EL2520
Control Theory and Practice

Lecture 14: Summary

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Outline

- Course objectives
- Checklist
- The exam and beyond
- Brief review of lectures

Preparations for Exam

- Old exams with answers (usually not complete solutions) in Canvas
- An old exam will be covered in the final exercise (tomorrow Friday)
- A discussion group will be opened in Canvas to answer questions that may come up before the exam. I will do my best to answer all questions, but all of you feel free to post answers!
- Remember to motivate all your answers on the exam!

Course Objectives

The course aims to provide the participants basic knowledge of approaches and methods in advanced control, in particular linear multivariable feedback systems.

Key ingredients:

- A modern view of control
  - enabling systematic trade-off between various performance goals, respecting fundamental limitations and ensuring robust stability
- Multivariable control
  - multivariable systems (poles, zeros, gains and directions), decentralized control (RGA) and decoupling, several design methods based on controller synthesis through optimization, robust loop shaping
- Dealing with hard constraints
  - anti-windup and model predictive control

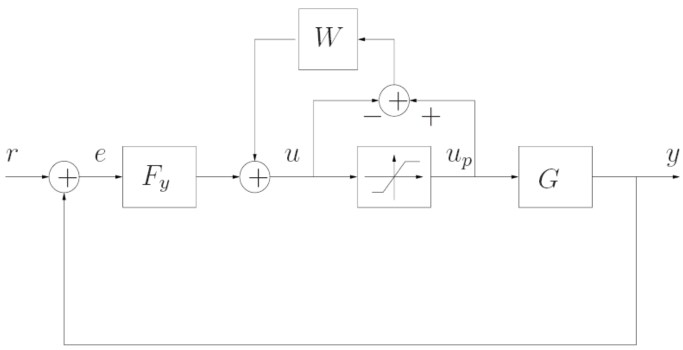
Checklist

The basics
- Matrix manipulations, eigenvalue computations, singular values
- Complex numbers
- Differential equations, state-space models, transfer functions and frequency response

A modern perspective on SISO control:
- Signal norms, system gain and the Small Gain Theorem
- The closed loop system and the central transfer functions
- Internal stability
- Fundamental limitations due to RHP poles/zeros, time delays
- Reasonable design goals and mapping to loop gain specifications
- Uncertainty sets and robust stability / performance

Lecture 13 – Dealing with hard constraints

Classical approach to deal with input constraints: input tracking, i.e., use feedback from difference between computed control and saturated control (Anti Reset Windup)



Lecture 12 – Model Predictive Control

Structured way of dealing with control and state constraints

All based on discrete time models!

1. Predict how state evolves (as function of future controls)

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u_j$$

2. Determine optimal control by minimizing LQ criterion w constraints

$$\begin{array}{ll} \text{minimize} & \sum_{i=0}^{N_p-1} [(x_i - x_i^{\text{ref}})^T Q_1 (x_i - x_i^{\text{ref}}) + (u_i - u_i^{\text{ref}})^T Q_2 (u_i - u_i^{\text{ref}})] \\ & + (x_{N_p} - x_{N_p}^{\text{ref}})^T S (x_{N_p} - x_{N_p}^{\text{ref}}) \\ \text{subject to} & u_{\text{min}} \leq u_i \leq u_{\text{max}} \\ & y_{\text{min}} \leq C x_i \leq y_{\text{max}} \end{array}$$

3. Implement first control, return to 1 at next sampling instant

Can be solved via efficient optimization (quadratic programming)

Checklist

Multivariable linear systems

- Transfer matrices and block diagram algebra
- Multivariable poles and zeros, directions
- Amplification, gain and directions
- Extending SISO results to MIMO

Multivariable control design techniques:

- H_2 and H_∞-optimal control: weighting functions and extended system, controller structure
- LQG: design, optimal control structure and disturbance models
- Loop shaping and Glover-McFarlane robustification
- The relative gain array for decentralized control structure design

Checklist

Dealing with hard constraints:

- Sampling of linear systems (continuous→discrete time)
- Model Predictive Control – MPC
  - finite horizon LQR w constraints –> Quadratic Program
- Anti-windup (tracking) to deal with actuator saturation

Lecture 11 – Comparing Design Methods, Case Study

Lecture 10 – Robust loop shaping

Glover Mc-Farlane: first perform classic loop shaping and then add robustifying controller in a second step

Solves the problem: find controller that stabilizes

$$G(s) = (M(s) + \Delta_M(s))^{-1}(N(s) + \Delta_N(s))$$

for all uncertainties

$$\|\Delta_M(s)\ \Delta_N(s)\|_\infty \leq \epsilon$$

Dynamic controller of high order (plant+nominal controller): model reduction based on balanced realization

About the exam

When? Tue May 31 at 08.00-13.00

How? Written on-campus exam (remember to register)

What? 5 problems, 5 hours. Grading criteria on homepage

Allowed aids?

Course book (Glad&Ljung *or* Skogestad&Postlethwaite)

Basic control book (Glad&Ljung)

Lecture notes and slides from this years course (printed!)

Mathematical handbook

Pocket calculator (not symbolic)

Note: all aids must be physical, i.e., not allowed to use any aids on computer or mobile phone

What is on the exam

- Signal and system norms, Small Gain Theorem
- Poles, zeros, directionality, singular values
- Closed-loop transfer functions
- Internal stability
- Performance limitations
- Input-output controllability analysis (requirements vs limitations)
- Robust stability
- RGA, decentralized control and decoupling
- LQG, H2- and Hinf-optimal control
- Glover-McFarlane robust loop shaping
- Discrete time systems, finite horizon LQR and MPC
- Anti reset windup

Lecture 8 – Linear Quadratic Control

Objective: minimize the cost

$$J = \mathbf{E}\left\{\lim_{T\to\infty}\int_0^T \left[z^T Q_1 z + u^T Q_2 u\right] dt\right\}$$

Optimal solution:

- State feedback and linear observer (separation principle)
- Feedback and observer gains by solving Riccati equations
- LQG controller can have very poor robustness margins
- Some equivalence to $\mathcal{H}_2$- optimal control

Question

- How do we choose suitable weights in $\mathcal{H}_2$-optimal design?
  - in the same way as in $\mathcal{H}_\infty$-optimal design?
  - is aim still to achieve e.g., $\|W_T T\|_2 \leq 1$?

The main difference between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ is that, in the latter case, the aim is to push down the peak value of $\bar{\sigma}(W_T T(i\omega))$ while in the former case the aim is to minimize the area under $\bar{\sigma}(W_T T(i\omega))$

Thus, $\|W_T T\|_2 \leq 1$ does not have a simple interpretation. Rather
- choose weight large at frequencies where you want T small
- if resulting T is too large in some frequency range, increase weight in that range and redo design

Beyond the exam

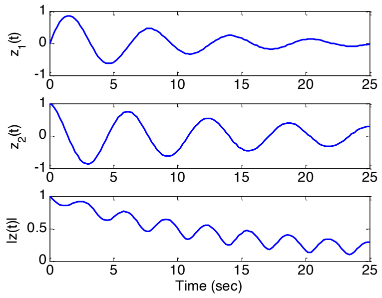
Learn more in our other advanced courses
- EL2820 Modeling of Dynamical Systems, per 1
- EL2700 Model Predictive Control, per 1
- EL2805 Reinforcement Learning, per 2
- EL2620 Nonlinear Control, per 2
- EL2450 Hybrid and Embedded Systems, per 3
- EL2810 Machine Learning Theory, per 3
- EL2425 Automatic Control Project Course, per 1-2

Put your skills to the test: do your master thesis at Decision and Control Systems
- see the web or come talk to us!

Contribute to frontline research: enroll in our PhD program!
- an exciting career – come and talk to us!

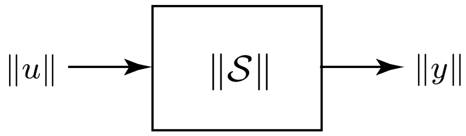
Lecture 1

- *Signal norms*: measure signal size across space (channels) and time



$\Rightarrow \|z\|$

- *System gain*: bounds signal amplification



Question

How choose weight W_u in objective $\|W_u G_{wu}\|_\infty < 1$

- We have

$$u = G_{wu}(s)w$$

- If system properly scaled so that $|w| < 1, \ |u| < 1 \ \forall\omega$, then we require

$$\bar{\sigma}(G_{wu}(i\omega)) < 1 \ \forall\omega \quad \Rightarrow \quad \|G_{wu}\|_\infty < 1$$

  - i.e., weight is $W_u = I$
- More generally
  - increase $|W_u(i\omega)|$ at frequencies where you want to use less input
  - e.g., if initial u(t) too large, increase the weight at higher frequencies
  - to limit derivative of u(t), use $W_u' = sW_u$

Lecture 7/9 – H_∞ / H_2 -optimal control

H_2-optimal

$$\min_{F_y} \|G_{ec}\|_2^2 = \min_{F_y} \int_{-\infty}^{\infty} \sum_i \sigma_j^2(G_{ec}(i\omega))\, d\omega$$

H_{inf}-optimal

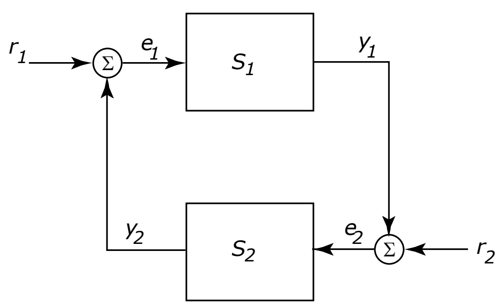
$$\min_{F_y} \|G_{ec}\|_\infty = \min_{F_y} \sup_\omega \overline{\sigma}(G_{ec}(i\omega))$$

Computed from state-space description of G_{ec}

- Solution is observer + static feedback from observed states

Small Gain Theorem

Theorem. Consider the interconnection



If $\mathcal{S}_1$ and $\mathcal{S}_2$ are input-output stable and

$$\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$$

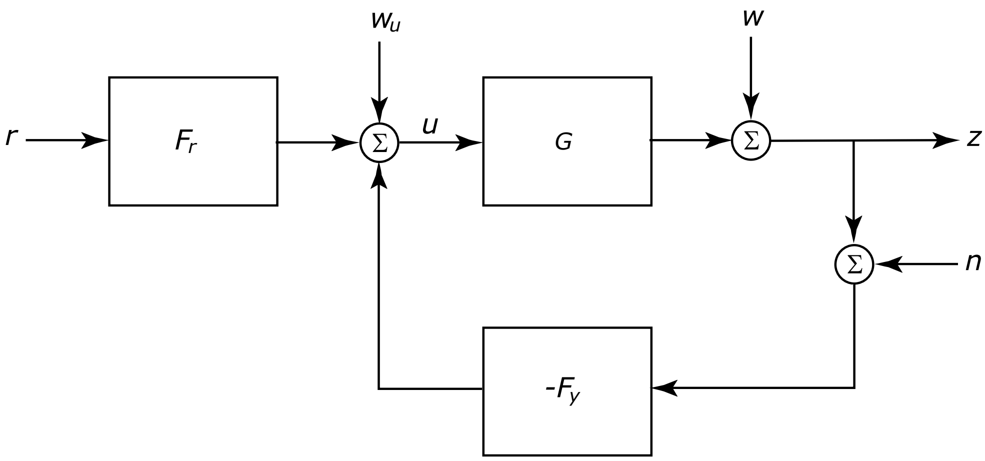
Then, the closed-loop system with r_1, r_2 as inputs and e_1, e_2, y_1, y_2 outputs is input-output stable.

Note: can use any norm that satisfies multiplicative property

$$\|AB\| \leq \|A\|\|B\|$$

e.g., inf-norm but not 2-norm

Lecture 2 - The closed-loop system



Controller: feedback F_y and feedforward F_r

Disturbances: w, w_u drive system from desired state

Measurement noise: corrupts information about z

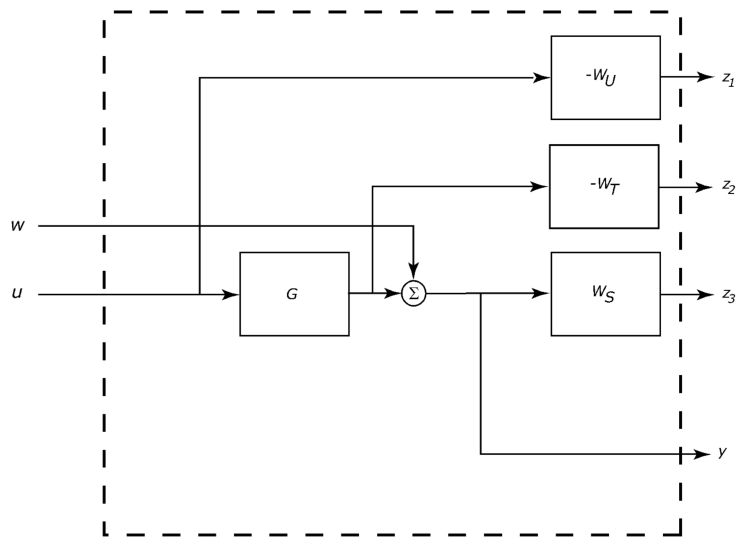
Aim: find controller such that z follows r.

Lecture 7/9 – H_∞ / H_2 -optimal control

Specification: optimize over all stabilizing controllers to achieve

$$\left\| \begin{pmatrix} W_S S \\ W_T T \\ W_U S F_y \end{pmatrix} \right\| \leq 1$$

Based on 'extended system'



Optimal controller:

- observer+linear feedback from estimated states

Question

- How to get the MIMO limitations in Lecture 6 from math?
  - Essentially combine interpolation constraints, e.g.,
  $$y_z^H S(z) = y_z^H$$
  with the Maximum Modulus Thm

  See also notes and slides from Lecture 6

Transfer functions and observations

$$S = \frac{1}{1+GF_y} \qquad (w \to z,\ w_u \to u) \quad \text{sensitivity function}$$

$$T = \frac{GF_y}{1+GF_y} \qquad (n \to z) \quad \text{complementary sensitivity}$$

$$G_c = \frac{GF_r}{1+GF_y} \qquad (r \to z) \quad \text{closed loop system}$$

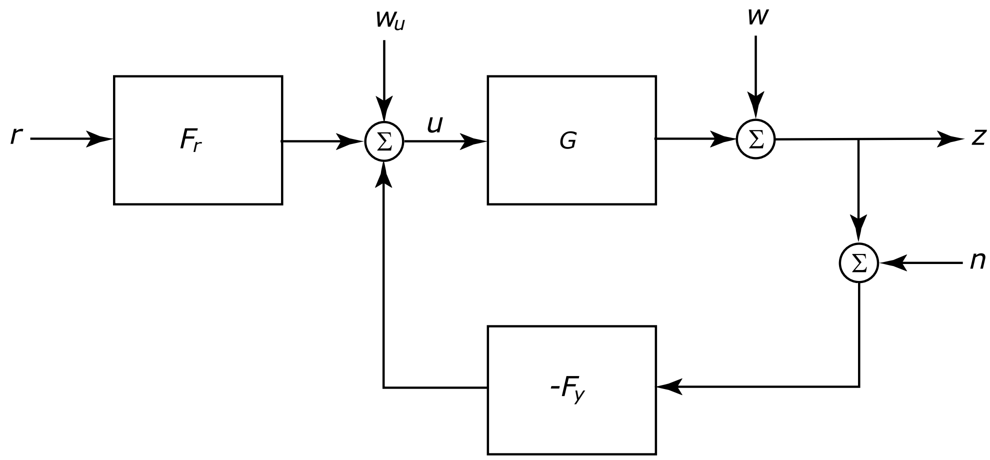
$$SG = \frac{G}{1+GF_y} \qquad (w_u \to z)$$

$$SF_y = \frac{F_y}{1+GF_y} \qquad (n \to u)$$

$$SF_r = \frac{F_r}{1+GF_y} \qquad (r \to u)$$

Observations: need to look at all! Many tradeoffs (e.g. S+T=1)

Internal Stability



Definition. The closed loop system above is *internally stable* iff it is input-output stable from all inputs r, w_u, w, n to all outputs u, z, y

Theorem. The closed-loop system is stable if and only if

$$S,\ SG,\ SF_y,\ F_r$$

are stable

Lecture 6 – MIMO Limitations

- Extend SISO limitations to MIMO case
- Essentially, we get interpolation constraints in certain directions (zero and pole directions)
- Otherwise derivations and limitations as for SISO case

Question

- How does decoupling affect robustness when the inputs are uncertain?
  - if we have input uncertainty then $u_p = (I + \Delta)u$ and we get for the compensated plant

$$G(s)(I + \Delta)d(s)G^{-1}(s) = d(s)(I + G(s)\Delta G^{-1}(s))$$

  - the term $G(s)\Delta G^{-1}(s)$ can become very large for ill-conditioned G(s)
  - Example with 10% input uncertainty

$$G = \frac{1}{s+1}\begin{pmatrix} 1 & -1 \\ 1.1 & -1 \end{pmatrix} \qquad \Delta = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.1 \end{pmatrix}$$

$$\Rightarrow \quad G(s)\Delta G^{-1}(s) = \begin{pmatrix} -2.1 & 2 \\ -2.2 & 2.1 \end{pmatrix}$$

Example

Example Let

$$G(s) = \frac{1}{s}\left(\frac{-s+2}{s+2}\right)$$

Determine a controller U(s)=F(s)[R(s)-Y(s)] that achieves

$$G_c(s) = \frac{1}{1+sT}\left(\frac{-s+2}{s+2}\right)$$

and show that the closed-loop system is internally stable

The Sensitivity Functions

The sensitivity function (S):

- Quantifies disturbance attenuation due to feedback

The complementary sensitivity function (T)

- Equals the closed-loop system G_c with 1-DOF control
- Quantifies the amplification of noise at the output
- Determines robust stability properties

A first trade-off: S+T=1

Decoupling

- Full decoupling

$$D(s) = d(s)G^{-1}(s) \qquad \Rightarrow \qquad G(s)D(s) = d(s)I$$

- Static decoupling

$$D = G^{-1}(0)$$

The Relative Gain Array

Definition. the *relative gain array, RGA,* of a square system is defined as

$$\Lambda(G) = G \times (G^{-1})^T$$

or, in Matlab notation, RGA(G)= G.*inv(G).'

Rules of thumb:

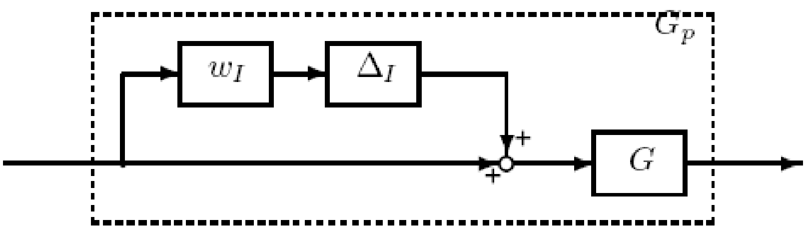
1. Avoid pairings with $\lambda_{ij}(0) < 0$ (why?)
2. Prefer pairings with $|\lambda_{ij}(i\omega_c)| \approx 1$

Lecture 3 - Robustness

Robustness = property maintained under uncertainty

Idea: specify uncertainty set and guarantee stability and performance for all possible models within set

We have focused on multiplicative (relative) uncertainty



SISO Robustness

Robust stability via small-gain theorem

$$|T(i\omega)| \leq |w_I^{-1}(i\omega)| \qquad \forall\omega$$

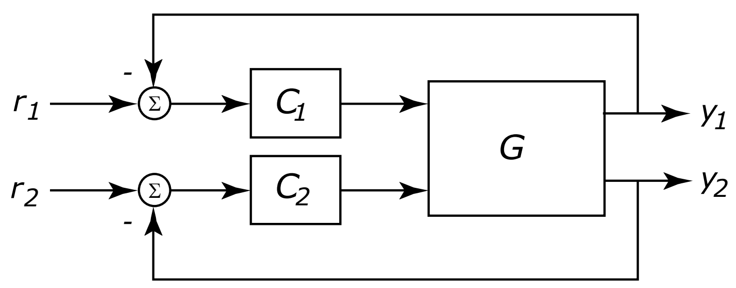
where

$$|w_I(i\omega)| \geq \left|\frac{G_p(i\omega) - G(i\omega)}{G(i\omega)}\right| \qquad \forall G_p \in \Pi_i$$

Robust performance puts simultaneous bound on S and T.

Decentralized Control

Decentralized control:



Interactions: each input affects multiple outputs

Qualitatively: the more interactions, the harder to control

- The relative gain array tries to quantify the degree of interactions

Question

Please repeat MIMO zeros and zero directions

- A zero z is a value of s where G(s) has less rank than normal.
  - Example:

$$G(s) = \frac{1}{s+1}\begin{pmatrix} s+1 & 1 \\ 3 & 1 \end{pmatrix} \quad \Rightarrow \quad G(2) = \frac{1}{3}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

- When a matrix is rank deficient it has a left and right null space; the output zero direction y_z is the left nullspace and the input zero direction u_z is the right nullspace for the zero

$$G(z)u_z = 0 \cdot y_z$$

  - for the example:

$$u_z^H = \begin{pmatrix} \frac{1}{3} & -1 \end{pmatrix} \quad y_z^H = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

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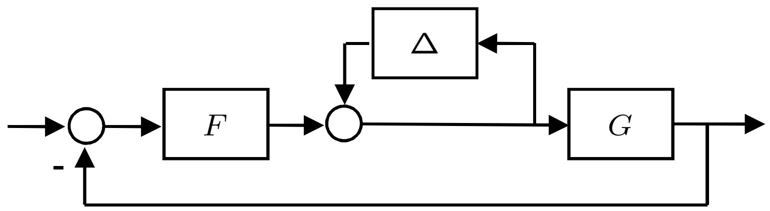
SISO Robustness

Example: Consider the uncertain system



Use the small-gain theorem to derive a robustness criterion

$$\|P\Delta\|_\infty \leq 1$$

for some transfer function P independent of Δ

Question

- Why is robustness important, because we can not operate at nominal operating where model is obtained?
  - partly, this usually gives rise to a model error
  - but, we always have uncertainty in actuators (inputs) and measurements, and model uncertainty even at nominal operating point
- How is uncertainty modeled in practice
  - from system identification, e.g., parametric uncertainty. See e.g., example in Lecture notes 3
  - from knowledge of uncertainty in actuators and sensors
  - by adding some generic uncertainty to achieve a reasonable robustness (similar to AM and PM in SISO control), e.g., robust loopshaping

Lecture 5 – Multivariable linear systems

Poles, zeros and gains.

Theorem. The *pole polynomial* of a system with transfer matrix G(s) is the least common denominator of all minors of G(s). The poles of G(s) are the roots of the pole polynomial.

Theorem. The *zero polynomial* of G(s) is the greatest common divisor of the maximal minors of G(s), normed so that they have the pole polynomial of G(s) as denominator. The *zeros* of G(s) are the roots of its zero polynomial.

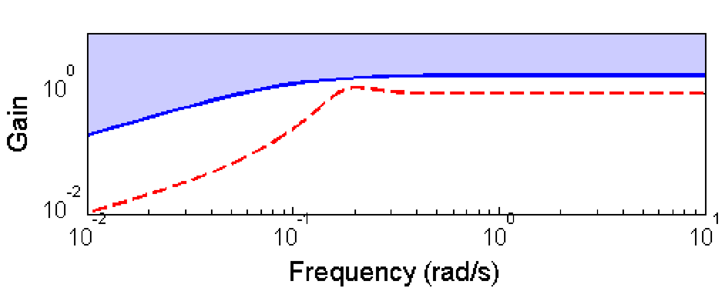
Theorem. The gain of a linear system G(s) is given by

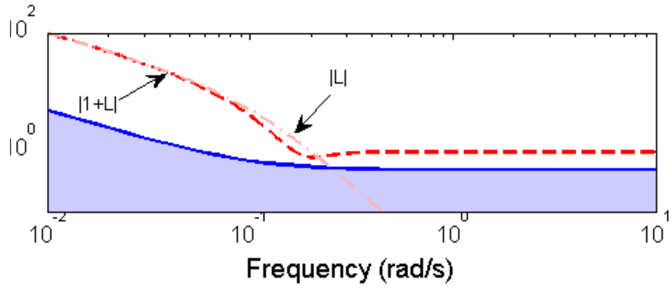
$$\|G\|_\infty = \sup_\omega |G(i\omega)| = \sup_\omega \overline{\sigma}(G(i\omega))$$

Classical Loop Shaping

In certain frequency ranges, there is a reasonable approximate mapping between constraints on S and T into requirements on the loop gain L.

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)| \Leftrightarrow |1 + L(i\omega)| \geq |W_S(i\omega)|$$

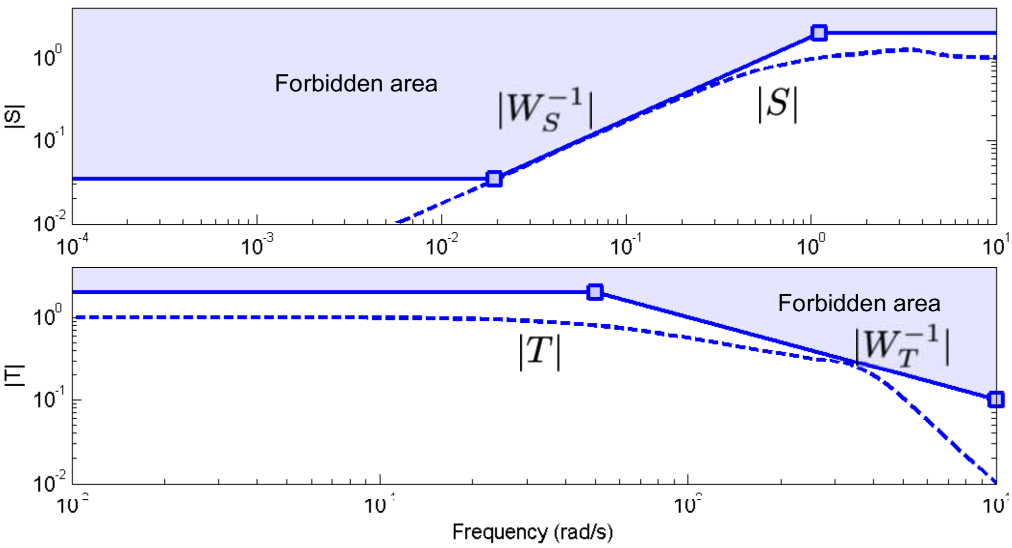




Problematic area is around cross-over frequency

- Put requirements on phase and amplitude margin

Lecture 4– Limitations and Conflicts



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)| \ \forall\omega \quad \Leftrightarrow \quad \|W_S S\|_\infty < 1$$

$$|T(i\omega)| \leq |W_T^{-1}(i\omega)| \ \forall\omega \quad \Leftrightarrow \quad \|W_T T\|_\infty < 1$$

Can we choose weights w_S, w_T (“forbidden areas”) freely?

– No, there are many constraints and limitations!

Limitations and conflicts

- Fundamental trade-offs in control systems design
  - S+T=1 (both cannot be small at the same frequency)
  - Cannot attenuate disturbances at all frequencies (Bode Sensitivity Integral)

- Fundamental limitations:
  - Unstable poles
  - Non-minimum phase zeros
  - Time delays

- Practical limitation:
  - Control input constraints

Rules of thumb

RHP zeros limit bandwidth (of S)

$$\boxed{\omega_{BS} \leq \frac{z}{2}}$$

Time-delays impose a similar bound

$$\boxed{\omega_{BS} \leq \frac{1}{T}}$$

RHP poles require a minimum bandwidth (for T)

$$\boxed{\omega_{BT} \geq 2p}$$