Transfer Matrix Method

Light propagation in a multilayered film structure (**Figure 1**) can be calculated based on transfer matrix method.

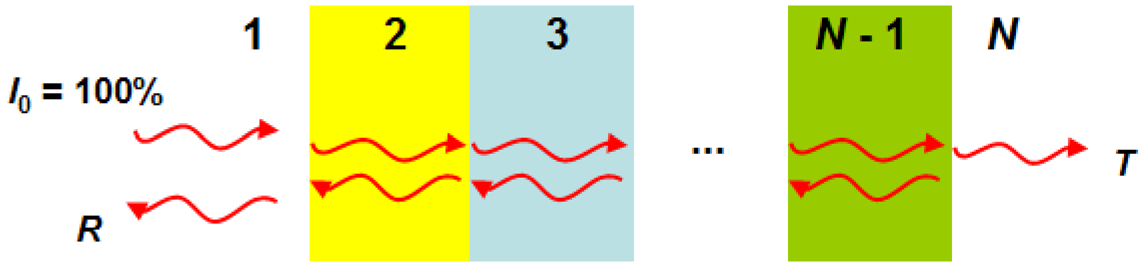

Figure 1. A multilayered structure.

Assume all the materials are non-magnetic ($\mu_r = 1$)

First define the propagation direction, which is illustrated in **Figure 2**. When wave propagates along the *x* axis from left to right, the expression of the electric field is

$$E_{1+} = |E_{1+}| \exp(ik_1 x) \quad \text{(S1)}$$

when wave propagates in the opposite direction, it becomes

$$E_{1-} = |E_{1-}| \exp(-ik_1 x) \quad \text{(S2)}$$

and

$$k_1 = \frac{2\pi}{\lambda} n_1 \quad \text{(S3)}$$

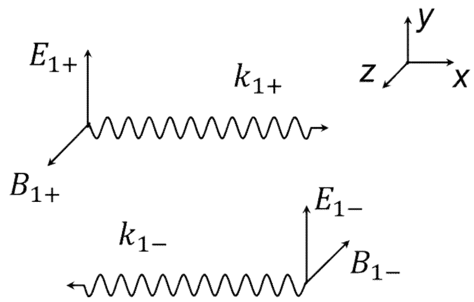

Figure 2. Defining the positive direction of propagation.

Thin-film filters consist of boundaries between different homogenous media, as shown in **Figure 3**. At the interface of media 1 and 2, the boundary conditions are

$$\begin{cases} E_1^{\parallel} = E_2^{\parallel} \\ B_1^{\parallel} = B_2^{\parallel} \end{cases} \tag{S4}$$

and $B = \dfrac{n}{c} E$

Therefore

$$\begin{cases} E_{1+} + E_{1-} = E_{2+} + E_{2-} \\ B_{1+} - B_{1-} = B_{2+} - B_{2-} \end{cases} \rightarrow \begin{cases} E_{1+} + E_{1-} = E_{2+} + E_{2-} \\ n_1(E_{1+} - E_{1-}) = n_2(E_{2+} - E_{2-}) \end{cases} \tag{S5}$$

which can be written in matrix form:

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1+\frac{n_2}{n_1}\right) & \frac{1}{2}\left(1-\frac{n_2}{n_1}\right) \\ \frac{1}{2}\left(1-\frac{n_2}{n_1}\right) & \frac{1}{2}\left(1+\frac{n_2}{n_1}\right) \end{bmatrix} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} = D_{12} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} \tag{S6}$$

D_{12} is a matrix to describe the process at the 1-2 interface.

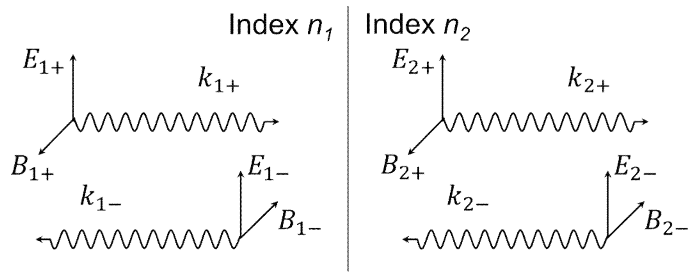

Figure 3. Plane wave incident on a single interface.

Light propagation in homogenous medium 2 (thickness d_2) has the relations (**Figure 4**)

$$\begin{cases} E'_{2+} = |E_{2+}|\exp(ik_2x + ik_2d_2) = E_{2+}\exp(i\frac{2\pi}{\lambda}n_2d_2) \\ E'_{2-} = |E_{2-}|\exp(-ik_2x - ik_2d_2) = E_{2-}\exp(-i\frac{2\pi}{\lambda}n_2d_2) \end{cases} \tag{S7}$$

so

$$\begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} = \begin{bmatrix} \exp(-i\frac{2\pi}{\lambda}n_2 d_2) & 0 \\ 0 & \exp(i\frac{2\pi}{\lambda}n_2 d_2) \end{bmatrix} \begin{bmatrix} E'_{2+} \\ E'_{2-} \end{bmatrix} = P_2 \begin{bmatrix} E'_{2+} \\ E'_{2-} \end{bmatrix} \tag{S8}$$

P_{12} is a matrix to describe the propagation in the medium 2.

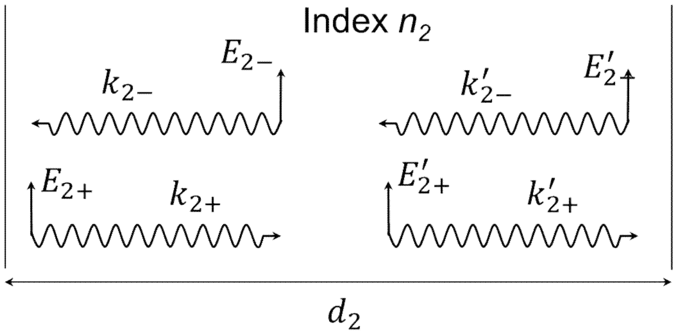

Figure 4. Plane wave propagating in a homogeneous medium.

Figure 5 shows model geometry of light traveling through N layers. The following matrix relation can be deduced referring to Equation S6 and S8:

$$\begin{aligned} \begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} &= D_{12} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} = D_{12}P_2 \begin{bmatrix} E'_{2+} \\ E'_{2-} \end{bmatrix} = D_{12}P_2D_{23} \begin{bmatrix} E_{3+} \\ E_{3-} \end{bmatrix} \\ &= \ldots = D_{12}P_2D_{23}P_3 \cdots P_{N-1}D_{(N-1)N} \begin{bmatrix} E_{N+} \\ E_{N-} \end{bmatrix} \\ &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} E_{N+} \\ E_{N-} \end{bmatrix} \end{aligned} \tag{S9}$$

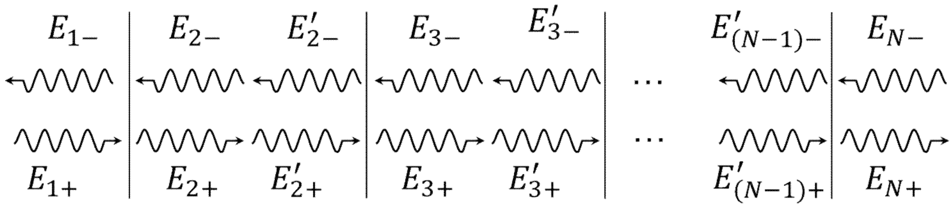

Figure 5. Plane wave traveling through a number of layers.

and there is no back reflected light at the last medium N, so

$$E_{N-} = 0 \tag{S10}$$

The irradiance intensity is related to the Poynting vector

$$I = \frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) \tag{S11}$$

or

$$I \propto n\left|E\right|^2 \tag{S12}$$

Reflectance R is defined as the ratio of the reflected and incident irradiances and transmittance T as the ratio of the transmitted and incident irradiances:

$$R = \left|\frac{E_{1-}}{E_{1+}}\right|^2 = \left|\frac{M_{21}}{M_{11}}\right|^2 \tag{S13}$$

$$T = \frac{n_N}{n_1}\left|\frac{E_{N+}}{E_{1+}}\right|^2 = \frac{n_N}{n_1}\left|\frac{1}{M_{11}}\right|^2 \tag{S14}$$

For absorptive materials, use the complex $\tilde{n}$ to replace n

$$\tilde{n} = n + i\kappa \text{, if we define wave function } E \sim \exp(ikx) \tag{S15}$$

note: use $\tilde{n} = n - i\kappa$, if we define wave function $E \sim \exp(-ikx)$

κ - extinction coefficient

For oblique angles (**Figures 6 and 7**), boundary conditions are

$$\begin{cases} {n_1}^2 E_1^{\perp} = {n_2}^2 E_2^{\perp} \\ E_1^{\parallel} = E_2^{\parallel} \\ B_1^{\perp} = B_2^{\perp} \\ B_1^{\parallel} = B_2^{\parallel} \end{cases} \tag{S16}$$

and from Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{S17}$$

For s-polarised light, in the notation of **Figure 6**, is

$$\begin{bmatrix} 1 & 1 \\ n_1\cos\theta_1 & -n_1\cos\theta_1 \end{bmatrix}\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ n_2\cos\theta_2 & -n_2\cos\theta_2 \end{bmatrix}\begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} \tag{S18}$$

and p-polarised light can be described as

$$\begin{bmatrix} \cos\theta_1 & \cos\theta_1 \\ n_1 & -n_1 \end{bmatrix} \begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & \cos\theta_2 \\ n_2 & -n_2 \end{bmatrix} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} \tag{S19}$$

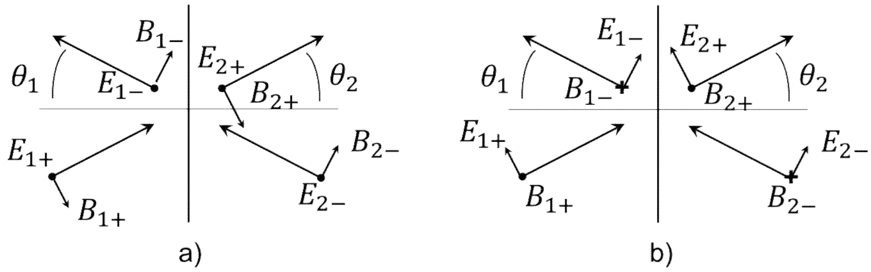

Figure 6. a) Convention defining the positive directions of electric and magnetic vectors for s-polarised light (TE waves). b) Convention defining the positive directions of electric and magnetic vectors for p-polarised light (TM waves).

Light of oblique incidence propagating in homogeneous medium (**Figure 7**) are analogous to that of normal incidence:

$$\begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} = \begin{bmatrix} \exp(-i\frac{2\pi}{\lambda}n_2 d_2\cos\theta_2) & 0 \\ 0 & \exp(i\frac{2\pi}{\lambda}n_2 d_2\cos\theta_2) \end{bmatrix} \begin{bmatrix} E'_{2+} \\ E'_{2-} \end{bmatrix} = P_2 \begin{bmatrix} E'_{2+} \\ E'_{2-} \end{bmatrix}. \tag{S20}$$

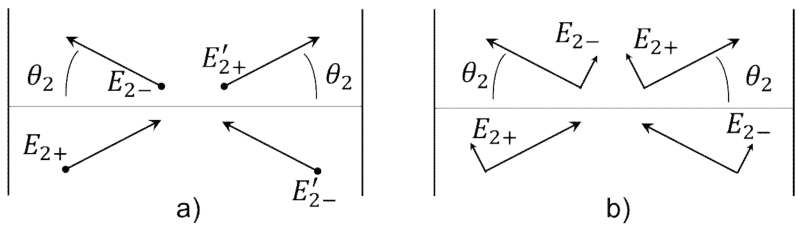

Figure 7. a) S-polarised light and b) p-polarised light propagating in homogeneous medium with oblique angles.