Probability on Graphs

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Contents

1 Basics of graph theory

Definition 1 (Graph).
A *graph* $G = (V, E)$ is a consists of

1) $V \neq \emptyset$;

2) $E \subseteq \{\{u, v\} \mid u, v \in V\}$.

We call elements of V *vertices* or *nodes* and elements of E *edges* or *bonds*. We write $v \sim u$ if $\{u, v\} \in E$. A graph $G = (V, E)$ is called *network* if for $e = \{u, v\} \in V$, we assign a weight $c(u, v) > 0$. For $v \in V$, we set $c(v) = \sum_{u \sim v} c(u, v)$. We further define

- A vertex v is called *incident* to an edge e if $v \in e$;
- the *degree* of a vertex v is the number of edges incident to it: $\deg v = |\{\{u, v\} \in E \mid u \sim v\}|$;
- the graph G is called *locally finite* if for all $v \in V$, $\deg v < \infty$;
- for $u, v \in V$, $\pi = (x_1, \dots, x_N)$ is called a *path* from u to v if $v = x_1$, $x_N = v$ and $\{x_{n-1}, x_n\} \in E$;
- a path from u to u is called a *cycle*;
- a graph G is called *connected* if for any $u, v \in V$ there is a finite path from u to v.

ede

Definition 2 (Product of graphs).
Let G_1, G_2 be graphs. We define

$$\begin{aligned} G_1 \boxtimes G_2 :=&(V_1 \times V_2, \\ &\{((x_1, y_1), (x_2, y_2)) \mid \text{either } (x_1 = x_2, (y_1, y_2) \in E_2) \text{ or } (y_1 = y_2, (x_1, x_2) \in E_1)\}) \end{aligned}$$

as the *product* of G_1 and G_2.

Definition 3 (Vertex simple and edge simple paths).
Let G be a graph. A *path* is a (possibly infinite) sequence of vertices $v_1, v_2, \dots$ such that $(v_n, v_{n+1}) \in E$.

1) If a path does not visit any vertex more that once, it is called a *vertex simple path.*

2) If a path does not visit any edge more than once, it is called an *edge simple path.*

Definition 4 (Network).
A *network* is a graph G combined with a function $c : E \to [0, \infty)$ with if $c(u, v) = 0$ if $(v, u) \notin E$. We call a network connected if $(V, \{e \in E \mid c(e) > 0\})$ is connected. A network is locally finite if for any $v \in V$, we have $c(v) = \sum_{u \sim v} c(u, v) < \infty$.

Lemma 1 (Example).
$V = \mathbb{Z}$, $E = \{(v, u) \mid u \neq v \in V\}$ with $c(v, u) = \frac{1}{|u-v|^2}$.

Definition 5 (Cycle).
A cycle is a path starting and ending in the same point. A cycle is simple if the only vertex appearing more than once is the first (and last) vertex.

Definition 6 (Induced graphs and networks).
Let $G = (V, E)$ be a graph and let $K \subseteq V$. The induced graph is the graph $G_K = (K, \{e \in E \mid e \subseteq K\})$. The induced network is defined similarly.

Definition 7 (Connected component).
Let G be a graph a graph or a network. A connected component is a set K of vertices such that G_K is connected, but for every $K \subsetneq H$, G_H is not connected.

Definition 8 (Contraction of graphs).
Let $G = (V, E, c)$ be a graph (or network) and let $K \subseteq G$. The connected graph (or network) G/K is the graph $(V \setminus \cup \{z\}, \{e \in E \mid e \subseteq V \setminus K\} \cup \{(v, z) \mid v \notin K, \exists u \in K : (v, u) \in E\})$. For all $u, v \notin K$, take conductance $c'(v, u) = c(v, u)$ for $v \notin K$ take $c(v, u) = \sum_{u \in K} c(v, u)$.

Definition 9 (Forest and tree)**.**
A forest is a graph with no (simple) cycles. A tree is a connected forest.

Remark.
A contraction of an edge lets it "disappear" in a sense that the vertices are merged.

Definition 10 (Homomorphisms)**.**
Let G_1, G_2 be graphs. Let $\phi : V_1 \to V_2$ be a function.

1. ϕ is called a *homomorphism* if for all $(u, v) \in E_1$, we have that $(\phi(u), \phi(v)) \in E_2$;
2. ϕ is called an *isomorphism* if ϕ is bijective, and ϕ, ϕ^{-1} are homomorphisms. In this case, G_1, G_2 are called *isomorpic*;
3. ϕ is called an *automorphism* if ϕ is an isomorphism with $G_1 = G_2$.

Definition 11 (Transitivity)**.**
A graph $G = (V, E)$ is called (vertex) transitive if for all $v, u \in V$ there is an automorhpism ϕ such that $\phi(v) = u$. G is called edge transitive if for every $e_1, e_2 \in E$ there is an automorphism $\phi(e_1) = e_2$.

Remark.
There are graphs that are vertex transitive but not edge transitive.

Remark.
Good source of example for transitive graphs: Let G be a graph, and let $Y \subseteq G \setminus \{\text{unit}\}$. Take the graph $V = G$ and $(u, v) \in E$ if there is $y \in Y$ such that $u = yv$ or $v = yu$.

1.1 Random walks on networks

Definition 12 (Simple rnadom walk on networks)**.**
Let $G = (V, E, c)$ be a network and assume that G is locally finite. The simple random walk on G is the (discrete time) Markov chain with transition matrix

$$\Pi(x, y) = \frac{c(x, y)}{\sum_z c(x, z)} \cdot \mathbb{1}\left[\sum_z c(x, z) > 0\right] + 1 \cdot \mathbb{1}\left[\sum_z c(x, z) = 0\right]$$

A random walk on a locally finite graph $G = (V, E)$ is a random walk on the network $(V, E, 1)$.

Remark.
Question: Does $\mathbb{Z}^d$ have a transient subgraph?
Answer: No, every subgraph of a recurrent graph is recurrent.
The main tool to provide the proof for the answer is based on the connection between the random walk and electrical networks.

Lemma 2.
Let $G = (V, E, c)$ be a finite connected network. For every $x \in V$, let p^x be the distribution of the random walk on G started on x. Let $a, z \in V$. Let τ_a (τ_z) be the first time hitting a (z), and let $\tau_{a,z}$ be first time hitting either of them. Then, $\tau_a, \tau_z, \tau_{a,z} < \infty$.

Fact.
Let $G = (V, E, c)$ be a finite connected network. We consider the simple random walk defined as above. Then, every state x is recurrent. We have $\mathbb{P}_x(\tau_a < \infty) = \mathbb{P}_x(\tau_z < \infty) = 1$.

Proof. For every x, let γ_x be a path from x to a. Let h_x be the probability that the random walk takes the path γ_x immediately starting in x. Let $h = \min_{x \in V} h_x > 0$. Let $\mathcal{F}_n) = \sigma(X_1, \ldots, X_N)$, $\mathcal{F} = \sigma(X_1, X_2, \ldots)$. Then, $\{\tau_a < \infty\} \in \mathcal{F}$. By Kolmogorov's 0-1-law, $\mathbb{P}_x(\tau_a < \infty \,|\, \mathcal{F}) \in \{0, 1\}$. We have $\mathbb{P}_x(\tau_a < \infty \,|\, \mathcal{F}) = \lim_{n\to\infty} \mathbb{P}_x(\tau_a < \infty \,|\, \mathcal{F}_n)$. We have $\mathbb{P}_x(\tau_a < \infty \,|\, \mathcal{F}_n) \geq h$. Thus, $\mathbb{P}_x(\tau_a < \infty \,|\, \mathcal{F}) \geq h$. It follows from the 0-1-law, $\mathbb{P}_x(\tau_a < \infty \,|\, \mathcal{F}) = 1$ and thus $\mathbb{P}_x(\tau_a < \infty) = \mathbb{E}\left[\mathbb{P}_x(\tau_a < \infty \,|\, \mathcal{F})\right]$. □

Definition 13 (Voltage function)**.**
The voltage function is defined as

$$\mathcal{U} : V \to \mathbb{R}, x \mapsto \mathcal{U}(x) := \mathbb{P}_x(\tau_{a,z} = \tau_a)$$

Lemma 3 (Properties of the voltage function)**.**
Let $\mathcal{U} : V \to \mathbb{R}, x \mapsto \mathbb{P}_x(\tau_{a,z} = \tau_a)$ be the voltage function. Then, the following statements hold:

1. $\mathcal{U}(a) = 1$;
2. $\mathcal{U}(z) = 0$;
3. For $x \neq a, z$, we have

$$\mathcal{U}(x) = \sum_{y \in V} \mathbb{P}_x(X_1 = y)\mathcal{U}(y) = \sum_{y \in V} \frac{c(x,y)}{\sum_{z \in V} c(x,z)} \mathcal{U}(y) = \sum_{y \in V} c(x,y)\frac{\mathcal{U}(y)}{c(x)}$$

   (harmonicity)

Proof. To show the harmonicity, we use the Markov property. Since $x \neq a, z$, $p^x(\tau_a \geq 1) = 1$. The Markov property implies

$$\begin{aligned} p^x(\tau_a \leq \tau_z) &= \mathbb{E}_x\left[p^x(\tau_a < \tau_z)\right] \\ &= \sum_y p^x(X_1 = y) \cdot p^y(\tau_a < \tau_z) \\ &= \sum_y p^x(X_1 = y) \cdot \mathcal{U}(y) \end{aligned}$$

□

Lemma 4 (Uniqueness of voltage function)**.**
The voltage function $\mathcal{U}$ is the only function satisfying the properties above.

Proof. Assume $\mathcal{U}_1$,$\mathcal{U}_2$ satisfy the properties of the voltage function. We show $h \equiv \mathcal{U}_1 - \mathcal{U}_2 = 0$.

1. By properties 1 and 2, $h(a) = h(z) = 0$.
2. For $x \neq a, z$, we have $h(x) = \sum_y \frac{c(x,y)}{c(x)} h(y)$ ($h(x)$ is a weighted average over its neighbors). Assume $h(x) \neq 0$. We assume Œ$h(x) > 0$. Let $M = \max\{h(x) \,|\, x \in V\} > 0$. Let $H = \{x \,|\, h(x) = M\}$. Note that $a, z \notin H$. Let x be an element of H with a neighbor w outside of H. Then, the following are true
   (a) $x \in H$, thus $h(x) = M$.
   (b)

$$\begin{aligned} h(x) &= \sum_y \frac{c(x,y)}{c(x)} h(y) \\ &= \frac{c(x,w)}{c(x)} h(w) + \sum_{y \neq w} \frac{c(x,y)}{c(x)} h(y) \\ &\leq h(w)\frac{c(x,w)}{c(x)} + M \sum_{y \neq w} \frac{c(x,y)}{c(x)} \\ & h(w)\frac{c(x,w)}{c(x)} + M\left(1 - \frac{c(x,w)}{c(x)}\right) \\ &= M + \underbrace{(h(w) - M)}_{<0} \cdot \underbrace{\frac{c(x,w)}{c(x)}}_{>0} < M \end{aligned}$$

   This is a contradiction to $h \neq 0$.

□

Remark.
The proof is similar to the proof of uniqueness of the solution to the Dirichlet problem.

Remark.
In the theory of electrical networks, the (physical) voltage function satisfies the same three properties (up to normalization of the boundary conditions). Therefore, everything that is known about physical electrical networks has the potential of being useful in the understanding of random walks on networks.

Definition 14 (Resistance)**.**
For an edge e, we define its resistance as

$$r(e) = \frac{1}{e}$$

1.2 Currents and flows

Definition 15 (Current)**.**
Let $e = (x, y) \in E$. For every x, y we define the current as

$$I(x, y) = \frac{\mathcal{U}(x) - \mathcal{U}(y)}{r(x, y} = (\mathcal{U}(x) - \mathcal{U}(y))\, c(x, y)$$

Lemma 5 (Properties of the current)**.**
Let I be the current. Then, the following statements hold:

1. $I(x, y) = -I(y, x)$ (Antisymmetry)
2. For $x \neq a, z$, $\sum_{y \neq x} I(x, y) = 0$ (Kirchhof's node law)
3. For a cycle $x_0, \ldots, x_N = x_0$, then $\sum_{n=1}^{N} I(x_{n-1}, x_n) r(x_{n-1}, x_n) = 0$ (Kirchhof's circuit law)

Proof.

1. follows from $c(x, y) = c(y, x)$.
2. Notice that

$$\begin{aligned}
\sum_{y \neq x} I(x, y) &= \sum_{y \neq x} c(x, y)\,(v(x) - v(y)) \\
&= v(x) \sum_{y \neq x} c(x, y) - \sum_{y \neq x} c(x, y) v(y) \\
&= v(x) c(x) - c(x) \sum_{y \neq x} \frac{c(x, y)}{c(x)} v(y) \\
&= 0
\end{aligned}$$

   where we used the harmonity of v.

3. By definition of I, we have $(I \cdot r)(x, y) = v(x) - v(y)$. Thus, we have

$$\sum_{n=1}^{N} I(x_{n-1,x_n)r(x_{n-1},x_n)} = \sum_{n=1}^{N} v(x_{n-1}) - v(x_n) = 0$$

∎

Definition 16 (Flow)**.**
A flow is a function $F : V^2 \to \mathbb{R}$, such that

1. $F(x, y) = -F(y, x)$
2. If $x \nsim y$, then $F(x, y) = 0$
3. For all $x \neq a, z$, we have $\sum_{y \neq x} F(x, y) = 0$ (Kirchhof's node law)

Definition 17 (Total current)**.**
The total current in a network is defined as

$$\sum_{x \neq a} I(a,x) = \sum_{x \neq z} I(x,z)$$

Proof. We prove equality for the two descriptions of total current.

$$\begin{aligned}
\sum_{x \neq a} I(a,x) - \sum_{x \neq z} I(x,z) &= \sum_{x \neq a} I(a,x) + \sum_{x \neq z} I(z,x) + \underbrace{\sum_{y \neq a,z} \sum_{x \neq y} I(y,x)}_{=0} \\
&\overset{\text{(I)}}{=} \sum_{y} \sum_{x \neq y} I(y,x) \\
&= \sum_{\{x,y\} \subseteq V} I(x,y) + I(y,x) \\
&= 0
\end{aligned}$$

$\square$

Definition 18 (Effective resistance)**.**
The effective resistance in a network is defined as

$$R_{\text{eff}}(a,z) = \frac{1}{\text{total current}}$$

Similarly, we denote

$$c_{\text{eff}}(a,z) = \text{total current}$$

Notation (Stopping times)**.**
We use the following notation for stopping times

$$\begin{aligned}
\tau_x &= \inf\{t \geq 0 \,|\, X_t = x\} \\
\sigma_x &= \inf\{t \geq 0 \,|\, X_t \neq x\} \\
\eta_x^0 &= 0 \\
\eta_x &= \inf\{t > 0 \,|\, X_t = x\} \\
\eta_x^{n+1} &= \inf\{t > \eta_x^n \,|\, X_t = x\}
\end{aligned}$$

Notice that $\tau_x \neq \eta_x$ under $\mathbb{P}_x$ and $\tau_x = \eta_x$ under $\mathbb{P}_y$ for $y \neq x$.

Lemma 6 (n-th return time of Markov chains)**.**
Notice that if X is a Markov chain, then we have

$$\eta_x^{n+1} = (\tau_x \circ \theta_{\sigma_x} + \sigma_x) \circ \theta_{\eta_x^n} + \eta_x^n$$

Theorem 1.
Let $G = (V, E, c)$ be a connected network, $a \neq z \in V$. Let $\tau_x := \inf\{n \geq 0 \,|\, X_n = x\}$ and $\eta_x = \inf\{n \geq 1 \,|\, X_n = 1\}$. Then,

$$\mathbb{P}_a\left(\tau_z < \eta_a\right) = \frac{c_{\text{eff}}(a,z)}{c(a)}$$

Proof. We have that

$$\begin{aligned}
\mathbb{P}_a(\tau_z < \eta_a) &= \sum_x \mathbb{P}_a(X_1 = x)\mathbb{P}_x(\tau_z < \eta_a) \\
&= \sum_x \frac{c(a,x)}{c(a)} \cdot (\underbrace{1}_{v(a)} - \underbrace{\mathbb{P}_x(\tau_a - \tau_z)}_{v(x)}) \\
&= \frac{1}{c(a)} \sum_x c(a,x)\,(v(a) - v(x)) \\
&= \frac{1}{c(a)} \sum_x I(a,x) \\
&= \frac{c_{\text{eff}}(a,z)}{c(a)}
\end{aligned}$$

□

Theorem 2 (Commute time formula)**.**
We have that

$$\mathbb{E}_a[\tau_z] + \mathbb{E}_z[\tau_a] = 2R_{\text{eff}}(a,z) + \sum_{e \in E} c(e)$$

Proof. We first provide facts about Markov chains.

1. Every Markov chain on a finite state space has an invariant distribution. If the Markov chain is a random walk on a finite connected network, then

$$\mu(x) = \frac{c(x)}{2\sum_{e \in E} c(e)}$$

   is the unique invariant distribution, since

   i) μ is a distribution: $\mu(x) \geq 0$ holds since $c \geq 0$, and

$$\begin{aligned}
\sum_x \mu(x) &= \frac{1}{2\sum_{e \in E} c(e)} \sum_x \left(\sum_y c(x,y)\right) \\
&= \frac{1}{2\sum_{e \in E} c(e)} \cdot 2\sum_{e \in E} c(e) \\
&= 1
\end{aligned}$$

   ii) μ is invariant: Let $X =_d \mu$, then for every x

$$\begin{aligned}
\mathbb{P}_\mu &= \sum_y \mathbb{P}_\mu(X_0 = y)\mathbb{P}_\mu(X_1 \mid X_0 = y) \\
&= \sum_y \mu(y) \cdot \frac{c(x,y)}{c(y)} \\
&= \frac{1}{2\sum_e c(e)} \sum_y c(y)\frac{c(x,y)}{c(y)} \\
&= \frac{1}{2\sum_e c(e)} c(x) \\
&= \mu(x)
\end{aligned}$$

2. $\mathbb{E}_x[\eta_x] = \frac{1}{\mu(x)}$. We sketch the proof of this claim: Let μ be stationary and invariant distribution, i.e. $X_n =_d \mu$ for all $n \geq 0$. Then, for large N, $\mu(x) \cdot N$ is approximately the number of times we are at x. Thus, for the average distance between appearances of x, we have $\mathbb{E}[\eta_x^{n+1} - \eta_x^n] = \frac{1}{\mu(x)}$. This can be shown using the Markov property to derive $\eta_x^1 =_d \eta_x^{n+1} - \eta_x^n$ and using the Law of Large Numbers.

3. Let $\eta_a^1, \eta_a^2, \dots$ be the consecutive return times to a. Let $m := \inf\left\{k \geq 0 \,\middle|\, \eta_a^{k-1} < \tau_z < \eta_a^k\right\}$. Then

$$\begin{aligned}
\mathbb{E}_a[\tau_z] + \mathbb{E}_z[\tau_a] &= \mathbb{E}_a[\eta_a^m] \\
&= \mathbb{E}_a[\eta_a^{m-1}] + \mathbb{E}_a[\eta_a^m - \eta_a^{m-1}] \\
&= \mathbb{E}_a[m-1]\mathbb{E}\left[\eta_a^1 \,\middle|\, \eta_a^1 < \tau_z\right] + \mathbb{E}_a\left[\eta_a^1 \,\middle|\, \tau_z < \eta_a^n\right] \\
&= \mathbb{E}[\mathrm{Geom}(\mathbb{P}_a(\tau_z < \eta_A) - 1] \cdot \mathbb{E}_a\left[\eta_a^1 \,\middle|\, \eta_a^1 < \tau_z\right] + \frac{\mathbb{P}_a(\tau_z < \eta_a^1)}{\mathbb{P}_a(\tau_z < \eta_a^1)}\mathbb{E}_a\left[\eta_a^1 \,\middle|\, \tau_z < \eta_a^1\right] \\
&= \left(\frac{1}{\mathbb{P}_a(\tau_z < \eta_a^1)} - 1\right)\mathbb{E}_a\left[\eta_a^1 \,\middle|\, \eta_a^1 < \tau_z\right] + \frac{\mathbb{P}_a(\tau_z < \eta_a^1)}{\mathbb{P}_a(\tau_z < \eta_a^1)}\mathbb{E}\left[\eta_a^1 \,\middle|\, \tau_z < \eta_a^1\right] \\
&= \frac{1}{\mathbb{P}_a(\tau_z < \eta_a^1)}\left[\mathbb{P}_a(\eta_a^1 < \tau_z)\mathbb{E}_a\left[\eta_a^1 \,\middle|\, \eta_a^1 < \tau_z\right] + \mathbb{P}_a(\tau_z < \eta_a^1)\mathbb{E}_a\left[\eta_a^1 \,\middle|\, \tau_z < \eta_a\right]\right] \\
&= \frac{1}{\mathbb{P}_a(\tau_z < \eta_a^1}\mathbb{E}_a[\eta_a^1] \\
&= \frac{2\sum_e c(e)}{c(a)} \Big/ \frac{c_{\mathrm{eff}(a,z)}}{c(a)} \\
&= 2\frac{\sum_e c(e)}{c_{\mathrm{eff}}(a,z)} \\
&= 2R_{\mathrm{eff}(a,z)} \sum_e c(e)
\end{aligned}$$

□

Exercise.
We have that

$$\mathbb{P}_x(\tau_a < \tau_z) \leq \frac{c_{\mathrm{eff}}(x,a)}{c_{\mathrm{eff}}(x,z)}$$

To make use of these theorem, we need to be able to estimate (or calculate) effective resistance/conductance. In some examples we are capable of calculating effective resistances.

Fact.
We have the following:

1) in $d = 1$, $R_{\mathrm{eff}}(0,n) = n$
2) in $d = 2$, $R_{\mathrm{eff}}\left((0,0),(n,n)\right) = \mathcal{O}(\ln n)$
3) in $d = 3$, $R_{\mathrm{eff}}\left((0,0,0),(n,n,n)\right) = \mathcal{O}(n)$

Thus, we have

$$\begin{aligned}
\mathbb{E}_{a,0}[\tau_{n,n}] &= \frac{1}{2}\left(\mathbb{E}_{a,0}[\tau_{n,n}] + \mathbb{E}_{n,n}[\tau_{0,0}]\right) \\
&= \frac{1}{2} 2R_{\mathrm{eff}}\left((0,0),(n,n)\right)\sum_e c(e) \\
&= \mathcal{O}(n^2 \cdot \ln n)
\end{aligned}$$

Definition 19 (Associated Hilbert space)**.**
Let $G = (V,E)$ be a graph of bounded degree, i.e. $\sup_{v \in V} \deg v < \infty$. We define the associated Hilbert space

$$\ell^2(V) := \left\{ f : V \to \mathbb{R} \,\middle|\, \sum_v f^2(v) < \infty \right\}$$

with the scalar product

$$\langle f, g \rangle = \sum_v f(v) \cdot g(v) \quad \text{for } f, g \in \ell^2(V).$$

For $e = (u, v)$, we define $\check{e} = u, \hat{e} = v$ and $-e = (v, u)$. We define the associated Hilbert space

$$\ell^2_-(E) := \left\{ \theta : E \to \mathbb{R} \,\middle|\, \sum_e \theta(e) < \infty,\, \theta(-e) = -\theta(e) \right\}$$

with the scalar product

$$\langle \theta, \eta \rangle = \frac{1}{2} \sum_e \theta(e) \cdot \eta(e)$$

Definition 20 (Boundary and coboundary operator)**.**
We define the *coboundary operator* as

$$d : \ell^2(V) \to \ell^2_-(E),\, df(e) = f(\check{e}) - f(\hat{e}) \quad \text{for } e \in E.$$

and the *boundary operator* as

$$d^* : \ell^2_-(E) \to \ell^2(V),\, d^*\theta(v) = \sum_{e:\check{e}=v} \theta(e) \quad \text{for } v \in V.$$

Exercise.
Show that d, d^* are well-defined.

Lemma 7.
d, d^* are adjoint linear maps, i.e. for all $f \in \ell^2(V), \theta \in \ell^2_-(E)$, we have $\langle \theta, df \rangle = \langle d^*\theta, f \rangle$.

Proof. We show the proof in two steps:

1. $-\sum_{e:\hat{e}=v} \theta(e) = \sum_{e:\hat{e}=v} \theta(-e) = \sum_{e:\check{e}=v} \theta(e) = d^*\theta(v)$.
2. By definition of the scalar product, we have

$$\begin{aligned}
\langle \theta, df \rangle &= \frac{1}{2} \sum_e \theta(e) \cdot (f(\check{e}) - f(\hat{e})) \\
&= \frac{1}{2} \sum_e \theta(e) f(\check{e}) - \frac{1}{2} \sum_e \theta(e) f(\hat{e}) \\
&= \frac{1}{2} \sum_v f(v) \sum_{e:\check{e}=v} \theta(e) - \frac{1}{2} \sum_v f(v) \sum_{e:\hat{e}=v} \theta(v) \\
&\overset{1.}{=} \sum_v f(v) \cdot d^*\theta(v) \\
&= \langle d^*\theta, f \rangle
\end{aligned}$$

□

Remark.
We have the following results:

- Ohm's law: $dU(e) = i(e) \cdot r_e$;
- Kirchhoff's node law: $d^*i(v) = 0$ for $v \in V \setminus (A \cup Z)$.

Definition 21 (Flow and strength)**.**
θ is called a *flow* ffrom A to Z if

1) $d^*\theta \geq 0$ on A;
2) $d^*\theta = 0$ on $V \setminus (A \cup Z)$;
3) $d^*\theta \leq 0$ on Z.

We define the *strength* of a flow θ as $\text{strength}(\theta) = \sum_{a \in A} d^*\theta(a)$.

Lemma 8 (Flow conversion)**.**
Let G be a finite graph and *theta* be a flow from A to Z with $A, Z \subseteq V$ and $A \cap Z = \emptyset$. Then,

$$\sum_{a \in A} d^*\theta(a) = -\sum_{z \in Z} d^*\theta(z)$$

Furthermore, if $f : V \to \mathbb{R}$ such that $f|_A = \alpha$ and $f|_Z = \zeta$, then $\langle \theta, df \rangle = \mathrm{strength}(\theta)(\alpha - \zeta)$.

Proof. We show the proof in two steps:

1. We have by defintion of the scalar product

$$\begin{aligned}\sum_{a \in A} d^*\theta(a) + \sum_{v \in V \setminus (A \cap Z)} d^*\theta(v) + \sum_{z \in Z} d^*\theta(z) &= \langle d^*\theta, \mathbb{1}_V \rangle \\ &= \langle \theta, d\mathbb{1}_V \rangle \\ &= 0\end{aligned}$$

2. To show the second statement, notice that

$$\begin{aligned}\langle \theta, df \rangle &= \langle d^*\theta, f \rangle \\ &= \sum_{v \in V} d^*\theta(v) \cdot f(v) \\ &= \sum_{a \in A} d^*\theta(a) \cdot f(a) + \sum_{z \in Z} d^*\theta(z) \cdot f(z) \\ &\overset{1.}{=} \mathrm{strength}(\theta)(\alpha - \zeta)\end{aligned}$$

$\square$

Definition 22 (Energy)**.**
For antisymmetric fucntions θ, η, we define the scalar product

$$\langle \theta, \eta \rangle_r = \frac{1}{2} \sum_e \theta(e) \cdot \eta(e) \cdot r_e$$

We define the *energy* of an antisymmetric function θ as $\mathcal{E}(\theta) = \|\theta\|_r^2 = \frac{1}{2} \sum_e \theta(e) \cdot r_e$.

Lemma 9.
Let $i : E \to \mathbb{R}$ be a unit current flow from A to Z such that the voltage $\mathcal{U}$ satisfies $\mathcal{U}|_A = \mathcal{U}_A$ and $\mathcal{U}|_Z = \mathcal{U}_Z$. Then, $\mathcal{E}(i) = \mathcal{U}_A - \mathcal{U}_Z = R_{\mathrm{eff}}(A \leftrightarrow Z)$.

Proof. By Ohm's law, we have

$$\begin{aligned}\mathcal{E}(i) &= \langle i, i \rangle_r \\ &= \langle i, ri \rangle \\ &= \langle i, dU \rangle \\ &= (\mathcal{U}_A - \mathcal{U}_Z) \underbrace{\mathrm{strength}(i)}_{=1} \\ &= \mathcal{U}_A - \mathcal{U}_Z\end{aligned}$$

Since all vertices in A have the same voltage, we can identify them as one point a. Then,

$$\begin{aligned}c_{\mathrm{eff}}(a \leftrightarrow Z) &= \sum_{x \neq a} \theta(a, x) \\ &= \sum_{x \neq a} \frac{i(a, x)}{\mathcal{U}_A - \mathcal{U}_Z} \\ &= \frac{1}{\mathcal{U}_A - \mathcal{U}_Z}\end{aligned}$$

This is equivalent to $R_{\mathrm{eff}} = \mathcal{U}_A \cdot \mathcal{U}_Z$. $\square$

Lemma 10.
Let $i : E \to \mathbb{R}$ be antisymmetric satisfying Kirchhoff's cycle law. Suppose i satisfies Kirchhoff's node law on $W \subseteq V$. Then, there exists a function (voltage) $\mathcal{U} : V \to \mathbb{R}$, such that U is harmonic on W and i is the current associated with $\mathcal{U}$. The voltage is unique up to addition.

Proof. We show the proof in two steps:

1. Existence: Let $v_0 \in V$ and set $\mathcal{U}(v_0) = 0$. For $u \in V$, let $(v_0, \dots, v_N = u)$ be a path from v_0 to u. We define
$$\mathcal{U}(u) = \sum_{n=1}^{N} r_{\{v_{n-1}, v_n\}} i(v_n, v_{n-1})$$
Let $(w_0 = v_0, \dots, w_m = v_n)$ be a second path, then $(v_0, \dots, v_n = w_m, \dots, w_0 = v_0)$ is a cycle, Thus,
$$\sum_{n=1}^{N} r_{\{v_{n-1}, v_n\}} i(v_n, v_{n-1}) = 0$$
Thus, $\mathcal{U}$ is well-defined. For $\{u, v\} \in E$, we can join the edge to the path
$$\mathcal{U}(v) = \mathcal{U} + r_{\{u,v\}} i(v, u)$$
Then, we have $i(v, u) = \frac{1}{r_{\{u,v\}}} (\mathcal{U}(v) - \mathcal{U}(u))$. Then, the current i is associated to $\mathcal{U}$. Harmonicity on W follows from Kirchhoff's node law.

2. Uniqueness: For a given $\mathcal{U}(v_0)$, we can define $\mathcal{U}(u)$ inductively by Ohm's law.

$\square$

1.2.1 Thomson's principle and corollaries

Theorem 3 (Thomson's principle)**.**
It holds
$$R_{\text{eff}}(a, z) = \inf\{\mathcal{E}(F) \mid F \text{ unit flow from } a \text{ to } z\}$$
Furthermore, the infimum is attained at the unit current flow.

Corollary 1.
Let $G_1 = (V, E_1, c_1)$ and $G_2 = (V, E_2, c_2)$, $a \neq z \in V$. Assume $E_1 \subseteq E_2$ and for all $e \in E_1$, we have $c_2(e) \geq c_1(e)$. Then,
$$C_{\text{eff}}^{G_1}(a, z) \leq C_{\text{eff}}^{G_2}(a, z)$$
$$\mathbb{P}^{G_2}(\tau_z < \eta_a) \geq \mathbb{P}^{G_2}(\tau < \eta_a).$$

Proof. Let I be the unit current flow from a to z in G_1. Then,
$$\begin{aligned}
R_{\text{eff}}^{G_2} &= \inf_F \mathcal{E}^{G_2}(F) \\
&\leq \mathcal{E}^{G_2}(I) \\
&= \sum_{e \in E_2} \frac{|I(e)|^2}{c_2(e)} \\
&\overset{(a)}{=} \sum_{e \in E_1} \frac{|I(e)|^2}{c_2(e)} \\
&\leq \sum_{e \in E_1} \frac{|I(e)|^2}{c_2(e)} \\
&= \mathcal{E}^{G_1}(I) \\
&= R_{\text{eff}}^{G_1}(a, z)
\end{aligned}$$

Hence, $C_{\text{eff}}^{G_1}(a,z) \leq C_{\text{eff}}^{G_2}(a,z)$. The second result follows from noting that

$$\begin{aligned}
\mathbb{P}^{G_2}(\tau_z < \eta_a) &= \frac{C_{\text{eff}}^{G_2}(a,z)}{c^{G_2}(a)} \\
&\geq \frac{C_{\text{eff}}^{G_1}(a,z)}{c^{G_1}(a)} \\
&= \mathbb{P}^{G_2}(\tau_z < \eta_a)
\end{aligned}$$

□

Corollary 2 (Monotonicity of effective conductance)**.**
Let $G = (V, E, c)$ be a network, $a \in V$, $\emptyset \neq Z_1 \subseteq Z_2 \subseteq V \setminus \{a\}$. Then, $C_{\text{eff}}(a, Z_1) \leq C_{\text{eff}}(a, Z_2)$.

Definition 23 (Exhaustion)**.**
Let $G = (V, E, c)$ be an infinite, locally finite, connected graph (including $\mathbb{Z}^d$, regular trees, Cayley graphs). A sequence $(V_n)_n$ of sets of vertices is called an *exhaustion* of the graph if

1) for all $n \in \mathbb{N}$, $|V_n| < \infty$;
2) for all $n \in \mathbb{N}$, $V_n \subseteq V_{n+1} \subseteq V$;
3) $\bigcup_{n=1}^{\infty} V_n = V$.

We define $G_n := G/(V \setminus C_n)$ as the graph with $V \setminus V_n$ reduced to a single vertex.

Definition 24 (Effective conductance to infinity)**.**
Let $a \in V$. We define the *effective conductance* between a and ∞ as

$$C_{\text{eff}}(a, \infty) = \lim_{n\to\infty} C_{\text{eff}}^{G_n}(a, V \setminus V_n).$$

Fact (Well-definedness of effective conductance to infinity)**.**
The conductance $C_{\text{eff}}(a,\infty) = \lim_{n\to\infty} C_{\text{eff}}^{G_n}(a, V \setminus V_n)$ is well-defined.

Proof. We have to show existence of the limit and independence of the choice of exhaustion.

1. Existence: By monotonicity of effective conductance, $C_{\text{eff}}(a, G \setminus V_n) \geq C_{\text{eff}}(a, G \setminus V_{n+1)}$. Hence $(C_{\text{eff}}(a, G\setminus)$ is monotonically decreasing. It is also bounded from below, hence convergent.

2. Let $(V_n)_n$ and $(W_n)_n$ be two exhaustions. Then, for each n there is some m such that $V_n \subseteq W_m$ and vice-versa. Thus,

$$\begin{aligned}
\lim_{m\to\infty} C_{\text{eff}}(a, G \setminus W_m) \leq C_{\text{eff}}(a, G \setminus V_n) \xrightarrow[n\to\infty]{} \lim_{n\to\infty} C_{\text{eff}}(a, G \setminus V_n) \\
\lim_{n\to\infty} C_{\text{eff}}(a, G \setminus V_n) \leq C_{\text{eff}}(a, G \setminus W_m) \xrightarrow[m\to\infty]{} \lim_{m\to\infty} C_{\text{eff}}(a, G \setminus W_m)
\end{aligned}$$

   Thus, we have equality.

□

Theorem 4.
The random walk starting at a is recurrent if and only if $R_{\text{eff}}(a,\infty) = \infty$ (i.e. $C_{\text{eff}}(a,\infty) = 0$).

Proof. Let $(V_n)_n$ be an exhaustion. Then,

$$\begin{aligned}
\mathbb{P}(\eta_a < \infty) &= \lim_{n\to\infty} \mathbb{P}(\eta < \tau_{G\setminus V_n}) \\
&= 1 - \lim_{n\to\infty} \mathbb{P}(\eta_a > \tau_{G\setminus V_n}) \\
&= 1 - \frac{1}{c(a)} \lim_{n\to\infty} C_{\text{eff}}(a, G \setminus V_n) \\
&= 1 - \frac{C_{\text{eff}}(a,\infty)}{\underbrace{c(a)}_{\in(0,\infty)}}
\end{aligned}$$

$R_{\text{eff}}(a,\infty) = \infty$ is equivalent to $C_{\text{eff}}(a,\infty) = 0$. By our calculations, this is equivalent to $\mathbb{P}(\eta_a < \infty) = 1$. □

Theorem 5 (Thomson's principle for infinite networks)**.**
A *unit flow* from a to ∞ in G is an antisymmetric edge function F such that for all $u \in V$, we have

$$\sum_{v \neq u} F(u,v) = \begin{cases} 1, & x = a \\ 0, & x \neq a \end{cases}$$

Then,

$$R_{\text{eff}}(a,\infty) = \inf\{\mathcal{E}(F) \,|\, F \text{ is unit flow from } a \text{ to } \infty\} \quad \text{for } \mathcal{E}(F) := \sum_e \frac{|F(e)|^2}{c(e)}.$$

Remark.
In order to prove that a graph (or network) is transient, all we need to find is a flow of finite energy

Example 1 (Binary tree)**.**
What is $\mathcal{E}(F)$ of a binary tree with three edges at the origin? At level n, we have $3 \cdot 2^n$ edges, each carries a flow of $(3 \cdot 2^n)^{-1}$. Thus, $\mathcal{E}(F) = \sum_{n=0}^{\infty} 3 \cdot 2^n \cdot 3 \cdot 2^{n-2} < \infty$.

From Thomson's principle we get monotonicity: If G is recurrent and G' is a subgraph of G, then G' is recurrent as well. The argument is as follows. Let F be a flow on G', then F is also a flow on G and $\mathcal{E}^{G'}(F) = \mathcal{E}^{G}(F) = \infty$ since G is recurrent. Thus, G' is recurrent. As a result we find a way to prove transience. Namely, we have to find a flow with finite energy. We now find a criterion for recurrence.

Definition 25 (Cutset)**.**
Let $G = (V, E, c)$ be an infinite network and let $a \in V$. A set $\Pi \subseteq E$ is called a *cutset* if every infinity simple path starting from a has non-empty intersection with Π. We define the conductance of Π as $\sum_{e \in \Pi} c(e)$.

Theorem 6 (Nash-Williams)**.**
Let $(\Pi_n)_n$ be a sequence of disjoint cutsets, then $R_{\text{eff}}(a,\infty) \leq \sum_{n=1}^{\infty} \frac{1}{c(\Pi_n)}$.

Proof. Let F be a unit flow from a to ∞. For all n, $\sum_{e \in \Pi_n} |F(e)| \leq 1$, since the amount of flow going through the cutset is at least 1). Thus,

$$\begin{aligned}
\mathcal{E}(F) &= \frac{1}{2} \sum_{e \in E} \frac{|F(e)|^2}{c(e)} \\
&= \sum_{n=1}^{\infty} \left(\sum_{e \in \Pi_n} \frac{|F(e)|^2}{c(e)} \right) \\
&\overset{(1)}{\geq} \sum_{n=1}^{\infty} \frac{1}{\sum_{e \in \Pi_n} c(e)} \\
&= \sum_{n=1}^{\infty} \frac{1}{c(\Pi_n)}.
\end{aligned}$$

(1) follows from Sedrakyan's lemma. □

Lemma 11 (Sedrakyan's lemma)**.**
Let $\sum_{n=1}^{N} x_n \geq 1$ and $c_1, \ldots, c_N > 0$. Then

$$\sum_{n=1}^{N} \frac{x_n^2}{c_n} \geq \frac{1}{\sum_{n=1}^{N} c_n}.$$

Theorem 7.
$\mathbb{Z}^3$ is transient.

Proof. We provide a flow of finite energy by Y. Peres. This is done in the following steps:

1. Define the dual of $\mathbb{Z}^3$. For every edge $e \in E(\mathbb{Z}^3)$, take the dual e' to the plaquette perpendicualr to e.

2. Consider a sphere with surface area of 1 around the origin.

3. The flow is directed away from the origin. The intensity of the flow is the area of the projection of the plaquette on the sphere.

We have to show that this in fact defines a flow (i) and find a bound of $\mathcal{E}(F)$ (ii).

(i) We need to show that for every $x \neq 0 =: a$, the amount going in is the same as the amount going out.

(ii) $|F(e)| =$ [size of projection]. Each axis of the projection is at most $\frac{1}{|e|}$. Thus, the area of the projection is at most $\frac{1}{|e|^2}$. Hence,

$$\begin{aligned}\mathcal{E}(F) &\leq \sum_{e\in E(\mathbb{Z}^3)} \left(\frac{1}{|e|^2}\right)^2 \\ &\leq \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)^2 \operatorname{card}\{e \mid n \leq |e| \leq n+1\} \\ &\leq c \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.\end{aligned}$$

□

1.3 Spherically symmetric trees

Definition 26 (Rooted tree)**.**

Definition 27 (Spherically symmetric tree)**.**
A rooted tree (T,ρ) is called *spherically symmetric* if for all $u, v \in T$, we have

$$d(v,\rho) = d(u,\rho) \implies \deg u = \deg v.$$

If $(T, \rho$ is such a spherically tree. We define

$$f(n) := [\text{outwards degree of vertices at distance } n \text{ from } \rho]$$

. Then, (T,ρ) is determined by $f(n))_n$ up to an isometry (rotation). We further set $g(N) := \sum_{n=1}^{N} f(n)$ as the number of vertices at distance at distance n from ρ.

Theorem 8.
Let T be a spherically symmetric tree. Then, the following statements are equivalent:

1) T is recurrent;

2) $\sum_{n=1}^{\infty} \frac{1}{g(n)} = \infty$.

Notice, that $[2) \implies 1]$ even if T is a general graph instead of a spherically symmetric tree. The converse implication fails.

Proof. 1. $[2) \implies 1)]$. Define $\Pi_n = \{(u,v) \mid d(v,\rho) = n-1,\, d(u,\rho) = n\}$. Then, $(\Pi_n)_n$ is a sequence of disjoint cutsets, and setting $c(\Pi_n) = g(n)$ yields $\sum_{n=1}^{\infty} \frac{1}{c(\Pi_n)} = \infty$. Thus T is recurrent.

2. Assume to the contrary that $\sum_{n=1}^{\infty} \frac{1}{g(n)}$. We define a flow F as follows. Let $e = (u,v)$ such that $d(u,\rho) = d(v,\rho) - 1$ (i.e. v is parent of u). We then set $|F(u,v)| = \frac{1}{g(n)}$ for $n = d(v,\rho)$ (Notice,

that this defines a flow only on a spherically symmetric tree!). We then obtain

$$\begin{aligned}
\mathcal{E}(F) &= \frac{1}{2}\sum_{e}|F(e)|^2 \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{u:d(u,\rho)=n}\left|F(\underbrace{u'}_{\text{parent of } u},u)\right|^2 \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\underbrace{g(n)}_{\text{number of vertices}}\underbrace{\left(\frac{1}{g(n)}\right)^2}_{\text{intensity}} \\
&= \infty
\end{aligned}$$

Thus, we have transience.

□

We now want to deal with random conductances on fixed graphs such as $\mathbb{Z}$ or $\mathbb{Z}^2$. We assume the following restrictions:

1) random conductances are in $(0,\infty)$;
2) the distribution of conductances is translation invariant, i.e. for all $x \in \mathbb{Z}$ ($\mathbb{Z}^2$), we have $(c(e))_e =_d (c(e+x))_e$.

We would like to know whether we can produce a transient network by choosing random conductances for the recurrent graph.

1.3.1 Random network is one-dimensional lattice

Theorem 9.
$\mathbb{Z}$ as a random network is always a recurrent.

Proof. Fix $\epsilon > 0$. We show that $\mathbb{P}(\text{Network is recurrent}) \geq 1-2\epsilon$. Take M usch that $\mathbb{P}(c(e) > M) < \epsilon$. Then we have that

$$\begin{aligned}
\mathbb{P}(\text{There are infinitely many } n \geq 0 \text{ such that } \underbrace{c(n,n+1) \leq M}_{A_n:=} &= \mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k\right) \\
&\overset{=}{\text{I}} \lim_{n\to\infty}\mathbb{P}\left(\bigcup_{k=n}^{\infty}\right) \\
&\geq \varliminf_{n\to\infty}\mathbb{P}(A_k) \\
&=\geq 1-\epsilon
\end{aligned}$$

(I) holds since the intersection contains decreasing sets. It follows that the probability that there are infinitely many $n \geq 0$ with $c(n,n+1) \leq M$ and infinitely many $n \leq 0$ with $c(n,n-1) \leq M$ is at least $1-2\epsilon$. Thus we can find infinitely many disjoint cutsets with individual conductance at most $2M$.

□

1.3.2 Random network is two-dimensional lattice

The two-dimensional case is more delicate and depends on further circumstances.

Theorem 10.
Consider $\mathbb{Z}^2$ as a random network. Then, the following statements hold:

1) If $\mathbb{E}[c(e)] < \infty$ for horizontal and vertical edges e, the network is recurrent;
2) If conductances are independent, the network is recurrent;

3) There exists a transient system of random conductances on $\mathbb{Z}^2$.

Proof. We show the result in a few steps:

1. Assign 1 to every edge;
2. For every vertex sample V_x, H_x (vertical, horizontal) iid, we set

$$\begin{cases} \mathbb{P}\left(V_x = n\right) = 3^{-n}, & n \geq 1 \\ \mathbb{P}\left(V_x = 0\right) = 1 - \sum_{n=1}^{\infty} 3^{-n} & \end{cases}$$

For every x, if $V_x \geq 1$, then add 4^{V_x} to the 2^{V_x} above x;

For every x, if $H_x \geq 1$, then add 4^{H_x} to the 2^{H_x} right to x.

By Borel-Cantelli, each edge will have infinitely many contributions. $\square$

2 Chapter two

3 Chapter three

4 Chapter four

5 Chapter five

6 Chapter six

A Measure theory

Theorem 11 (Layer-cake representation)**.**
Let $f : \Omega \to \mathbb{R}$ such that $f \geq 0, f' \geq 0$. Then

$$\mathbb{E}[f(X)] = \int_0^\infty \mathbb{P}(f(X) > t)\,\mathrm{d}t = \int_{f^{-1}(0)}^{f^{-1}(\infty)} \mathbb{P}(X > t) f'(t)\,\mathrm{d}t$$

A.1 Convergence

Theorem 12 (Fatou's lemma)**.**
Let $(X_n)_n$ consist of non-negative random variables. Then,

$$\mathbb{E}[\varliminf_{n\to\infty} X_n] \leq \varliminf_{n\to\infty} \mathbb{E}[X_n]$$

Theorem 13 (Monotone convergence theorem)**.**
Let $(X_n)_n$ consist of non-negative increasing random variables. Then,

$$\mathbb{E}[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} \mathbb{E}[X_n]$$

Theorem 14 (Dominated convergence theorem)**.**
Let $(X_n)_n$ consist of converging random variables that are absolutely dominated by $Y \in L^1(\Omega, \mathbb{P})$. Then,

$$\mathbb{E}[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} \mathbb{E}[X_n]$$

B Probability theory

B.1 Calculus of probability

Theorem 15 (Chebychev's inequality)**.**
Let $\phi : \mathbb{R} \to [0, \infty)$ be increasing. Then

$$\mathbb{P}(X \geq \lambda) \leq \frac{\mathbb{E}[\phi(X)]}{\phi(\lambda)}$$

Remark.
Chebychev's inequality can be used to find sharp estimations by considering parametrized functions such as $\phi(\xi) = \exp(t\xi))$ or $\phi(\xi) = (\xi + t)^2$ and optimizing t.

Definition 28 (Limit of events)**.**
We define

$$\varlimsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n$$
$$\varliminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_n$$

Theorem 16 (Calculus of probability)**.**

Theorem 17 (Borel-Cantelli)**.**
Let $A_n\Omega$. Then

1. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\varlimsup_{n\to\infty} A_n) = 0$ (A_n happens finitely).
2. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and $(A_n)_n$ are independent, then $\mathbb{P}(\varlimsup_{n\to\infty} \mathbb{P}(A_n) = 1)$

B.2 Conditional expectation

Definition 29 (Conditional expectation).
Y is called $\mathcal{G}$-conditional expectation of X if

1. Y is $\mathcal{G}$-measurable
2. for each $A \in \mathcal{G}$, $\mathbb{E}[Y \mathbb{1}_A] = \mathbb{E}[X \mathbb{1}_A]$

We write $Y = \mathbb{E}[X|\mathcal{G}]$.

B.3 Radon-Nikodym theorem

Theorem 18 (Radon-Nikodym).

B.4 Stochastic processes

Definition 30 (Martingale).

Theorem 19 (Doob's maximal inequality).

B.4.1 Stopping times

Theorem 20 (Optional stopping).

C Markov chains

Definition 31 (Discrete-time Markov chain).
A discrete-time Markov chain is a sequence $X = (X_n)_n$ of random variables with

1. $\mathbb{P}(X_{n+1} = y \mid X_1 = x_1, \ldots, X_n = x_n) = \mathbb{P}(X_{n+1} = y \mid X_n = x_n)$

One might also use the following characterization which extends more easily to $X = (X_t)_{t \geq 0}$:

1. $\mathbb{E}_x[g \circ \theta_s \mid \mathcal{F}_s] = \mathbb{E}_{X_s}[g]$ for all measurable and bounded g

where $\mathbb{P}_x(X_0 = x) = 1$.

Definition 32 (Time-homogeneity).
A Markov chain X is called time-homogeneous if

$$\mathbb{P}(X_{n+1} = y \mid X_n = x) = \mathbb{P}(X_1 = y \mid X_0 = x)$$

Definition 33 (Stationarity).
A Markov chain X is called stationary if

$$\mathbb{P}(X_k = x_0, \ldots, X_{n+k} = x_n) = \mathbb{P}(X_0 = x_0, \ldots, X_n = x_n)$$

Definition 34 (Standard stopping times).
For a (right-continuous) family of random variables $X = (X_t)_{t \in \mathcal{T}}$, where we allow $\mathcal{T} = \mathbb{R}, \mathbb{N}$, we write

1. $\tau_x = \inf \{t > 0 \mid X_t = x\}$
2. $\sigma_x = \inf \{t > 0 \mid X_t \neq x\}$
3. $R_x^n = 0$, $R_x^{n+1} = (\tau_x \circ \theta_{\sigma_x} + \sigma_x) \circ \theta_{R_x^n} + R_x^n$

Notice that a process X is a Markov chain if $(R_x^{n+1} - R_x^n) =_d R_x^1$ and $(R_x^{n+1} - R_x^n)_n$ forms independent and identically distributed random variables. (So called recurrence times)

Remark.
The standard approach to prove statements about probabilities of Markov chains is as follows: ....

Definition 35 (Invariant measure)**.**
A measure π is called invariant for the Markov chain X if $\pi\Pi = \pi$

1. $\pi(y) = \sum_x \pi(x)p(x,y)$

Definition 36 (Reversible measure)**.**
A measure π is called reversible for the Markov chain X if

1. $\pi(x)p(x,y) = \pi(y)p(y,x)$

Definition 37 (Irreducible Markov chain)**.**
A Markov chain X is called irreducible if for all x, y there is $n \geq 1$ such that $\mathbb{P}_x(X_n = y) > 0$.

Remark.
Irreducibility means there is the probability that a chain starting in x might reach y. It does not have to occur, though.

Definition 38 (Recurrence and transience)**.**
x is called recurrent if $\mathbb{P}_x(R_x^1 < \infty) = 1$, otherwise transient.

Theorem 21 (Invariant distribution implies recurrence)**.**
An irreducible Markov chain with an invariant distribution is recurrent.

Theorem 22 (Existence of an invariant measure)**.**
An irreducible and recurrent Markov chain has an invariant distribution which is unique and stricly positive.

Theorem 23 (Characterization of invariant distribution)**.**
Let X be a Markov chain with an invariant distribution π. Then

$$\pi(y) = \lim_{n\to\infty} \mathbb{P}_x(X_n = y)$$

regardless of the choice for x.

Theorem 24 (Cycle characterization of reversibility)**.**
A discrete-time Markov chain X is reversible if and only if for all states $x_1, \dots, x_N$ with $x_1 = x_N$, we have

$$\prod_{n=1}^{N} p(x_n, x_{n+1}) = \prod_{n=1}^{N} p(x_{N-n+1}, p_{N-n})$$

Proof. □