EL2520 - Control Theory and Practice - Advanced Course

Preliminary Solution/Answers – 220531 (Partly only answers, i.e., motivation as required not given)

1. (a) The minors are the elements and the LCD is $(s+1)(s-1)$, hence poles in $s=-1, s=1$. The maximal minors are the elements and the GCD is empty, thus no zeros. The input direction of the pole at $s=1$ is $u_p = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

   (b) The delay $\theta = 5$ gives an upper limit for the bandwidth of the sensitivity function at approx. $\omega = 2/5$. At this frequency $|G_d(i0.4)| = 1.94 > 1$. Thus, it is not possible to achieve $|SG_d| < 1 \; \forall \omega$.

   (c) Writing out the objective function

   $$J(U) = Q_1 x_0^2 + Q_2 u_0^2 + 0.25 Q_1 x_0^2 + Q_1 x_0 u_0 + Q_1 u_0^2 + Q_2 u_1^2$$

   Taking the derivative with respect to U we get

   $$dJ = (2Q_2 u_0 + Q_1 x_0 + 2Q_1 u_0) du_0 + 2Q_2 u_1 du_1$$

   and setting this to zero we get $u_0 = -\frac{Q_1}{2(Q_1+Q_2)} x_0$ and $u_1 = 0$. With $Q_2 = 0$ we get $u_0 = -0.5 x_0$ and $x_1 = 0$, i.e., steady-state reached in one sample.

2. (a) G has a RHP zero at $s=1$ and a RHP pole at $s=0$. To ensure internal stability none of these should be cancelled by the controller and hence $S(0)=0$ and $T(1)=0$, or $S(1)=1$. We have $S(0)=0$, $S(1)=1$ and no RHP poles in S, hence the closed-loop is internally stable.

   (b) The system has a RHP zero at $s=2$ and $|W_p(2)| = 2.4 > 1$ and the norm-bound is not feasible. As for the 2-norm it is not defined for this system since $W_p S$ is semi-proper only.

   (c) We check the diagonal elements of the RGA. $\lambda_{11}(0) = 1.6$ and $|\lambda_{11}(i1)| = 1.02$. Hence the suggested pairing corresponds to pairing on positive RGA-elements at steady-state and RGA-elements with absolute value close to 1 around expected crossover and hence reasonable.

3. (a) $G(s)$ has a RHP zero at $s = \sqrt{2}$ with output zero direction $y_z^H = \begin{bmatrix} -0.51 & 0.86 \end{bmatrix}$. The scaled $G_d = 10/(s+1) \begin{bmatrix} 5 & 3 \end{bmatrix}^T$ and $|y_z^H G_d(\sqrt(2))| = 0.025 < 1$ and hence acceptable disturbance attenuation is not prevented by the RHP zero. Also, the uncertainty in the control input exceeds 100% at a frequency where $|G_d| << 1$ and hence this should not pose a severe problem.

(b) Yes, no control, i.e., $F_y = 0$, will provide robust stability since the open-loop is stable.

(c) The RS condition with input uncertainty is $\|W_I T_I\|_\infty < 1$ where W_I is the uncertainty weight. An uncertainty weight fitting the given input uncertainty is $W_I(s) = 2\frac{s+20}{s+200}$. The requirement for disturbance attenuation is $\|SG_d\|_\infty < 1$. We stack the objectives into one matrix

$$J = \left\| \begin{matrix} SG_d \\ W_I T_I \end{matrix} \right\|_\infty$$

(d) For SG_d we choose d as the input and z as the output. For $W_I T_I$ we choose a disturbance at the input as input and the weighted controller output $W_I u$ as the output. For the norm minimization we should use 2-norm for the signals and then minimize the worst case amplification from input to output to reflect the $\mathcal{H}_\infty$-norm in (c).

4. (a) Since all matrices A, B, M, Q_1, Q_2 diagonal, we solve Riccati equation for diagonal $P > 0$

$$2AP + Q_1 - P^2 = 0$$

For the diagonal elements

$$p_1^2 + 4p_1 - q_1 = 0 \; ; \quad p_2^2 + 8p_2 - q_2 = 0$$

and positive definite solution

$$p_1 = -2 + \sqrt{4 + q_1} \; ; \quad p_2 = -4 + \sqrt{16 + q_2}$$

The optimal feedback $u = -Lx$, $L = Q_2^{-1} B^T P = P$ and we get the closed-loop poles in $-2 - p_1$ and $-4 - p_2$. Thus, $p_1 = 3$ and $p_2 = 1$ which corresponds to $q_1 = 21$ and $q_2 = 9$.

(b) The loop-gain at the input $L_I = L(sI - A)^{-1}B$ and the complementary sensitivity function

$$T_I = L_I(I + L_I)^{-1}$$

With all matrices diagonal, T_I is also diagonal and with elements $T_{I,11} = 3/(s+5)$ and $T_{I,22} = 1/(s+5)$.

(c) With all matrices diagonal we can analyze robustness in the two loops independently, and with the RS condition $|T_{I,ii}| < 1/|W_{I,i} \; \forall \omega$, where $W_{I,i}$ is the relative input uncertainty in input i, we directly see that we can tolerate more uncertainty in input 2.

(d) Increasing the measurement noise puts more emphasis on the complementary sensitivity in the optimization, hence increase R_2.

5. (a) $S = \frac{s}{s+1} I$ and $\|S\|_\infty = 1$.

(b) The poles are in $s = -1 - \delta_1$, $s = -1 - \delta_2$. Thus, stable for all $|\delta_i| < 0.2$.

(c) We get $T_I = \frac{1}{(s+1)} I$ and a sufficient robust stability condition is then $\|\Delta_I\|_\infty < 1$, where Δ_I is the relative input uncertainty, assuming Δ_I a full complex matrix. The uncertainty given corresponds to a diagonal Δ_I with real elements which represents a subset of a full complex matrix. The given uncertainty corresponds to $\|\Delta_I\|_\infty < |\delta_i| < 0.2$ and hence robust stability (with a large margin).

(d) The loop-gain $G_p F_y = G(1 + \Delta_I)F_y$ becomes

$$L_p = \frac{1}{s} \begin{pmatrix} 1 + 11\delta_2 - 10\delta_1 & 11(\delta_1 - \delta_2) \\ 10(\delta_2 - \delta_1) & 1 + 11\delta_1 - 10\delta_2 \end{pmatrix}$$

With $\delta_1 = \delta_2 = \delta$

$$L_p = \frac{1}{s} \begin{pmatrix} 1 + \delta & 0 \\ 0 & 1 + \delta \end{pmatrix} \quad \Rightarrow \quad S_p = (I + L_p)^{-1} = \frac{s}{s + 1 + \delta} I$$

With $\delta_1 = -\delta_2 = \delta$

$$L_p = \frac{1}{s} \begin{pmatrix} 1 + 21\delta & 22\delta \\ 20\delta & 1 - 21\delta \end{pmatrix} \quad \Rightarrow \quad S_p = (I + L_p)^{-1} = \frac{s}{s^2 + 2s + 1 - \delta^2} \begin{pmatrix} s + 1 - 21\delta & 22\delta \\ -20\delta & s + 1 + 21\delta \end{pmatrix}$$

for the first case $\|S_p\|_\infty = 1$ with peak as $\omega \to \infty$. For the second case it is more involved to find the peak value. However, we note that the poles of S_p are all close to 1 (1.2 and 0.8) and hence the peak should appear close to $\omega = 1$. We find $\bar{\sigma}(S_p(i1)) = \sqrt{\lambda_{max}(S_p(i1) S_p(i1)^H)} = 4.32$, and hence $\|S_p\|_\infty \geq 4.32$ (in fact, it is 4.32). Thus, the sensitivity function is relatively unrobust to input uncertainty (20% input uncertainty gives over 300% uncertainty in the peak of the sensitivity function.

(d) With output uncertainty we get the same robust stability condition as above since $T_I = T$ in this case. As for the sensitivity we now get

$$L_p = \frac{1}{s}(I + \Delta_I) \quad \Rightarrow \quad S_p = s \begin{pmatrix} \frac{1}{s+1+\delta_1} & 0 \\ 0 & \frac{1}{s+1+\delta_2} \end{pmatrix}$$

and $\|S_p\|_\infty = 1$ in both cases. Hence the sensitivity function is relative insensitive to output uncertainty.