EL2520
Control Theory and Practice

Multivariable systems

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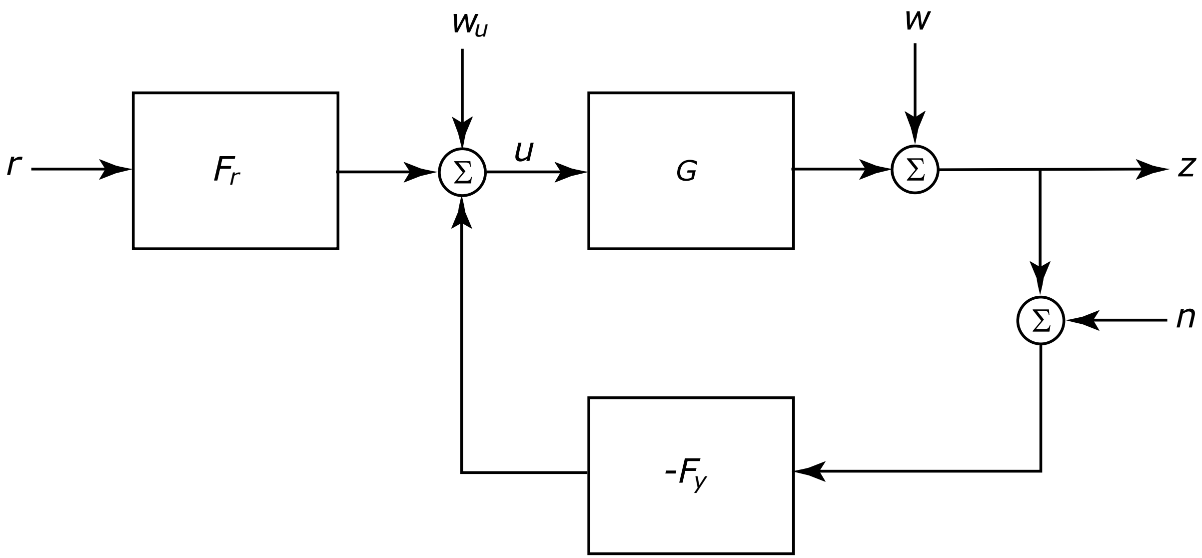
So far…



SISO control revisited:

- Signal norms, system gain and the small gain theorem
- Shaping the loop by minimizing weighted sensitivity functions
- The closed-loop system and the design problem
  - characterized by six transfer functions: need to look at all!
  - fundamental limitations (RHP zeros, RHP poles, time delay), conflicts and waterbed effect.

From now and on: MIMO

Linear systems with multiple inputs and multiple outputs

- Basic properties of multivariable systems (this video)
- Decentralized control and decoupling (video 6)
- Fundamental limitations and robustness in MIMO systems (video 7,8)
- H_∞-optimal control (video 9,10)
- State-space theory, state feedback and observers, LQG (video 11)
- H_2-optimal control (video 12)
- Robust loop shaping (video 13)

The final part of the course considers systems with constraints

Today's lecture

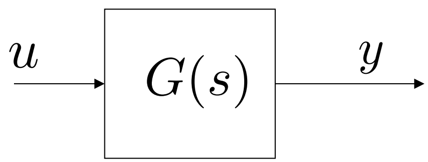
Basic properties of multivariable systems

- Transfer matrices
- Poles and zeros
- Amplification and gain

Chapters 2-3 and 8.3 in the textbook, Lecture notes 5

Multivariable Systems

Consider a MIMO system with m inputs and p outputs



- All signals are vectors

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \quad ; \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$$

- The transfer-matrix G(s) has elements

$$G_{ij}(s) = \frac{y_i(s)}{u_j(s)}$$

Transfer-Matrix from State-Space

Given a linear time-invariant system on state-space form

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \; ; \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \\ y(t) &= Cx(t) + Du(t) \; ; \quad y \in \mathbb{R}^p \end{aligned}$$

Laplace transform (assuming u(t)=0 for t<0 and x(0)=0)

$$Y(s) = \{C(sI - A)^{-1}B + D\}U(s) = G(s)U(s)$$

G(s) is a $p \times m$ transfer-matrix

Example

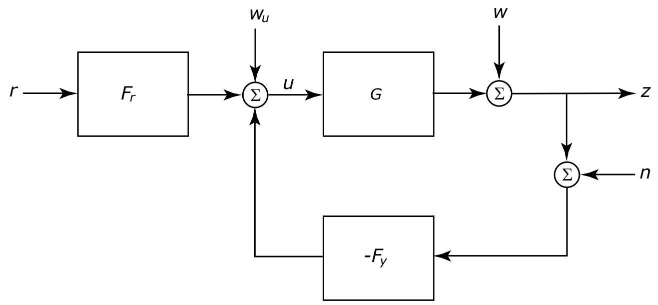
LTI system

$$\begin{aligned}\dot{x} &= \begin{pmatrix}-1 & -2\\ 0 & -2\end{pmatrix} x(t) + \begin{pmatrix}2 & 2\\ 0 & 1\end{pmatrix} u(t)\\ y(t) &= \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix} x(t) + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} u(t)\end{aligned}$$

Laplace transform yields

$$G(s) = \begin{pmatrix}\frac{2}{s+1} & \frac{1}{s+2}\\ \frac{s+3}{s+1} & \frac{2}{s+2}\end{pmatrix}$$

Closed-Loop Transfer-Matrices



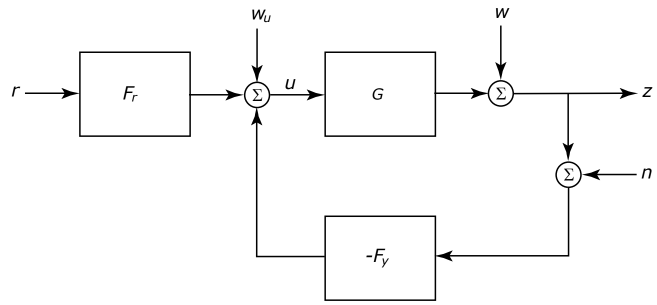
To derive transfer-function from an input to an output; use algebra (as before) or employ simple rule:

1. Start from output and move against signal flow towards input
2. Write down blocks, from left to right, as you meet them
3. When you exit a loop, add the term $(I+L)^{-1}$, where L is the loop transfer-function evaluated from the exit against the signal flow
4. Parallell paths should be added together

Also useful is the "push through" rule (for matrices of appropriate dimensions)

$$A(I+BA)^{-1} = (I+AB)^{-1}A$$

Closed-Loop Transfer-Matrices



- Examples:

$$z = (I + GF_y)^{-1} w = Sw$$

$$z = GF_y (I + GF_y)^{-1} n = Tn$$

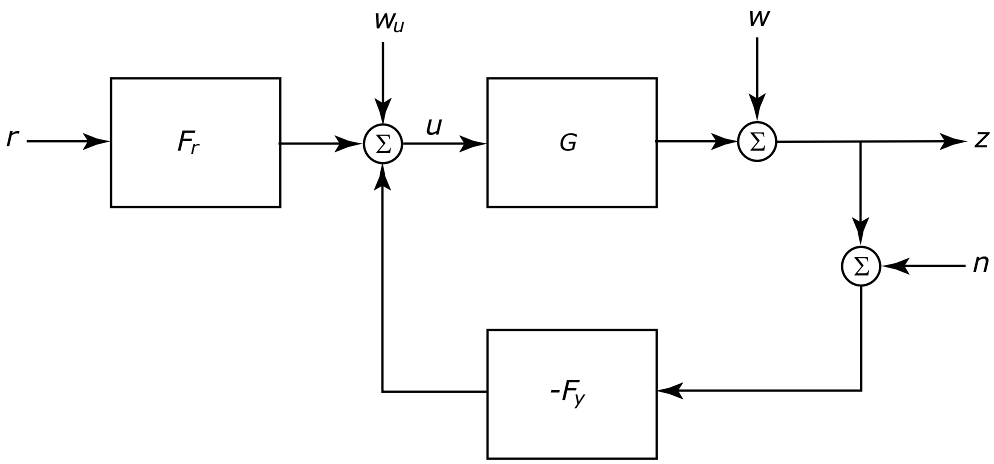
$$z = G(I + F_y G)^{-1} w_u = (I + GF_y)^{-1} G w_u = SG w_u$$

$$u = (I + F_y G)^{-1} w_u = S_u w_u$$

  – note:

$$(I + GF_y)^{-1} \neq (I + F_y G)^{-1}$$

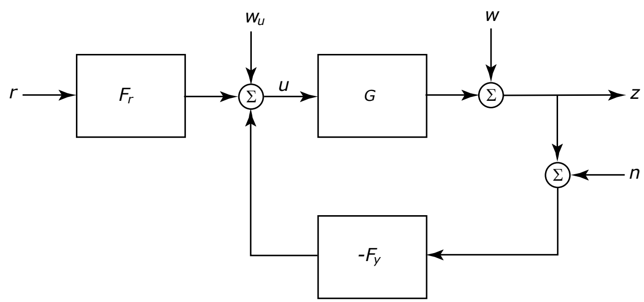
Quiz



- What is transfer-function from r to z?

- What is transfer-function from n to u?

Internal Stability



- Consider one input and one output at either side of the two blocks in the loop , e.g., w, w_u and z, u

$$z = \underbrace{(I + GF_y)^{-1}}_{S} w + \underbrace{G(I + F_y G)^{-1}}_{GS_u = SG} w_u$$

$$u = \underbrace{-F_y(I + GF_y)^{-1}}_{F_y S = S_u F_y} w + \underbrace{(I + F_y G)^{-1}}_{S_u} w_u$$

- Thus, require stability of $S, SG, S_u, S_u F_y$ and F_r

Poles

Definition. The *poles* of a linear system are the eigenvalues of the system matrix A in a minimal state-space realization.

Definition. The *pole polynomial* is the characteristic polynomial of the A matrix, $\lambda(s)$ = det(sI-A).

Alternatively, the poles of a linear system are the zeros of the pole polynomial, i.e., the values p_i such that $\lambda(p_i) = 0$

Poles from G(s)

Since the transfer matrix is given by

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{\det(sI - A)} r(s)$$

where r(s) is a polynomial matrix in s (see book for precise expression), the pole polynomial must be ”at least” the least common denominator of the elements of the transfer matrix.

Example: The system

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 2(s+2) & 3(s+1) \\ (s+2) & (s+2) \end{bmatrix}$$

must (at least) have poles in s=-1 and s=-2.

Poles from G(s)

Theorem. The pole polynomial of a system with transfer matrix G(s) is the least common denominator of all minors of G(s)

Recall: a minor of a matrix M is the determinant of a (smaller) square matrix obtained by deleting some rows and columns of M

Example: The minors of

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

are $\dfrac{2}{s+1}, \quad \dfrac{3}{s+2}, \quad \dfrac{1}{s+1}$ and $\det G(s) = \dfrac{1-s}{(s+1)^2(s+2)}$

Thus, the system has two poles in s=-1 and one pole in s=-2

Zeros

Zeros are essentially the values of s where G(s) looses rank

Theorem. The *zero polynomial* of G(s) is the greatest common divisor of the maximal minors of G(s), normed so that they have the pole polynomial of G(s) as denominator. The *zeros* of G(s) are the roots of its zero polynomial.

Example: The maximal minor of

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

is $\det G(s) = \dfrac{1-s}{(s+1)^2(s+2)}$ (already normed!).

Thus, G(s) has a zero at s=1 (and G(1) is rank 1)

Quiz: multivariable poles and zeros

What are the poles and zeros of the multivariable system

$$G(s) = \frac{1}{(s+1)} \begin{pmatrix} 1 & s+1 \\ s-1 & 1 \end{pmatrix}$$

Pole and Zero Directions

For scalar system G(s) with poles p_i and zeros z_i,

$$G(p_i) = \infty \; ; \quad G(z_i) = 0$$

But, for a multivariable system directions matter!

For a system with pole p, there exist vectors u_p, y_p:

$$G(p)u_p = \infty \cdot y_p$$

Similarly, a zero at z implies the existence of vectors u_z, y_z:

$$G(z)u_z = 0 \cdot y_z$$

Note: a transfer-matrix may have a pole and a zero at the same location without cancelling, provided they have different directions

Amplification and Frequency

- Recall: for a SISO system the amplification is frequency dependent

$$\frac{|Y(i\omega)|}{|U(i\omega)|} = |G(i\omega)|$$

- The maximum amplification over all frequencies is the system gain

$$\sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_\omega |G(i\omega)| = \|G\|_\infty$$

Direction Dependent Amplification

Linear mapping $y = Ax$

Since

$$|y|^2 = |Ax|^2 = (Ax)^H Ax = x^H A^H Ax$$

we get

$$|x|^2 \lambda_{min}(A^H A) \leq |y|^2 \leq |x|^2 \lambda_{max}(A^H A)$$

and so

$$\underbrace{\sqrt{\lambda_{min}(A^H A)}}_{\underline{\sigma}(A)} \leq \frac{|y|}{|x|} \leq \underbrace{\sqrt{\lambda_{max}(A^H A)}}_{\bar{\sigma}(A)}$$

where $\underline{\sigma}(A), \bar{\sigma}(A)$ are the minimum and maximum *singular values* of A, respectively

The Singular Value Decomposition

A $m \times r$ matrix (with r<m, rank(A)=r), can be represented by its singular value decomposition (SVD)

$$A = U\Sigma V^H = [u_1 \ u_2 \cdots u_r]\, \mathtt{diag}(\sigma_i)\, [v_1 \ v_2 \cdots v_r]^H = \sum_{I=1}^{r} \sigma_i u_i v_i^H$$

where

- the positive scalars σ_i are the *singular values* of A
- v_i are the *input singular vectors* of A, $V^H V = I \quad \Rightarrow \quad AV = U\Sigma$
- u_i are the *output singular vectors* of A, $U^H U = I$

Matlab: *[u,s,v]=svd(A)*

SVD interpretation

Consider static system

$$y = Au$$

- An input in the direction v_i gives an output in the direction u_i and the amplification is

$$\frac{|y|}{|u|} = \sigma_i(A)$$

- The maximum amplification is achieved for $u \parallel v_1$ which gives $y \parallel u_1$ and the amplification is

$$\frac{|y|}{|u|} = \bar{\sigma}(A)$$

The MIMO frequency response

For a linear multivariable system Y(s)=G(s)U(s), we have

$$Y(i\omega) = G(i\omega)U(i\omega)$$

Since this is a linear mapping, at any given frequency

$$\underline{\sigma}(G(i\omega)) \leq \frac{|Y(i\omega)|}{|U(i\omega)|} \leq \bar{\sigma}(G(i\omega))$$

The maximum amplification, at a given frequency is then

$$\frac{|Y(i\omega)|}{|U(i\omega)|} = \overline{\sigma}(G(i\omega))$$

The system gain

As for scalar systems, we have

$$\|y\|_2 \leq \|G\|_\infty \|u\|_2$$

where

$$\|G\|_\infty = \sup_\omega |G(i\omega)| = \underbrace{\sup_\omega}_{picks\ worst\ frequency} \overbrace{\overline{\sigma}}^{picks\ worst\ direction}(G(i\omega))$$

Note: the infinity norm is the maximum amplification across both frequencies and input directions

Summary

- Poles and zeros from transfer-matrix
- MIMO poles and zeros have directions
- The amplification of a system depends on input direction
  - the maximum amplification, i.e., the gain, is the maximum singular value of G
  - the minimum amplification is the smallest singular value of G
- The $\mathcal{H}_\infty$- norm of G is the peak value of the maximum singular value over all frequencies