Lecture 4: Model-Free Prediction

1. Introduction

Model-Free Reinforcement Learning

- Last lecture:
  - **Planning by dynamic programming**
  - Solve a known MDP
- This lecture:
  - **Model-free prediction**
  - Estimate the value function of an unknown MDP
- Next lecture:
  - **Model-free control**
  - Optimise the value function of an unknown MDP

2. Monte-Carlo Learning

Monte-Carlo Reinforcement Learning

- MC methods learn directly from episodes of experience
- MC is model-free: no knowledge of MDP transitions / rewards
- MC learns from complete episodes: no bootstrapping
- MC uses the simplest possible idea: value = mean return
- Caveat: can only apply MC to episodic MDPs
  - All episodes must terminate

Monte-Carlo Policy Evaluation

- Goal: learn v_π from episodes of experience under policy π

$$S_1, A_1, R_2, \ldots, S_k \sim \pi$$

- Recall that the return is the total discounted reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-1} R_T$$

- Recall that the value function is the expected return:

$$v_\pi(s) = E_\pi[G_t | S_t = s]$$

- Monte-Carlo policy evaluation uses empirical mean return instead of expected return

First-Visit Monte-Carlo Policy Evaluation

- To evaluate state s
- The **first** time-step t that state s is visited in an episode
- Increment counter $N(s) \leftarrow N(s) + 1$

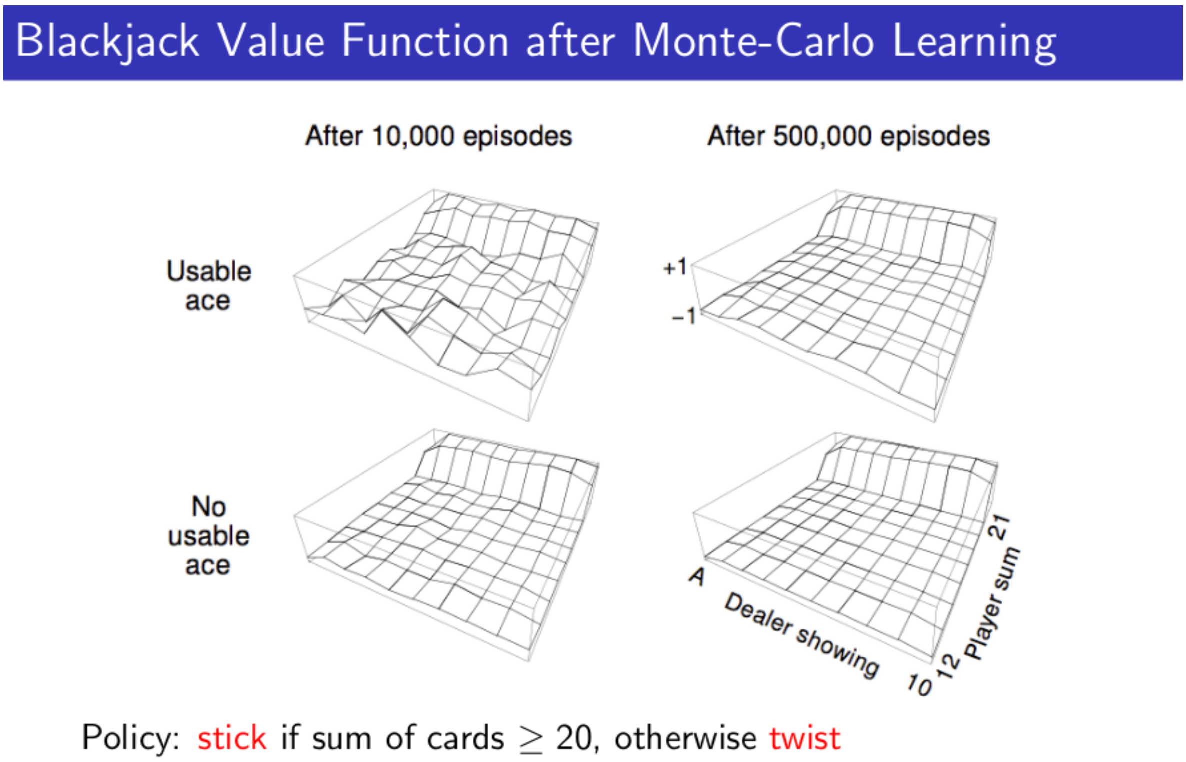
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return $V(s) = \frac{S(s)}{N(s)}$
- By law of large numbers, $V(s) \rightarrow v_\pi(s)$ as $N(s) \rightarrow \infty$

Every-Visit Monte-Carlo Policy Evaluation

- To evaluate state s
- Every time-step t that state s is visited in an episode,
- Increment counter $N(s) \leftarrow N(s) + 1$
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return $V(s) = \frac{S(s)}{N(s)}$
- Again , $V(s) \rightarrow v_\pi(s)$ as $N(s) \rightarrow \infty$

Blackjack Example

- States (200 of them):
    - Current sum $(12 - 21)$
    - Dealer's showing card $(ace - 10)$
    - Do I have a "useable" ace? (yes-no)
- Action **stick**: Stop receiving cards (and terminate)
- Action **twist**: Take another card (no replacement)
- Reward for **stick**:
    - $+1$ if sum of cards $>$ sum of dealer cards
    - 0 if sum of cards $=$ sum of dealer cards
    - -1 if sum of cards $<$ sum of dealer cards
- Reward for **twist**:
    - -1 if sum of cards > 21 (and terminate)
    - 0 otherwise
- Transitions: automatically **twist** if sum of cards < 12



Incremental Monte-Carlo

Incremental Mean

The mean $\mu_1, \mu_2, ...$ of a sequence $x_1, x_2, ...$ can be computed incrementally,

$$\begin{aligned}\mu_k &= \frac{1}{k}\sum_{j=1}^{k} x_j \\ &= \frac{1}{k}(x_k + \sum_{j=1}^{k-1} x_j) \\ &= \frac{1}{k}(x_k + (k-1)\mu_{k-1}) \\ &= \mu_{k-1} + \frac{1}{k}(x_k - \mu_{k-1})\end{aligned}$$

Incremental Monte-Carlo Updates

- Update $V(s)$ incrementally after episode $S_1, A_1, R_2, \ldots, S_T$
- For each state S_t with return G_t

$$\begin{aligned}N(S_t) &\leftarrow N(S_t) + 1 \\ V(S_t) &\leftarrow V(S_t) + \frac{1}{N(S_t)}(G_t - V(S_t))\end{aligned}$$

- In non-stationary problems, it can be useful to track a running mean, i.e. forget old episodes

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$

3. Temporal-Difference Learning

Temporal-Difference Learning

- TD methods learn directly from episodes of experience
- TD is model-free: no knowledge of MDP transitions / rewards
- TD learns from incomplete episodes, by bootstrapping
- TD updates a guess towards a guess

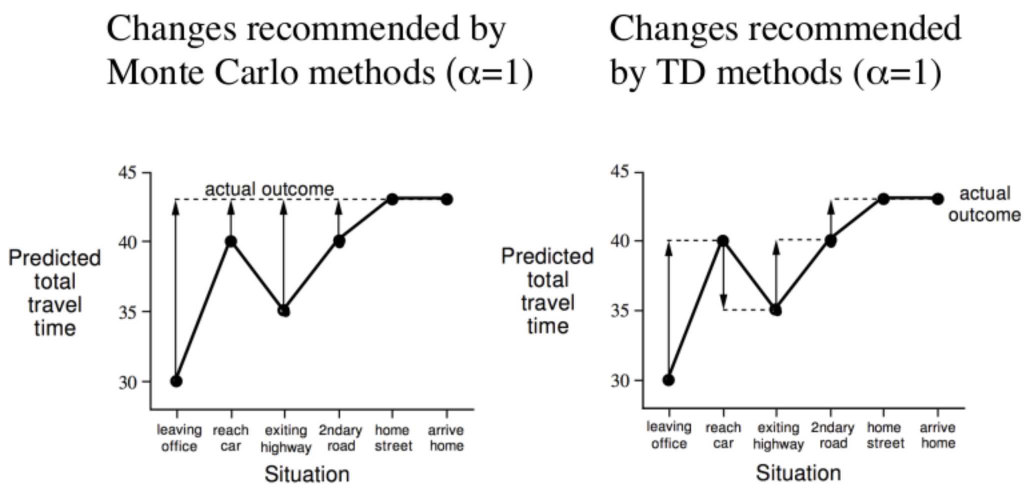
MC and TD

- Goal: learn v_π online from experience under policy π
- Incremental every-visit Monte-Carlo
  - Update value $V(S_t)$ toward actual return G_t
  - $$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$
- Simplest temporal-difference learning algorithm: TD(0)
  - Update value $V(S_t)$ toward estimated return $R_{t+1} + \gamma V(S_{t+1})$
  - $$V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$
  - $R_{t+1} + \gamma V(S_{t+1})$ is called the TD target
  - $\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$ is called the TD error

Driving Home Example

State	Elapsed Time (minutes)	Predicted Time to Go	Predicted Total Time
leaving office	0	30	30
reach car, raining	5	35	40
exit highway	20	15	35
behind truck	30	10	40
home street	40	3	43
arrive home	43	0	43

Driving Home Example: MC vs. TD



Advantages and Disadvantages of MC vs. TD

- TD can learn before knowing the final outcome
  - TD can learn online after every step
  - MC must wait until end of episode before return is known
- TD can learn without the final outcome
  - TD can learn from incomplete sequences
  - MC can only learn from complete sequences
  - TD works in continuing (non-terminating) environments
  - MC only works for episodic (terminating) environments

Bias/Variance Trade-Off

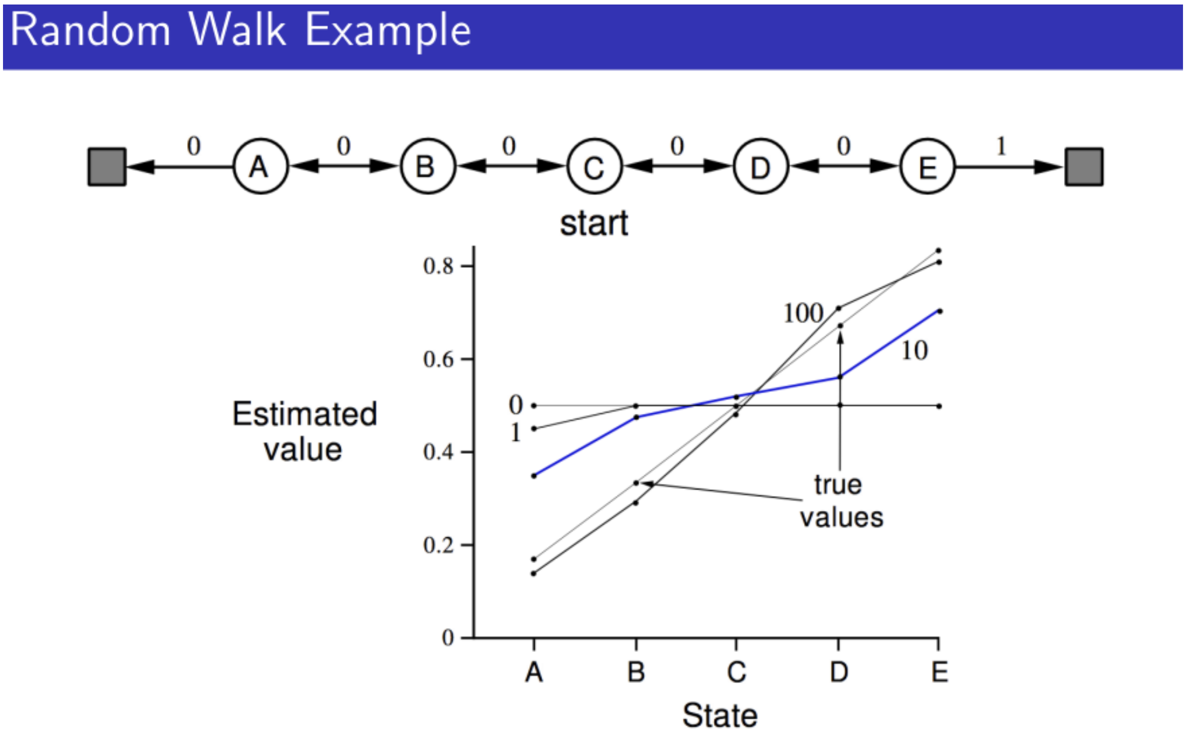
- Return $G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-1} R_T$ is unbiased estimate of $v_\pi(S_t)$
- True TD target $R_{t+1} + \gamma v_\pi(S_{t+1})$ is unbiased estimate of $v_\pi(S_t)$

- TD target $R_{t+1} + \gamma V(S_{t+1})$ is biased estimate of $v_\pi(S_t)$
- TD target is much lower variance than the return:
  - Return depends on many random actions, transitions, rewards
  - TD target depends on one random action, transition, reward

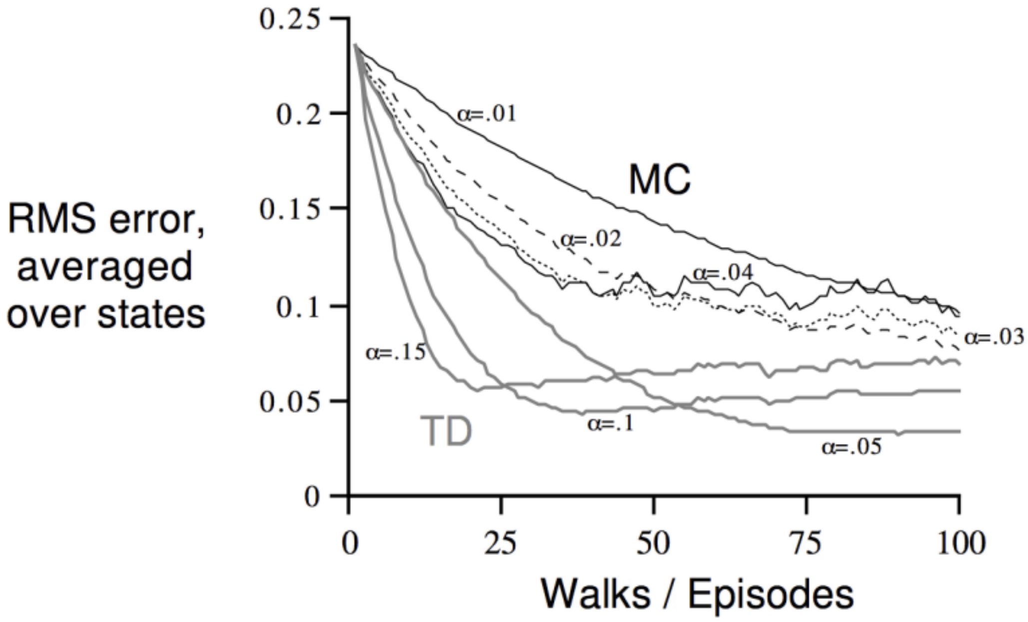
Advantages and Disadvantages of MC vs. TD (2)

- MC has high variance, zero bias
  - Good convergence properties
  - (even with function approximation)
  - Not very sensitive to initial value
  - Very simple to understand and use
- TD has low variance, some bias
  - Usually more efficient than MC
  - TD(0) converges to $v_\pi(s)$
  - (but not always with function approximation)
  - More sensitive to initial value

Random Walk Example



Random Walk: MC vs. TD



Batch MC and TD

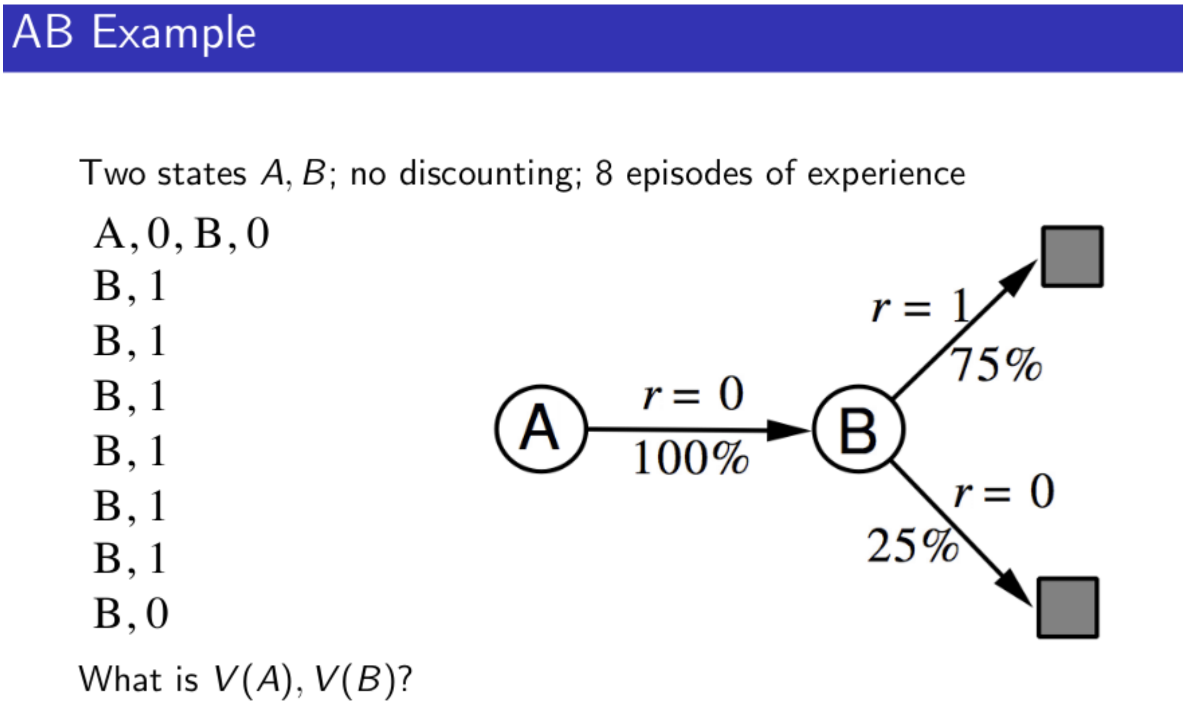
- MC and TD converge: $V(s) \to v_\pi(s)$ as experience $\to \infty$
- But what about batch solution for finite experience?
  - $$\begin{gathered} s_1^1, a_1^1, r_2^1, \ldots, s_{T_1}^1 \\ \vdots \\ s_1^k, a_1^k, r_2^k, \ldots, s_{T_1}^k \end{gathered}$$
  - e.g. Repeatedly sample episode $k \in [1, k]$
  - Apply MC or TD(0) to episode k

AB Example

Two states A, B; no discounting; 8 episodes of experience

A, 0, B, 0
B, 1
B, 1
B, 1
B, 1
B, 1
B, 1
B, 0

What is $V(A), V(B)$?



Certainty Equivalence

- MC converges to solution with minimum mean-squared error
  - Best fit to the observed returns
  - $$\sum_{k=1}^{K}\sum_{t=1}^{T_k}(G_t^k - V(s_t^k))^2$$
  - In the AB example, $V(A) = 0$
- TD(0) converges to solution of max likelihood Markov model
  - Solution to the MDP $< S, A, \hat{P}, \hat{R}, \gamma >$ that best fits the data
  - $$\begin{aligned} \hat{P}_{s,s'}^a &= \frac{1}{N(s,a)}\sum_{k=1}^{K}\sum_{t=1}^{T_k}\mathbf{1}(s_t^k, a_t^k, s_{t+1}^k = s, a, s') \\ \hat{R}_s^a &= \frac{1}{N(s,a)}\sum_{k=1}^{K}\sum_{t=1}^{T_k}\mathbf{1}(s_t^k, a_t^k = s, a) r_t^k \end{aligned}$$
  - In the AB example, $V(A) = 0.75$

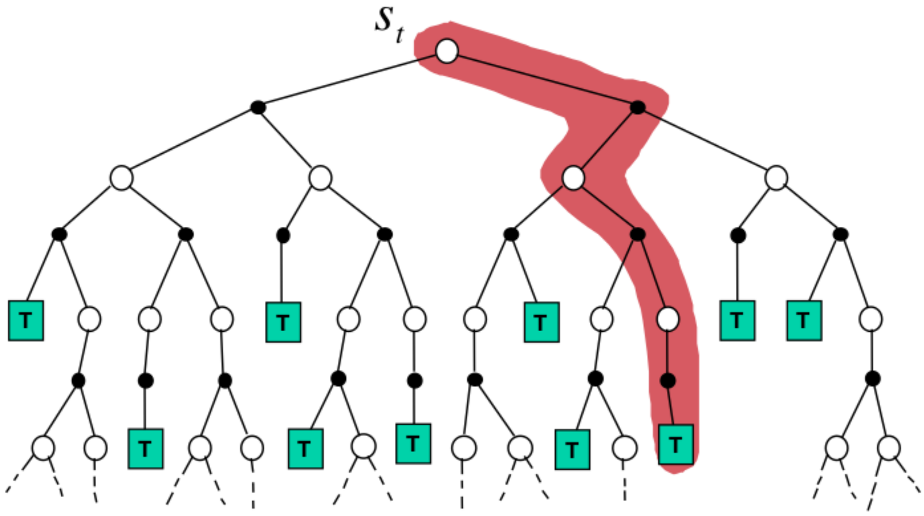
Advantages and Disadvantages of MC vs. TD (3)

- TD exploits Markov property
  - Usually more efficient in Markov environments
- MC does not exploit Markov property
  - Usually more effective in non-Markov environments

Unified View

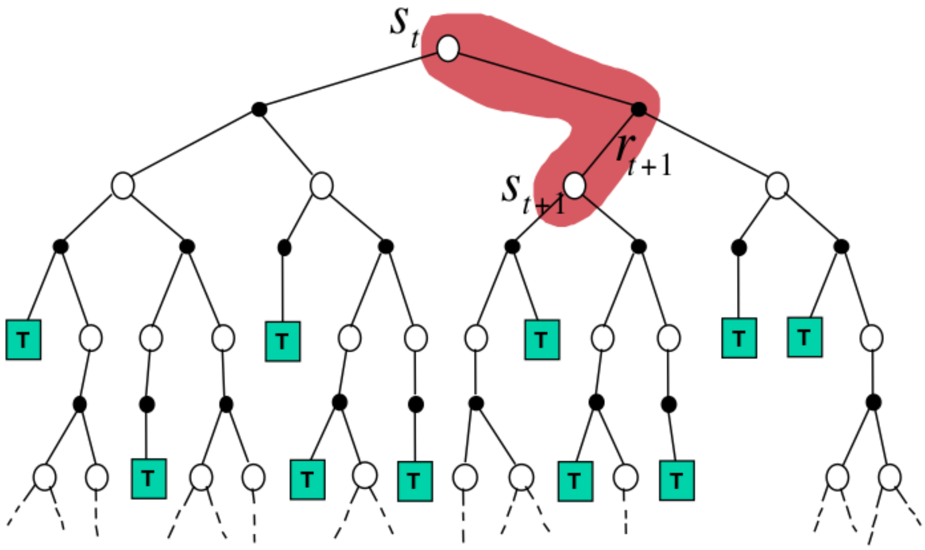
Monte-Carlo Backup

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t - V(S_t) \right)$$



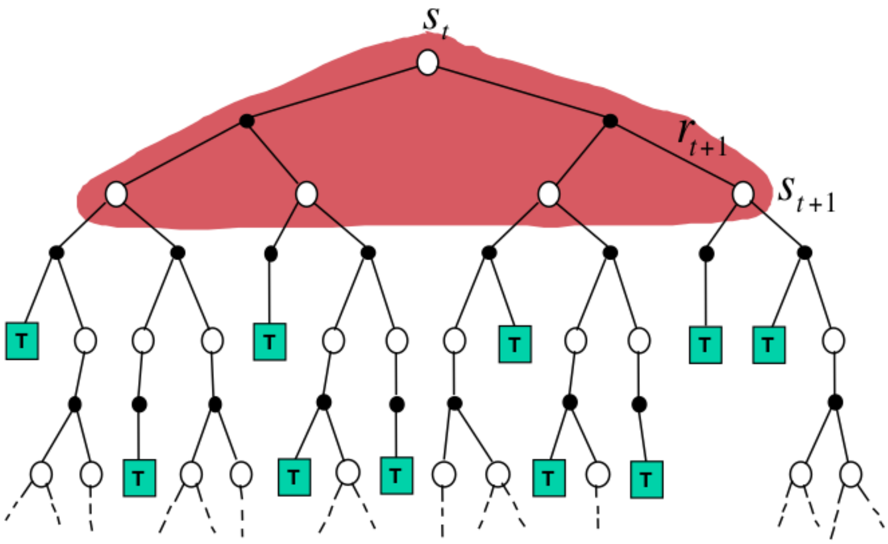
Temporal-Difference Backup

$$V(S_t) \leftarrow V(S_t) + \alpha \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \right)$$



Dynamic Programming Backup

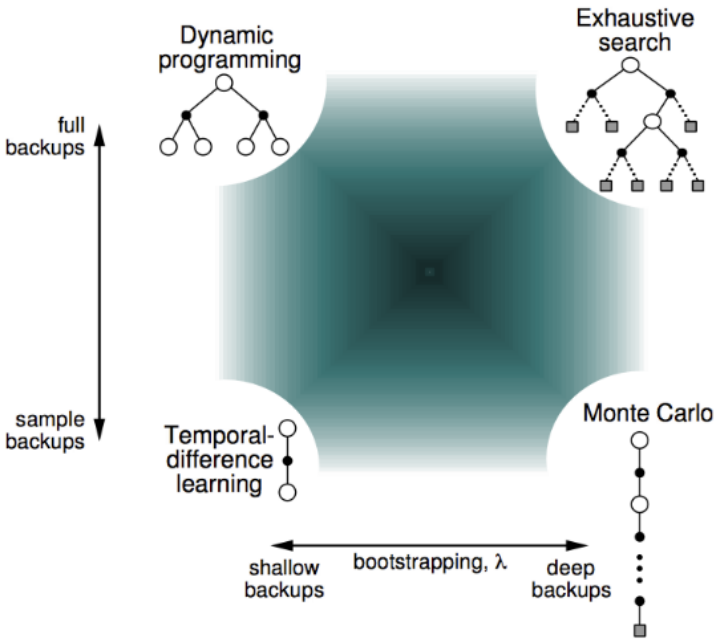
$$V(S_t) \leftarrow \mathbb{E}_\pi \left[R_{t+1} + \gamma V(S_{t+1})\right]$$



Bootstrapping and Sampling

- **Bootstrapping**: update involves an estimate
  - MC does not bootstrap
  - DP bootstraps
  - TD bootstraps
- **Sampling**: update samples an expectation
  - MC samples
  - DP does not sample
  - TD samples

Unified View of Reinforcement Learning

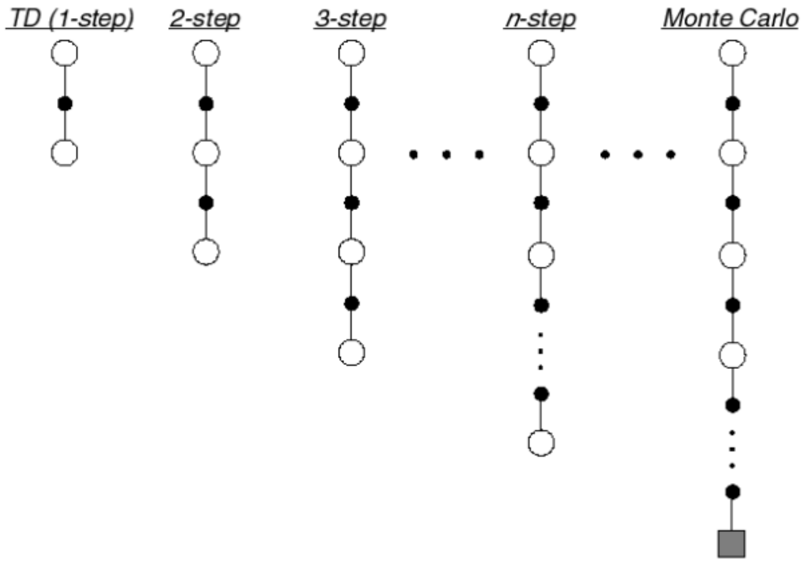


4. TD(λ)

n-Step TD

n-Step Prediction

- Let TD target look n steps into the future



n-Step Return

- Consider the following n-step returns for $n = 1, 2, \infty$:

$$\begin{array}{lll} n = 1 & (TD) & G_t^{(1)} = R_{t+1} + \gamma V(S_{t+1}) \\ n = 2 & & G_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2}) \\ \vdots & & \vdots \\ n = \infty & (MC) & G_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-1} R_T \end{array}$$

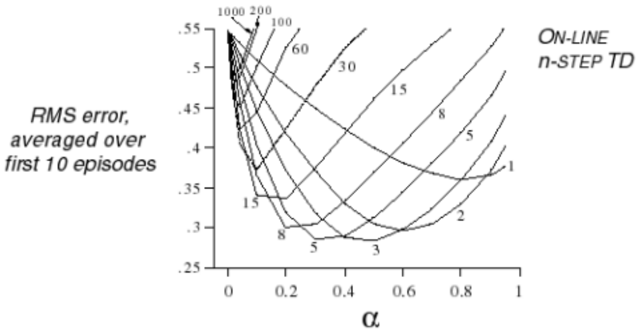
- Define the n-step return

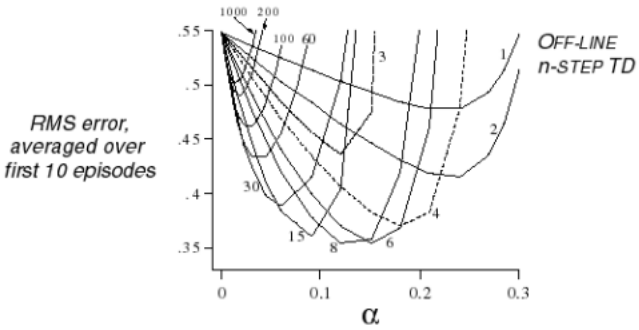
$$G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- n-step temporal-difference learning

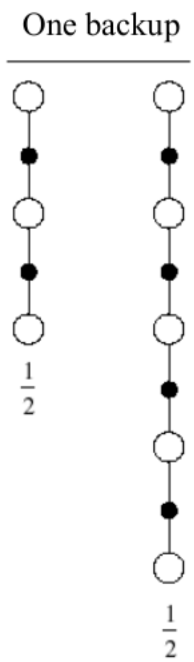
$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t^{(n)} - V(S_t) \right)$$

Large Random Walk Example





Averaging *n*-Step Returns

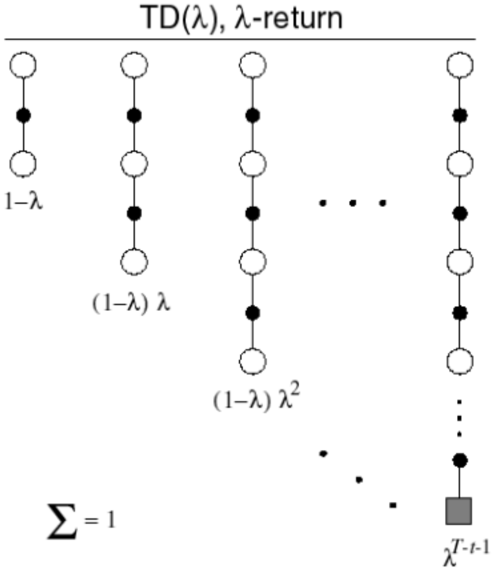


- We can average n-step returns over different n
- e.g. average the 2-step and 4-step returns

$$\frac{1}{2}G^{(2)} + \frac{1}{2}G^{(4)}$$

- Combines information from two different time-steps
- Can we efficiently combine information from all time-steps?

λ-return



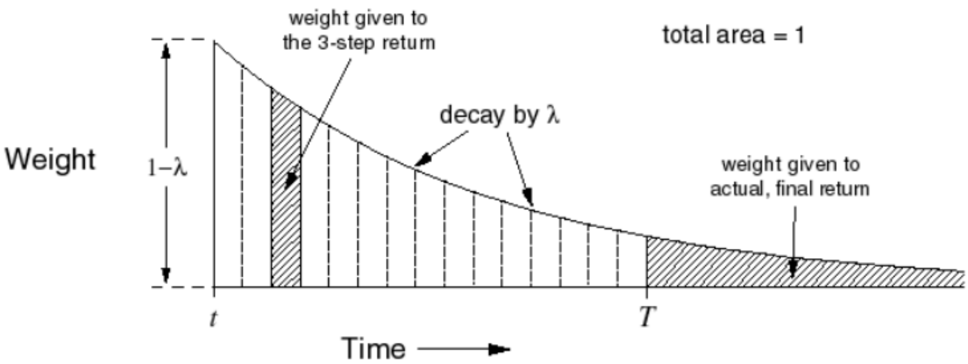
- The λ-*return* G_t^λ combines all n-step returns $G_t^{(n)}$
- Using weight $(1-\lambda)\lambda^{n-1}$

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{(n)}$$

- Forward-view TD(λ)

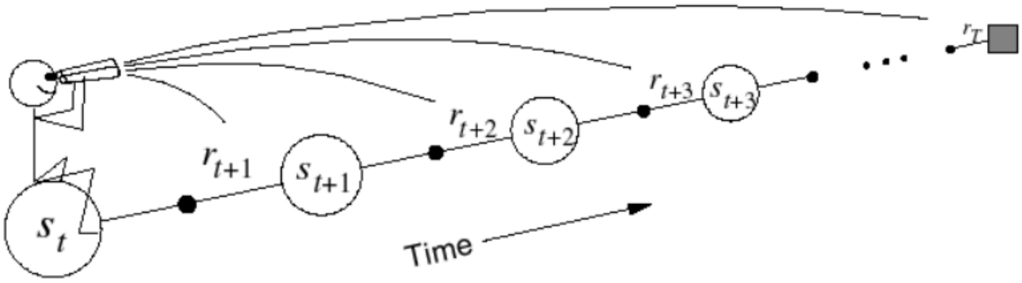
$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^\lambda - V(S_t)\right)$$

TD(λ) Weighting Function



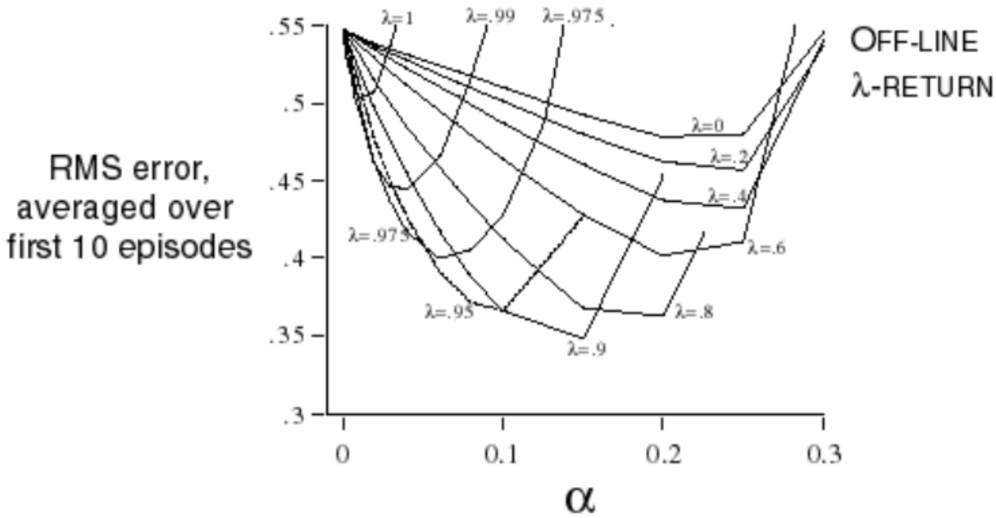
$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{(n)}$$

Forward-view TD(λ)



- Update value function towards the λ-return
- Forward-view looks into the future to compute G_t^λ
- Like MC, can only be computed from complete episodes

Forward-View TD(λ) on Large Random Walk



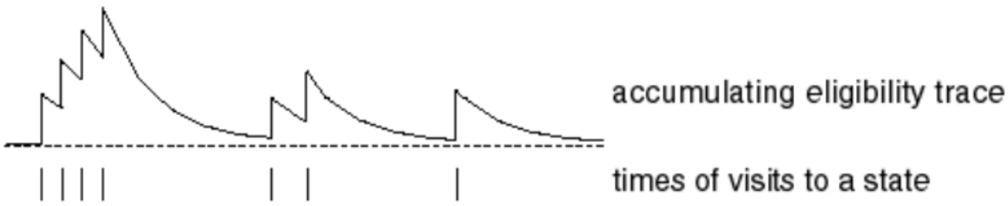
Backward View TD(λ)

- Forward view provides theory
- Backward view provides mechanism
- Update online, every step, from incomplete sequences

Eligibility Traces

- Credit assignment problem: did bell or light cause shock?
- Frequency heuristic: assign credit to most frequent states
- Recency heuristic: assign credit to most recent states
- *Eligibility traces* combine both heuristics

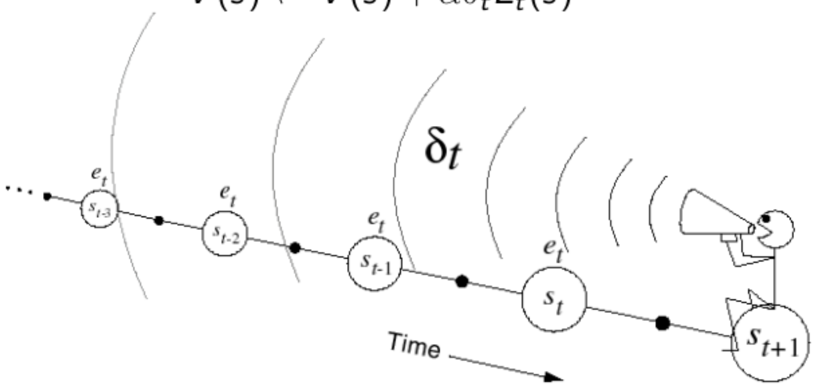
$$E_0(s) = 0$$
$$E_t(s) = \gamma\lambda E_{t-1}(s) + \mathbf{1}(S_t = s)$$



Backward View TD(λ)

- Keep an eligibility trace for every state s
- Update value $V(s)$ for every state s
- In proportion to TD-error δ_t and eligibility trace $E_t(s)$

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$
$$V(s) \leftarrow V(s) + \alpha\delta_t E_t(s)$$



Relationship Between Forward and Backward TD

TD(λ) and TD(0)

- When $\lambda = 0$, only current state is updated

$$E_t(s) = \mathbf{1}(S_t = s)$$
$$V(s) \leftarrow V(s) + \alpha\delta_t E_t(s)$$

- This is exactly equivalent to TD(0) update

$$V(S_t) \leftarrow V(S_t) + \alpha\delta_t$$

TD(λ) and MC

- When $\lambda = 1$, credit is deferred until end of episode
- Consider episodic environments with offline updates
- Over the course of an episode, total update for TD(1) is the same as total update for MC

The sum of offline updates is identical for forward-view and backward-view TD(λ)

$$\sum_{t=1}^{T} \alpha \delta_t E_t(s) = \sum_{t=1}^{T} \alpha (G_t^\lambda - V(S_t)) \mathbf{1}(S_t = s)$$

Forward and Backward Equivalence

MC and TD(1)

- Consider an episode where s is visited once at time-step k,
- TD(1) eligibility trace discounts time since visit,

$$\begin{aligned} E_t(s) &= \gamma E_{t-1}(s) + \mathbf{1}(S_t = s) \\ &= \begin{cases} 0, & \text{if} \quad t < k \\ \gamma^{t-k} & \text{if} \quad t \geq k \end{cases} \end{aligned}$$

- TD(1) updates accumulate error online

$$\sum_{t=1}^{T-1} \alpha \delta_t E_t(s) = \alpha \sum_{t=k}^{T-1} \gamma^{t-k} \delta_t = \alpha (G_k - V(S_k))$$

- By end of episode it accumulates total error

$$\delta_k + \gamma \delta_{k+1} + \gamma^2 \delta_{k+2} + \dots + \gamma^{T-1-k} \delta_{T-1}$$

Telescoping in TD(1)

When $\lambda = 1$, sum of TD errors telescopes into MC error,

$$\begin{aligned} & \delta_t + \gamma \delta_{t+1} + \gamma^2 \delta_{t+2} + \dots + \gamma^{T-1-t} \delta_{T-1} \\ = & R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \\ + & \gamma R_{t+2} + \gamma^2 V(S_{t+2}) - \gamma V(S_{t+1}) \\ + & \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3}) - \gamma^2 V(S_{t+2}) \\ & \vdots \\ + & \gamma^{T-1-t} R_T + \gamma^{T-t} V(S_T) - \gamma^{T-1-t} V(S_{T-1}) \\ = & R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} \dots + \gamma^{T-1-t} R_T - V(S_t) \\ = & G_t - V(S_t) \end{aligned}$$

TD(λ) and TD(1)

- TD(1) is roughly equivalent to every-visit Monte-Carlo
- Error is accumulated online, step-by-step
- If value function is only updated offline at end of episode
- Then total update is exactly the same as MC

Telescoping in TD(λ)

For general λ, TD errors also telescope to λ-error, $G_t^\lambda - V(S_t)$

$$
\begin{aligned}
G_t^\lambda - V(S_t) = -V(S_t) & + (1-\lambda)\lambda^0 \left(R_{t+1} + \gamma V(S_{t+1})\right) \\
& + (1-\lambda)\lambda^1 \left(R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2})\right) \\
& + (1-\lambda)\lambda^2 \left(R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3})\right) \\
& + \ldots \\
= -V(S_t) & + (\gamma\lambda)^0 \left(R_{t+1} + \gamma V(S_{t+1}) - \gamma\lambda V(S_{t+1})\right) \\
& + (\gamma\lambda)^1 \left(R_{t+2} + \gamma V(S_{t+2}) - \gamma\lambda V(S_{t+2})\right) \\
& + (\gamma\lambda)^2 \left(R_{t+3} + \gamma V(S_{t+3}) - \gamma\lambda V(S_{t+3})\right) \\
& + \ldots \\
= \quad & \quad (\gamma\lambda)^0 \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right) \\
& + (\gamma\lambda)^1 \left(R_{t+2} + \gamma V(S_{t+2}) - V(S_{t+1})\right) \\
& + (\gamma\lambda)^2 \left(R_{t+3} + \gamma V(S_{t+3}) - V(S_{t+2})\right) \\
& + \ldots \\
= \delta_t + \gamma\lambda\delta_{t+1} & + (\gamma\lambda)^2\delta_{t+2} + \ldots
\end{aligned}
$$

Forwards and Backwards TD(λ)

- Consider an episode where s is visited once at time-step k,
- TD(λ) eligibility trace discounts time since visit,

$$
\begin{aligned}
E_t(s) &= \gamma\lambda E_{t-1}(s) + \mathbf{1}(S_t = s) \\
&= \begin{cases} 0, & \text{if} \quad t < k \\ (\gamma\lambda)^{t-k} & \text{if} \quad t \geq k \end{cases}
\end{aligned}
$$

- Backward TD(λ) updates accumulate error online

$$
\sum_{t=1}^{T} \alpha\delta_t E_t(s) = \alpha \sum_{t=k}^{T} (\gamma\lambda)^{t-k}\delta_t = \alpha(G_k^\lambda - V(S_k))
$$

- By end of episode it accumulates total error for λ-return
- For multiple visits to s, $E_t(s)$ accumulates many errors

Offline Equivalence of Forward and Backward TD

Offline updates

- Updates are accumulated within episode
- but applied in batch at the end of episode

Onine Equivalence of Forward and Backward TD

Online updates

- TD(λ) updates are applied online at each step within episode
- Forward and backward-view TD(λ) are slightly different
- **NEW**: Exact online TD(λ) achieves perfect equivalence
- By using a slightly different form of eligibility trace
- Sutton and von Seijen, ICML 2014

Summary of Forward and Backward TD(λ)

Offline updates	$\lambda = 0$	$\lambda \in (0, 1)$	$\lambda = 1$
Backward view	TD(0)	TD(λ)	TD(1)
	$=$	$=$	$=$
Forward view	TD(0)	Forward TD(λ)	MC
Online updates	$\lambda = 0$	$\lambda \in (0, 1)$	$\lambda = 1$
Backward view	TD(0)	TD(λ)	TD(1)
	$=$	$\neq$	$\neq$
Forward view	TD(0)	Forward TD(λ)	MC
	$=$	$=$	$=$
Exact Online	TD(0)	Exact Online TD(λ)	Exact Online TD(1)

$=$ here indicates equivalence in total update at end of episode.