1}{n}\mathrm{tr}A.$$

Combining Theorem 2.2.4 and Lemma 6.3.2, we immediately obtain the following:

**Theorem 6.3.3** ([Mo, 2000]) *A Finsler manifold is Riemannian if and only if its Finsler bundle is minimal.*

**Corollary 6.3.4** *A Finsler manifold is a flat Riemannian manifold if and only if its Finsler bundle has minimal leaves.*

***Proof.*** Follows from Theorem 6.2.2 and Theorem 6.3.3. □

In general, we have:

**Proposition 6.3.5** ([Wood, 1986]) *Suppose $V$ is integrable. Then the foliation determined by $V$ is Riemannian if and only if the corresponding horizontal distribution is totally geodesic.*

By using this result, we give a unified description of Riemannian, Landsberg and weak Landsberg manifold in terms of the property of projective spheres as follows:

**Proposition 6.3.6** *A Finsler manifold is Riemannian (resp. Landsberg; weak Landsberg) if and only if its all projective spheres are Riemannian (resp. totally geodesic; minimal).*

Now we consider the Hilbert form. It is an important section in the Finsler bundle which arise from Hilbert's problem 23 of his famous Paris address of 1900. Note that, from (6.38) and (6.39), we have

$$\zeta_{\bar{\alpha}} \equiv 0 \mod \psi_\beta \quad \text{for} \quad \forall \quad \alpha.$$

Therefore we obtain the following

**Proposition 6.3.7** *The Hilbert form is an asymptotic direction of Finsler bundles, that is, the normal curvature of $\hat{\ell}$ (the dual section of Hilbert form) is vanishing.*

# Chapter 7

# Relation among Three Classes of Invariants

As we see above, there are three classes of geometry invariants on a Finsler manifold. Two classes are non-Riemannian, and one class is Riemannian. Recently, we find some important delicate relation between Riemannian invariants and non-Riemannian invariants. In this chapter, we establish the fundamental relation between tha Cartan tensor, the Landsberg curvature curvature and the flag curvature in spiring of the equation of Akbar-Zadeh in constant curvature manifolds. Then we give the intrinsic relation between the $S$-curvature, the flag curvature and the Cartan form by discussing subtle Ricci identities.

## 7.1 The relation between Cartan tensor and flag curvature

**Proposition 7.1.1** ([Mo, 1999]) *Let $(M, F)$ be a Finsler manifold and $H_{ij\alpha}$ (resp. $R_{\alpha\beta}$) its Cartan (resp. flag curvature) tensor. Then*

$$\ddot{H}_{\alpha\beta\gamma} + H_{\alpha\beta\epsilon}R^{\epsilon}{}_{\gamma} + \frac{1}{6}R^{\beta}{}_{\alpha;\gamma} + \frac{1}{6}R^{\alpha}{}_{\beta;\gamma} + \frac{1}{3}R^{\beta}{}_{\gamma;\alpha} + \frac{1}{3}R^{\alpha}{}_{\gamma;\beta} = 0 \qquad (7.1)$$

*where*

$$\ddot{H}_{\alpha\beta\gamma}\omega^{n} \equiv d\dot{H}_{\alpha\beta\gamma} - \sum \dot{H}_{i\beta\gamma}\omega_{\alpha}{}^{i} - \sum \dot{H}_{\alpha i\gamma}\omega_{\beta}{}^{i} - \sum \dot{H}_{\alpha\beta i}\omega_{\gamma}{}^{i}$$

$$mod \quad \omega^{\epsilon}, \quad \omega_{n}{}^{\epsilon}. \qquad (7.2)$$

***Proof.*** Using Lemma 6.2.1, we have

$$R_{\beta\gamma\alpha} = \frac{R^\beta{}_{\gamma;\alpha} - R^\beta{}_{\alpha;\gamma}}{3}, \tag{7.3}$$

where $R_{\beta\gamma\alpha} := \delta_{\beta\epsilon} R_n{}^\epsilon{}_{\gamma\alpha}$ and $R_i{}^j{}_{k\alpha}$ is the Riemannian curvature of $(M, F)$. On the other hand, from (6.38), we get

$$R^\alpha{}_{\beta;\gamma} = R^\alpha{}_{\beta\gamma} + L_{\alpha\beta\gamma|n} + R_\gamma{}^\alpha{}_{\beta n}, \tag{7.4}$$

where

$$L_{\alpha\beta\gamma} := -\dot{H}_{\alpha\beta\gamma} \tag{7.5}$$

is the Landsberg curvature of $(M, F)$. Substituting (7.3) and (7.5) into (7.4) we obtain that

$$\begin{aligned} R_\alpha{}^\beta{}_{\gamma n} &= R^\beta{}_{\gamma;\alpha} - R^\beta{}_{\gamma\alpha} - L_{\alpha\beta\gamma|n} \\ &= R^\beta{}_{\gamma;\alpha} + \tfrac{1}{3}(R^\beta{}_{\alpha;\gamma} - R^\beta{}_{\gamma;\alpha}) + \ddot{H}_{\alpha\beta\gamma} \\ &= \tfrac{2}{3} R^\beta{}_{\gamma;\alpha} + \tfrac{1}{3} R^\beta{}_{\alpha;\gamma} + \ddot{H}_{\alpha\beta\gamma}, \end{aligned} \tag{7.6}$$

together with (5.13) this gives

$$\begin{aligned} 0 &= R_{\alpha\beta\gamma n} + R_{\beta\alpha\gamma n} + 2H_{\alpha\beta\epsilon} R_n{}^\epsilon{}_{\gamma n} \\ &= \tfrac{2}{3} R^\beta{}_{\gamma;\alpha} + \tfrac{1}{3} R^\beta{}_{\alpha;\gamma} + \ddot{H}_{\alpha\beta\gamma} \\ &\quad + \tfrac{2}{3} R^\alpha{}_{\gamma;\beta} + \tfrac{1}{3} R^\alpha{}_{\beta;\gamma} + \ddot{H}_{\beta\alpha\gamma} + 2H_{\alpha\beta\epsilon} R^\epsilon{}_\gamma \\ &= 2(\ddot{H}_{\alpha\beta\gamma} + H_{\alpha\beta\epsilon} R^\epsilon{}_\gamma + \tfrac{1}{6} R^\beta{}_{\alpha;\gamma} + \tfrac{1}{6} R^\alpha{}_{\beta;\gamma} + \tfrac{1}{3} R^\beta{}_{\gamma;\alpha} + \tfrac{1}{3} R^\alpha{}_{\gamma;\beta}). \end{aligned}$$

□

In particular, when $M$ has constant flag curvature, then

$$R^\alpha{}_{\beta;\gamma} = 0, \tag{7.7}$$

$$R^\epsilon{}_\gamma = c\delta^\epsilon{}_\gamma, \tag{7.8}$$

we get a second-order differential equation for the Cartan tensor as follows (see [Akbar-Zadeh, 1988])

$$\ddot{H}_{\alpha\beta\gamma} + cH_{\alpha\beta\gamma} = 0. \tag{7.9}$$

Furthermore, if $M$ has Landsberg type (for details see Definition 4.3.1) and $c \neq 0$, then $M$ has vanishing Cartan tensor. Hence we obtain

**Theorem 7.1.2** ([Akbar-Zadeh, 1988]) *Suppose that $(M, F)$ is a Landsberg space with non-zero constant flag curvature. Then $(M, F)$ is a Riemann space with non-zero constant sectional curvature.*

## 7.2 Ricci identities

In this section we give some important Ricci identities for a Finsler space which will be used in later. Let $(M, F)$ be an $n$-dimensional Finsler manifold and $SM$ its projective sphere bundle. Let $f : SM \to R$ be a smooth function. From (4.4), we have

$$df = f_{|i}\omega^i + f_{;\alpha}\omega_n{}^\alpha, \tag{7.10}$$

where $|i$ (resp.$;\alpha$) denote horizontal (resp. vertical) covariant derivative. Differenting (7.10) and using the sructure equations one deduces that

$$Df_{|i} \wedge \omega^i + Df_{;\alpha} \wedge \omega_n{}^\alpha + f_{;\alpha}\Omega_n{}^\alpha = 0, \tag{7.11}$$

where

$$Df_{|i} = df_{|i} - f_{|j}\omega_i{}^j, \tag{7.12}$$

$$Df_{;\alpha} = df_{;\alpha} - f_{;\beta}\omega_\alpha{}^\beta. \tag{7.13}$$

Recall that the cuevature 2-form $\Omega_a{}^b$ have the following structure

$$\Omega_i{}^j = \frac{1}{2}R_i{}^j{}_{kl}\omega^k \wedge \omega^l + P_i{}^j{}_{k\alpha}\omega^k \wedge \omega_n{}^\alpha. \tag{7.14}$$

Substituting (7.12), (7.13) and (7.14) into (7.11) we have

$$f_{|i|j} = f_{|j|i} + f_{;\alpha}R_n{}^\alpha{}_{ij}, \tag{7.15}$$

$$f_{|i;\alpha} = f_{;\alpha|i} + f_{;\beta}P_n{}^\beta{}_{i\alpha}, \tag{7.16}$$

$$f_{;\alpha;\beta} = f_{;\beta;\alpha}. \tag{7.17}$$

We denote simply $f_{|n}$ by $\dot{f}$ because it is a globally defined function on $SM$. Let $\{\epsilon_i, \epsilon_{\bar{\alpha}}\}$ be the dual basis of $\{\omega_i, \omega_n{}^\alpha\}$ with respect to G. By (7.10) and (7.12) we have

$$\begin{aligned} f_{|n;\alpha} &= (Df_{|n})(\epsilon_{\bar{\alpha}}) \\ &= (df_{|n} - f_{|b}\omega_n{}^b)(\epsilon_{\bar{\alpha}}) = \dot{f}_{;\alpha} - f_{|\alpha}. \end{aligned} \tag{7.18}$$

Note that

$$P_n{}^\beta{}_{n\alpha} = -L^\beta{}_{n\alpha} = -\dot{H}^\beta{}_{n\alpha} = 0. \tag{7.19}$$

Together with (7.16) and (7.18) we get

$$f_{;\alpha|n} = \dot{f}_{;\alpha} - f_{|\alpha}. \tag{7.20}$$

By (7.10) and (7.12) we have

$$f_{|n|\alpha} = Df_{|n}(\epsilon_\alpha) = (df_{|n} - f_{|j}\omega_n{}^j)(\epsilon_\alpha) = \dot{f}_{|\alpha}. \tag{7.21}$$

Differentiate (7.20) along the Hilbert form and use (7.15),(7.16) and (7.21). We have

$$\begin{aligned} f_{;\alpha|n|n} &= \dot{f}_{;\alpha|n} - f_{|\alpha|n} \\ &= \dot{f}_{|n;\alpha} - f_{|n|\alpha} - f_{;\beta}R^\beta{}_\alpha \\ &= \dot{f}_{|n;\alpha} - \dot{f}_{|\alpha} - f_{;\beta}R^\beta{}_\alpha. \end{aligned}$$

Together with (7.20), we have the following

**Lemma 7.2.1** *For any smooth function $f$ on $SM$, we have*

$$f_{;\alpha|n} = \dot{f}_{;\alpha} - f_{|\alpha}, \tag{7.22}$$

$$f_{;\alpha|n|n} = \dot{f}_{|n;\alpha} - \dot{f}_{|\alpha} - f_{;\beta}R^\beta{}_\alpha. \tag{7.23}$$

**Corollary 7.2.2** *Let $(M, F)$ be a Finsler manifold. Then its Cartan form $\eta_\alpha$, S-curvature $S$ and Flag curvature tensor $R^\beta{}_\alpha$ satisfy*

$$\left(\frac{S}{F}\right)_{|n;\alpha} - \left(\frac{S}{F}\right)_{|\alpha} = \ddot{\eta}_\alpha + \eta_\beta R^\beta{}_\alpha. \tag{7.24}$$

***Proof.*** Recall that

$$\tau_{;\alpha} = \eta_\alpha; \quad \tau_{|n} = \frac{S}{F}, \tag{7.25}$$

where $\tau$ is the distortion of $(M, F)$. By using (7.23) and (7.25) we have

$$\begin{aligned} \ddot{\eta}_\alpha &= \tau_{;\alpha|n|n} \\ &= \dot{\tau}_{|n;\alpha} - \dot{\tau}_{|\alpha} - \tau_{;\beta}R^\beta{}_\alpha \\ &= (\tfrac{S}{F})_{|n;\alpha} - (\tfrac{S}{F})_{|\alpha} - \eta_\beta R^\beta{}_\alpha. \end{aligned}$$

□

## 7.3 The relation between $S$-curvature and flag curvature

**Lemma 7.3.1** *Let $(M, F)$ be a Finsler manifold. Then*

$$\ddot{\eta}_\alpha + \eta_\beta R^\beta{}_\alpha + \frac{1}{3}(Ric_{;\alpha} + 2R^\beta{}_{\alpha;\beta}) = 0. \tag{7.26}$$

***Proof.*** Contract (7.1). □

**Corollary 7.3.2**(Z. Shen) *Suppose that $(M, F)$ is a Finsler manifold with constant curvature c. Then*

$$\ddot{\eta}_\alpha + c\eta_\alpha = 0. \tag{7.27}$$

*Furthermore, if $F$ has weak Landsberg type and $c \neq 0$. Then $(M, F)$ is a Riemannian manifold with non-zero constant sectional curvature.*

***Proof.*** Suppose that $(M, F)$ is of constant curvature c. Then

$$Ric_{;\alpha} = 0.$$

Together with (7.7), (7.8) we have (7.27). If $F$ has vanishing mean Landsberg curvature, then $J_\alpha = -\dot{\eta}_\alpha = 0$. Combine with $c \neq 0$ we get $\eta_\alpha = 0$. By using Deicke's result we have $(M, F)$ is Riemannian (cf. Theorem 2.2.4). □

**Theorem 7.3.3** *Let $(M,\ F)$ be a Finsler space. Then its flag curvature tensor $R^\beta{}_\alpha$, $S$-curvature $\mathbf{S}$, Ricci scalar $Ric$ and Cartan form $\eta_\alpha$ satisfy*

$$\ddot{\eta}_\alpha + \eta_\beta R^\beta{}_\alpha = \left(\frac{\mathbf{S}}{F}\right)_{|n;\alpha} - \left(\frac{\mathbf{S}}{F}\right)_{|\alpha} = -\frac{1}{3}(Ric_{;\alpha} + 2R^\beta{}_{\alpha;\beta}). \tag{7.28}$$

***Proof.*** Combine Corollary 7.2.2 with Lemma 7.3.1. □

## 7.4 Finsler manifolds with constant $S$-curvature

Recall that a Finsler metric is called to be constant $S$-curvature if $S = (n+1)cF$ for some constant c, where $n = \dim$ M. Now we consider a Finsler manifold $(M, F)$ with constant $s$-curvature. By using (7.24) we have

$$\ddot{\eta}_\alpha + \eta_\beta R^\beta{}_\alpha = 0. \tag{7.29}$$

This equation deduce the following

**Theorem 7.4.1** ([Shen, 2001]) *Let $(M, F)$ be a compact Finsler manifold with constant $S$-curvature . If $F$ has non-positive flag curvature, then $F$ has weak Landberg type. In particular if its flag curvature is negative at $x \in M$, then $F$ is Riemannian at $x$, i.e. $F^2(x, y) = \sum_{i,j} g_{ij}(x)y^i y^j$.*

Denote the principal curvature of $(M, F)$ by $\kappa_1, \cdots, \kappa_{n-1}$, i.e.

$$R_{\alpha\beta} = \kappa_\alpha \delta_{\alpha\beta}. \tag{7.30}$$

If $(M, F)$ is a weak Landsberg manifold with constant $S$-curvature, then we get

$$\eta_\alpha \kappa_\alpha = 0, \quad \alpha = 1, \cdots, n-1.$$

Hence we get

**Theorem 7.4.2** ([Mo, 2005]) *Let $(M, F)$ be an $n$-dimensional weak Landsberg manifold with constant $S$-curvature. If $\kappa_1, \cdots, \kappa_{n-1} \neq 0$, then $(M, F)$ is a Riemannian manifold.*

Recall that a Finsler manifold is of Berwald type if it has vanishing Minkowskian curvature. It is easy to see locally Minkowski manifolds and Riemannian manifoldas are all Berwald manifolds. Furthermore, Berwald manifolds are of weak Landsberg type and satisfy $S = 0$[Shen, 1997]. we have following:

**Corollary 7.4.3** *Let $(M, F)$ be an $n$-dimensional Berwald manifold. Suppose that $\kappa_1, \cdots, \kappa_{n-1} \neq 0$, then $(M, F)$ is Riemannian.*

For a Finsler surface, its Gaussian curvature $\kappa$ takes the place of the flag curvature in the general case.

**Theorem 7.4.4** ([Mo, 2005]) *Suppose that $(M, F)$ is a Finsler surface with constant $S$-curvature. Then the Gauss curvature of $(M, F)$ lives on $M$.*

Note that a Finsler surface is locally Minkowski if and only if it has vanishing Gaussian curvature and Minkowskian curvature. Thus our corollary

7.4.4 and Theorem 7.4.5 are a natural extension of Szabó's theorem [Szabó, 1981]. Szabó's result tells us any Berwald surface is Riemannian or locally Minkowskian.

In Theorem 7.3.2, the condition of $(M, F)$ to be weak Landsberg can not be removed. In fact Bao and Shen recently constructed, by using Randers metrics, a family of non-Riemannian Finsler structrues $F_k$ on $S^3$ with constant flag curvature one and vanishing $S$-curvature [Bao and Shen, 2002]. From the well-known formula (7.9) due to Akbar-Zadeh, we see that the $F_k$ are not weak Landsberg structures. According to Corollary 7.3.2 (cf.[Shen, 2001], Theorem 9.1.1]), any weak Landsberg metric of non-zero constant flag curvature is Riemannian. Here we weaken Shen's condition on the constant flag curvature and impose the constancy of $S$-curvature on the Finsler metric instead.

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# Chapter 8

# Finsler Manifolds with Scalar Curvature

A Finsler manifold $(M, F)$ is called to be of scalar curvature if for any $y \in TM\backslash\{0\}$ the flag curature $K(P, y)$ is independent of the tangent $P$ containing $y$. When dim$M$=2, $(M, F)$ is automatically of scalar curvature. In this chapter we will discuss Finsler manifolds with scalar curvature by using the fundamental equations derived in previous chapter. We particular determine the flag curvature when certain non-Riemannian quantities are isotropic and give Akbar-Zadeh type differential equations of Matsumoto torsion. We also give a series of their applications.

## 8.1 Finsler manifolds with isotropic $S$-curvature

In this section we partically detemine the flag curvature of a Finsler manifold when its $S$-curvature is isotropic.

**Theorem 8.1.1** ([Chen, Mo and Shen, 2003]) *Let $(M, F)$ be an $n$-dimensional Finsler manifold of scalar curvature $\kappa(x, y)$. Suppose that the $S$-curvature is isotropic,*

$$S = (n+1)c(x)F(x, y),$$

*where $c(x)$ is a scalar function on $M$. Then there is a scalar function $\sigma(x)$ on $M$ such that*

$$\kappa = 3\dot{c} + \sigma.$$

*In particular, $c(x) = c$ is a constant if and only if $\kappa = \kappa(x)$ is a scalar function on $M$. In this case, $\kappa =$ constant when $n \geq 3$.*

***Proof.*** Suppose that $(M, F)$ is of scalar curvature $\kappa$. It is easy to show

$$R^\beta{}_\alpha = \kappa \delta^\beta{}_\alpha, \tag{8.1}$$

$$Ric_{;\alpha} = (n-1)\kappa_{;\alpha}, \tag{8.2}$$

$$R^\beta{}_{\alpha;\gamma} = \kappa_{;\gamma}\delta^\beta{}_\alpha. \tag{8.3}$$

Plug them into (7.28), we obtan

$$\left(\frac{S}{F}\right)_{|n;\alpha} - \left(\frac{S}{F}\right)_{|\alpha} = -\frac{1}{3}(n+1)\kappa_{;\alpha}. \tag{8.4}$$

Note that

$$\frac{S}{F} = (n+1)c(x).$$

Hence

$$c_{|n;\alpha} - c_{|\alpha} = -\frac{1}{3}\kappa_{;\alpha}. \tag{8.5}$$

By using (7.16) and (7.19), we have

$$c_{|n;\alpha} = c_{;\alpha|n} = 0. \tag{8.6}$$

(7.22) tells us

$$c_{|\alpha} = \dot{c}_{;\alpha} - c_{;\alpha|n} = \dot{c}_{;\alpha}. \tag{8.7}$$

Substituting (8.6) and (8.7) into (8.5) yields

$$(\kappa - 3\dot{c})_{;\alpha} = 0.$$

Thus

$$\sigma := \kappa - 3\dot{c}$$

is a scalar function on $M$. In terms of the local coordinat $(x^i, y^i)$ on $M$, we have

$$\dot{c} = \frac{\partial c}{\partial x^i}\frac{y^i}{F}.$$

Thus $\kappa = \kappa(x)$ if and only if $c = \mathit{constant}$. In this case, $\kappa = \mathit{constant}$ when $n \geq 3$ by the Schur theorem (cf. Proposition 5.2.5). $\square$

For any Finsler manifold $(M, F)$ with positive constant flag curvature and zero $S$-curvature, Kim and Yim have showed that if $F$ is reversible, then $F$ is Riemannian. Thus we obtain

**Corollary 8.1.2** *Let $(M, F)$ be an $m(\geq 3)$-dimensional reversible Finsler manifold with zero $S$-curvature and positive scalar curvature. Then $(M, F)$ is Riemannian.*

We have a lot of Finsler surfaces whose Gauss curvature are not scalar functions (see Section 8.3). In the case that $dimM \geq 3$, we also have many examples of Finsler manifolds of scalar curvatures, with a non-trivial dependence on $y$. Recall that a randers metric $\alpha+\beta$ is locally projectuively flat (and therefor of scalar curvature) if and only if $\alpha$ is of constant curvature and $\beta$ is a closed 1-form. Let $(M, \alpha)$ be a compact Riemannian manifold with constant sectional curvature $\bar{\kappa} = 1$ and $\beta$ be an arbitrary closed 1-form on $M$. Then the Randers metric $F_\epsilon = \alpha + \epsilon\beta$ is of scalar curvature with negative curvature for small $\epsilon$. If in addition, $\kappa = \kappa(x)$ is a scalar function on $M$ (in which case it is constant when $n \geq 3$ by Schur theorem) then $\beta = 0$ follows from Akbar-Zadeh's rigidity theorem: every Finsler metric on a compact manifold of negative constant flag curvature must be Riemannian. We can always conclude that the flag curvature $\kappa$ depends on $y$ if $n \geq 3$ and $\beta \neq 0$.

**Remark** The reversibility in Corollary 8.1.2 can not be dropped. Z.Shen has constructed many non-Riemannian Finsler metrics on $S^n$ with constant flag curvature 1 and zero $S$-curvature.

## 8.2 Fundamental equation on Finsler manifolds with scalar curvature

Let $(M, F)$ be a Finsler manifold. Let

$$M_{\alpha\beta\gamma} := H_{\alpha\beta\gamma} - \frac{1}{n+1}\eta_\alpha\delta_{\beta\gamma} - \frac{1}{n+1}\eta_\beta\delta_{\gamma\alpha} - \frac{1}{n+1}\eta_\gamma\delta_{\alpha\beta} \tag{8.8}$$

where $H_{\alpha\beta\gamma}$ is Cartan tensor and $\eta_\alpha$ is Cartan form. We obtain a totally symmetric tensor. Note that $M_{\alpha\beta\gamma} = 0$ for Finsler surfaces. The quantity $M_{\alpha\beta\gamma}$ is introduced by M.Matsumoto [Matsumoto, 1972].

**Definition 8.2.1** We call totally symmetric tensor $M_{\alpha\beta\gamma}$ the *Matsumoto torsion.*

Matsumoto proves that every Randers metric satisfies that $M_{\alpha\beta\gamma} = 0$. Later on, Matsumoto-Hōjō proves that the converse is true too.

**Theorem 8.2.2** ([Matsumoto and Hōjō, 1978]) *A Finsler manifold of dimension $n \geq 3$ is a Randers manifold if and only if it has zero Matsumoto torsion.*

**Remark** The Matsumoto torsion gives a measure of the failure of a Finsler manifold to be a Riemannian manifold.

**Proposition 8.2.3** ([Mo and Shen, 2005]) *Let $(M, F)$ be a Finsler manifold with scalar curvature $\kappa$. Then its Matsumoto torsion satisfies the following Fundamental Equation:*

$$\ddot{M}_{\alpha\beta\gamma} + \kappa M_{\alpha\beta\gamma} = 0 \tag{8.9}$$

***Proof.*** Plugging (8.1),(8.2) and (8.3) into (7.28) yields

$$\ddot{\eta}_\alpha + \eta_\alpha \kappa = -\frac{1}{3}(n+1)\kappa_{;\alpha} \tag{8.10}$$

that is,

$$\kappa_{;\alpha} = -\frac{3}{n+1}(\ddot{\eta}_\alpha + \kappa\eta_\alpha). \tag{8.11}$$

On the other hand, substitute (8.1),(8.2) and (8.3) into (7.1) we have

$$\ddot{H}_{\alpha\beta\gamma} + \kappa H_{\alpha\beta\gamma} + \frac{1}{3}\delta_{\alpha\beta}\kappa_{;\gamma} + \frac{1}{3}\delta_{\beta\gamma}\kappa_{;\alpha} + \frac{1}{3}\delta_{\gamma\alpha}\kappa_{;\beta} = 0. \tag{8.12}$$

Now (8.9) follows from (8.8), (8.11) and (8.12). □

Using the fundamental equation (8.9), Theorem 8.2.2 and ODE theory one can show the following

**Theorem 8.2.4** ([Mo and Shen, 2005]) *Let $(M, F)$ be a compact negatively curved Finsler manifold of dimension $n \geq 3$. Suppose that $F$ is of scalar curvature. Then $F$ is a Randers metric.*

Theorem 8.2.4 greatly narrows down the possibility of compact negatively curved Finsler manifolds of scalar curvature. Although Randers metrics are very special Finsler metrics, the classification of Randers metrics of scalar curvature has not been completely done yet. See [Yasuda and Shimada, 1977; Matsumoto, 1989; Matsumoto and Shimada, 2002; Bao and Robles, 2003; Bao, Robles and Shen, 2004] for the classification of Randers metric of constant flag curvature.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. Based on Theorem 8.2.4, we obtain following

**Theorem 8.2.5** *Let $(M, F)$ be a compact negatively curved Finsler manifold of dimension $n \geq 3$. $F$ is locally projectively flat if and only if $F = \alpha+\beta$ is a Randers metric where $\alpha$ is of constant sectional curvature and $\beta$ is a closed 1-form.*

***Proof.*** Since locally projectively flat Finsler metric $F$ has vanishing Weil curvature and Douglas curvature, we get $F$ has scalar curvature. Recall that the Weil curvature gives a measure of the failure of a Finsler metric to be of scalar curvature. By using Theorem 8.2.4, $F$ is a Randers metric. Vanishing Douglas curvature implies that $\beta$ is a closed form and $\alpha$ has vanishing Weyl curvature too. Thus $\alpha$ is of constant sectional curvature.

Conversely, if $d\beta = 0$ then $F = \alpha + \beta$ has vanishing Dougals curvature. Together with $\alpha$'s constant curvature, $F$ has vanishing Weil curvature. Thus $F$ is locally projectively flat. □

**Corollary 8.2.6** *Let $(M, F)$ be a compact negatively curved Finsler manifold of dimension $n \geq 3$. Suppose that $F$ is of scalar curvature and weakly Landsbergian. Then $F$ is Riemannian.*

***Proof.*** It follows from Theorem 8.2.4 that $F$ is a Randers metric. Thus $F$ has vanishing Matsumoto metric from Proposition 8.2.3. Since $F$ is of weak Landsberg type,

$$\begin{aligned} 0 &= \dot{M}_{\alpha\beta\gamma} \\ &= \dot{H}_{\alpha\beta\gamma} - \tfrac{1}{n+1}\dot{\eta}_\alpha \delta_{\beta\gamma} - \tfrac{1}{n+1}\dot{\eta}_\beta \delta_{\gamma\alpha} - \tfrac{1}{n+1}\dot{\eta}_\gamma \delta_{\alpha\beta} \\ &= \dot{H}_{\alpha\beta\gamma} = -L_{\alpha\beta\gamma}. \end{aligned}$$

That is, $F$ is a Landsberg metric. Then the follows from Numata's Theorem [Numata, 1975]. □

According to Numata [Numata, 1975], any Landsberg metric of scalar curvature with non-zero flag curvature is Riemannian. Corollary 8.2.6 weaken Numata's condition on the Landsberg curvature and impose the compactness on the manifold instead. We do not know if Corollary 8.2.8 is still true for weakly Landsberg metric of scalar curvature with positive flag curvature. It is known that every weakly Landsberg metric of non-zero constant flag curvature is Riemannian (see Corollary 7.3.2).

Let $S$ be the $S$-curvature of a Finsler manifold $(M, F)$, and let

$$E_{ij} := \frac{1}{2}\frac{\partial^2 S}{\partial y^i \partial y^j}, \quad E_y = E_{ij}(x,y)dx^i \otimes dx^j$$

**Definition 8.2.7** We call $E := \{E_y\}$ *mean Berwald curvature.* $F$ is said to be *weakly Berwaldian* if $E = 0$.

**Corollary 8.2.8** *Let $(M, F)$ be a compact negatively curved Finsler manifold of dimension $n \geq 3$. Suppose that $F$ is of scalar curvature and weakly Berwaldian. Then $F$ is Riemannian.*

***Proof.*** It follows from Theorem 8.2.4 that $F$ is a Randers metric. According [Chen and Shen, 2003] a Randers metric has constant Berwald curvature if and only if it has constant $S$-curvature. Thus $F$ has constant $S$-curvature. Using Theorem 8.1.1, every Finsler metric of scalar curvature with constant $S$-curvature must be of constant flag curvature in dimension greater than two. Thus the flag curvature of $F$ is a non-zero constant. By Akbar-Zadeh's Rigidity theorem [Akbar-Zadeh, 1988], $F$ must be Riemannian. □

The compactness in Corollary 8.2.8 can not be dropped. Example 3.3.3 in [Bao and Robles, 2003] is a weakly Berwaldian Randers metric on $S^2 \times (0, \tau)$ with constant curvature -1 where $\tau$ is a positive real number.

From the proof of Theorem 8.2.5, a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only of $\alpha$ is of conatant sectional curvature and $\beta$ is closed. In Theorem 8.1.1, we find $\beta$ such that $F = \alpha + \beta$ is locally projectively flat with $S = (n+1)c(x)F$ [Chen, Mo and Shen, 2003]. Hence we have completely classified the locally projectively flat Randers metric of isotropic $S$-curvature.

## 8.3 Finsler metrics with relatively isotropic mean Landsberg curvature

Let $(M, F)$ be a Finsler manifold and $J$ (resp. $\eta$) its mean Landsberg curvature (resp. Cartan form). $(M, F)$ is said to be of relatively isotropic mean Landsberg curvature if $J = c(x)\eta$ where $c = c(x)$ is a scalar function on $M$.

**Theorem 8.3.1** *Let $(M, F)$ be an n-dimendional Finsler manifold of scalar curvature. Suppose that $J$ is relatively isotropic*

$$J - c(x)\eta = 0 \tag{8.13}$$

*where $c = c(x)$ is a $C^\infty$ scalar function on $M$. Then the flag curvature $\kappa = \kappa(x, y)$ and the distortion $\tau = \tau(x, y)$ satisfy*

$$\frac{n+1}{3}\kappa_{;\alpha} + (\kappa + c^2 - \dot{c})\tau_{;\alpha} = 0. \tag{8.14}$$

*Moreover,*

*(a) If $c(x) = c$ is a constant, then there is a scalar function $\rho(x)$ on $M$ such that*

$$\kappa = -c^2 + \rho e^{-\frac{3\tau}{n+1}}. \tag{8.15}$$

*(b) Suppose that $F$ is non-Riemannian on any open subset of $M$. If $\kappa = \kappa(x)$ is a scalar function on $M$, then $c(x) = c$ is a constant, in which case $\kappa = -c^2 \le 0$.*

***Proof.*** By assumption (8.13),

$$J_\alpha - c\eta_\alpha = 0 \tag{8.16}$$

for $\forall \alpha \in \{1, 2, \cdots, n-1\}$. Note that (see Section 4.3)

$$J_\alpha = -\dot{\eta}_\alpha. \tag{8.17}$$

Plugging (8.17) into (8.13) yields

$$-\dot{J}_\alpha + \kappa\eta_\alpha = -\frac{1}{3}(n+1)\kappa_{;\alpha}. \tag{8.18}$$

By using (8.16) and (8.17) we have

$$\begin{aligned}\dot{J}_\alpha &= \dot{c}\eta_\alpha + c\dot{\eta}_\alpha \\ &= \dot{c}\eta_\alpha + c(-J_\alpha) \\ &= \dot{c}\eta_\alpha - c^2\eta_\alpha = (\dot{c} - c^2)\eta_\alpha.\end{aligned}$$

Substituting it into (8.18) we obtain

$$(c^2 - \dot{c})\eta_\alpha + \kappa\eta_\alpha + \frac{1}{3}(n+1)\kappa_{;\alpha} = 0.$$

Hence (8.14) holds. Recall that $\tau_{;\alpha} = \eta_\alpha$.

(a) Suppose that $c = c(x)$ is a constant. Then equation (8.14) simplifies to

$$\frac{n+1}{3}\kappa_{;\alpha} + (\kappa + c^2)\tau_{;\alpha} = 0.$$

This implies that

$$(log\rho^{\frac{n+1}{3}})_{;\alpha} = 0,$$

where $\rho = (K + c^2)e^{\frac{3\tau}{n+1}}$. Thus $\rho = \rho(x)$ is a scalar function on $M$ and (8.15) holds.

(b) Suppose that $\kappa = \kappa(x)$ is a scalar function on $M$. Then (8.14) simplifies to

$$(\kappa + c^2 - \dot{c})\tau_{;\alpha} = 0. \tag{8.19}$$

We claim that $c(x) = c$ is a constant. If this is false, then there is an open subset $\mathcal{U}$ such that $dc(x) \neq 0$ for any $x \in \mathcal{U}$. Clearly, at any $x \in \mathcal{U}$, $\kappa(x) \neq -c(x)^2 + \dot{c}(x,y)$ for almost all $y \in T_xM$. By (8.19), $\tau_{;\alpha} = \eta_\alpha = 0$. Thus $F$ is Riemannian on $\mathcal{U}$ by Deicke's theorem (cf. Theorem 2.2.4). This contradicts our assumption in the theorem. This proves the claim. By (8.15) and (8.17)

$$\rho(x)\tau_{;\alpha} = 0. \tag{8.20}$$

We claim that $\rho(x) = 0$. If this is false, then there is an open subset $\mathcal{U}$ such that $\rho(x) \neq 0$ for any $x \in \mathcal{U}$. By (8.20), we obtain that $\tau_{;\alpha} = \eta_\alpha = 0$ on $\mathcal{U}$. Thus $F$ is Riemannian on $\mathcal{U}$. This again contradicts the assumption in the theorem. Therefore $\rho(x) \equiv 0$. We conclude that $\kappa = -c^2$ by (8.15). $\square$

**Example** Consider the Randers metrics defined in (4.16) and (4.17), It is easy to verify that

$$J_\alpha - c\eta_\alpha = 0,$$

where $c = c(x)$ is defined in (4.18). when $\dim M = 2$, a direct calculation show the Gauss curvature is given by

$$\kappa = \frac{-5\kappa^2 - 4\zeta[\epsilon + (\kappa^2 + \epsilon\zeta)|x|^2]}{4[\epsilon + (\kappa^2 + \epsilon\zeta)|x|^2]^2} + \frac{3(\kappa^2 + \epsilon\zeta)(1 + \zeta|x|^2)\alpha}{[\epsilon + (\kappa^2 + \epsilon\zeta)|x|^2]^2 F}.$$

In fact, for any Randers metric $F = \alpha + \beta$, $F$ is of relatively isotropic mean Landsberg curvature if and only if $F$ has isotropic $S$-curvature and $\beta$ is closed. For a general Finsler metric, (8.13) does not imply that

$$S = (n + 1)c(x)F. \tag{8.21}$$

Now we combine two conditions (8.13) and (8.21) and prove the following:

**Theorem 8.3.2** *Let $(M, F)$ be an $n$-dimensional Finsler manifold of scalar curvature. Suppose that the $S$-curvature and the mean Landsberg curvature satisfy*

$$S = (n + 1)cF, \quad J = c\eta, \tag{8.22}$$

*where $c = c(x)$ is a scalar function on $M$. Then the flag curvature is given by*

$$\kappa = 3\dot{c} + \sigma = -\frac{3c^2 + \sigma}{2} + \mu e^{\frac{-2\tau}{n+1}},$$

*where $\sigma(x)$ and $\mu(x)$ are scalar function on $M$.*

*(a) Suppose that $F$ is not Riemannian on any open subset in $M$. If $c(x) = c$ is a constant, then $\kappa = -c^2$, $\sigma(x) = -c^2$ and $\mu(x) = 0$.*

*(b) If $c(x) \neq$ constant, then the distortion is given by*

$$\tau = \ln\left[\frac{2\mu}{6\dot{c} + 3(c^2 + \sigma)}\right]^{\frac{n+1}{2}}. \tag{8.23}$$

***Proof.*** By Theorem 8.1.1 we have

$$\dot{c} = \frac{1}{3}(\kappa - \sigma)$$

for some scalar function $\sigma$ on $M$. Plugging it into (8.14) yields

$$\frac{n+1}{3}\kappa_{;\alpha} + (\frac{2}{3}\kappa + c^2 + \frac{\sigma}{3})\tau_{;\alpha}. \tag{8.24}$$

We obtain

$$[(2\kappa + 3c^2 + \sigma)^{\frac{n+1}{2}} e^{\tau}]_{;\alpha} = 0. \tag{8.25}$$

Thus there is a scalar function $\mu(x)$ on $M$ such that

$$\kappa = \mu e^{-\frac{2\tau}{n+1}} - \frac{3c^2 + \sigma}{2}. \tag{8.26}$$

Comparing (8.26) and the formula in Theorem 8.1.1, we obtain

$$\dot{c} = -\frac{1}{2}(c^2 + \sigma) + \frac{\mu}{3} e^{-\frac{2\tau}{n+1}}. \tag{8.27}$$

(a) Suppose that $c(x) = c$ is a constant. We claim that $\mu(x) = 0$. If it false, then $\mathcal{U} := \{x \in M, \mu(x) \neq 0\} \neq \phi$. From (8.27) $\tau = \tau(x)$ is a scalar function on $\mathcal{U}$, hence $F$ is Riemannian on $\mathcal{U}$. This contradicts the assumption in (a). Now (8.27) is reduced to that $\sigma = -c^2$ and (8.26) is reduced to that $\kappa = -c^2$.

(b) If $c(x) \neq$ *constant*, then $\mu \neq 0$ by (8.27). In this case, we can solve (8.27) for $\tau$ and obtain (8.23). $\square$

# Chapter 9

# Harmonic Maps from Finsler Manifolds

A Finsler manifold is a Riemannian manifold without the quadratic restiction. In this chaper we introduce the energy function, the Euler-Lagrange operator, and the sress-energy tensor for a smooth map $\phi$ from a Finsler manifold to a Riemannian manifold. we show that $\phi$ is an extremal of the energy function if and only if $\phi$ satisfies the corresponding Euler-Lagrange equation and the fundamental existence theorem of harmonic maps on a Finsler space. We also characterize weak Landsberg manifolds in terms of harmonicity and horizontal conserativity. Using the representation of a tensor field in terms of geodesic coefficients, we construct new examples of harmonic maps from Berwald manifolds which are neither Riemannian nor Minkowskian.

## 9.1 Some definitions and lemmas

Let $(M, F)$ be an $m$-dimensional Finsler manifold and $\omega_1, \cdots, \omega_m$ the adapted frame fields on the dual Finsler bundle (cf. Lemma 3.1.1). Let $\omega_{ij}$ be chern connection 1-form. Recall that $g_{ij} = [\frac{1}{2}F^2]_{y^i y^j}$. From (4.1) it is easy to see that

$$\omega_1 \wedge \cdots \wedge \omega_m \wedge \omega_{m1} \wedge \cdots \wedge \omega_{m,m-1}$$

is the volume form with respect to the Riemannian metric on $SM$. We denote it by $\Pi$.

The following lemmas will be used later.

**Lemma 9.1.1** *If $M$ is a compact Finsler manifold, then for any function*

$f : SM \to \mathbb{R}$

$$\int_{SM} f\Pi = \int_M dx \int_{S_xM} f\sqrt{\det(g_{ij})}\chi$$

*where* $dx = dx^1 \wedge \cdots \wedge dx^m$ *and*

$$\chi \equiv \omega_{m1} \wedge \cdots \wedge \omega_{m,m-1} \mod dx^j.$$

*In particular, if* $M$ *is Riemannian and* $f$ *is defined on* $M$*, then*

$$\int_{SM} f\Pi = \mathrm{Vol}(S^{m-1}) \int_M f dv,$$

*where* $\mathrm{Vol}(S^{m-1})$ *is the volume of stand* $(m-1)$*-dimendional sphere.*

***Proof.*** Obvious. □

In this chapter, we set

$$1 \leq i, j, k, \cdots \leq m,$$

$$1 \leq \lambda, \mu, \tau, \cdots \leq m-1, \quad \bar{\lambda} = m + \lambda$$

$$1 \leq a, b, c, \cdots \leq 2m-1,$$

From (4.2) and (4.3), the first structure equation $(M, F)$ can be written as

$$d\omega_i = \sum \omega_j \wedge \omega_{ji}, \quad \omega_{ij} + \omega_{ji} = -2\sum H_{ij\lambda}\omega_{m\lambda}, \tag{9.1}$$

where $H_{ij\lambda} = A(e_i, e_j, e_\lambda)$. By using (5.4) the curvature 2-form $\Omega_{ij} := d\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj}$ can be expressed in the form

$$\Omega_{ij} = \frac{1}{2}\sum R_{ijkl}\omega_k \wedge \omega_l + \sum P_{ijk\lambda}\omega_k \wedge \omega_{m\lambda},$$

where $R_{ijkl} = -R_{ijlk}$. Set

$$L_{\lambda\mu\gamma} := P_{m\lambda\mu\gamma}$$

is the Landsberg curvature. From the proof of Proposition 5.1.2, we have

$$\sum_\lambda L_{\lambda\lambda\mu} = -\sum_\lambda \dot{H}_{\lambda\lambda\mu}, \quad L_{m\lambda\mu} = 0, \tag{9.2}$$

where the dot denotes the convariant derivative along the Hilbert form as usual.

Denote the Riemannian metric on $SM$ by $\Pi$. The divergence of a form $\Psi$ on $SM$ with respect to $\Pi$ is defined by

$$\operatorname{div}\Psi := \sum_a (D_{\epsilon_a}\Psi)(\quad, \epsilon_a)$$

where $\{\epsilon_a\}$ is the dual basis of $\{\omega_1, \cdots, \omega_m, \omega_{m1}, \cdots, \omega_{m,m-1}\}$ on $T(SM)$ and $D_{\epsilon_a}$ is the covariant derivative induced by $G$ along $\epsilon_a$.

**Lemma 9.1.2**

(i) *For* $S = \sum S_i\omega_i \in \Gamma(p^*T^*M)$, $\operatorname{div}S = \sum S_{i|i} + \sum S_\mu L_{\lambda\lambda\mu}$.

(ii) *For* $T = \sum T_{ij}\omega_i\omega_j \in \Gamma(\Theta^2 p^*T^*M)$,

$$\operatorname{div}T(e_i) = \sum T_{ij|j} + \sum T_{i\mu}L_{\lambda\lambda\mu}.$$

***Proof.*** With the abbreviations

$$\psi_i = \omega_i, \quad \psi_{\bar\lambda} = \omega_{m\lambda}$$

denote the Levi-Civita connection with respect to $\{\psi_a\}$ by $\{\psi_{ab}\}$. By using (6.18), we have

$$\psi_{ij} \equiv \omega_{ji} \mod \psi_{\bar\lambda}, \quad \psi_{i\bar\lambda} \equiv -\sum L_{i\lambda\mu}\psi_{\bar\mu} \mod \psi_j.$$

Thus

$$\begin{aligned}\operatorname{div}S &= \textstyle\sum_a (DS_a)(\epsilon_a)\\ &= \textstyle\sum (dS_i - \sum S_j\psi_{ji})(\epsilon_i) - \sum S_i\psi_{i\bar\lambda}(\epsilon_{\bar\lambda})\\ &= \textstyle\sum (ds_i - \sum s_j\omega_{ji})(\epsilon_i) - \sum S_\mu(-L_{\lambda\lambda\mu})\\ &= \textstyle\sum S_{i|i} + \sum S_\mu L_{\lambda\lambda\mu},\end{aligned}$$

where the covariant derivative of $S$ is defined by

$$DS_i = dS_i - \sum S_j\omega_{ij} = \sum S_{i|j}\omega_j + \sum S_{i;\lambda}\omega_{m\lambda}.$$

This proves (i).

Similarly, for $T$ we have

$$\begin{aligned}\operatorname{div}T(\epsilon_i) &= \textstyle\sum (dT_{ib} - \sum T_{cb}\psi_{ci} - \sum T_{ic}\psi_{cb})(\epsilon_b)\\ &= \textstyle\sum (dT_{ij} - \sum T_{kj}\omega_{ik} - \sum T_{ik}\omega_{jk})(\epsilon_j) + \sum T_{i\mu}L_{\lambda\lambda\mu}\\ &= \textstyle\sum T_{ij|j} + \sum T_{i\mu}L_{\lambda\lambda\mu}\end{aligned}$$

which proves (ii). □

The energy density of a map $\phi : (M, F) \to (N, h)$ from a Finsler manifold to a Riemannian manifold is the function $e(\phi) : SM \to \mathbb{R}^+ \cup \{0\}$ defined by

$$e(\phi)(x, [y]) = \frac{1}{2} \sum_j h(\phi_* e_j, \phi_* e_j), \tag{9.3}$$

where $\{e_j\}$ is the orthonormal basis with respect to $g$ (the fundamental tensor of $F$) at $(x, [y])$.

If $\Omega$ is a compact domain in $M$, we use the canonical volume element $\Pi$ associated with $F$ to define the energy of $\phi : (\Omega, F) \to (N, h)$ by

$$E(\phi, \Omega) = \frac{1}{c} \int_{S\Omega} e(\phi)\Pi,$$

where $c := \mathrm{Vol}(S^{m-1})$ is the volume of the standard $(m-1)$-dimensional sphere and $S\Omega$ the projective sphere bundle of $\Omega$. If $M$ is compact, we write $E(\phi) = E(\phi, M)$.

**Remark** By Lemma 9.1, our notion of energy reduces to the usual notion of energy if $M$ is a compact Riemannian manifold.

A smooth map $\phi : (M, F) \to (N, h)$ from a Finsler manifold to a Riemannian manifold is said to be *harmonic* if it is an extremal of the restriction of $E$ on every compact subdomain of $(M, F)$.

## 9.2 The first variation

Let $(M, F)$ be a smooth Finsler manifold and $g$ the fundermental tensor of $F$. Let $(N, h)$ be a Riemannain manifold. Let $\phi : (M, F) \to (N, h)$ be a smooth map. Set

$$h = \sum \theta_\alpha^2 \in \Gamma(\Theta^2 T^* N), \quad 1 \le \alpha, \beta, \gamma, \cdots \le n. \tag{9.4}$$

The first structure equation for $(N, h)$ is

$$d\theta_\alpha = \sum \theta_\beta \wedge \theta_{\beta\alpha}, \quad \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0. \tag{9.5}$$

A vector field $v$ along $\phi$ determines a variation $\phi_t$ by

$$\phi_t(x) = \exp_{\phi(x)}[tv(x)],$$

where $t \in I := (-\epsilon, \epsilon)$ for some $\epsilon > 0$. Noting that

$$\phi_t^* \theta_\alpha \in \Gamma(T^*M) \subset \Gamma(p^*T^*M),$$

we put

$$\phi_t^* \theta_\alpha = \sum a_{\alpha i} \omega_i, \tag{9.6}$$

where $a_{\alpha i} = a_{\alpha i}(t)$. It follows that

$$\begin{aligned} \phi_t^*(h) &= \phi_t^*(\textstyle\sum \theta_\alpha^2) \\ &= \textstyle\sum [\phi_t^* \theta_\alpha]^2 = \sum a_{\alpha i} a_{\alpha j} \omega_i \omega_j. \end{aligned} \tag{9.7}$$

Since $\{e_i\}$ is the dual frame field of $\{\omega_i\}$, from (9.3) and (9.7) we obtain

$$2e(\phi_t) = \sum_i (\sum a_{\alpha j} a_{\alpha k} \omega_j \omega_k)(e_i, e_i) = \sum a_{\alpha i}^2.$$

If $v$ has compact support $\Omega \subset M$, then

$$c \cdot \frac{d}{dt} E(\phi_t, \Omega)|_{t=0} = \int_{SM} \sum a_{\alpha i} \frac{\partial a_{\alpha i}}{\partial t}|_{t=0} \Pi. \tag{9.8}$$

Define $\Phi : M \times I \to N$ by

$$(x, t) \xrightarrow{\Phi} \phi_t(x).$$

It is easy to see that

$$\Phi^* \theta_\alpha \equiv \phi_t^* \theta_\alpha, \quad \Phi^* \theta_{\alpha\beta} \equiv \phi_t^* \theta_{\alpha\beta} \mod dt.$$

Put

$$\Phi^* \theta_\alpha = \phi_t^* \theta_\alpha + b_\alpha dt, \tag{9.9}$$

$$\Phi^* \theta_{\alpha\beta} = \phi_t^* \theta_{\alpha\beta} + B_{\alpha\beta} dt. \tag{9.10}$$

Then $\sum b_\alpha v_\alpha|_{t=0} := b$ is the deformation vector field, where $\{v_\alpha\}$ is the dual frame field of $\theta_\alpha$, snd $B_{\alpha\beta}$ satisfies

$$B_{\alpha\beta} = -B_{\beta\alpha}. \tag{9.11}$$

Using (9.5), (9.6), (9.9) and (9.10), we obtain

$$\begin{aligned} d(\Phi^*\theta_\alpha) &= \phi^*(d\theta_\alpha) \\ &= \Phi^*(\sum \theta_\beta \wedge \theta_{\beta\alpha}) \\ &= \sum \Phi^*\theta_\beta \wedge \Phi^*\theta_{\beta\alpha} \\ &= \sum(\sum a_{\beta i}\omega_i + b_\beta dt) \wedge (\phi_t^*\theta_{\beta\alpha} + B_{\beta\alpha}dt). \end{aligned} \tag{9.12}$$

On the other hand, from (9.6) and (9.9), we have

$$\begin{aligned} d(\Phi^*\theta_\alpha) &= d[\sum a_{\alpha i}\omega_i + b_\alpha dt] \\ &= \sum(da_{\alpha i}) \wedge \omega_i + \sum a_{\alpha i}d\omega_i + db_\alpha \wedge dt \\ &= \sum(d_{sm}a_{\alpha i} + \tfrac{\partial a_{\alpha i}}{\partial t}dt) \wedge \omega_i + \sum a_{\alpha i}d\omega_i + d_{SM}b_\alpha \wedge dt. \end{aligned} \tag{9.13}$$

Comparing the coefficients of $dt$ in (9.12) and (9.13), we obtain

$$\sum \frac{\partial a_{\alpha i}}{\partial t}\omega_i - d_{SM}b_\alpha = \sum_\beta \left( b_\beta \phi_t^*\theta_{\beta\alpha} - \sum_i B_{\beta\alpha}a_{\beta i}\omega_i \right). \tag{9.14}$$

Define the covariant derivative of $\{b_\alpha\}$ by

$$\begin{aligned} Db_\alpha &= d_{SM}b_\alpha - \sum b_\beta \phi_t^*\theta_{\alpha\beta} \\ &= \sum b_{\alpha|i}\omega_i + \sum b_{\alpha;\lambda}\omega_{m\lambda}. \end{aligned} \tag{9.15}$$

Substituting (9.15) into (9.14) we obtain

$$\frac{\partial a_{\alpha i}}{\partial t} = b_{\alpha|i} - \sum B_{\beta\alpha}a_{\beta i}, \quad b_{\alpha;\lambda} = 0. \tag{9.16}$$

From (9.8), (9.11) and (9.16) we have

$$\begin{aligned} c \cdot \tfrac{d}{dt}E(\phi_t)|_{t=0} &= \int \sum a_{\alpha i}(b_{\alpha|i} - \sum B_{\beta\alpha}a_{\beta i})\Pi \\ &= \int(\sum a_{\alpha i}b_{\alpha|i} - \sum a_{\alpha i}B_{\beta\alpha}a_{\beta i})\Pi \\ &= \int \textstyle\sum_i(\sum a_{\alpha i}b_\alpha)_{|i}\Pi - \int \sum a_{\alpha i|i}b_\alpha\Pi, \end{aligned}$$

where

$$a_{\alpha i|j} := [da_{\alpha i} - \sum a_{\alpha k}\omega_{ik} - \sum a_{\beta i}\phi^*\theta_{\alpha\beta}](e_j)$$

and

$$\sum_i(\sum a_{\alpha i}b_\alpha)\omega_i = \langle d\phi, b\rangle$$

is a global section on the dual Finsler bundle $p^*T^*M$. Using (9.2) and Lemma 9.2 we get

$$c \cdot \frac{d}{dt}E(\phi_t)|_{t=0} = \int_{SM} \operatorname{div}\langle d\phi, b\rangle\Pi - \int_{SM}\langle\tau(\phi), b\rangle\Pi \\ = -\int_{SM}\langle\tau(\phi), b\rangle\Pi,$$

where

$$\tau(\phi) := -\langle d\phi, \dot{\eta}\rangle + \mathrm{tr}\nabla d\phi \in \Gamma((\phi\circ p)^*TN) \tag{9.17}$$

and $\eta$ (resp. $\nabla d\phi$) denotes the Cartan form (resp. the second fundamental) form of $\phi$. The field $\tau(\phi)$ is called the *tension field* of $\phi$.

**Theorem 9.2.1** *Let $\phi$ be a smooth map from a Finsler manifold $M$ to a Riemannian manifold $N$. Then $\phi$ is harmonic if and only if it has vanishing tension field.*

The basic problem for harmonic maps can be formulated in the following manner: Let $u_0 : M \to N$ be a map of Riemannian manifolds. Can $u_0$ be deformed into a harmonic map $u : M \to N$? Assuming that $M$ and $N$ are compact and without boundary, J. Eells and J. H. Sampson have a positive answer when the Riemannian sectional curvature of $N$ is non-positive. This is so-called the fundamental existence theorem for harmonic maps.

Let $\phi : (M,\ F) \to (N,\ h)$ be a smooth map from a Finsler manifold to a Riemannian manifold. Shen and Zhang has obtained the second variation formula of this energy functional [Shen and Zhang, 2003]. On the 2000' Finsler geometry workshop Professor S. S. Chern conjectured the fundamental existence theorem of harmonic maps on a Finsler space is true. Recently we have proved the fundamental existence theorem of harmonic maps on a Finsler space. Precisely we show the following:

**Theorem 9.2.2** ([Mo and Yang, 2004]) *Let $(M,\ F)$ be a compact Finsler space and $(N,\ h)$ be a compact Riemannian manifold with non-positive sectional curvature. Then any map $\phi : (M,\ F) \to (N,\ h)$ is homotopic to a harmonic map which has minimum energy in its homotopy class.*

Let us now express the tensor field in local coordinates $(x^i)$ on $M$ and $(u^\alpha)$ on $N$. We denote by $g_{ij}$ and ${}^M\Gamma^i_{jk}$ the components of the fundamental tensor and the Christoffel symbols of the Chern connection on $(M, F)$, and by $h_{\alpha\beta}$ and ${}^N\Gamma^\alpha_{\beta\gamma}$ the corresponding objects on $(N, h)$. Note that ${}^N\Gamma^\alpha_{\beta\gamma}$ are just the christoffel symbols of Levi-Civita on $(N, h)$ because $h_{\alpha\beta}$ are Riemannian.

Let $\nabla$ denote the covariant differentiation (of sections of tensor product of $p^*TM$ and $p^*T^*M$) on $SM$ with respect to the Chern connection. Then, by (2.46) and (2.47a) in [Bao and Chern, 1996], we have

$$\nabla \frac{\partial}{\partial x^k} = {}^M\Gamma^i_{kl} dx^l \otimes \frac{\partial}{\partial x^i},$$

where

$${}^M\Gamma^i_{kl} = g^{ij\,M}\Gamma_{jkl}, \tag{9.18}$$

$${}^M\Gamma_{jkl} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k}\right) + \frac{1}{2}(M_{jkl} - M_{klj} + M_{ljk}), \tag{9.19}$$

$$M_{ijk} = -\frac{\partial g_{ij}}{\partial y^l}\frac{\partial G^l}{\partial y^k} \tag{9.20}$$

and $G^l$ are the geodesic coefficients of $(M, F)$ (cf. Definition 3.3.8). Using the Leibniz rule, we obtain (cf. [Bao, Chern and Shen, 2000].p.41)

$$\nabla dx^i = -{}^M\Gamma^i_{kl} dx^k \otimes dx^l. \tag{9.21}$$

Suppose that $\phi : (M, F) \to (N, h)$ is a smooth map. Locally, we can write $\phi = (\phi^\alpha)$ where each $\phi^\alpha$ is a smooth function defined on open subset in $M$. Let $\nabla$ denote the covariant differentiation on $p^*T^*M \otimes (\phi \circ p)^*TN$. Then (cf. [Eells and Lemaire, 1983])

$$\begin{aligned}\nabla_{\partial/\partial x^i}(d\phi) &= \nabla_{\partial/\partial x^i}\left(\phi^\alpha_j dx^j \tfrac{\partial}{\partial u^\alpha}\right)\\ &= \phi^\alpha_{ij} dx^j \tfrac{\partial}{\partial u^\alpha} + \phi^\alpha_j \nabla^{P^*T^*M}_{\partial/\partial x^i} dx^j \tfrac{\partial}{\partial u^\alpha}\\ &+ \phi^\alpha_j dx^j \nabla^{(\phi\circ P)^*TN}_{\partial/\partial x^i} \tfrac{\partial}{\partial u^\alpha},\end{aligned}$$

where

$$\phi^\alpha_i = \frac{\partial \phi^\alpha}{\partial x^i}, \quad \phi^\alpha_{ij} = \frac{\partial^2 \phi^\alpha}{\partial x^i \partial x^j}.$$

Now

$$\nabla^{P^*T^*M}_{\partial/\partial x^i} dx^j = -{}^M\Gamma^j_{ki} dx^k$$

and

$$\nabla^{P^*\phi^*TN}_{\partial/\partial x^i}\frac{\partial}{\partial u^\alpha} = \phi_i^\beta {}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial u^\gamma},$$

so that

$$\nabla_{\partial/\partial x^i}(d\phi) = (\phi^\alpha_{ij} - {}^M\Gamma^k_{ij}\phi^\alpha_k + {}^N\Gamma^\alpha_{\beta\gamma}\phi_i^\beta\phi_j^\gamma)dx^j\frac{\partial}{\partial u^\alpha},$$

where we have used the fact (cf. [Bao and Chern, 1996])

$${}^M\Gamma^k_{ij} = {}^M\Gamma^k_{ji}.$$

It follows that the components of the second fundamental form $Dd\phi$ satisfy

$$(\nabla d\phi)^\alpha_{ij} = \phi^\alpha_{ij} - {}^M\Gamma^k_{ij}\phi^\alpha_k + {}^N\Gamma^\alpha_{\beta\gamma}\phi_i^\beta\phi_j^\gamma.$$

Now consider a smooth function $f$ defined on an open subset in $M$. Set

$$f_j = \frac{\partial f}{\partial x^j}, \quad f_{ij} = \frac{\partial f_j}{\partial x^j}$$

and

$$\nabla_{\partial/\partial x^i}(df) = (\nabla df)_{ij}dx^j.$$

Then

$$df = f_j dx^j,$$

$$(\nabla df)_{ij} = (\nabla df)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = f_{ij} - {}^M\Gamma^k_{ij}f_k.$$

Thus (9.17) reduces to

$$\begin{aligned}\tau(f) &= -\langle df, \dot\eta\rangle + \mathrm{Tr}\nabla df \\ &= g^{ij}[f_{ij} - {}^M\Gamma^k_{ij}f_k - \xi_i f_j],\end{aligned} \tag{9.22}$$

where

$$\xi_j = \dot\eta(\frac{\partial}{\partial x^j}). \tag{9.23}$$

Suppose that $\phi : (M, F) \to (N, h)$ is a smooth map. By (9.22) we have

$$\tau(\phi^\alpha) = g^{ij}[\phi^\alpha_{ij} - {}^M\Gamma^k_{ij}\phi^\alpha_k - \xi_i\phi^\alpha_j]. \tag{9.24}$$

Hence the tension field of $\phi$ is

$$\begin{aligned}\tau_\phi^\alpha &= du^\alpha(-\langle d\phi, \dot{\eta}\rangle + \mathrm{Tr}\nabla d\phi) \\ &= g^{ij}[-\xi_i\phi_j^\alpha + \phi_{ij}^\alpha - {}^M\Gamma_{ij}^k\phi_k^\alpha + {}^N\Gamma_{\beta\gamma}^\alpha\phi_i^\beta\phi_j^\gamma] \\ &= \tau(\phi^\alpha) + g^{ij\,N}\Gamma_{\beta\gamma}^\alpha\phi_i^\beta\phi_j^\gamma,\end{aligned} \tag{9.25}$$

A direct calculation using (9.2), (9.18),(9.19) and (9.20) yields (cf. [Bao, Chern and Shen, 2000], (3.3.3))

$$\xi_i = -y^j\frac{\partial^M\Gamma_{jk}^k}{\partial y^i}$$

and

$${}^M\Gamma_{ki}^i = \left(\frac{\partial}{\partial x^k} - \frac{\partial G^i}{\partial y^k}\frac{\partial}{\partial y^i}\right)\log\sqrt{\det(g_{jl})}.$$

## 9.3 Composition properties

Let $\phi : (M, F) \to (N, h)$ be a smooth map, Define on $M$

$$\langle\theta_\alpha, d\phi\rangle = \sum a_{\alpha i}\omega_i$$

where $\theta_\alpha$ is an orthonormal coframe of $h$. Then

$$\begin{aligned}d(\textstyle\sum_i a_{\alpha i}\omega_i) &= d(\phi^*\theta_\alpha) \\ &= \phi^* d\theta_\alpha \\ &= \phi^*(\textstyle\sum \theta_\alpha \wedge \theta_{\beta\alpha}) \\ &= \textstyle\sum \phi^*\theta_\beta \wedge \phi^*\theta_{\beta\alpha}\end{aligned} \tag{9.26}$$

by (9.5), consider (9.26) as a two-form defined on the projective sphere bundle $SM$. We have

$$\begin{aligned}d(\textstyle\sum_i a_{\alpha i}\omega_i) &= \textstyle\sum da_{\alpha i}\wedge\omega_i + \sum a_{\alpha i}d\omega_i \\ &= \textstyle\sum da_{\alpha i}\wedge\omega_i + \sum a_{\alpha i}\omega\wedge\omega_{ji} \\ &= \textstyle\sum a_{\beta i}\omega_i \wedge \phi^*\theta_{\beta\alpha}.\end{aligned}$$

It follows that

$$\sum Da_{\alpha i}\wedge\omega_i = 0, \tag{9.27}$$

where

$$\begin{aligned}Da_{\alpha i} &:= da_{\alpha i} - \textstyle\sum a_{\alpha j}\omega_{ij} + \sum a_{\beta i}\phi^*\theta_{\beta\alpha} \\ &:= \textstyle\sum a_{\alpha i|j} + \sum a_{\alpha i;\lambda}\omega_{m\lambda}.\end{aligned} \tag{9.28}$$

Substituting (9.28) into (9.27) yields the following result.

**Proposition 9.3.1** *The second fundamental form of $\phi : (M, F) \to (N, g)$ satisfies*

$$a_{\alpha i|j} = a_{\alpha j|i} \quad \text{and} \quad a_{\alpha i;\lambda} = 0.$$

Denote the second fundamental form of $\phi$ by

$$\nabla d\phi = \Sigma_{i,\alpha}(Da_{\alpha\, i})\omega_i v_\alpha,$$

where $v_\alpha$ is the dual frame of $\theta_\alpha$. It is easy to see that (9.28) is equivalent to

$$(\nabla d\phi)(e_i,\, e_j) = \nabla_{\phi_* e_i}(\phi_* e_j) - \phi_*(\nabla_{e_i} e_j),$$

where $e_i$ is the dual frame of $\omega_i$ and the last $\nabla$ denotes covariant differentiation (of sections of tensor products of $\pi^*TM$ and $\pi^*T^*M$), on $SM$, relative to the connection $\omega_{ij}$. Combine with (9.28) we have

$$(\nabla d\phi)(X,\, Y) = \nabla_{\phi_* X}(\phi_* Y) - \phi_*(\nabla_X Y) = (\nabla d\phi)(Y,\, X) \tag{9.29}$$

for $\forall X,\, Y \in \Gamma(\pi^*TM)$.

**Proposition 9.3.2** *If $(M,\, F)$ is a Finsler manifold, $(N,\, h)$ and $(P,\, k)$ are two Riemannian manifolds and $\phi \in C(M,\, N)$, $\psi \in C(N,\, P)$, then*

$$\tau(\psi \circ \phi) = d\psi \circ \tau(\phi) + tr\nabla d\psi(d\phi,\, d\phi). \tag{9.30}$$

***Proof.*** By using (9.29) we have

$$\begin{aligned}
\nabla d(\psi \circ \phi)(X,\, Y) &= \nabla_{(\psi\circ\phi)_* X}[\psi \circ \phi)_* Y] - (\psi \circ \phi)_*(\nabla_X Y) \\
&= \nabla_{(\psi\circ\phi)_* X}[(\psi \circ \phi)_* Y] - \psi_*(\nabla_{\phi_* X}\phi_* Y) \\
&\quad + \psi_*(\nabla_{\phi_* X}\phi_* Y) - (\psi \circ \phi)_*(\nabla_X Y) \\
&= (\nabla d\psi)(\phi_* X,\, \phi_* Y) \\
&\quad + \psi_*[\nabla_{\phi_* X}(\phi_* Y) - \phi_*(\nabla_X Y)] \\
&= \nabla d\psi(d\phi,\, d\phi)(X,\, Y) + d\psi \circ \nabla d\phi(X,\, Y).
\end{aligned} \tag{9.31}$$

Put $d\phi = \Sigma\xi_\alpha v_\alpha$ where $\{v_\alpha\}$ is the dual frame field of $\{\theta_\alpha\}$, then

$$\begin{aligned} d\psi(< d\phi,\, \dot\eta >) &= d\psi(< \Sigma\xi_\alpha v_\alpha,\, \dot\eta >) \\ &= \Sigma d\psi(< \xi_\alpha,\, \dot\eta > v_\alpha) \\ &= \Sigma < \xi_\alpha,\, \dot\eta > d\psi(v_\alpha) \\ &= \Sigma < \xi_\alpha d\psi(v_\alpha),\, \dot\eta > \\ &= \Sigma < d\psi(\xi_\alpha v_\alpha),\, \dot\eta >=< d\psi\circ d\phi,\, \dot\eta > . \end{aligned} \tag{9.32}$$

Taking traces for (9.31) and using (9.17) and (9.32) yields

$$\begin{aligned} \tau(\psi\circ\phi) &= \mathrm{tr}\nabla d(\psi\circ\phi) - < d(\psi\circ\phi),\, \dot\eta > \\ &= d\psi\circ(\mathrm{tr}\nabla d\phi) + \mathrm{tr}\nabla d\psi(d\phi,\, d\phi) \\ &\quad - d\psi(< d\phi,\, \dot\eta >) \\ &= d\psi\circ\tau(\phi) + \mathrm{tr}\nabla d\psi(d\phi,\, d\phi). \end{aligned}$$

□

A smooth map $\phi : (M,\, F) \to (N,\, h)$ from a Finsler space to a Riemannian space is said to be *totally geodesic* if it has vanishing second fundamental form.

**Corollary 9.3.3** *Let $(M,\, F)$ be a Finsler manifold, $(N,\, h)$ and $(P,\, k)$ two Riemannian manifolds and $\phi \in \mathcal{C}(M,\, N)$, $\psi \in \mathcal{C}(N,\, P)$. If $\phi$ is harmonic and $\psi$ totally geodesic, then $\psi\circ\phi$ is harmonic.*

If $(M,\, F)$ is a Finsler manifold, $(N,\, h)$ and $(P,\, k)$ are two Riemannian manifolds and $\phi \in \mathcal{C}(M,\, N)$. Suppose that $N$ isometrically immersed in $P$ by a mapping $i$. (A Euclidean space would be a good choice because every Riemannian manifold can be isometrically immersed into a Euclidean space.) The second fundamental form of $i$ as a map coincides with that of $i$ as a submanifold inclusion. Write $\Phi : M \to P$ for the composition of $\phi$ with $i$. Then at each point of $SM$, $\tau(\phi)$ is the orthogonal projection of $\tau(\Phi)$ onto $(\phi\circ\pi)^*TN$. More precisely, $\tau(\Phi) = \tau(\phi) + trace\beta(d\phi,\, d\phi)$, where $\beta$ is the second fundamental form of $i$. In particular, we get

**Proposition 9.3.4** *Let $\phi : (M,\, F) \to (N,\, h)$ be a smooth map. Then $\phi$ is harmonic if and only if $\tau(\Phi)$ lies in the normal bundle of $N$ in $P$ where $\Phi = i\circ\phi$.*

The following result is a natural generalization of Takahashi's Theorem:

**Proposition 9.3.5** *Let $i : S^n \to R^{n+1}$ be the standard inclusion. A map $\phi : (M, F) \to (S^n, i^*(ds^2_{R^{n+1}}))$ is harmonic if and only if*

$$\tau(\Phi) = -2e(\phi)\Phi$$

*where $e(\phi)$ is the energy density of $\phi$.*

***Proof.*** Because $i$ is an isometric immersion, Proposition 3.3 tells us that $\phi$ is harmonic if and only if $\tau(\Phi) = A\Phi$ for some function $A : SM \to R$. Notice that $i$ is a totally umbilical isometric immersion with constant mean curvature one, then

$$\mathrm{tr}\nabla di(d\phi, d\phi) = \mathrm{tr}\phi^* hH = -2e(\phi)\Phi,$$

where $H$ is the mean curvature of $i$. Thus $A = -2e(\phi)$. □

Say that a function $k$ defined on an open set $U$ of $(M, F)$ is *subharmonic* if $\tau(k) \geq 0$ at every point on $SU$.

**Theorem 9.3.6** *A map $\phi : (M, F) \to (N, h)$ is harmonic if and only if it carries germs of convex functions to germs of subharmonic functions.*

***Proof.*** For $k$ a function on an open set $U$ of $N$, we shall start from the composition law $\tau(k \circ \phi) = dk \circ \tau(\phi) + trace\nabla dk(d\phi, d\phi)$. If $\phi$ is harmonic, we get $\tau(k \circ \phi) \geq 0$ on $S\phi^{-1}(U)$ so that $k \circ \phi$ is subharmonic. Conversely, if at a point $(x_0, [y_0]) \in SM$, $\tau(\phi)(x_0, [y_0]) = w \neq 0$. Because $(N, h)$ is a Riemannian manifold, we can choose normal coordinates $(v^\alpha)$ near $\phi(x_0)$. Define a function in the normal coordinates

$$k = b_\alpha v^\alpha + \Sigma(v^\alpha)^2 \tag{9.33}$$

where $\{b_\alpha\}$ satisfy that

$$\Sigma b_\alpha w^\alpha < -4e(\phi)(x_0, [y_0]), \quad w^\alpha = dv^\alpha(w), \quad Hess(k)|_{\phi(x_0)} = 2h. \tag{9.34}$$

Then $k$ is a convex function near $\phi(x_0)$. Using (3.4) and (3.5) we have

$$dk(w)|_{\phi(x_0)} = \Sigma b_\alpha w^\alpha.$$

This gives

$$\begin{aligned}\tau(k \circ \phi)_{(x_0, [y_0])} &= dk \circ \tau(\phi) + \mathrm{tr}\nabla dk(d\phi, d\phi) \\ &= dk(w) + 4e(\phi) < 0\end{aligned}$$

contradicting the hypothesis. □

## 9.4 The stress-energy tensor

Let $\phi : (M, F) \to (N, h)$ be a smooth map from a Finsler manifold $(M, F)$ to a Riemannian manifold $(N, h)$. The stress-energy tensor $S_\phi$ is a tensor on $SM$ defined by

$$S_\phi := e(\phi)g - \phi^* h$$

where $e(\phi)$ (resp. $g$) denotes the energy density (resp. the fundamental tensor) of $\phi$ and $\phi^* h$ denotes the pull back of the tensor $h$ to a tensor on $SM$. We say the $S_\phi$ is horizontally divergence-free if $\sum_{i=1}^{m}(D_{\epsilon_i}, S_\phi)(\epsilon_i, Y) = 0$ for all $Y \in H_p$, where $\{\epsilon_i\}$ is any orthonormal basis for for the horizontal space $H_p$ and $H_p := \{X \in T_pSM, \omega_{m\lambda}(X)\} = 0$ (see Section 3.4).

Denote the stress-energy $S_\phi$ of $\phi$ by

$$S_\phi = \sum S_{ij}\omega_i \otimes \omega_j,$$

then

$$S_{ij} = e(\phi)\delta_{ij} - \sum a_{\alpha i}a_{\alpha j}, \tag{9.35}$$

where $e(\phi)$ is the energy density of $\phi$. Then the horizontal divergence of $S_\phi$ is

$$\begin{aligned}
\mathrm{div} S_\phi &= \sum S_{ijj}\omega_i \\
&= \textstyle\sum_i (\sum S_{ij|j} + \sum S_{i\mu}L_{\lambda\lambda\mu})\omega_i \\
&= \textstyle\sum_i \{\sum_j [e(\phi)\delta_{ij} - \sum a_{\alpha i}a_{\alpha j}]_{|j} \\
&\quad + \textstyle\sum [e(\phi)\delta_{i\mu} - \sum a_{\alpha i}a_{\alpha\mu}]L_{\lambda\lambda\mu}\}\omega_i \\
&= \textstyle\sum [e(\phi)_{|i} - \sum a_{\alpha i|j}a_{\alpha j} - \sum a_{\alpha i}a_{\alpha j|j} \\
&\quad + e(\phi)L_{\lambda\lambda i} - \textstyle\sum a_{\alpha i}a_{\alpha\mu}L_{\lambda\lambda\mu}]\omega_i \\
&= -\textstyle\sum_i [\sum a_{\alpha i}a_{\alpha j|j} + \sum a_{\alpha i}a_{\alpha\mu}L_{\lambda\lambda\mu}]\omega_i + e(\phi)\sum L_{\lambda\lambda\mu}\omega_\mu \\
&= -\langle \tau(\phi), d\phi\rangle - e(\phi)\dot{\eta},
\end{aligned} \tag{9.36}$$

where $\dot{\eta}$ denotes the covariant derivative of the Cartan form along the Hilbert form, and where we have used (9.2) and (ii) of Lemma 9.1.2. Recall that a Finsler manifold is said to be of weak Landsberg type if $\dot{\eta} = 0$ (see Definition 4.3.3).

The following theorems are immediate consequences of (9.36).

**Theorem 9.4.1** *Let $\phi : (M, F) \to (N, h)$ be a non-constant harmonic map from a Finsler manifold to a Riemannian manifold. Then $S_\phi$ is horizontally divergence-free if and only if $(M, F)$ is of weak Landsberg type.*

Combing this with Theorem 4.3.4 we obtain the following Wood type result (cf. [Wood, 1986], Theorem 2.9).

**Theorem 9.4.2** *Let $\phi : (M, F) \to (N, h)$ be a submersion from a Finsler manifold to a Riemannian manifold. Then any two of the following condition imply the third condition:*

(i) $\phi$ *is harmonic;*

(ii) $S_\phi$ *is horizontally divergence-free;*

(iii) $p : SM \to M$ *has minimal fibres.*

**Lemma 9.4.3** *Let $\phi : (M,\, F) \to (N,\, h)$ be a smooth map and $\Psi$ a smooth section on $p^*T^*M$. Then*

$$\begin{aligned} \operatorname{div}(e(\phi)\Psi) &= \operatorname{div}(\Sigma\langle\Psi,\, \phi^*\theta_\alpha\rangle\phi^*\theta_\alpha) \\ &\quad +\langle \operatorname{div}_{\mathcal{H}} S_\phi,\, \Psi\rangle + \langle S_\phi,\, D\Psi\rangle, \end{aligned} \tag{9.37}$$

*where $div_{\mathcal{H}}$ is the horizontal divergence.*

***Proof.***

$$\operatorname{div}(e(\phi)\Psi) = \Sigma e(\phi)_{|j}\Psi_j + e(\phi)\Sigma\Psi_{j|j} + e(\phi)\Sigma\Psi_\nu L_{\lambda\lambda\nu} \tag{9.38}$$

by using Lemma 9.1.2 in [10]. On the other hand,

$$\begin{aligned} \operatorname{div}(\Sigma\langle\Psi,\, \phi^*\theta_\alpha\rangle\phi^*\theta_\alpha) &= \operatorname{div}(\Sigma\Psi_j a_{\alpha j} a_{\alpha i}\omega_i) \\ &= \Sigma(\Psi_j a_{\alpha j} a_{\alpha i})_{|i} + \Sigma\Psi_j a_{\alpha j} a_{\alpha\nu} L_{\lambda\lambda\nu} \\ &= \Sigma\Psi_{j|i} a_{\alpha j} a_{\alpha i} + \Sigma\Psi_j a_{\alpha j|i} a_{\alpha i} \\ &\quad +\Sigma\Psi_j a_{\alpha j} a_{\alpha i|i} + \Sigma\Psi_j a_{\alpha j} a_{\alpha\nu} L_{\lambda\lambda\nu} \\ &= \Sigma\Psi_{i|j} a_{\alpha j} a_{\alpha i} + \Sigma\Psi_j a_{\alpha i|j} a_{\alpha i} \\ &\quad +\Sigma\Psi_j a_{\alpha j}\tau_\alpha(\phi). \end{aligned} \tag{9.39}$$

Put

$$\begin{aligned} \langle S_\phi,\, D\Psi\rangle_{\pi^*T^*M} :&= \Sigma S_{ij}\Psi_{i|j} \\ &= \Sigma\left(e(\phi)\delta_{ij} - \Sigma a_{\alpha i} a_{\alpha j}\right)\Psi_{i|j} \\ &= e(\phi)\Sigma\Psi_{j|j} - \Sigma\Phi_{i|j} a_{\alpha j} a_{\alpha i} \end{aligned} \tag{9.40}$$

(9.38)-(9.40) implies (9.37). □

**Proposition 9.4.4** *If supp$\Psi$ (supporting set of $\Psi$) is compact, then*

$$\int_{SM} \langle \mathrm{div}_{\mathcal{H}} S_\phi,\ \Psi\rangle \Pi + \int_{SM} \langle S_\phi,\ D\Psi\rangle \Pi = 0$$

***Proof.*** Integrating both sides of (9.3.7) and using Green's theorem. □

We denote the angle between 2-tensor $\Phi$ and $\Psi$ on $SM$ by $\mathcal{L}(\Phi,\ \Psi)$. By using Theorem 4.4 in [Mo, 2001] we have

**Corollary 9.4.5** *Let $\phi : (M,\ F) \to (N,\ h)$ be a smooth map from a closed Finsler manifold. Then*

$$\mathcal{L}(S_\phi,\ D\mathrm{div}_{\mathcal{H}} S_\phi) \geq \frac{\pi}{2},$$

*where equality holds if and only if $u$ is horizontally divergence-free. In particular, if $F$ is of weak Landsberg type, then equality holds if and only if $\phi$ is harmonic.*

## 9.5 Harmonicity of the identity map

In this section we present the harmonic equation of the identity map from a Finsler manifold to a Riemannian manifold in terms of their geodesic coefficients, and we construct harmonic maps from Berwald manifolds which are neither Riemannian nor Minkowskian to Riemannian manifolds.

Let $I : (M, F) \to (M, h)$ be the identity map from a Finsler manifold $(M, F)$ to a Riemannian manifold $(M, h)$. As usual we put

$$I = (I^i) : U(\subset M) \to \mathbb{R},$$

where locally $I^i(x^1, \cdots, x^m) = x^i$. It follows that

$$I^i_j = \frac{\partial I^i}{\partial x^j}, \quad I^i_{jk} = 0.$$

Using (9.24) and (9.25) we have

$$\tau(I^k) = -g^{ij\,F}\Gamma^k_{ij} - g^{kj}\xi_j$$

and hence

$$\tau_I^k = g^{ij}[{}^h\Gamma_{ij}^k - {}^F\Gamma_{ij}^k] - g^{kj}\xi_j. \tag{9.41}$$

Denote the geodesic coefficients of $(M, F)$ and $(N, h)$ by ${}^FG^i$ and ${}^hG^i$ respectively. By ([Bao, Chern and Shen, 2000], (3.8.3)) we have

$$\frac{1}{2}({}^FG^i)_{y^j y^k} = {}^F\Gamma_{jk}^i + \dot{H}^i{}_{jk},$$

where $\dot{H}^i{}_{jk}$ is the covariant derivative of the Cartan tensor along the Hilbert form. It follows that

$$\begin{aligned} g^{ij}\Gamma_{ij}^k + g^{ki}\xi_i &= g^{ij}[\tfrac{1}{2}({}^FG^k)_{y^i y^j} - \dot{H}_{ij}^k] + g^{ki}\dot{H}_i \\ &= \tfrac{1}{2}g^{ij}({}^FG^k)_{y^i y^j} - \dot{H}^k + \dot{H}^k = \tfrac{1}{2}g^{ij}({}^FG^k)_{y^i y^j}, \end{aligned} \tag{9.42}$$

where in the second step we used (cf. [Bao, Chern and Shen, 2000], (2.5.11)). Similarly, for the Riemannian metric $h$ we have

$$\frac{1}{2}({}^FG^i)_{y^j y^k} = {}^h\Gamma_{jk}^i. \tag{9.43}$$

Substituting (9.42) and (9.43) into the harmonic equation (9.31) gives

$$\tau_I^k = \frac{1}{2}g^{ij}({}^hG^k - {}^FG^k)_{y^i y^j}. \tag{9.44}$$

Thus we have the following result.

**Proposition 9.5.1** *Let $(M, h)$ be a flat Riemannian space. Then, for any local Minkowski structure $F$ on $M$, the identity map*

$$I : (M, F) \to (M, h)$$

*is harmonic.*

***Proof.*** By (4.9) we have

$${}^FG^i = \frac{1}{4}\sum_{i,k,l} g^{jl}[2\frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l}]y^i y^k. \tag{9.45}$$

On the other hand, $F$ is local Minkowskian if and only if (see Definition 1.3.1)

$$g_{ij}(x, y) = g_{ij}(y). \tag{9.46}$$

The conclusion is now immediate from (9.44)-(9.46). □

Recall that a Finsler manifold $(M, F)$ is said to be of Berwald type if $F$ has vanishing Minkowski curvature, i.e., if $P_{ijk\lambda} = 0$ for all $i, j, k, \lambda$ (cf. Definition 5.1.3).

Recall that a Finsler manifold $(M, F)$ is said to be of Randers type if $F = \alpha + \beta$, where $\alpha$ is a Riemannian metric on $M$, and $\beta = \beta_i dx^i$ is a 1-form. It is easy to see that a Randers manifold if and only if $\beta_{i|j} = 0$, where $\beta_{i|j}$ is the covariant derivative of $\beta$ with respect to the parallel with respect to $\alpha$. In this case, the Randers metric $\alpha + \beta$ and Riemannian metric $\alpha$ have same geodesic coefficients (cf. [Bao, Chern and Shen, 2000], 11.311). Combining this with (9.44) we obtain:

**Proposition 9.5.2** *Let $(M, \alpha + \beta)$ be a Randers manifold. If $\beta$ is parallel with respect to the Riemannian metric $\alpha$, then the identity*

$$I : (M, \alpha + \beta) \to (M, \alpha)$$

*is harmonic.*

Antonelli, Ingarden, and Matsumoto (cf. [AIM, 1993]) showed that Berwald nor Minkowskian can be constructed using certain Randers metrics. In view of this, our results gives examples of harmonic maps from Berwald manifolds which are neither Riemannian nor Minkowskian.

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