Generative Models for Dimensionality Reduction: Probabilistic PCA and Factor Analysis

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Machine Learning (CS771A)

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- When $\epsilon_n \sim \mathcal{N}(0, \Psi)$, Ψ is diagonal, it is called "Factor Analysis" (FA)

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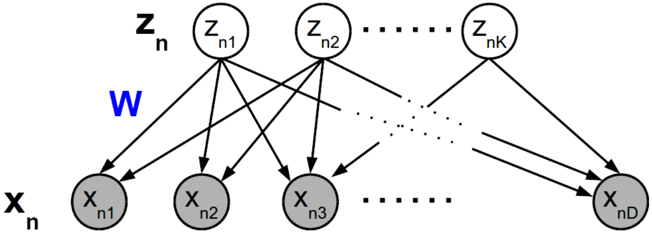
- Zooming in at the relationship between each $\boldsymbol{x}_n \in \mathbb{R}^D$ and each $\boldsymbol{z}_n \in \mathbb{R}^K$



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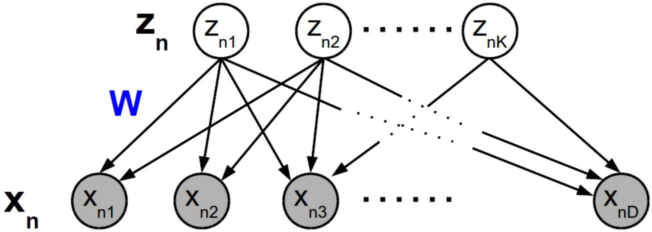

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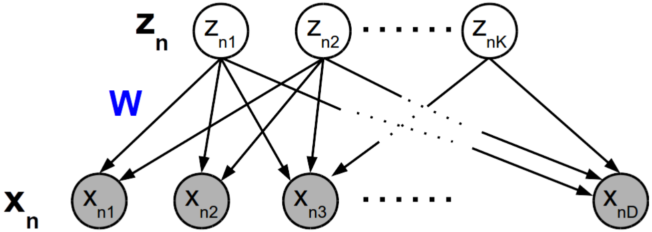

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- This view also helps in thinking about "deep" generative models that have many layers of latent variables or "hidden units"

Linear Gaussian Systems

- Note that PPCA and FA are special cases of linear Gaussian Systems which have the following general form

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p(\boldsymbol{z}) &= \mathcal{N}(\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_z) \\
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(Chapter 4 of Murphy and Chapter 2 of Bishop have various useful results on properties of multivar. Gaussians)

PPCA or FA = Low-Rank Gaussian

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Parameter Estimation for PPCA

- Data: $\mathbf{X} = \{\boldsymbol{x}_n\}_{n=1}^N$, latent vars: $\mathbf{Z} = \{\boldsymbol{z}_n\}_{n=1}^N$, parameters: $\mathbf{W}, \sigma^2$

† Probabilistic Principal Component Analysis (Tipping and Bishop, 1999)

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$$\log p(\mathbf{X}, \mathbf{Z}|\mathbf{W}, \sigma^2) = \log \prod_{n=1}^{N} p(\boldsymbol{x}_n, \boldsymbol{z}_n|\mathbf{W}, \sigma^2) = \log \prod_{n=1}^{N} p(\boldsymbol{x}_n|\boldsymbol{z}_n, \mathbf{W}, \sigma^2)p(\boldsymbol{z}_n)$$

$$= \sum_{n=1}^{N} \{\log p(\boldsymbol{x}_n|\boldsymbol{z}_n, \mathbf{W}, \sigma^2) + \log p(\boldsymbol{z}_n)\}$$

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- As we'll see, it leads to much simpler expressions and efficient solutions

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- We will need the expected value of this quantity in M step of EM

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- We will need the expected value of this quantity in M step of EM
  - This requires computing the posterior distribution of $\boldsymbol{z}_n$ in E step (which is Gaussian; recall the result from earlier slide on linear Gaussian systems)

EM based Parameter Estimation for PPCA

- The expected complete data log-likelihood $\mathbb{E}[\log p(\mathbf{X}, \mathbf{Z}|\mathbf{W}, \sigma^2)]$

$$= -\sum_{n=1}^{N} \left\{ \frac{D}{2} \log \sigma^2 + \frac{1}{2\sigma^2} ||\boldsymbol{x}_n||^2 - \frac{1}{\sigma^2} \mathbb{E}[\boldsymbol{z}_n]^\top \mathbf{W}^\top \boldsymbol{x}_n + \frac{1}{2\sigma^2} \text{tr}(\mathbb{E}[\boldsymbol{z}_n \boldsymbol{z}_n^\top] \mathbf{W}^\top \mathbf{W}) + \frac{1}{2} \text{tr}(\mathbb{E}[\boldsymbol{z}_n \boldsymbol{z}_n^\top]) \right\}$$

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- Taking the derivative of $\mathbb{E}[\log p(\mathbf{X},\mathbf{Z}|\mathbf{W},\sigma^2)]$ w.r.t. $\mathbf{W}$ and setting to zero

$$\mathbf{W} = \left[\sum_{n=1}^{N}\boldsymbol{x}_n\mathbb{E}[\boldsymbol{z}_n]^\top\right]\left[\sum_{n=1}^{N}\mathbb{E}[\boldsymbol{z}_n\boldsymbol{z}_n^\top]\right]^{-1}$$

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$$p(\boldsymbol{z}_n|\boldsymbol{x}_n, \mathbf{W}) = \mathcal{N}(\mathbf{M}^{-1}\mathbf{W}^\top\boldsymbol{x}_n, \sigma^2\mathbf{M}^{-1}) \qquad \text{where } \mathbf{M} = \mathbf{W}^\top\mathbf{W} + \sigma^2\mathbf{I}_K$$

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- Note: The noise variance σ^2 can also be estimated (take deriv., set to zero..)

The Full EM Algorithm for PPCA

- Specify K, initialize $\mathbf{W}$ and σ^2 randomly. Also center the data

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\end{aligned}
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\end{aligned}
$$

- **M step:** Re-estimate $\mathbf{W}$ and σ^2

$$
\mathbf{W}_{new} = \left[\sum_{n=1}^{N}\boldsymbol{x}_n\mathbb{E}[\boldsymbol{z}_n]^\top\right]\left[\sum_{n=1}^{N}\mathbb{E}[\boldsymbol{z}_n\boldsymbol{z}_n^\top]\right]^{-1} = \left[\sum_{n=1}^{N}\boldsymbol{x}_n\mathbb{E}[\boldsymbol{z}_n]^\top\right]\left[\sum_{n=1}^{N}\mathbb{E}[\boldsymbol{z}_n]\mathbb{E}[\boldsymbol{z}_n]^\top + \sigma^2\mathbf{M}^{-1}\right]^{-1}
$$

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The Full EM Algorithm for PPCA

- Specify K, initialize $\mathbf{W}$ and σ^2 randomly. Also center the data
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- If not converged, go back to E step (can monitor the incomplete/complete log-likelihood to assess convergence)

EM for Factor Analysis

- Similar to PPCA except that the Gaussian conditional distribution $p(\boldsymbol{x}_n|\boldsymbol{z}_n)$ has diagonal instead of spherical covariance, i.e., $\boldsymbol{x}_n \sim \mathcal{N}(\mathbf{W}\boldsymbol{z}_n, \boldsymbol{\Psi})$, where $\boldsymbol{\Psi}$ is a diagonal matrix

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$$\boldsymbol{\Psi}_{new} = \text{diag}\left\{\mathbf{S} - \mathbf{W}_{new}\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[\boldsymbol{z}_n]\boldsymbol{x}_n^\top\right\} \qquad (\mathbf{S} \text{ is the cov. matrix of data})$$

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- Can also handle missing data as additional latent variables in E step. Just write each data point as $\boldsymbol{x}_n = [\boldsymbol{x}_n^{obs} \;\; \boldsymbol{x}_n^{miss}]$ and treat $\boldsymbol{x}_n^{miss}$ as latent vars.

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- Can learn the model very efficiently using "online EM"
- Possible to give it a fully Bayesian treatment (which has many other benefits such as inferring K using nonparametric Bayesian modeling)

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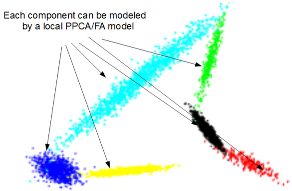

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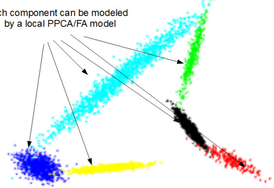
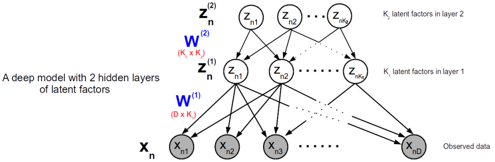


- Deep models for feature learning and dimensionality reduction



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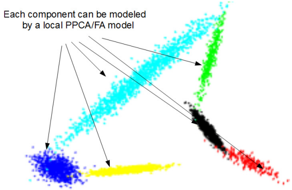

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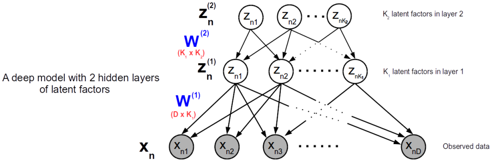
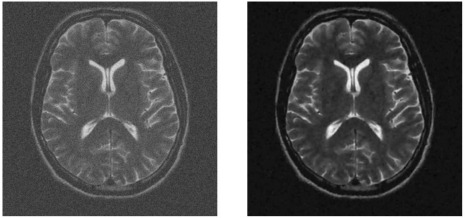


- Supervised extensions, e.g., by jointly modeling labels y_n as conditioned on latent factors, i.e., $p(y_n = 1|\boldsymbol{z}_n, \theta)$ using a logistic model with weights $\theta \in \mathbb{R}^K$

Some Applications of PPCA

- Learning the noise variance allows "image denoising"



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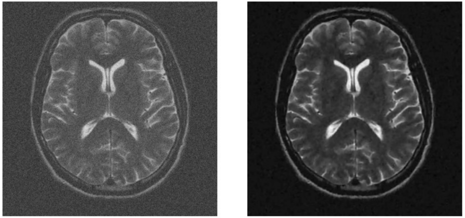
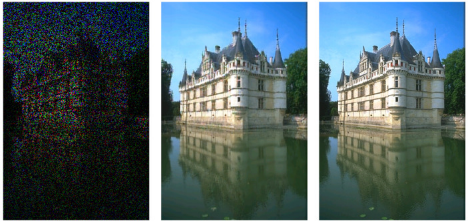


- Ability to fill-in missing data allows "image inpainting" (left: image with 80% missing data, middle: reconstructed, right: original)



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- Thus EM can also be used to efficiently solve the standard non-probabilistic PCA without doing eigendecomposition

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&= \mathcal{N}(\mathbf{0}, \mathbf{W}\mathbf{R}\mathbf{R}^\top\mathbf{W}^\top + \sigma^2 \mathbf{I}_D) \\
&= \mathcal{N}(\mathbf{0}, \mathbf{W}\mathbf{W}^\top + \sigma^2 \mathbf{I}_D)
\end{aligned}
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- Thus PPCA doesn't give a unique solution (for every $\mathbf{W}$, there is another $\tilde{\mathbf{W}} = \mathbf{W}\mathbf{R}$ that gives the same solution)
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Identifiability

- Note that $p(\boldsymbol{x}_n) = \mathcal{N}(\mathbf{0}, \mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}_D)$
- If we replace $\mathbf{W}$ by $\tilde{\mathbf{W}} = \mathbf{W}\mathbf{R}$ for some orthogonal rotation matrix $\mathbf{R}$ then

$$\begin{aligned} p(\boldsymbol{x}_n) &= \mathcal{N}(\mathbf{0}, \tilde{\mathbf{W}}\tilde{\mathbf{W}}^\top + \sigma^2\mathbf{I}_D) \\ &= \mathcal{N}(\mathbf{0}, \mathbf{W}\mathbf{R}\mathbf{R}^\top\mathbf{W}^\top + \sigma^2\mathbf{I}_D) \\ &= \mathcal{N}(\mathbf{0}, \mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}_D) \end{aligned}$$

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- Usually this is not a problem, unless we want to very strictly interpret $\mathbf{W}$
- To ensure identifiability, we can impose some more structure on $\mathbf{W}$, e.g., constrain it to be a lower-triangular or sparse matrix

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  - E.g., Hidden Markov Models and Kalman Filters can be seen as generalization of mixture models and Gaussian latent factor models, respectively, for sequential data ($\boldsymbol{z}_n$ correspond to the "state" of $\boldsymbol{x}_n$)
  - We will look at these and other related models (e.g., LSTM) when talking about learning from seqential data