EL2520
Control Theory and Practice

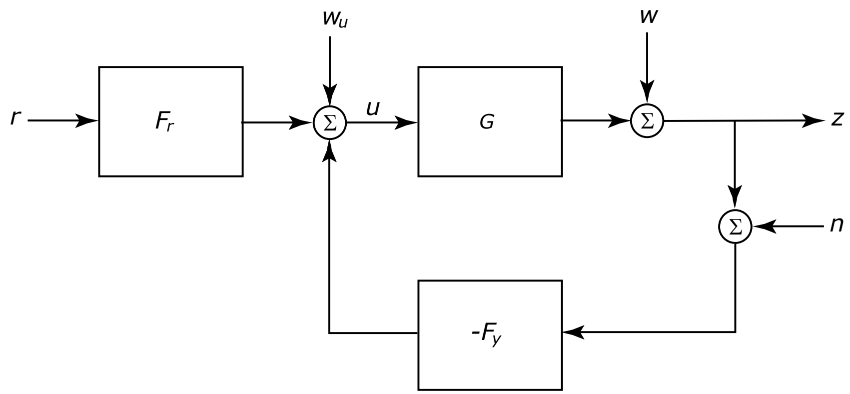
Fundamental Limitations in MIMO Control

Elling W. Jacobsen
School of Electrical Engineering and Computer Science
KTH, Stockholm, Sweden

Today’s Lecture

- Performance specifications for MIMO systems
- Performance limitations in MIMO systems
  - S+T=I
  - Generalized Bode sensitivity integral
  - RHP zeros and poles
  - Disturbances and RHP zeros

Performance Objectives



- We have

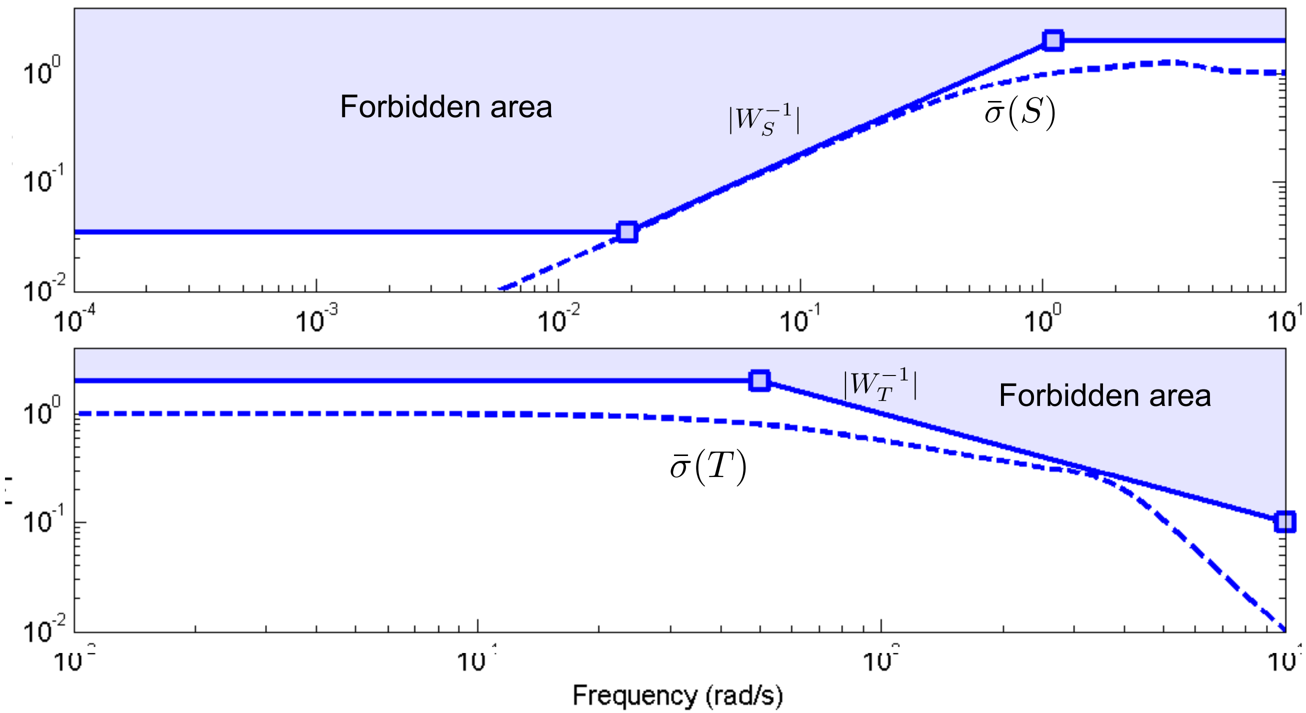
$$z = (I + GF_y)^{-1} w - (I + GF_y)^{-1} GF_y n = Sw - Tn$$

- Thus, make S "small" for disturbance attenuation and T "small" for noise damping
- At each frequency we have

$$\underline{\sigma}(S(i\omega)) \leq \frac{|z|}{|w|} \leq \bar{\sigma}(S(i\omega)) \ ; \quad \underline{\sigma}(T(i\omega)) \leq \frac{|z|}{|n|} \leq \bar{\sigma}(T(i\omega))$$

  - Thus, to bound |z|, need to bound $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$

Frequency Domain Specifications



$$\bar{\sigma}(S) \leq |W_S^{-1}| \ \ \forall\omega \quad \Leftrightarrow \quad \|W_S S\|_\infty \leq 1 \quad ; \quad \bar{\sigma}(T) \leq |W_T^{-1}| \ \ \forall\omega \quad \Leftrightarrow \quad \|W_T T\|_\infty \leq 1$$

- What limits the choices for W_S and W_T ?
- Note: we assume scalar weights throughout

A Note on Scaling

- Before considering performance specifications and limitations it is useful to scale the problem so that the expected or allowed size of any signal is 1
- See Lecture notes 6 on how to scale.

Trade-off 1: S+T=I

$$\underbrace{(I+GF_y)^{-1}}_{S}+\underbrace{(I+GF_y)^{-1}GF_y}_{T}=I$$

- Fan's Theorem:

$$\sigma_i(A+B)\geq\sigma_i(A)-\bar{\sigma}(B)\quad\forall i$$

- Thus,

$$|1-\bar{\sigma}(T)|\leq\bar{\sigma}(S)\leq 1+\bar{\sigma}(T)$$

- Hence, at any frequency
  - can not make both $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ small (at the same frequency)
  - peak in S implies peak in T:

$$\bar{\sigma}(S)>>1\;\Leftrightarrow\;\bar{\sigma}(T)>>1$$

- Hence, we can not choose $|W_S|$ and $|W_T|$ large at the same frequency

Trade-off 2: (Bode) Sensitivity Integral

- Extension of the Bode Sensitivity Integral (see lec 4) to MIMO systems yields

$$\int_0^\infty \ln|\det(S(i\omega))|d\omega = \Sigma_j \int_0^\infty \ln\sigma_j(S(i\omega))d\omega = \pi\Sigma_i \operatorname{Re}(p_i)$$

  - proof based on Cauchy integral formula.
    - det(S) is essentially a sensitivity function and $\det(S) = \prod_j \sigma_j(S)$
  - trade-off between frequencies as well as between directions
  - cannot make $|W_S|$ large at all frequencies and in all directions (note that a scalar weight usually is preferred and then all directions are weighted equal)

Limitation 1: RHP Zeros

Thm: Assume G(s) has a RHP zero at s=z>0. Then

$$\|W_S S\|_\infty \geq |W_S(z)|$$

– generalization of result for SISO case (see video/lec notes 4)

Proof: By definition G(z) is rank deficient, i.e.,

$$y_z^H G(z) = 0 \;\Rightarrow\; y_z^H T(z) = 0$$

Since T=S-I we get

$$y_z^H S(z) = y_z^H \;\Rightarrow\; S^H(z) y_z = y_z$$

and since $\bar{\sigma}(S) = \bar{\sigma}(S^H)$

$$\bar{\sigma}(S(z)) \geq 1$$

RHP Zeros cont'd

Then, by Maximum Modulus Thm

$$\|W_S S\|_\infty \geq \bar{\sigma}(W_S(z)S(z)) \geq |W_S(z)|$$

where we have assumed the weight W_S is scalar

– same restriction on $\bar{\sigma}(S)$ as on $|S|$ in SISO case (see lec notes 4)

Limitation 2: RHP Poles

Thm: Assume G(s) has a RHP pole at s=p>0, then

$$\|W_T T\|_\infty \geq |W_T(p)|$$

Proof: as above for RHP zeros, but with $S(p)y_p = 0 \;\Rightarrow\; T(p)y_p = y_p$

– same restriction on $\bar{\sigma}(T)$ as on $|T|$ in SISO case (see lec notes 4)

Requirements for Disturbance Attenuation

- Consider a scalar disturbance d such that

$$w = g_d(s)d \;\Rightarrow\; z = S(s)g_d(s)d, \quad |d| < 1 \;\forall\omega$$

- Requirement $|z| < 1 \;\forall\omega$ implies

$$\bar{\sigma}(Sg_d) < 1 \;\forall\omega \;\Rightarrow\; \|Sg_d\|_\infty < 1$$

- Define the *disturbance direction*

$$y_d(i\omega) = \frac{g_d(i\omega)}{|g_d(i\omega)|}$$

- Then, requirement is

$$\bar{\sigma}(Sy_d) < \frac{1}{|g_d|} \quad \forall\omega$$

  - note: requirement on S is only in direction y_d

Requirements cont'd

- Consider high-gain and low-gain directions of sensitivity S (from SVD of S)

$$S\bar{v} = \bar{\sigma}(S)\bar{u} \ ; \qquad S\underline{v} = \underline{\sigma}(S)\underline{u}$$

  - If $y_d = \bar{u} \ \Rightarrow \ \bar{\sigma}(S) < \dfrac{1}{|g_d|}$ ('worst' direction)
  - if $y_d = \underline{u} \ \Rightarrow \ \underline{\sigma}(S) < \dfrac{1}{|g_d|}$ ('best' direction)

Disturbances and RHP Zeros

- Assume G(s) has a RHP zero at s=z, then

$$y_z^H S(z) = y_z^H \;\; \Rightarrow \;\; y_z^H S(z) g_d(z) = y_z^H g_d(z)$$

- From Maximum Modulus Thm we get

$$\|S g_d\|_\infty \geq |y_z^H g_d(z)|$$

- Thus, we get requirement

$$|y_z^H g_d(z)| < 1$$

  - otherwise no controller exists that will provide acceptable performance corresponding to keeping $|z| < 1$ in the presence of disturbances $|d| < 1$
  - note that this depends on the system only

Disturbances and RHP Zeros

- "Extreme" cases
  - If $y_z \perp y_d \;\Rightarrow\; y_z^H g_d(z) = 0$ (RHP zero has no impact on disturbance attenuation)
  - if $y_z \parallel y_d \;\Rightarrow\; y_z^H g_d(z) = |g_d(z)|$ ("worst-case" alignment)
- Example:

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{1}{s+2} \\ \frac{s+3}{s+1} & \frac{2}{s+2} \end{pmatrix} \; ; \quad \det G(s) = \frac{1-s}{(s+1)(s+2)} \;\Rightarrow\; z = 1$$

$$\Downarrow$$

$$G(1) = \begin{pmatrix} 1 & 1/3 \\ 2 & 2/3 \end{pmatrix} \;\Rightarrow\; y_z^H = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \end{pmatrix}$$

Example cont'd

1. Disturbance d_1

$$g_{d1}(s) = \frac{2}{s+1}\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow |y_z^H g_{d1}(1)| = \frac{1}{\sqrt{5}} < 1$$

2. Disturbance d_2

$$g_{d2}(s) = \frac{2}{s+1}\begin{pmatrix}-1\\1\end{pmatrix} \Rightarrow |y_z^H g_{d2}(1)| = \frac{3}{\sqrt{5}} > 1$$

– Thus, can attenuate disturbance d_1 but not d_2 such that $|z| < 1$ when $|d| < 1$ (with any controller!)

Summary

- Results on performance requirements and limitations carry more or less directly over from SISO to MIMO by considering the maximum singular values of the transfer-matrices we want to make small, e.g., S and T
- By using scalar weights W_S, W_T we impose same bound on sensitivity in all directions
- Disturbances have specific directions and therefore impose requirements on the sensitivity only in these directions
- To what extent a RHP zero imposes a limitation for disturbance attenuation depends on how the disturbance direction is aligned with the zero output direction
- Multivariable systems: directions matter!

- Next time: controller design