Lecture 6: Value Function Approximation

1. Introduction

Large-Scale Reinforcement Learning

Reinforcement learning can be used to solve large problems, e.g.

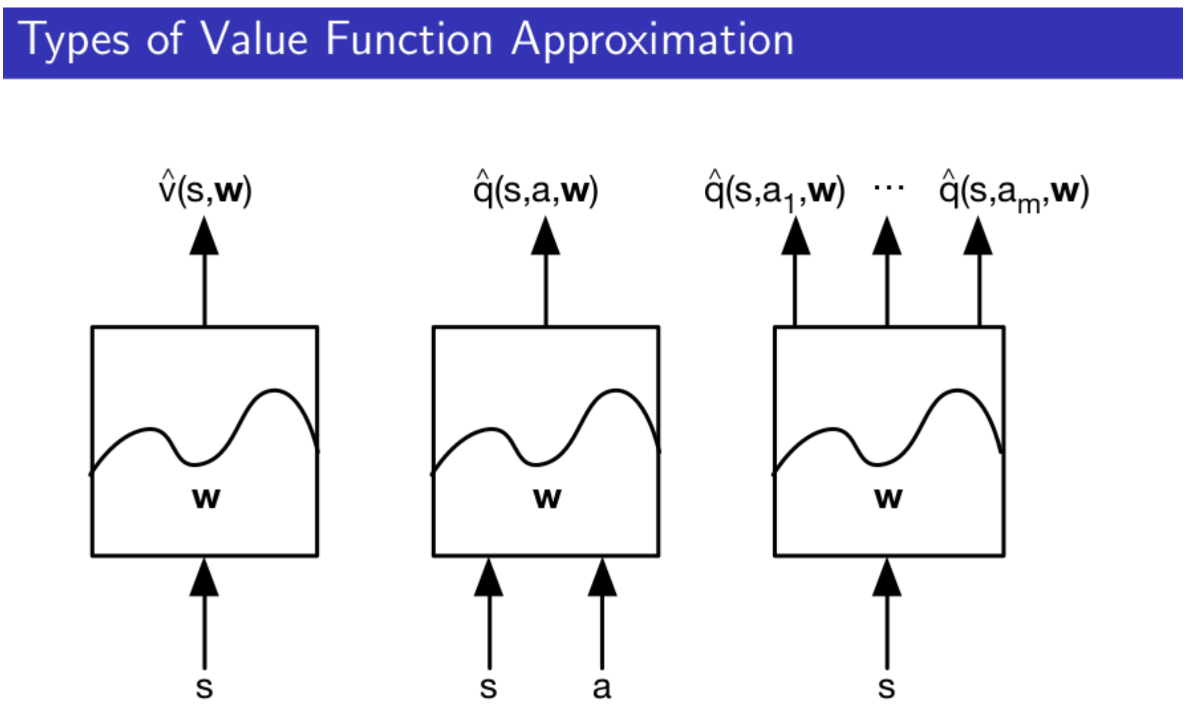
- Backgammon: 10^{20} states
- Computer Go: 10^{170} states
- Helicopter: continuous state space

Value Function Approximation

- So far we have represented value function by a lookup table
  - Every state s has an entry $V(s)$
  - Or every state-action pair s, a has an entry $Q(s, a)$
- Problem with large MDPs:
  - There are too many states and/or actions to store in memory
  - It is too slow to learn the value of each state individually
- Solution for large MDPs:
  - Estimate value function with function approximation

$$\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$$
$$or \quad \hat{q}(s, a, \mathbf{w}) \approx q_\pi(s, a)$$

  - Generalise from seen states to unseen states
  - Update parameter $\mathbf{w}$ using MC or TD learning



Which Function Approximator?

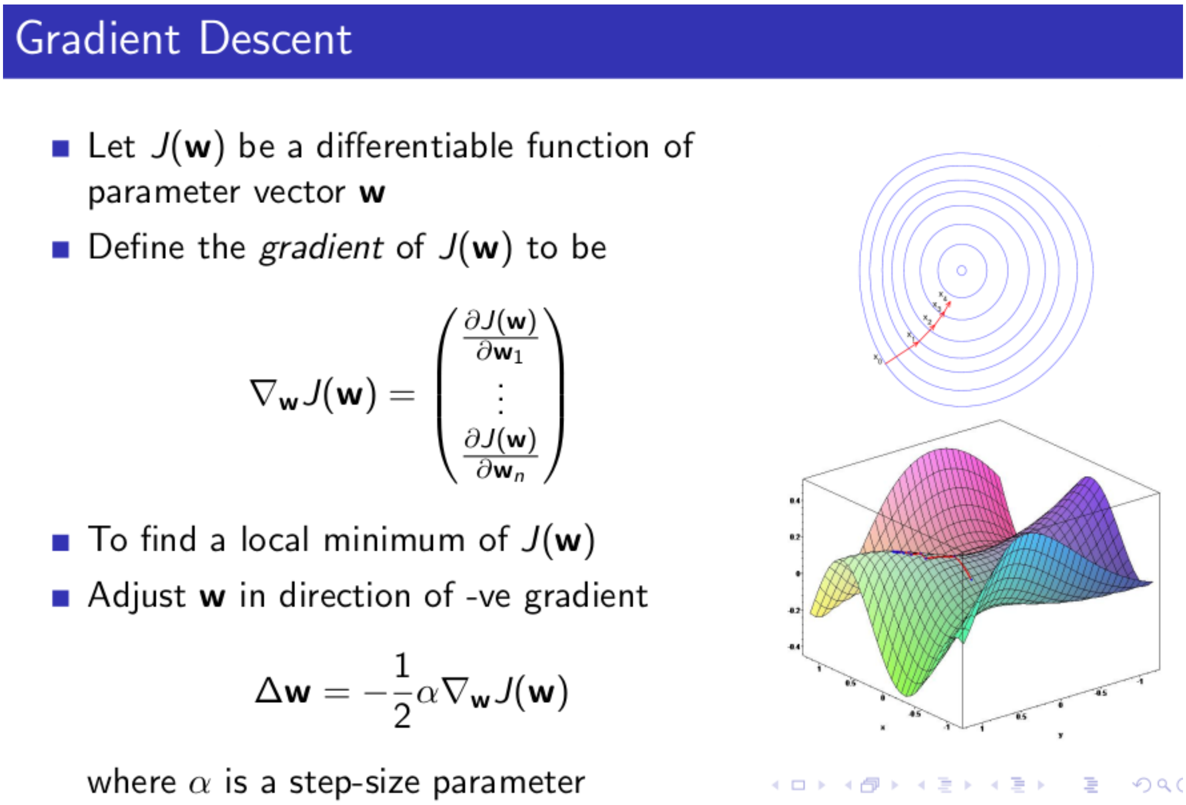
We consider differentiable function approximators, e.g.

- **Linear combinations of features**
- **Neural network**
- Decision tree
- Nearest neighbour
- Fourier / wavelet bases

Furthermore, we require a training method that is suitable for **non-stationary, non-iid** data

2. Incremental Methods

Gradient Descent



Value Function Approx. By Stochastic Gradient Descent

- Goal: find parameter vector $\mathbf{w}$ minimising mean-squared error between approximate value $\hat{v}(s, \mathbf{w})$ and true value fn $v_\pi(s)$

$$J(\mathbf{w}) = E_\pi[(v_\pi(S) - \hat{v}(S, \mathbf{w}))^2]$$

- Gradient descent finds a local minimum

$$\begin{aligned} \triangle \mathbf{w} &= -\frac{1}{2}\alpha \nabla_{\mathbf{w}} J(\mathbf{w}) \\ &= \alpha E_\pi[(v_\pi(S) - \hat{v}(S, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w})] \end{aligned}$$

- Stochastic gradient descent samples the gradient

$$\triangle \mathbf{w} = \alpha(v_\pi(S) - \hat{v}(S, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w})$$

- Expected update is equal to full gradient update

Linear Function Approximation

Feature Vectors

- Represent state by a feature vector

$$\mathbf{x}(S) = \begin{pmatrix} \mathbf{x}_1(S) \\ \vdots \\ \mathbf{x}_n(S) \end{pmatrix}$$

- For example:
  - Distance of robot from landmarks
  - Trends in the stock market
  - Piece and pawn configurations in chess

Linear Value Function Approximation

- Represent value function by a linear combination of features

$$\hat{v}(S, \mathbf{w}) = \mathbf{x}(S)^T \mathbf{w} = \sum_{j=1}^{n} \mathbf{x}_j(S)\mathbf{w}_j$$

- Objective function is quadratic in parameters $\mathbf{w}$

$$J(\mathbf{w}) = E_\pi[(v_\pi(S) - \mathbf{x}(S)^T \mathbf{w})^2]$$

- Stochastic gradient descent converges on global optimum
- Update rule is particularly simple

$$\begin{aligned} \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) &= \mathbf{x}(S) \\ \triangle \mathbf{w} &= \alpha(v_\pi(S) - \hat{v}(S, \mathbf{w}))\mathbf{x}(S) \\ \text{Update} &= \text{step-size} \times \text{prediction error} \times \text{feature value} \end{aligned}$$

Table Lookup Features

- Table lookup is a special case of linear value function approximation
- Using table lookup features

$$\mathbf{x}^{table}(S) = \begin{pmatrix} \mathbf{1}(S = s_1) \\ \vdots \\ \mathbf{1}(S = s_n) \end{pmatrix}$$

- Parameter vector $\mathbf{w}$ gives value of each individual state

$$\hat{v}(S, \mathbf{w}) = \begin{pmatrix} \mathbf{1}(S = s_1) \\ \vdots \\ \mathbf{1}(S = s_n) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{pmatrix}$$

Incremental Prediction Algorithms

- Have assumed true value function $v_\pi(s)$ given by supervisor
- But in RL there is no supervisor, only rewards
- In practice, we substitute a target for $v_\pi(s)$
  - For MC, the target is the return G_t

$$\triangle \mathbf{w} = \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w})$$

- For TD(0), the target is the TD target $R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w})$

$$\triangle \mathbf{w} = \alpha(R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}) - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w})$$

- For TD(λ), the target is the λ-return G_t^λ

$$\triangle \mathbf{w} = \alpha(G_t^\lambda - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w})$$

Monte-Carlo with Value Function Approximation

- Return G_t is an unbiased, noisy sample of true value $v_\pi(S_t)$
- Can therefore apply supervised learning to "training data":

$$< S_1, G_1 >, < S_2, G_2 >, \ldots, < S_T, G_T >$$

- For example, using linear Monte-Carlo policy evaluation

$$\begin{aligned} \triangle \mathbf{w} &= \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w}) \\ &= \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\mathbf{x}(S_t) \end{aligned}$$

- Monte-Carlo evaluation converges to a local optimum
- Even when using non-linear value function approximation

TD Learning with Value Function Approximation

- The TD-target $R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w})$ is a biased sample of true value $v_\pi(S_t)$
- Can still apply supervised learning to "training data":

$$< S_1, R_2 + \gamma \hat{v}(S_2, \mathbf{w}) >, < S_2, R_3 + \gamma \hat{v}(S_3, \mathbf{w}) >, \ldots, < S_{T-1}, R_T >$$

- For example, using linear TD(0)

$$\begin{aligned} \triangle \mathbf{w} &= \alpha(R + \gamma \hat{v}(S', \mathbf{w}) - \hat{v}(S, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) \\ &= \alpha \delta \mathbf{x}(S) \end{aligned}$$

- Linear TD(0) converges (close) to global optimum

TD(λ) with Value Function Approximation

- The λ-return G_t^λ is also a biased sample of true value $v_\pi(s)$
- Can again apply supervised learning to "training data":

$$< S_1, G_1^\lambda >, < S_2, G_2^\lambda >, \ldots, < S_{T-1}, G_{T-1}^\lambda >$$

- Forward view linear TD(λ)

$$\begin{aligned} \triangle \mathbf{w} &= \alpha(G_t^\lambda - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w}) \\ &= \alpha(G_t^\lambda - \hat{v}(S_t, \mathbf{w}))\mathbf{x}(S_t) \end{aligned}$$

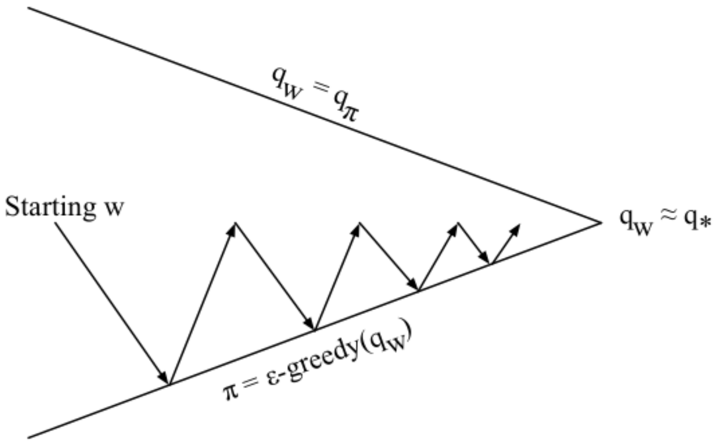
- Backward view linear TD(λ)

$$\begin{aligned} \delta_t &= R_{t+1} + \gamma \hat{v}(S_{t+1}) - \hat{v}(S_t, \mathbf{w}) \\ E_t &= \gamma \lambda E_{t-1} + \mathbf{x}(S_t) \\ \triangle \mathbf{w} &= \alpha \delta_t E_t \end{aligned}$$

Forward view and backward view linear TD(λ) are equivalent

Invremental Control Algorithms

Control with Value Function Approximation



Policy evaluation **Approximate** policy evaluation, $\hat{q}(\cdot, \cdot, \mathbf{w}) \approx q_\pi$

Policy improvement ϵ-greedy policy improvement

Action-Value Function Approximation

- Approximate the action-value function

$$\hat{q}(S, A, \mathbf{w}) \approx q_\pi(S, A)$$

- Minimise mean-squared error between approximate action-value fn $\hat{q}(S, A, \mathbf{w})$ and true action-value fn $q_\pi(S, A)$

$$J(\mathbf{w}) = E_\pi[(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))^2]$$

- Use stochastic gradient descent to find a local minimum

$$-\frac{1}{2}\nabla_\mathbf{w} J(\mathbf{w}) = (q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))\nabla_\mathbf{w}\hat{q}(S, A, \mathbf{w})$$
$$\triangle\mathbf{w} = \alpha(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))\nabla_\mathbf{w}\hat{q}(S, A, \mathbf{w})$$

Linear Action-Value Function Approximation

- Represent state and action by a feature vector

$$\mathbf{x}(S, A) = \begin{pmatrix} \mathbf{x}_1(S, A) \\ \vdots \\ \mathbf{x}_n(S, A) \end{pmatrix}$$

- Represent action-value fn by linear combination of features

$$\hat{q}(S, A, \mathbf{w}) = \mathbf{x}(S, A)^T\mathbf{w} = \sum_{j=1}^{n} \mathbf{x}_j(S, A)\mathbf{w}_j$$

- Stochastic gradient descent update

$$\nabla_\mathbf{w}\hat{q}(S, A, \mathbf{w}) = \mathbf{x}(S, A)$$
$$\triangle\mathbf{w} = \alpha(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))\mathbf{x}(S, A)$$

Incremental Control Algorithms

- Like prediction, we must substitute a target for $q_\pi(S, A)$

- For MC, the target is the return G_t

$$\triangle \mathbf{w} = \alpha(G_t - \hat{q}(S_t, A_t, \mathbf{w}))\nabla_{\mathbf{w}}\hat{q}(S_t, A_t, \mathbf{w})$$

- For TD(0), the target is the TD target $R_{t+1} + \gamma Q(S_{t+1}, A_{t+1})$

$$\triangle \mathbf{w} = \alpha(R_{t+1} + \gamma\hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}))\nabla_{\mathbf{w}}\hat{q}(S_t, A_t, \mathbf{w})$$

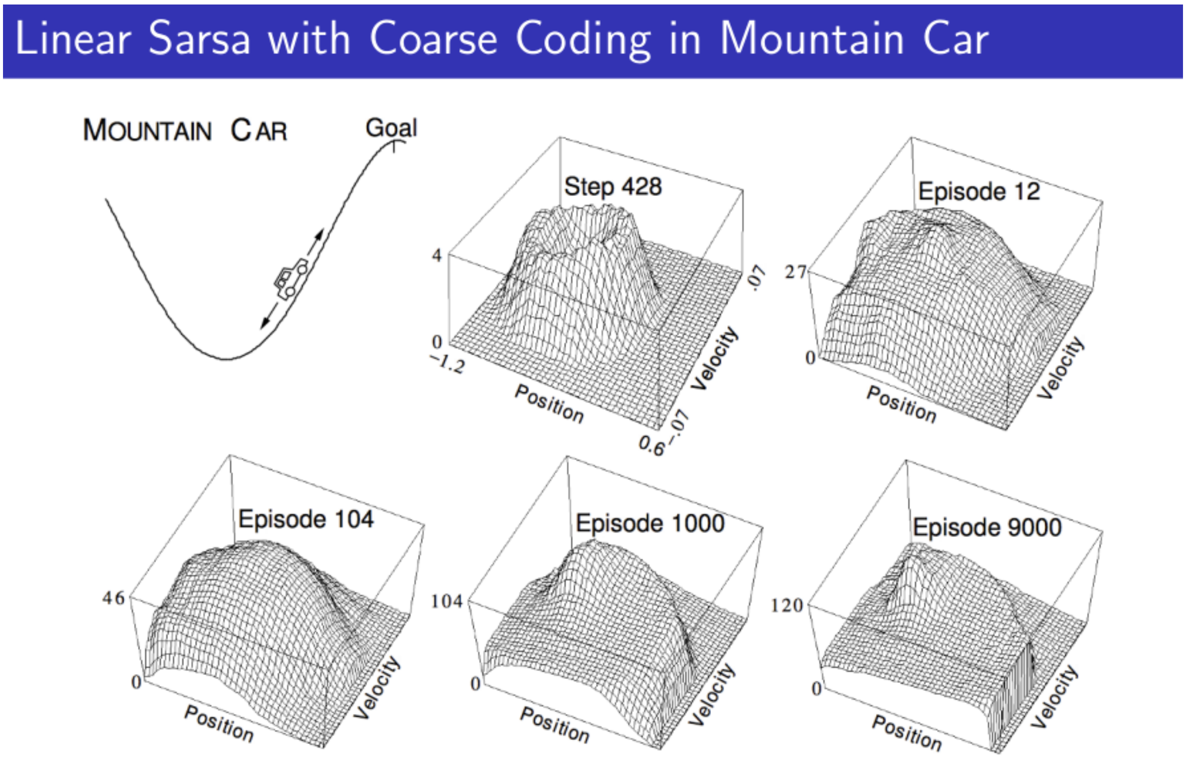
- For forward-view TD(λ), target is the action-value λ-return

$$\triangle \mathbf{w} = \alpha(q_t^\lambda - \hat{q}(S_t, A_t, \mathbf{w}))\nabla_{\mathbf{w}}\hat{q}(S_t, A_t, \mathbf{w})$$

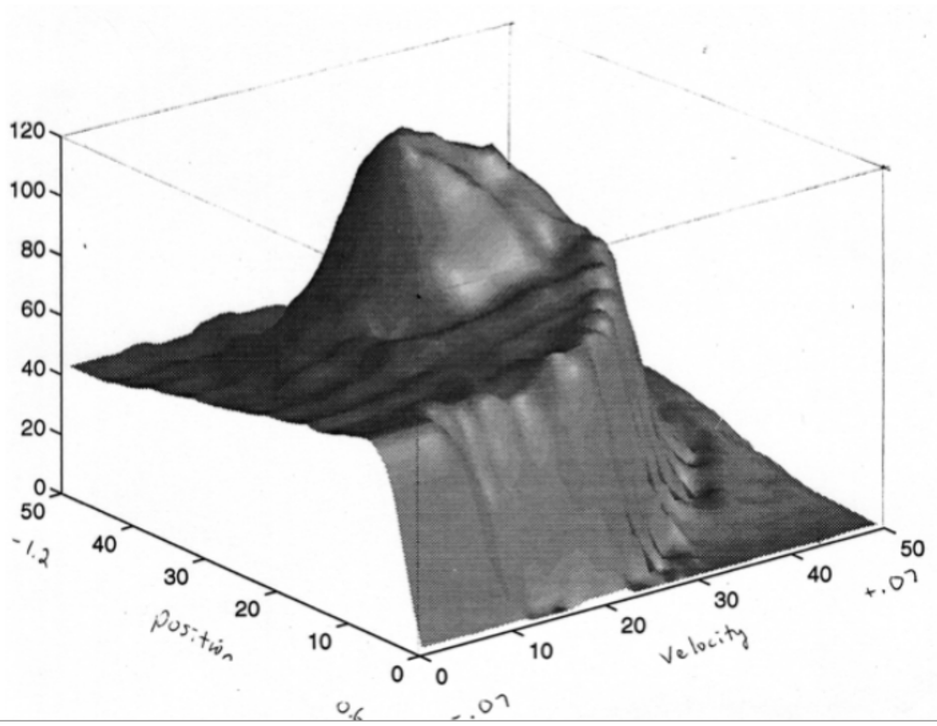
- For backward-view TD(λ), equivalent update is

$$\begin{aligned}
\delta_t &= R_{t+1} + \gamma\hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}) \\
E_t &= \gamma\lambda E_{t-1} + \nabla_{\mathbf{w}}\hat{q}(S_t, A_t, \mathbf{w}) \\
\triangle \mathbf{w} &= \alpha\delta_t E_t
\end{aligned}$$

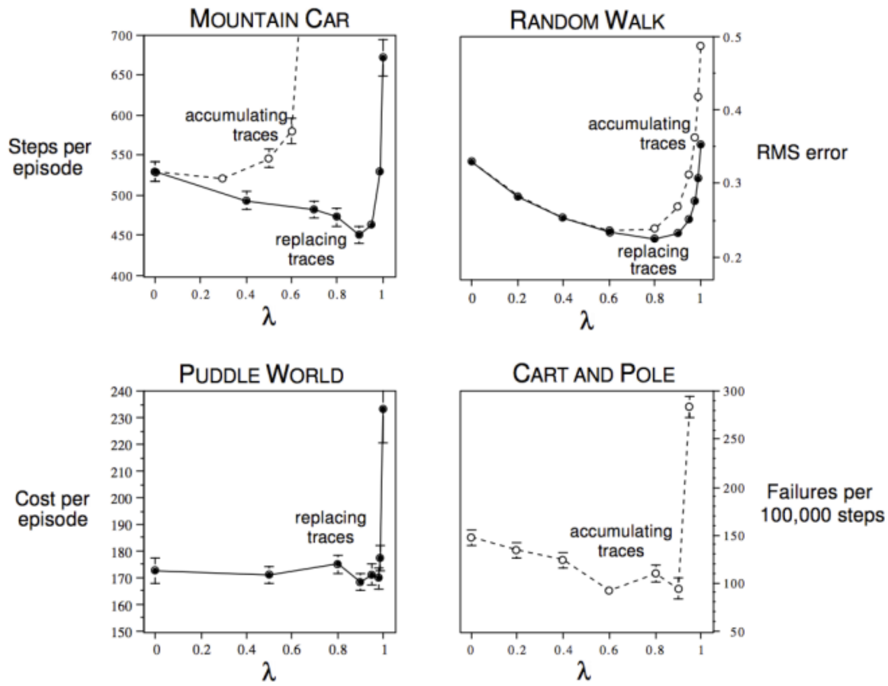
Mountain Car



Linear Sarsa with Radial Basis Functions in Mountain Car

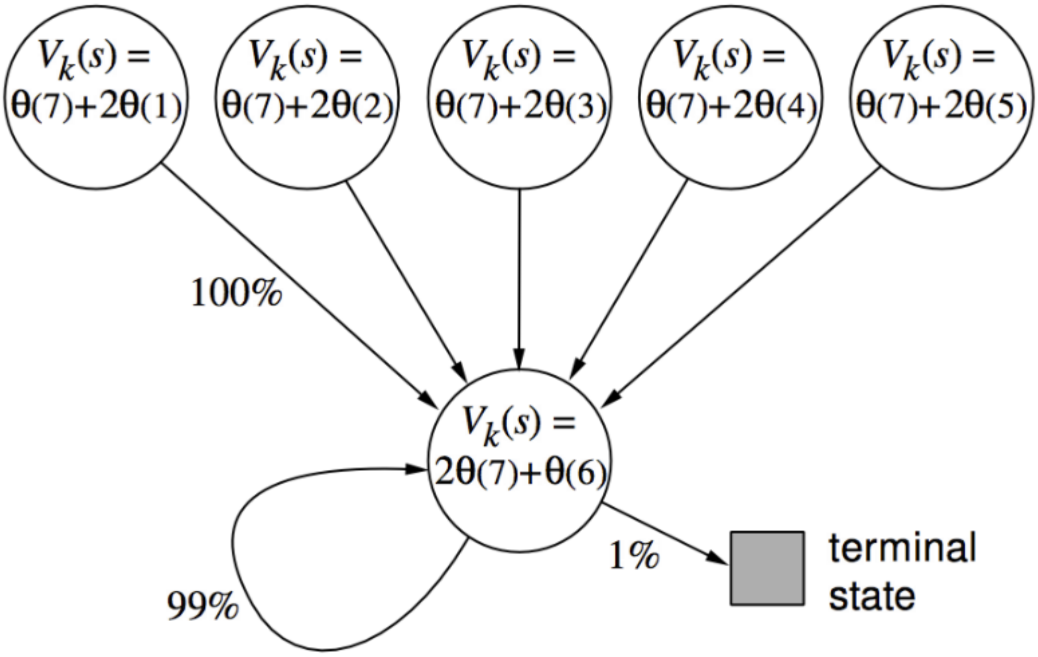


Study of λ: Should We Bootstrap?

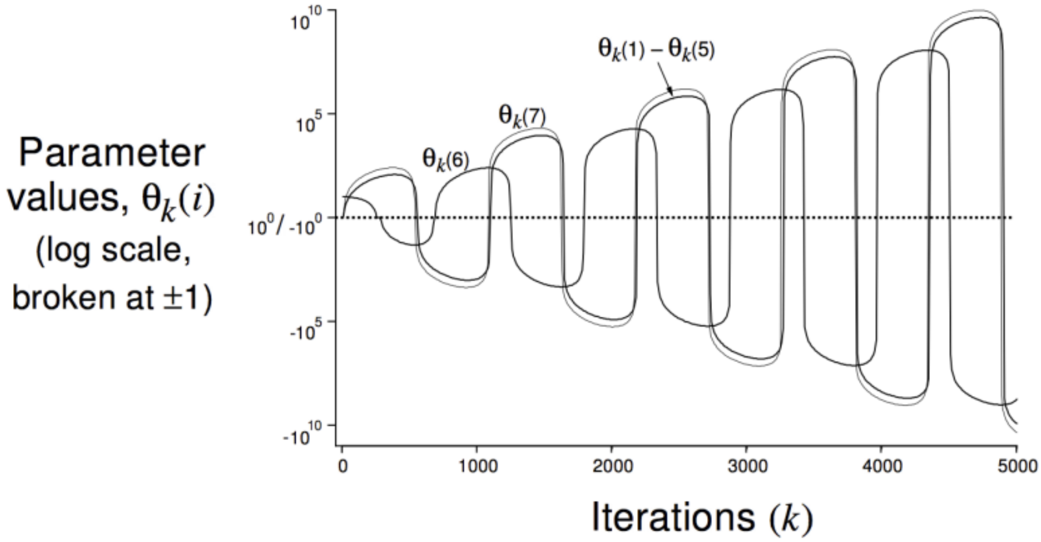


Convergence

Baird's Counterexample



Parameter Divergence in Baird's Counterexample



Convergence of Prediction Algorithms

On/Off-Policy	Algorithm	Table Lookup	Linear	Non-Linear
On-Policy	MC	✓	✓	✓
	TD(0)	✓	✓	✗
	TD(λ)	✓	✓	✗
Off-Policy	MC	✓	✓	✓
	TD(0)	✓	✗	✗
	TD(λ)	✓	✗	✗

Gradient Temporal-Difference Learning

- TD does not follow the gradient of any objective function
- This is why TD can diverge when off-policy or using non-linear function approximation
- **Gradient TD** follows true gradient of projected Bellman error

On/Off-Policy	Algorithm	Table Lookup	Linear	Non-Linear
On-Policy	MC	✓	✓	✓
	TD	✓	✓	✗
	Gradient TD	✓	✓	✓
Off-Policy	MC	✓	✓	✓
	TD	✓	✗	✗
	Gradient TD	✓	✓	✓

Convergence of Control Algorithms

Algorithm	Table Lookup	Linear	Non-Linear
Monte-Carlo Control	✓	(✓)	✗
Sarsa	✓	(✓)	✗
Q-learning	✓	✗	✗
Gradient Q-learning	✓	✓	✗

(✓) = chatters around near-optimal value function

3. Batch Methods

Batch Reinforcement Learning

- Gradient descent is simple and appealing
- But it is not sample efficient
- Batch methods seek to find the best fitting value function
- Given the agent's experience ("training data")

Least Squares Prediction

- Given value function approximation $\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$
- And experience D consisting of $< state, value >$ pairs

$$D = \{< s_1, v_1^\pi >, < s_2, v_2^\pi >, \ldots, < s_T, v_T^\pi >\}$$

- Which parameters $\mathbf{w}$ give the best fitting value fn $\hat{v}(s, \mathbf{w})$?
- **Least squares** algorithms find parameter vector $\mathbf{w}$ minimising sum-squared error between $\hat{v}(s_t, \mathbf{w})$ and target values v_t^π ,

$$\begin{aligned} \mathrm{LS}(\mathbf{w}) &= \sum_{t=1}^{T} (v_t^\pi - \hat{v}(s_t, \mathbf{w}))^2 \\ &= E_D[(v^\pi - \hat{v}(s, \mathbf{w}))^2] \end{aligned}$$

Stochastic Gradient Descent with Experience Replay

Given experience consisting of $< state, value >$ pairs

$$D = \{< s_1, v_1^\pi >, < s_2, v_2^\pi >, \ldots, < s_T, v_T^\pi >\}$$

Repeat:

1. Sample state, value from experience

$$< s, v^\pi > \sim D$$

2. Apply stochastic gradient descent update

$$\triangle \mathbf{w} = \alpha(v^\pi - \hat{v}(s, \mathbf{w}))\nabla_{\mathbf{w}}\hat{v}(s, \mathbf{w})$$

Converges to least squares solution

$$\mathbf{w}^\pi = \arg\min_{\mathbf{w}} \mathrm{LS}(\mathbf{w})$$

Experience Replay in Deep Q-Networks (DQN)

DQN uses **experience replay** and **fixed Q-targets**

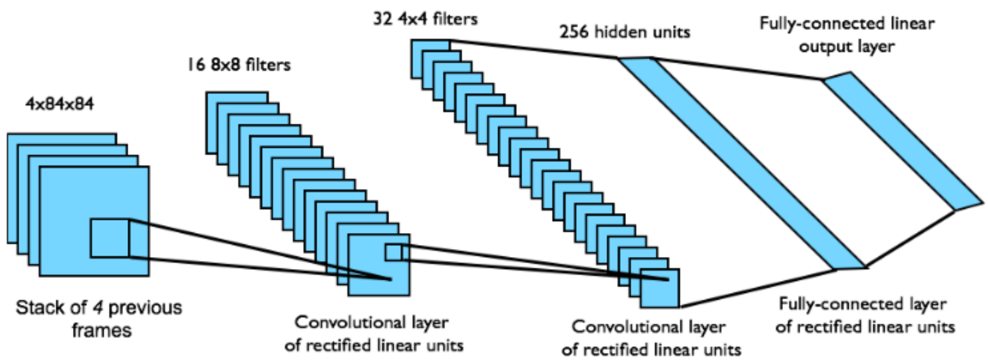
- Take action a_t according to ϵ-greedy policy
- Store transition $(s_t, a_t, r_{t+1}, s_{t+1})$ in replay memory D
- Sample random mini-batch of transitions (s, a, r, s') from D
- Compute Q-learning targets w.r.t. old, fixed parameters w^-
- Optimise MSE between Q-network and Q-learning targets

$$L_i(w_i) = E_{s,a,r,s' \sim D_i}\left[\left(r + \gamma \max_{a'} Q(s', a'; w_i^-) - Q(s, a; w_i)\right)^2\right]$$

- Using variant of stochastic gradient descent

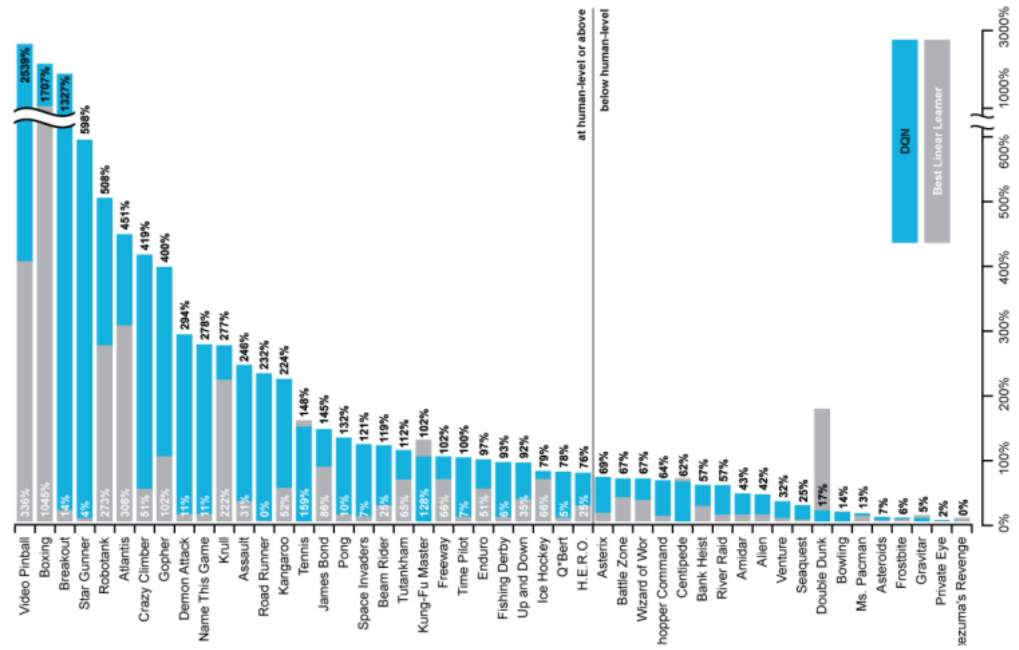
DQN in Atari

- End-to-end learning of values $Q(s, a)$ from pixels s
- Input state s is stack of raw pixels from last 4 frames
- Output is $Q(s, a)$ for 18 joystick/button positions
- Reward is change in score for that step



Network architecture and hyperparameters fixed across all games

DQN Results in Atari



How much does DQN help?

	Replay Fixed-Q	Replay Q-learning	No replay Fixed-Q	No replay Q-learning
Breakout	316.81	240.73	10.16	3.17
Enduro	1006.3	831.25	141.89	29.1
River Raid	7446.62	4102.81	2867.66	1453.02
Seaquest	2894.4	822.55	1003	275.81
Space Invaders	1088.94	826.33	373.22	301.99

Linear Least Squares Prediction

- Experience replay finds least squares solution
- But it may take many iterations
- Using linear value function approximation $\hat{v}(s, \mathbf{w}) = \mathbf{x}(s)^T \mathbf{w}$
- We can solve the least squares solution directly
- At minimum of $\mathrm{LS}(\mathbf{w})$, the expected update must be zero
- For N features, direct solution time is $O(N^3)$
- Incremental solution time is $O(N^2)$ using Shermann-Morrison

Linear Least Squares Prediction Algorithms

- We do not know true values v_t^π
- In practice, our "training data" must use noisy or biased samples of v_t^π
  - LSMC: Least Squares Monte-Carlo uses return

    - $v_t^\pi \approx G_t$
  - LSTD: Least Squares Temporal-Difference uses TD target
    - $v_t^\pi \approx R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w})$
  - LSTD(λ) Least Squares TD(λ) uses λ-return
    - $v_t^\pi \approx G_t^\lambda$
- In each case solve directly for fixed point of MC/TD/TD(λ)

LSMC

$$0 = \sum_{t=1}^{T} \alpha (G_t - \hat{v}(S_t, \mathbf{w}))\mathbf{x}(S_t)$$

$$\mathbf{w} = \left(\sum_{t=1}^{T} \mathbf{x}(S_t)\mathbf{x}(S_t)^\top \right)^{-1} \sum_{t=1}^{T} \mathbf{x}(S_t) G_t$$

LSTD

$$0 = \sum_{t=1}^{T} \alpha (R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}) - \hat{v}(S_t, \mathbf{w}))\mathbf{x}(S_t)$$

$$\mathbf{w} = \left(\sum_{t=1}^{T} \mathbf{x}(S_t)(\mathbf{x}(S_t) - \gamma \mathbf{x}(S_{t+1}))^\top \right)^{-1} \sum_{t=1}^{T} \mathbf{x}(S_t) R_{t+1}$$

LSTD(λ)

$$0 = \sum_{t=1}^{T} \alpha \delta_t E_t$$

$$\mathbf{w} = \left(\sum_{t=1}^{T} E_t(\mathbf{x}(S_t) - \gamma \mathbf{x}(S_{t+1}))^\top \right)^{-1} \sum_{t=1}^{T} E_t R_{t+1}$$

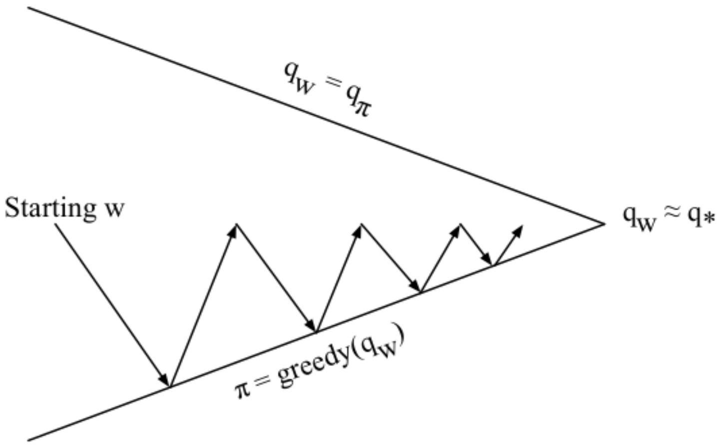
Convergence of Linear Least Squares Prediction Algorithms

On/Off-Policy	Algorithm	Table Lookup	Linear	Non-Linear
On-Policy	MC	✓	✓	✓
	LSMC	✓	✓	-
	TD	✓	✓	✗
	LSTD	✓	✓	-
Off-Policy	MC	✓	✓	✓
	LSMC	✓	✓	-
	TD	✓	✗	✗
	LSTD	✓	✓	-

Least Squares Control

Least Squares Policy Iteration

Least Squares Policy Iteration



Policy evaluation Policy evaluation by least squares Q-learning

Policy improvement Greedy policy improvement

Least Squares Action-Value Function Approximation

- Approximate action-value function $q_\pi(s, a)$
- using linear combination of features $\mathbf{x}(s, a)$

$$\hat{q}(s, a, \mathbf{w}) = \mathbf{x}(s, a)^T \mathbf{w} \approx q_\pi(s, a)$$

- Minimise least squares error between $\hat{q}(s, a, \mathbf{w})$ and $q_\pi(s, a)$
- from experience generated using policy π
- consisting of $< (state, action), value >$ pairs

$$D = \{< (s_1, a_1), v_1^\pi >, < (s_2, a_2), v_2^\pi >, \ldots, < (s_T, a_T), v_T^\pi >\}$$

Least Squares Control

- For policy evaluation, we want to efficiently use all experience
- For control, we also want to improve the policy
- This experience is generated from many policies
- So to evaluate $q_\pi(S, A)$ we must learn **off-policy**
- We use the same idea as Q-learning:
  - Use experience generated by old policy $S_t, A_t, R_{t+1}, S_{t+1} \sim \pi_{old}$
  - Consider alternative successor action $A' = \pi_{new}(S_{t+1})$
  - Update $\hat{q}(S_t, A_t, \mathbf{w})$ towards value of alternative action $R_{t+1} + \gamma \hat{q}(S_{t+1}, A', \mathbf{w})$

Least Squares Q-Learning

- Consider the following linear Q-learning update

$$\begin{aligned} \delta &= R_{t+1} + \gamma \hat{q}(S_{t+1}, \pi(S_{t+1}), \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}) \\ \triangle \mathbf{w} &= \alpha \delta \mathbf{x}(S_t, A_t) \end{aligned}$$

- LSTDQ algorithm: solve for total update = zero

$$0 = \sum_{t=1}^{T} \alpha(R_{t+1} + \gamma\hat{q}(S_{t+1}, \pi(S_{t+1}), \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}))\mathbf{x}(S_t, A_t)$$

$$\mathbf{w} = (\sum_{t=1}^{T} \mathbf{x}(S_t, A_t)(\mathbf{x}(S_t, A_t) - \gamma\mathbf{x}(S_{t+1}, \pi(S_{t+1})))^T)^{-1} \sum_{t=1}^{T} \mathbf{x}(S_t, A_t)R_{t+1}$$

Least Squares Policy Iteration Algorithm

- The following pseudocode uses LSTDQ for policy evaluation
- It repeatedly re-evaluates experience D with different policies

function LSPI-TD$(\mathcal{D}, \pi_0)$
    $\pi' \leftarrow \pi_0$
    repeat
        $\pi \leftarrow \pi'$
        $Q \leftarrow$ **LSTDQ**$(\pi, \mathcal{D})$
        for all $s \in \mathcal{S}$ **do**
            $\pi'(s) \leftarrow \underset{a \in \mathcal{A}}{\text{argmax}}\ Q(s, a)$
        end for
    until $(\pi \approx \pi')$
    return π
end function

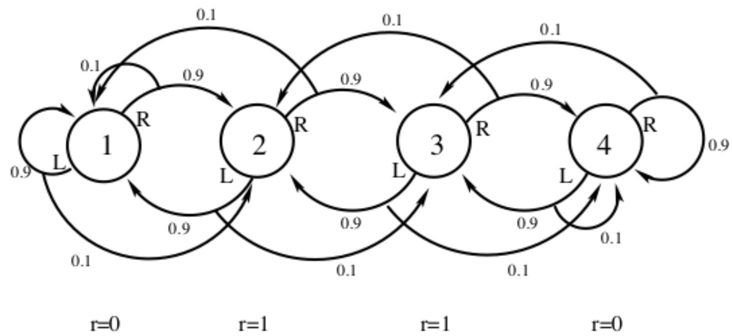
Convergence of Control Algorithms

Algorithm	Table Lookup	Linear	Non-Linear
Monte-Carlo Control	✓	(✓)	✗
Sarsa	✓	(✓)	✗
Q-learning	✓	✗	✗
LSPI	✓	(✓)	-

(✓) = chatters around near-optimal value function

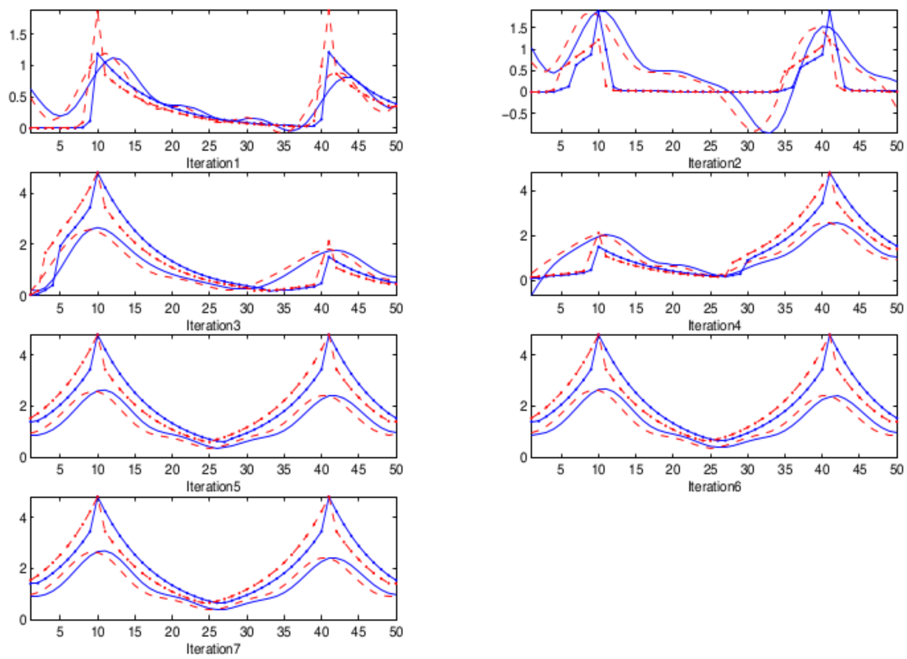
Chain Walk Example

Chain Walk Example



- Consider the 50 state version of this problem
- Reward $+1$ in states 10 and 41, 0 elsewhere
- Optimal policy: R (1-9), L (10-25), R (26-41), L (42, 50)
- Features: 10 evenly spaced Gaussians ($\sigma = 4$) for each action
- Experience: 10,000 steps from random walk policy

LSPI in Chain Walk: Action-Value Function



LSPI in Chain Walk: Policy