

Convolutional Neural Operators for robust and accurate learning of PDEs

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ABSTRACT. Although very successfully used in conventional machine learning, convolution based neural network architectures – believed to be inconsistent in function space – have been largely ignored in the context of learning solution operators of PDEs. Here, we present novel adaptations for convolutional neural networks to demonstrate that they are indeed able to process functions as inputs and outputs. The resulting architecture, termed as convolutional neural operators (CNOs), is designed specifically to preserve its underlying continuous nature, even when implemented in a discretized form on a computer. We prove a universality theorem to show that CNOs can approximate operators arising in PDEs to desired accuracy. CNOs are tested on a novel suite of benchmarks, encompassing a diverse set of PDEs with possibly multi-scale solutions and are observed to significantly outperform baselines, paving the way for an alternative framework for robust and accurate operator learning.

1 Introduction.

Partial Differential Equations (PDEs) [13] are ubiquitous as mathematical models in the sciences and engineering. Solving a PDE amounts to (approximately) computing the so-called *solution operator* that maps function space inputs such as initial and boundary conditions, coefficients, source terms etc, to the PDE solution which also belongs to a suitable function space. Well-established numerical methods such as finite differences, finite elements, finite volumes and spectral methods (see [47]) have been very successfully used for many decades to approximate PDE solution operators. However, the prohibitive computational cost of these methods, particularly in high dimensions and for *many query* problems such as UQ, inverse problems, PDE-constrained control and optimization, necessitates the design of *fast, robust and accurate* surrogates. This provides the rationale for the use of *data-driven machine learning* methods for solving PDEs [20].

Implementation is available at <https://github.com/bogdanraonic3/ConvolutionalNeuralOperator.git>

As *operators* are the objects of interest in solving PDEs, learning such operators from data, which is loosely termed as *operator learning*, has emerged as a dominant paradigm in recent years for the applications of machine learning to PDEs. A very partial list of architectures for operator learning include operator networks [9], DeepONets [38] and its variants [40, 7], PCA-net [5], neural operators [24] such as graph neural operator [33], Multipole neural operator [34] and the very popular Fourier Neural Operator [32] and its variants [35, 44], VIDON [46], spectral neural operator [14], LOCA [22], NOMAD [51] and transformer based operator learning architectures [8].

Despite the considerable success of the recently proposed operator learning architectures, several pressing issues remain to be addressed. These include, but are by no means restricted to, limited expressivity for some of these algorithms [27] to aliasing errors for others [14] to the very fundamental issue of possible *lack of consistency in function spaces*. As argued in a recent paper [2] (see also [14, 24]), a structure-preserving operator learning algorithm or *representation equivalent neural operator* has to respect some form of *continuous-discrete equivalence* (CDE) in order to learn the underlying operator, rather than just a discrete representation of it. Failure to respect such a CDE can lead to the so-called *aliasing errors* [2] and affect model performance at multiple discrete resolutions.

The absence of a suitable CDE has also plagued the naive use of convolutional neural networks (CNNs) in the context of operator learning, see [64, 32, 2] on how using CNNs for operator learning leads to results that heavily rely on the underlying grid resolution. This very limited use of Convolution (in physical space) based architectures for operator learning stands in complete contrast to the fact that CNNs [29] and their variants are widely used architectures for image classification and generation and in other contexts in machine learning [28, 36, 62]. Moreover, CNNs can be thought of as natural generalizations of the foundational finite difference methods for discretizing PDEs [17, 37]. Given their innate locality, computational and data efficiency and the availability of a wide variety of successful CNN architectures in other fields, it could be very advantageous to bring CNN-based algorithms back into the reckoning for operator learning. This is precisely the central point of the current paper where we make the following contributions,

- We propose novel modifications to CNNs in order to enforce structure-preserving continuous-discrete equivalence and enable the genuine learning of operators. The resulting architecture, termed as *Convolutional Neural Operator* (CNO), is instantiated as a novel *operator* adaptation of the widely used U-Net architecture.
- In addition to showing that CNO is a *representation equivalent neural operator* in the sense of [2], we also prove a universality result to rigorously demonstrate that CNOs can approximate the operators, corresponding to a large class of PDEs, to desired accuracy.
- We test CNO on a *novel* set of benchmarks, that we term as *Representative PDE Benchmarks* (RPB), that span across a variety of PDEs ranging from linear elliptic and hyperbolic to nonlinear parabolic and hyperbolic PDEs, with possibly *multiscale*

solutions. We find that CNO is either on-par or outperforms the tested baselines on all the benchmarks, both when testing in-distribution as well as in *out-of-distribution* testing.

Thus, we present a new CNN-based operator learning model, with desirable theoretical properties and excellent empirical performance, with the potential to be widely used for learning PDEs.

2 Convolutional Neural Operators.

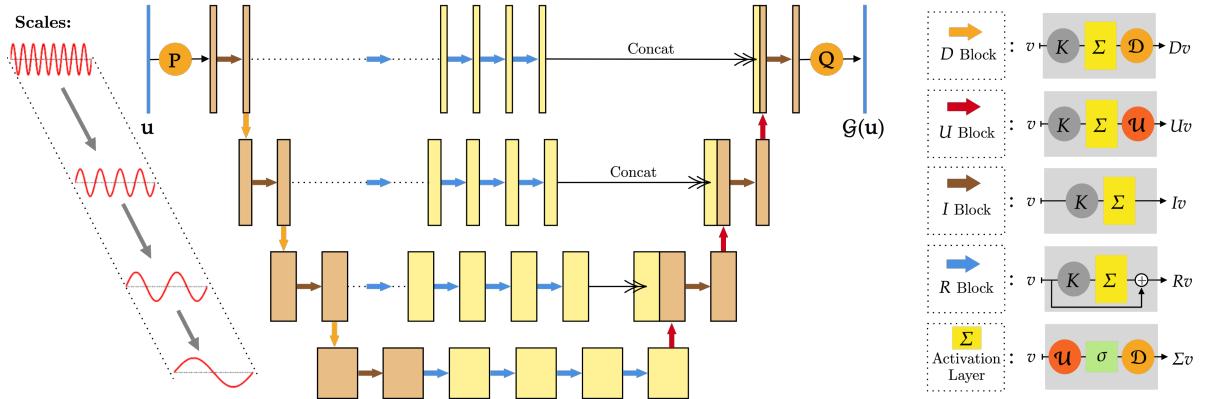


Figure 1: Schematic representation of CNO (2.3) as a modified U-Net with a sequence of layers (each identified with the relevant operators on the right, see Section 2) mapping between bandlimited functions. Rectangles represent multi-channel signals. Larger the height, larger is the resolution. Wider the rectangles, more channels are present.

Setting. For simplicity of the exposition, we will focus here on the two-dimensional case by specifying the underlying domain as $D = \mathbb{T}^2$, being the 2-d torus. Let $\mathcal{X} = H^r(D, \mathbb{R}^{d_x}) \subset \mathcal{Z}$ and $\mathcal{Y} = H^s(D, \mathbb{R}^{d_y})$ be the underlying function spaces, where $H^{r,s}(D, \cdot)$ are Sobolev spaces of order r and s . Without loss of generality, we set $r = s$ hereafter. Our aim would be to approximate *continuous operators* $\mathcal{G}^\dagger : \mathcal{X} \rightarrow \mathcal{Y}$ from data pairs $(u_i, \mathcal{G}^\dagger(u_i))_{i=1}^M \in \mathcal{X} \times \mathcal{Y}$. We further assume that there exists a *modulus of continuity* for the operator i.e.,

$$\|\mathcal{G}^\dagger(u) - \mathcal{G}^\dagger(v)\|_{\mathcal{Y}} \leq \omega(\|u - v\|_{\mathcal{Z}}), \quad \forall u, v \in \mathcal{X}, \quad (2.1)$$

with $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ being a monotonically increasing function with $\lim_{y \rightarrow 0} \omega(y) = 0$. The underlying operator \mathcal{G}^\dagger can correspond to solution operators for PDEs (see Section 3 for the exact setting) but is more general than that and encompasses examples such as those arising in inverse problems, for instance in imaging [4].

Bandlimited Approximation. As argued in a recent paper [2], Sobolev spaces such as H^r are, in a sense, *too large* to allow for any form of *continuous-discrete equivalence* (CDE), i.e., equivalence between the underlying operator and its discrete representations, which is necessary

for robust operator learning. Consequently, one has to consider smaller subspaces of H^r which allow for CDEs. In this respect, we choose the space of *bandlimited functions* [58] defined by,

$$\mathcal{B}_w(D) = \{f \in L^2(D) : \text{supp } \hat{f} \subseteq [-w, w]^2\}, \quad (2.2)$$

for some $w > 0$ and with \hat{f} denoting the Fourier transform of f . It is straightforward to show using (2.1) (see SMA.1) that for any $\varepsilon > 0$, there exists a w , large enough depending on r , and a continuous operator $\mathcal{G}^* : \mathcal{B}_w(D) \rightarrow \mathcal{B}_w(D)$, such that $\|\mathcal{G}^\dagger - \mathcal{G}^*\| < \varepsilon$, with $\|\cdot\|$ denoting the corresponding operator norm. In other words, the underlying operator \mathcal{G}^\dagger can be approximated to arbitrary accuracy by the operator \mathcal{G}^* that maps between band-limited spaces. Consequently, as shown in SMA.2, one can readily define discrete versions of \mathcal{G}^* using the underlying basis for bandlimited functions and establish a continuous-discrete equivalence for it.

Definition of CNO. Given the above context, our goal will be to approximate the operator \mathcal{G}^* in a *structure-preserving manner* i.e., as the underlying operator maps between spaces of bandlimited functions, we will construct our operator approximation architecture to also map bandlimited functions to bandlimited functions, thus respecting the continuous-discrete equivalence. To this end, we denote the operator $\mathcal{G} : \mathcal{B}_w(D) \rightarrow \mathcal{B}_w(D)$ as a *convolutional neural operator* (CNO) which we define as a compositional mapping between functions (see also Figure 1) as

$$\mathcal{G} : u \mapsto P(u) = v_0 \mapsto v_1 \mapsto \dots v_L \mapsto Q(v_L) = \bar{u}, \quad (2.3)$$

where

$$v_{l+1} = \mathcal{P}_l \circ \Sigma_l \circ \mathcal{K}_l(v_l), \quad 1 \leq l \leq L-1. \quad (2.4)$$

From (2.3), we see that first, the input function $u \in \mathcal{B}_w(D)$ is lifted to the latent space of bandlimited functions through a *lifting layer*:

$$P : \left\{ u \in \mathcal{B}_w(D, \mathbb{R}^{d_x}) \right\} \rightarrow \left\{ v_0 \in \mathcal{B}_w(D, \mathbb{R}^{d_0}) \right\}.$$

Here, $d_0 > d_x$ is the number of channels in the lifted, latent space. The lifting operation is performed by a convolution operator which will be defined below.

Then, the lifted function is processed through the composition of a series of mappings between functions (layers), with each layer consisting of three elementary mappings, i.e., \mathcal{P}_l is either the *upsampling* or *downsampling* operator, \mathcal{K}_l is the convolution operator and Σ_l is the activation operator. These elementary operators are defined below and are inspired by the modifications of CNNs for image generation in [21]. Finally, the last output function in the iterative procedure v_L is projected to the output space with a *projection operator* Q , defined as

$$Q : \left\{ v_L \in \mathcal{B}_w(D, \mathbb{R}^{d_L}) \right\} \rightarrow \left\{ \bar{u} \in \mathcal{B}_w(D, \mathbb{R}^{d_y}) \right\}.$$

The projection operation is also performed by a convolution operator defined below.

Convolution Operator. For simplicity of exposition, we will present the *single-channel* version of the convolution operator \mathcal{K}_l here. See SM A.3 for the *multi-channel* version for this and other operators considered below. Convolution operations are performed with discrete kernels

$$K_w = \sum_{i,j=1}^k k_{ij} \cdot \delta_{z_{ij}}$$

defined on the $s \times s$ uniform grid on D with grid size $\leq 1/2w$, in-order to satisfy the requirements of the Whittaker-Shannon-Kotelnikov sampling theorem [57], and z_{ij} being the resulting grid points, $k \in \mathbb{N}$ being the kernel size and δ_x denoting the Dirac measure at point $x \in D$. The convolution operator for a *single-channel* $\mathcal{K}_w : \mathcal{B}_w(D) \rightarrow \mathcal{B}_w(D)$ is defined by

$$\mathcal{K}_w f(x) = (K_w \star f)(x) = \int_D K_w(x-y) f(y) dy = \sum_{i,j=1}^k k_{ij} f(x - z_{ij}), \quad \forall x \in D,$$

where the last identity arises from the fact that $f \in \mathcal{B}_w$. Thus, our convolution operator is directly parametrized in physical space, in contrast to the Fourier space parametrization of a convolution in the FNO architecture of [32]. Hence, our parametrization is of a *local* nature.

Upsampling and Downsampling Operators. For some $\bar{w} > w$, we can *upsample* a function $f \in \mathcal{B}_w$ to the *higher band* $\mathcal{B}_{\bar{w}}$ by simply setting,

$$\mathcal{U}_{w,\bar{w}} : \mathcal{B}_w(D) \rightarrow \mathcal{B}_{\bar{w}}(D), \quad \mathcal{U}_{w,\bar{w}} f(x) = f(x), \quad \forall x \in D.$$

On the other hand, for some $\underline{w} < w$, we can *downsample* a function $f \in \mathcal{B}_w$ to the *lower band* $\mathcal{B}_{\underline{w}}$ by setting $\mathcal{D}_{w,\underline{w}} : \mathcal{B}_w(D) \rightarrow \mathcal{B}_{\underline{w}}(D)$, defined by

$$\mathcal{D}_{w,\underline{w}} f(x) = \left(\frac{\underline{w}}{w}\right)^2 (h_{\underline{w}} \star f)(x) = \left(\frac{\underline{w}}{w}\right)^2 \int_D h_{\underline{w}}(x-y) f(y) dy, \quad \forall x \in D,$$

where \star is the convolution operation on functions defined above and $h_{\underline{w}}$ is the so-called *interpolation sinc filter*:

$$h_w(x_0, x_1) = \text{sinc}(2wx_0) \cdot \text{sinc}(2wx_1), \quad (x_0, x_1) \in \mathbb{R}^2. \quad (2.5)$$

Activation Layer. Naively, one can apply the activation function pointwise to any function. However, it is well-known that such an application will no longer respect the band-limits of the underlying function space and generate *aliasing errors* [21, 14, 2]. In particular, nonlinear activations can generate features at arbitrarily high frequencies. As our aim is to respect the underlying CDE, we will modulate the application of the activation function so that the resulting outputs fall within desired band limits. To this end, we first upsample the input function $f \in \mathcal{B}_w$ to a higher bandlimit $\bar{w} > w$, then apply the activation and finally downsample the result back to the original bandlimit w (See Figure 1). Implicitly assuming that \bar{w} is large enough such that $\sigma(\mathcal{B}_w) \subset \mathcal{B}_{\bar{w}}$, we define the activation layer in (2.3) as,

$$\Sigma_{w,\bar{w}} : \mathcal{B}_w(D) \rightarrow \mathcal{B}_w(D), \quad \Sigma_{w,\bar{w}} f(x) = \mathcal{D}_{\bar{w},w}(\sigma \circ \mathcal{U}_{w,\bar{w}} f)(x), \quad \forall x \in D. \quad (2.6)$$

Instantiation through an Operator U-Net architecture. The above ingredients are assembled together in the form of an Operator U-Net architecture that has bandlimited functions as inputs and outputs. In addition to the blocks that have been defined above, we also need additional ingredients, namely incorporate *skip connections* through *ResNet* blocks of the form, $\mathcal{R}_{w,\bar{w}} : \mathcal{B}_w(D, \mathbb{R}^d) \rightarrow \mathcal{B}_w(D, \mathbb{R}^d)$ such that

$$\mathcal{R}_{w,\bar{w}}(v) = v + (\Sigma_{w,\bar{w}} \circ \mathcal{K}_w)^r v, \quad \forall v \in \mathcal{B}_w(D, \mathbb{R}^d), \quad (2.7)$$

where $r \in \mathbb{N}$ and d is the number of channels. In most cases, we set $r \in \{1, 2\}$. We also need the so-called *Invariant blocks* of the form, $\mathcal{I}_{w,\bar{w}} : \mathcal{B}_w(D, \mathbb{R}^d) \rightarrow \mathcal{B}_w(D, \mathbb{R}^d)$ such that

$$\mathcal{I}_{w,\bar{w}}(v) = \Sigma_{w,\bar{w}} \circ \mathcal{K}_w v, \quad \forall v \in \mathcal{B}_w(D, \mathbb{R}^d). \quad (2.8)$$

Finally, all these ingredients are assembled together in a modified Operator U-Net architecture which is graphically depicted in Figure 1. As seen from this figure, the input function, say $u \in \mathcal{B}_w(D, \mathbb{R}^{d_x})$ is first lifted and then processed through a series of layers. Four types of blocks are used i.e., downsampling (D) block corresponding to using the downsampling operator \mathcal{D} as the \mathcal{P} in (2.4), upsampling (U) block corresponding to using the upsampling operator \mathcal{U} as the \mathcal{P} in (2.4), ResNet (R) block corresponding to (2.7) and Invariant (I) block corresponding to (2.8). Each block takes a band-limited function as input and returns another band-limited function (with the same band) as the output. Finally, U-Net style patching operators, which concatenate outputs for different layers as additional channels are also used. As these operations act only in the channel width and leave the spatial resolution unchanged, they conform to the underlying bandlimits. Thus, CNO takes a function input and passes it through a set of encoders, where the input is downsampled in space but expanded in channel width and then processed through a set of decoders, where the channel width is reduced but the space resolution is increased. At the same time, encoder and decoder layers (at the same spatial resolution or band limit) are connected through additional ResNet blocks. Thus, this architectural choice allows for transferring high frequency content via the skip connections, before filtering them out with the *sinc* filter as we go deeper into the Encoder. Hence, the high frequency content is not just recreated with the activation function, but also modified through the intermediate networks. Consequently, we build a genuinely *multiscale operator learning architecture*.

Continuous-Discrete Equivalence for CNO. We have defined CNO (2.3) as an operator that maps bandlimited functions to bandlimited functions. In practice, like any computational algorithm, CNO has to be implemented in a discrete manner, with *discretized versions* of each of the above-defined elementary operations being specified in SM A.4. Given how each of the elementary blocks (convolution, up- and downsampling, activation, ResNets etc) are constructed, we prove the following proposition (in SM A.5):

Proposition 2.1. *Convolutional Neural Operator $\mathcal{G} : \mathcal{B}_w(D, \mathbb{R}^{d_x}) \rightarrow \mathcal{B}_w(D, \mathbb{R}^{d_y})$ (2.3) is a Representation equivalent neural operator or ReNO, in the sense of [2], Definition 3.4 and Remark 3.5.*

Hence, we establish correspondence between the definition of CNO at the continuum level and its realization in terms of computations at the discrete level. In particular, following [2], this implies that CNO satisfies a form of *resolution invariance*, allowing it to be evaluated on multiple resolutions.

3 Universal Approximation by CNOs.

We want to prove that a large class of operators, stemming from PDEs, can be approximated to desired accuracy by CNOs. To this end, we consider the following abstract PDE in the domain $D = \mathbb{T}^2$,

$$\mathcal{L}(u) = 0, \quad \mathcal{B}(u) = 0, \quad (3.1)$$

with \mathcal{L} being a differential operator and \mathcal{B} a boundary operator. We assume that the differential operator \mathcal{L} only depends on the coordinate x through a *coefficient* function $a \in H^r(D)$. The corresponding *solution* operator is denoted by $\mathcal{G}^\dagger : \mathcal{X}^* \subset H^r(D) \rightarrow H^r(D) : a \mapsto u$, with u being the solution of the PDE (3.1). We assume that \mathcal{G}^\dagger is continuous. Moreover, we also assume the following modulus of continuity,

$$\left\| \mathcal{G}^\dagger(a) - \mathcal{G}^\dagger(a') \right\|_{L^p(\mathbb{T}^2)} \leq \omega \left(\|a - a'\|_{H^\sigma(\mathbb{T}^2)} \right), \quad (3.2)$$

for some $p \in \{2, \infty\}$ and $0 \leq \sigma \leq r - 1$, and where $\omega : [0, \infty) \rightarrow [0, \infty)$ is a monotonously increasing function with $\lim_{y \rightarrow 0} \omega(y) = 0$. (3.2) is automatically satisfied if \mathcal{X}^* is compact and \mathcal{G}^\dagger is continuous. Under these assumptions, we have the following *universality theorem* for CNOs (2.3),

Theorem 3.1. *Let $\sigma \in \mathbb{N}_0$ and $p \in \{2, \infty\}$ as in (3.2), $r > \max\{\sigma, 2/p\}$ and $B > 0$. For any $\varepsilon > 0$ and any operator \mathcal{G}^\dagger , as defined above, there exists a CNO \mathcal{G} such that for every $a \in \mathcal{X}^*$ with $\|a\|_{H^r(D)} \leq B$ it holds,*

$$\|\mathcal{G}^\dagger(a) - \mathcal{G}(a)\|_{L^p(D)} < \varepsilon. \quad (3.3)$$

In fact, we will prove a more general version of this theorem in **SM B**, where we also include additional source terms in the PDE (3.1).

4 Experiments.

Training Details and Baselines. We provide a detailed description of the implementation of CNO and the training (and test) protocol for CNO as well as all the baselines in **SM C.1**. To ensure a *level playing field* among all the tested models for each benchmark, we follow an *ensemble training procedure* by specifying a range for the underlying hyperparameters *for each model* and randomly selecting a subset of the hyperparameter space. For each such hyperparameter configuration, the corresponding models are trained on the benchmark and the configuration with smallest validation error is selected and the resulting test errors are

reported, allowing us to identify and compare the *best performing* version of each model for every benchmark. We compare CNO with the following baselines: two very popular operator learning architectures, namely DeepONet [38] and FNO [32], a feedforward neural network (FFNN) with residual connections [18] and the U-Net [49] architecture, which is widely used in image processing.

Table 1: Relative median L^1 test errors, for both in- and out-of-distribution testing, for different benchmarks and models.

	In/Out	FFNN	UNet	DeepONet	FNO	CNO
Poisson Equation	In	5.74%	0.71%	12.92%	4.78%	0.23%
	Out	5.35%	1.27%	9.15%	8.89%	0.27%
Wave Equation	In	2.51%	1.51%	2.26%	1.10%	0.83%
	Out	3.01%	2.03%	2.83%	1.6%	1.48%
Smooth Transport	In	7.09%	0.49%	1.14%	0.40%	0.30%
	Out	650.57%	1.28%	157.22%	13.83%	0.45%
Discontinuous Transport	In	13.0%	1.31%	5.78%	1.29%	1.17%
	Out	257.3%	1.35%	117.11%	8.78%	1.61%
Allen-Cahn	In	18.27%	0.82%	13.63%	0.57%	0.83%
	Out	46.93%	2.18%	19.86%	2.36%	3.67%
Navier-Stokes	In	8.05%	3.54%	11.64%	3.93%	3.01%
	Out	16.12%	10.93%	15.05%	13.45%	7.06%
Compressible Euler	In	0.78%	0.38%	1.93%	0.47%	0.35%
	Out	1.34%	0.76%	2.88%	0.85%	0.62%

Representative PDE Benchmarks (RPB). Given the lack of consensus on a *standard* set of benchmarks for machine learning of PDEs, we propose a *new suite of benchmarks* here. Our aims in this regard are to ensure i) sufficient diversity among the types of PDE considered, ii) access to training and test data is readily available for rapid prototyping and reproducibility and iii) *intrinsic computational complexity* of problem to make sure that it is worthwhile to design fast surrogates to classical PDE solvers for a particular problem. In other words, we will only consider PDEs where classical PDE solvers can *only* resolve the underlying operator on fine enough grids. To meet these requirements, we will not consider PDEs in one space dimension as traditional numerical methods are already quite fast for them. On the other hand, it is hard to obtain data for problems in three dimensions, due to computational expense of traditional methods. The *sweet spot* is achieved by considering PDEs in two space dimensions. We further restrict to Cartesian domains here as all models can be readily evaluated in this setting. In addition to including a diverse set of PDEs, we only consider problems with sufficiently *many spatial and temporal scales*. Otherwise, traditional numerical solvers can approximate the underlying PDE on very coarse grids and it is not worthwhile to design surrogates (see

SM C.3.7 for a discussion in this context on a widely used Navier-Stokes benchmark). With these considerations in mind, we present the following set of *Representative PDE Benchmarks* or **RPB**,

Poisson Equation. This prototypical *linear elliptic PDE* is given by,

$$-\Delta u = f, \text{ in } D, \quad u|_{\partial D} = 0. \quad (4.1)$$

The solution operator $\mathcal{G}^\dagger : f \mapsto u$, maps the source term f to the solution u . With source term,

$$f(x, y) = \frac{\pi}{K^2} \sum_{i,j=1}^K a_{ij} \cdot (i^2 + j^2)^{-r} \sin(\pi i x) \sin(\pi j y), \quad \forall (x, y) \in D, \quad (4.2)$$

with $r = -0.5$, the corresponding exact solution can be analytically computed (see **SM C.3.1**) and represents K - spatial scales. For training the models, we fix $K = 16$ in (4.2) and choose a_{ij} to be i.i.d. uniformly distributed from $[-1, 1]$ (See **SM D** for a representation of the inputs and outputs of \mathcal{G}^\dagger). This *multiscale solution* needs fine enough grid size to be approximated accurately by finite element methods, fitting our complexity criterion for benchmarks. In addition to *in-distribution* testing, we also consider an *out-of-distribution* testing task by setting $K = 20$ in (4.2). This will enable us to evaluate the ability of the models to *generalize* to inputs (and outputs) with frequencies higher than those encountered during training.

Wave Equation. This prototypical *linear hyperbolic PDE* is given by

$$u_{tt} - c^2 \Delta u = 0, \text{ in } D \times (0, T), \quad u_0(x, y) = f(x, y), \quad (4.3)$$

with a constant propagation speed $c = 0.1$. The underlying operator $\mathcal{G}^\dagger : f \mapsto u(., T)$ maps the initial condition f into the solution at the final time. If we consider initial conditions to be given by (4.2) with $r = 1$, then one can explicitly compute the exact solution (see **SM C.3.2**) to represent a *multiscale standing wave* with periodic pulsations (depending on K) in time. The training and *in-distribution* test samples are generated by setting $T = 5$, $K = 24$ and a_{ij} to be i.i.d. uniformly distributed from $[-1, 1]$ (See **SM D** for input and output samples). For *out-of-distribution* testing, we change the exponent of decay of the modes in (4.2) to $r = 0.85$ and $K = 32$, in order to test the ability of the models to generalize to learn the effect of higher frequencies, than those present in the training data.

Transport Equation. The transport of scalar quantities of interest is modeled by PDE,

$$u_t + v \cdot \nabla u = 0, \quad u(t = 0) = f, \quad (4.4)$$

with a given velocity field and initial data f . The underlying operator $\mathcal{G}^\dagger : f \mapsto u(., T = 1)$ maps the initial condition f into the solution at the final time. We set a constant velocity field $v = (v_x, v_y) = (0.2, 0.2)$ leading to solution $u(x, y, t) = f(x - v_x t, y - v_y t)$. Two different

types of training data are considered, i.e., *smooth* initial data which takes the form of a radially symmetric Gaussian, with centers randomly and uniformly drawn from $(0.2, 0.4)^2$ and corresponding variance drawn uniformly from $(0.003, 0.009)$ and a *discontinuous* initial data in the form of the indicator function of radial disk with centers, uniformly drawn from $(0.2, 0.4)^2$ and radii uniformly drawn from $(0.1, 0.2)$ (See **SM C.3.3** for details and **SM D** for illustrations). For *out-of-distribution* testing in the smooth case, the centers of the Gaussian inputs are sampled uniformly from $(0.4, 0.6)^2$ and in the discontinuous case, the centers of the disk are drawn uniformly from $(0.4, 0.6)^2$, while keeping the variance and the radii, respectively, the same as that of *in-distribution testing*. This *out-of-distribution* task tests the model's ability to cope with input translation-equivariance.

Allen-Cahn Equation. It is a prototype for *nonlinear parabolic PDEs*,

$$u_t = \Delta u - \varepsilon^2 u(u^2 - 1), \quad (4.5)$$

with a reaction rate of $\varepsilon = 220$ and underlying operator $\mathcal{G}^\dagger : f \mapsto u(., T)$, mapping initial conditions f to the solution u at a final time $T = 0.0002$. The initial conditions for training and *in-distribution* testing are of the form (4.2), with $r = 1$ and $K = 24$ and coefficients a_{ij} drawn uniformly from $[-1, 1]$. For *out-of-distribution* testing, we set $K = 16$ and randomly select the initial decay r , uniformly from the range $[0.85, 1.15]$ of the modes in (4.2), which allows us to test the ability of the model to generalize to different dynamics of the system. Both training and test data are generated by using a finite difference scheme [63] on a grid at 64^2 resolution (see **SM D** for illustrations).

Navier-Stokes Eqns. These PDEs model the motion of incompressible fluids by,

$$u_t + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \quad \text{div } u = 0, \quad (4.6)$$

in the torus $D = \mathbb{T}^2$ with periodic boundary conditions and viscosity $\nu = 4 \times 10^{-4}$, only applied to high-enough Fourier modes (those with amplitude ≥ 12) to model fluid flow at *very high Reynolds-number*. The solution operator $\mathcal{G}^\dagger : f \mapsto u(., T)$, maps the initial conditions $f : D \rightarrow \mathbb{R}^2$ to the solution at final time $T = 1$. We consider initial conditions representing the well-known *thin shear layer* problem [3, 26] (See **SM C.3.5** for details), where the shear layer evolves via vortex shedding to a complex distribution of vortices (see **SM D** for samples). The training and *in-distribution testing* samples are generated, with a spectral viscosity method [26], from an initial sinusoidal perturbation of the shear layer [26], with layer thickness $\rho = 0.1$ and 10 perturbation modes, each sampled uniformly from $[-1, 1]$. For *out-of-distribution* testing, the layer thickness is reduced to $\rho = 0.09$ and the layers are shifted up in the domain to test the ability of the models to *generalize* to a flow regime with an increased number and different locations of the shed vortices.

Flow past airfoils. We model this flow by the compressible Euler equations,

$$u_t + \text{div } F(u) = 0, \quad u = [\rho, \rho v, E]^\perp, \quad F = [\rho v, \rho v \otimes v + p\mathbf{I}, (E + p)v]^\perp, \quad (4.7)$$

with density ρ , velocity v , pressure p and total Energy E related by an ideal gas equation of state. The airfoils we consider are described by perturbing the shape of a well-known RAE2822 airfoil [39] by Hicks-Henne Bump functions [41] (see SM C.3.6). Freestream boundary conditions are imposed and the solution operator maps the shape function onto the steady state density distribution (see SM D for samples) and training data are obtained with a compressible flow solver (NUWTUN) with shapes corresponding to 20 bump functions, with coefficients sampled uniformly from $[0, 1]$. *Out-of-distribution* testing is performed 30 bump functions.

Results. The test errors, for both *in-distribution* and *out-of-distribution* testing for all the models on the **RPB** benchmarks are shown in Table 1. Starting with the *in-distribution* results, we see that among the baselines, FNO clearly outperforms both FFNN and DeepONet on all the **RPB** benchmarks. On the other hand, the convolution-based U-Net model is quite competitive vis-a-vis FNO, with comparable performances on most benchmarks, while outperforming FNO by a factor of 7 for the Poisson test case. This already indicates that convolution-based architectures can perform very well. Moreover, we observe from Table 1 that CNO is the best performing architecture on every task except Allen-Cahn. It readily outperforms FNO, for instance by almost a factor of 20 on the Poisson test case but more moderately but significantly on other tasks. It also outperforms U-Net on all tasks for *in-distribution* testing. This trend is further reinforced when we consider *out-of-distribution* testing. CNO generalizes well to unseen data, with test errors increasing by approximately a factor of 2, at most (with Allen-Cahn being an outlier) and still outperforms the baselines significantly in all cases other than Allen-Cahn and discontinuous transport, where U-Net generalizes the best. FNO shows decent generalization for most problems but generalizes very poorly on the transport problems. This can be attributed to its lack of translation equivariance, in contrast to U-Net and CNO. Moreover, all models, with the possible exception of U-Net, generalize relatively poorly for the Allen-Cahn equation as the solutions are very sensitive to perturbations of initial data.

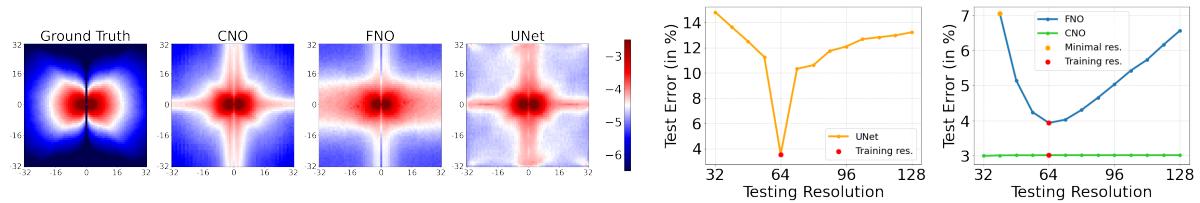


Figure 2: Thin Shear Layer Left: Averaged *logarithmic* amplitude spectra comparing Ground Truth, CNO, FNO and UNet. Right: Test error vs. Resolution for UNet, FNO and CNO.

A detailed comparison of the models, especially CNO, U-Net and FNO is presented in SM C. However, to provide some insight into the relative performance of these three models, we highlight differences between them for the Navier-Stokes test case here. To this end, we start with Figure 2 (left), where we present the averaged (log) spectra for the ground truth (reference solution computed with the spectral viscosity method) and those computed with CNO, FNO and U-Net. We observe from this figure that i) the spectrum of the exact solution is very rich with

representations of many frequencies, attesting to the *multi-scale* nature of the underlying problem and ii) there are significant differences in how CNO and FNO approximate the underlying solution in Fourier space. In particular, the decay in CNO’s spectrum is more accurate. On the other hand, the FNO spectrum is amplified along the horizontal axis, possibly on account of *aliasing errors* that add incorrect frequency content. The U-Net spectra are similar to that of CNO but with high-frequency modes being amplified, which leads to a higher test error. Next, in Figure 2 (right), we compare CNO, FNO and U-Net vis-a-vis the metric of *how the test error varies across resolutions*, see [SM C.4](#) for details, which is an important aspect for robust operator learning that been highlighted in [32, 24], see also [2] for a discussion with respect to representation equivalent neural operators or ReNOs. We find from Figure 2 (right) that for the Navier-Stokes benchmark, the FNO error is not *invariant* with respect to resolution, with an increase in error of around 70% on both grids at lower- as well as at higher-resolution than the training resolution of 64². Similarly, the increase of U-Net test error with respect to varying resolutions is even more pronounced, with a maximum increase of a factor of 3, indicating neither FNO nor U-Net are resolution (representation) equivalent in this case. In contrast, CNO error is invariant with respect to test resolution, verifying that it respects *continuous-discrete equivalence*. Further ablation studies for CNO are presented in [SM C.5](#).

5 Discussion.

Summary. We propose CNO, a *novel* convolution-based architecture for learning operators. The basic design principle was to enforce a form of *continuous-discrete equivalence* in order to genuinely learn the underlying operators, rather than discrete representations of them. To this end, we modified the elementary operators of convolution, up-and downsampling and particularly nonlinear activations to realize CNO as a representation equivalent neural operator or ReNO in the sense of [2]. Consequently, we show that there is an equivalence between continuous and discrete operations inside CNO. We also prove a universality theorem to show that CNO can approximate a large class of operators arising in PDEs to desired accuracy. A novel suite of experiments, termed as representative PDE benchmarks (**RPB**) is also proposed. It encompasses a wide variety of PDEs, with multiple scales in the corresponding solutions, which are hard to resolve with traditional numerical methods. The experiments demonstrate that CNO outperforms the baselines, including FNO, significantly on most benchmarks. This also holds for the considered *out-of-distribution* testing tasks which ascertain the ability of the models to *generalize* to unseen data. Thus, we present a novel CNN-based and representation equivalent neural operator for robust and accurate operator learning.

Comparison to Related Work. We emphasize that our construction of CNO follows the theoretical prescription of recent paper [2] on enforcing structure preserving *continuous-discrete equivalence*. CNO is a *representation equivalent neural operator*, with respect to spaces of bandlimited functions, in the sense of [2]. Another motivating work for us is [21], see also [59], where the authors modify CNNs to eliminate (or reduce) aliasing errors in the context of image

generation. We adapt the construction of [21] to our setting. In contrast to [21], we rewrite the operators in a continuous manner to realize CNO as a neural operator. Also, we deploy the architecture in a very different context from that of [21], namely that of operator learning for PDEs rather than image generation. In addition to the corresponding theoretical results (universality) and numerical experiments, we also instantiate CNO with a very different operator UNet architecture than that proposed in [21]. We would also like to mention related work on using CNNs for solving PDEs such as [1, 16] and emphasize that in contrast to CNO, they lack suitable notions of continuous-discrete equivalence. Finally comparing CNO to the widely used FNO model, we observe that unlike FNO which can fail to enforce CDE (see [14, 2] and Figure 2(right)), CNO preserves continuous-discrete equivalence. Moreover, the convolution operator in CNO is local in space, in contrast to convolution in Fourier space for FNO. See **SM C**) for detailed comparison between FNO and CNO.

Limitations and Future Work. We have presented CNO for operators on an underlying two-dimensional Cartesian domain. The extension to three-space dimensions is conceptually straightforward but computationally demanding. Similarly, extending to non-Cartesian domains will require some form of transformation maps between domains, for instance reworking those suggested for FNO in [31, 56] can be readily considered. Moreover, CNO, as presented here, maps between spaces of bandlimited functions, which suffice for universal approximation of PDEs considered herein. Extending CNO to other small enough spaces, where CDE can be realized, for instance those suggested in [2] is certainly worth considering. Adapting CNO to approximate trajectories (in time) of time-dependent PDEs, for instance by employing it in an *auto-regressive* manner, is another possible extension of this paper. At the level of theoretical results, we believe that the generic framework of [11] can be adapted to show that not only does CNO approximate a large class of PDEs universally, it does so without incurring any curse of dimensionality, as shown for DeepONets in [25] and FNOs in [23]. Finally, adapting and testing CNO for learning operators, beyond the forward solution operator of PDEs is also interesting. One such direction lies in efficiently approximating *PDE inverse problems*, for instance those considered in [43]. Finally, we would like to emphasize that CNO is a framework and naturally allows us for the straightforward adaptation of innovative CNN architectures in other fields to the context of operator learning.

Supplementary Material for:

Convolutional Neural Operators for Robust and Accurate Learning of PDEs.

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A Technical Details for Section 2 of main text.

A.1 Approximation of Operators mapping between Sobolev spaces by operators mapping between spaces of bandlimited functions.

We prove that one can approximate any continuous operator $\mathcal{G}^\dagger : \mathcal{X} \rightarrow \mathcal{Y}$ (as introduced in Section 2) of the main text by an operator mapping between spaces of bandlimited functions to arbitrary accuracy. We obtain this result by discarding the high-frequency components, e.g. higher than frequency w , of both the input and output of \mathcal{G}^\dagger . This can be performed by a Fourier projection P_w . For orthogonal Fourier projections and also trigonometric polynomial interpolation [19, 23] the following result on the accuracy of the projection holds,

Lemma A.1. *Given $\sigma, r \in \mathbb{N}_0$ with $r > d/2$ and $r \geq \sigma$, and $f \in C^r(\mathbb{T}^d)$ it holds for every $w \in \mathbb{N}$ that,*

$$\|f - P_w(f)\|_{H^\sigma(\mathbb{T}^d)} \leq C(r, d) w^{-(r-\sigma)} \|f\|_{H^r(\mathbb{T}^d)}, \quad (\text{A.1})$$

for a constant $C(r, d) > 0$ that only depends on r and d .

Using this result, we show that by discarding the high frequencies of the input and output of \mathcal{G}^\dagger one can approximate \mathcal{G}^\dagger to arbitrary accuracy by choosing an appropriate frequency cutoff.

Lemma A.2. *For any $\varepsilon, B > 0$ there exist $w \in \mathbb{N}$ such that $\|\mathcal{G}^\dagger(a) - P_w \mathcal{G}^\dagger(P_w a)\|_{L^2(D)} \leq \varepsilon$ for all $a \in H^r(D)$ with $\|a\|_{H^r(D)} \leq B$.*

Proof. We follow [23] and use Lemma A.1 repeatedly together with the stability of \mathcal{G}^\dagger (2.1) to obtain,

$$\begin{aligned} \|\mathcal{G}^\dagger(a) - P_w \mathcal{G}^\dagger(P_w a)\|_{L^2} &\leq \|\mathcal{G}^\dagger(a) - P_w \mathcal{G}^\dagger(a)\|_{L^2} + \|P_w \mathcal{G}^\dagger(a) - P_w \mathcal{G}^\dagger(P_w a)\|_{L^2} \\ &\lesssim w^{-r} \|\mathcal{G}^\dagger(a)\|_{H^r} + \|\mathcal{G}^\dagger(a) - \mathcal{G}^\dagger(P_w a)\|_{L^2} \\ &\lesssim w^{-r} \|\mathcal{G}^\dagger\|_{op} \|a\|_{H^r} + \omega(\|a - P_w a\|_{H^\sigma}) \\ &\lesssim w^{-r} \|\mathcal{G}^\dagger\|_{op} \|a\|_{H^r} + \omega(C w^{-(r-\sigma)} \|a\|_{H^r}). \end{aligned} \quad (\text{A.2})$$

It follows immediately that for large enough w ,

$$\sup_{\|a\|_{H^r} \leq B} \|\mathcal{G}^\dagger(a) - P_w \mathcal{G}^\dagger(P_w a)\|_{L^2} \leq \varepsilon. \quad (\text{A.3})$$

This proves the statement of the lemma. \square

Given that both $P_w a \in \mathcal{B}_w(D)$ and $P_w \mathcal{G}^\dagger(P_w a) \in \mathcal{B}_w(D)$, a consequence of the above lemma is the existence of an operator $\mathcal{G}^* : \mathcal{B}_w(D) \rightarrow \mathcal{B}_w(D) : a \mapsto P_w \mathcal{G}^\dagger(a)$ that can approximate \mathcal{G}^\dagger arbitrarily well. It follows from the lemma and its proof that $\|\mathcal{G}^\dagger - \mathcal{G}^*\|_{op} \leq \varepsilon$, where the operators are considered as mappings from and to $\mathcal{B}_w(D) \cap H^r(D)$ equipped with the $H^r(D)$ -norm.

A.2 Continuous-Discrete Equivalence for Operator \mathcal{G}^* from Section 2.1

For every $w > 0$, we denote by $\mathcal{B}_w(\mathbb{R}^2)$ the space of multivariate bandlimited functions

$$\mathcal{B}_w(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : \text{supp } \widehat{f} \subseteq [-w, w]^2\},$$

where \widehat{f} denotes the Fourier transform on $L^1(\mathbb{R})$

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx, \quad \xi \in \mathbb{R},$$

which extends to $L^2(\mathbb{R})$ by a classical density argument. The set $\Psi_w = \{\text{sinc}(2wx_1 - m) \cdot \text{sinc}(2wx_2 - n)\}_{m,n \in \mathbb{Z}}$ constitutes an orthonormal basis for $\mathcal{B}_w(\mathbb{R}^2)$. The bounded operator

$$T_{\Psi_w} : \ell^2(\mathbb{Z}^2) \rightarrow \mathcal{B}_w(\mathbb{R}^2), \quad T_{\Psi_w}(c_{m,n}) = \sum_{m,n \in \mathbb{Z}} c_{m,n} \text{sinc}(2w \cdot -m) \cdot \text{sinc}(2w \cdot -n),$$

which reconstructs a function from its basis coefficients, is called *synthesis operator*, and its adjoint

$$T_{\Psi_w}^* : \mathcal{B}_w(\mathbb{R}^2) \rightarrow \ell^2(\mathbb{Z}^2), \quad T_{\Psi_w}^* f = \left\{ f\left(\frac{m}{2w}, \frac{n}{2w}\right) \right\}_{m,n \in \mathbb{Z}},$$

which extract basis coefficients from an underlying function, is called *analysis operator*. Every bandlimited function can be uniquely and stably recovered from its sampled values $\{f\left(\frac{m}{2w}, \frac{n}{2w}\right)\}_{m,n \in \mathbb{Z}}$ via the reconstruction formula

$$f(x_1, x_2) = T_{\Psi_w} T_{\Psi_w}^* f(x_1, x_2) = \sum_{m,n \in \mathbb{Z}} f\left(\frac{m}{2w}, \frac{n}{2w}\right) \text{sinc}(2wx_1 - m) \cdot \text{sinc}(2wx_2 - n), \quad (\text{A.4})$$

and we say that there is a *continuous-discrete equivalence (CDE)* between f and its samples $\{f\left(\frac{m}{2w}, \frac{n}{2w}\right)\}_{m,n \in \mathbb{Z}}$. More in general, every bandlimited function $f \in \mathcal{B}_w(\mathbb{R}^2)$ can be uniquely and stably recovered from its values $\{f(mT, nT)\}_{m,n \in \mathbb{Z}}$ if the *sampling rate* or reciprocal of grid size, $1/T$ is greater or equal than the *Nyquist rate* $2w$. This simply follows from the fact that $\mathcal{B}_w(\mathbb{R}^2) \subset \mathcal{B}_{w'}(\mathbb{R}^2)$ for every $w' > w$. On the contrary, reconstructing $f \in \mathcal{B}_w$ at a sampling rate below the Nyquist rate, i.e. $1/T < 2w$, results in a non-zero value for the *aliasing error function*:

$$\varepsilon(f) = f - T_{\Psi_{\frac{1}{2T}}} T_{\Psi_{\frac{1}{2T}}}^* f,$$

and the associated *aliasing error* $\|\varepsilon\|_2$ (cfr. Definition 2.1 in [2]).

Let \mathcal{G}^* be a (possibly) non-linear operator between band-limited spaces, i.e. $\mathcal{G}^* : \mathcal{B}_w(\mathbb{R}^2) \rightarrow \mathcal{B}_{w'}(\mathbb{R}^2)$, for some $w, w' > 0$. As argued in [2], the concepts of continuous-discrete equivalence (CDE) and aliasing error can be adapted to the operator \mathcal{G}^* . The continuous operator \mathcal{G}^* is uniquely determined by a map $\mathbf{g}_{\Psi_w, \Psi_{w'}} : \ell(\mathbb{Z}^2) \rightarrow \ell^2(\mathbb{Z}^2)$ if the aliasing error operator

$$\varepsilon = \mathcal{G}^* - T_{\Psi_{w'}} \circ \mathbf{g}_{\Psi_w, \Psi_{w'}} \circ T_{\Psi_w}^* \quad (\text{A.5})$$

is identically zero, and we say that \mathcal{G}^* and $\mathbf{g}_{\Psi_w, \Psi_{w'}}$ satisfy a continuous-discrete equivalence (cfr. Definition 3.1 in [2]). Equivalently, the diagram

$$\begin{array}{ccc} \mathcal{B}_w & \xrightarrow{\mathcal{G}^*} & \mathcal{B}_{w'}, \\ \downarrow T_{\Psi_w}^* & & \uparrow T_{\Psi_{w'}} \\ \ell^2(\mathbb{Z}^2) & \xrightarrow{\mathbf{g}_{\Psi_w, \Psi_{w'}}} & \ell^2(\mathbb{Z}^2) \end{array}$$

commutes, i.e. the black and the blue directed paths in the diagram lead to the same result. In this latter case, since $T_{\Psi_w}^* \circ T_{\Psi_w}$ is the identity operator from $\ell^2(\mathbb{Z}^2)$ onto itself, equation (A.5) forces the discretization $\mathbf{g}_{\Psi_w, \Psi_{w'}}$ to be defined as

$$\mathbf{g}_{\Psi_w, \Psi_{w'}} = T_{\Psi_{w'}}^* \circ \mathcal{G}^* \circ T_{\Psi_w}, \quad (\text{A.6})$$

i.e. the diagram

$$\begin{array}{ccc} \mathcal{B}_w & \xrightarrow{\mathcal{G}^*} & \mathcal{B}_{w'} \\ \uparrow T_{\Psi_w} & & \downarrow T_{\Psi_{w'}}^* \\ \ell^2(\mathbb{Z}^2) & \xrightarrow{\mathbf{g}_{\Psi_w, \Psi_{w'}}} & \ell^2(\mathbb{Z}^2) \end{array}$$

also commutes. In other words, once we fix the discrete representations associated to the input and output functions, there exists a unique way to define a discretization $\mathbf{g}_{\Psi_w, \Psi_{w'}}$ that is consistent with the continuous operator \mathcal{G}^* and this is given by (A.6). In practice, we may have access to different discrete representations of the input and output functions, e.g. point samples evaluated on different grids, which in the theory amounts to a change of reference systems in the function spaces. For instance, sampling a function $f \in \mathcal{B}_w$ on a finer grid $\left\{ \left(\frac{m}{2\bar{w}}, \frac{n}{2\bar{w}} \right) \right\}_{m,n \in \mathbb{Z}}$, $\bar{w} > w$, amounts to representing the function f with respect to the system $\Psi_{\bar{w}} = \{ \text{sinc}(2\bar{w}x_1 - m) \cdot \text{sinc}(2\bar{w}x_2 - n) \}_{m,n \in \mathbb{Z}}$, which constitutes an orthonormal basis for $\mathcal{B}_{\bar{w}} \supset \mathcal{B}_w$. Then, one can define the associated CDE discretization $\mathbf{g}_{\Psi_{\bar{w}}, \Psi_{\bar{w}'}}$ as in (A.6), and by equation (A.5), one readily obtains the change of basis formula

$$\mathbf{g}_{\Psi_{\bar{w}}, \Psi_{\bar{w}'}} = T_{\Psi_{\bar{w}'}}^* \circ T_{\Psi_{\bar{w}}} \circ \mathbf{g}_{\Psi_w, \Psi_{w'}} \circ T_{\Psi_w}^* \circ T_{\Psi_{\bar{w}}}, \quad (\text{A.7})$$

see also Remark 3.5 in [2] for a more general *change of frame* formula. Finally, all the above concepts generalize to every pair of frame sequences (Ψ, Φ) that span respectively the input and output function spaces, and we refer to [2] for a complete exposition. Appendix A.2 can be adapted to bandlimited periodic functions, i.e. periodic functions with a finite number of non-zero Fourier coefficients, with the Dirichlet kernel as a counterpart of the sinc function, see [58, Section 5.5.2] for further details.

A.3 Multi-channel versions of elementary operators for CNO (2.3)

In this section, we will define *multi-channel* versions of the elementary mappings which define CNO (2.3). Note that the single-channel versions were defined in the main text.

Convolution Operator. In the multi-channel settings, discrete kernels K_w are defined on the $d_{in} \times d_{out} \times s^2$ uniform grids on D , where d_{in} is the number of input channels and d_{out} is the number of output channels. Formally, the kernels are defined as

$$K_{w,cl} = \sum_{i,j=1}^k k_{ij,cl} \cdot \delta_{z_{ij}}.$$

where c is the channel index in the input space, while l is the channel index in the output space. Each pair of channels defines corresponding single-channel convolution operation $\mathcal{K}_{w,cl} : \mathcal{B}_w(D) \rightarrow \mathcal{B}_w(D)$. For $a \in \mathcal{B}_w(D, \mathbb{R}^{d_{in}})$, the multi-channel convolution operation \mathcal{K}_w is defined as

$$(\mathcal{K}_w a(x))_l = \sum_{c=1}^{d_{in}} \mathcal{K}_{w,cl} a_c(x), \quad l = 1 \dots d_{out}.$$

Upsampling and Downsampling Operators. To upsample a signal $a \in \mathcal{B}_w(D, \mathbb{R}^d)$ with d channels from the bandlimit $w > 0$ to the bandlimit $\bar{w} > w$, one should apply the single-channel upsampling operator $\mathcal{U}_{w,\bar{w}}$ to each individual channel of the input signal, independently. Formally, for $a \in \mathcal{B}_w(D, \mathbb{R}^d)$, the multi channel upsampling $\mathcal{U}_{w,\bar{w}} : \mathcal{B}_w(D, \mathbb{R}^d) \mapsto \mathcal{B}_{\bar{w}}(D, \mathbb{R}^d)$ is defined as

$$(\mathcal{U}_{w,\bar{w}} a(x))_c = a_c(x), \quad \forall x \in D, \quad c = 1 \dots d.$$

The downsampling operator of a signal $a \in \mathcal{B}_w(D, \mathbb{R}^d)$ from the bandlimit $w > 0$ to the bandlimit $\underline{w} > w$ is defined in a similar manner (independent applications of the single-channel downsampling operators).

Activation layer. The multi-channel version of the activation layer, namely $\Sigma_{w,\bar{w}} : \mathcal{B}_w(D, \mathbb{R}^d) \mapsto \mathcal{B}_w(D, \mathbb{R}^d)$, is realized by applying the single-channel activation layer to each of the d channels, independently.

A.4 Discrete operators for CNO

In this section, we will define the *discrete versions* of the elementary mappings in (2.3). Given a *discrete*, multi-channel signal $a_s \in \mathbb{R}^{s \times s \times d}$ on $s \times s \times d$ uniform grid, we will use the notation $a_s[i, j, c]$ to refer to the (i, j) -th coordinate of the c -th channel of the signal, where $i, j = 1 \dots s$ and $c = 1 \dots d$.

Convolution operator. Assume that instead of a continuous, single-channel signal $a \in \mathcal{B}_w(D)$, one has an access only to its sampled version $a_s \in \mathbb{R}^{s \times s}$ on $s \times s$ uniform grid on D . Assume that a_s is to be convolved with a *discrete* kernel $K_w \in \mathbb{R}^{k \times k}$ with $k = 2\hat{k} + 1$. Let $\hat{a}_s \in \mathbb{R}^{s+2\hat{k} \times s+2\hat{k}}$ be an extended version of a_s obtained by circular-padding or zero-padding of a_s . The discrete,

single-channel convolution $\mathcal{K}_s : \mathbb{R}^{s \times s} \mapsto \mathbb{R}^{s \times s}$ of the signal a_s and the kernel K_w is given by

$$\mathcal{K}_s(a_s) = (a_s \star K_w)[i, j] = \sum_{m,n=-\hat{k}}^{\hat{k}} K_w[m, n] \cdot \hat{a}_s[i - m, j - n], \quad i, j = 1 \dots s,$$

where indices of \hat{a}_s outside the range $1 \dots s$ correspond to the padded samples. By performing the convolution in a described way, we ensure that the input and the output signals have the same spatial dimension $s \times s$.

Let $a_s \in \mathbb{R}^{s \times s \times d_{in}}$ be a discrete, multi-channel signal and $K_w \in \mathbb{R}^{k \times k \times d_{in} \times d_{out}}$ a discrete kernel with $k = 2\hat{k} + 1$. The multi-channel convolution of a_s and K_w is defined by

$$(a_s \star K_w)[i, j, l] = \sum_{m,n=-\hat{k}}^{\hat{k}} \sum_{c=1}^{d_{in}} K_w[m, n, c, l] \cdot \hat{a}_s[i - m, j - n, c], \quad i, j = 1 \dots s,$$

where l corresponds to the index of the output channel and c to the index of the input channel.

Upsampling and Downsampling Operators. In this section, we will define the discrete upsampling and downsampling operators. For $w > 0$, let h_w be the interpolation *sinc* filter defined in 2.5. For a discrete, single-channel signal $a_s \in \mathbb{R}^{s \times s}$, let $(\tilde{a}_s[n])_{n \in \mathbb{Z}}$ be its periodic extension into infinite length. In other words, $\tilde{a}_s[n] = a_s[n \bmod s]$ for $n \in \mathbb{Z}$. The discrete upsampling $\mathcal{U}_{s,N} : \mathbb{R}^{s \times s} \rightarrow \mathbb{R}^{Ns \times Ns}$ by an *integer factor* $N \in \mathbb{N}$ of the signal $a_s \in \mathbb{R}^{s \times s}$ is done in *two phases*:

1. First step is to increase the number of samples of the signal a_s from s^2 to $(Ns)^2$. One transforms the signal a_s into the signal $a_{s,\uparrow Ns}$ obtained by separating each two signal samples of a_s with $N-1$ zero-valued samples. In other words, it holds that $a_{s,\uparrow Ns} \in \mathbb{R}^{Ns \times Ns}$ and

$$a_{s,\uparrow Ns}[i, j] = \mathbb{1}_S(i) \cdot \mathbb{1}_S(j) \cdot a_s[i \bmod s, j \bmod s], \quad i, j = 1 \dots Ns,$$

where $S = \{1, s+1, \dots, (N-1)s+1\}$ and $\mathbb{1}_S$ is the indicator function.

2. Second step is to convolve the periodic extension of $a_{s,\uparrow Ns}$ with the $h_{s/2}$ interpolation filter to eliminate high frequency components. The upsampled signal is formally obtained by

$$\mathcal{U}_{s,N}(a_s)[i, j] = \sum_{n,m \in \mathbb{Z}} \tilde{a}_{s,\uparrow Ns}[n, m] \cdot h_{s/2}(is - ns, js - ms), \quad i, j = 1 \dots Ns.$$

The discrete downsampling $\mathcal{D}_{s,N} : \mathbb{R}^{s \times s} \rightarrow \mathbb{R}^{s/N \times s/N}$ by an *integer factor* $N \in \mathbb{N}$ of the signal $a_s \in \mathbb{R}^{s \times s}$ is also done in *two phases* (under the assumption that $s/N \in \mathbb{N}$):

1. First step is to convolve the periodic extension of a_s with the $h_{s/(2N)}$ interpolation filter to eliminate high frequency content. Formally, the first step is defined by

$$a_{s,s/N}[i, j] = \sum_{n,m \in \mathbb{Z}} \tilde{a}_s[n, m] \cdot h_{s/(2N)}(is - ns, js - ms), \quad i, j = 1 \dots s/N.$$

2. Second step is to decrease the sampling rate of $a_{s,N/s}$ by keeping *every N-th* sample of the signal. The downsampled signal is formally defined by

$$\mathcal{D}_{s,N}(a_s)[i,j] = a_{s,N/s}[(i-1)s+1, (j-1)s+1], \quad i,j = 1 \dots s/N.$$

Multi-channel discrete upsampling and downsampling are performed by independent applications of the corresponding single-channel operators.

Since perfect filters h_w have infinite impulse response and cause ringing artifacts around high-gradient points (e.g. discontinuities) due the Gibbs phenomenon, one usually uses *windowed-sinc* filters in the implementation. We will describe these filters later in the text (see C.1.3)

Activation layer. Given the definitions of the discrete operators, the discrete, single-channel activation layer is defined as

$$\Sigma_s : \mathbb{R}^{s \times s} \rightarrow \mathbb{R}^{s \times s}, \quad \Sigma_s(a_s) = \mathcal{D}_{s,N} \circ \sigma \circ \mathcal{U}_{s,N}(a_s),$$

where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is an activation function applied point-wise and $N \in \mathbb{N}$ is a fixed constant. In our experiments, we noticed that $N = 2$ is sufficient for accurate predictions. The multi-channel activation layer is performed by independent applications of the single-channel activation layer.

A.5 Proof of Proposition 2.1 of Main Text

We use the same notation as in Section 2 and Appendix A.2. The layers of a convolutional neural operator (2.3) are given by,

$$v_{l+1} = \mathcal{P}_l \circ \Sigma_l \circ \mathcal{K}_l(v_l), \quad 0 \leq l \leq L-1, \tag{A.8}$$

Hence, they consist of three elementary mappings between spaces of bandlimited functions, i.e., \mathcal{K}_l is a convolution operator, Σ_l is a non-linear operator whose definition depends on the choice of an activation function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, and \mathcal{P}_l is a projection operator. We now show that CNO layers, whose discrete versions are outlined in the previous section, respect equation (A.6) and consequently CNOs are Representation equivalent Neural Operators (ReNOs) in the sense of [2, Definiton 3.4] and [2, Remark 3.5]. We recall that the convolutional operator appearing in (A.8) takes the form

$$\mathcal{K}_w f(x) = \sum_{m,n=-k}^k k_{m,n} f(x - z_{m,n}), \quad x \in \mathbb{R},$$

for some $w > 0$, where $k \in \mathbb{N}$, $k_{m,n} \in \mathbb{C}$ and $z_{m,n} = \left\{ \left(\frac{m}{2w}, \frac{n}{2w} \right) \right\}_{m,n \in \mathbb{Z}}$. By definition, \mathcal{K}_w is a well-defined operator from $\mathcal{B}_w(\mathbb{R}^2)$ into itself. Moreover, its discretized version is defined by the mapping

$$\left\{ f \left(\frac{m}{2w}, \frac{n}{2w} \right) \right\}_{m,n \in \mathbb{Z}} \mapsto \left\{ \mathcal{K}_w f \left(\frac{m}{2w}, \frac{n}{2w} \right) \right\}_{m,n \in \mathbb{Z}} = \left\{ \sum_{m',n'=-k}^k k_{m',n'} f(z_{m,n} - z_{m',n'}) \right\}_{m,n \in \mathbb{Z}},$$

and thus results in the commutative diagram

$$\begin{array}{ccc} \mathcal{B}_w & \xrightarrow{\mathcal{K}_w} & \mathcal{B}_w \\ T_{\Psi_w} \uparrow & & \downarrow T_{\Psi_w}^* \\ \ell^2(\mathbb{Z}^2) & \longrightarrow & \ell^2(\mathbb{Z}^2) \end{array}$$

Equivalently, the discretized version of \mathcal{K}_w is defined via (A.6), which was to be shown. In order to define the activation layer Σ_l , we first assume that the activation function $\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is such that for every $f \in \mathcal{B}_w(\mathbb{R}^2)$

$$\sigma(f) \in \mathcal{B}_{\bar{w}}(\mathbb{R}^2), \quad (\text{A.9})$$

for some $\bar{w} > w$. In fact, in Section 2 we assume that the pointwise activation can be approximated by an operator between bandlimited spaces and consequently (A.9) is satisfied up to negligible frequencies. Thus, the activation layer $\Sigma_{w,\bar{w}}: \mathcal{B}_w(\mathbb{R}^2) \rightarrow \mathcal{B}_w(\mathbb{R}^2)$ in (A.8) is defined by the composition

$$\Sigma_{w,\bar{w}} = P_{\mathcal{B}_w(\mathbb{R}^2)} \circ \sigma \circ P_{\mathcal{B}_{\bar{w}}(\mathbb{R}^2)}, \quad (\text{A.10})$$

where $P_{\mathcal{B}_w(\mathbb{R}^2)}: \mathcal{B}_{\bar{w}}(\mathbb{R}^2) \rightarrow \mathcal{B}_w(\mathbb{R}^2)$ denotes the orthogonal projection onto $\mathcal{B}_w(\mathbb{R}^2)$ and $P_{\mathcal{B}_{\bar{w}}(\mathbb{R}^2)}: \mathcal{B}_w(\mathbb{R}^2) \rightarrow \mathcal{B}_{\bar{w}}(\mathbb{R}^2)$ denotes the natural embedding of $\mathcal{B}_w(\mathbb{R}^2)$ into $\mathcal{B}_{\bar{w}}(\mathbb{R}^2)$. The discretized version of each mapping in (A.10) is defined in order to guarantee a continuous-discrete equivalence (CDE) between the continuous and discrete levels. More precisely, $P_{\mathcal{B}_w(\mathbb{R}^2)}$ and $P_{\mathcal{B}_{\bar{w}}(\mathbb{R}^2)}$ are discretized via (A.6) as

$$\mathcal{D}_{\bar{w},w} = T_{\Psi_w}^* \circ P_{\mathcal{B}_w(\mathbb{R}^2)} \circ T_{\Psi_{\bar{w}}}, \quad \mathcal{U}_{w,\bar{w}} = T_{\Psi_{\bar{w}}}^* \circ P_{\mathcal{B}_{\bar{w}}(\mathbb{R}^2)} \circ T_{\Psi_w},$$

which are respectively called downsampling and upsampling. Consequently, the discretized version of the activation layer is given by the composition

$$\mathcal{D}_{\bar{w},w} \circ \sigma \circ \mathcal{U}_{w,\bar{w}},$$

which yields the commutative diagram

$$\begin{array}{ccccccc} \mathcal{B}_w & \xrightarrow{P_{\mathcal{B}_{\bar{w}}(\mathbb{R}^2)}} & \mathcal{B}_{\bar{w}} & \xrightarrow{\sigma} & \mathcal{B}_{\bar{w}} & \xrightarrow{P_{\mathcal{B}_w(\mathbb{R}^2)}} & \mathcal{B}_w \\ T_{\Psi_w} \uparrow & & \downarrow T_{\Psi_{\bar{w}}}^* & & T_{\Psi_{\bar{w}}} \uparrow & & \downarrow T_{\Psi_w}^* \\ \ell^2(\mathbb{Z}^2) & \xrightarrow{\mathcal{U}_{w,\bar{w}}} & \ell^2(\mathbb{Z}^2) & \xrightarrow{\sigma} & \ell^2(\mathbb{Z}^2) & \xrightarrow{\mathcal{D}_{\bar{w},w}} & \ell^2(\mathbb{Z}^2) \end{array}$$

which we wanted to show. Finally, the activation layer might be followed by an additional projective operator, i.e., by a downsampling or an upsampling. Thus, this exact correspondence between its constituent continuous and discrete operators establishes CNO as an example of Representation equivalent neural operators or ReNOs in the sense of [2, Definiton 3.4] and [2, Remark 3.5], thus proving Proposition 2.1 of the main text. As in Appendix A.2, the above proofs can be readily adapted to bandlimited periodic functions, i.e. periodic functions with a finite number of non-zero Fourier coefficients.

B Proof of Theorem 3.1 of Main Text.

We present the proof of a generalization of the universality result of Theorem 3.1. The theorem in the main text only holds when the differential operator \mathcal{L} only depends on the coordinate x through a coefficient function $a \in H^r(D)$. Although all benchmark PDEs in Section 4 satisfy this requirement, there are other important PDEs that do not, such as the standard elliptic PDE $\nabla \cdot (a \nabla u) = f$. We therefore generalize this requirement in the following setting,

Setting B.1. *We set $D = \mathbb{T}^2$ and assume that the following is true,*

1. *\mathcal{L} only depends on the coordinate x through functions $a, f_1, \dots, f_\ell \in H^r(\mathbb{T}^2)$.*
2. *The solution of the PDE characterized by a and $f = (f_1, \dots, f_\ell)$ is given by a continuous operator $\tilde{\mathcal{G}} : \tilde{\mathcal{X}} \subset (H^r(\mathbb{T}^2))^{\ell+1} \rightarrow H^r(\mathbb{T}^2) : (a, f) \mapsto u$ or $u(T)$, depending on the PDE. The operator of interest \mathcal{G}^\dagger is a restriction of $\tilde{\mathcal{G}}$ for fixed f_1, \dots, f_ℓ i.e., $\mathcal{G}^\dagger : \mathcal{X}^* \subset H^r(\mathbb{T}^d) \rightarrow H^r(\mathbb{T}^d) : a \mapsto \tilde{\mathcal{G}}(a, f_1, \dots, f_\ell)$.*
3. *Similar to (2.1), it holds for all $(a, f), (a', f') \in \mathcal{X}^*$ it holds that*

$$\left\| \tilde{\mathcal{G}}(a, f) - \tilde{\mathcal{G}}(a', f') \right\|_{L^p(\mathbb{T}^2)} \leq \omega \left(\|a - a'\|_{H^\sigma(\mathbb{T}^2)} + \max_i \|f_i - f'_i\|_{H^\sigma(\mathbb{T}^2)} \right), \quad (\text{B.1})$$

for some $p \in \{2, \infty\}$ and $\sigma \in \mathbb{N}_0$ with $\sigma < r$. This is automatically satisfied if \mathcal{X}^* is compact and $\tilde{\mathcal{G}}$ is continuous [23].

4. *It holds that the activation function σ is at least r times continuously differentiable and not a polynomial. (See Remark B.4 for a generalization.)*

In addition, we will use the following notation in the proof.

- For $J \in \mathbb{N}$ we define for every $j \in \{0, \dots, J-1\}^2$ the grid $\mathbf{x}_j^J = (2\pi j_1/J, 2\pi j_2/J)$.
- We denote the Fourier basis by $\{\mathbf{e}_\mathbf{k}\}_{\mathbf{k} \in \mathbb{Z}^2}$, following the notation of [23]. For $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_d) \in \mathbb{Z}^d$, we let $\sigma(\mathbf{k})$ be the sign of the first non-zero component of \mathbf{k} and we define

$$\mathbf{e}_\mathbf{k} := C_\mathbf{k} \begin{cases} 1, & \sigma(\mathbf{k}) = 0, \\ \cos(\mathbf{k} \cdot \mathbf{x}), & \sigma(\mathbf{k}) = 1, \\ \sin(\mathbf{k} \cdot \mathbf{x}), & \sigma(\mathbf{k}) = -1, \end{cases} \quad (\text{B.2})$$

where the factor $C_\mathbf{k} > 0$ ensures that $\mathbf{e}_\mathbf{k}$ is properly normalized, i.e. that $\|\mathbf{e}_\mathbf{k}\|_{L^2(\mathbb{T}^d)} = 1$.

- For $N \in \mathbb{N}$ let P_N denote a trigonometric polynomial interpolation operator as in (B.16) in **SM B.1**.

Assuming Setting B.1 we can now prove the following theorem on the universality of CNOs. In the proof we will construct an operator $\mathcal{G} : H^r(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2)$, mapping between function spaces,

and we will therefore allow to apply the activation function to the continuous representation of the signal rather than an upsampled version. We then make the link to the discrete implementation of the CNO by considering an encoder \mathcal{E}_K that maps the input function a to the evaluation of a on a grid, enhanced by some Fourier features [54] in case $\ell > 0$ in Setting B.1.

Theorem B.2. *Let $\sigma \in \mathbb{N}_0$ and $p \in \{2, \infty\}$ as in (B.1), $r > \max\{\sigma, 2/p\}$ and $B > 0$. For any $\varepsilon > 0$ and any operator \mathcal{G}^\dagger satisfying Setting B.1, there exist $K, N \in \mathbb{N}_0$ and a CNO $\mathcal{G} : H^r(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2)$ such that for every $a \in \mathcal{X}^*$ with $\|a\|_{H^r(\mathbb{T}^2)} \leq B$ it holds,*

$$\left\| \mathcal{G}^\dagger(a) - \mathcal{G}(a) \right\|_{L^p(\mathbb{T}^2)} < \varepsilon. \quad (\text{B.3})$$

The CNO is implemented through an encoder

$$\mathcal{E}_K : H^r(\mathbb{T}^2) \rightarrow (\mathbb{R}^{N \times N})^{(K+1)^2} : a \mapsto (a(\mathbf{x}^N), (\cos(\mathbf{k} \cdot \mathbf{x}^N), \sin(\mathbf{k} \cdot \mathbf{x}^N))_{1 \leq \|\mathbf{k}\|_\infty \leq K}) \quad (\text{B.4})$$

and a single invariant block $\widehat{\Phi} : (\mathbb{R}^{N \times N})^{(K+1)^2} \rightarrow \mathbb{R}^{N \times N}$ such that $\mathcal{G}(a)(\mathbf{x}^N) = (\widehat{\Phi} \circ \mathcal{E}_K)(a)$. If $\ell = 0$ (see Setting B.1) then $K = 0$, meaning that no Fourier features are needed.

Proof. Let $M, N \in \mathbb{N}$ with $N/M \in \mathbb{N}$. We will construct a CNO with input $a(\mathbf{x}^N)$ and the Fourier features $\mathbf{e}_k(\mathbf{x}^N)$ for $\mathbf{k} \in \mathcal{K} := \{-M/2, -M/2 + 1, \dots, M/2\}^2 \setminus \{0, 0\}$, summarized in the tensor $(a(\mathbf{x}^N), \mathbf{e}^{M/2}(\mathbf{x}^N)) := (a(\mathbf{x}^N), (\mathbf{e}_k(\mathbf{x}^N))_{\mathbf{k} \in \mathcal{K}})$. In the proof, we will use the property that bandlimited functions can be represented by their function values on a fine enough grid. We will therefore first construct a continuous operator $\mathcal{G} : H^r(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2)$ that is a good approximation of \mathcal{G}^\dagger . In the second step, we will then prove that $\mathcal{G}(a)(\mathbf{x}^N)$ indeed corresponds to a CNO.

Step 1: construction of \mathcal{G} . First, since $a, f \in H^r(\mathbb{T}^2)$ we can use Lemma A.1 and assumption (B.1) on the stability of $\tilde{\mathcal{G}}$ to find that,

$$\left\| \tilde{\mathcal{G}}(a, f) - \tilde{\mathcal{G}}(P_M(a, f)) \right\|_{L^p(\mathbb{T}^2)} \leq \omega \left(C_{B,f} M^{-(r-\sigma)} \right). \quad (\text{B.5})$$

Next, we define for any $J \in \mathbb{N}$ the set

$$\mathcal{A}_J = \{ \mathbf{y} \in (\mathbb{R}^{J \times J})^{(M+1)^2} \mid \exists a \in H^r(\mathbb{T}^2) : \mathbf{y} = (a(\mathbf{x}^J), \mathbf{e}^{M/2}(\mathbf{x}^J)) \text{ and } \|a\|_{H^r(\mathbb{T}^2)} \leq B \}, \quad (\text{B.6})$$

and the map,

$$G : \mathcal{A}_M \subset (\mathbb{R}^{M \times M})^{(M+1)^2} \rightarrow \mathbb{R} : (a(\mathbf{x}^M), \mathbf{e}^{M/2}(\mathbf{x}^M)) \mapsto \tilde{\mathcal{G}}(P_M(a, f))(\mathbf{x}_{0,0}). \quad (\text{B.7})$$

The existence of the map G can be justified as follows. Let P_M denote a trigonometric polynomial interpolation operator as in (B.16) in **SM B.1**. By the Nyquist–Shannon sampling theorem and the Whittaker–Shannon interpolation formula there is a bijection between the discrete values $(a(\mathbf{x}^M), \mathbf{e}^{M/2}(\mathbf{x}^M))$ and $P_M a$ and $\mathbf{e}^{M/2}$, and therefore also $P_M a$ and $P_M f_i$ for all $1 \leq i \leq \ell$. Hence, the mapping G is equivalent to the mapping $P_M(a, f) \mapsto P_M u$, and therefore well-defined. The continuity of G follows from that of $\tilde{\mathcal{G}}$. By the universal approximation theorem (Theorem

[B.6](#)) there exists a shallow neural network Ψ such that $|\Psi(\mathbf{y}) - G(\mathbf{y})| < \varepsilon$ for all $\mathbf{y} \in \mathcal{A}_M$. Note that Ψ only provides an approximation in the point $\mathbf{x}_{0,0}$. We can expand Φ to the whole \mathbb{T}^2 by defining the operator Ψ^* as follows,

$$\Psi^* : \mathcal{X}^* \rightarrow C(\mathbb{T}^2) : a \mapsto \left[\mathbb{T}^2 \ni \mathbf{z} \mapsto \Psi \left(a(\mathbf{z} + \mathbf{x}^M), \mathbf{e}^{M/2}(\mathbf{z} + \mathbf{x}^M) \right) \right]. \quad (\text{B.8})$$

For the intuition of the reader: the extension from Ψ to Ψ^* is similar to the extension from the local stencil of a finite difference scheme to its corresponding global approximation. As a result, Ψ^* has the same accuracy as Ψ ,

$$\left\| \mathcal{G}^\dagger(P_M a) - \Psi^*(P_M a) \right\|_{C^0(\mathbb{T}^2)} < \varepsilon. \quad (\text{B.9})$$

We finalize our construction by projecting $\Psi^*(P_M a)$ on to the space of trigonometric polynomials. The accuracy of such a projection is given by [Lemma A.1](#),

$$\|(P_N - \text{Id})\Psi^*(a)\|_{L^p(\mathbb{T}^2)} \leq \|(P_N - \text{Id})\Psi^*(a)\|_{H^{1-2/p}(\mathbb{T}^2)} \leq CN^{-(r-2/p)} \|\Psi^*(a)\|_{H^r(\mathbb{T}^2)}, \quad (\text{B.10})$$

where we used that either $p = 2$ or $p = \infty$. It is important to note that $\|\Psi^*(a)\|_{H^r(\mathbb{T}^2)}$ is independent of N . We then define the operator \mathcal{G} as,

$$\mathcal{G}(a)(\mathbf{z}) = (P_N \circ \Psi^*)(P_M a)(\mathbf{z}). \quad (\text{B.11})$$

Finally, we can put all obtained estimates together to find,

$$\begin{aligned} & \left\| \mathcal{G}^\dagger(a) - \mathcal{G}(a) \right\|_{L^p(\mathbb{T}^2)} \\ & \leq \left\| \mathcal{G}^\dagger(a) - \mathcal{G}^\dagger(P_M a) \right\|_{L^p(\mathbb{T}^2)} + \left\| \mathcal{G}^\dagger(P_M a) - \Psi^*(P_M a) \right\|_{C^0(\mathbb{T}^2)} \\ & \quad + \left\| \Psi^*(P_M a) - \mathcal{G}(a) \right\|_{L^p(\mathbb{T}^2)} \\ & \leq \omega \left(C_{B,f} M^{-(r-\sigma)} \right) + \varepsilon + C_{B,M,\varepsilon} N^{-(r-2/p)}. \end{aligned} \quad (\text{B.12})$$

It then follows that one can make this upper bound arbitrarily small by choosing ε sufficiently small and M, N sufficiently large (in that order).

Step 2: \mathcal{G} corresponds to a CNO. We will now show that the operator \mathcal{G} is in agreement with our definition of a convolutional neural operator (CNO). To do so, we will use that trigonometric polynomials up to a certain degree can be exactly retrieved based on their values on a grid (see [SM B.1](#) and [\[19\]](#)).

First of all, given that $N/M \in \mathbb{N}$ we find that the functions $P_M a$ and $\mathbf{e}^{M/2}$ can be exactly recovered from their discrete values on the grid \mathbf{x}^N . We therefore will look for a CNO with input $\mathbf{y} = (a(\mathbf{x}^N), \mathbf{e}^{M/2}(\mathbf{x}^N)) \in \mathcal{A}_N$ for which the continuous representation of the output agrees with $\mathcal{G}(a)$.

A crucial next observation is that G is equivariant with respect to translations in space of the input (simultaneously across all channels). By [\[45, Theorem 1.1\]](#) there then exists a

shallow CNN Φ such that $\pi \circ \Phi = \Psi$, where π is the projection on the first coordinate $\pi : (\mathbb{R}^{N \times N})^{(M+1)^2} \rightarrow \mathbb{R}^{(M+1)^2} : X \mapsto (X_{1,1}^1, \dots, X_{1,1}^\ell)$ (as in [45]). For simplicity we will assume that the CNN is of the form $\Phi(\mathbf{y}) = K_2 * \sigma(K_1 * \mathbf{y})$, i.e. that it only has one channel at every layer. The proof of the general case is completely analogous, but much heavier on notation.

We then lift the convolution filter $K_1 \in \mathbb{R}^{M \times M}$ to the grid $\mathbb{R}^{N \times N}$ by using a stride of N/M and filling up the rest by zeroes. More rigorously, we consider the matrix $\hat{K}_1 := K_1 \otimes E$ with $E_{ij} = \delta_{i1}\delta_{j1}$. Similarly we define $\hat{K}_2 := K_2 \otimes E$. We can then define a new CNN $\hat{\Phi} : \mathcal{A}_N \rightarrow \mathbb{R}^{N \times N} : \mathbf{y} \mapsto \hat{K}_2 * \sigma(\hat{K}_1 * \mathbf{y})$. The output of $\hat{\Phi}$ then consists of approximations of $\mathcal{G}^\dagger(a)$ at $(N/M)^2$ different $M \times M$ subgrids of \mathbf{x}^N , i.e. all possible translations of \mathbf{x}^M within \mathbf{x}^N . More precisely, it holds that

$$\Psi^*(\mathbf{x}^N) = \hat{\Phi} \left(P_M a(\mathbf{x}^N), \mathbf{e}^{M/2}(\mathbf{x}^N) \right) \in \mathbb{R}^{N \times N}. \quad (\text{B.13})$$

Moreover, since the operator P_N only uses the values of Ψ^* on the grid \mathbf{x}^N it follows that applying P_N to the right-hand side of the above equation or applying an interpolation sinc filter with corresponding frequency gives the exact same result,

$$\mathcal{G}(a)(\mathbf{x}^N) = (P_N \circ \Psi^*)(P_M a)(\mathbf{x}^N) = h_N * \hat{\Phi} \left(P_M a(\mathbf{x}^N), \mathbf{e}^{M/2}(\mathbf{x}^N) \right). \quad (\text{B.14})$$

The right-hand side exactly corresponds to our definition of a CNO, thereby concluding the universality proof. \square

Remark B.3 (Alternative proof). *We stress that it is crucial in the proof that M can be chosen independently of N . A straightforward application of [45, Theorem 1.1] on the map G with $N = M$ would not lead to an accurate CNO approximation as the $\|\Psi^*(a)\|_{H^r(\mathbb{T}^2)}$ will depend on $N = M$ such that $N^{-(r-2/p)} \|\Psi^*(a)\|_{H^r(\mathbb{T}^2)}$ might not converge to zero. In addition, because of the used trick we obtain convolution kernels with a stride of N/M and therefore a sparse kernel. An alternative strategy could be to replace the universal approximation (Theorem B.6) by an approximation theorem that provides explicit control on the network size and upper bounds on the weights such as those in [10]. Other than a much more complicated proof, one will also need to put stronger regularity conditions on \mathcal{G}^\dagger .*

Remark B.4 (Polynomial and rational activations). *The CNO constructed in the above theorem is exactly equivariant. As suggested in [21], it can be sufficient in practice to break this perfect equivariance by applying the activation function σ to an upsampled discrete version of the signal rather than the continuous representation of the signal. We do the same in our implementation of CNO. Note that if one would use polynomial activation functions one could still recover exact equivariance by choosing a high enough upsampling rate. In this case the universality of CNOs can be proven by replacing the universal approximation theorem for neural networks (Theorem B.6) by the Weierstrass approximation theorem. The rest of the proof of Theorem B.2 remains unchanged. Similarly, one could consider using Padé and rational approximants as activation functions [55, 42, 12, 6]. The computation of a rational activation function $\sigma(x) = p(x)/q(x)$ can then be approximated by iteratively minimizing $\|p(x)\sigma(x) - q(x)\|_2^2$, following the idea of [61].*

Methods such as Newton-Raphson only involve multiplications and can therefore be completely applied in an alias-free way through proper upsampling before, and downsampling after each multiplication.

Remark B.5 (Physics-informed CNOs). *Physics-informed learning employs a PDE residual-based loss to circumvent the need for training data. Examples of such frameworks include physics-informed neural networks (PINNs) [48], physics-informed DeepONets [60] and physics-informed FNOs [35]. Using the continuous representation of the CNO output, one can use automatic differentiation to create a physics-informed CNO loss. In order to use the tools provided in [11, Theorem 3.9] to obtain a bound on the approximation error for physics-informed CNOs, one needs to prove that the CNO error converges at a certain rate in terms of its size. Although Theorem B.2 does not provide such a rate, its proof does give a hint of which stronger assumptions are needed to obtain this result. The most notable ingredients include a stronger stability result (B.1) with a continuity modulus ω decreasing at least at a polynomial rate, and a stronger regularity assumption on G (B.7), and hence \mathcal{G}^\dagger .*

B.1 Auxiliary results

We list the auxiliary results that are used in the proof of Theorem B.2. First, we state a well-known version of the universal approximation theorem for feedforward neural networks [30]:

Theorem B.6 ([30]). *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a function that is locally essentially bounded with the property that the closure of the set of points of discontinuity has zero Lebesgue measure. For $1 \leq j \leq n$, let $\alpha_j, \theta_j \in \mathbb{R}$ and $y_j \in \mathbb{R}^d$. Then finite sums of the form*

$$G(x) = \sum_{j=1}^N \alpha_j \sigma(y_j^T x + \theta_j), \quad x \in \mathbb{R}^d \tag{B.15}$$

are dense in $C(\mathbb{R}^d)$ if and only if σ is not an algebraic polynomial.

Next, we demonstrate the equivalence of using the interpolation sinc filter (2.5) and trigonometric polynomial interpolation. If you sample a function $f \in C(\mathbb{T} = [0, 2\pi))$ with sampling frequency $\frac{2\pi}{N}$, the result obtained through trigonometric polynomial interpolation $P_N g$ is given by [19],

$$P_N f(x) = \begin{cases} \sum_{|n| \leq (N-1)/2} \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \exp(in(x - x_j)), & \text{for } n \text{ odd}, \\ \sum_{|n| \leq N/2} \frac{1}{N c_n} \sum_{j=0}^{N-1} f(x_j) \exp(in(x - x_j)), & \text{for } n \text{ even}, \end{cases} \tag{B.16}$$

where $x_j = 2\pi j/N$, $c_n = 1$ for $|n| < N/2$ and $c_n = 2$ for $|n| = N/2$. We will prove that one obtains the exact same result by using an interpolation sinc filter with the same frequency on the periodic extension of f . We prove this result in the one-dimensional case for odd N . The result for even N follows in an identical way, the result for the multi-dimensional case through tensorisation.

Lemma B.7. *For any $N \in 2\mathbb{N} + 1$ and $f \in C(\mathbb{T})$ it holds that,*

$$P_N f(x) = \frac{1}{N} \sum_{|n| \leq (N-1)/2} \sum_{j=0}^{N-1} f(x_j) \exp(in(x - x_j)) = \sum_{n \in \mathbb{Z}} f(x_n) \operatorname{sinc}\left(N \cdot \frac{x - x_n}{2\pi}\right) \quad (\text{B.17})$$

Proof. As a first step, it follows from [19, Section 2.2.2] that

$$P_N f(x) = \frac{1}{N} \sum_{|n| \leq (N-1)/2} \sum_{j=0}^{N-1} f(x_j) \exp(in(x - x_j)) = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \frac{\sin(N(x - x_n)/2)}{\sin((x - x_n)/2)}. \quad (\text{B.18})$$

Then we use the result from [50], where we replace their N -periodic signal $x(t)$ by the function $f(x)$ through the transformation $t = Nx/2\pi$ and $x(t) = f(2\pi t/N)$. In their notation, but with the change that here we use the *normalized* sinc function ($\operatorname{sinc}(x) = \sin(\pi x)/\pi x$ for $x \neq 0$), [50] shows that

$$\sum_{n \in \mathbb{Z}} x(n) \operatorname{sinc}(t - n) = \frac{\sin(\pi t)}{N} \sum_{n=-L}^{M-1} x(n) (-1)^n \csc(\pi(t - n)/N) \quad (\text{B.19})$$

with $L, M \in \mathbb{N}_0$ such that $L + M = N$. We will take $L = 0$ and $M = N$, and use that $\csc(z) = 1/\sin(z)$ and that $\cos(\pi n) = (-1)^n$ and $\sin(\pi n) = 0$ to obtain,

$$\sum_{n \in \mathbb{Z}} x(n) \operatorname{sinc}(t - n) = \frac{1}{N} \sum_{n=-L}^{M-1} x(n) \frac{\sin(\pi(t - n))}{\sin(\pi(t - n)/N)}, \quad (\text{B.20})$$

which is equivalent to,

$$\sum_{n \in \mathbb{Z}} f(x_n) \operatorname{sinc}(N(x - x_n)/2\pi) = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \frac{\sin(N(x - x_n)/2)}{\sin((x - x_n)/2)}. \quad (\text{B.21})$$

Combining all obtained equalities proves the claim. \square

C Technical Details for Section 4 of Main Text.

Implementation of all the models is available at

■ <https://github.com/bogdanraonic3/ConvolutionalNeuralOperator.git>

In the *readme.md* file, it is explained how to download the data sets, as well as best performing models. Alternatively, the data and the models can be downloaded from

■ <https://zenodo.org/record/7963379>

C.1 Training and Implementation Details

We start with a succinct description of the baselines that were used in the main text.

C.1.1 Feed Forward Dense Neural Networks

Given an input $u \in \mathbb{R}^m$, a feedforward neural network (also termed as a multi-layer perceptron), transforms it to an output, through a layer of units (neurons) which compose of either affine-linear maps between units (in successive layers) or scalar nonlinear activation functions within units [15], resulting in the representation,

$$\bar{u}_\theta(y) = C_L \circ \sigma \circ C_{L-1} \dots \circ \sigma \circ C_2 \circ \sigma \circ C_1(u). \quad (\text{C.1})$$

Here, \circ refers to the composition of functions and σ is a scalar (nonlinear) activation function. For any $1 \leq \ell \leq L$, we define

$$C_\ell z_\ell = W_\ell z_\ell + b_\ell, \text{ for } W_\ell \in \mathbb{R}^{d_{\ell+1} \times d_\ell}, z_\ell \in \mathbb{R}^{d_\ell}, b_\ell \in \mathbb{R}^{d_{\ell+1}}, \quad (\text{C.2})$$

and denote,

$$\theta = \{W_\ell, b_\ell\}_{\ell=1}^L, \quad (\text{C.3})$$

to be the concatenated set of (tunable) weights for the network. Thus in the terminology of machine learning, a feed forward neural network (C.1) consists of an input layer, an output layer, and L hidden layers with d_ℓ neurons, $1 < \ell < L$. In all numerical experiments, we consider a uniform number of neurons across all the layer of the network $d_\ell = d_{\ell-1} = d$, $1 < \ell < L$. The first baseline model consists into a feed forward neural network with *residual blocks* which use *skip or shortcut connections* [18]. A residual block spanning k layers is defined as follows,

$$r(z_\ell, z_{\ell-k}) = \sigma(W_\ell z_\ell + b_\ell) + z_{\ell-k}. \quad (\text{C.4})$$

The residual network takes as input a sample function $u \in \mathcal{X}$, encoded at $m = s \times s$ *Cartesian grid* points (x_1, \dots, x_m) , $\mathcal{E}(u) = (u(x_1), \dots, u(x_m)) \in \mathbb{R}^m$, and outputs the output sample $\mathcal{G}(u) \in \mathcal{Y}$ encoded at the same set of points, $\mathcal{E}(\mathcal{G}(u)) = (\mathcal{G}(u)(x_1), \dots, \mathcal{G}(u)(x_m)) \in \mathbb{R}^m$. In all the experiments, but the compressible Euler, $s = 64$. Instead, for the compressible Euler equation, the sampling rate is $s = 128$. The number of layers L , neurons d are chosen through cross-validation, whereas the activation function σ corresponds to a Leaky ReLU and the depth of the residual block k is fixed and equal to 2.

C.1.2 UNet

For the UNet baseline we use the model architecture proposed in [49]. However, we slightly modify the proposed architecture by varying the number of output channels c of the first convolutional layer, which is chosen through cross validation. We ensure that the number of channels used in the subsequent layers align with the chosen value of c . Specifically, we respect the progressive increase or decrease in the number of channels as established in the original architecture across different layers.

C.1.3 Convolutional Neural Operator

Design of the filters. As we noted before, perfect *sinc* interpolation filters h_w have infinite impulse response and cause ringing artifacts around high-gradient points due the Gibbs phenomenon. In practice, one uses *windowed-sinc* filters which serve as convenient approximations of h_w . They have finite impulse response and weakened ringing effect [58].

The *windowed-sinc* filters are constructed by multiplying the ideal filter h_w by a corresponding *window function* of *finite* length. That is equivalent to convolving the filter with the window function in the frequency domain. To design the windowed filter, one can use standard Python libraries and their functions such as `scipy.signal.firwin`. By using this function, we are enabled to manually control the cutoff frequency w_c and the half-width of the transition band w_h of the designed filters. We design discrete filters with a prescribed compact support $N_{tap} \in \mathbb{N}$. In this case, we say that a designed filter has N_{tap} *taps*. Implementation of the filters is borrowed from [21] (CUDA programming model).

We show several *1D* designed filters in the Figure 3, where we set $w_c = s/(2 + \varepsilon)$, for $\varepsilon \ll 1$. We control the half-width of the filter $w_h = c_h \cdot s$ by controlling the coefficient c_h . When c_h is set to 0.5, one would anticipate the design of an almost perfect *sinc* filter. However, the presence of undesirable oscillations in the frequency domain can be observed due to the finite impulse response of windowed filters, as depicted in Figure 3. That is why we set c_h to be at least 0.6. One can implement a *2D* filter by first convolving a *1D* filter with each row and then with each column.

The activation layer $\Sigma_{w, \bar{w}}$ plays a vital role in the CNO model. It is essential to closely examine the ratio $N_\sigma = \bar{w}/w$ as a significant parameter. To facilitate implementation of the CNO, we make the assumption that $N_\sigma \in \mathbb{N}$ and $N_\sigma \geq 2$. Throughout the entire architecture, we make the assumption that the value of N_σ remains fixed. In our implementation of the CNO, it is worth noting that the value of N_σ can also be a rational number if the sampling rate of an input signal requires it (e.g. if one wants to upsample a signal from the sampling rate 11 to the sampling rate 20).

We choose to fix the coefficient $w_c = s/2.0001$, so that the cutoff frequency is very close to the *Nyquist critical frequency*. It remains to choose the number of taps N_{tap} , the coefficient related to the half-width of the filter c_h and the ratio related to the activation layer N_σ .

Choice of parameters. Throughout our experiments, we maintained a consistent configuration, setting N_σ to 2, c_h to 1, and N_{tap} to 12. Prior to finalizing the filter parameters, we conducted experiments using various filters; however, no significant differences were observed. To further validate this assumption, we conducted the Navier-Stokes experiment using different filter designs. First, we selected the best-performing CNO model based on the criteria described in C.2 using filter parameters $N_\sigma = 2$, $c_h = 1$ and $N_{tap} = 12$. For this chosen model, we conducted training with identical model settings as outlined in 10, but with different values for coefficients c_h , N_σ and N_{tap} .

In the first set of experiments, we set $N_{tap} = 12$ and vary c_h and N_σ . Note that increasing the coefficient N_σ leads to a significant increase in computational time. We show different test errors in the Table 2. Once the coefficient c_h reaches a sufficiently high value (i.e. $c_h \geq 0.8$), we observe no significant difference in test errors. Additionally, we do not find a high correlation between the error and the coefficient N_σ . We set the N_σ as low as possible, to a fixed value of $N_\sigma = 2$. Similarly, we set the coefficient c_h to a fixed value of 1.

In the second experiment, we fix $N_\sigma = 2$ and $c_h = 1$ and vary the number of taps N_{tap} . By increasing the number of taps, the computational time also increases. We show different test errors in the Table 3. Although there is an improvement of approximately 5% in the test error when $N_{tap} = 24$ compared to when $N_{tap} = 12$, it comes at the cost of increased training time. Specifically, the training time for 1000 epochs increases from 0.7 hours to 1.7 hours, representing an almost 150% increase. Due to this significant increase in training time, but not very significant improvement in performance, we decide to fix the number of taps at $N_{tap} = 12$.

Remark C.1. *Given the above description, it is important to emphasize that, although in principle, the activation layer of CNO (2.3) needs to be exactly equivariant, i.e., $\sigma(\mathcal{B}_\omega) \subset \mathcal{B}_{\omega'}$ for the pair (ω, ω') , for the CNO architecture to be representation equivariant in the sense of [2], definition 3.4, see also section A.5, several approximations are used in practice that might be lead to this condition to hold only approximately. However, as the above results show, once the upsampling frequency is choosen high enough, this approximation of equivariance seems to suffice in practice, see also results in Section C.4. Nevertheless, if exact equivariance is sought for, it can be realized through either polynominal or rational activation functions as suggested in Remark B.4 although this choice might be of little practical utility.*

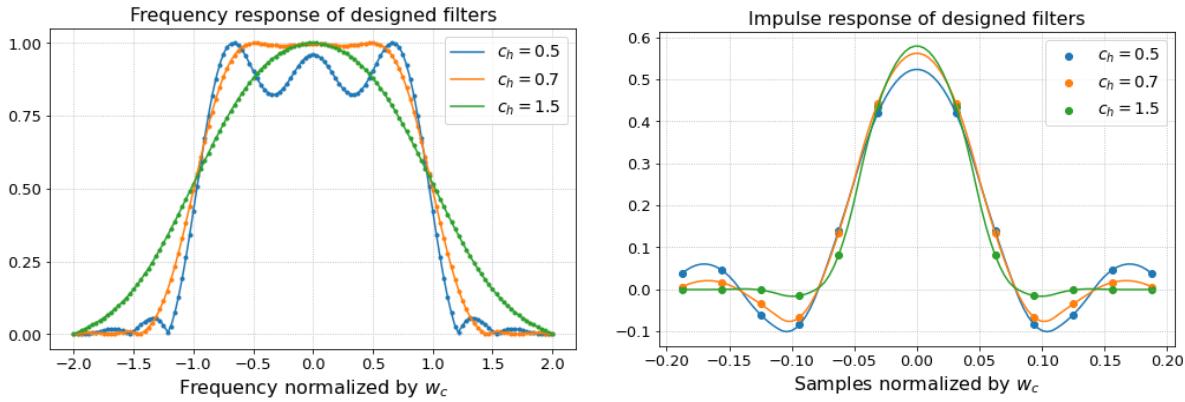


Figure 3: On the left: Frequency responses of different designed filters. On the right: Impulse responses of different designed filters. The sampling rate is $s = 128$, the cutoff frequency is $w_c = s/2.0001$, while the half-width of each filter is $w_h = c_h \cdot s$. Each filter has $N_{tap} = 12$ taps.

Building blocks. In the convolutional neural operator, the architecture consist of 4 different blocks, i.e. the *downsampling block* (D), the *upsampling block* (U), the *invariant block* (I) and the *ResNet block* (R).

Table 2: CNO model. Navier-Stokes Equations. Relative median L^1 -error computed over 128 in-distribution testing samples for different filter designs. The error of the model with original filter parameters $c_h = 1$, $N_\sigma = 2$ and $N_{tap} = 12$ is marked in blue.

	$c_h = 0.6$	$c_h = 0.8$	$c_h = 1.0$	$c_h = 1.5$	$c_h = 2.0$
$N_\sigma = 2$	3.18%	2.98%	3.01%	2.97%	3.01%
$N_\sigma = 3$	3.17%	2.89%	2.91%	3.06%	3.14%
$N_\sigma = 4$	2.96%	3.30%	2.94%	3.00%	3.05%
$N_\sigma = 5$	3.13%	2.94%	2.95%	2.98%	3.03%
$N_\sigma = 6$	3.27%	3.26%	3.03%	2.96%	3.09%

Table 3: CNO model. Navier-Stokes Equations. Relative median L^1 -error computed over 128 in-distribution testing samples for different number of taps N_{tap} . The error of the model with original filter parameters $c_h = 1$, $N_\sigma = 2$ and $N_{tap} = 12$ is marked in blue.

	$N_{tap} = 12$	$N_{tap} = 16$	$N_{tap} = 20$	$N_{tap} = 24$
$N_\sigma = 2 \& c_h = 1$	3.01%	2.94%	2.96%	2.86%

In the simplest scenario, the architecture consists only of the lifting layer, number of (D) and (U) blocks, and the projection layer. In this simple scenario, once the input is lifted to higher dimensional space (in the channel width), one performs first T iterations of (D) blocks. These T iterations define the *encoder*, namely

$$v_{l+1} = \mathcal{D}_{s_l, s_{l+1}} \circ \Sigma_{s_l, s_{l+1}} \circ \mathcal{K}_{s_l} v_l, \quad v_l \in \mathcal{B}_{s_l}(D, \mathbb{R}^{d_l}), \quad l = 0 \dots T-1,$$

where $s_l = s/2^l$ is the current bandlimit and d_l is the current number of channels. The next T iterations are (U) blocks and are devoted to the *decoder*. Let $\tilde{s}_l = s_{2T-l}$. The decoder is defined as

$$v_{l+1} = \mathcal{U}_{\tilde{s}_l, \tilde{s}_{l+1}} \circ \Sigma_{\tilde{s}_l, \tilde{s}_{l+1}} \circ \mathcal{K}_{\tilde{s}_l} v_l, \quad v_l \in \mathcal{B}_{\tilde{s}_l}(D, \mathbb{R}^{d_l}), \quad l = T \dots 2T-1.$$

The last output of the decoder is projected to the output space (in the channel width). In all the experiments, we use $d_l = 32$ as the lifting dimension. In the encoder, the number of channels increases as per

$$32 \mapsto d_e \mapsto 2d_e \mapsto \dots \mapsto 2^{T-1}d_e.$$

The number d_e is a hyperparameter. In this simple case where no UNet style patching is present in the architecture, the number of channels in the decoder decreases as per

$$2^{T-1}d_e \mapsto 2^{T-2}d_e \mapsto \dots \mapsto d_e \mapsto d_e.$$

When the patching is present in the architecture (see Figure 1), number of channels in the decoder changes differently (as a certain number of transferred channels is concatenated).

Operator UNet architecture. We add an (I) block 2.8 before each up/downsampling block. Additionally, we add a few (R) blocks 2.7 between the encoder and decoder. We denote the number of residual blocks as a hyperparameter N_{res} . There is an additional hyperparameter r in the definition of the (R) block and it corresponds to the number of consecutive applications of (I) blocks to an input signal before the skip connection is employed (see 2.7 for clarification).

Remark C.2. *The last hyperparameter to be chosen is the size of the convolution kernel k . Usually, we set $k = 3$ or $k = 5$. Hence, the objectives of cross-validation are N_{tap} , d_e , N_{res} , r and k .*

C.1.4 DeepONet

Let $x := (x_1, \dots, x_m) \in D$ be a fixed set of *sensor points*. Given an input function $u \in \mathcal{X}$, we encode it by the point values $\mathcal{E}(u) = (u(x_1), \dots, u(x_m)) \in \mathbb{R}^m$. DeepONet is formulated in terms of two neural networks [38]: (1) a *branch-net* β , which maps the point values $\mathcal{E}(u)$ to coefficients $\beta(\mathcal{E}(u)) = (\beta_1(\mathcal{E}(u)), \dots, \beta_p(\mathcal{E}(u)))$, resulting in a mapping

$$\beta : \mathbb{R}^m \rightarrow \mathbb{R}^p, \quad \mathcal{E}(\bar{u}) \mapsto (\beta_1(\mathcal{E}(\bar{u})), \dots, \beta_p(\mathcal{E}(\bar{u}))). \quad (\text{C.5})$$

and (2) a *trunk-net* $\tau(y) = (\tau_1(y), \dots, \tau_p(y))$, which is used to define a mapping

$$\tau : U \rightarrow \mathbb{R}^p, \quad y \mapsto (\tau_1(y), \dots, \tau_p(y)). \quad (\text{C.6})$$

While the branch net provides the coefficients, the trunk net provides the “basis” functions in an expansion of the output function of the form

$$\mathcal{G}(u)(y) = \sum_{k=1}^p \beta_k(\bar{u}) \tau_k(y), \quad \bar{u} \in \mathcal{X}, \quad y \in U, \quad (\text{C.7})$$

with $\beta_k(\bar{u}) = \beta_k(\mathcal{E}(\bar{u}))$. The resulting mapping $\mathcal{G} : \mathcal{X} \rightarrow \mathcal{Y}$, $u \mapsto \mathcal{G}$ is a *DeepONet*.

In the numerical experiments, for the trunk-net we use simple feed-forward neural networks. On the other hand the branch consists of a convolutional network of the following form:

$$\mathcal{G} : \mathcal{X} \mapsto \mathcal{Y} : \quad \mathcal{G} = Q \circ Fl \circ R_{N_{res}} \circ \dots \circ R_1 \circ D_M \circ I_M \circ \dots \circ D_1 \circ I_1 \quad (\text{C.8})$$

where I , D and R are the *invariant*, *downsampling* and *ResNet* blocks defined in 2, where the downsampling in D and Σ is instead performed by average pooling with kernel size 2. The parameter r in the residual block is set to 1. The output is then flattened through Fl and linearly transformed by $Q : \mathbb{R}^n \mapsto \mathbb{R}^p$, with n being the number of units after flattening. The

convolution is performed with a kernel of size 3 and stride 1, whereas the number of channels across the layers is

$$32 \mapsto 64 \mapsto 128 \mapsto \dots \mapsto 2^{M-1} 32.$$

The activation function is chosen as Leaky ReLU. The number of layers L and units d of the trunk, the number of layers M and residual blocks N_{res} of the branch, and the number of bases p , are chosen through cross-validation.

C.1.5 Fourier Neural Operator

A *Fourier neural operator* (FNO) \mathcal{G} [32] is a composition

$$\mathcal{G} : \mathcal{X} \mapsto \mathcal{Y} : \quad \mathcal{G} = Q \circ \mathcal{L}_T \circ \dots \circ \mathcal{L}_1 \circ R. \quad (\text{C.9})$$

It has a ‘‘lifting operator’’ $u(x) \mapsto R(u(x), x)$, where R is represented by a linear function $R : \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_v}$ where d_u is the number of components of the input function and d_v is the ‘‘lifting dimension’’. The operator Q is a non-linear projection, instantiated by a shallow neural network with a single hidden layer, 128 neurons and *GeLU* activation function, such that $v^{L+1}(x) \mapsto \mathcal{G}(u)(x) = Q(v^{L+1}(x))$.

Each *hidden layer* $\mathcal{L}_\ell : v^\ell(x) \mapsto v^{\ell+1}(x)$ is of the form

$$v^{\ell+1}(x) = \sigma \left(W_\ell \cdot v^\ell(x) + \left(K_\ell v^\ell \right) (x) \right),$$

with $W_\ell \in \mathbb{R}^{d_v \times d_v}$ a trainable weight matrix (residual connection), σ an activation function, corresponding to GeLU, and the *non-local Fourier layer*,

$$K_\ell v^\ell = \mathcal{F}_N^{-1} \left(P_\ell(k) \cdot \mathcal{F}_N v^\ell(k) \right),$$

where $\mathcal{F}_N v^\ell(k)$ denotes the (truncated)-Fourier coefficients of the discrete Fourier transform (DFT) of $v^\ell(x)$, computed based on the given s grid values in each direction. Here, $P_\ell(k) \in \mathbb{C}^{d_v \times d_v}$ is a complex Fourier multiplication matrix indexed by $k \in \mathbb{Z}^d$, and \mathcal{F}_N^{-1} denotes the inverse DFT.

The lifting dimension d_v , the number of Fourier layers L and k_{max} , defined in 2, are objectives of cross-validation.

C.2 Training Details

The training of the models, including the baselines, is performed with the ADAM optimizer, with a learning rate η for 1000 epochs and minimizing the L^1 -loss function. We also use a step learning rate scheduler and reduce the learning rate of each parameter group by a factor γ every epoch. We train FFNN, UNet, and DeepONet in mini-batches of size 10 and FNO and CNO in batches of 32. A weight decay of magnitude w is used.

All the parameters mentioned above (η , γ , w) are chosen through cross-validation.

At every epoch, the relative L^1 error is computed on the validation set, and the set of trainable parameters resulting in the lowest error during the entire process is saved for testing. Early stopping is used to interrupt the training if the best validation error does not improve after 50 epochs.

The cross-validation is performed by running a random search over a chosen range of hyperparameters values and selecting the configuration, realizing the lowest relative L^1 error on the validation set. Overall, 30 hyperparameters configurations are tested for the FFNN, UNet and DeepONet , 24 to 48 configurations for CNO and 36 to 72 configurations for FNO. The model size (minimum and maximum number of trainable parameters) covered in this search are reported in Table 5.

The results of the random search, i.e., the best-performing hyperparameter configurations for each model and each benchmark, are reported in tables 6, 8 and 7, 9 and 10.

Different Initialization. After selecting the models and computing the test median errors, we proceed to train the CNO, FNO, and UNet models again using the same settings but different initializations for the model parameters (by changing the random seeds). Each model is trained for each experiment a total of 10 times. We report the means and the standard deviations of the 10 different test median errors for each benchmark experiment in the Table 4. We observe from this table that CNO is very robust with respect to random initializations, with very low standard deviation to mean ratio for all the benchmarks in the **RPB** dataset.

Table 4: Means and standard deviations for the 10 relative median L^1 test errors, for both in-distribution testing, for the CNO, FNO and U-Net models. The format is $mean \pm std$.

	CNO	FNO	UNet
Poisson Equation	$0.26 \pm 0.02\%$	$5.27 \pm 0.29\%$	$0.76 \pm 0.16\%$
Wave Equation	$0.79 \pm 0.03\%$	$1.12 \pm 0.04\%$	$1.67 \pm 0.12\%$
Smooth Transport	$0.32 \pm 0.06\%$	$0.45 \pm 0.06\%$	$0.79 \pm 0.21\%$
Discontinuous Transport	$1.20 \pm 0.03\%$	$1.35 \pm 0.04\%$	$1.40 \pm 0.09\%$
Allen-Cahn	$0.88 \pm 0.06\%$	$0.57 \pm 0.09\%$	$1.84 \pm 0.33\%$
Navier-Stokes	$2.97 \pm 0.13\%$	$4.07 \pm 0.12\%$	$3.48 \pm 0.07\%$
Compressible Euler	$0.37 \pm 0.02\%$	$0.48 \pm 0.03\%$	$0.39 \pm 0.01\%$

C.3 Details about the description and numerical results in each benchmark

This section provides details about all the experiments that are a part of the **RPB** benchmarks of the main text.

Table 5: Minimum (Top sub-row) and maximum (Bottom sub-row) number of trainable parameters among the random-search hyperparameters configurations for all the models in every problem reported in Table 1 in main text.

	FFNN	UNet	DeepONet	FNO	CNO
Poisson Equation	0.3M	0.5M	0.8M	0.2M	0.5M
	8.2M	31.0M	48.1M	52.5M	26.8M
Wave Equation	0.3M	0.5M	0.8M	0.9M	0.6M
	6.0M	31.0M	48.1M	13.2M	23.6M
Smooth Transport	0.3M	0.5M	0.7M	0.2M	0.4M
	8.2M	7.7M	49.2M	16.4M	23.6M
Discontinuous Transport	0.3M	0.5M	0.7M	0.6M	0.4M
	5.5M	7.7M	49.2M	13.2M	23.6M
Allen Cahn	0.3M	0.5M	1.0M	0.9M	0.4M
	7.1M	31.0M	47.9M	65.6M	23.6M
Navier Stokes	0.3M	0.5M	1.0M	0.2M	0.5M
	7.1M	31.0M	47.9M	65.6M	26.8M
Compressible Euler	1.1M	0.5M	0.8M	0.9M	0.5M
	18.6M	31.0M	49.4M	65.6M	26.8M

Table 6: FFNN best-performing hyperparameters configuration for different benchmark problems.

	η	γ	w	L	d	Trainable Params
Poisson Equation	0.0005	0.98	1e-06	10	512	6.6M
Wave Equation	0.001	0.98	1e-06	4	256	2.3M
Continuous Translation	0.001	1.0	0.0	16	256	3.1M
Discontinuous Translation	0.0005	1.0	0.0	6	512	5.5M
Allen Cahn	0.0005	0.98	0.0	8	512	6.0M
Navier Stokes	0.001	1.0	1e-06	16	256	3.1M
Compressible Euler	0.0005	1.0	0.0	16	32	1.1M

C.3.1 Poisson Equation

In this experiment, we study Poisson equation 4.1 with the source term given by

$$f(x, y) = \frac{\pi}{K^2} \sum_{i,j}^K a_{ij} \cdot (i^2 + j^2)^r \sin(\pi i x) \sin(\pi j y), \quad \forall (x, y) \in D,$$

Table 7: UNet best-performing hyperparameters configuration for different benchmark problems.

	η	γ	w	c	Trainable Params
Poisson Equation	0.001	0.98	0.0	32	7.8M
Wave Equation	0.001	1.0	1e-06	64	31.0M
Continuous Translation	0.001	0.98	1e-06	16	1.9M
Discontinuous Translation	0.001	0.98	1e-06	32	7.8M
Allen Cahn	0.0005	0.98	1e-06	64	31.0M
Navier Stokes	0.0005	0.98	1e-06	64	31.0M
Compressible Euler	0.001	0.98	1e-06	32	7.8M

Table 8: DeepONet best-performing hyperparameters configuration for different benchmark problems.

	η	γ	w	p	L	d	M	N_{res}	Trainable Params
Poisson Equation	0.001	0.98	0.0	500	8	128	4	4	5.2M
Wave Equation	0.0005	0.98	0.0	100	4	512	4	4	4.2M
Continuous Translation	0.0005	0.98	0.0	500	8	128	4	0	2.8M
Discontinuous Translation	0.0005	0.98	0.0	100	8	512	4	4	5.3M
Allen Cahn	0.0005	0.98	1e-06	50	8	512	4	4	5.0M
Navier Stokes	0.0005	0.98	1e-06	100	8	512	6	2	30.3M
Compressible Euler	0.0005	0.98	1e-06	500	8	256	4	4	11.7M

where $K = 16$, $r = 0.5$ and a_{ij} are i.i.d. uniformly distributed from $[-1, 1]$. Given the source term above, the *exact* solution u of the Poisson equation is given by

$$u(x, y) = \frac{1}{\pi K^2} \sum_{i,j}^K a_{ij} \cdot (i^2 + j^2)^{r-1} \sin(\pi i x) \sin(\pi j y), \quad \forall (x, y) \in D.$$

During the out-of-distribution testing, we augment the number of modes to $K = 20$ and evaluate the models' ability to generalize to inputs with frequencies higher than those encountered during training. We approximate the operator \mathcal{G}^\dagger , which maps f to u . An illustration of the operator \mathcal{G}^\dagger is given in the Figure 26. For training purposes, we generate 1024 samples and for testing, we generate 256 samples for both in-distribution and out-of-distribution testing, by sampling the exact solution u at a resolution of 64×64 points on $D = [0, 1]^2$. We also create a

Table 9: FNO best-performing hyperparameters configuration for different benchmark problems.

	η	γ	w	k_{max}	d_v	L	Trainable Params
Poisson Equation	0.0007	0.98	1e-10	12	32	2	0.6M
Wave Equation	0.001	0.95	1e-07	20	32	3	2.5M
Smooth Transport	0.001	0.98	1e-07	16	64	5	10.5M
Discontinuous Transport	0.001	0.98	1e-10	16	64	4	8.4M
Allen Cahn	0.001	0.98	1e-07	20	32	3	2.5M
Navier Stokes	0.001	0.98	1e-07	20	64	5	16.4M
Compressible Euler	0.001	0.98	1e-09	16	64	3	6.3M

Table 10: CNO best-performing hyperparameters configuration for different benchmark problems.

	η	γ	w	M	d_e	N_{res}	r	k	Trainable Params
Poisson Equation	0.0007	0.98	1e-10	2	32	8	1	5	1.7M
Wave Equation	0.0005	0.98	1e-10	2	48	4	2	5	3.4M
Smooth Transport	0.00075	0.98	1e-10	2	48	8	1	5	3.4M
Discontinuous Transport	0.0005	0.98	1e-10	3	32	8	2	3	3.9M
Allen Cahn	0.00075	0.98	1e-10	3	48	6	2	3	6.7M
Navier Stokes	0.0007	0.98	1e-10	3	32	6	2	3	3.1M
Compressible Euler	0.0007	0.98	1e-10	4	48	6	2	3	26.8M

validation set consisting of 128 samples for model selection. The training data is normalized to the interval $[0, 1]$. The testing data is normalized with the same normalization constants as the training data. In Figure 4, we show empirical test error distributions for UNet, FNO and CNO models (in-distribution in the left Figure and out-of-distribution in the right Figure). We show a random in-distribution testing sample and an out-of-distribution testing sample, as well as predictions made by CNO, FNO and UNet in Figure 5. As was already evidenced in Table 1 of the main text, Figure 4 demonstrates that CNO is clearly the best performing model here with UNet a distant second. FNO performs very poorly on this problem, with test errors that are more than an order of magnitude higher than CNO. A closer perusal of Figure 5 reveals that FNO approximates the multiple scales in the exact solution very poorly and this is particularly striking for the out of distribution testing example shown in this figure. On the other hand, CNO approximates the multiple frequencies in the solution very accurately.

Finally, to further investigate the poor performance of FNO, as compared to CNO, for this problem, we present in Figure 6, the averaged logarithmic amplitude spectra, which compare the ground truth, CNO, FNO, and UNet models. We see from this spectrogram that i) the ground truth solution contains multiple scales, corresponding to a range of frequencies ii) the CNO model successfully captures the complete spectra with high accuracy, iii) FNO (and to some extent UNet) resolves the underlying spectrum with quite a lot of error, particularly in the high-frequency components, perhaps attributable to aliasing errors in this case.

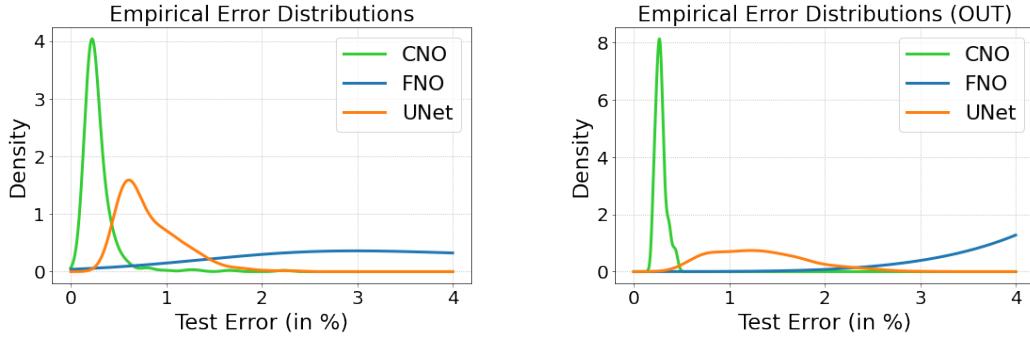


Figure 4: Poisson equation. Empirical test error distributions for UNet, FNO and CNO. Left: In-distribution testing. Right: Out-of-distribution testing.

C.3.2 Wave Equation

In this experiment, we study Wave equation 4.3 with constant speed of propagation $c = 0.1$ and the initial condition given by 4.2 with $K = 24$ and $r = 1$. The exact solution at time $t > 0$ is given by

$$u(x, y, t) = \frac{\pi}{K^2} \sum_{i,j}^K a_{ij} \cdot (i^2 + j^2)^{-r} \sin(\pi i x) \sin(\pi j y) \cos\left(c\pi t \sqrt{i^2 + j^2}\right), \quad \forall (x, y) \in D.$$

The objective is to approximate the operator $\mathcal{G}^\dagger : f \mapsto u(\cdot, T = 5)$. An illustration of \mathcal{G}^\dagger is given in Figure 27. During the out-of-distribution testing, we decrease the decay parameter to $r = 0.85$. This adjustment changes the ratio between the amplitudes of different modes, which alters the dynamics of the solution. For the training set, we generate a total of 512 samples. In addition, we generate 256 samples for both in-distribution and out-of-distribution testing, all by sampling the above exact solution at a resolution of 64×64 . Furthermore, we create a validation set comprising 128 samples. The training data is normalized to the interval $[0, 1]$. The testing data is normalized with the same normalization constants as the training data.

In Figure 7, we present the empirical test error distributions for UNet, FNO and CNO models during in-distribution and out-of-distribution testing. We also show a random in-distribution testing sample and a random out-of-distribution testing sample, as well as predictions made by CNO, FNO and UNet in Figure 8. Both these figures demonstrate that CNO is the best performing model in this case, reinforcing the conclusion of Table 1 of the main text.

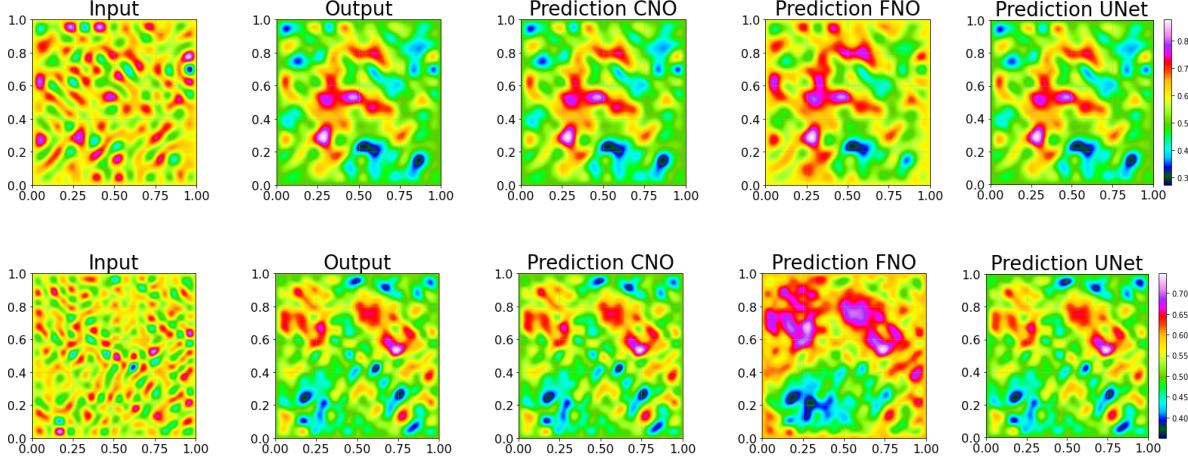


Figure 5: Poisson equation. Exact and predicted coefficients for an in-distribution (top row) and an out-of-distribution (bottom row) samples and for different models (columns). From left to right: input, output (ground truth), CNO, FNO and UNet.

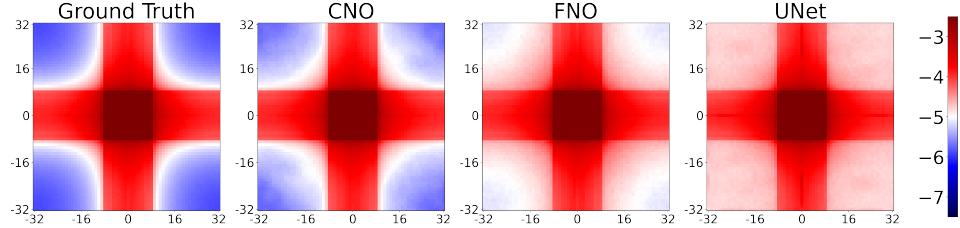


Figure 6: Poisson equation. Averaged logarithmic amplitude spectra comparing Ground Truth, CNO, FNO and UNet.

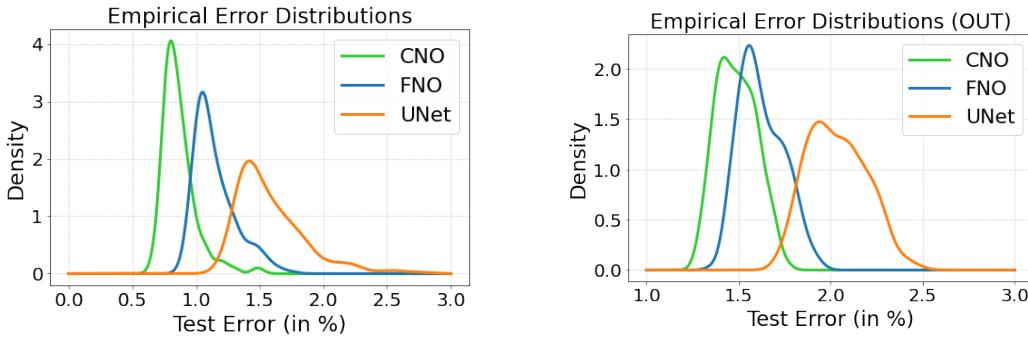


Figure 7: Wave equation. Empirical test error distributions for UNet, FNO and CNO. Left: In-distribution testing. Right: Out-of-distribution testing.

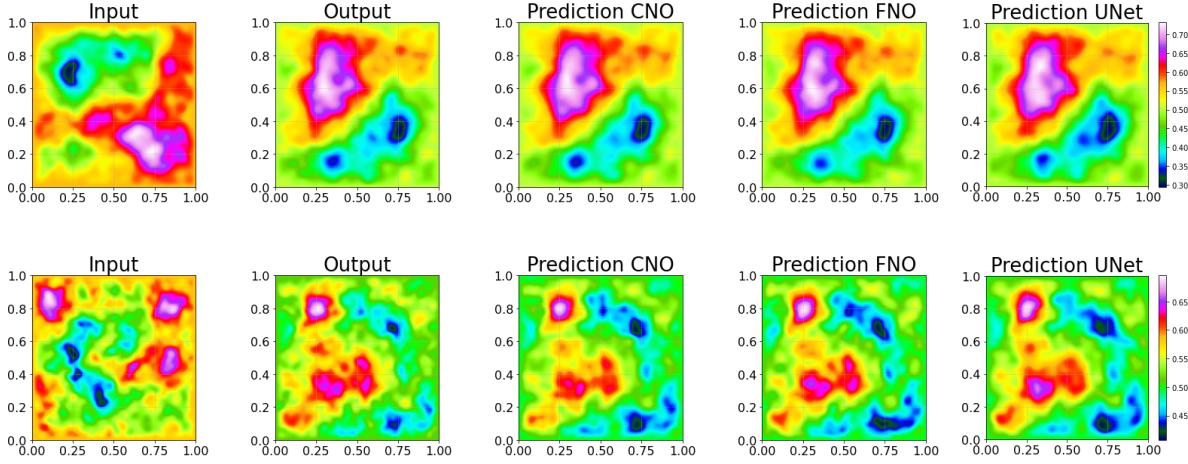


Figure 8: Wave equation. Exact and predicted coefficients for an in-distribution (top row) and an out-of-distribution (bottom row) samples and for different models (columns). From left to right: input, output (ground truth), CNO, FNO and UNet.

C.3.3 Transport Equation

In this experiment, we study Transport equation 4.4. We fix the velocity field to $v = (v_x, v_y) = (0.2, -0.2)$ leading to solution $u(x, y, t) = f(x - v_x t, y - v_y t)$. We conduct two different experiments, i.e., Smooth Transport and Discontinuous Transport. In both cases, the goal is to approximate the operator $\mathcal{G}^\dagger : f \mapsto u(\cdot, T = 1)$. Moreover, in both cases, we generate 512 training samples, 256 validation samples and 256 in-distribution and out-of-distribution testing samples, all from the exact solution. Each sample is normalized to the interval $[0, 1]$.

Smooth Transport. In this case, the data takes form of a radially symmetric Gaussian. The data is drawn from a Gaussian distribution with centers randomly and uniformly drawn from $(0.2, 0.4)^2$ and corresponding variance drawn uniformly from $(0.003, 0.009)$. Formally, the initial conditions are given by

$$f(\underline{x}) = \frac{1}{\sqrt{(2\pi)^2 \det(\Sigma)}} \exp\left(-\frac{1}{2}(\underline{x} - \mu)^T \Sigma^{-1}(\underline{x} - \mu)\right), \quad \underline{x} = (x, y), \quad \mu = (\mu_x, \mu_y),$$

where $\Sigma = \sigma I$ such that $\sigma \sim \mathcal{U}(0.003, 0.009)$ and $\mu_x, \mu_y \sim \mathcal{U}(0.2, 0.4)$. Here, I is the identity matrix and $\mathcal{U}(\cdot)$ is the uniform distribution. Finally, each initial condition is normalized to $(0, 1)$.

For *out-of-distribution* testing, the centers of the Gaussian inputs are sampled uniformly from $(0.4, 0.6)^2$ (i.e. $\mu_x, \mu_y \sim \mathcal{U}(0.4, 0.6)$). The data is generated at 64×64 resolution. An illustration of the operator \mathcal{G}^\dagger for the Smooth Transport experiment is shown in Figure 28. We show empirical test error distributions for UNet, FNO and CNO models (in-distribution and out-of-distribution testing) in Figure 9. We show a random in-distribution testing sample and

an out-of-distribution testing sample, as well as predictions made by CNO, FNO and UNet in Figure 10. The figures reinforce the conclusions drawn from Table 1 i.e., CNO is slightly superior to UNet and FNO for in-distribution testing. However, there is a significant advantage for CNO over UNet on out-of-distribution testing. On the other hand, FNO generalizes poorly out-of-distribution, as clearly seen from the sample shown in Figure 10. Similarly, DeepONet and FFNN are even poorer in terms of their generalization abilities, justifying the very high errors seen in Table 1. An example of this very poor generalization for DeepONet and FFNN can be seen in Figure 11.

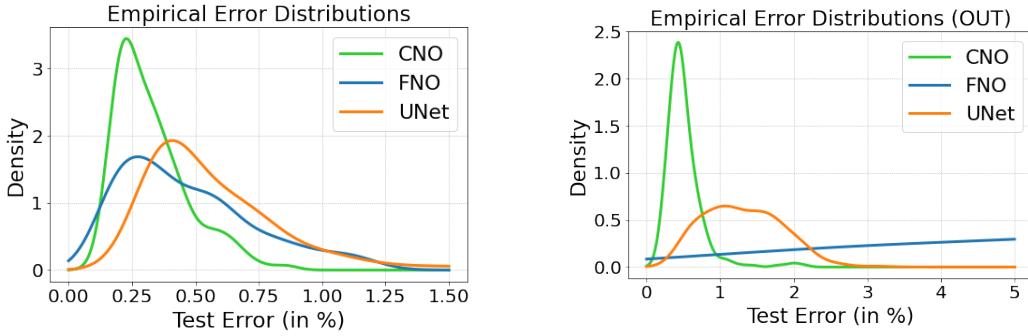


Figure 9: Smooth Transport. Empirical test error distributions for UNet, FNO and CNO. Left: In-distribution testing. Right: Out-of-distribution testing.

Discontinuous Transport. In this case, initial data in the form of the indicator function of radial disk with centers, uniformly drawn from $(0.2, 0.4)^2$ and radii uniformly drawn from $(0.1, 0.2)$. For *out-of-distribution* testing, the centers of the disk are drawn uniformly from $(0.4, 0.6)^2$. Formally, the initial conditions are given by

$$f(\underline{x}) = \mathbb{1}_{S_r(\mu)}(\underline{x}), \quad \underline{x} = (x, y), \quad \mu = (\mu_x, \mu_y),$$

where $r \sim \mathcal{U}(0.1, 0.2)$ and $\mu_x, \mu_y \sim \mathcal{U}(0.2, 0.4)$. Also, $\mathbb{1}_\cdot$ is an indicator function and $S_r(\mu)$ is the sphere of radius r with the center μ , defined by

$$S_r(\mu) = \{\underline{x} : \|\underline{x} - \mu\|_2 \leq r\}.$$

Note that discontinuous data has infinite spectral content, so the aliasing error is always present when the data is sampled. For that reason, we first generate the samples at 128×128 resolution, to reduce the aliasing error that emerges in data generation. We get our actual samples by downsampling the generated data in the frequency domain to the resolution 64×64 . As the Gibbs phenomenon is strongly present when discontinuous data is downsampled in this way, we reduce the impact of this phenomenon by applying a Gaussian filter with a standard deviation $\sigma = 1.75$ to the generated samples, before downsampling them to the final resolution. An example of a random sample with horizontal cut plots is shown in the Figure 12.

We also plot empirical test error distributions for UNet, CNO and FNO models (in-distribution and out-of-distribution testing) in Figure 13. We plot a random in-distribution testing sample

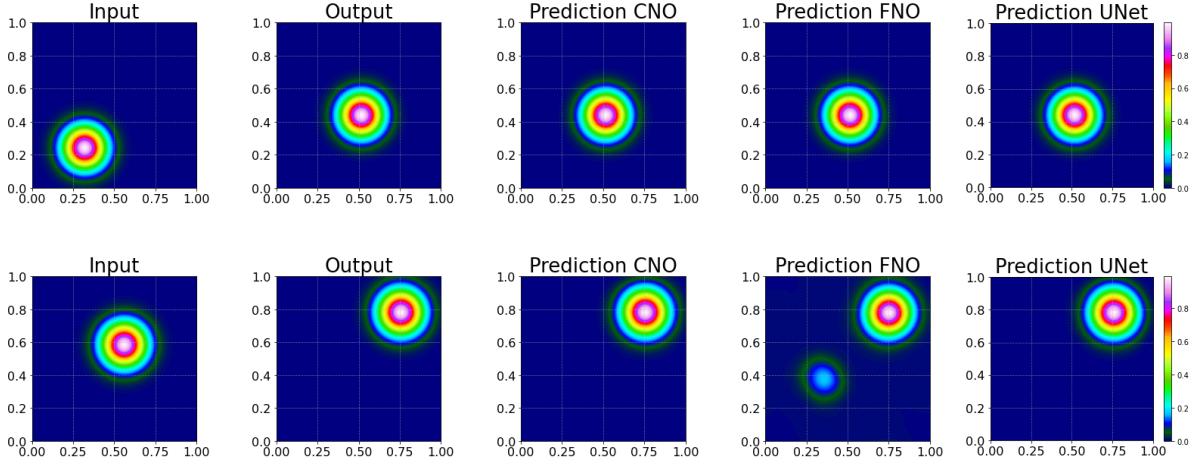


Figure 10: Smooth Transport. Exact and predicted coefficients for an in-distribution (top row) and an out-of-distribution (bottom row) samples and for different models (columns). From left to right: input, output (ground truth), CNO, FNO and UNet.

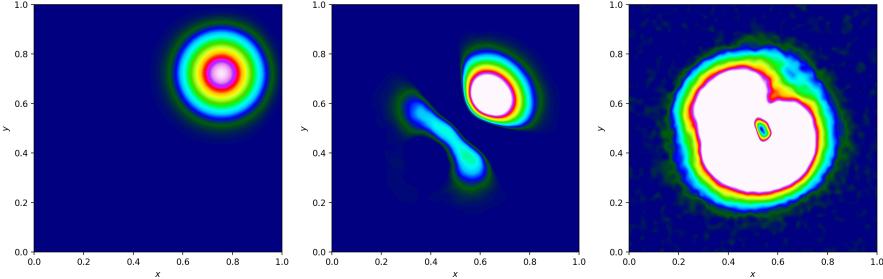


Figure 11: Smooth Transport. An out-of-distribution sample and predictions for DeepONet and FFNN. From left to right: output (ground truth), DeepONet and FFNN.

and an out-of-distribution testing sample, as well as predictions made by CNO, FNO and UNet in Figure 14. These figures clearly reinforce the conclusions from Table 1 that UNet is slightly superior in performance on out-of-distribution testing, when compared to CNO. On the other hand, FNO, DeepONet and FFNN (in that order) generalize very poorly as they fail to be translation equivariant.

C.3.4 Allen-Cahn Equation

In this experiment, we study Allen-Cahn equation 4.5 with fixed reaction rate $\varepsilon = 220$ and initial condition given by 4.2 with $K = 24$ and $r = 1$. The goal is to approximate the operator $\mathcal{G}^\dagger : f \mapsto u(\cdot, T = 0.0002)$ (see Figure 30 for illustrations).

As exact solutions are no longer available, we generate the training and test data using a standard finite difference discretization of the Allen-Cahn equation. We uniformly discretize space at the

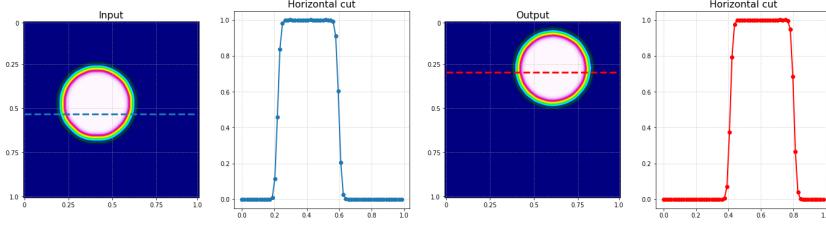


Figure 12: Discontinuous Transport. An example with horizontal cut plots of the disks.

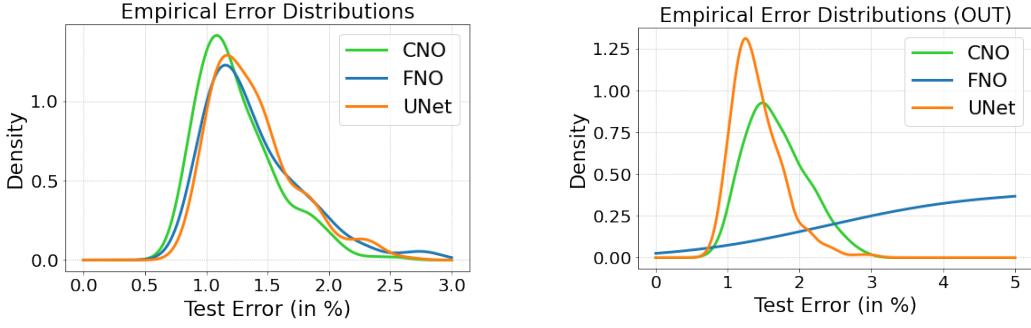


Figure 13: Discontinuous Transport. Empirical test error distributions for UNet, FNO and CNO. Left: In-distribution testing. Right: Out-of-distribution testing.

resolution $s^2 = 64 \times 64$ and set $\Delta x = 1/s$. As we are using an explicit method, we uniformly discretize the time domain with the time step $\Delta t \approx 5.47 \cdot 10^{-7}$ and set $N = \lfloor T/\Delta t \rfloor + 1$. We denote $U_{i,j}^n = u(i\Delta x, j\Delta x, n\Delta t)$ for $i, j = 0, 1, \dots, s$ and $n = 0, 1, \dots, N$. Additionally, we also add the zero-valued ghost cells at the boundaries. The Finite Difference scheme is given by

$$U_{i,j}^{n+1} = U_{i,j}^n + \frac{\Delta t}{\Delta x} (U_{i+1,j}^n + U_{i,j+1}^n + U_{i-1,j}^n U_{i,j-1}^n - 4U_{i,j}^n) - \Delta t \varepsilon^2 U_{i,j}^n (U_{i,j}^n \cdot U_{i,j}^n - 1),$$

for $i, j = 0, 1, \dots, s$ and $n = 0, 1, \dots, N$. With our choice of Δt , the CFL condition $\Delta t < \frac{(\Delta x)^2}{2\varepsilon}$ is satisfied. We generate 256 training samples, 128 validation samples and 128 in-distribution and out-of-distribution testing samples, all at the 64×64 resolution. The training data is normalized to the interval $[0, 1]$. The testing data is normalized with the same normalization constants as the training data. In Figure 15, we present the empirical test error distributions for UNet, FNO and CNO models during in-distribution and out-of-distribution testing. We also plot a random in-distribution testing sample and an out-of-distribution testing sample, as well as predictions made by CNO, FNO and UNet in Figure 16. Again, these figures reinforce the conclusions of Table 1 as FNO is marginally superior to CNO and UNet on in-distribution testing whereas UNet is the best model on out-of-distribution testing.

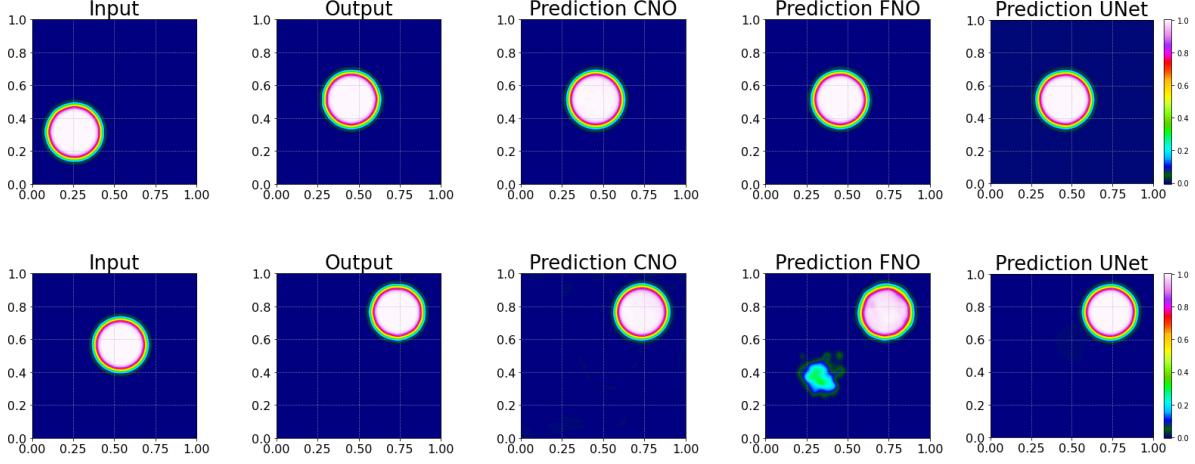


Figure 14: Discontinuous Transport. Exact and predicted coefficients for an in-distribution (top row) and an out-of-distribution (bottom row) samples and for different models (columns). From left to right: input, output (ground truth), CNO, FNO and UNet.

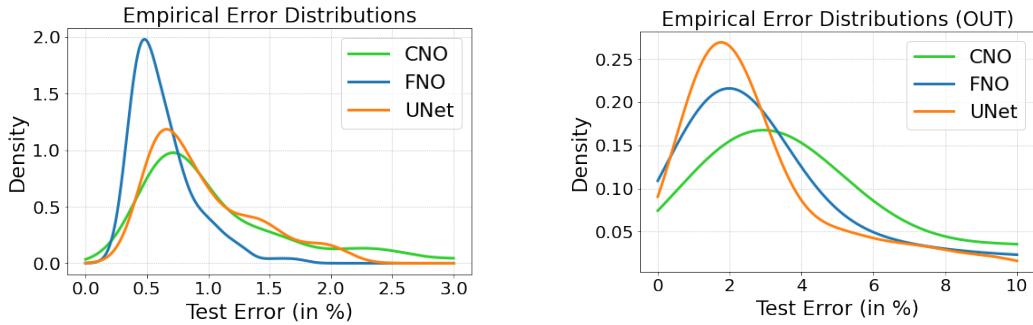


Figure 15: Allen-Cahn equation. Empirical test error distributions for UNet, FNO and CNO. Left: In-distribution testing. Right: Out-of-distribution testing.

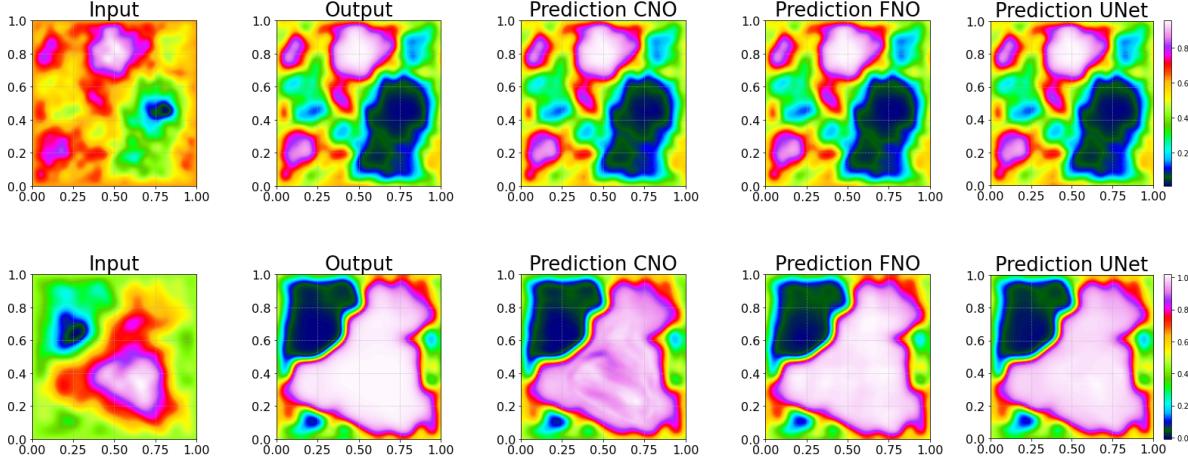


Figure 16: Allen-Cahn equation. Exact and predicted coefficients for an in-distribution (top row) and an out-of-distribution (bottom row) samples and for different models (columns). From left to right: input, output (ground truth), CNO, FNO and UNet.

C.3.5 Navier-Stokes

In this experiment, we study a motion of an incompressible fluid with high Reynolds number. We study Navier-Stokes equations 4.6 in the torus $D = \mathbb{T}^2$ with periodic boundary conditions and, for stabilization, viscosity $\nu = 4 \times 10^{-4}$ only applied to high-enough Fourier modes. We take as initial conditions

$$u_0(x, y) = \begin{cases} \tanh\left(2\pi \frac{y-0.25}{\rho}\right) & \text{for } y + \sigma_\delta(x) \leq \frac{1}{2} \\ \tanh\left(2\pi \frac{0.75-y}{\rho}\right) & \text{otherwise} \end{cases} \quad (C.10)$$

$$v_0(x, y) = 0$$

where $\sigma_\delta : [0, 1] \rightarrow \mathbb{R}$ is a perturbation of the initial data given by

$$\sigma_\delta(x) = \delta \sum_{k=1}^p \alpha_k \sin(2\pi kx - \beta_k). \quad (C.11)$$

The random variables α_k and β_k are i.i.d. uniformly distributed on $[0, 1]$ and $[0, 2\pi]$ respectively. The parameters δ and p are chosen to be $\delta = 0.025$ and $p = 10$. For the smoothing parameter we choose $\rho = 0.1$. (see Figure 31 for illustrations). For the out-of-distribution experiments, we reduced ρ to $\rho = 0.09$ and shifted the location of the shear layers towards the middle of the domain so that they were located at $y = 0.3$ and $y = 0.7$ instead of $y = 0.25$ and $y = 0.75$ like in the original initial condition.

Fix a mesh width $\Delta = \frac{1}{N}$ for some $N \in \mathbb{N}$. We consider the following discretization of the

Navier-Stokes equations 4.6 in the Fourier domain

$$\begin{cases} \partial_t u^\Delta + \mathcal{P}_N(u^\Delta \cdot \nabla u^\Delta) + \nabla p^\Delta = \varepsilon_N |\nabla|^{2s}(Q_N * u^\Delta) \\ \nabla \cdot u^\Delta = 0 \\ u^\Delta|_{t=0} = \mathcal{P}_N u_0 \end{cases} \quad (\text{C.12})$$

where \mathcal{P}_N is the spatial Fourier projection operator mapping a function $f(x, t)$ to its first N Fourier modes: $\mathcal{P}_N = \sum_{|k|_\infty \leq N} \hat{f}_k(t) e^{ik \cdot x}$. We additionally have the hyperviscosity parameter $s \geq 1$ which can be used to dampen the higher Fourier modes strongly, thus allowing for a larger part of the spectrum to be free of numerical dissipation. The artificial viscosity term we use for the stabilization of the solver consists of a resolution-dependent viscosity ε_N and a Fourier multiplier Q_N controlling the strength at which different Fourier modes are damped. This allows us to not dampen the low frequency modes, while applying some diffusion to the problematic higher frequencies. The Fourier multiplier Q_N is of the form

$$Q_N(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d, |\mathbf{k}| \leq N} \hat{Q}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (\text{C.13})$$

In order to have convergence, the Fourier coefficients of Q_N need to fulfill [26], [52] [53]

$$\hat{Q}_k = 0 \text{ for } |k| \leq m_N, 1 - \left(\frac{m_N}{|k|} \right)^{\frac{2s-1}{\theta}} \leq \hat{Q}_k \leq 1 \quad (\text{C.14})$$

where we have introduced an additional parameter $\theta > 0$. The quantities m_N and ε_N are required to scale as

$$m_N \sim N^\theta, \varepsilon_N \sim \frac{1}{N^{2s-1}}, 0 < \theta < \frac{2s-1}{2s}. \quad (\text{C.15})$$

For the experiment described here, we choose $s = 1$, $m_N = \sqrt{N}$, $\varepsilon_N = \frac{0.05}{N}$, and $N = 128$. This gives rise to the viscosity $\nu \approx 4 \cdot 10^{-4}$ mentioned above.

Applying the Fourier projection operator to the PDE C.12 causes the solutions to be bandlimited functions and therefore they only have finitely many nonzero basis function coefficients (at most N). By writing the above discretization in the Fourier basis, we transform the spatial derivatives into multiplications with the wave vectors k and obtain

$$\partial_t \hat{u}_k + ik^T \cdot \hat{B}_k + ik \hat{p}_k = -\nu |k|^2 \hat{u}_k \quad (\text{C.16})$$

where we have substituted $B = u \otimes u$. By requiring \hat{u}_k (and $\partial_t \hat{u}_k$) to be divergence free, we can compute the pressure \hat{p}_k to be

$$\hat{p}_k = -\frac{k^T \cdot \hat{B}_k \cdot k}{|k|^2}. \quad (\text{C.17})$$

Note that the pressure can be computed from local quantities only. This is in contrast to numerical methods solving the equations in physical space where the pressure is obtained as the solution to a Poisson equation. Finally, we can solve the incompressible Euler equations by computing

$$\partial_t \hat{u}_k + \left(\text{Id} - \frac{kk^T}{|k|^2} \right) \cdot \hat{b}_k = -\nu |k|^2 \hat{u}_k \quad (\text{C.18})$$

where $\hat{b}_k = ik^T \cdot \hat{B}_k$. Timestepping is done using a third-order strong stability preserving Runge-Kutta scheme (SSPRK3)

$$\begin{aligned} u^{(1)} &= u(t) + \Delta t \partial_t u(t) \\ u^{(2)} &= \frac{3}{4}u(t) + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t \partial_t u^{(1)} \\ u(t + \Delta t) &= \frac{1}{3}u(t) + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t \partial_t u^{(2)}. \end{aligned} \quad (\text{C.19})$$

Note that through the construction of the pressure field, the numerical scheme is not exactly divergence-free. It merely preserves the divergence of the initial conditions u_0 . We therefore implicitly project all the initial conditions onto divergence free vector fields. This operation is described by the Leray projection $\mathbb{P} : L^2(\Omega) \rightarrow \{u \in L^2(\Omega) \mid \operatorname{div} u = 0\}$ mapping $u \mapsto u - \nabla \Delta^{-1}(\operatorname{div} u)$. In Fourier space, this can again be simplified to the local equation

$$\mathbb{P}\hat{u}_k = \left(\operatorname{Id} - \frac{kk^T}{|k|^2} \right) \cdot \hat{u}_k. \quad (\text{C.20})$$

For the training set, we generate a total of 750 samples. In addition, we generate 128 samples for validation set, in-distribution and out-of-distribution testing. To generate the training and test data, we simulate the Navier-Stokes equations with a spectral viscosity method on a 128×128 resolution and downsample the data to a 64×64 resolution. The goal is to learn the operator mapping the initial velocity to velocity at $T = 1$. The training data is normalized to the interval $[0, 1]$. The testing data is normalized with the same normalization constants as the training data. In Figure 17, we present the empirical test error distributions for UNet, FNO and CNO models during in-distribution and out-of-distribution testing. We also plot a random in-distribution testing sample and an out-of-distribution testing sample, as well as predictions made by CNO, FNO and UNet in Figure 18. These figures demonstrate that CNO is clearly the best performing model, for both in-distribution and out-of-distribution testing, outperforming UNet and FNO significantly. Moreover, given the highly multiscale nature of this problem (see Figure 2 of Main Text for spectrograms), it is not surprising that the errors with all the models are higher than in the other **RPB** benchmarks.

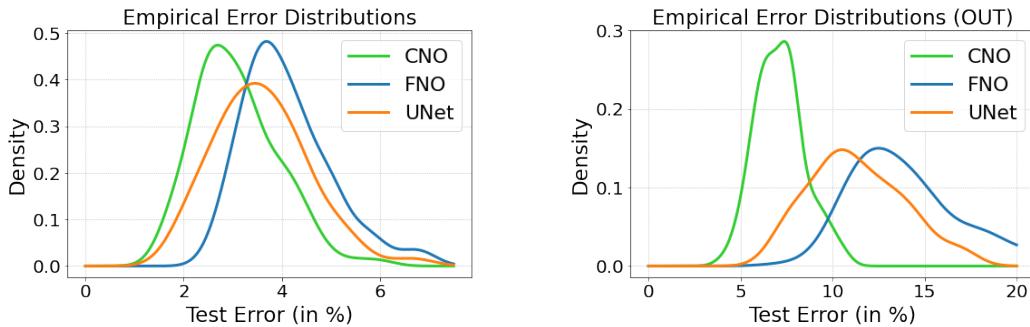


Figure 17: Navier-Stokes equations. Empirical test error distributions for UNet, FNO and CNO. Left: In-distribution testing. Right: Out-of-distribution testing.

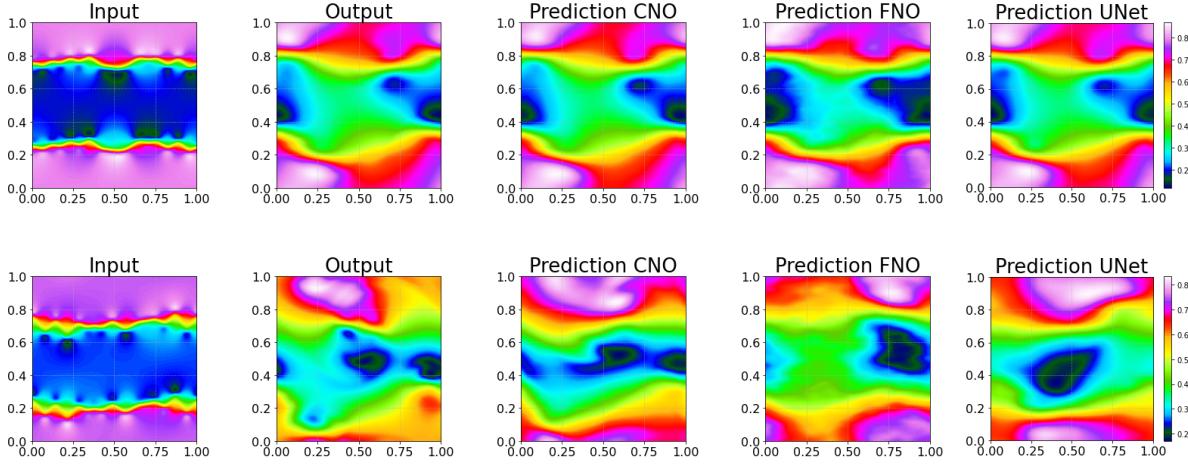


Figure 18: Navier-Stokes equations. Exact and predicted coefficients for an in-distribution (top row) and an out-of-distribution (bottom row) samples and for different models (columns). From left to right: input, output (ground truth), CNO, FNO and UNet.

C.3.6 Flow past airfoils

The flow past the airfoil is modeled by the two-dimensional compressible Euler equations

$$u_t + \operatorname{div} F(u) = 0, \quad u = [\rho, \rho v, E]^\perp, \quad F = [\rho v, \rho v \otimes v + p\mathbf{I}, (E + p)v]^\perp, \quad (\text{C.21})$$

with density ρ , velocity v , pressure p and total Energy E related by the ideal gas equation of state:

$$E = \frac{1}{2}\rho|u|^2 + \frac{p}{\gamma - 1}, \quad (\text{C.22})$$

where $\gamma = 1.4$. Additional important variables associated with the flow include the speed of sound $a = \sqrt{\frac{\gamma p}{\rho}}$ and the Mach number $M = \frac{|u|}{a}$.

We follow standard practice in aerodynamic shape optimization and consider a reference airfoil shape with upper and lower surface of the airfoil are located at $(x, y_{\text{ref}}^U(x/c))$ and $(x, y_{\text{ref}}^L(x/c))$ where c is the chord length and y_{ref}^U and y_{ref}^L corresponding to the well-known RAE2822 airfoil. The reference shape is then perturbed by *Hicks-Henne Bump functions* [39] :

$$\begin{aligned} y^L(\xi) &= y_{\text{ref}}^L(\xi) + \sum_{i=1}^{10} a_i^L B_i(\xi), & y^U(\xi) &= y_{\text{ref}}^U(\xi) + \sum_{i=1}^{10} a_i^U B_i(\xi), \\ B_i(\xi) &= \sin^3(\pi \xi^{q_i}), & q_i &= \frac{\ln 2}{\ln 14 - \ln i}, & \xi &= \frac{x}{c}, \\ a_i^L &= 2(\psi_i - 0.5)(i+1) \times 10^{-3}, & a_i^U &= 2(\psi_{i+10} - 0.5)(11-i) \times 10^{-3}, & i &= 1, \dots, 10 \end{aligned}$$

with $\psi \in [0, 1]^d$.

We can now formally define the airfoil shape as $\mathcal{S} = \{(x, y) \in D : x \in [0, c], y^L \leq y \leq y^U\}$ and accordingly the shape function $f = \chi_{[\mathcal{S}]}(x, y)$, with χ being the *characteristic function*. The

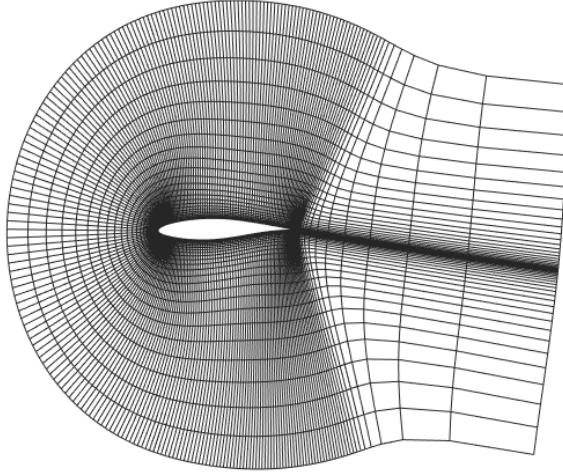


Figure 19: Elliptic mesh for the airfoil problem

underlying operator of interest $\mathcal{G}^\dagger : f \mapsto \rho$ maps the shape function f into the density of the flow at steady state of the compressible Euler equations.

The equations are solved with the solver NUWTUN on 243×43 elliptic mesh (Fig. 19) given the following free-stream boundary conditions,

$$T^\infty = 1, \quad M^\infty = 0.729, \quad p^\infty = 1, \quad \alpha = 2.31^\circ.$$

The data is ultimately interpolated onto a Cartesian grid of dimensions 128×128 on the underlying domain $D = [-0.75, 1.75]^2$, and unit values are assigned to the density $\rho(x, y)$ for all (x, y) in the set \mathcal{S} .

The shapes of the training data samples correspond to 20 bump functions, with coefficients ψ sampled uniformly from $[0, 1]^{20}$. Out-of-distribution testing is performed with 30 bump functions. During the training and evaluation processes, the difference between the learned solution and the ground truth is exclusively calculated for the points (x, y) that do not belong to the airfoil shape \mathcal{S} .

We generate 750 samples for the training set and 128 samples for validation set, in-distribution testing set and out-of-distribution testing set. In this experiment, the data is *not* normalized. In Figure 20, we show the empirical test error distributions for UNet, FNO and CNO models during in-distribution and out-of-distribution testing. We also show a random in-distribution testing sample and an out-of-distribution testing sample, as well as predictions made by CNO, FNO and UNet in Figure 21. The latter figure clearly shows the superiority of CNO and UNet over FNO when it comes to out-of-distribution testing.

C.3.7 On the Choice of the RPB benchmarks.

As noted in the main text, the rationale for the inclusion of benchmark experiments in the **RPB** dataset presented here is three-fold. First, we would like to span a variety of PDEs, ranging from

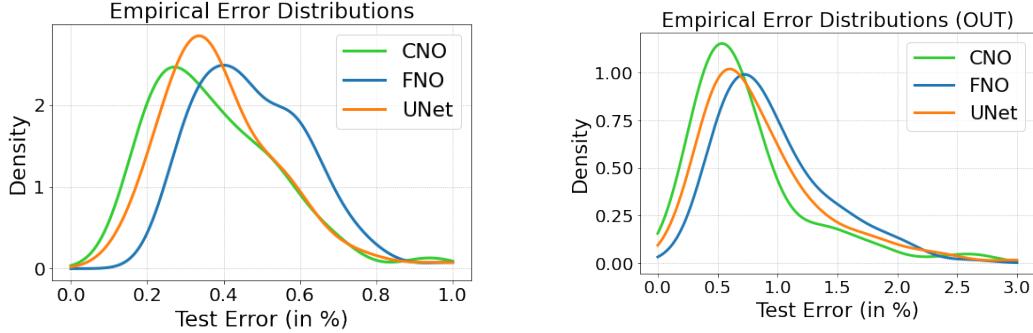


Figure 20: Airfoil experiment. Empirical test error distributions for UNet, FNO and CNO. Left: In-distribution testing. Right: Out-of-distribution testing.

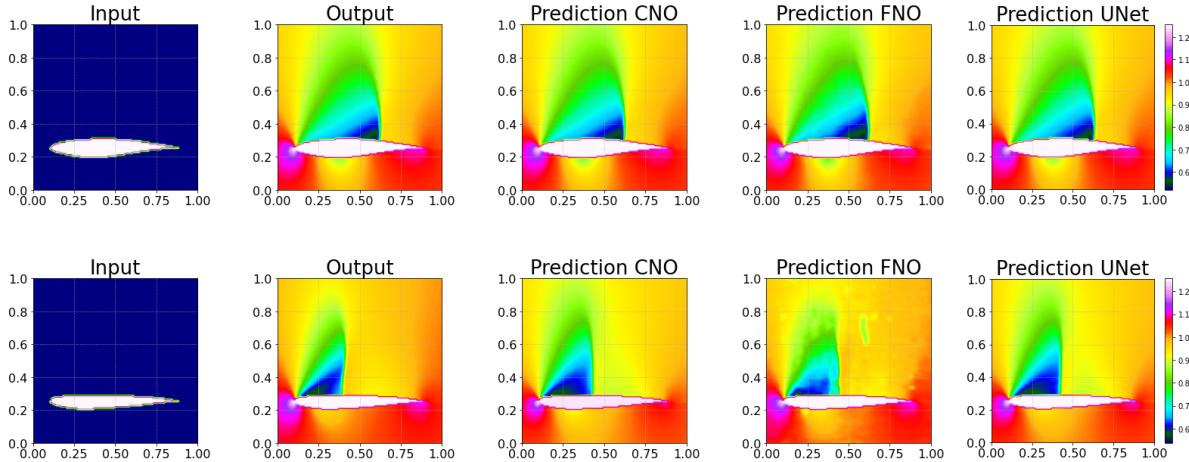


Figure 21: Airfoil experiment. Exact and predicted coefficients for an in-distribution (top row) and an out-of-distribution (bottom row) samples and for different models (columns). From left to right: input, output (ground truth), CNO, FNO and UNet.

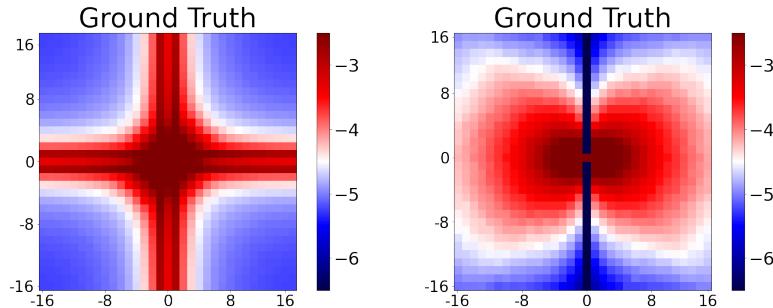


Figure 22: Comparison of the 32 central frequencies of averaged logarithmic amplitude spectra for the two Navier-Stokes experiments. Left: Old NS experiment. Right: Thin shear layer experiment.

linear elliptic (Poisson) to linear hyperbolic (wave, transport) to nonlinear parabolic (Allen-Cahn) to nonlinear hyperbolic (Compressible Euler) to non-local advection-diffusion (Incompressible Navier-Stokes). Second, we would like the underlying data to be readily available for rapid prototyping and reproducibility. This limits the use of three-dimensional data-sets as data access can be cumbersome. This requirement also leads us to prioritize problems with available analytical solutions. Finally, the selected benchmarks should be *sufficiently computationally complex* such that traditional numerical methods for approximating them are expensive and there is a potential pay-off for the design of efficient machine learning based surrogates. This criterion rules out one-dimensional (in space) problems as traditional numerical methods are very fast in this case on modern computers and there is little reason to discard them for ML surrogates. Even among two-dimensional problems, one has to be careful in selecting appropriate benchmarks to ensure that they entails sufficient computational complexity.

We illustrate this issue by comparing and contrasting two possible benchmarks. First, we consider a *Navier-Stokes* data-set, considered in [32] and widely used in the recent literature on machine learning for PDEs. In this problem, the incompressible Navier-Stokes equations (4.6) are recast in the so-called *velocity-vorticity* formulation by considering the vorticity $\omega = \nabla \times u$ of the fluid. In two space dimensions, the following evolution equation for the vorticity can be readily derived from (4.6),

$$\omega_t + (u \cdot \nabla)\omega = \nu \Delta \omega, \quad \omega(0, \cdot) = \omega_0. \quad (\text{C.23})$$

We consider the above evolution of the vorticity with periodic boundary conditions. The underlying solution operator maps the initial vorticity ω_0 to the vorticity $\omega(\cdot, T)$ at a final time T . Following [32], we choose the initial conditions $\omega_0 \sim \mu$ where $\mu = \mathcal{N}(0, 7^{\frac{3}{2}}(-\Delta + 49\text{Id})^{-2.5})$ and extend (C.23) with a forcing term $f(x) = 0.1(\sin(2\pi(x_1 + x_2)) + \cos(2\pi(x_1 + x_2)))$. Furthermore, the viscosity is chosen to be $\nu = 10^{-3}$. To generate the training and test data, we use a spectral method such as the one suggested in [26] and references therein. A rough estimate on the computational complexity of this problem can already be formed by observing Figure 22 (Left) where we present the averaged logarithmic amplitude spectra corresponding to the ground truth output (vorticity at time $T = 30$ as considered in [32]). We clearly see from this figure that only very few frequency modes (2-3) in each direction have relatively high amplitude and the spectrum decays quite fast for higher frequencies. Thus, this problem could be potentially approximated to high accuracy on fairly coarse grids.

To provide a quantitative elaboration of the above argument, we write $u_i^{N_f} = \mathcal{P}_{N_f}(u_i)$ where u_i is the solution corresponding to the i -th drawn initial conditions and \mathcal{P}_N is the spatial Fourier projection operator mapping a function $f(x, t)$ to its first N Fourier modes: $\mathcal{P}_N = \sum_{|k|_\infty \leq N} \widehat{f}_k(t) e^{ik \cdot x}$. For each sample u_i we compute the relative L^1 error against the downsampled solution $u_i^{N_f}$. This provides us with an estimate how many Fourier modes need to be accurately approximated in order to achieve reasonable errors. The supremum and median of the errors over 128 samples, at time $T = 30$, are plotted in Figure 23. One can observe from this figure that even after $t = 30$ time units, only a maximum of 20 Fourier modes (in each direction) are needed to approximate the solution with an error of approximately 1%. Hence, a standard

numerical method would only need to simulate it on a grid of 20×20 points will suffice in order to achieve the same error. Consequently, the time requirements for solving the problem on very coarse mesh with traditional spectral or finite difference methods are in the range of 10^{-3} seconds or lower. In contrast, we tested both FNO and CNO on this dataset to obtain test errors of 1.15% and 0.96%, respectively. Moreover, the inference time for both FNO and CNO in this case are of the order of 10^{-4} secs on a NVIDIA quadro t2000 GPU. Thus to achieve similar test errors, FNO and CNO are almost only one order of magnitude faster than a traditional numerical method. Given the training time and data generation overheads, it is clear that there is very little payoff on using such a relatively simple two-dimensional problem as a benchmark for ML surrogates for PDEs.

On the other hand, we perform exactly the same analysis for the *thin shear layer* problem for the incompressible Navier-Stokes equation that is described in the main text. First, from Figure 22 (Right), we see that the ground truth output (horizontal velocity at time $T = 1$) has much more of a multiscale structure than in the previous experiment (compare with Figure 22 (Left)) with at least non-trivial frequencies upto 32 modes, suggesting that it is much more challenging to approximate it numerically. This is indeed verified from Figure 23 (Right) where we present the averaged (over 128 samples) L^1 -error for the velocity as a function of the number of modes to observe that almost 100 Fourier modes are needed to get an L^1 -error of 2%. This corresponds to a 100×100 spatial grid and even a state-of-the-art GPU implementation of the spectral viscosity method of [26] would require 10^{-1} seconds of run time. When compared to a CNO inference time of 10^{-4} secs for an error of approximately 3%, we see that the ML surrogate (CNO) provides *three orders of magnitude* or more of speedup in this case, making its deployment worthwhile. Thus, we have demonstrated the rationale for the choice of this benchmark, rather than the Navier-Stokes benchmark of [32], in our proposed **RPB** dataset.

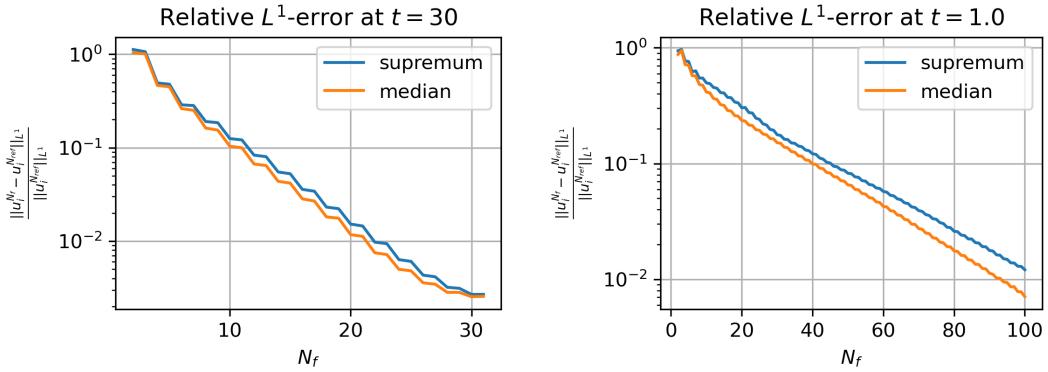


Figure 23: Relative L^1 -error of the vorticity experiment when restricting the solution to N_f Fourier modes.

C.4 Testing at Different Resolutions.

We have emphasized repeatedly that CNO upholds the principle of continuous-discrete equivalence (CDE), which implies that there is an equivalence between the underlying operator and its discrete representations. As a reminder, the CNO models are operators denoted as $\mathcal{G}^* : \mathcal{B}_w(D) \rightarrow \mathcal{B}_w(D)$ and are designed to ensure that the continuous representations of functions align with their discrete samples on a uniform grid. This holds true when the sampling rate s of the grid is sufficiently high, specifically $s \geq 2w$. It is important to note that the implemented CNO models are specified on a predefined *computational grid* with a sampling rate of $s \geq 2w$. Hence following [2] Remark 3.7, the input functions must be *compatible* with this grid. If the input function is not compatible with the computational grid, one needs transform it to an appropriate representation. Once the model is applied, the output is transformed back to the original representation (see Remark 3.5 of [2] for a formal explanation) and also Formula (A.7) for a precise description of these transformations.

Hence to apply an implemented CNO model to a continuous function $f \in \mathcal{B}_{w'}(D)$, it is necessary to employ a discrete representation of the function on a computational grid with a sampling rate of s . Essentially, it means that one needs to sample f on that grid. If the band limit w' exceeds half the sampling rate $s/2$, it is crucial to first filter out frequencies above $s/2$ to prevent aliasing effects, which involves applying a downsampling filter. Once the function's representation and the computational grid are compatible with each other, the model can be applied.

To apply an implemented CNO model to a discrete representation $f_{s'} \in \mathbb{R}^{s' \times s'}$, it is necessary to follow (A.7) and transform $f_{s'}$ into a compatible representation $f_s \in \mathbb{R}^{s \times s}$. If $s' \leq s$, the signal needs to be upsampled to the sampling rate s by using an appropriate upsampling filter. However, if $s' > s$, it is necessary to filter out frequencies above $s/2$ that are present in the signal. One should downsample the signal to the sampling rate s by applying an appropriate downsampling filter.

As highlighted in the main text, an important characteristic of an operator learning model is to maintain a relatively consistent test error when evaluated on various resolutions or discretizations. To assess this aspect, we evaluate the performance of UNet, FNO, and CNO models on different resolutions for three problems: Navier-Stokes equations, Poisson equation, and Wave equation. For the Navier-Stokes equations, the original data is generated at a resolution of 128×128 . To obtain data at any lower resolution $s < 128$, we downsample the original data to the desired resolution. As for the Wave equation and the Poisson equation, exact solutions are available. To generate data for these equations, we sample the exact solutions on the desired grid, allowing us to obtain data at different resolutions. The models that we use to make predictions are the ones that we trained on 64×64 resolution. The configurations of all the models are reported in C.2.

We apply the afore-described strategy to practically realize Formula (A.7) and apply CNO to different resolutions. In contrast, we follow the approach outlined in [32] to evaluate the FNO or UNet models at different resolutions by applying the underlying model directly to the original, unresized input.

We show the variations of the test errors across resolutions for the three benchmarks in Figure 2, right (Navier-Stokes) and Figure 24 (left for Poisson equation, right for Wave equation). The CNO model demonstrates the highest stability when it comes to resolution changes and is (approximately) invariant to resolution, unlike the other two models which exhibit notable fluctuations at different resolutions. Specifically, the UNet model displays a strong reliance on the training resolution, whereas the FNO model exhibits a slightly less pronounced dependence. These examples show that the CNO model respects continuous-discrete equivalence, while the other two models are not resolution (or representation) equivalent.

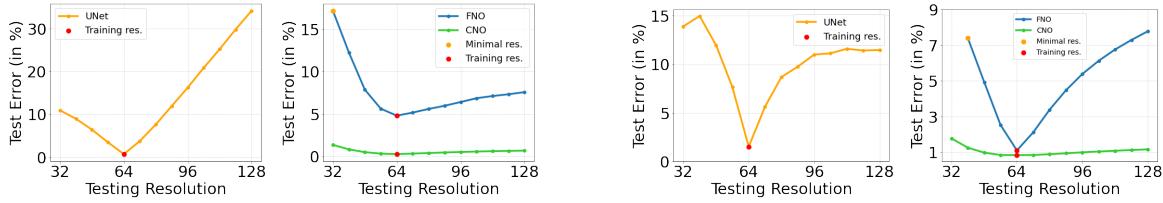


Figure 24: Test error vs. Resolution for UNet, FNO and CNO. Left: Poisson equation. Right: Wave equation.

C.5 Ablation Studies.

We conduct two ablation studies focusing on two key aspects of CNO. Firstly, we examine the impact of modified operations, assessing how they affect the overall performance. Secondly, we investigate the influence of ResNets that connect the Encoder and Decoder components within the Operator UNet architecture (refer to Figure 1). These studies aim to provide valuable insights into the effects of these key elements.

In our first ablation study, we aim to evaluate the effects of modifying operations, including upsampling operators, downsampling operators, and activation layers, on performance and training time. It is worth reiterating that the modified operations enable *continuous-discrete equivalence* (CDE). Specifically, we replace the upsampling operator in the Operator UNet architecture with a discrete, nearest neighbor upsampling method, while the downsampling operator is substituted with average pooling. Additionally, we replace the activation layer with a simple pointwise application of the activation function. As a result, the model takes on a structure resembling a regular UNet architecture, but with the inclusion of additional ResNets that establish connections between the Encoder and the Decoder components. We will refer to this model as *CNO w/o Filters*.

The second ablation study focuses on evaluating the influence of additional ResNets that connect the Encoder and the Decoder components on both the overall performance and training time. In this study, we remove these ResNets while retaining the UNet-like concatenations between corresponding levels of the Encoder and the Decoder. It is important to note that the ResNet between the deepest levels of the Encoder and the Decoder is preserved within the model. This ablation model respects the *continuous-discrete equivalence* (CDE).

Performance. We train two ablation models for every benchmark experiment that we studied in the main text. In order to maintain consistency, we use the same hyperparameter configurations for the ablation models as those of the best-performing CNO models (refer to Table 10 for the specific values). We report the in-distribution and out-of-distribution test errors in the Table 11.

Among the 14 tests conducted, the original CNO model outperforms the others in 12 of them. In all of the tests conducted, the first ablation model exhibits inferior performance compared to the original CNO model. This observation indicates that the aliasing errors resulting from regular CNN operations like average pooling, nearest neighbor upsampling and a regular application of the activation function have an impact on the test error. Furthermore, it is important to note that the first ablation study does not adhere to the continuous-discrete equivalence (CDE) property, resulting in the model’s resolution dependence, similar to the UNet model (see Figure 2 and Section C.4).

In two out-of-distribution tests, the second ablation model demonstrates slightly superior performance compared to the original CNO model. It is worth noting that in most cases, the original CNO model exhibits significantly better performance than the second ablation model. This disparity in performance ranges from a 10% in the Compressible Euler benchmark to a 40% in the Allen-Cahn benchmark, and even up to a 95% in the out-of-distribution Poisson benchmark. While it is true that the second ablation model maintains the continuous-discrete equivalence (CDE) property, we observe that the inclusion of ResNets is vital for achieving good performance and decent generalization.

Training time. Since the first ablation model does not utilize any interpolation filters, it is reasonable to anticipate that it will have a faster training time than the original CNO model. Specifically, it trains approximately 1.5 times faster for the Compressible Euler, Discontinuous Transport, and Wave Equation benchmarks. For the Allen-Cahn Equation, it trains around 1.75 times faster, while for the Continuous Transport and Poisson Equation benchmarks, it trains approximately 2 times faster. Finally, for the Navier-Stokes Equations, it trains about 2.5 times faster.

The second ablation model, which excludes the middle ResNets from the architecture, is expected to have a significantly faster training process. Specifically, it trains approximately 1.5 times faster for the Discontinuous Transport and Poisson Equation benchmarks. For the Continuous Transport. For the Compressible Euler benchmark, it trains around 2 times faster, while for the Wave Equation, it trains approximately 2.25 times faster. Finally, for the Navier-Stokes Equations and Allen-Cahn Equation, it trains about 3 times faster.

However, it is important to note that even the CNO models themselves have notably fast training times, comparable to the training times of the best-performing FNO models, with the shortest training time observed in the Navier-Stokes benchmark (approximately 0.7 hours for CNO and 0.6 hours for FNO for 1000 epochs) and the longest training time observed in the Compressible Euler benchmark (approximately 3.5 hours for CNO and 2.7 hours for FNO

for 1000 epochs). Our observations indicate that there exists a trade-off between two aspects: performance and the continuous-discrete equivalence (CDE) property on one side, and training time on the other.

Table 11: Relative median L^1 test errors, for both in- and out-of-distribution testing, for the CNO models and two ablation models.

	In/Out	CNO	CNO w/o Filters	CNO w/o ResNets
Poisson Equation	In	0.23%	0.31%	0.64%
	Out	0.27%	0.54%	5.37%
Wave Equation	In	0.83%	1.18%	1.01%
	Out	1.48%	1.68%	1.61%
Smooth Transport	In	0.30%	0.43%	0.71%
	Out	0.45%	2.08%	0.91%
Discontinuous Transport	In	1.17%	1.29%	1.29%
	Out	1.61%	1.68%	1.55%
Allen-Cahn	In	0.83%	0.97%	1.29%
	Out	3.67%	5.61%	4.30%
Navier-Stokes	In	3.01%	3.51%	4.34%
	Out	7.06%	9.35%	6.87%
Compressible Euler	In	0.35%	0.40%	0.39%
	Out	0.62%	0.86%	0.69%

C.6 Error vs. number of training samples.

Once again, we revisit the best-performing CNO and FNO model architectures for the Poisson equation and Wave equation, as reported in C.2. This time, we focus on varying the number of training samples and retraining the selected CNO and FNO models accordingly. Consequently, we generate a plot that illustrates the in-distribution test error as we change the cardinality of the training set, as shown in Figure 25 (left for Poisson equation, right for Wave equation). In the case of Poisson equation, the CNO model outperforms by far the FNO model in all the data regimes. In the case of Wave equation, we notice that the FNO performs better than the CNO in low data regime, with the opposite behaviour in the large data regime. Moreover, CNO shows an approximately error decay rate of 0.5 with respect to the number of training samples.

D Depiction of the Datasets.

In the following figures, we illustrate the different PDE forward problems considered in the main text.

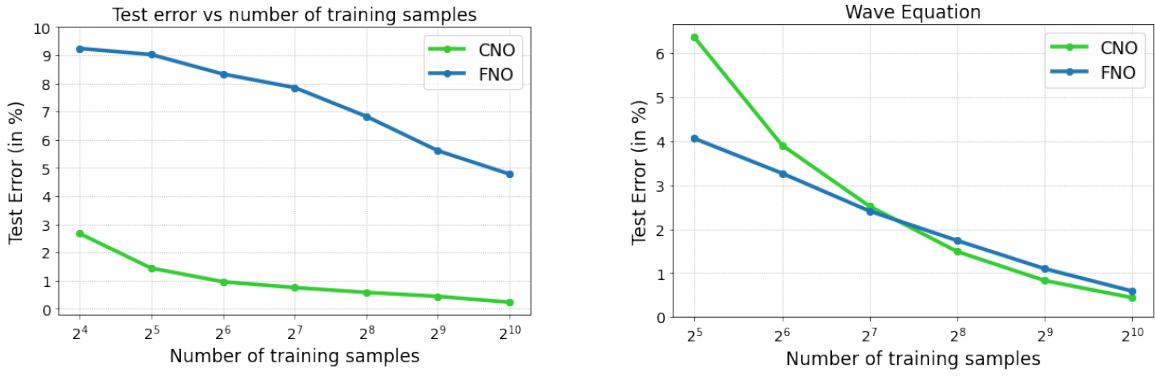


Figure 25: In-distribution testing errors for different cardinalities of the training set for FNO and CNO. Left: Poisson equation. Right: Wave equation.

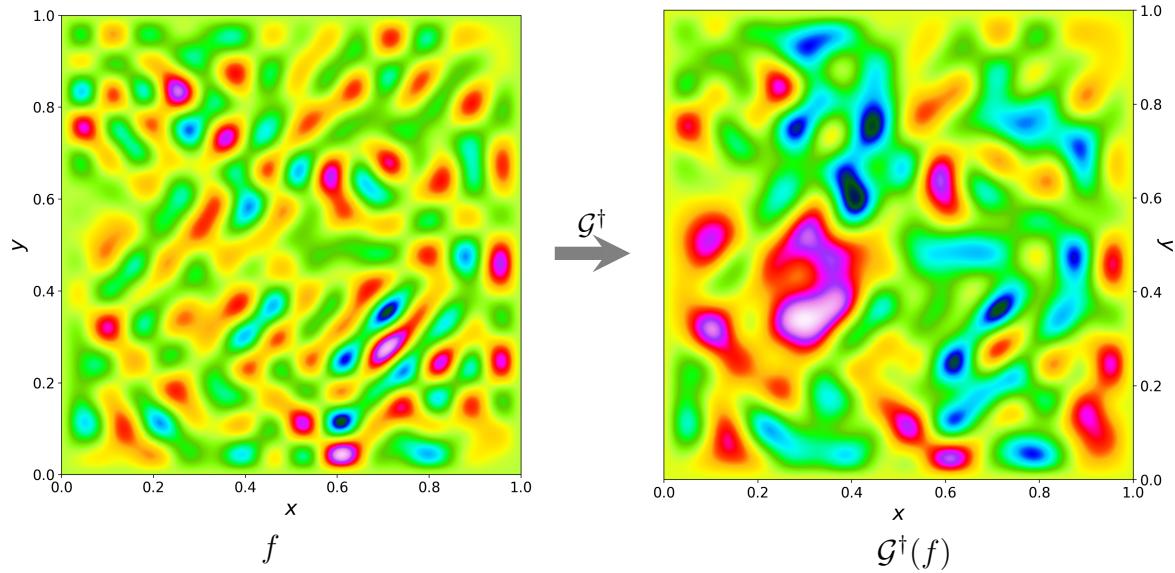


Figure 26: Illustration of input (left) and output (right) samples for the Poisson Equation.

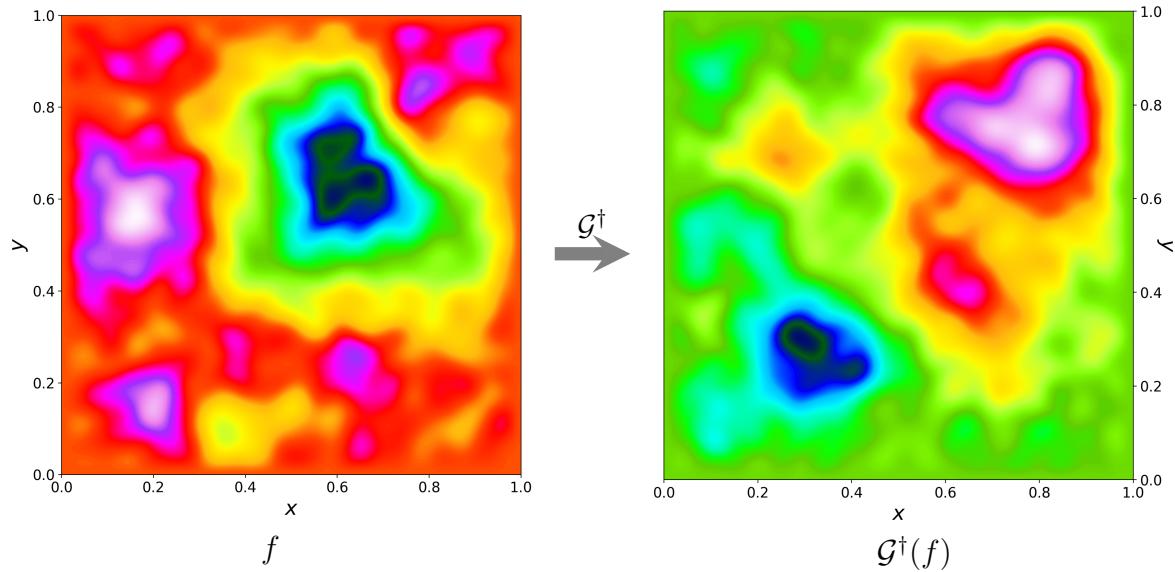


Figure 27: Illustration of input (left) and output (right) samples for the Wave Equation.

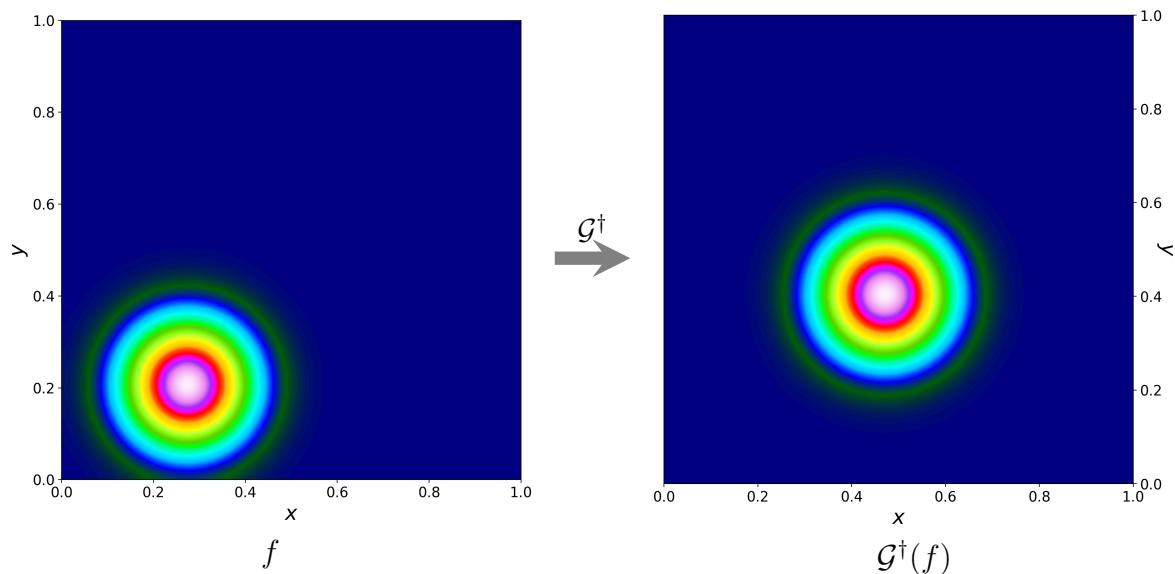


Figure 28: Illustration of input (left) and output (right) samples for the Continuous Transport.

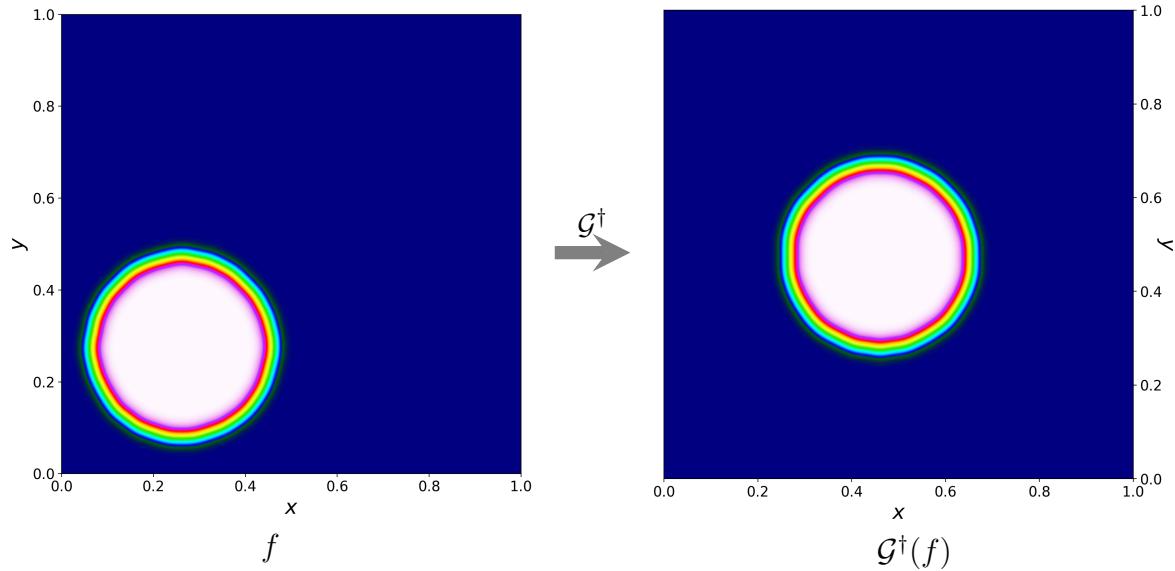


Figure 29: Illustration of input (left) and output (right) samples for the discontinuous transport problem.

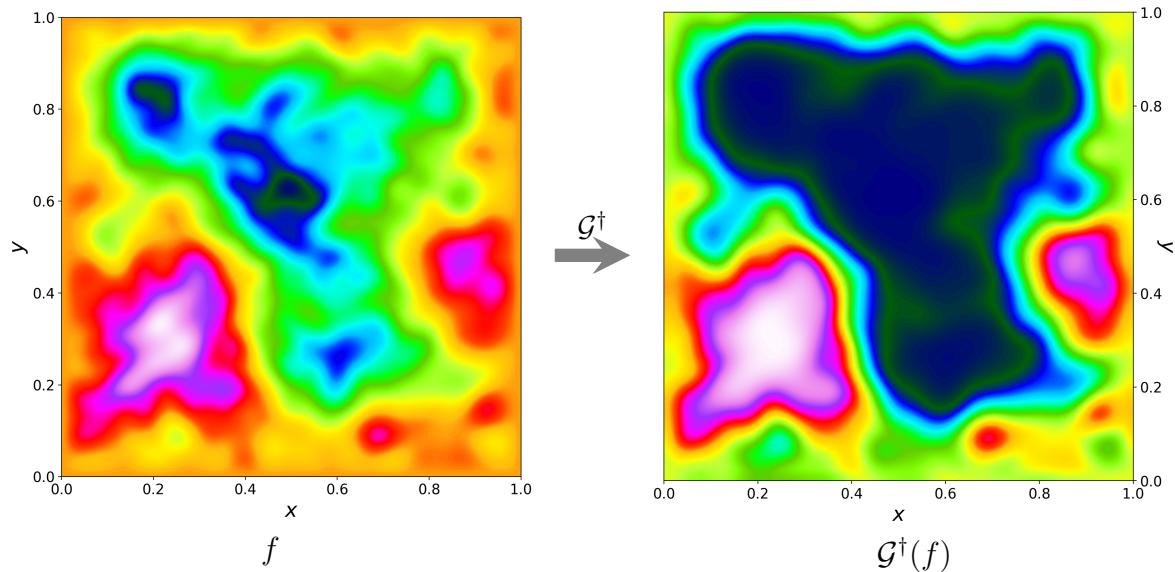


Figure 30: Illustration of input (left) and output (right) samples for the Allen-Cahn.

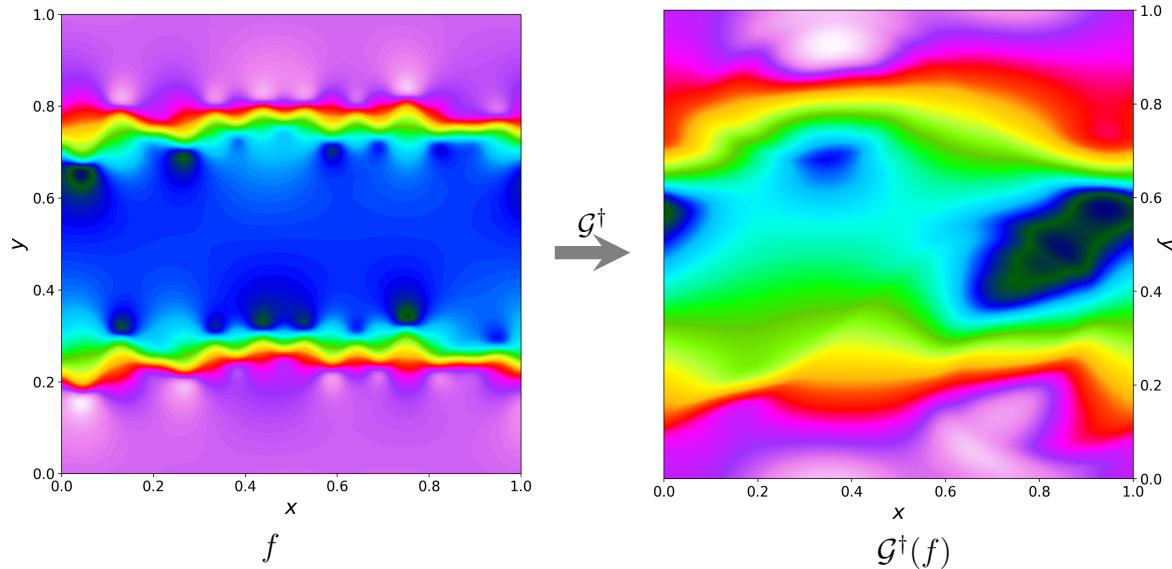


Figure 31: Illustration of input (left) and output (right) samples for the Navier Stokes equation.

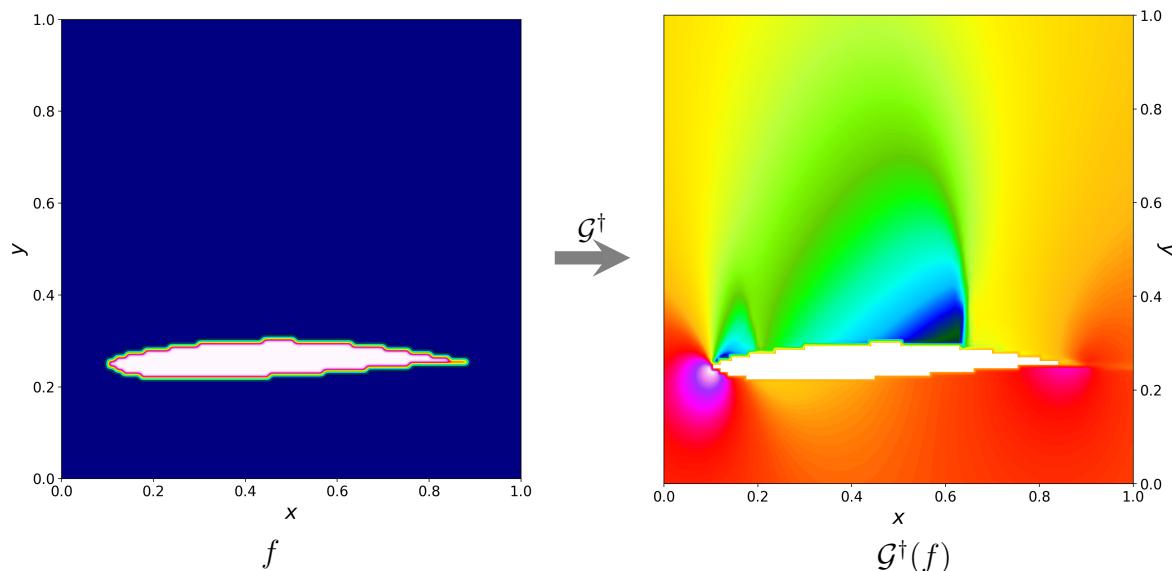


Figure 32: Illustration of input (left) and output (right) samples for the compressible Euler equations.

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