# DECISIONS AND CONTROL

KTH

## EL2520 Control Theory and Practice - Advanced Course

Exam 08.00–13.35    June 8, 2021

Part I (problems 1-2) 08.00-10.00, hand in 10.00-10.20

Part II (problems 3-5) 10.35-13.35, hand in 13.35-13.55

**Aid:**

Printed course book *Glad and Ljung, Control Theory / Reglerteori* **or** *Skogestad and Postlethwaite, Multivariable Control*, basic control course book *Glad and Ljung, Reglerteknik*, or equivalent if approved by examiner beforehand, printed copies of slides from lectures 2021 or 2020, printed lecture notes from 2021 or 2020, mathematical tables, pocket calculator (graphing, not symbolic). Any notes related to solutions of problems are not allowed. **No digital aids allowed.**

Note that separate notes, exercise material and old exams etc are NOT allowed.

**Observandum:**

- Name and social security number (*personnummer*) on every page.
- Only one solution per page.
- Each answer has to be motivated.
- The exam consists of five problems worth a total of 50 credits
- The answers to Part I should be uploaded on Canvas no later than 10.20
- The answers to Part II should be uploaded on Canvas no later than 13.55

**Grading:**

Grade A: $\geq 43$, Grade B: $\geq 38$

Grade C: $\geq 33$, Grade D: $\geq 28$

Grade E: $\geq 23$, Grade Fx: $\geq 20$

All grades require at least 18p on problems 1-3

**Results:** The results will be available about 3 weeks after the exam on "My Pages".

**Responsible:** Elling W. Jacobsen 0703722244 / Yu Wang 076 4269870

*Good Luck!*

1. (a) Consider the $2 \times 2$ system

$$G(s) = \frac{1}{s+1} \begin{pmatrix} 1 & 2 \\ \frac{s+1}{s+3} & 1 \end{pmatrix}$$

   (i) Determine the poles and zeros, and the corresponding input direction for those in the RHP. (3p)

   (ii) How many states will there be in a minimal realization of $G$? (1p)

   (b) Determine the $\mathcal{H}_\infty$-norm of

$$S(s) = \frac{s}{2s+1} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

(3p)

   (c) Given a $2 \times 2$ system with model

$$G(s) = \frac{1}{4s+1} \begin{pmatrix} 1 & -1 \\ 2 & -2.1 \end{pmatrix}$$

The aim is to design a feedback controller so that the bandwidth for the sensitivity function is around 1 $rad/s$. You are asked to determine if decentralized control is a reasonable choice for this system, and if so, which pairing of inputs and outputs is preferable. Motivate your answer! (3p)

2. (a) Consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_1(t) + 2x_2(t) + 2u(t) + w_1(t) \\
\dot{x}_2 &= x_1(t) \\
z(t) &= x_1(t) - x_2(t) + w_2(t) \\
y(t) &= z(t) + n(t)
\end{aligned}
$$

We are given the task of designing a controller that attenuates the effect of the disturbance $w_2$ on the output $z$ and at the same time do not amplify the measurement noise $n$ too much in $z$.

(i) What is the smallest amplification $|z|/|w_2|$ we can achieve if we consider sinusoidal disturbances $w_2$ of any frequency? (4p)

(ii) To attenuate the measurement noise $n$ it is proposed to include the objective

$$
\|W_T T\|_\infty < 1 \ , \quad W_T = \frac{1}{M_T} + \frac{s}{\omega_{BT}}, \quad M_T = 2, \ \omega_{BT} = 1
$$

in the controller synthesis. Is it possible to find a stabilizing controller satisfying this objective? (2p)

(b) We shall design a one-degree of freedom decoupling controller for the system

$$
G = \frac{1}{(s+1)} \begin{pmatrix} 1 & s+1 \\ 1 & 2 \end{pmatrix}
$$

(i) One of your co-workers has proposed to design a controller such that the complementary sensitivity becomes

$$
T(s) = \frac{1}{0.1s+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Explain why this is a bad idea. (2p)

(ii) You suggest a modification of the proposed $T(s)$ on the form

$$
T(s) = t(s) \frac{1}{0.1s+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Determine an appropriate choice of $t(s)$. (2p)

3. We shall consider control design for the following system

$$z = \frac{1}{10s+1}\begin{pmatrix} 1 & 3 \\ 2 & s+2 \end{pmatrix} u + \frac{1}{10s+1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} d$$

The aim is to keep $|z| < 1$ in the presence of disturbances $|d| < 5$, for all frequencies $\omega$. The control inputs are uncertain and the uncertainty is 20% at low frequencies, becomes 100% at frequency $\omega = 5$ and can be assumed to be more than 100% at higher frequencies.

(a) Determine if it is feasible to find a controller that provides the required disturbance attenuation in the nominal case, i.e., when not considering input uncertainty. (3p)

(b) Determine if it is feasible to find a controller that provides robust stability in the presence of the given input uncertainty. (1p)

(c) A controller that satisfies the control objectives can be determined by solving an optimization problem

$$F_y = \arg\min_{F_y} J$$

where $J$ is the objective function to be minimized. Determine an objective function $J$ that reflects the requirements on disturbance attenuation and robust stability. All weights in $J$ should be specified. (4p)

(d) In order to solve the optimization problem in (c) we need to determine an extended system. What should the inputs and outputs of this extended system be? (2p)

4. Consider a simple servo with model

$$Z(s) = \frac{3}{s}U(s) \tag{1}$$

We shall consider designing a feedback controller based on measurement of the position $z$ and manipulating the input $u$ to the motor.

(a) Assume first we can measure $z$ without uncertainty, and determine the controller that minimizes the criterion

$$J = \int_0^\infty z(t)^2 + u(t)^2 dt$$

(2p)

(b) When applying the controller from (a), one finds that the output becomes too noisy due to measurement noise being amplified by the controller. To deal with this, one decides to employ an observer. Assume the measurement $y$ is noisy

$$y = z + n \; ; \quad R_n = 0.4$$

where $R_n$ is the variance of $n$. Also assume the input is uncertain such that true input $u_p$ is given by

$$u_p = u + w \; ; \quad R_w = 0.2$$

That is, the input uncertainty is modeled as white noise with variance $R_w$. Determine the optimal observer, i.e., the Kalman filter, for estimating $z$. How will the use of the observer affect the controller in (a)? (3p)

(c) Upon implementation of the controller in (a), one runs into problems with constraints in the input which is constrained as $|u| < 0.2 \; \forall t$. To deal with this, it is decided to employ an MPC controller

(i) Consider the sampling time $T = 1s$ and determine the discrete time state space model for the system (1). You can assume zero order hold on the input. (1p)

(ii) The MPC optimization problem is

$$\min_u \sum_{i=k}^{i=k+N} (Q_z z_k^2 + u_k^2)$$

subject to $|u_k| < 0.2$. Consider the prediction horizon $N = 1$ and write the MPC problem in the form of a QP problem

$$\min_u u^T H u + h^T u \; ; \quad Lu \leq b$$

That is, determine $H, h, L$ and $b$ for this problem. (4p)

5. We shall consider how we can ensure robust stability for systems with uncertain open-loop stability using either $\mathcal{H}_\infty$-optimal control or LQ.

   A fellow engineer has determined a model of a system using system identification. The model is given by

$$\begin{aligned} \dot{x} &= -(a + \Delta_a)x(t) + bu(t) \\ y(t) &= x(t) \end{aligned} \tag{2}$$

   where $\Delta_a \in \mathbb{R}$ denotes the uncertainty in the estimation of $a$.

   (a) Design a controller that minimizes the criterion

$$\int_0^\infty Q_1 y^2 + Q_2 u^2 dt \tag{3}$$

   for the nominal model, i.e., assuming $\Delta_a = 0$. (1p)

   (b) Show that the controller from (a) can stabilize the uncertain model (2), for arbitrary $\Delta_a$, for a suitable choice of weights $Q_1$ and $Q_2$ in (3). (3p)

   (c) Now use a robust stability condition, based on the $\mathcal{H}_\infty$-norm of some suitable transfer-function, to derive a robust stability criteria for $Q_1$ and $Q_2$ in the controller design in (a) given $|\Delta_a| < \delta$ in (2). (4p)

   (d) Compare the results from (b) and (c) and discuss differences, or absence thereof. (2p)