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# Learning Value Equivalent Model on Policy Improvement Path

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## 1 Introduction

The basic intuition of value-aware model learning [4, 3] is to assign different weights to the transitions according to the *optimal* state value. However, the optimal value function is not known and the VAML loss cannot be readily minimized. Thus, as shown in [6, 5], we may find a model using the value equivalence principle. As more value functions are augmented to be considered, the model will less compromise the performance, and eventually collapse to the real environment model. The key is to determine the **representative** value functions (or the value function of representative policies [5]).

In this work, we show that learning value equivalent models through the policy improvement path gives a  $T^{-\frac{1}{2}}$  rate of convergence. We first give convergence rates of policy gradient in terms of gradient bias and variance for finite MDPs with direct policy parameterization and for more general function approximations settings with natural gradient. Then we provide the bounds for model-based likelihood ratio gradient estimator and pathwise gradient estimator, respectively. We show that for both methods, the model learner is equipped with the value equivalence objective in an online learning manner:

$$\min_{\hat{f}_1, \dots, \hat{f}_T} \sum_{t=1}^T \left\| \mathcal{T}_{\pi_t}^H \hat{V}_{\phi}^{\pi_t} - \hat{\mathcal{T}}_{\pi_t}^H \hat{V}_{\phi}^{\pi_t} \right\|.$$

The above online learning problem is thus to select the value functions that are used to obtain the PVE model. Then loss  $\mathcal{L}_t$  depends on how representative the selected value functions are. If  $\hat{V}_{\phi}^{\pi_{1:t-1}}$  are representative enough, we may expect  $\hat{f}_t$  can perfectly describe the environment, and thus suffering a low loss  $\mathcal{L}_t(\hat{f}_t)$  even under a novel value function  $\hat{V}_{\phi}^{\pi_t}$ .

For tabular settings and liner function approximations, the model learning process becomes a *convex* problem, where online gradient descent can be adopted to obtain an iterative style of VAML algorithm [3]. However, online GD is not enough for more general nonconvex settings. We bring the family of *follow the leader* algorithms to tackle this model learning problem. That is, vanilla FTL indicates choosing the past policies on the policy improvement path as representative ones, which also shares similarities with the implementations in [2, 5]. We also propose several principled representative policy selection procedure by considering its predictable nature. Then optimistic FTL can be leveraged to obtain tighter regret bound.

There are wide applications of value equivalence principle in modern deep RL agents, *e.g.*, MuZero [12], Predictron [13], VPN [9]. Besides, it is experimentally shown in [5] that MuZero integrating past policies outperforms MuZero. We may expect the modifications can further bring improvements.

## 2 Policy Gradient Convergence Rates

### 2.1 Finite MDPs

First consider discounted finite MDPs where projected gradient ascent on the direct policy parameterization is performed.

direct policy parameterization:  $\pi_\theta(a|s) = \theta_{s,a}$

projected gradient ascent:  $\pi_{t+1} = P_{\Delta(\mathcal{A})|S|}(\theta_t + \eta \nabla_\theta \hat{J}(\pi_{\theta_t}))$

Here,  $J(\pi) = \mathbb{E}_{s_0 \sim \zeta(\cdot)} [V_\pi(s_0)]$  and  $\hat{J}(\pi) = \mathbb{E}_{s_0 \sim \zeta(\cdot)} [\hat{V}_\pi(s_0)]$ .

**Assumption 1.**  $V_{\pi_\theta}$  is  $L$ -smooth in  $\theta$ .

This assumption holds when the reward  $r$  and transition function  $f$  are both Lipschitz continuous and smooth (i.e., twice continuously differentiable with Lipschitz continuous first derivative) [1, 14, 11].

**Fact 1.** (Bastani, 2021, Lemma D.2). Denote  $L_h$  as the Lipschitz constant for function  $h$  and  $\bar{L}_h = \max\{L_h, L_{\nabla h, 1}\}$ . Then  $\nabla V_{\pi_\theta}$  is  $L$ -Lipschitz, where  $L = 44H^5 \bar{L}_r \bar{L}_f^{4H}$ . Particularly,  $\nabla_\theta V_{\pi_\theta}$  is Lipschitz continuous in  $\theta$  with Lipschitz constant  $24H^5 \bar{L}_r \bar{L}_f^{4H}$ .

gradient mapping:  $\rho_t = \frac{1}{\eta} [P_{\Delta(\mathcal{A})|S|}(\theta_t + \eta \nabla_\theta J(\pi_{\theta_t}))]$

gradient bias:  $b_t = \left\| \nabla_\theta J(\pi_{\theta_t}) - \mathbb{E}[\nabla_\theta \hat{J}(\pi_{\theta_t})] \right\|_2$

upper bound of squared variance:  $v_t = \mathbb{E} \left[ \left\| \nabla_\theta \hat{J}(\pi_{\theta_t}) - \mathbb{E}[\nabla_\theta \hat{J}(\pi_{\theta_t})] \right\|_2^2 \right]$

With a similar proof in [14], we have the following bound.

**Lemma 1.** For  $\eta \leq \frac{1}{L}$ ,

$$\sum_{t=1}^T \mathbb{E} [\|\rho_t\|_2^2] \leq 4 \left( \sum_{t=1}^T (\eta - L\eta^2)^{-1} (\eta b_t + \frac{\eta}{2} v_t) + b_t^2 + v_t \right) + 4 (\eta - L\eta^2)^{-1} \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_1})]$$

**Theorem 1.** For  $\eta \leq \frac{1}{L}$ ,

$$\min_{t \in [T]} J(\pi^*) - J(\pi_{\theta_t}) \leq \frac{4}{T} \left\| \frac{d_\zeta^{\pi^*}}{d_\zeta^{\pi_{\theta_t}}} \right\|_\infty \left( (\eta - L\eta^2)^{-1} \left( \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_1})] + \eta \sum_{t=1}^T (b_t + \frac{v_t}{2}) \right) + \sum_{t=1}^T (b_t^2 + v_t) \right)^{\frac{1}{2}}. \quad (1)$$

*Proof.* See Appendix 6.2. □

### 2.2 Function Approximations

Next, we consider discounted MDPs beyond finite settings, which enables function approximations. In particular, we focus on the convergence rates of Natural Policy Gradient.

$$\theta_{t+1} - \theta_t = \eta \tilde{\nabla}_\theta J(\theta_t)$$

$$\tilde{\nabla}_\theta J(\theta) = F(\theta)^{-1} \nabla_\theta J(\theta) = \frac{1}{1-\gamma} F(\theta)^{-1} \mathbb{E}_{s \sim d_{\mu_1}^{\pi_\theta}} \mathbb{E}_{a \sim \pi_\theta} [A^{\pi_\theta}(s, a) \nabla_\theta \log \pi_\theta(a|s)].$$

**Theorem 2.** Assume  $\log \pi_\theta(a|s)$  is a  $\iota$ -smooth function of  $\theta$ . For  $\eta \leq \frac{1}{L}$ ,

$$\min_{t \in [T]} J(\pi^*) - J(\pi_{\theta_t}) \leq \frac{\log |\mathcal{A}|}{T\eta(1-\gamma)} + \frac{\iota}{T(1-\gamma)(1-L\eta)} \left( \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_1})] + \eta \sum_{t=1}^T (b_t + \frac{v_t}{2}) \right). \quad (2)$$

*Proof.* See Appendix 6.3. □

What remains is to bound  $b_t$  and  $v_t$  in the convergence rate.

### 3 Likelihood Ratio Gradient Estimator

In the following sections, we assume the initial state  $s_0$  is deterministic and known for simplicity, which implies  $J(\pi) = V_\pi(s_0)$ . We analyze finite-horizon  $\gamma$ -discounted MDPs.

Denote the value by unrolling  $H$  steps of model  $\hat{f}$  with policy  $\pi$  as  $V_{\pi,H}^{\hat{f}}$ . One potential way to estimate  $V_\pi(s_0)$  is simply calculating  $V_{\pi,H}^{\hat{f}}(s_0)$  and setting  $H$  as the task horizon. However, such naive estimation will suffer from large bias due to model compounding error, or even cause training failure when adopting pathwise gradient estimator due to the curse of chaos, which we will formalize later. To tackle these challenges, we may adopt another value function  $\hat{V}_\phi$ , which can be either parametric or non-parametric, and obtain the *H-step model value expansion*:

$$\hat{V}_{\pi,H}(s_0) = V^{\hat{f}}(s_0) + \gamma^H \hat{V}_\phi(\hat{s}_H).$$

**Proposition 1.** *With function approximations, for a stochastic model  $\hat{f}$  with form  $\hat{s}_{h+1} = \hat{f}(\hat{s}_h, \hat{a}_h) + \xi_h$ , where  $\xi_h \sim p(\xi)$  and  $p(\xi)$  is  $\sigma_\xi$ -subgaussian, we have*

$$v_t \leq O\left((\gamma L_{\hat{f}})^{2H}\right). \quad (3)$$

*Proof.* See Appendix 6.4.  $\square$

Denote  $\mathcal{P}_\pi^H$  as the  $H$ -step environment transition operator under policy  $\pi$ :  $\mathcal{P}_\pi^H[x](s_0) = \mathbb{E}[x(s_H)|f^*, \pi]$  and  $\hat{\mathcal{P}}_\pi^H$  is the counterpart using the model  $\hat{f}$  instead of the environment  $f^*$ .

Let  $\mathcal{T}_\pi^H$  be  $H$  applications of the policy's Bellman operator  $\mathcal{T}_\pi$ , defined as  $\mathcal{T}_\pi[v](s) = \mathbb{E}[r(s, a) + \gamma v(s')|f^*, \pi]$ . Similarly,  $\hat{\mathcal{T}}_\pi$  is the Bellman operator induced by model  $\hat{f}$  and policy  $\pi$ .

**Proposition 2.** *The gradient bias is bounded by*

$$b_t \leq \left\| \mathcal{T}_\pi^H \hat{V}_\phi - \hat{\mathcal{T}}_\pi^H \hat{V}_\phi \right\| |\mathcal{A}| + \gamma^H \left\| \mathcal{P}_\pi^H V - \mathcal{P}_\pi^H \hat{V}_\phi \right\| |\mathcal{A}|. \quad (4)$$

*Proof.* See Appendix 6.5.  $\square$

### 4 Pathwise Gradient Estimator

Pathwise gradient estimator is another model-based approach that leverages the first-order gradient information in the function approximations. The pathwise gradient is calculated by backpropagating through the *H-step model value expansion*, but with a parameterized value function  $\hat{V}_\phi$ . Denote the  $H$ -step value expansion as  $\hat{V}_{\pi,H}$ .

**Proposition 3.** *Denote  $\epsilon = \max \nabla_\theta \pi(a|s)$ .*

$$v_t \leq \left( \frac{2L_r(\epsilon + K^{H-1})}{L_{\hat{f}} - 1} \frac{\gamma^H L_{\hat{f}}^{H+1}}{\gamma L_{\hat{f}} - 1} + \gamma^H \left( 2L_{\hat{Q}} K^{H+1} + 2L_{\hat{Q}} L_1(\epsilon + K^H) L_{\hat{f}} \frac{L_{\hat{f}}^H - 1}{L_{\hat{f}} - 1} \right) \right)^2 \quad (5)$$

*Proof.* See Appendix 6.6.  $\square$

We can see the necessity of introducing  $\hat{V}_\phi$ . A large truncation step  $H$  will lead to the curse of chaos and failure in training [8, 10]. Specifically,  $v_t$  depends exponentially on  $L_{\hat{f}}$ , i.e., small changes in initial conditions can result in diverging states and the gradient variance is also hopelessly large. (One potential way to mitigate it is to constrain the Jacobian of the learned dynamical system [8]. In other words, we can learn a more well-behaved non-chaotic version of physics instead of the rigid body dynamics with sharp changes.)

**Proposition 4.** *The gradient bias is bounded by*

$$b_t \leq \left\| \nabla_\theta \left( \mathcal{T}_\pi^H \hat{V}_\phi - \hat{\mathcal{T}}_\pi^H \hat{V}_\phi \right) \right\|_2 + \gamma^H \left\| \nabla_\theta \left( \mathcal{P}_\pi^H V - \mathcal{P}_\pi^H \hat{V}_\phi \right) \right\|_2. \quad (6)$$

*Proof.* See Appendix 6.7.  $\square$

## 5 Value Equivalent Model Learning Through Policy Improvement Path

Procedure: first learn  $\hat{V}_\phi^{\pi_{t-1}}$ , then learn model  $\hat{f}_t$ , last update policy to obtain  $\pi_t$ .

In [5], the Proper Value Equivalence (PVE) objective is

$$\begin{aligned} \|V^\pi - \hat{\mathcal{T}}_\pi^H V^\pi\| &\leq \|V^\pi - \mathcal{T}_\pi^H \hat{V}_\phi\| + \|\mathcal{T}_\pi^H \hat{V}_\phi - \hat{\mathcal{T}}_\pi^H \hat{V}_\phi\| + \|\hat{\mathcal{T}}_\pi^H \hat{V}_\phi - \hat{\mathcal{T}}_\pi^H V^\pi\| \\ &\leq \gamma^H \|\mathcal{P}_\pi^H V^\pi - \mathcal{P}_\pi^H \hat{V}_\phi\| + \gamma^H \|\hat{\mathcal{P}}_\pi^H V^\pi - \hat{\mathcal{P}}_\pi^H \hat{V}_\phi\| + \|\mathcal{T}_\pi^H \hat{V}_\phi - \hat{\mathcal{T}}_\pi^H \hat{V}_\phi\| \end{aligned} \quad (7)$$

Thus, the identical terms we want to minimize in Proposition 2 and 4 are

$$(??) \|\mathcal{T}_{\pi_t}^H \hat{V}_\phi^{\pi_t} - \hat{\mathcal{T}}_{\pi_t}^H \hat{V}_\phi^{\pi_t}\| + \gamma^H \|\mathcal{P}_{\pi_t}^H V^{\pi_t} - \mathcal{P}_{\pi_t}^H \hat{V}_\phi^{\pi_t}\| \leq \|V^{\pi_t} - \hat{\mathcal{T}}_{\pi_t}^H V^{\pi_t}\|. \quad (8)$$

As the lower bound of  $\|V^{\pi_t} - \hat{\mathcal{T}}_{\pi_t}^H V^{\pi_t}\|$ , minimizing the LHS of Eq. (??) can be done by minimizing the PVE loss with respect to a single policy  $\pi_t$  in the policy improvement path at iteration  $t$ .

However, in the convergence rate in Theorem 1, we need the accumulate  $\sum_{t=1}^T b_t$  to be small. That is, the true objective for the model learner is

$$\min_{\hat{f}_1, \dots, \hat{f}_T} \sum_{t=1}^T \mathcal{L}_t(\hat{f}_t) \triangleq \min_{\hat{f}_1, \dots, \hat{f}_T} \sum_{t=1}^T \|\mathcal{T}_{\pi_t}^H \hat{V}_\phi^{\pi_t} - \hat{\mathcal{T}}_{\pi_t}^H \hat{V}_\phi^{\pi_t}\|$$

This model learning problem can be cast as an online learning process. During each iteration  $t$ , the learner is tasked with learning a model  $\hat{f}_t$ . Then the model-based policy gradient is returned and a new policy  $\pi_t$  is obtained. The loss function  $\mathcal{L}_t$  is revealed to the model learner afterwards. This procedure is repeated across  $T$  rounds. Since  $\mathcal{L}_t(f^*) = 0$  for the real model  $f^*$ , we have the following equivalence

$$\sum_{t=1}^T \mathcal{L}_t(\hat{f}_t) \iff \sum_{t=1}^T \mathcal{L}_t(\hat{f}_t) - \sum_{t=1}^T \mathcal{L}_t(f^*) \iff \mathfrak{R} := \sum_{t=1}^T \mathcal{L}_t(\hat{f}_t) - \inf_f \sum_{t=1}^T \mathcal{L}_t(f)$$

In other words, it is equivalent to solve a regret  $\mathfrak{R}$  minimization problem, where many existing online learning approaches can be adopted.

The most straightforward algorithm is *Follow the Leader* (FTL) and its variants including *Follow the Regularized Leader* (FTRL) and *Follow the Perturbed Leader* (FTPL). For tabular settings and linear function approximations, the model learning process becomes an online convex learning problem. In this convex case, online gradient descent can be used to obtain  $O(T^{\frac{1}{2}})$  regret. The iterative VAML [3] can be viewed as a special case of convex model learning with online gradient descent.

$$\begin{aligned} \text{FTL: } \hat{f}_t &= \operatorname{argmin}_{\hat{f} \in \mathcal{F}} \sum_{i=1}^{t-1} \mathcal{L}_i(\hat{f}) \\ \text{Online GD: } \hat{f}_t &= \hat{f}_{t-1} - \eta \nabla \mathcal{L}_{i-1}(\hat{f}_{t-1}) \\ \text{FTPL: } \hat{f}_t &= \operatorname{argmin}_{\hat{f} \in \mathcal{F}} \sum_{i=1}^{t-1} \mathcal{L}_i(\hat{f}) + \sigma_t \\ \text{OFTPL: } \hat{f}_t &= \operatorname{argmin}_{\hat{f} \in \mathcal{F}} \sum_{i=1}^{t-1} \mathcal{L}_i(\hat{f}) + M_t(\hat{f}) + \sigma_t \end{aligned}$$

However, online GD loses its attractiveness in nonconvex setting, where the loss functions encountered by the model learner can potentially be nonconvex. For example, when we use nonlinear function approximation. Or Approximate Value Iteration (or Fitted Value) is adopted when value iteration cannot be performed exactly (e.g., due to large state space). In general non-convex settings,

FTPL can still achieve  $O(T^{\frac{1}{2}})$  regret with access to an offline optimization oracle which takes as input a function and returns an approximate minimizer of this function.

We also give the insights behind using FTL-style algorithms in the model learning process. The basic intuition of value-aware model learning is to assign different weights to the transitions according to the *optimal* state value. However, the optimal value function is not known and the VAML loss cannot be readily minimized. Thus, as shown in [6, 5], we may find models using value equivalence principle. As more value functions are augmented to be considered, the model will less compromise the performance, and eventually collapse to the real environment model. The key is to determine the representative value functions (or the value function of representative policies [5]). The above online learning problem is thus to select the value functions that are used to obtain the PVE model. Then loss  $\mathcal{L}_t$  depends on how representative the selected value functions are. If  $\hat{V}_\phi^{\pi_{1:t-1}}$  are representative enough, we may expect  $\hat{f}_t$  can perfectly describe the environment, and thus suffering a low loss  $\mathcal{L}_t(\hat{f}_t)$  even under a novel value function  $\hat{V}_\phi^{\pi_t}$ .

That is, choose the past policies on the policy improvement path as representative ones, which also shares similarities with [2, 5]. For non-oblivious loss which is not adversarially chosen, we may expect the losses predictable from the past. Specifically, our model learning process is predictable as successive losses depend on the policies obtained by applying policy gradient. Thus, using a predictor  $M_t$  might provide useful information and the resulting optimistic algorithms will enjoy tighter bounds. For example, we may simply set  $M_t = \hat{f}_{t-1}$  to emphasise the policy similarities between consecutive iterations, or set  $M_t = \frac{1}{t-1} \sum_{i=1}^{t-1} \hat{f}_i$  as the past average, or adding importance weights for fading memory statistics.

**Proposition 5.** *Denote the  $\alpha$ -approximate optimization oracle of function  $g$  as  $\mathcal{O}_\alpha(g)$  in the sense that if  $x^* = \mathcal{O}_\alpha(g)$ , then  $g(x^*) \leq \inf_x f(x) + \alpha$ . We have for FTPL and OFTPL that*

$$\mathfrak{R} = O(T^{\frac{1}{2}}).$$

Recall that  $\mathfrak{R} = \sum_{t=1}^T \mathcal{L}_t(\hat{f}_t) = \sum_{t=1}^T \left\| \mathcal{T}_{\pi_t}^H \hat{V}_\phi^{\pi_t} - \hat{\mathcal{T}}_{\pi_t}^H \hat{V}_\phi^{\pi_t} \right\|$ . Plugging the above results into the two theorems, we get a  $T^{-\frac{1}{2}}$  rate of convergence for model-based RL with both gradient estimators.

## 6 Proofs

### 6.1 Proof of Lemma 1

*Proof.* Denote  $\beta_t = \frac{\theta_{t+1} - \theta_t}{\eta}$ . By Assumption 1, we have

$$\begin{aligned} J(\pi_{\theta_{t+1}}) - J(\pi_{\theta_t}) &\geq \nabla_{\theta} J(\pi_{\theta_t})^{\top} (\theta_{t+1} - \theta_t) - \frac{L}{2} \|\theta_{t+1} - \theta_t\|_2^2 \\ &= \eta \nabla_{\theta} J(\pi_{\theta_t})^{\top} \beta_t - \frac{L\eta^2}{2} \|\beta_t\|_2^2. \end{aligned} \quad (9)$$

Rewrite the exact gradient  $\nabla_{\theta} J(\pi_{\theta_t})$  as

$$\nabla_{\theta} J(\pi_{\theta_t}) = \left( \nabla_{\theta} J(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] \right) - \left( \nabla_{\theta} \hat{J}(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] \right) + \nabla_{\theta} \hat{J}(\pi_{\theta_t}).$$

Then we bound  $\nabla_{\theta} J(\pi_{\theta_t})^{\top} \beta_t$  in Eq. (9) by bounding the resulting three terms.

$$\left| \left( \nabla_{\theta} J(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] \right)^{\top} \beta_t \right| \leq \|\beta_t\|_2 \|\nabla_{\theta} J(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})]\|_2 = \|\beta_t\|_2 b_t \quad (10)$$

$$\left( \nabla_{\theta} \hat{J}(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] \right)^{\top} \beta_t \leq \frac{\|\nabla_{\theta} \hat{J}(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})]\|_2^2}{2} + \frac{\|\beta_t\|_2^2}{2} \quad (11)$$

$$\nabla_{\theta} \hat{J}(\pi_{\theta_t})^{\top} \beta_t \geq \|\beta_t\|_2^2, \quad (12)$$

where Eq. (12) holds due to the fact that  $\left( \theta_{t+1} - (\theta_t + \eta \nabla_{\theta} \hat{J}(\theta_t)) \right)^{\top} (\theta_{t+1} - \theta_t) \leq 0$ .

Thus, we can bound Eq. (9) by

$$J(\pi_{\theta_{t+1}}) - J(\pi_{\theta_t}) \geq \eta \left( -\|\beta_t\|_2 b_t - \frac{\|\nabla_{\theta} \hat{J}(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})]\|_2^2}{2} + \frac{\|\beta_t\|_2^2}{2} \right) - \frac{L\eta^2}{2} \|\beta_t\|_2^2. \quad (13)$$

By taking expectation in Eq. (13), we have

$$\begin{aligned} \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \mathbb{E}[\|\beta_t\|_2^2] &\leq \mathbb{E}[J(\pi_{\theta_{t+1}}) - J(\pi_{\theta_t})] + \eta \mathbb{E}[\|\beta_t\|_2] b_t + \frac{\eta}{2} v_t \\ &\leq \mathbb{E}[J(\pi_{\theta_{t+1}}) - J(\pi_{\theta_t})] + b_t + \frac{\eta}{2} v_t \end{aligned} \quad (14)$$

Besides,

$$\begin{aligned} \|\rho_t - \beta_t\|_2 &= \frac{1}{\eta} \left\| P_{\Delta(\mathcal{A})|S|}(\theta_t + \eta \nabla_{\theta} J(\pi_{\theta_t})) - P_{\Delta(\mathcal{A})|S|}(\theta_t + \eta \nabla_{\theta} \hat{J}(\pi_{\theta_t})) \right\|_2 \\ &\leq \left\| \nabla_{\theta} J(\pi_{\theta_t}) - \nabla_{\theta} \hat{J}(\pi_{\theta_t}) \right\|_2. \end{aligned}$$

Then due to the fact that  $\|x + y\|_2^2 \leq 2\|x\|_2^2 + 2\|y\|_2^2$ , we have

$$\begin{aligned} \mathbb{E}[\|\rho_t - \beta_t\|_2^2] &\leq \mathbb{E} \left[ \left\| \nabla_{\theta} J(\pi_{\theta_t}) - \nabla_{\theta} \hat{J}(\pi_{\theta_t}) \right\|_2^2 \right] \\ &= \mathbb{E} \left[ \left\| \nabla_{\theta} J(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] + \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] - \nabla_{\theta} \hat{J}(\pi_{\theta_t}) \right\|_2^2 \right] \\ &\leq 2 \mathbb{E} \left[ \left\| \nabla_{\theta} J(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] \right\|_2^2 \right] + 2 \mathbb{E} \left[ \left\| \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] - \nabla_{\theta} \hat{J}(\pi_{\theta_t}) \right\|_2^2 \right] \\ &= 2 \left\| \nabla_{\theta} J(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] \right\|_2^2 + 2 \mathbb{E} \left[ \left\| \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] - \nabla_{\theta} \hat{J}(\pi_{\theta_t}) \right\|_2^2 \right] \\ &\leq 2b_t^2 + 2v_t. \end{aligned} \quad (15)$$

For  $\eta \leq \frac{1}{L}$ ,  $\frac{\eta}{2} - \frac{L\eta^2}{2} > 0$ . From Eq. (14) and Eq. (15), we have

$$\begin{aligned} \sum_{t=1}^T \mathbb{E} [\|\rho_t\|_2^2] &\leq 2 \sum_{t=1}^T (\mathbb{E} [\|\beta_t\|_2^2] + \mathbb{E} [\|\rho_t - \beta_t\|_2^2]) \\ &\leq 4 (\eta - L\eta^2)^{-1} \left( \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_1})] + \sum_{t=1}^T (b_t + \frac{\eta}{2} v_t) \right) + \frac{4}{T} \sum_{t=1}^T (b_t^2 + v_t) \\ &= 4 \left( \sum_{t=1}^T (\eta - L\eta^2)^{-1} (\eta b_t + \frac{\eta}{2} v_t) + b_t^2 + v_t \right) + 4 (\eta - L\eta^2)^{-1} \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_1})] \end{aligned}$$

□

## 6.2 Proof of Theorem 1

We first give two lemmas before the proof of Theorem 1.

**Lemma 2.** (Agarwal, 2021, Lemma 4.1). *Value function  $V$  satisfies gradient domination property. For all state distribution  $\mu_1, \mu_2 \in \Delta(\mathcal{S})$ , we have*

$$\mathbb{E}_{s_0 \sim \mu_1} [V_{\pi^*}(s_0)] - \mathbb{E}_{s_0 \sim \mu_1} [V_{\pi}(s_0)] \leq \left\| \frac{d_{\mu_1}^{\pi^*}}{d_{\mu_2}^{\pi}} \right\|_{\infty} \max_{\bar{\pi}} (\bar{\pi} - \pi)^{\top} \nabla_{\pi} \mathbb{E}_{s_0 \sim \mu_2} [V_{\pi}(s_0)].$$

**Lemma 3.** (Agarwal, 2021, Proposition B.1). *Define  $G^{\eta}(\pi) = \frac{1}{\eta} [P_{\Delta(\mathcal{A})|\mathcal{S}}(\theta_t + \eta \nabla_{\pi} \mathbb{E}_{s_0} [V_{\pi}(s_0)])]$ . If  $\|G^{\eta}(\pi)\|_2 \leq \epsilon$ , then*

$$\max_{\pi + e \in \Delta(\mathcal{A})^{|\mathcal{S}|}, \|e\|_2 \leq 1} e^{\top} \nabla_{\pi} \mathbb{E}_{s_0} [V_{\pi}(s_0)] \leq \epsilon(\eta L + 1).$$

*Proof.* From Lemma 2, we know that for initial distribution  $\zeta$ ,

$$J(\pi^*) - J(\pi_{\theta_t}) \leq \left\| \frac{d_{\zeta}^{\pi^*}}{d_{\zeta}^{\pi_{\theta_t}}} \right\|_{\infty} \max_{\bar{\pi}} (\bar{\pi} - \pi_{\theta_t})^{\top} \nabla_{\pi} J(\pi_{\theta_t}).$$

Then by Lemma 3, Lemma 1 and  $\eta L + 1 \leq 2$ , we know

$$\begin{aligned} &\min_{t \in [T]} J(\pi^*) - J(\pi_{\theta_t}) \\ &\leq \frac{1}{T} \sum_{t=1}^T (J(\pi^*) - J(\pi_{\theta_t})) \\ &\leq \frac{1}{T} \left\| \frac{d_{\zeta}^{\pi^*}}{d_{\zeta}^{\pi_{\theta_t}}} \right\|_{\infty} (\eta L + 1) \sum_{t=1}^T \mathbb{E} [\|\rho_t\|_2] \\ &\leq \frac{1}{T} \left\| \frac{d_{\zeta}^{\pi^*}}{d_{\zeta}^{\pi_{\theta_t}}} \right\|_{\infty} (\eta L + 1) \sum_{t=1}^T \sqrt{\mathbb{E} [\|\rho_t\|_2^2]} \\ &\leq \frac{2}{T} \left\| \frac{d_{\zeta}^{\pi^*}}{d_{\zeta}^{\pi_{\theta_t}}} \right\|_{\infty} (\eta L + 1) \left( \left( \sum_{t=1}^T (\eta - L\eta^2)^{-1} (\eta b_t + \frac{\eta}{2} v_t) + b_t^2 + v_t \right) + (\eta - L\eta^2)^{-1} \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_1})] \right)^{\frac{1}{2}} \\ &\leq \frac{4}{T} \left\| \frac{d_{\zeta}^{\pi^*}}{d_{\zeta}^{\pi_{\theta_t}}} \right\|_{\infty} \left( (\eta - L\eta^2)^{-1} \left( \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_1})] + \eta \sum_{t=1}^T (b_t + \frac{v_t}{2}) \right) + \sum_{t=1}^T (b_t^2 + v_t) \right)^{\frac{1}{2}}, \end{aligned}$$

where the second inequality follows Jensen's inequality with the concave square root function. □

### 6.3 Proof of Theorem 2

*Proof.* By smoothness, we have

$$\log \frac{\pi_{t+1}(a|s)}{\pi_t(a|s)} - \nabla_{\theta} \log \pi_t(a|s)(\theta_{t+1} - \theta_t) \geq -\frac{\iota}{2} \|\theta_{t+1} - \theta_t\|_2^2. \quad (16)$$

To match the definition in Lemma 1, denote  $\beta_t = \frac{\theta_{t+1} - \theta_t}{\eta} = \tilde{\nabla}_{\theta} \hat{J}(\theta_t)$ .

$$\begin{aligned} \mathbb{E}_{s \sim d^{\pi^*}} [D_{\text{KL}}(\pi^* || \pi_t) - D_{\text{KL}}(\pi^* || \pi_{t+1})] &= \mathbb{E}_{s \sim d^{\pi^*}} \mathbb{E}_{a \sim \pi^*} \left[ \log \frac{\pi_{t+1}(a|s)}{\pi_t(a|s)} \right] \\ &\geq \mathbb{E}_{s \sim d^{\pi^*}} \mathbb{E}_{a \sim \pi^*} \left[ \nabla_{\theta} \log \pi_t(a|s)(\theta_{t+1} - \theta_t) - \frac{\iota}{2} \|\theta_{t+1} - \theta_t\|_2^2 \right] \\ &= \mathbb{E}_{s \sim d^{\pi^*}} \mathbb{E}_{a \sim \pi^*} \left[ \eta A^{\pi_t}(s, a) - \frac{\iota \eta^2}{2} \|\beta_t\|_2^2 \right] \\ &= (1 - \gamma) \eta (J(\pi^*) - J(\pi_t)) - \frac{\iota \eta^2}{2} \mathbb{E}[\|\beta_t\|_2^2]. \end{aligned} \quad (17)$$

where the last equality follows the performance difference lemma [7].

Using similar techniques in the proof of Lemma 1, we have for  $\eta \leq \frac{1}{L}$  that

$$\mathbb{E}[\|\beta_t\|_2^2] \leq 2(\eta - L\eta^2)^{-1} \left( \mathbb{E}[J(\pi_{\theta_{t+1}}) - J(\pi_{\theta_t})] + \eta b_t + \frac{\eta}{2} v_t \right). \quad (18)$$

Thus,

$$\begin{aligned} &\min_{t \in [T]} J(\pi^*) - J(\pi_{\theta_t}) \\ &\leq \frac{1}{T} \sum_{t=1}^T (J(\pi^*) - J(\pi_{\theta_t})) \\ &\leq \frac{1}{T\eta(1-\gamma)} \sum_{t=1}^T \mathbb{E}_{s \sim d^{\pi^*}} [D_{\text{KL}}(\pi^* || \pi_t) - D_{\text{KL}}(\pi^* || \pi_{t+1})] + \frac{\iota \eta^2}{2} \mathbb{E}[\|\beta_t\|_2^2] \\ &\leq \frac{\mathbb{E}_{s \sim d^{\pi^*}} [D_{\text{KL}}(\pi^* || \pi_1)]}{T\eta(1-\gamma)} + \frac{\iota}{T(1-\gamma)(1-L\eta)} \left( \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_1})] + \eta \sum_{t=1}^T (b_t + \frac{v_t}{2}) \right) \\ &\leq \frac{\log |\mathcal{A}|}{T\eta(1-\gamma)} + \frac{\iota}{T(1-\gamma)(1-L\eta)} \left( \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_1})] + \eta \sum_{t=1}^T (b_t + \frac{v_t}{2}) \right). \end{aligned}$$

□

### 6.4 Proof of Proposition 1

*Proof.* First, we have

$$\begin{aligned} \|\hat{s}_{h+1} - \mathbb{E}[\hat{s}_{h+1}]\|_2 &= \left\| \hat{f}(\hat{s}_h, \hat{a}_h) + \xi_h - \mathbb{E}[\hat{f}(\hat{s}_h, \hat{a}_h)] \right\|_2 \\ &\leq L_{\hat{f}} \sqrt{\|\hat{s}_h - \mathbb{E}[\hat{s}_h]\|_2^2 + |\mathcal{A}|^2 + \sigma_{\xi}^2} \\ &\leq L_{\hat{f}} \|\hat{s}_h - \mathbb{E}[\hat{s}_h]\|_2 + L_{\hat{f}} |\mathcal{A}| + \sigma_{\xi}. \end{aligned} \quad (19)$$

Since  $\|\hat{s}_0 - \mathbb{E}[\hat{s}_0]\|_2 = 0$ , we have

$$\|\hat{s}_h - \mathbb{E}[\hat{s}_h]\|_2 \leq \left( L_{\hat{f}} |\mathcal{A}| + \sigma_{\xi} \right) \frac{L_{\hat{f}}^{h-2} - 1}{L_{\hat{f}} - 1}$$



Besides,

$$\begin{aligned}
\|\hat{V} - \mathbb{E}[\hat{V}]\| &= \left\| \sum_{h=0}^{H-1} \gamma^h (\hat{r}(\hat{s}_h, \hat{a}_h) - \mathbb{E}[\hat{r}(\hat{s}_h, \hat{a}_h)]) + \gamma^H (\hat{V}_\phi(\hat{s}_H) - \mathbb{E}[\hat{V}_\phi(\hat{s}_H)]) \right\|_2 \\
&\leq \sum_{h=0}^{H-1} \gamma^h L_{\hat{r}} (\|\hat{s}_h - \mathbb{E}[\hat{s}_h]\|_2 + |\mathcal{A}|) + \gamma^H L_{\hat{V}_\phi} \|\hat{s}_H - \mathbb{E}[\hat{s}_H]\|_2 \\
&= \frac{L_{\hat{r}}(L_{\hat{f}}|\mathcal{A}| + \sigma_\xi)}{L_{\hat{f}} - 1} \left( \frac{\gamma^H L_{\hat{f}}^{H-2} - \gamma^2}{\gamma L_{\hat{f}} - 1} + \gamma^H L_{\hat{V}_\phi} (L_{\hat{f}}^{H-2} - 1) \right) + \frac{L_{\hat{r}}|\mathcal{A}|}{1 - \gamma}.
\end{aligned} \tag{20}$$

Since for finite MDP,

$$\nabla_\theta J(\pi_{\theta_t}) = \sum_{s \in \mathcal{S}} d^\pi(s) \sum_{a \in \mathcal{A}} Q^\pi(s, a),$$

where  $d^\pi(s) = \lim_{h \rightarrow \infty} P(s_h = s | s_0, \pi_\theta)$ .

Finally,

$$\begin{aligned}
v_t &= \mathbb{E} \left[ \left\| \nabla_\theta \hat{J}(\pi_{\theta_t}) - \mathbb{E}[\nabla_\theta \hat{J}(\pi_{\theta_t})] \right\|_2^2 \right] \\
&\leq \left( \frac{L_{\hat{r}}(L_{\hat{f}}|\mathcal{A}| + \sigma_\xi)}{L_{\hat{f}} - 1} \left( \frac{\gamma^H L_{\hat{f}}^{H-2} - \gamma^2}{\gamma L_{\hat{f}} - 1} + \gamma^H L_{\hat{V}_\phi} (L_{\hat{f}}^{H-2} - 1) \right) + \frac{L_{\hat{r}}|\mathcal{A}|}{1 - \gamma} \right)^2 |\mathcal{A}|^2 \\
&= O((\gamma L_{\hat{f}})^{2H}).
\end{aligned}$$

□

## 6.5 Proof of Proposition 2

*Proof.* Let  $R_\theta = \mathbb{E}[r(s, a)]$ .

For H-step model-based rollout, we have

$$\begin{aligned}
\|V - \mathbb{E}[\hat{V}]\| &= \left\| \sum_{h=0}^{H-1} \gamma^h (\mathbb{E}[r(s_h, a_h)] - \mathbb{E}[\hat{r}(\hat{s}_h, \hat{a}_h)]) + \gamma^H (\mathbb{E}[V(s_H)] - \mathbb{E}[\hat{V}_\phi(\hat{s}_H)]) \right\| \\
&= \left\| \sum_{h=0}^{H-1} \gamma^h (\mathcal{P}_\pi^h R_\theta - \hat{\mathcal{P}}_\pi^h \hat{R}_\theta) + \gamma^H (\mathcal{P}_\pi^H V - \hat{\mathcal{P}}_\pi^H \hat{V}_\phi) \right\| \\
&= \left\| \sum_{h=0}^{H-1} \gamma^h (\mathcal{P}_\pi^h R_\theta - \hat{\mathcal{P}}_\pi^h \hat{R}_\theta) + \gamma^H (\mathcal{P}_\pi^H \hat{V}_\phi - \hat{\mathcal{P}}_\pi^H \hat{V}_\phi + \mathcal{P}_\pi^H V - \mathcal{P}_\pi^H \hat{V}_\phi) \right\| \\
&\leq \left\| \mathcal{T}_\pi^H \hat{V}_\phi - \hat{\mathcal{T}}_\pi^H \hat{V}_\phi \right\| + \gamma^H \mathcal{L}(\phi),
\end{aligned} \tag{21}$$

where the loss w.r.t. value parameter  $\phi$  is  $\mathcal{L}(\phi) = \left\| \mathcal{P}_\pi^H V - \mathcal{P}_\pi^H \hat{V}_\phi \right\| = \left\| \mathbb{E}[V(s_H)] - \mathbb{E}[\hat{V}_\phi(s_H)] \right\|$ .

The gradient bias is thus

$$\begin{aligned}
b_t &= \left\| \nabla_\theta J(\pi_{\theta_t}) - \mathbb{E}[\nabla_\theta \hat{J}(\pi_{\theta_t})] \right\|_2 \\
&\leq \left\| \mathcal{T}_\pi^H \hat{V}_\phi - \hat{\mathcal{T}}_\pi^H \hat{V}_\phi \right\| |\mathcal{A}| + \gamma^H \mathcal{L}(\phi) |\mathcal{A}|.
\end{aligned} \tag{22}$$

□

### 6.6 Proof of Proposition 3

*Proof.* For  $\hat{V}_{\pi,H}(s_0) = V^{\hat{f}}(s_0) + \gamma^H \mathbb{E}_a[\hat{Q}_\phi(\hat{s}_H, \hat{a}_H)]$ , we have

$$\nabla_\theta \hat{V}_{\pi,H}(s_0) = \nabla_\theta V^{\hat{f}}(s_0) + \gamma^H \nabla_a \hat{Q}_\phi(\hat{s}_H, \hat{a}_H) \nabla_\theta \hat{a}_H + \gamma^H \nabla_s \hat{Q}_\phi(\hat{s}_H, \hat{a}_H) \nabla_\theta \hat{s}_H.$$

Thus,  $v_t$  is bounded by

$$\begin{aligned} & \left\| \nabla_\theta \hat{J}(\pi_{\theta_t}) - \mathbb{E}[\nabla_\theta \hat{J}(\pi_{\theta_t})] \right\|_2 \\ & \leq \left\| \nabla_\theta V_H^{\hat{f}}(s_0) - \mathbb{E}[\nabla_\theta V_H^{\hat{f}}(s_0)] \right\|_2 + \gamma^H \left\| \nabla_a \hat{Q}_\phi(\hat{s}_H, \hat{a}_H) \nabla_\theta \hat{a}_H - \mathbb{E}_{\hat{s}_H, \hat{a}_H} [\nabla_a \hat{Q}_\phi(\hat{s}_H, \hat{a}_H) \nabla_\theta \hat{a}_H] \right\|_2 \\ & \quad + \gamma^H \left\| \nabla_s \hat{Q}_\phi(\hat{s}_H, \hat{a}_H) \nabla_\theta \hat{s}_H - \mathbb{E}_{\hat{s}_H, \hat{a}_H} [\nabla_s \hat{Q}_\phi(\hat{s}_H, \hat{a}_H) \nabla_\theta \hat{s}_H] \right\|_2 \\ & = \left\| \nabla_\theta V_H^{\hat{f}}(s_0) - \mathbb{E}[\nabla_\theta V_H^{\hat{f}}(s_0)] \right\|_2 + \gamma^H \left\| \nabla \hat{Q}_\phi \nabla_\theta \hat{x}_H - \mathbb{E}_{\hat{x}_H} [\nabla \hat{Q}_\phi \nabla_\theta \hat{x}_H] \right\|_2 \\ & \leq \frac{2L_r(\epsilon + K^{H-1})}{L_{\hat{f}} - 1} \frac{\gamma^H L_{\hat{f}}^{H+1}}{\gamma L_{\hat{f}} - 1} + \gamma^H \left\| \nabla \hat{Q}_\phi \nabla_\theta \hat{x}_H - \mathbb{E}_{\hat{x}_H} [\nabla \hat{Q}_\phi \nabla_\theta \hat{x}_H] \right\|_2, \end{aligned} \tag{23}$$

where the last inequality follows Lemma 4. The second term is bounded by

$$\begin{aligned} & \left\| \nabla \hat{Q}_\phi \nabla_\theta \hat{x}_H - \mathbb{E}_{\hat{x}_H} [\nabla \hat{Q}_\phi \nabla_\theta \hat{x}_H] \right\|_2 = \left\| \nabla \hat{Q}_\phi \mathbb{E}_{\hat{x}_H} [\nabla_\theta \hat{x}_H] - \mathbb{E}_{\hat{x}_H} [\nabla \hat{Q}_\phi \nabla_\theta \hat{x}_H] + \nabla \hat{Q}_\phi \nabla_\theta \hat{x}_H - \nabla \hat{Q}_\phi \mathbb{E}_{\hat{x}_H} [\nabla_\theta \hat{x}_H] \right\|_2 \\ & \leq 2L_{\hat{Q}} \left\| \mathbb{E}_{\hat{x}_H} [\nabla_\theta \hat{x}_H] \right\|_2 + L_{\hat{Q}} \left\| \nabla_\theta \hat{x}_H - \mathbb{E}_{\hat{x}_H} [\nabla_\theta \hat{x}_H] \right\|_2 \\ & \leq 2L_{\hat{Q}} K^{H+1} + 2L_{\hat{Q}} L_1 (\epsilon + K^H) L_{\hat{f}} \frac{L_{\hat{f}}^H - 1}{L_{\hat{f}} - 1}, \end{aligned} \tag{24}$$

where the last inequality follows Equation (29).

Finally,

$$\begin{aligned} v_t &= \mathbb{E} \left[ \left\| \nabla_\theta \hat{J}(\pi_{\theta_t}) - \mathbb{E}[\nabla_\theta \hat{J}(\pi_{\theta_t})] \right\|_2^2 \right] \\ &\leq \left( \frac{2L_r(\epsilon + K^{H-1})}{L_{\hat{f}} - 1} \frac{\gamma^H L_{\hat{f}}^{H+1}}{\gamma L_{\hat{f}} - 1} + \gamma^H \left( 2L_{\hat{Q}} K^{H+1} + 2L_{\hat{Q}} L_1 (\epsilon + K^H) L_{\hat{f}} \frac{L_{\hat{f}}^H - 1}{L_{\hat{f}} - 1} \right) \right)^2. \end{aligned} \tag{25}$$

□

**Lemma 4.** For  $H$ -step model rollout value, we have the following for its pathwise gradient

$$\left\| \nabla_\theta V_H^{\hat{f}}(s_0) - \mathbb{E}[\nabla_\theta V_H^{\hat{f}}(s_0)] \right\|_2 \leq \frac{2(\epsilon + K^{H-1})L_r}{L_{\hat{f}} - 1} \left( \frac{\gamma^H L_{\hat{f}}^{H+1}}{\gamma L_{\hat{f}} - 1} \right). \tag{26}$$

*Proof.*

$$\left\| \nabla_\theta V_H^{\hat{f}}(s_0) - \mathbb{E}[\nabla_\theta V_H^{\hat{f}}(s_0)] \right\|_2 \leq \sum_{h=0}^{H-1} \gamma^h \left\| \nabla_\theta r(\hat{s}_h, \hat{a}_h) - \mathbb{E}[\nabla_\theta r(\hat{s}_h, \hat{a}_h)] \right\|_2 \tag{27}$$

Here,  $\hat{s}_{h+1} = \hat{f}(\hat{s}_h, \hat{a}_h) + \xi_h$  and  $\hat{a}_h \sim \pi(\cdot | \hat{s}_h)$ .

Assume  $\|\nabla_{\theta}\hat{x}_h\|_2 \leq K^{h+1}$ , where  $K$  is related to the Lipschitz constants.

$$\begin{aligned}
\left\| \nabla_{\theta}\hat{s}_{h+1} - \mathbb{E}_{\hat{s}_{h+1}} [\nabla_{\theta}\hat{s}_{h+1}] \right\|_2 &= \left\| \nabla_s \hat{f} \nabla_{\theta} \hat{s}_h + \nabla_a \hat{f} \nabla_{\theta} \hat{a}_h - \mathbb{E}_{\hat{x}_h} [\nabla_s \hat{f} \nabla_{\theta} \hat{s}_h + \nabla_a \hat{f} \nabla_{\theta} \hat{a}_h] \right\|_2 \\
&\leq \left\| \nabla_s \hat{f} \nabla_{\theta} \hat{s}_h - \mathbb{E}_{\hat{x}_h} [\nabla_s \hat{f} \nabla_{\theta} \hat{s}_h] \right\|_2 + \left\| \nabla_a \hat{f} \nabla_{\theta} \hat{a}_h - \mathbb{E}_{\hat{x}_h} [\nabla_a \hat{f} \nabla_{\theta} \hat{a}_h] \right\|_2 \\
&\leq \left\| \nabla_s \hat{f} \nabla_{\theta} \hat{s}_h - \mathbb{E}_{\hat{x}_h} [\nabla_s \hat{f}] \nabla_{\theta} \hat{s}_h \right\|_2 + \left\| \mathbb{E}_{\hat{x}_h} [\nabla_s \hat{f}] \nabla_{\theta} \hat{s}_h - \mathbb{E}_{\hat{x}_h} [\nabla_s \hat{f} \nabla_{\theta} \hat{s}_h] \right\|_2 \\
&\quad + \left\| \nabla_a \hat{f} \nabla_{\theta} \hat{a}_h - \mathbb{E}_{\hat{x}_h} [\nabla_a \hat{f}] \nabla_{\theta} \hat{a}_h \right\|_2 + \left\| \mathbb{E}_{\hat{x}_h} [\nabla_a \hat{f}] \nabla_{\theta} \hat{a}_h - \mathbb{E}_{\hat{x}_h} [\nabla_a \hat{f} \nabla_{\theta} \hat{a}_h] \right\|_2 \\
&\leq 2L_{\hat{f}} \|\nabla_{\theta}\hat{x}_h\|_2 + L_{\hat{f}} \left\| \nabla_{\theta}\hat{s}_h - \mathbb{E}_{\hat{x}_h} [\nabla_{\theta}\hat{s}_h] \right\|_2 + L_{\hat{f}} \left\| \nabla_{\theta}\hat{a}_h - \mathbb{E}_{\hat{x}_h} [\nabla_{\theta}\hat{a}_h] \right\|_2 \\
&\leq 2L_{\hat{f}} K^{h+1} + L_{\hat{f}} \left\| \nabla_{\theta}\hat{s}_h - \mathbb{E}_{\hat{x}_h} [\nabla_{\theta}\hat{s}_h] \right\|_2 + 2L_{\hat{f}} \epsilon
\end{aligned} \tag{28}$$

Applying the recursion we have

$$\begin{aligned}
\left\| \nabla_{\theta}\hat{s}_h - \mathbb{E}_{\hat{s}_h} [\nabla_{\theta}\hat{s}_h] \right\|_2 &\leq 2L_{\hat{f}}(\epsilon + K^h) \sum_{k=0}^{h-1} L_{\hat{f}}^{h-1-k} \\
&= 2(\epsilon + K^h) \sum_{k=0}^{h-1} L_{\hat{f}}^{h-k} \\
&= 2(\epsilon + K^h) L_{\hat{f}} \frac{L_{\hat{f}}^h - 1}{L_{\hat{f}} - 1}
\end{aligned} \tag{29}$$

Then

$$\|r(\hat{s}_h, \hat{a}_h) - \mathbb{E}[r(\hat{s}_h, \hat{a}_h)]\|_2 \leq L_r \|(\hat{s}_h - \mathbb{E}[\hat{s}_h], \hat{a}_h - \mathbb{E}[\hat{a}_h])\|_2 \tag{30}$$

$$\begin{aligned}
\|\nabla_{\theta} r(\hat{s}_h, \hat{a}_h) - \mathbb{E}[\nabla_{\theta} r(\hat{s}_h, \hat{a}_h)]\|_2 &\leq L_r \sqrt{\left( 2(\epsilon + K^{h-1}) L_{\hat{f}} \frac{L_{\hat{f}}^h - 1}{L_{\hat{f}} - 1} \right)^2 + (2\epsilon)^2} \\
&\leq 2(\epsilon + K^h) L_r \left( \frac{L_{\hat{f}}^{h+1} - L_{\hat{f}}}{L_{\hat{f}} - 1} + 1 \right) \\
&= 2(\epsilon + K^h) L_r \frac{L_{\hat{f}}^{h+1} - 1}{L_{\hat{f}} - 1}
\end{aligned} \tag{31}$$

Thus,

$$\begin{aligned}
\left\| \nabla_{\theta} V_H^{\hat{f}}(s_0) - \mathbb{E}[\nabla_{\theta} V_H^{\hat{f}}(s_0)] \right\|_2 &\leq \sum_{h=0}^{H-1} \gamma^h \left\| \nabla_{\theta} r(\hat{s}_h, \hat{a}_h) - \mathbb{E}[\nabla_{\theta} r(\hat{s}_h, \hat{a}_h)] \right\|_2 \\
&\leq \frac{2(\epsilon + K^{H-1})L_r}{L_{\hat{f}} - 1} \sum_{h=0}^{H-1} \gamma^h (L_{\hat{f}}^{h+1} - 1) \\
&\leq \frac{2(\epsilon + K^{H-1})L_r}{L_{\hat{f}} - 1} \left( \frac{\gamma^H L_{\hat{f}}^{H+1} - 1}{\gamma L_{\hat{f}} - 1} - \frac{\gamma^H - 1}{\gamma - 1} \right) \\
&\leq \frac{2(\epsilon + K^{H-1})L_r}{L_{\hat{f}} - 1} \left( \frac{\gamma^H L_{\hat{f}}^{H+1} - 1}{\gamma L_{\hat{f}} - 1} - \gamma^{H-1} \right) \\
&\leq \frac{2(\epsilon + K^{H-1})L_r}{L_{\hat{f}} - 1} \left( \frac{\gamma^H L_{\hat{f}}^{H+1}}{\gamma L_{\hat{f}} - 1} \right)
\end{aligned} \tag{32}$$

□

## 6.7 Proof of Proposition 4

*Proof.* The gradient bias is

$$\begin{aligned}
b_t &= \left\| \nabla_{\theta} J(\pi_{\theta_t}) - \mathbb{E}[\nabla_{\theta} \hat{J}(\pi_{\theta_t})] \right\|_2 \\
&= \left\| \sum_{h=0}^{H-1} \gamma^h (\mathbb{E}[\nabla_{\theta} r(s_h, a_h)] - \mathbb{E}[\nabla_{\theta} \hat{r}(\hat{s}_h, \hat{a}_h)]) + \gamma^H (\mathbb{E}[\nabla_{\theta} V(s_H)] - \mathbb{E}[\nabla_{\theta} \hat{V}_{\phi}(\hat{s}_H)]) \right\|_2 \\
&= \left\| \sum_{h=0}^{H-1} \gamma^h \nabla_{\theta} (\mathcal{P}_{\pi}^h R_{\theta} - \hat{\mathcal{P}}_{\pi}^h \hat{R}_{\theta}) + \gamma^H \nabla_{\theta} (\mathcal{P}_{\pi}^H V - \hat{\mathcal{P}}_{\pi}^H \hat{V}_{\phi}) \right\|_2 \\
&= \left\| \sum_{h=0}^{H-1} \gamma^h \nabla_{\theta} (\mathcal{P}_{\pi}^h R_{\theta} - \hat{\mathcal{P}}_{\pi}^h \hat{R}_{\theta}) + \gamma^H \nabla_{\theta} (\mathcal{P}_{\pi}^H \hat{V}_{\phi} - \hat{\mathcal{P}}_{\pi}^H \hat{V}_{\phi}) + \gamma^H \nabla_{\theta} (\mathcal{P}_{\pi}^H V - \mathcal{P}_{\pi}^H \hat{V}_{\phi}) \right\|_2 \\
&\leq \left\| \nabla_{\theta} (\mathcal{T}_{\pi}^H \hat{V}_{\phi} - \hat{\mathcal{T}}_{\pi}^H \hat{V}_{\phi}) \right\|_2 + \gamma^H \left\| \nabla_{\theta} (\mathcal{P}_{\pi}^H V - \mathcal{P}_{\pi}^H \hat{V}_{\phi}) \right\|_2.
\end{aligned} \tag{33}$$

□

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