EL2520
Control Theory and Practice

Robustness

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Todays Learning Outcomes

You should

- understand what robustness implies
- be able to quantify uncertainty using model sets
- be able to analyze robust stability and robust performance, given a model set, for SISO systems

Reference

Mainly based on Chapter 7 in “Multivariable Feedback Control”,

S. Skogestad, I. Postlethwaite, Wiley (but slides + ch.6 in course book + Lecture notes 3 suffices)

Classes of uncertainty

Parametric uncertainty:

- Model structure known, but some parameters are uncertain

Dynamic uncertainty / unmodelled dynamics:

- Some (often high frequency) dynamics is missing, either
  by lack of knowledge/information or in order to get a simpler model

Often, we have a combination of the two.

- Convenient to represent in “lumped” form

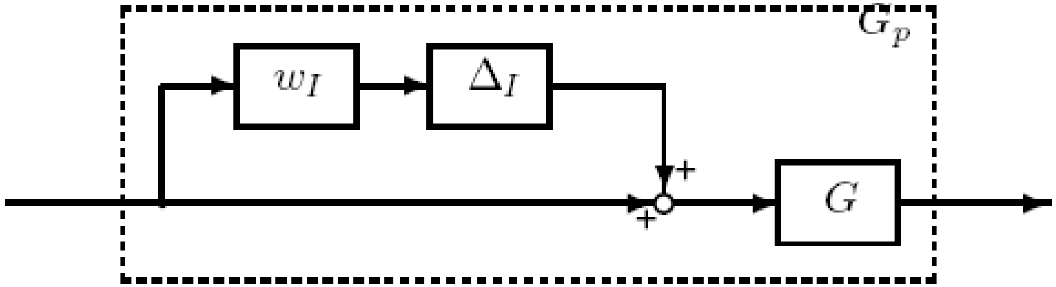
Uncertainty set – relative uncertainty

The true system $\tilde{G}$ is assumed to belong to the model set

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \leq 1\}$$

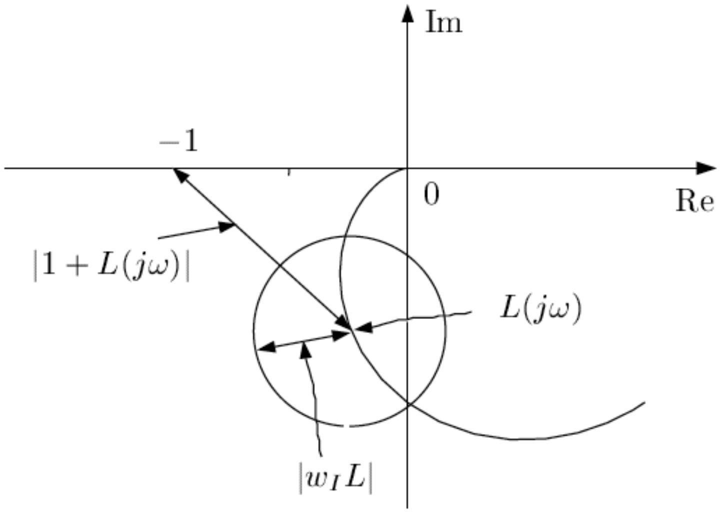
Here,

- Π_I is a *family* of possible behaviours of the physical plant
- Δ_I is *any* stable transfer function with gain less than one ($|\Delta_I(i\omega)| \leq 1 \ \forall\omega$)
- W_I is the *multiplicative uncertainty weight*



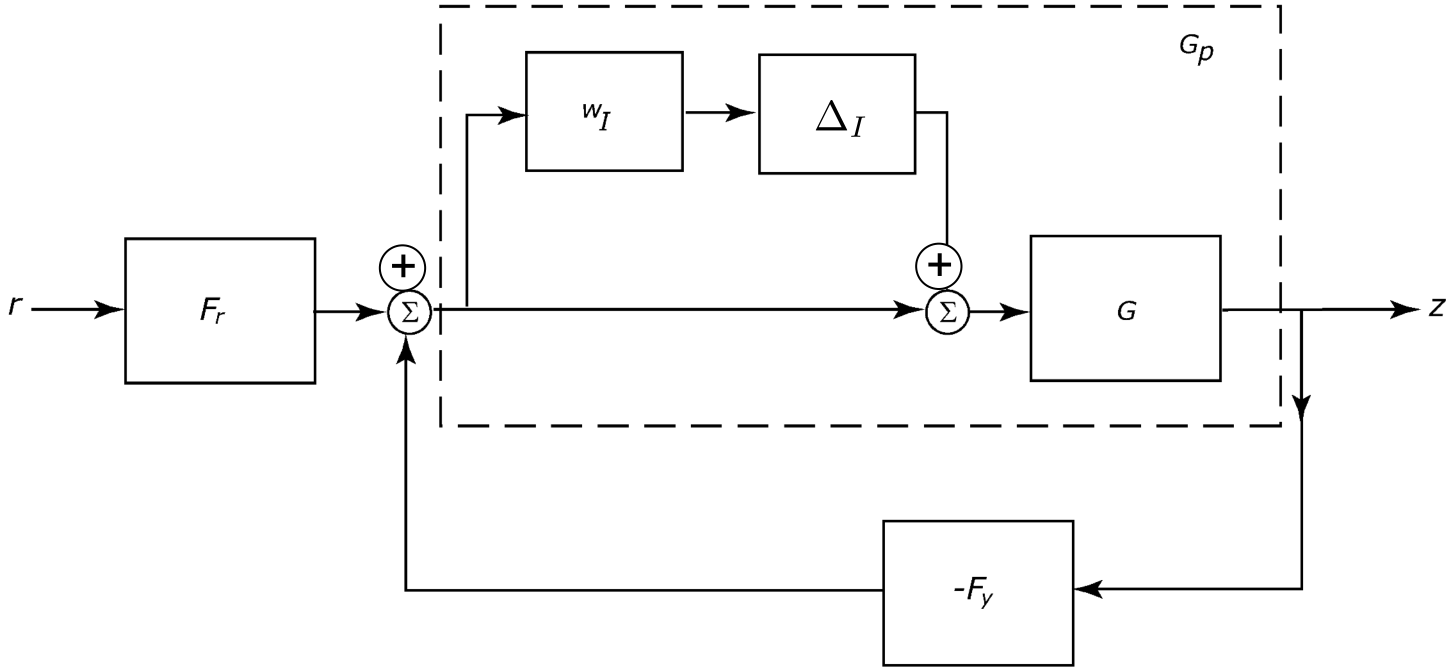
Robust stability: closed-loop stability for all $G_p \in \Pi_I$

Uncertainty in Nyquist Plot



- The perturbation generates at disc of radius $|W_I L|$ at each frequency
- The nominal loop-gain $L(i\omega)$ is at the center of the disc while the true system is assumed to be within the disc, i.e., the true system $\tilde{L}$ belongs to the set Π_I

Robust stability: relative uncertainty



The global system is stable if w_I and T stable, Δ_I stable, and

$$\|W_I T\|_\infty \leq 1 \quad \Leftrightarrow \quad |T(i\omega)| \leq |W_I^{-1}(i\omega)| \qquad \forall\omega$$

- follows from Small Gain Theorem (see lecture 2)

Example: uncertain gain

Consider the set of possible models

$$\Pi_I = \{G_p : G_p(s) = kG(s), \quad k \in [1-\delta, 1+\delta]\}$$

Standard form representation (with real Δ_I)

$$k = (1 + W_I \Delta_I),\ \Delta_I \in [-1, 1] \quad \Rightarrow \quad W_I(s) = \delta$$

– note that Δ_I strictly should be real in this case, but we allow it to be complex since we consider a bounded norm $\|\Delta_I\|_\infty$ which is defined for all complex Δ_I

Example: uncertain zero location

Nominal plant $G(s) = (s_0 + s)G_0(s)$

Set of possible plants

$$\Pi_I = \{G_p : G(s) = (s_0' + s)G_0(s), s_0' \in [-\delta + s_0, s_0 + \delta]\}$$

Standard form representation (with real Δ_I)

$$(s_0' + s)G_0(s) = (1 + W_I(s)\Delta_I(s))(s_0 + s)G_0(s) \quad \Rightarrow \quad W_I(s)\Delta_I(s) = \frac{s_0' - s_0}{s_0 + s}$$

$$\Rightarrow \quad W_I(s) = \frac{\delta}{s_0 + s}$$

Alternative approach to obtain weight

Note that the relative uncertainty set

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \leq 1\}$$

can be re-written as

$$\Pi_I = \left\{G_p(s) \;\middle|\; |W_I^{-1}G^{-1}(G_p - G)| \leq 1\right\}$$

Thus, the uncertainty about the system captured by W_I if

$$|W_I(i\omega)| \geq \left|\frac{G_p(i\omega) - G(i\omega)}{G(i\omega)}\right| \qquad \forall G_p \in \Pi_i$$

(must first pick a nominal model G)

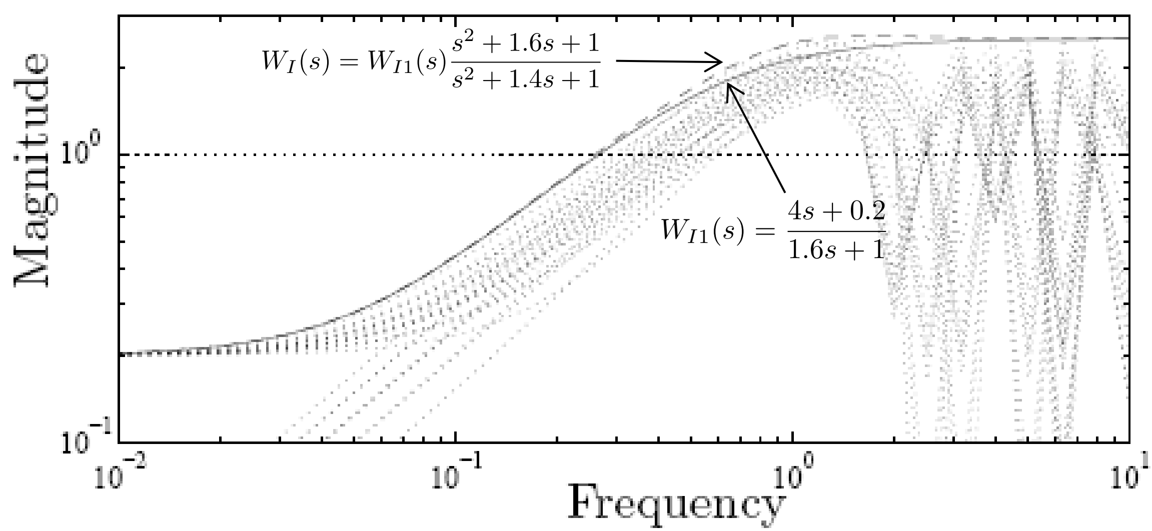
Example 1

Consider the uncertain system: $G_p(s) = \dfrac{k}{\tau s + 1} e^{-\theta s}, \quad k, \theta, \tau \in [2, 3]$

with nominal plant $G(s) = \dfrac{\bar{k}}{\bar{\tau} s + 1}$

Sample uncertainties (dotted) and corresponding W_I (dashed)

$$\left| \frac{G_p - G}{G} \right|$$



Example 2

Consider the following nominal plant and controller

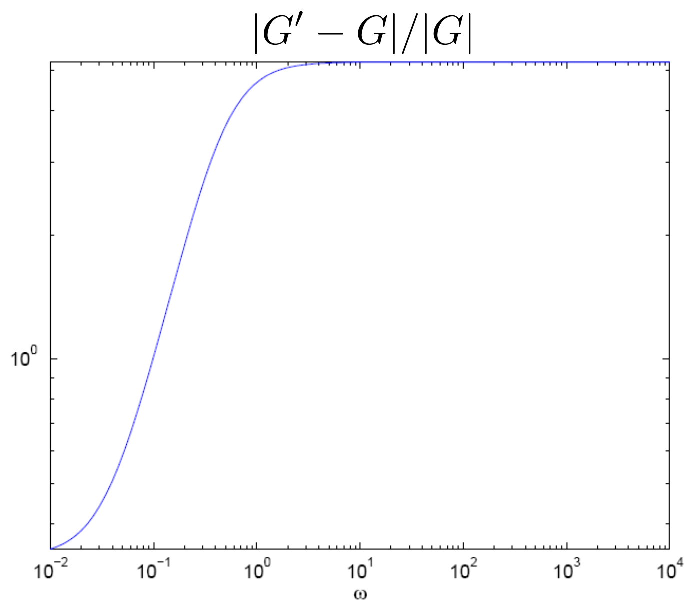
$$G(s) = \frac{3(1-2s)}{(5s+1)(10s+1)}, \qquad K(s) = K_c \frac{12.7s+1}{12.7s}$$

and assume that one "extreme" possible plant is

$$G'(s) = \frac{4(1-3s)}{(4s+1)^2}$$

Is the closed-loop robustly stable?

Example 2

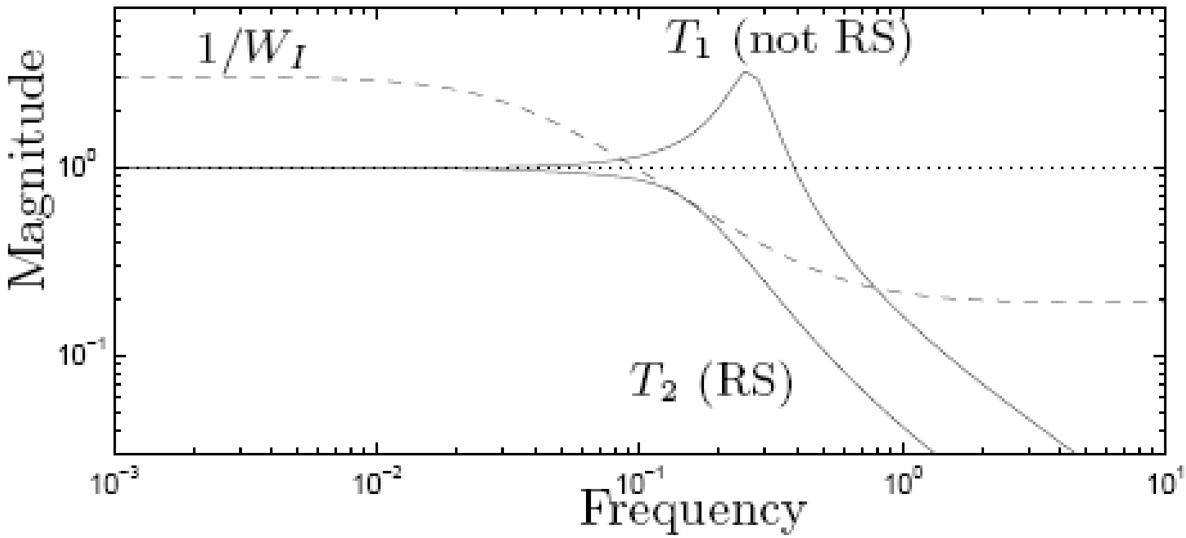


The relative error is around 0.33 for low frequencies and 5.25 at high frequencies. Fitted weight

$$W_I(s) = \frac{10s + 0.33}{(10/5.25)s + 1}$$

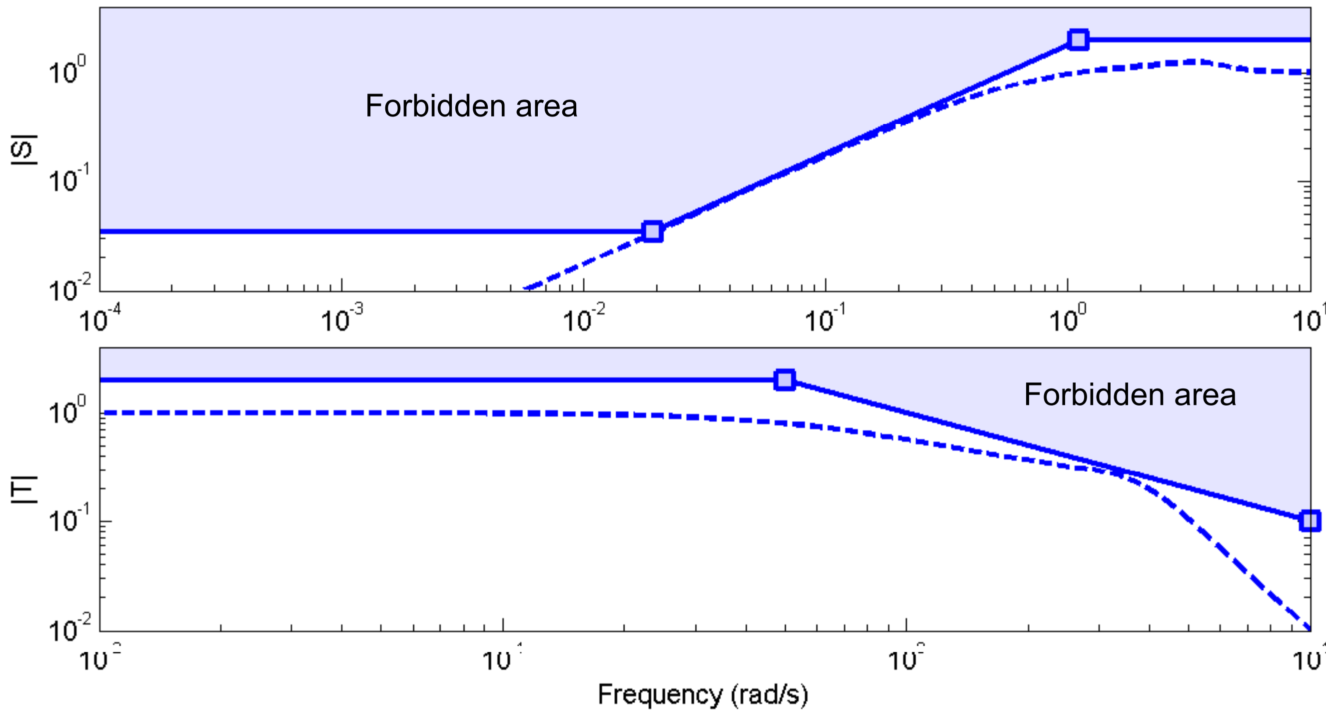
Example 2

Uncertainty weight W_I and complementary sensitivities for two different controller gains



The system with K_c=1.13 (T1) is not robustly stable, whereas it is robustly stable with K_c=0.31 (T2).

Frequency domain specifications



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|$$
$$|T(i\omega)| \leq |W_T^{-1}(i\omega)|$$

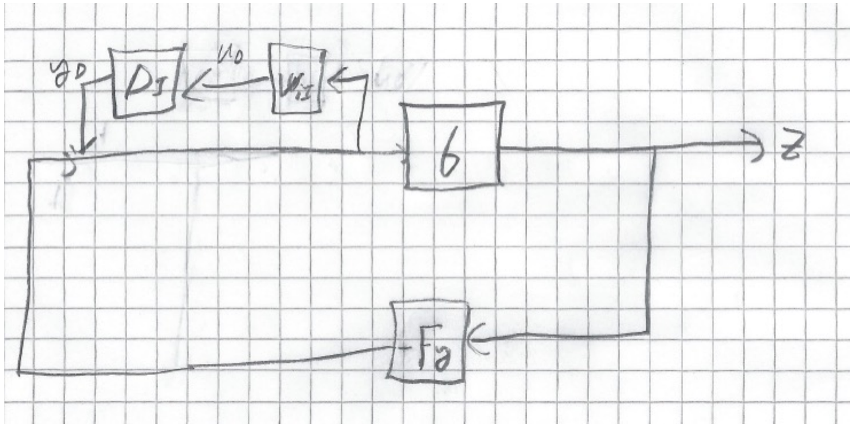
- Include relative uncertainty in weight $W_T(s)$

Uncertain number of RHP poles

- The requirement that the perturbation $\Delta_I(s)$ is stable implies that $G(s)$ and $G_p(s)$ must have the same number of RHP poles in Π_I
- To model uncertain number of RHP poles, define model set

$$\Pi_{iI} = \{G_p(s) = G(s)(1 + W_{iI}(s)\Delta_I(s))^{-1} \mid \|\Delta_I\|_\infty \leq 1\}$$

- Corresponds to feedback around $\Delta_I(s)$



  – feedback can make poles cross the imaginary axis even if the perturbation $\Delta_I(s)$ itself is stable

Example 3

- Uncertain pole location

$$G_p(s) = \frac{1}{s-p}, \quad p \in [-3, 1]$$

- Select nominal model

$$G(s) = \frac{1}{s+1}$$

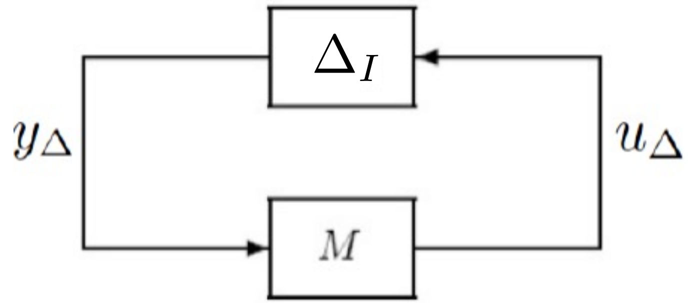
- With $\Delta_I \in [-1, 1]$ and

$$W_{iI} = \frac{2}{s+1}$$

  we cover all $G_p(s)$ in the model set Π_{iI}
  - note that it also in this case would suffice with a real Δ_I but that we must allow it to be complex to use the Small Gain Theorem

Robust stability with inverse uncertainty

- As before, we write the block-diagram with uncertainty on the form



- From the block diagram on slide 16 we find $u_\Delta = W_{iI}(s)S(s)y_\Delta$,
- hence $M = W_{iI}S$ and the Small Gain Theorem then gives RS if

$$\|W_{iI}S\|_\infty < 1$$

Robust performance

Assume nominal performance specified in terms of the sensitivity function

$$|W_P S| \leq 1 \quad \forall\omega$$

Robust performance if

$$|W_P S_p| \leq 1 \qquad \text{for all } \omega \text{ and all } S_p$$

We have

$$W_P S_p = W_P \frac{1}{1+L_p} = \frac{W_P}{1+L+W_I \Delta L}$$

Worst-case Δ is such that 1+L and $W_I \Delta L$ point in opposite directions

$$|W_P S_p| \leq \frac{|W_P|}{|1+L| - |W_I L|} = \frac{|W_P S|}{1-|W_I T|} \quad \forall\omega$$

Robust performance cont' d

Robust performance

$$|W_P S_p| = \frac{|W_P S|}{1 - |W_I T|} \leq 1$$

Can be expressed as

$$|W_P S| + |W_I T| \leq 1 \quad \forall \omega$$

Sometimes approximated by the *mixed* sensitivity constraint

$$\left\| \begin{pmatrix} W_P S \\ W_I T \end{pmatrix} \right\|_\infty \leq 1$$

Robust stability and performance

In summary

$$\begin{aligned} \text{nominal performance} \quad & |W_P S| \leq 1 \quad \forall \omega \\ \text{robust stability} \quad & |W_I T| \leq 1 \quad \forall \omega \\ \text{robust performance} \quad & |W_P S| + |W_I T| \leq 1 \quad \forall \omega \end{aligned}$$

Note that nominal performance and robust stability with rel. unc. implies

$$|W_P S| + |W_I T| \leq 2 \quad \forall \omega$$

(i.e. robust performance cannot be "too bad").

Holds only in SISO case!

Summary

Robust stability

closed-loop stability holds for all models within uncertainty set, including the true system

Can guarantee robustness if we model (or bound) uncertainty

- usually model as relative uncertainty
- but, also other uncertainty models, e.g., inverse multiplicative uncertainty, possible

Relative uncertainty imposes constraints on T, inverse uncertainty constraints on S

Robust performance: satisfy bounds on sensitivity function S for all plants in the model set