Lecture 5: Model-Free Control

1. Introduction

Model-Free Reinforcement Learning

- Last lecture:
  - **Model-free prediction**
  - Estimate the value function of an unknown MDP
- This lecture:
  - **Model-free control**
  - Optimise the value function of an unknown MDP

Uses of Model-Free Control

Some example problems that can be modelled as MDPs

Elevator	**Robocup Soccer**
Parallel Parking	**Quake**
Ship Steering	**Portfolio management**
Bioreactor	**Protein Folding**
Helicopter	**Robot walking**
Aeroplane Logistics	**Game of Go**

For most of these problems, either:

- MDP model is unknown, but experience can be sampled
- MDP model is known, but is too big to use, except by samples

Model-free control can solve these problems

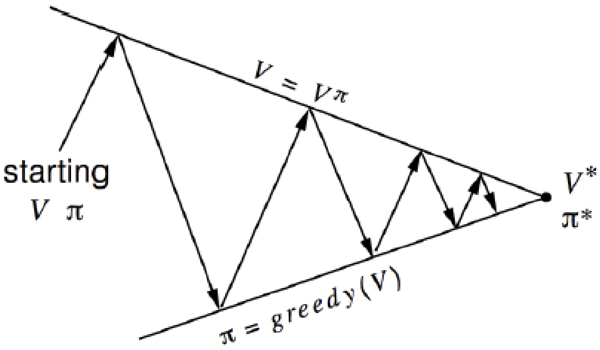
On and Off-Policy Learning

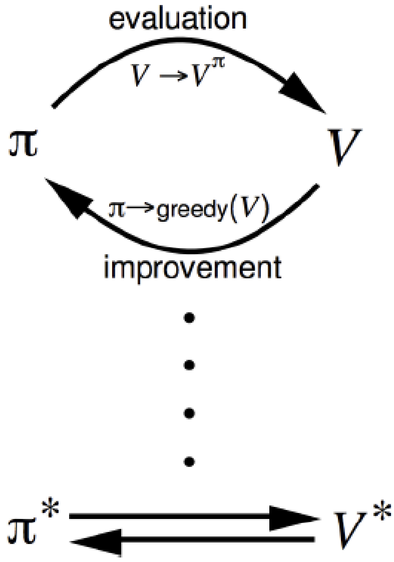
- **On-policy** learning
  - "Learn on the job"
  - Learn about policy π from experience sampled from π
- **Off-policy** learning
  - "Look over someone's shoulder"
  - Learn about policy π from experience sampled from μ

2. On-Policy Monte-Carlo Control

Generalised Policy Iteration

Generalised Policy Iteration (Refresher)





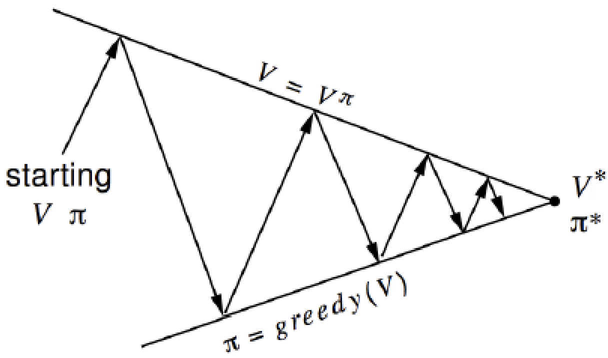
Policy evaluation Estimate v_π

e.g. Iterative policy evaluation

Policy improvement Generate $\pi' \geq \pi$

e.g. Greedy policy improvement

Generalised Policy Iteration With Monte-Carlo Evaluation



Policy evaluation Monte-Carlo policy evaluation, $V = v_\pi$?

Policy improvement Greedy policy improvement?

Model-Free Policy Iteration Using Action-Value Function

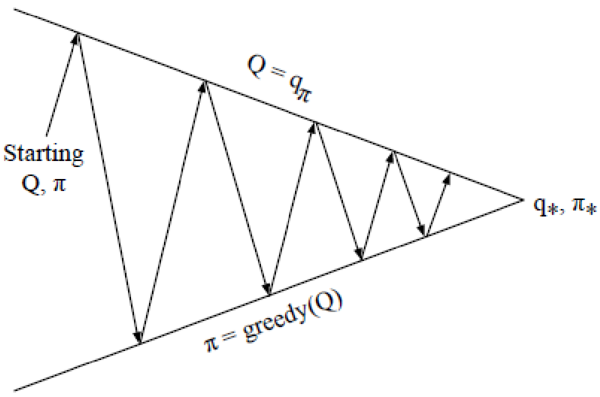
- Greedy policy improvement over $V(s)$ requires model of MDP

$$\pi'(s) = \arg\max_{a \in A} R_s^a + P_{ss'}^a V(s')$$

- Greedy policy improvement over $Q(s, a)$ is model-free

$$\pi'(s) = \arg\max_{a \in A} Q(s, a)$$

Generalised Policy Iteration with Action-Value Function



Policy evaluation Monte-Carlo policy evaluation, $Q = q_\pi$

Policy improvement Greedy policy improvement?

Exploration

Example of Greedy Action Selection

"Behind one door is tenure - behind the other is flipping burgers at McDonald's."

- There are two doors in front of you.
- You open the left door and get reward 0
  $V(left) = 0$
- You open the right door and get reward +1
  $V(right) = +1$
- You open the right door and get reward +3
  $V(right) = +2$
- You open the right door and get reward +2
  $V(right) = +2$

  ⋮
- Are you sure you've chosen the best door?

ϵ-Greedy Exploration

- Simplest idea for ensuring continual exploration
- All m actions are tried with non-zero probability
- With probability $1 - \epsilon$ choose the greedy action
- With probability ϵ choose an action at random

$$\pi(a|s) = \begin{cases} \frac{\epsilon}{m} + 1 - \epsilon & \text{if} \quad a^* = \arg\max_{a \in A} Q(s,a) \\ \frac{\epsilon}{m} & \text{otherwise} \end{cases}$$

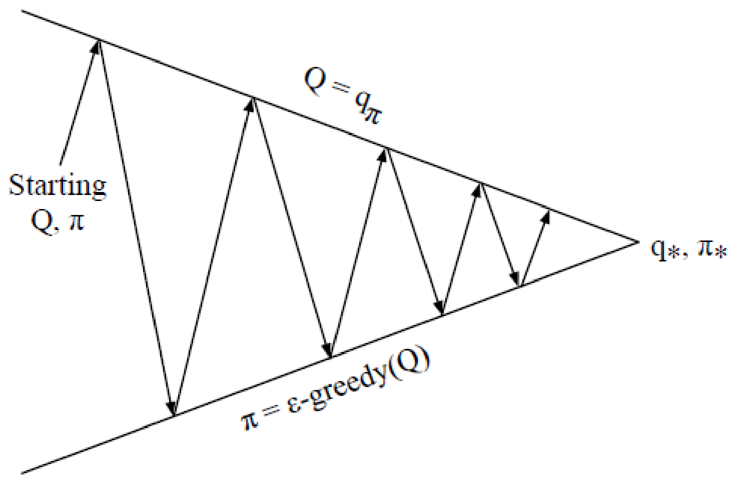
ϵ-Greedy Policy Improvement

For any ϵ-greedy policy π, the ϵ-greedy policy π' with respect to q_π is an improvement, $v_{\pi'} \geq v_\pi(s)$

$$
\begin{aligned}
q_\pi(s, \pi'(s)) &= \sum_{a \in A} \pi'(a|s) q_\pi(s,a) \\
&= \frac{\epsilon}{m} \sum_{a \in A} q_\pi(s,a) + (1-\epsilon) \max_{a \in A} q_\pi(s,a) \\
&\geq \frac{\epsilon}{m} \sum_{a \in A} q_\pi(s,a) + (1-\epsilon) \sum_{a \in A} \frac{\pi(a|s) - \frac{\epsilon}{m}}{1-\epsilon} q_\pi(s,a) \\
&= \sum_{a \in A} \pi(a|s) q_\pi(s,a) \\
&= v_\pi(s)
\end{aligned}
$$

Therefore from policy improvement theorem, $v_{\pi'}(s) \geq v_\pi(s)$

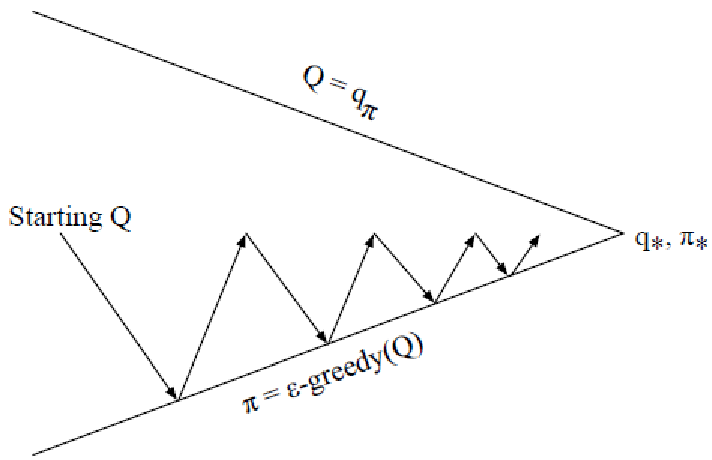
Monte-Carlo Policy Iteration



Policy evaluation Monte-Carlo policy evaluation, $Q = q_\pi$

Policy improvement ϵ-greedy policy improvement

Monte-Carlo Control



Every episode:

Policy evaluation Monte-Carlo policy evaluation, $Q \approx q_\pi$

Policy improvement ϵ-greedy policy improvement

GLIE

Greedy in the Limit with Infinite Exploration (GLIE)

- All state-action pairs are explored infinitely many times,

$$\lim_{k \to \infty} N_k(s, a) = \infty$$

- The policy converges on a greedy policy,

$$\lim_{k \to \infty} \pi_k(a|s) = \mathbf{1}(a = \arg\max_{a' \in A} Q_k(s, a'))$$

- For example, ϵ-greedy is GLIE if ϵ reduces to zero at $\epsilon_k = \frac{1}{k}$

GLIE Monte-Carlo Control

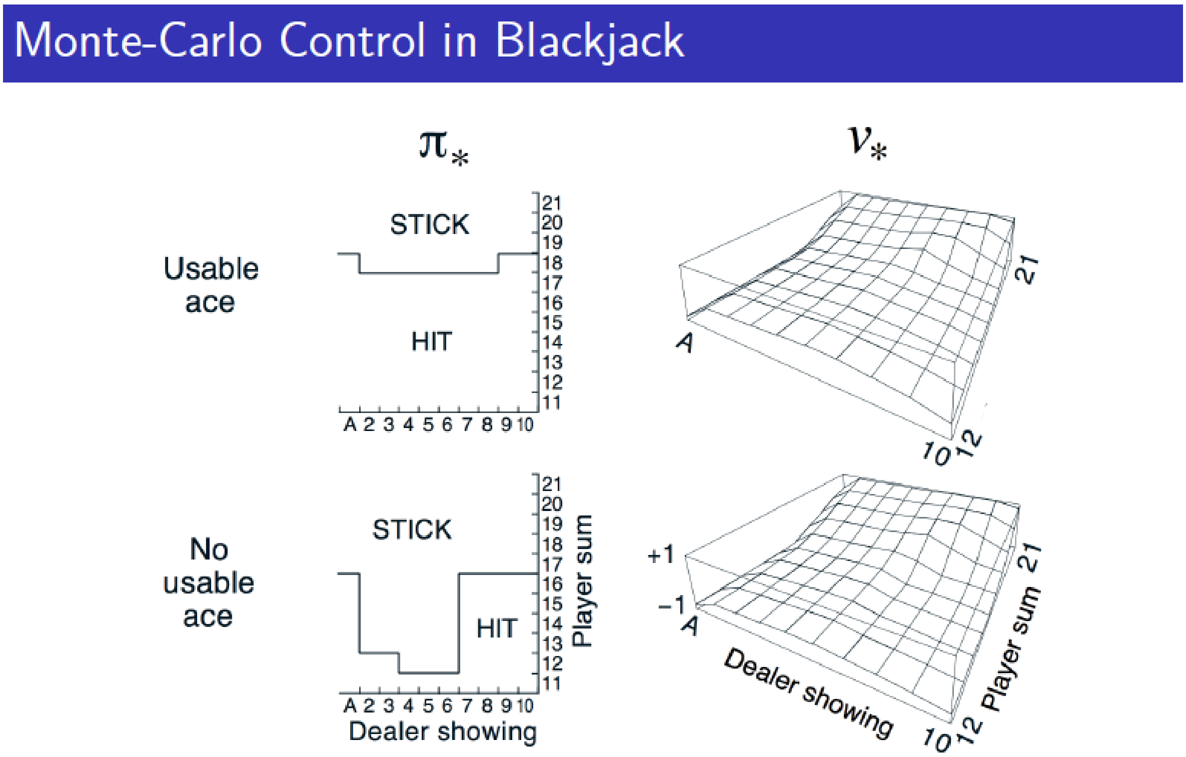
- Sample kth episode using $\pi: \{S_1, A_1, R_2, \ldots, S_T\} \sim \pi$
- For each state S_t and action A_t in the episode,

$$N(S_t, A_t) \leftarrow N(S_t, A_t) + 1$$
$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{1}{N(S_t, A_t)}(G_t - Q(S_t, A_t))$$

- Improve policy based on new action-value function

$$\epsilon \leftarrow \frac{1}{k}$$
$$\pi \leftarrow \epsilon - greedy(Q)$$

GLIE Monte-Carlo control converges to the optimal action-value function, $Q(s, a) \to q_*(s, a)$



3. On-Policy Temporal-Difference Learning

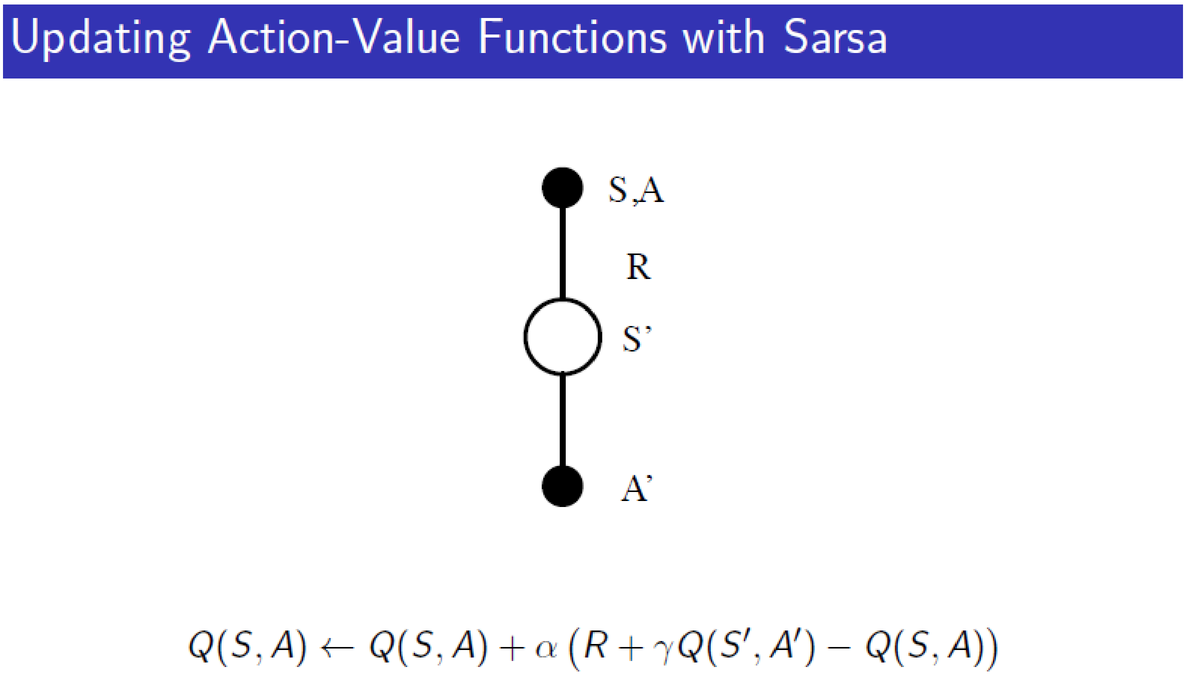
MC vs. TD Control

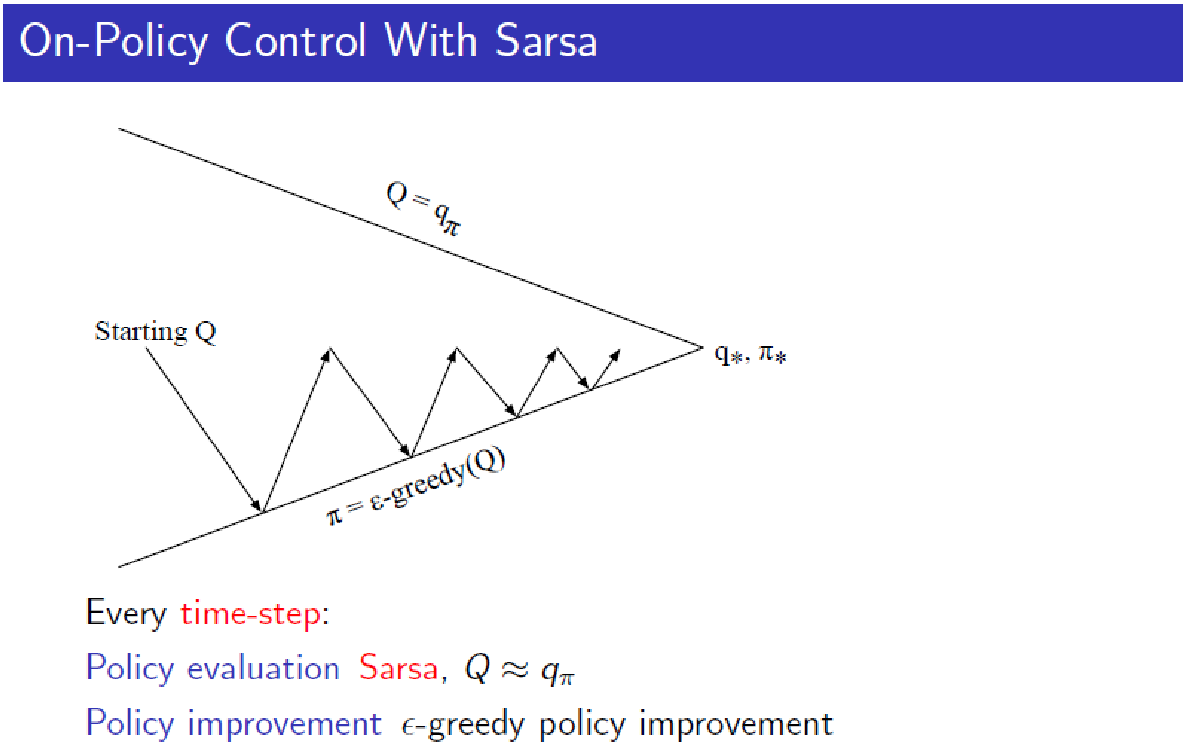
- Temporal-difference (TD) learning has several advantages over Monte-Carlo (MC)
  - Lower variance
  - Online
  - Incomplete sequences

- Natural idea: use TD instead of MC in our control loop
  - Apply TD to $Q(S, A)$
  - Use ϵ-greedy policy improvement
  - Update every time-step

Sarsa(λ)

Updating Action-Value Functions with Sarsa





Sarsa Algorithm for On-Policy Control

Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Choose A from S using policy derived from Q (e.g., ε-greedy)
  Repeat (for each step of episode):
    Take action A, observe R, S'
    Choose A' from S' using policy derived from Q (e.g., ε-greedy)
    $Q(S,A) \leftarrow Q(S,A) + \alpha\big[R + \gamma Q(S',A') - Q(S,A)\big]$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until S is terminal

Convergence of Sarsa

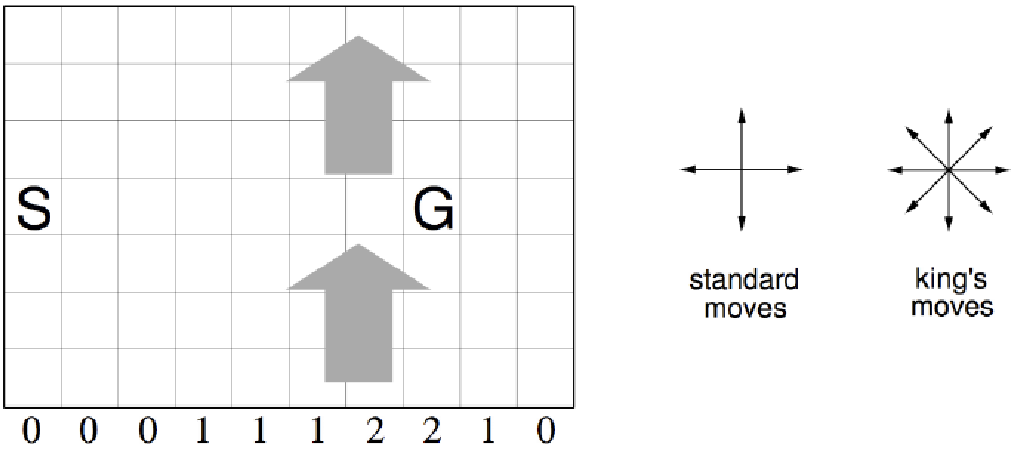
Sarsa converges to the optimal action-value function, $Q(s,a) \to q_*(s,a)$, under the following conditions:

- GLIE sequence of policies $\pi_t(a|s)$
- Robbins-Monro sequence of step-sizes α_t

$$\sum_{t=1}^{\infty} \alpha_t = \infty$$

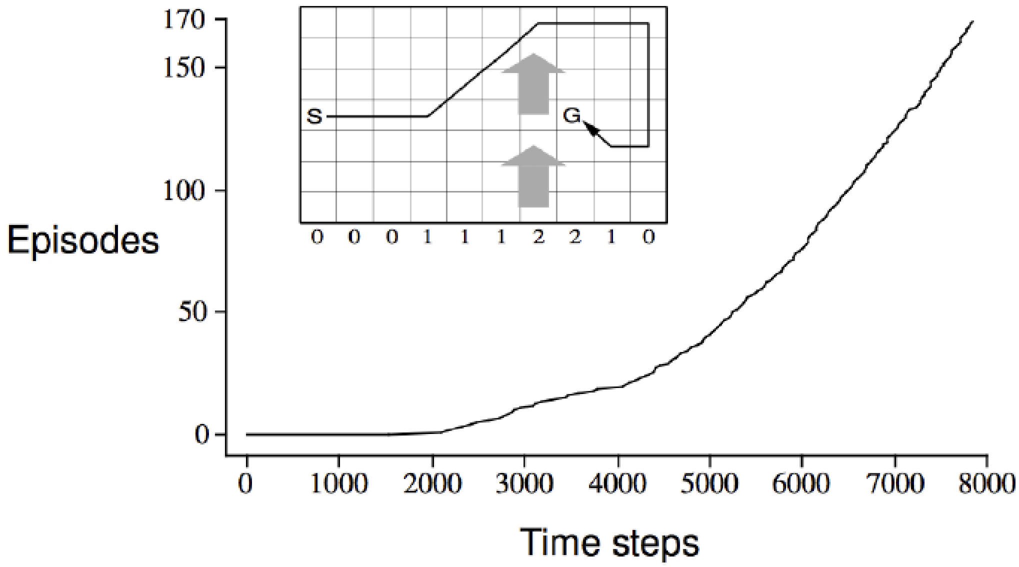
$$\sum_{t=1}^{\infty} \alpha_t^2 < \infty$$

Windy Gridworld Example



- Reward = -1 per time-step until reaching goal
- Undiscounted

Sarsa on the Windy Gridworld



n-Step Sarsa

- Consider the following n-step returns for $n = 1, 2, \infty$:

$$\begin{array}{lll} n = 1 & (Sarsa) & q_t^{(1)} = R_{t+1} + \gamma Q(S_{t+1}) \\ n = 2 & & q_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 Q(S_{t+2}) \\ \vdots & & \vdots \\ n = \infty & (MC) & q_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{T-1} R_T \end{array}$$

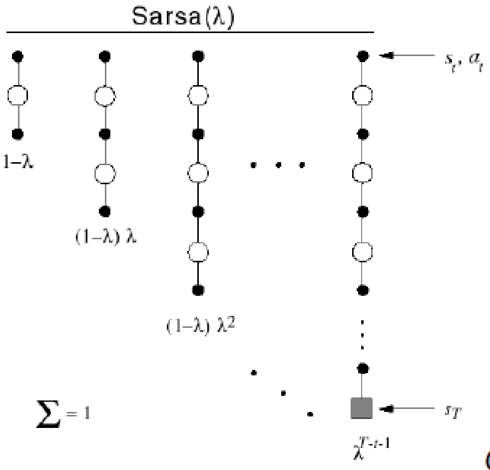
- Define the n-step Q-return

$$q_t^{(n)} = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{n-1} R_{t+n} + \gamma^n Q(S_{t+n})$$

- n-step Sarsa updates $Q(s, a)$ towards the n-step Q-return

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \left(q_t^{(n)} - Q(S_t, A_t) \right)$$

Forward View Sarsa(λ)



- The q^λ *return* combines all n-step Q-returns $q_t^{(n)}$
- Using weight $(1-\lambda)\lambda^{n-1}$

$$q_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} q_t^{(n)}$$

- Forward-view Sarsa(λ)

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \left(q_t^\lambda - Q(S_t, A_t) \right)$$

Backward View Sarsa(λ)

- Just like TD(λ), we use **eligibility traces** in an online algorithm
- But Sarsa(λ) has on eligibility trace for each state-action pair

$$E_0(s,a) = 0$$
$$E_t(s,a) = \gamma\lambda E_{t-1}(s,a) + \mathbf{1}(S_t = s, A_t = a)$$

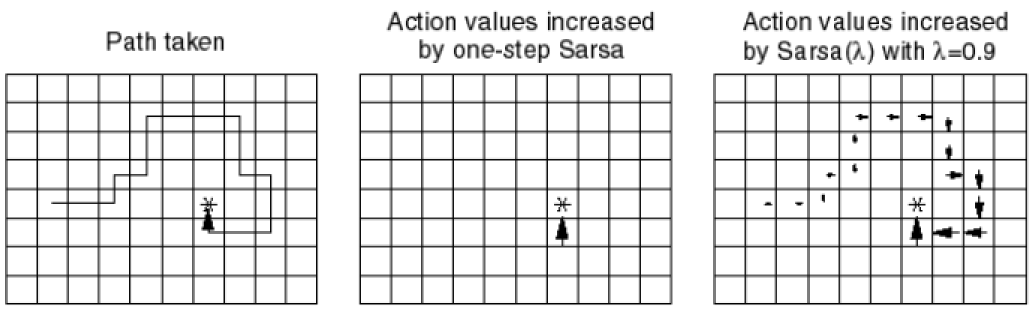
- $Q(s,a)$ is updated for every state s and action a
- In proportion to TD-error δ_t and eligibility trace $E_t(s,a)$

$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$
$$Q(s,a) \leftarrow Q(s,a) + \alpha \delta_t E_t(s,a)$$

Sarsa(λ) Algorithm

Initialize $Q(s,a)$ arbitrarily, for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
Repeat (for each episode):
    $E(s,a) = 0$, for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
    Initialize S, A
    Repeat (for each step of episode):
        Take action A, observe R, S'
        Choose A' from S' using policy derived from Q (e.g., ε-greedy)
        $\delta \leftarrow R + \gamma Q(S', A') - Q(S, A)$
        $E(S,A) \leftarrow E(S,A) + 1$
        For all $s \in \mathcal{S}, a \in \mathcal{A}(s)$:
            $Q(s,a) \leftarrow Q(s,a) + \alpha \delta E(s,a)$
            $E(s,a) \leftarrow \gamma \lambda E(s,a)$
        $S \leftarrow S'$; $A \leftarrow A'$
    until S is terminal

Sarsa(λ) Gridworld Example



4. Off-Policy Learning

- Evaluate target policy $\pi(a|s)$ to compute $v_\pi(s)$ or $q_\pi(s,a)$
- While following behaviour policy $\mu(a|s)$

$$\{S_1, A_1, R_2, \ldots, S_T\} \sim \mu$$

- Why is this important?
- Learn from observing humans or other agents
- Re-use experience generated from old policies □1; □2; :::; □tô€€€1
- Learn about optimal policy while following exploratory policy
- Learn about multiple policies while following one policy

Importance Sampling

- Estimate the expectation of a different distribution

$$\begin{aligned} E_{X\sim P}[f(X)] &= \sum P(X)f(X) \\ &= \sum Q(X)\frac{P(X)}{Q(X)}f(X) \\ &= E_{X\sim Q}[\frac{P(X)}{Q(X)}f(X)] \end{aligned}$$

Importance Sampling for Off-Policy Monte-Carlo

- Use returns generated from μ to evaluate π
- Weight return G_t according to similarity between policies
- Multiply importance sampling corrections along whole episode

$$G_t^{\frac{\pi}{\mu}} = \frac{\pi(A_t|S_t)}{\mu(A_t|S_t)} \frac{\pi(A_{t+1}|S_{t+1})}{\mu(A_{t+1}|S_{t+1})} \ldots \frac{\pi(A_T|S_T)}{\mu(A_T|S_T)} G_t$$

- Update value towards corrected return

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t^{\frac{\pi}{\mu}} - V(S_t))$$

- Cannot use if μ is zero when π is non-zero
- Importance sampling can dramatically increase variance

Importance Sampling for Off-Policy TD

- Use TD targets generated from μ to evaluate π
- Weight TD target $R + \gamma V(S')$ by importance sampling
- Only need a single importance sampling correction

$$V(S_t) \leftarrow V(S_t) + \alpha(\frac{\pi(A_t|S_t)}{\mu(A_t|S_t)}(R_{t+1} + \gamma V(S_{t+1})) - V(S_t))$$

- Much lower variance than Monte-Carlo importance sampling
- Policies only need to be similar over a single step

Q-Learning

- We now consider off-policy learning of action-values $Q(s, a)$
- **No** importance sampling is required
- Next action is chosen using behaviour policy $A_{t+1} \sim \mu(\cdot|S_t)$
- But we consider alternative successor action $A' \sim \pi(\cdot|S_t)$
- And update $Q(S_t, A_t)$ towards value of alternative action

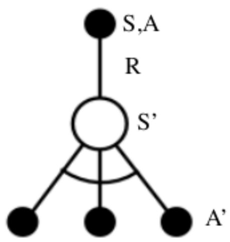
$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma Q(S_{t+1}, A') - Q(S_t, A_t))$$

Off-Policy Control with Q-Learning

- We now allow both behaviour and target policies to **improve**
- The target policy π is greedy w.r.t. $Q(s, a)$
- The behaviour policy μ is e.g. ϵ-greedy w.r.t $Q(s, a)$
- The Q-learning target then simplifies:

$$\begin{aligned} & R_{t+1} + \gamma Q(S_{t+1}, A') \\ &= R_{t+1} + \gamma Q(S_{t+1}, \arg\max_{a'} Q(S_{t+1}, a')) \\ &= R_{t+1} + \max_{a'} \gamma Q(S_{t+1}, a') \end{aligned}$$

Q-Learning Control Algorithm



$$Q(S, A) \leftarrow Q(S, A) + \alpha \left(R + \gamma \max_{a'} Q(S', a') - Q(S, A) \right)$$

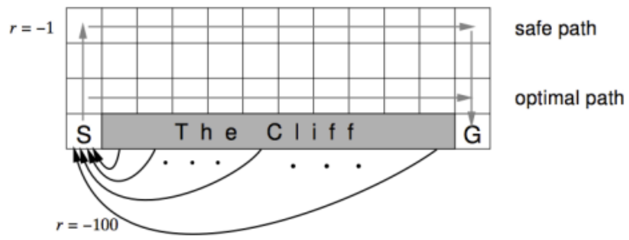
Theorem

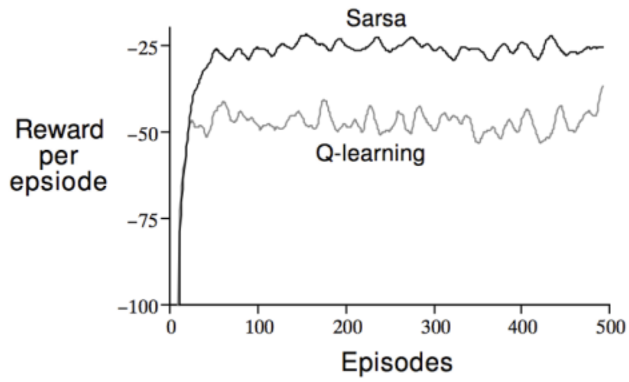
Q-learning control converges to the optimal action-value function, $Q(s, a) \rightarrow q_*(s, a)$

Q-Learning Algorithm for Off-Policy Control

Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
    Initialize S
    Repeat (for each step of episode):
        Choose A from S using policy derived from Q (e.g., ε-greedy)
        Take action A, observe R, S'
        $Q(S,A) \leftarrow Q(S,A) + \alpha\big[R + \gamma \max_a Q(S',a) - Q(S,A)\big]$
        $S \leftarrow S'$;
    until S is terminal

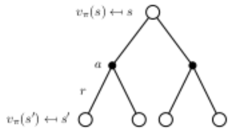
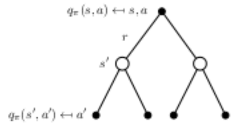
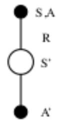
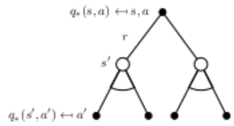
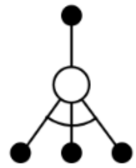
Cliff Walking Example





5. Summary

Relationship Between DP and TD

	Full Backup (DP)	*Sample Backup (TD)*
Bellman Expectation Equation for $v_\pi(s)$	 Iterative Policy Evaluation	 TD Learning
Bellman Expectation Equation for $q_\pi(s, a)$	 Q-Policy Iteration	 Sarsa
Bellman Optimality Equation for $q_*(s, a)$	 Q-Value Iteration	 Q-Learning

Relationship Between DP and TD (2)

Full Backup (DP)	*Sample Backup (TD)*
Iterative Policy Evaluation $V(s) \leftarrow \mathbb{E}[R + \gamma V(S') \mid s]$	TD Learning $V(S) \xleftarrow{\alpha} R + \gamma V(S')$
Q-Policy Iteration $Q(s,a) \leftarrow \mathbb{E}[R + \gamma Q(S', A') \mid s, a]$	Sarsa $Q(S,A) \xleftarrow{\alpha} R + \gamma Q(S', A')$
Q-Value Iteration $Q(s,a) \leftarrow \mathbb{E}\left[R + \gamma \max_{a' \in \mathcal{A}} Q(S', a') \mid s, a\right]$	Q-Learning $Q(S,A) \xleftarrow{\alpha} R + \gamma \max_{a' \in \mathcal{A}} Q(S', a')$

where $x \xleftarrow{\alpha} y \equiv x \leftarrow x + \alpha(y - x)$