EL2520 - Control Theory and Practice - Advanced Course
Solution/Answers – 2019-05-29

1. (a) (i) The determinant is $\det G = \frac{s+3}{(s+1)^2(s+2)}$ and the denominator is the LCD for all minors. Hence there are two poles in $s = -1$, one pole in $s = -2$ and one zero at $s = -3$. With 3 poles we need at least 3 states in a state-space realization.

   (b) The (only) singular value is given by $\sigma(i\omega) = \sqrt{\lambda(S(i\omega)^H S(i\omega))} = \frac{\omega}{\sqrt{\omega^2+1}}\sqrt{5}$. The peak value occurs as $\omega \to \infty$ and $\|S\|_\infty = \sup_\omega \bar{\sigma}(S) = \sqrt{5}$.

   (c) The 1,1-element of the RGA is $\lambda_{11} = \frac{1}{1-\frac{G_{12}G_{21}}{G_{11}G_{22}}} = \frac{s+1}{s+0.1}$. At $\omega = 0$ we get $\lambda_{11}(i0) = \lambda_{22}(i0) = 10$ and $\lambda_{12}(i0) = \lambda_{21}(i0) = 1 - \lambda_{11}(i0) = -9$. Since we should pair on positive steady-state RGA elements, only the diagonal pairing should be considered. At $\omega = 1$ we get $|\lambda_{11}(i1)| = |\lambda_{22}(i1)| = \sqrt{2}/\sqrt{1.01} = 1.41$. Around the bandwidth the RGA-elements are close to 1, indicating weak interactions and hence decentralized control should work well if we pair on the diagonal.

2. (a) Consider possible limitations. There are no RHP poles or time delays. Determine zeros from

$$\det G(s) = \frac{1-3s}{(0.5s+1)^2}$$

   Hence there is a RHP zero at $s = 1/3$. From maximum modulus thm we then have that

$$\|W_P S\|_\infty \geq |W_P(z)|$$

   We have

$$W_P(1/3) = 100(61/20)/13 = 23.46 > 1$$

   Since $W_P(z) > 1$ it is not possible to satisfy the given bounds on $\bar{\sigma}(S)$.

   (b) The sensivity function becomes

$$S = \frac{s}{s + K_c(2-s)} = \frac{s}{(1-K_c)s + 2K_c}$$

   which is stable for $0 < K_c < 1$. Since there are no RHP zero-pole cancellations, the closed-loop is also internally stable for these controller gains. Including the weight we obtain

$$w_P S = 0.5\frac{s+1}{(1-K_c)s + 2K_c}$$

Since there is one pole and one zero, the maximum value of $|w_P S(i\omega)|$ will occur either at frequency $\omega = 0$ or at $\omega = \infty$. At $\omega = 0$

$$|w_P S(0)| = 0.5\frac{1}{2K_c}$$

and at $\omega = \infty$

$$|w_P S(i\infty)| = 0.5\frac{1}{1-K_c}$$

The value at $\omega = 0$ is monotonically decreasing while the value at $\omega = \infty$ is monotonically decreasing with $K_c \in [0, 1]$, and the minimum is hence achieved when they are equal

$$\frac{1}{2K_c} = \frac{1}{1-K_c} \Rightarrow K_c = 1/3$$

which gives $\|w_P S\|_\infty = 3/4$. The lower bound for any stabilizing controller is, due to the RHP zero at $s = 2$

$$\|w_P S\|_\infty > |w_P(2)| = 3/4$$

That is, we can not do better, in terms of the $\mathcal{H}_\infty$ objective, with a more advanced controller in this case.

(c) Disturbances are attenuated at frequencies where $|S(i\omega)| < 1$. When the open-loop has a real RHP zero at $s = z$, then disturbance attenuation can only be achieved up to a frequency $\omega = z$. i) RHP zero at $s = 1$, hence maximum frequency $\omega = 1$, ii) RHP zero at $s = 2$, hence maximum frequency $\omega = 2$, iii) Greatest common divisor of the two elements is 1, hence no RHP zeros and no limitation on the frequency.

3. (a) Having a disturbance at the input u_2 corresponds to

$$G_d = \frac{10}{(3s+1)^2}\begin{pmatrix}-1\\-2\end{pmatrix}$$

where we also have scaled so that $|u_d| < 1$ and $|y| < 1$ for acceptable performance. No RHP poles or deadtimes in $G(s)$, check for RHP zeros

$$\det G(s) = \frac{2-2s}{(3s+1)^4}$$

and hence a RHP zero at $s = 1$. A condition for acceptable disturbance attenuation is that $|y_z^H G_d(z)| < 1$ where y_z is the output zero direction for the RHP zero. By definition of the output zero direction $y_z^H G(z) = 0$ and hence it follows that for a disturbance at one input we always have $y_z^H G_d(z) = 0$. Thus, acceptable disturbance attenuation is feasible.

(b) (i) We require $\bar{\sigma}(S(0)) < 0.01$ and $\bar{\sigma}(S(i\omega)) < 0.1\ \forall \omega < 0.1$ rad/s for acceptable disturbance attenuation. For acceptable noise attenuation we require $\bar{\sigma}(T(i\omega)) < 0.1\ \forall \omega > 10$ rad/s.

(ii) For $\bar{\sigma}(S) << 1$ we have $\bar{\sigma}(S) \approx 1/\underline{\sigma}(L)$ and hence the requirements on the sensitivity translate into the approximate requirements on the loop gain L; $\underline{\sigma}(L(0)) > 100$ and $\underline{\sigma}(L(i\omega)) > 10\ \forall \omega < 0.1$ rad/s. For $\bar{\sigma}(T) << 1$ we have that $\bar{\sigma}(T) \approx \bar{\sigma}(L)$, and hence $\bar{\sigma}(L(i\omega)) < 0.1\ \forall \omega > 10$ rad/s.

(iii) $\bar{\sigma}(L)$ must decrease by a factor of at least 100 over two decades, corresponding to a slope of about -1 in a log-log plot. This should normally not be a problem to achieve. However, if there is a large difference in magnitude between $\bar{\sigma}(L)$ and $\underline{\sigma}(L)$, the slope will need to be larger and this may potentially cause stability problems.

(iv) The weight $W_S = \frac{100}{100s+1}$ has $W_S(0) = 100$ and $|W_S(i0.1)| \approx 10$. The weight $W_T = 0.5(2s+1)$ has $|W_T(i\omega)| > 10\ \forall \omega > 10$ rad/s. Note that these two weights indicate that the conditions are tight since both weights are close to 1 in magnitude around frequency 1 rad/s, but this is in principle feasible to achieve. Also, by using higher order weights we can separate the weights more between $\omega = 0.1$ and $\omega = 10$.

4. (a) (i) The LQ controller $u(t) = -Lx(t)$ where $L = Q_2^{-1}B^TP$ where $P \geq 0$ solves

$$-Q_2^{-1}P^2 + 2P + Q_1 = 0$$

which yields $P = Q_2(1+\sqrt{1+Q_1/Q_2})$ and $L = 1+\sqrt{1+Q_1/Q_2} = 1+\sqrt{1+q}$. The Kalman filter has observer gain $K_f = PC^TR_2^{-1}$ where $P \geq 0$ solves

$$-R_2^{-1}P^2 + 2P + R_1 =$$

which yields $P = R_2(1+\sqrt{1+R_1/R_2}$ and $K_f = 1+\sqrt{1+R_1/R_2} = 1+\sqrt{1+r}$.

(ii) The closed-loop can be written, with $\tilde{x} = \hat{x} - x$,

$$\dot{x} = (A - BL)x(t) + BL\tilde{x}(t) + v_1(t)$$

$$\dot{\tilde{x}} = (A - KC)\tilde{x}(t) + v_1(t) - K_f v_2(t)$$

and the eigenvalues are then the union of the eigenvalues of $A - BL$ and $A - KC$ which are $-\sqrt{1+q}$ and $-\sqrt{1+r}$.

(iii) By increasing the size of the measurement noise v_2 relative to the amount of process noise v_1, we will put more emphasis on the transfer-function from v_2 to z which is the complementary sensitivity. Hence, by decreasing r we should reduce the size of $|T|$ and thereby increase robust stability for relative input uncertainty.

(b) The closed-loop Jacobian becomes

$$A - BL = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

and the eigenvalues are -2 and -1. Since we have eigenvalues outside the unit circle in the complex plane, the system is unstable.

5. Relative input uncertainty

$$G_p = G_{diag}(I + W_I \Delta_I), \ \|\Delta_I\|_\infty \leq 1$$

where G_p should include $G(s)$ and $G_{diag}(s)$ is the transfer-matrix obtained from $G(s)$ by neglecting the off-diagonal terms. We get

$$W_I = G_p G_{diag}^{-1} - I = \begin{pmatrix} 1 & \frac{-0.9}{s+1} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - I$$

$$= \begin{pmatrix} 0 & \frac{0.9}{s+1} \\ 1 & 0 \end{pmatrix}$$

The robust stability criterion for input uncertainty is

$$\|T_I W_I \Delta_I\|_\infty \leq 1$$

or, since $\|\Delta_I\|_\infty < 1$,

$$\|T_I W_I\|_\infty \leq 1$$

where $T_I = F_y G_{diag}(I + F_y G_{diag})^{-1}$ is the complementary sensitivity at the input. We get

$$T_I = \begin{pmatrix} \frac{K_{c1}}{s+K_{c1}} & 0 \\ 0 & \frac{K_{c2}}{s+K_{c2}} \end{pmatrix}$$

which is stable for all $K_{c1}, K_{c2} > 0$. This gives

$$T_I W_I = \begin{pmatrix} 0 & \frac{0.9K_{c2}}{(s+1)(s+K_{c2})} \\ \frac{K_{c1}}{s+K_{c1}} & \end{pmatrix}$$

and the $\mathcal{H}_\infty$-norm is the peak-value of the magnitude of the elements (since it can be re-arranged into a diagonal matrix) which is

$$\|T_I W_I\|_\infty = 1$$

independent of the controller gain K_{c1}, K_{c2}. Thus, we have robust stability for all $K_{c1}, K_{c2} > 0$.