2 EL2520 Lecture notes 3: Robustness

In Lecture 2 we derived a robustness condition for the stability of a SISO feedback loop when the model of the plant is uncertain. In this lecture we will discuss modeling of uncertainty in more detail and derive robustness conditions for stability under alternative uncertainty descriptions We will also derive a condition for robust performance for SISO systems. As in the previous lectures, we focus for now on SISO systems but approach the problem with a framework that will allow us to more or less directly extend the results to the MIMO case later.

By the term *robust* we understand that a system is insensitive to uncertainty, in the sense that small changes in the system does not result in abrupt changes in the system behavior. Robustness can concern different system properties such as stability, in which case we talk about *robust stability*, or performance, for which we refer to *robust performance*. Robust stability is clearly a prerequisite for robust performance.

A system is either robust or not, and the quantitative measure of robustness is then the size of the uncertainty that the system can tolerate. To be able to quantify robustness we must hence define the uncertainty the system should be able to tolerate to be robust, i.e., what differences between the nominal model G and the true system $\tilde{G}$ should be considered. In general, there may be different sources of model uncertainty

- *Parametric uncertainty:* when certain parameters in a model, such as a time-constant or a time-delay, are uncertain. This is typically the case if we e.g., have identified a model from experimental data using methods from the field of *system identification*.
- *Unmodelled dynamics:* when our model is not sufficiently rich to capture the full dynamics of the system. For instance, it is often hard to describe the fast dynamics of many systems and these are then typically left out of the model. In many cases it is also desirable to work with a simple nominal model G and leave out dynamics that are considered less important for control.

Typically both sources of uncertainty are present in a given model, and it is then often reasonable to lump them together into what is termed a *lumped uncertainty description* and which should then be quite general. Below we will introduce such a lumped uncertainty description, based on model sets, which can accomodate for different sources of uncertainty.

2.1 Model Sets

One way of representing model uncertainty is to define a set of models rather than a single model. To generate such a model set, centered around a nominal model, we introduce a stable perturbation $\Delta_I(s)$ such that

$$\|\Delta_I\|_\infty \leq 1 \quad \Rightarrow \quad |\Delta_I(i\omega)| \leq 1 \;\forall\omega \tag{1}$$

Recall that the $\mathcal{H}_\infty$-norm $\|\cdot\|_\infty$ is only defined for stable systems. Thus, $\Delta_I(s)$ can be any stable transfer-function that has amplitude less than one (and any phase) at all frequencies. Examples of such transfer-functions are $\Delta_I(s) = e^{-\theta s}$ for any delay $\theta > 0$ and $\Delta_I(s) = \frac{1}{(\tau s+1)^n}$ for any $\tau > 0$ and $n > 0$. However, we will not consider any specific transfer-functions in the model set but rather allow any stable perturbation $\Delta_I(s)$ satisfying (1).

Given the above definition of a model perturbation $\Delta_I(s)$, we can define the uncertainty set

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)), \quad \|\Delta_I\|_\infty < 1\} \tag{2}$$

where $W_I(s)$ is an *uncertainty weight* that defines the magnitude of the relative perturbation at each frequency. We will assume that the true system $\tilde{G}$ belongs to the set Π_I, such that if we ensure that the closed-loop system is stable for all models within the set Π_I then so will the true closed-loop system also be stable. Since the uncertainty set in (2) is on the form of relative model uncertainty introduced in Lecture 2, the robust stability condition derived using the small gain theorem applies to (2), i.e.,

$$\|W_I T\|_\infty \leq 1$$

See also Lecture notes 2.

Note that the model set (2) implies that, at each frequency ω, G_p is within a disc centered at $G(i\omega)$ and with radius $|W_I(i\omega)G(i\omega)|$ in the complex plane. See also Figure 2.1. The assumption is then that the frequency reponse of the true system at the given frequency is somewhere within the disc.

Example 1: uncertain gain. Consider a model $G_p(s) = k_p G(s)$ where the gain k is uncertain and can vary between $1-\delta$ and $1+\delta$ (nominal value 1). Thus,

$$\Pi_I = \{G_p(s) = k_p G(s), \quad k_p \in [1-\delta, 1+\delta]\}$$

which can be represented by (2) with a real $\Delta_I \in [-1, 1]$ and $W_I = \delta$.

Example 2: uncertain zero. Consider the uncertain set

$$\Pi_I = \{G_p = G_0(s)(s + z_p), \quad z_p \in [z_0 - \delta, z_0 + \delta]\}$$

which can be represented by (2) with $G(s) = G_0(s)(s + z_0)$, a real $\Delta_I \in [-1, 1]$ and the uncertainty weight $W_I(s) = \delta/(s + z_0)$.

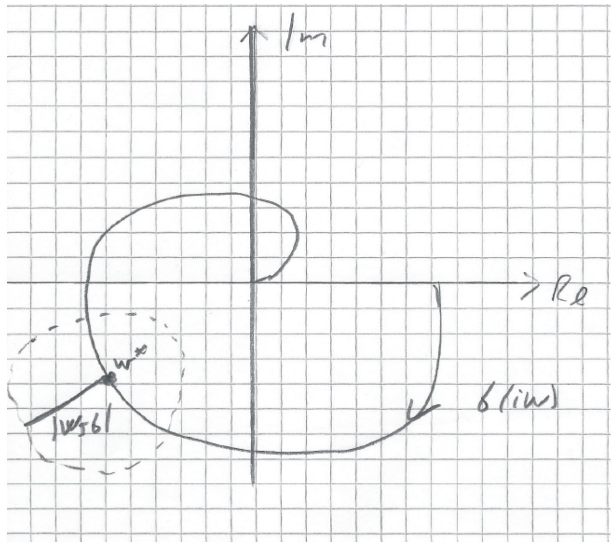

Figure 2.1: Nyquist plot of nominal model G. The dashed circle shows the uncertainty disc for G_p centered at $G(i\omega^*)|$ and with radius $|W_I(i\omega^*)G(i\omega^*)|$ at the frequency ω^*. The frequency response of the true system $\tilde{G}$ at ω^* is assumed to lie within the disc.

Note that in both examples above we should strictly use a real perturbation. However, since we will employ the small gain theorem for stability analysis of the set, we can only bound the magnitude of the uncertainty and hence we must allow a complex perturbation Δ_I, as defined by (1). Thereby we include models in the model set that are not part of the original model set and hence introduce some conservativeness in the robustness analysis.

Example 3: Fitting response data. According to the model set (2) we have

$$G_p(s) = G(1 + W_I \Delta_I) \ ; \quad \|\Delta_I\|_\infty \leq 1$$

From this we derive

$$\|W_I^{-1} G^{-1}(G_p - G)\|_\infty \leq 1$$

Hence, a model G_p is in the set Π_I if

$$|W_I(i\omega)| \geq \left| \frac{G_p(\omega) - G(i\omega)}{G(i\omega)} \right|, \quad \forall \omega \tag{3}$$

Thus, to obtain the uncertainty weight W_I one can simply plot the magnitude of $(G_p - G)/G$ as a function of frequency for all possible model candidates G_p and then fit a transfer-function such that $|W_I|$ is just larger than the maximum of these at all frequencies. Consider for instance the uncertain model

$$G_p = \frac{k e^{-\theta s}}{\tau s + 1} \ , \quad k, \theta, \tau \in [2, 3] \tag{4}$$

We first pick a nominal model, using the average values of k and τ while we choose the time-delay $\theta = 0$ to have a simple delay free nominal model

$$G(s) = \frac{2.5}{2.5s + 1}$$

We then evaluate G_p for values of k, τ and θ in the range $[2,3]$ and plot the corresponding $|G_p(i\omega) - G(i\omega)|/|G(i\omega)|$ as a function of frequency. The result is shown in Figure 2.2. We first fit the weight

$$W_{I1}(s) = \frac{4s + 0.2}{1.6s + 1}$$

The magnitude of this weight is shown by the solid line in Figure 2.2, and as can be seen it satisfies (3) except for in a small frequency range around $\omega = 1$. To make sure the weight covers all model candidates, we add a small modification to the weight above to obtain

$$W_I(s) = W_{I1}(s)\frac{s^2 + 1.6s + 1}{s^2 + 1.4s + 1}$$

which magnitude is shown by the dashed line in Figure 2.2. We see that the weight now satisfies (3) at all frequencies. Hence, we have obtained a model set on the form (2) that covers the uncertainty in (4). Note again that the fitted lumped model set includes more models than the ones given by the original model set (4), hence introducing some conservativeness in the robustness analysis.

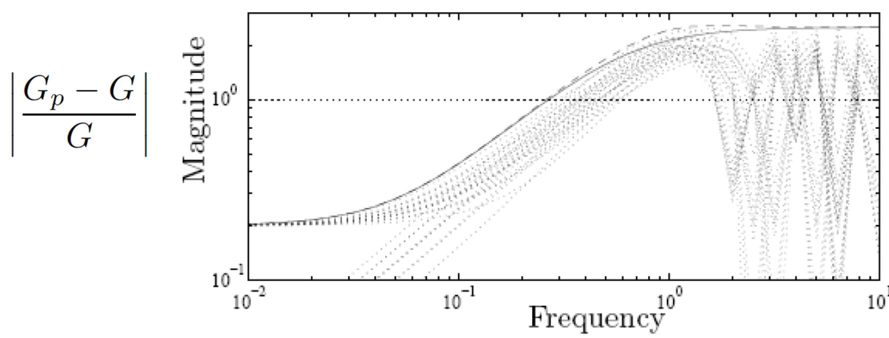

Figure 2.2: Example 3: amplitude of relative uncertainty $|G_p - G|/|G|$ for various parameter values of G_p. The solid and dashed lines shows fitted uncertainty weights W_I.

Let us now consider the controller

$$F_y(s) = K_c\frac{2.5s + 1}{2.5s}$$

According to the robust stability condition derived in Lecture 2 for model uncertainty on the form (2) we have robust stability if we have nominal stability, a stable perturbation $\Delta_G(s)$ and $\|W_I T\|_\infty < 1$. We first check nominal stability, i.e., closed-loop stability with the nominal model $G(s) = 2.5/(2.5s + 1)$. With the controller above we get

$$T(s) = \frac{F_y G}{1 + F_y G} = \frac{K_c}{s + K_c}$$

which is stable for any $K_c > 0$. For internal stability we need to check four transfer-functions (see Lec 2), but in general we have internal stability if one of them is stable and there are no cancellations of poles and zeros in the RHP between controller and plant. In this case, there are no such cancellations, and hence we have internal stability. The perturbation Δ_G is stable by definition (1). To check the robust stability condition $\|W_I T\|_\infty < 1$ we can either directly compute the norm (using e.g., the function `norm` in Matlab), or plot $|T|$ vs $|W_I^{-1}|$ in the frequency domain. With controller gain $K_c = 1$ we compute $\|W_I T\|_\infty = 1.75$ and hence we do not have robust stability. Reducing the controller gain to $K_c = 1/3$ we find $\|W_I T\|_\infty = 0.93$ and hence we have robust stability, i.e., we can guarantee closed-loop stability with any of the models in (4). The corresponding plots of $|T|$ vs $|W_I^{-1}|$ for the two values of K_c are shown in Figure 2.3 and as can be seen $|T|$ violates the bound for $K_c = 1$ but not for $K_c = 1/3$.

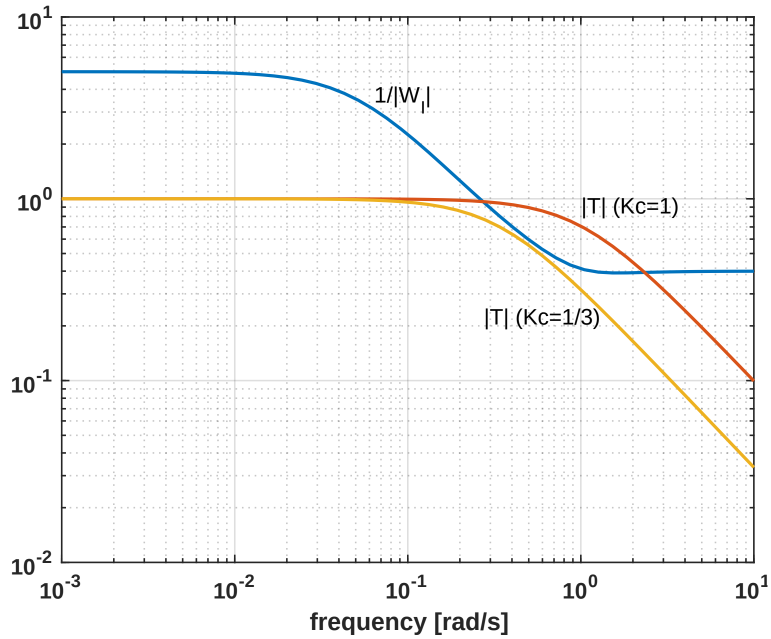

Figure 2.3: Example 3: plot of $|T|$ vs $|W_I^{-1}|$ for two different values of the controller gain K_c.

2.2 Uncertain Number of RHP Poles

One drawback with requiring the perturbation $\Delta_I(s)$ (1) to be stable is that we then can not allow poles to move between the LHP and RHP in the model set (2), i.e., all $G_p(s)$ (including $\tilde{G}$) must have the same number of RHP poles as the nominal model $G(s)$. Thus, if we for instance are uncertain about the open-loop stability of a system this can not be covered by the model set (2). However, it is possible to model an uncertain number of RHP poles by using a different model set. Define

$$\Pi_{iI} = \left\{ G_p(s) = G(s)(1 + W_{iI}(s)\Delta_I(s))^{-1}, \quad \|\Delta_I\|_\infty < 1 \right\} \tag{5}$$

The block diagram representation of (5) is shown in Figure 2.4. Note that we still only allow stable perturbations $\Delta_I(s)$, but by placing the perturbation within a feedback loop we can induce instabilty and thereby model poles crossing the imaginary axis. That is, $(1 + W_{iI}(s)\Delta_I(s))^{-1}$ can be unstable even if $\Delta_I(s)$ (and $W_{iI}(s)$) is stable.

Example 4: Uncertain stability. Assume we have a model with an uncertain pole

$$G_p(s) = \frac{1}{s - p}, \quad p \in [-3, 1]$$

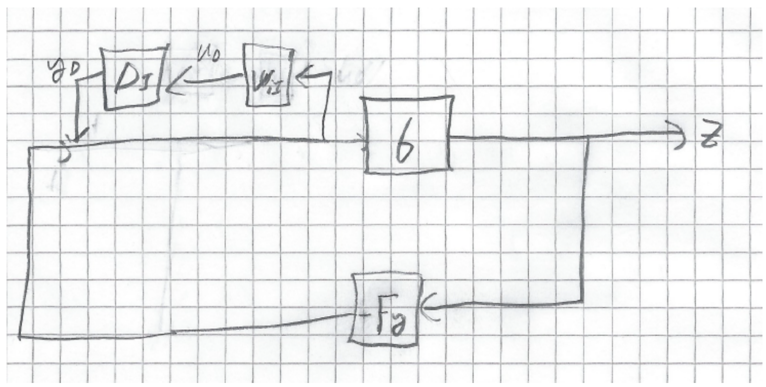

Figure 2.4: Block diagram with feedback around the uncertainty block.

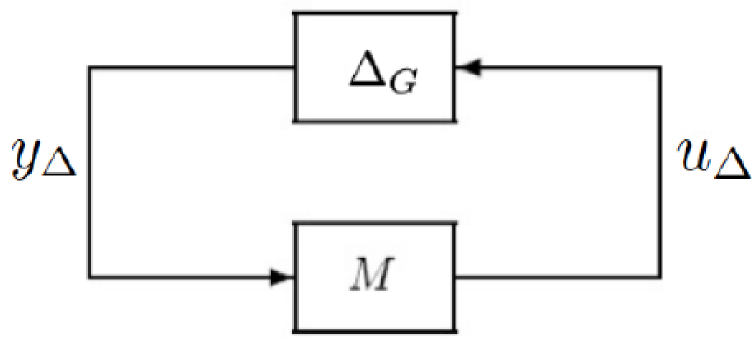

Figure 2.5: Block diagram in Figure 2.4 written on M-Δ-form.

We choose $p = -1$ in the nominal model, i.e., $G(s) = 1/(s+1)$, and the model set is then covered by (5) by choosing

$$W_{iI} = \frac{2}{s+1}$$

and $\Delta_I \in [-1, 1]$. Again we strictly only need a real perturbation Δ_I but must allow a complex perturbation with $|\Delta_I| < 1 \forall\omega$ to enable the use of the small gain theorem for robust stability analysis.

Let us now derive a robust stability condition for the model set (5). As before we rewrite the block-diagram in Figure 2.4 on M-Δ form (Figure 2.5), where M is a nominal transfer-function and Δ is a stable perturbation. Letting u_Δ and y_Δ be the input and outputs, respectively, of the Δ_I block in Figure 2.4, we identify

$$u_\Delta = W_{iI} y_\Delta - G F_y u_\Delta \;\Rightarrow\; u_\Delta = \underbrace{W_{iI} \frac{1}{1 + G F_y}}_{M} y_\Delta$$

Hence, $M = W_{iI} S$ and the robust stability condition is then W_{iI}, S stable, Δ_I stable and

$$\|W_{iI} S\|_\infty < 1$$

Thus, the robust condition now involves the sensitivity function S rather than the complementary sensitivity T, and we need to make $|S|$ small for frequencies where the uncertainty

$|W_{iI}|$ is large. Note that $|S|$ small means tight control and is as expected since uncertain open-loop instability implies that we need relatively high bandwidth to stabilize the system.

2.3 Robust Performance

Above we only considered closed-loop stability for all models in the respective model sets, i.e., robust stability. The assumption that the true system is within the considered model set then implies that we can guarantee closed-loop stability when the controller is applied to the real system. This is in some sense a least requirement on a controller; the system should not become unstable when we close the loop on the real system. However, it would obviously be an advantage if we also could guarantee that the performance specifications could be met for all models in the model set, and thereby the real system. This is called robust performance. It turns out that it is relatively easy to both analyze and achieve robust performance for SISO systems. We show this next.

Consider the performance specification

$$\|W_S S\|_\infty < 1 \quad \Rightarrow \quad |S(i\omega)| < |W_S^{-1}(i\omega)| \; \forall\omega$$

where S is the nominal sensitivity function, i.e., $S = 1/(1+GF_y)$ with G being the nominal model. For robust performance with the model set (2) we get

$$|W_S S_p(i\omega)| < 1 \; \forall\omega; \quad S_p = \frac{1}{1+G_pF_y} \quad \forall G_p \in \Pi_I$$

Inserting the relative model uncertainty of (2) we have

$$S_p = \frac{1}{1+L+W_I\Delta_I L}\;; \quad L = GF_y$$

At each frequency the worst case, i.e., the maximum $|S_p|$, is obtained when $1+L$ and $w_I\Delta_I L$ point in opposite directions in the complex plane. Then

$$\sup_{G_p\in\Pi_I} |S_p(i\omega)| = \frac{1}{|1+L(i\omega)| - |W_I(i\omega)L(i\omega)|}$$

and the requirement $|W_S S_p| < 1$ becomes

$$\frac{|W_S|}{|1+L| - |W_I L|} < 1 \; \forall\omega$$

Dividing numerator and denominator by $|1+L|$ we get

$$\frac{|W_S S|}{1-|W_I T|} < 1 \forall\omega \quad \Leftrightarrow \quad |W_S S| + |W_I T| < 1 \; \forall\omega \tag{6}$$

Note that $|W_S S| < 1$ is the nominal performance requirement and $|W_I T| < 1$ is the robust stability criterion for the model set (2). Thus, we can conclude that if we just satisfy nominal performance and robust stability with some margin then we also get robust performance.

Note that the robust performance criterion (6) is valid for SISO systems only. In contrast to most other results we present in this course, this analysis can (unfortunately) not be easily extended to MIMO systems.