# 9 EL2520 Lecture notes 9: $\mathcal{H}_2$-Optimal Control

In lecture 7 we showed how the problem of minimizing the $\mathcal{H}_\infty$-norm of weighted closed-loop transfer-functions could be solved by translating the system norm minimization problem, in input-output space, into an equivalent signal norm minimization problem in state space. In LQG, introduced in lecture 8, the starting point was a signal minimization problem in state space that could be solved explicitly using the separation principle. In this lecture we will show that the signal minimization problem treated in LQG under certain conditions is equivalent to the problem of minimizing the $\mathcal{H}_2$-norm of weighted closed-loop transfer-functions. Thus, LQG can in principle be used as a machinery to shape closed-loop transfer-functions, but then based on the $\mathcal{H}_2$-norm. At the end of the lecture we will discuss some important differences between $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control.

We start by formulating the $\mathcal{H}_2$-optimal control problem and then show that it can be formulated as a signal minimization problem equivalent to the one treated in LQG.

The $\mathcal{H}_2$-optimal control problem is[1]

$$F_y = \arg\min_{F_y} \|P\|_2 \tag{1}$$

where $P$ is the transfer-function we want to minimize and $\|\cdot\|_2$ denotes the $\mathcal{H}_2$-norm of $P$. Similar to what we considered for $\mathcal{H}_\infty$-optimal control in lecture 7, the matrix $P$ is typically a stacked matrix of weighted transfer-functions, e.g.,

$$P = \begin{pmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{pmatrix} \tag{2}$$

Recall that the $\mathcal{H}_2$-norm of a scalar (SISO) transfer-function $G(s)$ is defined as

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|^2 d\omega$$

Note that the $\mathcal{H}_2$-norm is defined for strictly proper[2] and stable $G(s)$ only. Thus, if we for instance consider minimizing the weighted sensitivity $P = W_S S$, then we must choose the weight $W_S$ strictly proper since $S$ is always semi-proper only. Compared to the $\mathcal{H}_\infty$-norm, which is the peak value of $|G(i\omega)|$, we see that the $\mathcal{H}_2$-norm essentially is a measure of the area under $|G(i\omega)|$ when plotted against frequency. For a multivariable (MIMO) system, the $\mathcal{H}_2$-norm is defined as

$$\|G\|_2^2 = \sum_{ij} \|G_{ij}\|_2^2 = \frac{1}{2\pi}\sum_{ij}\int_{-\infty}^{\infty} |G_{ij}(i\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G(i\omega)^H G(i\omega)) d\omega$$

[1] We also include the prefilter $F_r$ in the design when $P$ depends on it

[2] Recall that a proper transfer-function has at least as many poles as zeros, while strictly proper implies more poles than zeros. A system with as many poles as zeros is called semi-proper.

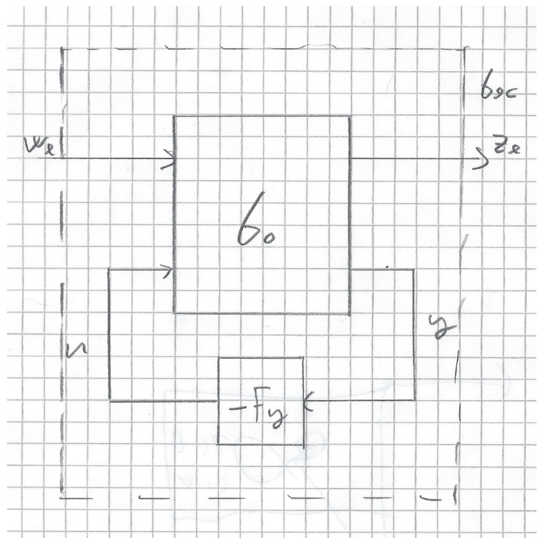


Figure 9.1: Extended system with closed-loop transfer-function $z_e = G_{ec}(s)w_e$ and corresponding open-loop $(z_e, y) = G_0(s)(w_e, u)$.

To solve (1), we formulate an extended system which has $P$ as the closed-loop transfer-matrix from the input $w_e$ to the output $z_e$. See Figure 9.1. In the same fashion as when solving the $\mathcal{H}_\infty$-optimal control problem in lecture 7, we pick the input $w_e$ and output $z_e$ from the signals of the original feedback system such that the closed-loop transfer-function $G_{ec}$ from $w_e$ to $z_e$ equals $P$. As an example, assume that we want to minimize the norm of the weighted sensitivity function, i.e., $P = W_S S$. Since $z = Sw$ in the original feedback problem, we pick $w_e = w$ and $z_e = W_S z$ in the extended system, such that $z_e = W_S S w_e$.

The solution of the $\mathcal{H}_2$-optimal control problem is based on a state-space realization of the corresponding open-loop of the extended system, i.e., $G_0(s)$

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) + Nw_e(t) \\ z_e(t) &= Mx(t) + Du(t) \\ y(t) &= Cx(t) + w_e(t) \end{aligned} \tag{3}$$

where the realization is chosen such that $D^T M = 0$ and $D^T D = I$ (for reasons that will become obvious below). Note that the formulation of the extended system to represent $P$, and the state-space realization of the corresponding open-loop system $G_0(s)$, is identical to the one used in $\mathcal{H}_\infty$-optimal control in lecture 7. However, while we in the latter case considered worst-case disturbances $w_e$, to reflect $\|P\|_\infty$, we now consider the case with $w_e$ being white noise with covariance

$$E\{w_e w_e^T\} = I \quad \Rightarrow \quad \Phi_{w_e} = I$$

Note that this corresponds to each scalar signal $w_{ei}$ in the vector $w_e$ having a flat frequency spectrum $\Phi_{w_{ei}} = 1$, i.e., the noise has the same energy (amplitude) at all frequencies. Now, using Parseval's theorem we get for the output

$$\|z_e(t)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Z_e(i\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_{ec}(i\omega)W_e(i\omega)|^2 d\omega \tag{4}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G_{ec}^H(i\omega)G_{ec}(i\omega))d\omega = \|G_{ec}\|_2^2 = \|P\|_2^2 \tag{5}$$

Thus, minimizing the 2-norm of the output $\|z_e\|_2$ when the input $w_e$ is white noise with spectrum $\Phi_{w_e} = I$ corresponds to minimizing the 2-norm of the closed-loop transfer-function from $w_e$ to $z_e$, i.e., $\|G_{ec}\|_2$ ($\|P\|_2$).

With $D^T M = 0$ and $D^T D = I$ we get

$$\|z_e(t)\|_2^2 = ||Mx(t) + Du(t)\|_2^2 = \|Mx(t)\|_2^2 + \|u(t)\|_2^2 = \int_0^\infty x^T M^T M x + u^T u dt$$

Note that this is equivalent to the objective function in the LQG-problem, discussed in lecture 8, with weights $Q_1 = Q_2 = I$. Also note that the model (3) corresponds to the model used in LQG with disturbance and noise covariances $R_1 = R_2 = I$. Thus, we conclude that solution of (1) corresponds to the solution of an LQG problem based on the extended system (3) with weights $Q_1 = Q_2 = I$ and covariances $R_1 = R_2 = I$. Note that the weighting of the problem is included in $P$ and hence in the model (3).

As shown in lecture 8, the solution to the LQG problem is an LQ-controller combined with a Kalman filter. Since (3) is on innovation form, i.e., we essentially measure the process disturbance $w_e$, the Kalman filter is

$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + N(y(t) - C\hat{x}(t))$$

and the feedback law is

$$u = -L\hat{x}(t)$$

where $L = B^T S$ and $S \geq 0$ solves the Riccati equation

$$A^T S + SA + M^T M - SBB^T S = 0$$

And the controller $F_y(s)$ is obtained by Laplace transformation to yield

$$F_y(s) = L(sI - A + BB^T S + NC)^{-1} N$$

Note that we require the observer above to be stable, i.e., $A - NC$ stable, otherwise the Kalman filter must be computed in a more involved manner not treated here. However, the most common situation is that $A - NC$ has one or more zero eigenvalues due to pure integrators in the weights used in $P$ and this can be avoided by using weights with poles close to zero but slightly inside the LHP. For instance, rather than using a weight $W_S = \frac{s+1}{s(0.001s+1)}$ for the sensitivity, we can use a weight $W_S = \frac{s+1}{(s+\epsilon)(0.001s+1)}$, $\epsilon > 0$ with $\epsilon$ being a small positive constant.

## 9.1 $\mathcal{H}_2$- vs $\mathcal{H}_\infty$-optimal control

Recall that the $\mathcal{H}_\infty$-norm of a transfer-function $G_{ec}$ is equal to the peak value of the maximum singular value over all frequencies, i.e.,

$$\|G_{ec}\|_\infty = \sup_\omega \bar{\sigma}(G_e c)$$

Thus, by minimizing the $\mathcal{H}_\infty$-norm we focus on the worst frequency and worst direction of the disturbance input $w_e$.

In terms of the singular values, the $\mathcal{H}_2$-norm is

$$\|G_{ec}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i \sigma_i^2(G_{ec}(i\omega)d\omega$$

Thus, when minimizing the $\mathcal{H}_2$-norm of $G_{ec}$ we consider all frequencies and all directions of the disturbance input $w_e$, and we determine the optimal trade-off between them.

From a performance point of view it may seem more reasonable to optimize with respect to some form of average over all frequencies and all directions, as is done in $\mathcal{H}_2$-optimization and LQG. Putting all effort into improving the worst case disturbance attenuation, as is done in $\mathcal{H}_\infty$-optimization, may seem somewhat irrational. However, since we include weights in the problem, i.e., the weights $W_S$ on the sensitivity $S$ and $W_T$ on the complementary sensitivity $T$, it makes sense to try to satisfy the corresponding bounds $|W_S^{-1}|$ and $|W_T^{-1}|$ for all frequencies and directions. Furthermore, the main motivation for introducing the $\mathcal{H}_\infty$-norm in the first place was that we thereby could include robustness as a criterion in the control design. It is quite obvious that stability can not be dealt with on an average basis; if the closed-loop is not stable for the worst case model within the uncertainty set then it is not robustly stable. However, it is of course possible to design controllers using $\mathcal{H}_2$-optimal control or LQG, and then analyze the robustness of the resulting closed-loop using $\mathcal{H}_\infty$.

We will return to a comparison between $\mathcal{H}_\infty$-optimal control, $\mathcal{H}_2$-optimal control and LQG in lecture 11 where we will consider control design for an example process.