Model-Based First-Order Policy Gradient for Contact Dynamics

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Abstract

In model-based reinforcement learning (RL), the learned models are typically smooth approximators of the environment dynamics. This is problematic in robotic systems that experience hard contact and have stiff or discontinuous local behaviors. Despite the unrealistic state predictions in contact-rich tasks, the inaccurate gradient estimation can lead to poor performance when applying First-Order Policy Gradient (FOPG). Therefore, we present a physics-guided complementarity-based contact model. However, our theory implies that a stiff contact model can cause the gradient variance to explode, which further leads to slow convergence and optimization difficulties. For this reason, we establish the relationship between the model stiffness and the centering parameter when solving the complementarity problem. Based on this result, we propose an Analytic Barrier Smoothed FOPG that avoids the large variance resulting from contact. Besides, we propose to use a contact-aware adaptive centering parameter to control the bias of analytic smoothing, such that the exact complementarity problem is solved when the body is far away from the contact, and analytic smoothing is performed at local contact regions. We show that analytic smoothing with a proper adaptive centering parameter is the best linear approximation of the original unsmoothed system. Moreover, we provide the bias upper bound of analytic smoothing. We also prove the equivalence between analytic smoothing and randomized smoothing and discuss the practical advantages of analytic smoothing.

1. Background

1.1. Reinforcement Learning

Consider learning to optimize a finite H-horizon Markov Decision Process (MDP) over repeated episodes of interaction. Denote the state space and action space as $\mathcal{X}$ and $\mathcal{U}$, respectively. When taking action $u \in \mathcal{U}$ at state $x \in \mathcal{X}$, the agent receives reward $r(x, u)$ and the MDP transitions to a new state according to probability $s' \sim f^*(\cdot \,|\, x, u)$.

We are interested in controlling the system by finding a policy π_θ that maximizes the expected cumulative reward. Denote by ζ the initial state distribution. The objective is

$$\mathcal{J}(\pi) = \mathbb{E}_{x_0 \sim \zeta}\big[V_0^\pi(x_0)\big] = \mathbb{E}_{p_\pi(\alpha)}\Bigg[\sum_{t=0}^{H-1} r(x_t, u_t)\Bigg],$$

where $p_\pi(\alpha)$ is the distribution over rollouts $\alpha := ((x_0, u_0), \cdots, (x_{H-1}, u_{H-1}))$ when executing π, formally, $x_0 \sim \zeta(\cdot)$, $u_i \sim \pi(\cdot \,|\, s_i)$, and $x_{i+1} \sim f^*(\cdot \,|\, x_i, u_i)$.

1.2. Stochastic Gradient Estimation

The general underlying problem of policy gradient, i.e., computing the gradient of a probabilistic objective with respect to the parameters of the sampling distribution, takes the form $\nabla_\theta \mathbb{E}_{p(x;\theta)}[y(z)]$. In RL, we set $p(z;\theta)$ as the trajectory distribution conditioned on policy parameter θ, and $y(z)$ as the cumulative reward. In the sequel, we introduce two commonly used gradient estimators in RL.

Zeroth-Order (or Likelihood Ratio) Gradient. By leveraging the *score function*, zeroth-order gradient estimators only require samples of the function values. Specifically, since the score function satisfies $\nabla_\theta \log p(z;\theta) = \nabla_\theta p(z;\theta)/p(z;\theta)$, the zeroth-order gradient has the form:

$$\begin{aligned}\nabla_\theta \mathbb{E}_{p(z;\theta)}\big[y(x)\big] &= \int y(z) \nabla_\theta p(z;\theta) \mathrm{d}z \\ &= \mathbb{E}_{p(z;\theta)}\big[y(z) \nabla_\theta \log p(z;\theta)\big]. \qquad (1.1)\end{aligned}$$

First-Order (or Reparameterization) Gradient. First-order gradient benefits from the structural characteristics of the objective, i.e., how the overall objective is affected by the operations applied to the sources of randomness as they pass through the measure and into the cost function

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(Mohamed et al., 2020). From the simulation property of continuous distribution, we have the following equivalence between direct and indirect ways of drawing samples:

$$\widehat{z} \sim p(z;\theta) \;\equiv\; \widehat{z} = g(\epsilon;\theta), \quad \epsilon \sim p. \tag{1.2}$$

Derived from the *law of the unconscious statistician* (LOTUS) (Grimmett & Stirzaker, 2020), i.e., $\mathbb{E}_{p(x;\theta)}[y(z)] = \mathbb{E}_{p(\epsilon)}[y(g(\epsilon;\theta))]$, the first-order gradient takes the form:

$$\begin{aligned}\nabla_\theta \mathbb{E}_{p(z;\theta)}\big[y(z)\big] &= \nabla_\theta \int p(\epsilon) y\big(g(\epsilon;\theta)\big)\mathrm{d}\epsilon \\ &= \mathbb{E}_{p(\epsilon)}\Big[\nabla_\theta y\big(g(\epsilon;\theta)\big)\Big].\end{aligned}$$

1.3. Rigid Body Dynamics

We consider a standard approach to modeling robotic systems – the framework of rigid-body systems with contacts. The continuous-time equation of motion is

$$M(q)\mathrm{d}v = \big(n(q,v)+u\big)\mathrm{d}t + J(q)^\top \lambda,$$

where we let q denote the generalized coordinates, v the generalized velocities, $u \in \mathbb{R}^{n_u}$ the applied control force, $M(q)$ the generalized inertia matrix, $n(q,v)$ the passive forces (e.g., Coriolis, centrifugal, and gravity), and $J(q)$ the Jacobian of the active contacts. Here, we define $\lambda := (\gamma^{(1)}, \beta^{(1)}, \cdots, \gamma^{(c)}, \beta^{(c)}) \in \mathbb{R}^{n_\lambda}$ as the (unknown) contact space impulse, where γ and β are the normal impact forces and friction forces, respectively, and c denotes the number of contact points.

Using Euler approximation and multiplying by M_t^{-1}, the discrete-time dynamics can be modeled in contact space by

$$\begin{aligned}&v_{t+1} = v_t + M_t^{-1}(n_t+u_t)h + M_t^{-1}J_t^\top \lambda_t, \\ &q_{t+1} = q_t + hv_{t+1}\end{aligned} \tag{1.3}$$

where h is the discretization step size and t is the timestep.

The friction and impacts are constrained by the system's configuration and the applied contact impulses. The impact problem is encoded with the following constraints:

$$\gamma_{t+1} \circ \phi(q_{t+1}) = \vec{0}, \qquad \gamma_{t+1}, \phi(q_{t+1}) \geq \vec{0}, \tag{1.4}$$

where $\circ$ is the element-wise (Hadamard) product, ϕ is the signed-distance function, $\vec{0}$ is the zero vector, and the equality, inequality are also element-wise. The intuition behind (1.4) is that the magnitude of the normal forces must be non-negative and can only be non-zero if there is a contact to maintain non-negative gaps (non-penetration).

Moreover, the Coulomb friction can be modeled using the maximum-dissipation principle and a linearized friction cone, which has the set of constraints:

$$\begin{aligned}&\beta_{t+1} \circ \xi_{t+1} = \vec{0}, \qquad \beta_{t+1}, \xi_{t+1} \geq \vec{0}, \\ &B(q_{t+1})v_{t+1} + \omega_{t+1}\vec{1} - \xi_{t+1} = \vec{0}, \\ &\omega_{t+1} \cdot (\alpha\gamma_{t+1} - \beta_{t+1}) = \vec{0},\end{aligned} \tag{1.5}$$

where $\alpha \geq 0$ is the friction coefficient, matrix B maps from the generalized coordinate velocity to tangential velocity in the contact frame, and $\omega_{t+1} \in \mathbb{R}$, ξ_{t+1} are dual variables associated with the linearized friction-cone and nonnegative constraint, respectively.

2. Complementarity-Based Contact Models

2.1. Linear Complementarity Systems

The dynamic (1.3) describes a hybrid system where different modes are controlled by the contact force λ under the complementarity constraints (1.4), (1.5). To simplify our analysis, we study the more abstracted Linear Complementarity Systems (LCS), which effectively capture the local behaviors of the transition and are widespread in the robotics community (Aydinoglu et al., 2021; Tassa & Todorov, 2010; Drumwright & Shell, 2012).

We first define the LCS model f_μ as a class of softened approximations of the exact LCS $f_{\mu=0}$.

Definition 2.1 (LCS Model)**.** A model $x_{t+1} = f_\mu(x_t, u_t)$ is an LCS model if the evolution of state $x \in \mathbb{R}^{d_x}$ is governed by a linear dynamics and a μ-softened LC problem (LCP):

$$\begin{aligned}&x_{t+1} = Ax_t + Bu_t + C\lambda_t + c, \\ &\lambda_t \circ (Dx_t + Eu_t + F\lambda_t + d) = \mu\vec{1}, \\ &\lambda_t \geq \vec{0}, \quad Dx_t + Eu_t + F\lambda_t + d \geq \vec{0},\end{aligned} \tag{2.1}$$

where $A \in \mathbb{R}^{d_x \times d_x}, B \in \mathbb{R}^{d_x \times d_u}, C \in \mathbb{R}^{d_x \times d_\lambda}, D \in \mathbb{R}^{d_\lambda \times d_x}, E \in \mathbb{R}^{d_\lambda \times d_u}, F \in \mathbb{R}^{d_\lambda \times d_\lambda}$, and the scalar $\mu \geq 0$. Denote S_μ as the solver of the μ-softened LCP (the last two lines of (2.1)), which returns the solution $\lambda_t = S_\mu(Dx_t + Eu_t + d) \in \mathbb{R}^{d_\lambda}$.

In simulation, $\mu = 0$ corresponds to the exact LCP where the system $f_{\mu=0}$ resembles the reality. Obviously, solving for the contact space impulse λ_t is the main problem, as x_{t+1} is readily obtained from the dynamics. Next, we introduce the assumption and method for solving the exact LCP.

Assumption 2.2 (P-Matrix)**.** Assume F in the LCS (2.1) is a P-matrix, defined as a matrix whose principal minors are all positive, i.e., the determinants of its principal submatrices $\det(F_{\alpha\alpha}) > 0$, $\forall \alpha \subseteq \{1, \cdots, d_\lambda\}$.

Assumption 2.2 guarantees that the solution λ_t exists and is unique, which is commonly assumed in contact dynamics problems (Aydinoglu et al., 2020; Jin et al., 2022).

2.2. Smoothed Objective with Barrier Function

To efficiently and accurately solve the convex constrained optimization problem (2.1), we adopt the *Interior-Point Method* (IPM) that leads to a sequence of relaxed problems by choosing a positive $\mu > 0$ that converges to zero to reliably converge to a solution of the original LCS ($\mu = 0$).

We show that the LCS model is the optimality condition of a barrier-smoothed objective with the following lemma. We defer all the proofs to Appendix A.

Lemma 2.3 (Primal Problem with Log-Barrier Function)**.** The (softened) LCS model (2.1) with $\mu \geq 0$ is the first-order optimality condition of the following program

$$\begin{aligned} \min_{\lambda_t, \epsilon_t} \quad & \lambda_t^\top \epsilon_t - \mu \sum_{i=1}^{d_\lambda} \big(\log \lambda_t^{(i)} + \log \epsilon_t^{(i)}\big) \\ \text{s.t.} \quad & Dx_t + Eu_t + F\lambda_t + d = \epsilon_t, \\ & Ax_t + Bu_t + C\lambda_t + c = x_{t+1}, \end{aligned} \tag{2.2}$$

where $\lambda_t^{(i)}, \epsilon_t^{(i)}$ are the i-th elements of vector $\lambda_t, \epsilon_t \in \mathbb{R}^{d_\lambda}$.

Lemma 2.3 indicates that the LCS model in (2.1) is in fact the perturbed Karush–Kuhn–Tucker (KKT) conditions, where the perturbation corresponds to smoothing the objective with barrier functions. By replacing the hard contact constraints in the LCS $f_{\mu=0}$, the logarithmic barrier functions in (2.2) discourages the solution to reach the boundary of the polytope constructed by the hard constraints. Therefore, μ is a centering parameter as it restrains the solution within the analytic center of the constraint polytope.

The barrier terms can be thought of as the potential of a force field whose strength is inversely proportional to the distance to the constraint boundary (Boyd et al., 2004; Pang et al., 2022). When applying IPM with a sequence of centering parameters, the intermediate problems with $\mu > 0$ achieve a smoothing effect similar to the "force-at-a-distance" relaxation of the complementarity constraints. In other words, μ controls both the *stiffness* and the *accuracy* of the LCS model dynamics f_μ. In the following section, we show that both of them are determining factors for the quality of first-order gradient estimation and the convergence of the policy gradient algorithm.

3. Model-Based First-Order Policy Gradient

In this section, we first provide a general framework of model-based First-Order Policy Gradient (FOPG). Then we establish the convergence of model-based FOPG and study the relationship between its convergence rate and the gradient bias, variance. Based on our analysis on the gradient variance and the stiffness of the complementarity-based models, we find that non-smooth local behaviors at contact points can lead to optimization difficulties, which motivates us to analytically smooth the system.

3.1. Framework

The pseudocode of model-based FOPG is presented in Algorithm 1, where two update procedures are performed iteratively. Namely, the model and policy are updated in every iteration $n \in [N]$, which give us sequences of $\{f_{\psi_n}\}_{n\in[N]}$ and $\{\pi_{\theta_t}\}_{t\in[T+1]}$, respectively.

Algorithm 1 Model-Based First-Order Policy Gradient

```
Input: Number of iterations $N$, transition data set $\mathcal{D} = \varnothing$
for iteration $n \in [N]$ do
    Update the model parameter $\psi_n$ by minimizing (3.1)
    Update the policy parameter $\theta_n$ by (3.3), where
    $f(x, u)$ is returned by the IPM solver (App. Alg. 2)
    Execute $\pi_{\theta_{n+1}}$ and update $\mathcal{D}$
end for
Output: $\{\pi_{\theta_n}\}_{n\in[N]}$
```

Model Update. A forward state-predictive model is learned from real-world data $\mathcal{D} = \{(x_t^*, u_t^*, x_{t+1}^*)\}_{t=1}^T$. For rigid-body systems that experience hard contact, we learn a physically grounded model $x_{t+1} = f(x_t, u_t; \psi)$ where the state $x_t \in \mathbb{R}^{d_x}$ is the system's configuration (including velocity v_t, coordinate q_t, etc.), and f returns the solution of (1.3) constrained by (1.4), (1.5). Instead of parameterized by a black-box neural network, the ψ contains all *estimated* physics parameters in (1.3), (1.4), (1.5), e.g., the inertia matrix M, Jacobian matrix J, friction coefficient α, and signed-distance function ϕ. The model training loss is given by

$$L(\psi; \mathcal{D}) = \sum_{t=1}^{T} \frac{1}{2} \big\| f(x_t^*, u_t^*; \psi) - x_{t+1}^* \big\|_2^2. \tag{3.1}$$

Policy Update. Consider optimizing a stochastic policy $u \sim \pi_\theta(\cdot|x)$ in continuous action spaces, or equivalently $u = \pi_\theta(x, \varsigma)$ with noise $\varsigma \sim p(\varsigma)$. The first-order policy gradient at iteration n is given by linking together the reward, model, policy, and differentiating through the model-generated trajectories:

$$\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) = \frac{1}{M} \sum_{m=1}^{M} \nabla_\theta \Big(\sum_{t=0}^{H-1} \gamma^t \cdot r(x_{t,m}, u_{t,m}) \Big), \tag{3.2}$$

where M is the sample size, $x_{0,m} \sim \zeta$, $u_{t,m} = \pi(x_{t,m}, \varsigma_m)$, $\varsigma_m \sim p(\varsigma)$, and $x_{t+1,m} = f(x_{t,m}, u_{t,m})$.

The update rule forthe policy parameter θ with learning rate η is as follows:

$$\theta_{n+1} \leftarrow \theta_n + \eta \cdot \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}). \tag{3.3}$$

3.2. Convergence of Model-Based FOPG

To begin with, we impose a common regularity condition on the policy functions following previous works (Xu et al., 2019; Pirotta et al., 2015; Zhang et al., 2020; Agarwal et al., 2021). The assumption below essentially ensures the smoothness of the objective $\mathcal{J}(\pi_\theta)$, which is required by most existing analyses of policy gradient methods (Wang et al., 2019; Bastani, 2020; Agarwal et al., 2020).

Assumption 3.1 (Lipschitz Continuous Policy Gradient)**.** Assume that $\nabla_\theta \mathcal{J}(\pi_\theta)$ is L-Lipschitz continuous in θ, such that $\|\nabla_\theta \mathcal{J}(\pi_{\theta_1}) - \nabla_\theta \mathcal{J}(\pi_{\theta_2})\|_2 \le L\|\theta_1 - \theta_2\|_2$.

We characterize the convergence of model-based FOPG by first providing the following proposition.

Theorem 3.2 (Convergence to Stationary Points)**.** Define the gradient bias b_n and variance v_n at iteration n as

$$b_n := \left\|\nabla_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}\big[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\big]\right\|_2,$$
$$v_n := \mathbb{E}\Big[\left\|\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}\big[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\big]\right\|_2^2\Big].$$

Denote $\delta := \sup \|\theta\|_2$ and $c := (\eta - L\eta^2)^{-1}$. It then holds for $N \ge 4L^2$ that

$$\min_{n\in[N]} \mathbb{E}\Big[\left\|\nabla_\theta \mathcal{J}(\pi_{\theta_n})\right\|_2^2\Big] \le \frac{4c}{N} \cdot \mathbb{E}\big[\mathcal{J}(\pi_{\theta_N}) - \mathcal{J}(\pi_{\theta_1})\big] + \frac{4}{N}\left(\sum_{n=0}^{N-1} c(2\delta \cdot b_n + \frac{\eta}{2} \cdot v_n) + b_n^2 + v_n\right).$$

Theorem 3.2 shows the reliance between the convergence error and the variance, bias of the gradient estimators. In general, to guarantee the convergence of model-based FOPG, we have to control both the variance and the bias to the sublinear growth rate. Before studying the upper bound of b_n and v_n, we make the following Lipschitz assumption, which is adopted in various previous works (Pirotta et al., 2015; Clavera et al., 2020; Li et al., 2021).

Assumption 3.3 (Lipschitz Continuity)**.** Assume the policy, model, and reward are L_π, L_f, L_r Lipschitz continuous.

3.3. Gradient Variance and LCS Stiffness

Denote $\widetilde{L}_g := \max\{L_g, 1\}$ for any function g. We have the following result for the variance of FOPG.

Proposition 3.4 (Gradient Variance)**.** Under Assumption 3.3, at any iteration $n \in [N]$, the gradient variance of FOPG is bounded by

$$v_n \le O\Big(H^4 \widetilde{L}_f^{4H} \widetilde{L}_\pi^{4H} / M\Big). \tag{3.4}$$

We observe that the variance upper bound has polynomial dependence on the Lipschitz of the model and policy, where the degrees are linear in the effective horizon. This makes intuitive sense as the system is chaotic: The stochasticity during training can lead to diverging trajectories and stochastic gradient directions, causing large gradient variance. The optimization difficulties imposed by non-smooth models, e.g. the hard contact models, result in slow convergence or training failure even in simple toy tasks (Parmas et al., 2018; Suh et al., 2022a).

It's worth noting that the above analysis holds for general model-based FOPG. Studying the model stiffness is especially important when adopting complementarity-based contact models since they are inherently non-smooth or discontinuous at local mode-switching points. We characterize the stiffness of the LCS models with the following proposition.

Proposition 3.5 (LCS Model Stiffness)**.** Let $\|\cdot\|_F$ denote the matrix Frobenius norm and define $\varepsilon := \sup \|Dx_t + Eu_t + d\|_2^2 / (2\|F\|_F^2)$. Under Assumption 2.2, the Lipschitz L_f of the LCS model f_μ defined in (2.1) satisfies

$$L_f \le (\|A\|_F + \|B\|_F) + d_\lambda^2 \|C\|_F (\|D\|_F + \|E\|_F) \cdot l(\mu),$$

where $l(\mu)$ is determined by μ and is lower bounded by

$$l(\mu) \ge \frac{\varepsilon}{\mu} + \frac{1}{\|F\|_F} + \varepsilon\sqrt{\frac{1}{\mu^2} + \frac{2}{\varepsilon\mu\|F\|_F}}.$$

Proposition 3.5 indicates that the stiffness of the LCS model is largely determined by the centering parameter μ: The upper bound of L_f (and thus of the variance (3.4)) is at least inversely proportional to μ. This is problematic when performing first-order policy gradient based on the LCS model: The accurate dynamics is obtained when solving the exact LCP ($\mu \to 0$), which, however, causes the gradient variance to explode since $l(\mu) \to \infty$. The optimization challenges, e.g. chaotic and non-smooth landscapes, are posed even when contact occurs occasionally in a full model unroll.

4. Contact-Aware Analytic Barrier Smoothing

4.1. Method

A natural idea to alleviate the exploding FOPG variance issue is to prevent μ from reaching 0. The solutions correspond to trajectories that do *not* well obey the physics laws. According to Lemma 2.3, setting μ to positive values is equivalent to analytically smoothing the complementarity constraints with log-barrier functions. For this reason, we call this vanilla method *analytic barrier smoothing*.

Unfortunately, simply softening the complementarity system with a constant μ can lead to large model simulation error and gradient bias. Therefore, to achieve a good convergence in Thm. 3.2, additional care must be taken to trade-off between the variance and bias.

In this work, we propose the *contact-aware* analytic barrier-smoothed FOPG: When calculating first-order gradients in (3.2), we differentiate through $f_{\mu(x_t,u_t)}$ — the *intermediate* solution of the IPM solver that correspond to a contact-aware *adaptive* $\mu(x_t, u_t)$, whose value is close to 0 if the body is far from contact, and is positive if the contact is nearby.

This design is based on the observation that the fundamental reason for the stiffness of complementarity-based models is the existence of contact. Therefore, when performing FOPG, we *only* need to smooth the *local* dynamics at contact points to avoid large variance, while preferring *globally* accurate simulation for a small overall bias.

We don't fix the choice of function $\mu(\cdot,\cdot)$ since it should be problem-dependent. Instead, we show that the proposed analytic smoothing has close relationship with randomized smoothing and, when the contact-aware $\mu(\cdot,\cdot)$ takes certain forms, enjoy small gradient bias.

4.2. Analysis

Studying the bias of analytic barrier smoothing requires more fine-grained analysis. In this section, we focus on the frictionless setting, which reduces d_λ to 1 corresponding to the contact impulses. Although the results might generalize to broader settings, their forms are prohibitive for analysis.

As a first step, we build the connection between the proposed *contact-aware analytic barrier smoothing* and *randomized smoothing* (Suh et al., 2022a;b; Pang et al., 2022), which samples and averages the stochastic gradient. We show that these two smoothing techniques are identical in principle.

Proposition 4.1 (Equivalence with Randomized Smoothing)**.** Denote $z_t := Dx_t + Eu_t + d \in \mathbb{R}$. Recall that the solution of the exact LCP is $S_{\mu=0}(z_t)$ and the analytically smoothed LCP solution is $S_{\mu(z_t)}(z_t)$ (c.f. Defn. 2.1). For any centering function $\mu(z_t)$, analytic smoothing is equivalent to randomized smoothing: $S_{\mu(z_t)}(z_t) = \mathbb{E}_{w\sim\rho(w)}[S_{\mu=0}(z_t+w)]$ where $\rho(w) = \nabla_w^2 S_{\mu(z_t)}(w)$.

The above proposition shows that the analytic barrier smoothing inherently smooths the contact impulse λ_t (with respect to z_t), and thus smoothing the dynamics $x_{t+1} = f_\mu(x_t, u_t)$ since x_t, u_t are prefixed. More importantly, by choosing a proper (adaptive) centering parameter $\mu(z_t)$, the proposed method can cover any randomized smoothing method while avoiding its drawbacks when calculating first-order gradients, which we will discuss in more detail.

As a benefit of Proposition 4.1, we can work directly on the randomization-smoothed model when studying the bias of analytic smoothing. This gives us the following results.

Proposition 4.2 (Smoothing as Linearization Minimizer)**.** Define the error function as the σ-Gaussian tail integral $\text{erf}(y;\sigma^2) := \int_y^\infty 1/(\sqrt{2\pi}\sigma)e^{-y^2/\sigma^2}$. Set the z_t-adaptive centering parameter as $\mu(z_t) = \kappa\cdot(z_t + F\kappa)$, where

$$\kappa := z_t \cdot \text{erf}(z_t,\sigma) + e^{-z_t^2/(2\sigma)}/\sqrt{\pi} + c_1 z_t + c_2, \quad (4.1)$$

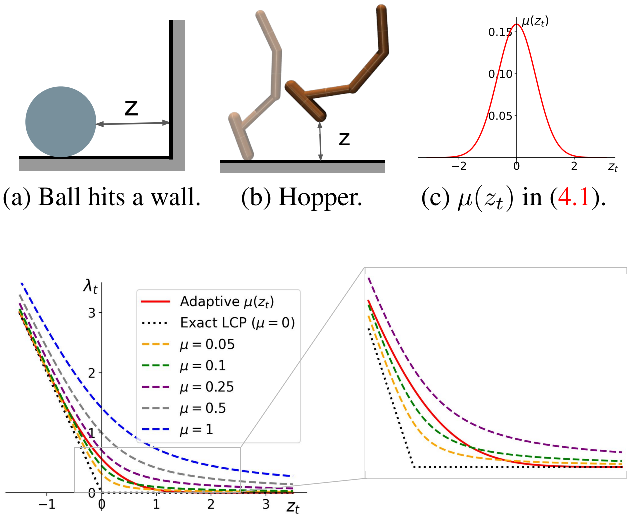
and $c_1, c_2 \in \mathbb{R}$ are tunable constants. Consider the problem of regressing the exact LCP solution $S_{\mu=0}$ with parameters (K, W) such that the residual around z_t distributed according to Gaussian is minimized, formally:

$$\delta = \min_{K,W} \mathbb{E}_{w\sim\mathcal{N}(0,\sigma)}\Big[\big|S_{\mu=0}(z_t+w) - Ww - K\big|\Big].$$

The solution K^*, W^* that achieves the minimum is the analytically smoothed surrogate and its gradient:

$$K^* = S_{\mu(z_t)}(z_t), \quad W^* = \nabla_z S_{\mu(z_t)}(z_t).$$

The above proposition shows that analytic smoothing is the best linear approximation of the LCP solution around z_t. Therefore, with a small approximation error, we can conclude the model gradient bias of analytic barrier smoothing.



(a) Ball hits a wall. (b) Hopper. (c) $\mu(z_t)$ in (4.1).

(d) Contact impulse of the exact LCP and of the smoothed system: Constant μ and the contact-aware adaptive $\mu(z_t)$.

Figure 1. 1(a), 1(b): Two 1D examples. 1(c): Plot of the adaptive $\mu(z_t)$. 1(d): Contact impulse comparison. The proposed $\mu(z_t)$ is contact-aware and has better balance of stiffness smoothing and bias reduction: At contact point around $z = 0$, adaptive $\mu(z_t)$ gives smoother dynamics (compared to $\mu \leq 0.1$) and meanwhile best approximates the exact LCP at contactless regions (i.e. $z > 0$).

We provide an example system and the plot of $\mu(z_t)$, λ_t in Figure 4.2. We observe that μ that adapts with z_t is contact-aware: μ is positive only when around the contact point around 0. This contact-aware design supports our intuition: When the body is away from contact, we can safely solve the exact LCP and get accurate simulation; When experiencing contact, the proposed method smooths the LCP to obtain non-stiff local dynamics and small gradient variance.

Proposition 4.3 (Bias of Analytic Smoothing)**.** With the same definition of $\mu(z_t)$ in Proposition 4.2, the gradient of the LCS model $f_{\mu(z_t)}$ approximately matches the gradient of LCS $f_{\mu=0}$, with the bias upper bounded by

$$\begin{aligned}&\left\|\nabla f_{\mu=0} - \nabla f_{\mu(z_t)}\right\|_2 \\ &\quad \leq \|C\|_F(\|D\|_F + \|E\|_F)\cdot\Big(\frac{\sigma F^2 Q(3/4)}{2} + \frac{12\delta + \varsigma}{\sigma Q(2/3)}\Big),\end{aligned}$$

where we define $\varsigma := 1/\sqrt{\pi} + c_2$ and $\mathcal{Q} : [0, 1] \to \mathbb{R}$ is the inverse of the cumulative distribution function (or quantile function) of the standard normal distribution, $\mathcal{Q}(3/4) \approx 0.67$, $\mathcal{Q}(2/3) \approx 0.43$.

In Proposition 4.3, we bound the bias of gradient when analytically smoothing the LCS model with an adaptive centering parameter $\mu(z_t)$. Therefore, when the model parameters ψ are accurately fitted with supervised learning, i.e. $f_{\mu=0} \approx f^*$, then the smoothed system $f_{\mu(z_t)}$ (i.e. the intermediate solution of IPM solver) and its gradient achieves the best linearization error and has small gradient bias b_n.

Discussion. Although equivalence can be established between the analytically smoothed system $f_{\mu>0}(x)$ and the randomized smoothed surrogate $\mathbb{E}_{w\sim\rho}[f_{\mu=0}(x + w)]$, applying randomized smoothing in FOPG suffers from the empirical bias (Suh et al., 2022b;a) and the noisy gradients (Howell et al., 2022).

Randomized smoothing estimates the FOPG by averaging the stochastic gradients of multiple noise-induced samples, each of which is calculated by differentiating through $f_{\mu=0}$. The empirical bias phenomenon happens under discontinuities or stiffness. Consider the Heaviside step dynamics when experiencing contact (e.g. pushing an object under friction), differentiating through which will give zero gradient. Therefore, the randomized smoothed FOPG is also zero, causing large bias. Even if the system is non-stiff, sampling and averaging the stochastic gradients is noisy and computationally expensive. In contrast, analytic smoothing by softening the complementarity constraints and solving the program, directly differentiates through the smoothed system $f_{\mu>0}$ and prevents the above issues.

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A. Proofs

A.1. Proof of Lemma 2.3

Proof. Corresponding to the constrained optimization problem (2.2) we can introduce the multipliers ι and form the Lagrangian function by

$$L(\lambda_t, \epsilon_t, \iota) = \lambda_t^\top \epsilon_t - \mu \sum_{i=1}^{n_\lambda} \big(\log \lambda_t^{(i)} + \log \epsilon_t^{(i)}\big) + \iota^\top (Dx_t + Eu_t + F\lambda_t + d - \epsilon_t).$$

Here, we omit the last equality constraint in (2.2) since x_{t+1} can be directly calculated when λ_t is obtained.

We have from the Karush–Kuhn–Tucker (KKT) conditions that the optimal solution must satisfy

$$\frac{\partial}{\partial \lambda_t^{(i)}} L(\lambda_t, \epsilon_t, \iota) = \epsilon_t^{(i)} - \mu \cdot \frac{1}{\lambda_t^{(i)}} + (\iota^\top F)^{(i)} - \iota_2^{(i)} = 0, \tag{A.1}$$

$$\frac{\partial}{\partial \epsilon_t^{(i)}} L(\lambda_t, \epsilon_t, \iota) = \lambda_t^{(i)} - \mu \cdot \frac{1}{\epsilon_t^{(i)}} - \iota_1^{(i)} - \iota_3^{(i)} = 0, \tag{A.2}$$

$$Dx_t + Eu_t + F\lambda_t + d = \epsilon_t, \tag{A.3}$$

where (A.1), (A.2) follow from the stationarity of the optimal solution, and (A.3) follows from the primal feasibility.

Combining the above equations, we know that $\epsilon_t^{(i)} \lambda_t^{(i)} = \mu$ and $\lambda_t \circ (Dx_t + Eu_t + F\lambda_t + d) = \mu \vec{1}$. □

A.2. Proof of Theorem 3.2

As a preparation before proving Theorem 3.2, we first present the following lemma stating that the RL objective is Lipschitz smooth under Assumption 3.1.

Lemma A.1 (Smooth Objective)**.** The objective $\mathcal{J}(\pi_\theta)$ is L-smooth in θ, such that $\|\nabla_\theta \mathcal{J}(\pi_{\theta_1}) - \nabla_\theta \mathcal{J}(\pi_{\theta_2})\|_2 \le L\|\theta_1 - \theta_2\|_2$, where

$$L := H^2 r_{\mathrm{m}} \cdot L_1 + 2H^3 r_{\mathrm{m}} \cdot B_\theta^2.$$

Proof. We refer to Lemma 3.2 in (Zhang et al., 2020) for detailed proof. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. From the policy update rule, we know that $\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) = (\theta_{n+1} - \theta_n)/\eta$. By the Lipschitz Assumption 3.3, we have

$$\begin{aligned} \mathcal{J}(\pi_{\theta_{n+1}}) - \mathcal{J}(\pi_{\theta_n}) &\ge \nabla_\theta \mathcal{J}(\pi_{\theta_n})^\top (\theta_{n+1} - \theta_n) - \frac{L}{2}\big\|\theta_{n+1} - \theta_n\big\|_2^2 \\ &= \eta \nabla_\theta \mathcal{J}(\pi_{\theta_n})^\top \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) - \frac{L\eta^2}{2}\big\|\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\big\|_2^2. \end{aligned} \tag{A.4}$$

We rewrite the exact gradient $\nabla_\theta \mathcal{J}(\pi_{\theta_n})$ as

$$\nabla_\theta \mathcal{J}(\pi_{\theta_n}) = \Big(\nabla_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}\big[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\big]\Big) - \Big(\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}\big[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\big]\Big) + \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}).$$

In order to lower-bound $\nabla_\theta \mathcal{J}(\pi_{\theta_n})^\top \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})$, we turn to bound the resulting three terms:

$$\begin{aligned} \left|\Big(\nabla_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}\big[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\big]\Big)^\top \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\right| &\le \Big\|\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\Big\|_2 \cdot \Big\|\nabla_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}\big[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\big]\Big\|_2 \\ &= \Big\|\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\Big\|_2 \cdot b_n, \\ \Big(\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}\big[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\big]\Big)^\top \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) &\le \frac{\Big\|\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}\big[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\big]\Big\|_2^2}{2} + \frac{\Big\|\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\Big\|_2^2}{2}, \\ \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})^\top \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) &\ge \Big\|\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\Big\|_2^2. \end{aligned}$$

Thus, we have the following inequality for (A.4):

$$\mathcal{J}(\pi_{\theta_{n+1}}) - \mathcal{J}(\pi_{\theta_n}) \geq \frac{\eta}{2} \cdot \left(- \left\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \right\|_2 \cdot 2b_n - \left\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})] \right\|_2^2 + \left\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \right\|_2^2 \right) \\ - \frac{L\eta^2}{2} \cdot \left\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \right\|_2^2. \tag{A.5}$$

By taking expectation in (A.5), we obtain

$$\mathbb{E}[\mathcal{J}(\pi_{\theta_{n+1}}) - \mathcal{J}(\pi_{\theta_n})] \geq -\eta \cdot \mathbb{E}\Big[\big\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2 \Big] \cdot b_n - \frac{\eta}{2} \cdot v_n + \frac{\eta - L\eta^2}{2} \cdot \mathbb{E}\Big[\big\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2^2 \Big].$$

By rearranging terms,

$$\frac{\eta - L\eta^2}{2} \cdot \mathbb{E}\Big[\big\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2^2 \Big] \leq \mathbb{E}[\mathcal{J}(\pi_{\theta_{n+1}}) - \mathcal{J}(\pi_{\theta_n})] + \eta \mathbb{E}[\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \|_2] b_n + \frac{\eta}{2} v_n. \tag{A.6}$$

We now turn our attention to characterize $\|\nabla_\theta \mathcal{J}(\pi_{\theta_n}) - \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})\|_2$.

$$\begin{aligned} \mathbb{E}\Big[\big\| \nabla_\theta \mathcal{J}(\pi_{\theta_n}) - \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2^2 \Big] &= \mathbb{E}\Big[\Big\| \nabla_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})] + \mathbb{E}[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})] - \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \Big\|_2^2 \Big] \\ &\leq 2 \Big\| \nabla_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})] \Big\|_2^2 + 2\mathbb{E}\Big[\Big\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) - \mathbb{E}[\widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n})] \Big\|_2^2 \Big] \\ &= 2b_n^2 + 2v_n, \end{aligned} \tag{A.7}$$

where the second inequality holds since for any vector $y, z \in \mathbb{R}^d$,

$$\|y + z\|_2^2 \leq \|y\|_2^2 + \|z\|_2^2 + 2\|y\|_2 \cdot \|z\|_2 \leq 2\|y\|_2^2 + 2\|z\|_2^2. \tag{A.8}$$

Then we are ready to bound the minimum expected gradient norm by relating it to the average norm over T iterations. Specifically,

$$\begin{aligned} \min_{t \in [T]} \mathbb{E}\Big[\big\| \nabla_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2^2 \Big] &\leq \frac{1}{N} \cdot \sum_{n=0}^{N-1} \mathbb{E}\Big[\big\| \nabla_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2^2 \Big] \\ &\leq \frac{2}{N} \cdot \sum_{n=0}^{N-1} \left(\mathbb{E}\left[\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \|_2^2 \right] + \mathbb{E}\Big[\big\| \nabla_\theta \mathcal{J}(\pi_{\theta_n}) - \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2^2 \Big] \right), \end{aligned}$$

where the second inequality follows from (A.8).

For $N \geq 4L^2$, by setting $\eta = 1/\sqrt{N}$, we have $\eta < 1/L$ and $(\eta - L\eta^2)/2 > 0$. Therefore, following the results in (A.6) and (A.7), we further have

$$\begin{aligned} &\min_{n \in [N]} \mathbb{E}\Big[\big\| \nabla_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2^2 \Big] \\ &\leq \frac{4c}{N} \cdot \left(\mathbb{E}[\mathcal{J}(\pi_{\theta_N}) - \mathcal{J}(\pi_{\theta_1})] + \sum_{n=0}^{N-1} \left(\eta \cdot \mathbb{E}\Big[\big\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2 \Big] \cdot b_n + \frac{\eta}{2} \cdot v_n \right) \right) + \frac{4}{N} \cdot \sum_{n=0}^{N-1} (b_n^2 + v_n) \\ &= \frac{4}{N} \cdot \left(\sum_{n=0}^{N-1} c \cdot \left(\eta \cdot \mathbb{E}\Big[\big\| \widehat{\nabla}_\theta \mathcal{J}(\pi_{\theta_n}) \big\|_2 \Big] \cdot b_n + \frac{\eta}{2} \cdot v_n \right) + b_n^2 + v_n \right) + \frac{4c}{N} \cdot \mathbb{E}\big[\mathcal{J}(\pi_{\theta_N}) - \mathcal{J}(\pi_{\theta_1}) \big], \end{aligned}$$

where the last step holds due to the definition $c := \big(\eta - L\eta^2\big)^{-1}$.

By noting that $\eta\widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n}) = \theta_{n+1} - \theta_n$, we conclude the proof by

$$\begin{aligned}
&\min_{n\in[N]} \mathbb{E}\Big[\big\|\nabla_\theta\mathcal{J}(\pi_{\theta_n})\big\|_2^2\Big] \\
&\leq \frac{4}{N}\cdot\bigg(\sum_{n=0}^{N-1} c\cdot\Big(\mathbb{E}\Big[\big\|\theta_{n+1}-\theta_n\big\|_2\Big]\cdot b_n + \frac{\eta}{2}\cdot v_n\Big) + b_n^2 + v_n\bigg) + \frac{4c}{N}\cdot\mathbb{E}\big[\mathcal{J}(\pi_{\theta_N}) - \mathcal{J}(\pi_{\theta_1})\big] \\
&\leq \frac{4}{N}\cdot\bigg(\sum_{n=0}^{N-1} c\cdot(2\delta\cdot b_n + \frac{\eta}{2}\cdot v_n) + b_n^2 + v_n\bigg) + \frac{4c}{N}\cdot\mathbb{E}\big[\mathcal{J}(\pi_{\theta_N}) - \mathcal{J}(\pi_{\theta_1})\big].
\end{aligned}$$

where the second inequality holds since $\|\theta\|_2 \leq \delta$ for any $\theta\in\Theta$. □

A.3. Proof of Proposition 3.4

In what follows, we interchangeably write $\nabla_a b$ and $\mathrm{d}b/\mathrm{d}a$ as the derivative, and use the notation $\partial b/\partial a$ to denote the partial derivative. With slight abuse of notation, for vector s and vector w, we denote the Jacobian matrix consisting of entries $\partial s^{(i)}/\partial w^{(j)}$ as $\partial s/\partial w$.

Proof. In order to upper-bound the gradient variance $v_n = \mathbb{E}[\|\widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n}) - \mathbb{E}[\widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n})]\|_2^2]$, we turn to find the supremum of the norm inside the outer expectation, which serves as a loose yet acceptable variance upper bound.

We start with the case when the sample size $M = 1$, which can naturally generalize to $N > 1$. Specifically, consider an *arbitrary* trajectory obtained by unrolling the model under policy π_{θ_n}. Denote the pathwise gradient $\widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n})$ of this trajectory as g'. Then we have

$$v_n \leq \max_{g'}\Big\|g' - \mathbb{E}\big[\widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n})\big]\Big\|_2^2 = \Big\|g - \mathbb{E}\big[\widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n})\big]\Big\|_2^2 = \Big\|\mathbb{E}\big[g - \widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n})\big]\Big\|_2^2,$$

where we let g denote the pathwise gradient $\widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n})$ of a *fixed* (but unknown) trajectory $(x_0, u_0, x_1, u_1, \cdots)$ such that the maximum is achieved.

Using the fact that $\|\mathbb{E}[\cdot]\|_2 \leq \mathbb{E}[\|\cdot\|_2]$, we further obtain

$$v_n \leq \mathbb{E}\Big[\big\|g - \widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n})\big\|_2\Big]^2. \tag{A.9}$$

Denote $y_t := (x_t, u_t)$. By triangular inequality, we have

$$\mathbb{E}\Big[\big\|g - \widehat{\nabla}_\theta\mathcal{J}(\pi_{\theta_n})\big\|_2\Big] \leq \sum_{t=0}^{H-1}\mathbb{E}_{\overline{y}_t}\Big[\big\|\nabla_\theta r(y_t) - \nabla_\theta r(\overline{y}_t)\big\|_2\Big]. \tag{A.10}$$

For $t \geq 1$, we have the following relationship according to the chain rule:

$$\frac{\mathrm{d}u_t}{\mathrm{d}\theta} = \frac{\partial u_t}{\partial x_t}\cdot\frac{\mathrm{d}x_t}{\mathrm{d}\theta} + \frac{\partial u_t}{\partial\theta}, \tag{A.11}$$

$$\frac{\mathrm{d}x_t}{\mathrm{d}\theta} = \frac{\partial x_t}{\partial x_{t-1}}\cdot\frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta} + \frac{\partial x_t}{\partial u_{t-1}}\cdot\frac{\mathrm{d}u_{t-1}}{\mathrm{d}\theta}. \tag{A.12}$$

Plugging $\mathrm{d}u_{t-1}/\mathrm{d}\theta$ in (A.11) into (A.12), we get

$$\frac{\mathrm{d}x_t}{\mathrm{d}\theta} = \left(\frac{\partial x_t}{\partial x_{t-1}} + \frac{\partial x_t}{\partial u_{t-1}}\cdot\frac{\partial u_{t-1}}{\partial x_{t-1}}\right)\cdot\frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta} + \frac{\partial x_t}{\partial u_{t-1}}\cdot\frac{\partial u_{t-1}}{\partial\theta}. \tag{A.13}$$

By the Cauchy-Schwarz inequality and the Lipschitz Assumption 3.3, we have

$$\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta}\right\|_2 \leq L_f\widetilde{L}_\pi\cdot\left\|\frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta}\right\|_2 + L_f L_\theta.$$

Applying the above recursion gives us

$$\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta}\right\|_2 \le L_f L_\theta \cdot \sum_{j=0}^{t-1} L_f^j \widetilde{L}_\pi^j \le i \cdot L_\theta L_f^{t+1} \widetilde{L}_\pi^t, \tag{A.14}$$

where the first inequality follows from the induction

$$z_n = a z_{t-1} + b = a \cdot (a z_{i-2} + b) + b = a^t \cdot z_0 + b \cdot \sum_{j=0}^{t-1} a^j, \tag{A.15}$$

for the real sequence $\{z_j\}_{0 \le j \le i}$ satisfying $z_j = a z_{j-1} + b$. For $\mathrm{d}u_t/\mathrm{d}\theta$ defined in (A.11), we further have

$$\left\|\frac{\mathrm{d}u_t}{\mathrm{d}\theta}\right\|_2 \le L_\pi \cdot \left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta}\right\|_2 + L_\theta \le t \cdot L_\theta L_f^{t+1} \widetilde{L}_\pi^{t+1} + L_\theta. \tag{A.16}$$

Combining (A.14) and (A.16), we obtain

$$\left\|\frac{\mathrm{d}y_t}{\mathrm{d}\theta}\right\|_2 = \left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta}\right\|_2 + \left\|\frac{\mathrm{d}u_t}{\mathrm{d}\theta}\right\|_2 \le K(t) := 2t \cdot L_\theta L_f^{t+1} \widetilde{L}_\pi^{t+1} + L_\theta, \tag{A.17}$$

where $K(t)$ is introduced for notation simplicity.

By the chain rule, (A.10) can be decomposed and bounded by

$$\begin{aligned}
&\mathbb{E}_{\overline{y}_t}\Big[\big\|\nabla_\theta r(y_t) - \nabla_\theta r(\overline{y}_t)\big\|_2\Big] \\
&\quad = \mathbb{E}_{\overline{y}_t}\Big[\big\|\nabla r(y_t)\nabla_\theta y_t - \nabla r(\overline{y}_t)\nabla_\theta \overline{y}_t\big\|_2\Big] \\
&\quad \le \mathbb{E}_{\overline{y}_t}\Big[\big\|\nabla r(y_t)\nabla_\theta y_t - \nabla r(y_t)\nabla_\theta \overline{y}_t\big\|_2\Big] + \mathbb{E}\Big[\big\|\nabla r(y_t)\nabla_\theta \overline{y}_t - \nabla r(\overline{y}_t)\nabla_\theta \overline{y}_t\big\|_2\Big] \\
&\quad \le L_r \cdot \left(\mathbb{E}_{\overline{x}_n}\left[\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_t}{\mathrm{d}\theta}\right\|_2\right] + \mathbb{E}_{\overline{u}_n}\left[\left\|\frac{\mathrm{d}u_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{u}_t}{\mathrm{d}\theta}\right\|_2\right]\right) + 2L_r \cdot K(t),
\end{aligned} \tag{A.18}$$

where the last step follows from the Cauchy-Schwartz inequality and the Lipschitz reward assumption.

Plugging (A.18) into (A.10) and (A.9), we have

$$\begin{aligned}
v_n &\le L_r \cdot \left(\sum_{t=0}^{H-1}\left(\mathbb{E}_{\overline{x}_t}\left[\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_t}{\mathrm{d}\theta}\right\|_2\right] + \mathbb{E}_{\overline{u}_t}\left[\left\|\frac{\mathrm{d}u_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{u}_t}{\mathrm{d}\theta}\right\|_2\right] + 2K(t)\right)\right)^2 \\
&\le O\left(\left(\sum_{t=0}^{H-1} t^2 \widetilde{L}_f^{2t} \widetilde{L}_\pi^{2t}\right)^2\right) = O\left(H^4 \widetilde{L}_f^{4H} \widetilde{L}_\pi^{4H}\right),
\end{aligned} \tag{A.19}$$

where the second inequality follows from the results from Lemma A.2 and by plugging the definition of K in (A.17). Since the analysis above considers batch size $M = 1$, the bound of gradient variance v_n is established by dividing M, which concludes the proof. □

Lemma A.2. Denote $e := \sup \mathbb{E}_{\overline{x}_0}[\|\mathrm{d}x_0/\mathrm{d}\theta - \mathrm{d}\overline{x}_0/\mathrm{d}\theta\|_2]$, which is a constant that only depends on the initial state distribution[1]. For any timestep $t \ge 1$ and the corresponding state x_t, control input u_t, we have the following inequality results:

$$\begin{aligned}
\mathbb{E}_{\overline{x}_t}\left[\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_t}{\mathrm{d}\theta}\right\|_2\right] &\le \widetilde{L}_f^t \widetilde{L}_\pi^t \Big(e + 4t \cdot \widetilde{L}_f \widetilde{L}_\pi \cdot K(t-1) + 2t \cdot \widetilde{L}_f L_\theta\Big), \\
\mathbb{E}_{\overline{u}_n}\left[\left\|\frac{\mathrm{d}u_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{u}_t}{\mathrm{d}\theta}\right\|_2\right] &\le \widetilde{L}_f^t \widetilde{L}_\pi^{t+1} \Big(e + 4i \cdot \widetilde{L}_f \widetilde{L}_\pi \cdot K(t-1) + 2t \cdot \widetilde{L}_f L_\theta\Big) + 2L_\pi K(t) + 2L_\theta.
\end{aligned}$$

[1] We define e to account for the stochasticity of the initial state distribution. $e = 0$ when the initial state is deterministic.

Proof. Firstly, we obtain from (A.12) that $\forall t \geq 1$,

$$\begin{aligned}
&\mathbb{E}_{\overline{x}_t}\left[\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_t}{\mathrm{d}\theta}\right\|_2\right] \\
&\quad = \mathbb{E}\left[\left\|\frac{\partial x_t}{\partial x_{t-1}} \cdot \frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta} + \frac{\partial x_t}{\partial u_{t-1}} \cdot \frac{\mathrm{d}u_{t-1}}{\mathrm{d}\theta} - \frac{\partial \overline{x}_t}{\partial \overline{x}_{t-1}} \cdot \frac{\mathrm{d}\overline{x}_{t-1}}{\mathrm{d}\theta} - \frac{\partial \overline{x}_t}{\partial \overline{u}_{t-1}} \cdot \frac{\mathrm{d}\overline{u}_{t-1}}{\mathrm{d}\theta}\right\|_2\right]
\end{aligned}$$

According to the triangle inequality, we continue with

$$\begin{aligned}
&\leq \mathbb{E}\left[\left\|\frac{\partial x_t}{\partial x_{t-1}} \cdot \frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta} - \frac{\partial \overline{x}_t}{\partial \overline{x}_{t-1}} \cdot \frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta}\right\|_2\right] + \mathbb{E}\left[\left\|\frac{\partial \overline{x}_t}{\partial \overline{x}_{t-1}} \cdot \frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta} - \frac{\partial \overline{x}_t}{\partial \overline{x}_{t-1}} \cdot \frac{\mathrm{d}\overline{x}_{t-1}}{\mathrm{d}\theta}\right\|_2\right] \\
&\quad + \mathbb{E}\left[\left\|\frac{\partial x_t}{\partial u_{t-1}} \cdot \frac{\mathrm{d}u_{t-1}}{\mathrm{d}\theta} - \frac{\partial \overline{x}_t}{\partial \overline{u}_{t-1}} \cdot \frac{\mathrm{d}u_{t-1}}{\mathrm{d}\theta}\right\|_2\right] + \mathbb{E}\left[\left\|\frac{\partial \overline{x}_t}{\partial \overline{u}_{t-1}} \cdot \frac{\mathrm{d}u_{t-1}}{\mathrm{d}\theta} - \frac{\partial \overline{x}_t}{\partial \overline{u}_{t-1}} \cdot \frac{\mathrm{d}\overline{u}_{t-1}}{\mathrm{d}\theta}\right\|_2\right] \\
&\leq 2L_f \cdot \left(\left\|\frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta}\right\|_2 + \left\|\frac{\mathrm{d}u_{t-1}}{\mathrm{d}\theta}\right\|_2\right) + L_f \cdot \mathbb{E}_{\overline{x}_{t-1}}\left[\left\|\frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_{t-1}}{\mathrm{d}\theta}\right\|_2\right] \\
&\quad + L_f \cdot \mathbb{E}_{\overline{u}_{t-1}}\left[\left\|\frac{\mathrm{d}u_{t-1}}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{u}_{t-1}}{\mathrm{d}\theta}\right\|_2\right]. \qquad \text{(A.20)}
\end{aligned}$$

Similarly, we have from (A.11) that

$$\begin{aligned}
&\mathbb{E}_{\overline{u}_n}\left[\left\|\frac{\mathrm{d}u_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{u}_t}{\mathrm{d}\theta}\right\|_2\right] \\
&\quad = \mathbb{E}\left[\left\|\frac{\partial u_t}{\partial x_t} \cdot \frac{\mathrm{d}x_t}{\mathrm{d}\theta} + \frac{\partial u_t}{\partial \theta} - \frac{\partial \overline{u}_t}{\partial \overline{x}_t} \cdot \frac{\mathrm{d}\overline{x}_t}{\mathrm{d}\theta} - \frac{\partial \overline{u}_t}{\partial \theta}\right\|_2\right] \\
&\quad \leq \mathbb{E}\left[\left\|\frac{\partial u_t}{\partial x_t} \cdot \frac{\mathrm{d}x_t}{\mathrm{d}\theta} - \frac{\partial \overline{u}_t}{\partial \overline{x}_t} \cdot \frac{\mathrm{d}x_t}{\mathrm{d}\theta}\right\|_2\right] + \mathbb{E}\left[\left\|\frac{\partial \overline{u}_t}{\partial \overline{x}_t} \cdot \frac{\mathrm{d}x_t}{\mathrm{d}\theta} - \frac{\partial \overline{u}_t}{\partial \overline{x}_t} \cdot \frac{\mathrm{d}\overline{x}_t}{\mathrm{d}\theta}\right\|_2\right] + \mathbb{E}\left[\left\|\frac{\partial u_t}{\partial \theta} - \frac{\partial \overline{u}_t}{\partial \theta}\right\|_2\right] \\
&\quad \leq 2L_\pi \cdot \mathbb{E}\left[\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta}\right\|\right] + L_\pi \cdot \mathbb{E}\left[\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_t}{\mathrm{d}\theta}\right\|_2\right] + 2L_\theta. \qquad \text{(A.21)}
\end{aligned}$$

Plugging (A.21) back to (A.20),

$$\begin{aligned}
&\mathbb{E}_{\overline{x}_t}\left[\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_t}{\mathrm{d}\theta}\right\|_2\right] \\
&\quad \lesssim 4L_f\widetilde{L}_\pi \cdot \left(\left\|\frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta}\right\|_2 + \left\|\frac{\mathrm{d}u_{t-1}}{\mathrm{d}\theta}\right\|_2\right) + L_f\widetilde{L}_\pi \cdot \mathbb{E}_{\overline{x}_{t-1}}\left[\left\|\frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_{t-1}}{\mathrm{d}\theta}\right\|_2\right] + 2L_fL_\theta \\
&\quad \leq 4L_f\widetilde{L}_\pi \cdot K(t-1) + L_f\widetilde{L}_\pi \cdot \mathbb{E}_{\overline{x}_{t-1}}\left[\left\|\frac{\mathrm{d}x_{t-1}}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_{t-1}}{\mathrm{d}\theta}\right\|_2\right] + 2L_fL_\theta,
\end{aligned}$$

where the last inequality follows from the definition of K in (A.17).

Applying this recursion gives us

$$\begin{aligned}
\mathbb{E}_{\overline{x}_t}\left[\left\|\frac{\mathrm{d}x_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{x}_t}{\mathrm{d}\theta}\right\|_2\right] &= e\big(L_f\widetilde{L}_\pi\big)^t + \big(4L_f\widetilde{L}_\pi \cdot K(t-1) + 2\widetilde{L}_fL_\theta\big) \cdot \sum_{j=0}^{t-1}\big(\widetilde{L}_f\widetilde{L}_\pi\big)^j \\
&\leq \widetilde{L}_f^t\widetilde{L}_\pi^t\Big(e + 4t \cdot \widetilde{L}_f\widetilde{L}_\pi \cdot K(t-1) + 2t \cdot \widetilde{L}_fL_\theta\Big),
\end{aligned}$$

where the first equality follows from (A.15).

As a consequence, we have from (A.21) that

$$\mathbb{E}_{\overline{u}_t}\left[\left\|\frac{\mathrm{d}u_t}{\mathrm{d}\theta} - \frac{\mathrm{d}\overline{u}_t}{\mathrm{d}\theta}\right\|_2\right] \leq \widetilde{L}_f^t\widetilde{L}_\pi^{t+1}\Big(e + 4t \cdot \widetilde{L}_f\widetilde{L}_\pi \cdot K(t-1) + 2t \cdot \widetilde{L}_fL_\theta\Big) + 2L_\pi K(t) + 2L_\theta.$$

This concludes the proof. □

A.4. Proof of Proposition 3.5

In the following proof, we use the notation $\|z\|_2$ to represent the Euclidean l_2 norm for vector z, and $\|Z\|_2$ to represent the induced 2-norm for matrix Z, i.e. $\|Z\|_2 := \max_{\|x\|_2=1} \|Zx\|_2$. Recall that $\|Z\|_F$ denotes the Frobenius norm of matrix Z, i.e. $\|Z\|_F = \sqrt{\operatorname{tr}(ZZ^\top)}$.

To characterize the Lipschitz of the LCS model, we need the partial derivatives of x_{t+1} with respect to x_t and u_t, which, however, further depend on the partial derivatives of λ_t with respect to x_t and u_t and cannot be expressed in closed form. Instead, they are implicitly defined by the LCP. Therefore, we introduce the following implicit function theorem.

Theorem A.3 (Implicit Function Theorem)**.** An implicit function $g : \mathbb{R}^{d_s} \times \mathbb{R}^{d_w} \to \mathbb{R}^{d_s}$ is defined as $g(s, w) = 0$ for solution $s \in \mathbb{R}^{d_s}$ and problem data $w \in \mathbb{R}^{d_w}$. Then the Jacobian $\partial s/\partial w$, i.e. the sensitivity of the solution with respect to the problem data, is given by

$$\frac{\partial s}{\partial w} = -\Big(\frac{\partial g}{\partial s}\Big)^{-1}\frac{\partial g}{\partial w}.$$

Proof. Differentiating g with respect to the problem data w gives:

$$\frac{\mathrm{d}g}{\mathrm{d}w} = \frac{\partial g}{\partial w} + \frac{\partial g}{\partial s}\frac{\partial s}{\partial w}.$$

Since for any w, $g(s, w) = 0$ always holds, the above total derivative is also always 0. This observation allows us to calculate the Jacobian

$$\frac{\partial s}{\partial w} = -\Big(\frac{\partial g}{\partial s}\Big)^{-1}\frac{\partial g}{\partial w}.$$

□

Proof of Proposition 3.5. To begin with, we first study the Jacobian $\partial x_{t+1}/\partial x_t$, and the Jacobian $\partial x_{t+1}/\partial u_t$ can be analyzed using similar techniques.

Denote $C^{(i)} \in \mathbb{R}^{d_x}$ as the i-th column of the matrix $C \in \mathbb{R}^{d_x \times d_\lambda}$. Similarly, denote $D^{(i)} \in \mathbb{R}^{d_x}$, $E^{(i)} \in \mathbb{R}^{d_u}$, $F^{(i)} \in \mathbb{R}^{d_\lambda}$ as the i-the rows of matrices D, E, F, respectively. Then we have the Jacobian with the form

$$\frac{\partial x_{t+1}}{\partial x_t} = A + \sum_{i=1}^{d_\lambda} C^{(i)}\frac{\partial \lambda^{(i)}}{\partial x_t}. \tag{A.22}$$

We rewrite the contact equation $\lambda_t \circ (Dx_t + Eu_t + F\lambda_t + d) = \mu\vec{1}$ in (2.1) as

$$\lambda_t^{(i)}(D^{(i)\top}x_t + E^{(i)\top}u_t + F^{(i)\top}\lambda_t + d^{(i)}) = \mu, \;\; \forall i \in [1, d_\lambda]. \tag{A.23}$$

By the Implicit Function Theorem A.3, we have

$$\begin{aligned}\frac{\partial \lambda^{(i)}}{\partial x_t} &= -\Big(D^{(i)\top}x_t + E^{(i)\top}u_t + \frac{\partial}{\partial \lambda_t^{(i)}}\lambda_t^{(i)}F^{(i)\top}\lambda_t + d^{(i)}\Big)^{-1}\lambda_t^{(i)}D^{(i)\top}\\ &= -(D^{(i)\top}x_t + E^{(i)\top}u_t + F^{(i)\top}\lambda_t + \lambda_t^{(i)}F^{(i)(i)} + d^{(i)})^{-1}\lambda_t^{(i)}D^{(i)\top}, \;\; \forall i \in [1, d_\lambda],\end{aligned} \tag{A.24}$$

where $F^{(i)(i)} \in \mathbb{R}$ is the i-th element of $F^{(i)}$.

Since F is a P-matrix, we know that all its first order principal sub-matrices are positive, i.e., $F^{(i)(i)} > 0$.

Plugging (A.24) into (A.22) and take the induced 2-norm, we obtain

$$\begin{aligned}\left\|\frac{\partial x_{t+1}}{\partial x_t}\right\|_2 &= \left\|A - \sum_{i=1}^{d_\lambda} C^{(i)}\big(D^{(i)\top}x_t + E^{(i)\top}u_t + F^{(i)\top}\lambda_t + \lambda_t^{(i)}F^{(i)(i)} + d^{(i)}\big)^{-1}\lambda_t^{(i)}D^{(i)\top}\right\|_2 \\ &\le \|A\|_2 + \sum_{i=1}^{d_\lambda}\lambda_t^{(i)}\|C^{(i)}\|_2\cdot\|D^{(i)}\|_2\cdot\big|D^{(i)\top}x_t + E^{(i)\top}u_t + F^{(i)\top}\lambda_t + \lambda_t^{(i)}F^{(i)(i)} + d^{(i)}\big|^{-1} \\ &\le \|A\|_2 + \sum_{i=1}^{d_\lambda}\|C^{(i)}\|_2\cdot\|D^{(i)}\|_2\cdot(\lambda_t^{(i)})^2/\mu, \end{aligned} \tag{A.25}$$

where the first inequality holds due to the Cauchy–Schwarz inequality, the second inequality holds since $F^{(i)(i)} > 0$ and $D^{(i)\top}x_t + E^{(i)\top}u_t + F^{(i)\top}\lambda_t + d^{(i)} \ge 0$.

By the definition of Frobenius norm, we know that

$$\begin{aligned}\|C\|_F = \sqrt{\sum_{i=1}^{d_\lambda}\|C^{(i)}\|_2^2} &= \sqrt{d_\lambda}\cdot\sqrt{\sum_{i=1}^{d_\lambda}\frac{1}{d_\lambda}\|C^{(i)}\|_2^2} \\ &\ge \sqrt{d_\lambda}\cdot\sum_{i=1}^{d_\lambda}\frac{1}{d_\lambda}\sqrt{\|C^{(i)}\|_2^2} = \frac{1}{\sqrt{d_\lambda}}\sum_{i=1}^{d_\lambda}\|C^{(i)}\|_2, \end{aligned} \tag{A.26}$$

where we adopt the Jensen's inequality in the second line.

Besides, define the diagonal matrix $\Lambda_t := \operatorname{diag}(\lambda_t^{(1)}, \cdots, \lambda_t^{(d_\lambda)}) \in \mathbb{R}^{d_\lambda \times d_\lambda}$. By definition, $\|\Lambda_t\|_2 = \max_i \lambda^{(i)}$ and thus

$$\|\lambda_t\|_2^2 = \sum_{i=1}^{d_\lambda}(\lambda_t^{(i)})^2 \le d_\lambda\cdot\|\Lambda_t\|_F^2. \tag{A.27}$$

Therefore, we can further bound (A.25) by

$$\begin{aligned}\left\|\frac{\partial x_{t+1}}{\partial x_t}\right\|_2 &\le \|A\|_2 + \frac{1}{\mu}\left(\sum_{i=1}^{d_\lambda}\|C^{(i)}\|_2\right)\cdot\left(\sum_{i=1}^{d_\lambda}\|D^{(i)}\|_2\right)\cdot\left(\sum_{i=1}^{d_\lambda}(\lambda_t^{(i)})^2\right) \\ &\le \|A\|_2 + \frac{d_\lambda}{\mu}\|C\|_F\|D\|_F\|\lambda_t\|_2^2 \\ &\le \|A\|_F + \frac{d_\lambda^2}{\mu}\|C\|_F\|D\|_F\|\Lambda_t\|_F^2, \end{aligned} \tag{A.28}$$

where the first inequality holds since $\sum_i y_i \cdot z_i \le (\sum_i y_i)\cdot(\sum_i z_i)$ for any non-negative scalar sequences y_i, z_i and the second inequality follows from (A.26). The third inequality follows from (A.27) and the fact that $\|A\|_2 \le \|A\|_F$.

The final step is to characterize the magnitude of $\|\Lambda_t\|_F^2$. This can be done by rewriting the contact equation $\lambda_t \circ (Dx_t + Eu_t + F\lambda_t + d) = \mu\vec{1}$ in (2.1) as

$$\Lambda_t(Dx_t + Eu_t + F\Lambda_t\vec{1} + d) = \mu\vec{1}$$

By the Cauchy-Schwartz inequality we have

$$\|\Lambda_t\|_F\cdot\big(\|Dx_t + Eu_t + d\|_2 + \|F\|_F\|\Lambda_t\|_F\big) \ge \mu.$$

Denote $e := \sup\|Dx_t + Eu_t + d\|_2$. The above inequality can be simplified as

$$\|F\|_F\cdot\|\Lambda_t\|_F^2 + e\cdot\|\Lambda_t\|_F - \mu \ge 0. \tag{A.29}$$

Solving (A.29) gives

$$\|\Lambda_t\|_F \geq \frac{\sqrt{e^2 + 4\mu\|F\|_F} - e}{2\|F\|_F}$$

Since $\varepsilon = e^2/(2\|F\|_F^2)$, we further have

$$\begin{aligned} l(\mu) :=& \frac{\|\Lambda_t\|_F^2}{\mu} \geq \frac{2e^2 + 4\mu\|F\|_F - 2e\sqrt{e^2 + 4\mu\|F\|_F}}{4\mu\|F\|_F^2} \\ =& \frac{e^2}{2\mu\|F\|_F^2} + \frac{1}{\|F\|_F} + \frac{e^2\sqrt{\frac{1}{\mu^2} + \frac{4\|F\|_F}{\mu e^2}}}{2\|F\|_F^2} \\ =& \frac{\varepsilon}{\mu} + \frac{1}{\|F\|_F} + \varepsilon\sqrt{\frac{1}{\mu^2} + \frac{2}{\varepsilon\mu\|F\|_F}}. \end{aligned} \tag{A.30}$$

Plug (A.30) into (A.28), we get the Jacobian norm

$$\left\|\frac{\partial x_{t+1}}{\partial x_t}\right\|_2 \leq \|A\|_F + d_\lambda^2\|C\|_F\|D\|_F \cdot l(\mu).$$

Using the same proof steps, the norm of Jacobian $\partial x_{t+1}/\partial u_t$ satisfies

$$\left\|\frac{\partial x_{t+1}}{\partial u_t}\right\|_2 \leq \|B\|_F + d_\lambda^2\|C\|_F\|E\|_F \cdot l(\mu).$$

We conclude the proof by noticing the relationship between the norm of Jacobian and the Lipschitz of the LCS model. □

A.5. Proof of Proposition 4.1

Proof. We first consider the original unsmoothed system $\lambda_t(Dx_t + Eu_t + F\lambda_t + d) = 0$. Since $\lambda_t \geq 0$, we know that the solution λ_t is a piece-wise linear function with the form:

$$\lambda_t = \begin{cases} -(Dx_t + Eu_t + d)/F & \text{if } Dx_t + Eu_t + d < 0 \\ 0 & \text{else} \end{cases}.$$

By rewriting the above function as a function of $z_t := Dx_t + Eu_t + d$, we can express the solver $S_{\mu=0}$ of the unsmoothed LCP as follows:

$$S_{\mu=0}(z_t) = \begin{cases} -z_t/F & \text{if } z_t < 0 \\ 0 & \text{else} \end{cases}. \tag{A.31}$$

Now our goal is to find the noise distribution $\rho(w)$ such that the following holds:

$$S_{\mu(z_t)}(z_t) = \mathbb{E}_{w\sim\rho(w)}[S_{\mu=0}(z_t + w)] = \int S_{\mu=0}(z_t + w)\rho(w)\mathrm{d}w. \tag{A.32}$$

Define H(x) as a Heaviside-like step function:

$$H(x) := \begin{cases} -1/F & \text{if } x < 0 \\ 0 & \text{else} \end{cases}.$$

We observe that the derivative of $S_{\mu=0}(z_t)$ is in fact $H(z_t)$. This allows us to write

$$\begin{aligned}\nabla_{z_t} S_{\mu(z_t)}(z_t) &= \nabla_{z_t} \int S_{\mu=0}(z_t+w)\rho(w)\mathrm{d}w \\ &= \int \nabla_{z_t} S_{\mu=0}(z_t+w)\rho(w)\mathrm{d}w \\ &= \int H(z_t+w)\rho(w)\mathrm{d}w.\end{aligned}$$

Since the derivative of the Heaviside step function is the dirac delta function $\delta(\cdot)$, we have

$$\begin{aligned}\nabla^2_{z_t} S_{\mu(z_t)}(z_t) &= \nabla_{z_t} \int H(z_t+w)\rho(w)\mathrm{d}w \\ &= \int \delta(z_t+w)\rho(w)\mathrm{d}w = \rho(z_t).\end{aligned}$$

This concludes the proof. □

A.6. Proof of Proposition 4.2

Recall that Proposition 4.1 connects the proposed analytic barrier smoothing with the randomized smoothing. Therefore, we first provide the following lemma established in randomized smoothing as a preparation before proving Proposition 4.2.

Lemma A.4 (Randomized Smoothing as Linearization Minimizer (Pang et al., 2022))**.** Let $\rho(w) = \mathcal{N}(0, \Sigma)$ be a zero-mean, Σ-covariance Gaussian. Consider the problem of regressing a function g with parameters (K, W) such that the residual around $\overline{x}$ distributed according to ρ is minimized:

$$\mathcal{L}(K, W) = \min_{K,W} \frac{1}{2}\mathbb{E}_{w\sim\rho(w)}\left[\left\|g(\overline{x}+w) - Ww - K\right\|_2^2\right]. \tag{A.33}$$

The solution is the linearization of the smoothed surrogate:

$$\begin{aligned}K^* &= \mathbb{E}_{w\sim\rho(w)}[g(\overline{x}+w)], \\ W^* &= \frac{\partial}{\partial x}\mathbb{E}_{w\sim\rho(w)}[g(x+w)]|_{x=\overline{x}}.\end{aligned}$$

Proof. The proof is originally provided in (Pang et al., 2022). We adapt it here for completeness.

Since (A.33) is a linear regression problem and is convex, the first-order stationarity condition implies optimality. By calculating the gradients and setting them to zero, we have

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial K} &= \mathbb{E}_{w\sim\rho(w)}[g(\overline{x}+w)] - K^* = 0 \\ \frac{\partial \mathcal{L}}{\partial W} &= \mathbb{E}_{w\sim\rho(w)}[ww^\top]W^* - \mathbb{E}_{w\sim\rho(w)}[g(\overline{x}+w)w^\top] = 0.\end{aligned}$$

Therefore, we obtain the solution

$$\begin{aligned}K^* &= \mathbb{E}_{w\sim\rho(w)}[g(\overline{x}+w)], \\ W^* &= \mathbb{E}_{w\sim\rho(w)}[ww^\top]^{-1}\mathbb{E}_{w\sim\rho(w)}[g(\overline{x}+w)w^\top] \\ &= \frac{\partial}{\partial x}\mathbb{E}_{w\sim\rho(w)}[g(x+w)]|_{x=\overline{x}},\end{aligned} \tag{A.34}$$

where the last step follows from the likelihood ratio gradient with the form (1.1), as well as the fact that the score function of the Gaussian is $\Sigma^{-1}w$. □

Proof of Proposition 4.2. By applying Lemma A.4, we know that Proposition 4.2 holds once the following equivalence is established:

$$S_{\mu(z_t)}(z_t) = \mathbb{E}_{w\sim\rho(w)}[S_{\mu=0}(z_t+w)], \tag{A.35}$$

where $\rho(w)$ is any zero-mean Gaussian distribution.

This is a direct result from Proposition 4.1. Specifically, when $\mu(z_t) = \kappa\cdot(z_t+F\kappa)$, the corresponding softened LCP is

$$\lambda_t(z_t+F\lambda_t) = \mu(z_t) = \kappa\cdot(z_t+F\kappa).$$

The solution of the above equation is given by

$$S_{\mu(z_t)}(z_t) = \lambda_t = \kappa = z_t\cdot \text{erf}(z_t,\sigma) + e^{-z_t^2/(2\sigma)}/\sqrt{\pi} + c_1 z_t + c_2. \tag{A.36}$$

Proposition 4.1 states that when $\rho(w) = \nabla_w^2 S_{\mu(z_t)}(w)$, then $S_{\mu(z_t)}(z_t) = \mathbb{E}_{w\sim\rho(w)}[S_{\mu=0}(z_t+w)]$. For $S_{\mu(z_t)}(z_t)$ satisfying (A.36), its second-order derivative is the Gaussian $\mathcal{N}(0,\sigma)$, due to the definition of the error function. Therefore, $S_{\mu(z_t)}(z_t) = \mathbb{E}_{w\sim\mathcal{N}(w;0,\sigma)}[S_{\mu=0}(z_t+w)]$, which concludes the proof of (A.35) and the proposition. □

A.7. Proof of Proposition 4.3

Proof. According to Taylor's theorem, we know that

$$\left|\frac{S_{\mu=0}(z_r+w)-S_{\mu=0}(z_t)}{w} - \nabla_z S_{\mu=0}(z_t)\right| \le |w|\cdot \sup\frac{|\nabla_z^2 S_{\mu=0}(z_t)|}{2} = \frac{F^2|w|}{2}, \tag{A.37}$$

where the second inequality follows from (A.31).

We define the linearization residual at point z_t+w as

$$\nu(w) := \left|S_{\mu=0}(z_r+w) - \nabla_z S_{\mu(z_t)}(z_t)\cdot w - S_{\mu(z_t)}(z_t)\right|.$$

Then we have from (A.37) that

$$\left|\frac{\nu(w)+S_{\mu(z_t)}(z_t)-S_{\mu=0}(z_t)}{w} + \nabla_z S_{\mu(z_t)}(z_t) - \nabla_z S_{\mu=0}(z_t)\right| \le \frac{F^2|w|}{2}.$$

Since $|S_{\mu(z_t)}(z_t) - S_{\mu=0}(z_t)| \le 1/\sqrt{\pi}+c_2 := \varsigma$, achieved at $z=0$, we obtain from the triangle inequality that the bias of gradient satisfies

$$\left|\nabla_z S_{\mu(z_t)}(z_t) - \nabla_z S_{\mu=0}(z_t)\right| \le \frac{F^2|w|}{2} + \frac{\nu(w)+\varsigma}{|w|}. \tag{A.38}$$

From Proposition 4.2, we know that

$$\mathbb{E}_{w\sim\mathcal{N}(0,\sigma)}[\nu(w)] = \delta. \tag{A.39}$$

We claim that there exists $\sigma\mathcal{Q}(2/3) \le w \le \sigma\mathcal{Q}(3/4)$ such that $\nu(w)\le 12\delta$.

This can be proved by contradiction: Suppose $\forall w \in [\sigma\mathcal{Q}(2/3), \sigma\mathcal{Q}(3/4)]$, $\nu(w) > 12\delta$. Then the expectation $\mathbb{E}_{w\sim\mathcal{N}(0,\sigma)}[\nu(w)] > (3/4-2/3)\cdot 12\delta = \delta$. This contradicts with (A.39). Therefore, this claim is correct.

Using the above claim, we have from (A.38) that

$$\left|\nabla_z S_{\mu(z_t)}(z_t) - \nabla_z S_{\mu=0}(z_t)\right| \le \frac{F^2\sigma\mathcal{Q}(3/4)}{2} + \frac{12\delta+\varsigma}{\sigma\mathcal{Q}(2/3)}.$$

We conclude the proof by applying chain rule in the LCS model (2.1):

$$\begin{aligned}
\left\|\nabla_x f_{\mu=0} - \nabla_x f_{\mu(z_t)}\right\|_2 &\leq \|C\|_F \|D\|_F \cdot \Big(\frac{\sigma F^2 \mathcal{Q}(3/4)}{2} + \frac{12\delta + \varsigma}{\sigma \mathcal{Q}(2/3)}\Big), \\
\left\|\nabla_u f_{\mu=0} - \nabla_u f_{\mu(z_t)}\right\|_2 &\leq \|C\|_F \|E\|_F \cdot \Big(\frac{\sigma F^2 \mathcal{Q}(3/4)}{2} + \frac{12\delta + \varsigma}{\sigma \mathcal{Q}(2/3)}\Big).
\end{aligned}$$

□