ver 1.1, 190408 Elling W. Jacobsen, KTH

7 EL2520 Lecture notes 7: Robust Stability in MIMO systems, Loop Shaping

We start this lecture by deriving a robust stability condition for MIMO systems. We then move on to discuss how to design controllers that satisfy robustness and performance specifications in terms of norm bounds on weighted closed-loop transfer-functions, e.g., $\|W_S S\|_\infty < 1$ and $\|W_T T\|_\infty < 1$. As we show, this can either be done using classical shaping of the open-loop $L = GF_y$, or shaping the closed-loop transfer-functions directly using optimization.

7.1 Robust Stability

We consider modeling uncertainty in the same way as we did for SISO systems in Lecture 3. That is, we consider a set of models G_p obtained by perturbing the nominal model $G(s)$ by a stable perturbation $\Delta_o(s)$

$$\Pi_o = \{G_p(s) = (I + W_o(s)\Delta_o(s))G(s), \quad \|\Delta_o\|_\infty < 1\} \tag{1}$$

Note that this corresponds to modeling the uncertainty at the output side. See also Figure 7.1. The stable perturbation $\Delta_o(s)$ is in general a full matrix of the same dimension as $G(s)$ and such that $\bar{\sigma}(\Delta_o) < 1 \ \forall\omega \Leftrightarrow \|\Delta_o\|_\infty < 1$. The uncertainty weight $W_o(s)$ can be a matrix of compatible dimension, but is often chosen to be a scalar weight. The assumption is, as before, that the true plant is within the model set Π_o (1), such that if we ensure closed-loop stability for all plants in the set Π_o then also the true system will be closed-loop stable.

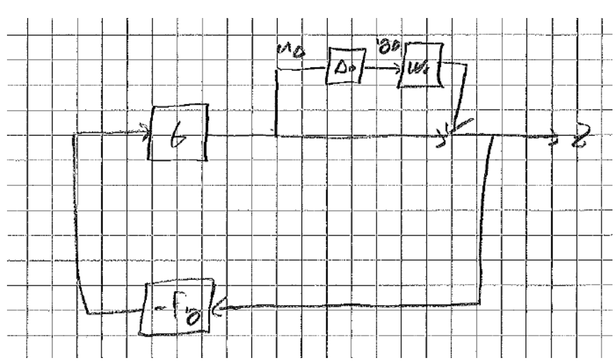

Figure 7.1: Closed-loop system with output uncertainty.

To analyze if the loop in Figure 7.1 is stable for all possible perturbations $\Delta_o(s)$, such that $\|\Delta_o\|_\infty < 1$, we employ the Small Gain Theorem. First rewrite the block-diagram in Figure 7.1 on M-Δ-form as shown in Figure 7.2. The block M is identified by considering the transfer-function from y_Δ to u_Δ in Figure 7.1.

$$u_\Delta = -GF_y(I + GF_y)^{-1}W_o y_\Delta = -TW_o y_\Delta$$

Thus, $M = -TW_o$ in Figure 7.2, where T is the complementary sensitivity function and W_o is the uncertainty weight. Applying the small gain theorem to the M-Δ loop then gives that the closed-loop is robustly stable if $T(s)$ is stable (nominal stability), $\Delta_o(s)$ stable (by assumption) and

$$\|TW_o\|_\infty < 1$$

Note that we must also choose the weight W_o to be stable, but this is no limitation since it is only the magnitude of the weight that matters in the uncertainty model (1). Thus, we arrive at a similar robust stability condition for MIMO systems as derived for SISO systems before.

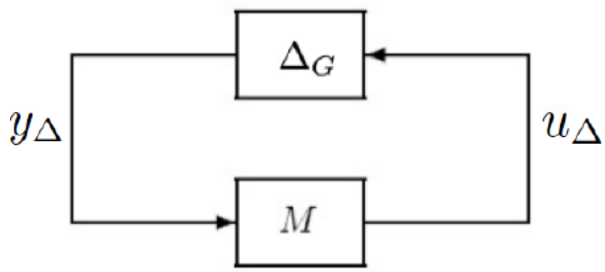

Figure 7.2: M-Δ-block for analyzing robust stability of system in Figure 7.1

For SISO systems it does not make any difference if we model the uncertainty on the input or output side of $G(s)$ since we can just move the uncertainty through $G(s)$, i.e.,

$$(1 + W(s)\Delta(s))G(s) = G(s)(1 + W(s)\Delta(s))$$

However, this is not true for MIMO systems and it makes a difference if we model the uncertainty on the input or output side. Let us consider modeling the uncertainty on the input side

$$\Pi_i = \{G_p(s) = G(s)(I + W_i(s)\Delta_i(s)), \quad \|\Delta_i\|_\infty < 1\} \tag{2}$$

Repeating the analysis done for output uncertainty above, we now obtain for M in the M-Δ_i-loop in Figure 7.2

$$u_\Delta = -F_yG(I + F_yG)^{-1}W_i y_\Delta = -T_u W_i y_\Delta$$

and hence the small gain theorem yields the robust stability criterion

$$\|T_u W_i\|_\infty < 1$$

where T_u is the complementary sensitivity function at the input side. Recall that in general $T_u \neq T$, and hence we get a different robust stability condition when we model the uncertainty on the input side (2) as compared to when we model it on the output side (1).

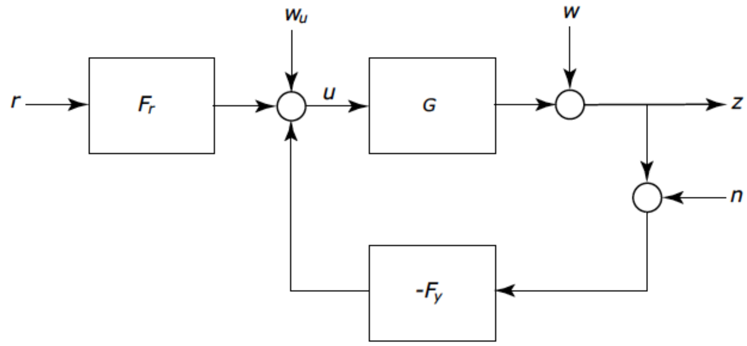

Figure 7.3: Two-degree of freedom control system.

7.2 Controller Design

Consider the closed-loop system in Figure 7.3. Recall that the aim of the control system is to make the output z follow the reference r despite disturbances on the input w_u, disturbances on the output w and measurement noise n. In addition, it is usually preferable to limit the use of the input u. Finally, we want to make sure the closed-loop is stable even if our model is uncertain, i.e., we want to make the closed-loop robustly stable. These performance and robustness specifications can all be formulated as requirements on the relevant closed-loop transfer-functions. For instance, to attenuate disturbances w in the output z we should make S small and to make the system robustly stable wrt output uncertainty and to attenuate measurement noise we should make T small. As discussed previously, we can not make a given transfer-function small att all frequencies, e.g., recall the Bode sensitivity integral, and we must also make a trade off between different transfer-functions, e.g., $S+T=I$. Finally, we should make sure that all specifications are feasible, that is, satisfy all limitations as discussed in Lecture 6. The aim of the control design is then to shape the relevant closed-loop transfer-functions, e.g., S and T, to fulfill the specifications using the controllers F_y and F_r. The shaping is done in two steps:

1. Define boundaries that a given closed-loop transfer-function should stay within, e.g.,

$$\bar{\sigma}(S) < |W_S^{-1}| \; \forall \omega \;\Rightarrow\; \|W_S S\|_\infty < 1 \tag{3}$$

$$\bar{\sigma}(T) < |W_T^{-1}| \; \forall \omega \;\Rightarrow\; \|W_T T\|_\infty < 1 \tag{4}$$

   The boundaries, or weights, should reflect the control objectives while respecting all fundamental trade-offs and limitations such that a controller satisfying all bounds exists.

2. Determine controller $F_y(s)$ (and $F_r(s)$) such that all specified bounds are satisfied.

The second step above can be approached either using classical loop shaping, i.e., shaping the open-loop $L = GF_y$, or using an optimization based approach, i.e., $\mathcal{H}_\infty$-synthesis, to directly shape the closed-loop transfer-functions. We discuss the two different approaches below.

7.2.1 Classic Loop Shaping

In classical loop shaping, the idea is to translate specifications on the closed-loop transfer-functions into specifications on the open-loop transfer-function (loop gain) $L = GF_y$ and then use F_y to shape L. The main motivation behind this approach is that the relation between the controller $F_y(s)$ and the loop gain $L = GF_y$ is much more transparant than the relationship between the controller and closed-loop transfer-functions like the sensitivity S and the complementary sensitivty T.

For MIMO plants, the specifications on the closed-loop (3) and (4) are in terms of bounds on the maximum singular values of S and T. Thus, we must translate the specifications on these singular values into specifications on the singular values of the loop gain $L = GF_y$. Starting with the sensitivity function S we have

$$S = (I + L)^{-1}$$

For a square matrix A we have $\bar{\sigma}(A^{-1}) = 1/\underline{\sigma}(A)$ and we get

$$\bar{\sigma}(S) = \frac{1}{\underline{\sigma}(I + L)}$$

Now, from Fan's Theorem (see Lec 6)

$$|\sigma_i(L) - 1| \leq \sigma_i(I + L) \leq \sigma_i(L) + 1 \tag{5}$$

If $\underline{\sigma}(L) >> 1$ then $\underline{\sigma}(I + L) \approx \underline{\sigma}(L)$ and

$$\bar{\sigma}(S) \approx \frac{1}{\underline{\sigma}(L)}, \quad \underline{\sigma}(L) >> 1$$

Thus, in the frequency range where we want to make the sensitivity function small, corresponding to $|W_S|$ large, we get

$$\bar{\sigma}(S) \leq |W_S^{-1}| \quad \Rightarrow \quad \underline{\sigma}(L) \geq |W_S| \ , \quad |W_S| >> 1 \tag{6}$$

Similarly, for T we use the fact that $\bar{\sigma}(A)\underline{\sigma}(B) \leq \bar{\sigma}(AB) \leq \bar{\sigma}(A)\bar{\sigma}(B)$ with $A = L$ and $B = (I + L)^{-1}$ together with (5) to derive

$$\bar{\sigma}(T) \approx \bar{\sigma}(L), \quad \bar{\sigma}(L) << 1$$

Hence, for frequencies where $|W_T| >> 1$ we get

$$\bar{\sigma}(T) \leq |W_T^{-1}| \quad \Rightarrow \quad \bar{\sigma}(L) \leq |W_T^{-1}| \ , \quad |W_T| >> 1 \tag{7}$$

Typically, we want to make the sensitivity small at low frequencies and the complementary sensitivity small at high frequencies. Thus, at low frequencies the loop gain should satisfy the bound (6) while at high frequencies it should satisfy the bound (7). This is illustrated in Figure 7.4.

Note that it is difficult to adress stability in MIMO loop shaping; there is no phase lag defined for MIMO systems, and hence no gain and phase margins. Thus, there is a need to adress stability after having shaped the loop to satisfy the given bounds. We will return to this problem in Lecture 10 when discussing robust loopshaping.

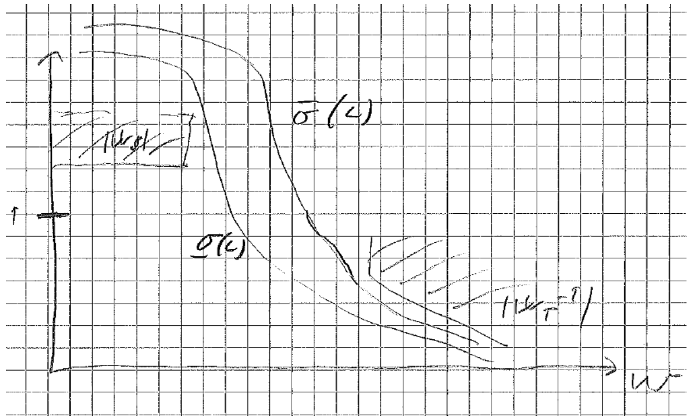

Figure 7.4: Illustration of shaping singular values of loop gain $L = GF_y$ to shape bounds $|W_S|$ and $|W_T^{-1}|$.

7.2.2 Shaping the closed-loop - $\mathcal{H}_\infty$-Synthesis

An attractive alternative to "manually" shaping the open-loop, is to employ an optimization algorithm to directly shape the closed-loop according to the specifications. With such an approach we can also solve more general problems, i.e., with specifications on closed-loop transfer-functions other than S and T. As an example, assume we want to attenuate disturbances on the output w and measurement noise n in z while keeping the input usage u small. The corresponding transfer-functions are

$$z = Sw + Tn \ ; \quad u = -F_y(I + GF_y)(w + n) = G_{wu}(w + n)$$

As before, we introduce weights to make a suitable trade-off between different frequencies and different transfer-functions, and to respect any fundamental limitations, and thereby get the performance objectives

$$\|W_S S\|_\infty \leq 1 \ , \quad \|W_T T\|_\infty \leq 1 \ , \quad \|W_u G_{wu}\|_\infty \leq 1 \tag{8}$$

Note that we can also include robust stability for output uncertainty, as discussed above, by including the uncertainty weight in W_T. In this course we will, unless stated otherwise, assume that all weights are scalar transfer-functions.

For optimization purposes we need a single objective function and this can be obtained by stacking all objectives in (8) into one big matrix and minimizing the norm of the stacked matrix

$$F_y = \arg\min_{F_y} \left\| \begin{pmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{pmatrix} \right\|_\infty \tag{9}$$

Note that if the $\mathcal{H}_\infty$-norm of the stacked matrix is less than some positive number γ, then so is the $\mathcal{H}_\infty$ norm of each of the elements $W_S S, W_T T$ and $W_u G_{wu}$ less than γ (with some margin).

Ideally, we could give the problem in (9) to an optimization algorithm to determine the feedback controller $F_y(s)$ that minimizes the objective function. Unfortunately, this is usually not possible for several reasons. Rather, we need to solve an associated signal minimization problem that reflects the objective function in (9). The signal minimization problem we can solve is, with reference to the system shown in Figure 7.5,

$$\min_{F_y} \sup_{w} \frac{\|z_e\|_2}{\|w\|_2} \tag{10}$$

where $z_e = G_{ec}w$. Note that

$$\sup_{w} \frac{\|z_e\|_2}{\|w\|_2} = \|G_{ec}\|_\infty \tag{11}$$

and hence solving (10) corresponds to minimizing the $\mathcal{H}_\infty$-norm of the closed-loop G_{ec} in Figure 7.5. The system in Figure 7.5 is called *the extended system* for reasons that will come obvious below. An important aspect of the extended system is that we can separate the controller $F_y(s)$, to be determined through optimization, from the open-loop system $G_0(s)$.

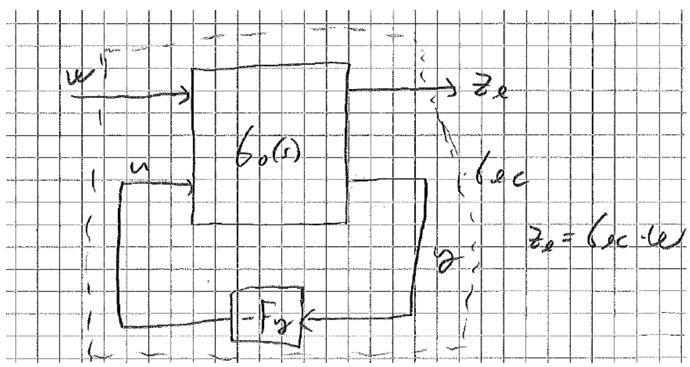

Figure 7.5: Extended system for $\mathcal{H}_\infty$-optimization.

The fact that the signal minimization problem in (10) corresponds to

$$\min_{F_y} \|G_{ec}\|_\infty \tag{12}$$

implies that we can solve the original problem (9) if we choose the signals z_e and w of the extended system such that

$$G_{ec} = \begin{pmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{pmatrix} \tag{13}$$

That is, we need to choose the output z_e of the extended system G_{ec} such that

$$z_e = \begin{pmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{pmatrix} w \tag{14}$$

where w is the disturbance on the output in the original block-diagram in Figure 7.3 and also the input to the extended system in Figure 7.5[1]. For the optimization, we need to provide the corresponding open-loop system $G_0(s)$ in Fig. 7.5.

[1] Depending on what problem we want to solve, we may choose different inputs to the exended system. For the problem considered here, use of w as the input is sufficient.

Let us derive the outputs $z_{ei}, i = 1, 3$, and corresponding open-loop transfer-functions G_{0i}, needed to reflect each of the three transfer-functions in (14).

- The first output we seek should reflect the first transfer-function in (14), i.e,

  $$z_{e1} = W_S S w$$

  To relate this to the orignal block-diagram in Figure 7.3, we have that $z = Sw$. Hence we choose the output

  $$z_{e1} = W_S z$$

  In open-loop we have $z = Gu + w$, and hence for G_0 we get

  $$z_{e1} = W_S(Gu + w)$$

- The second output z_{e2} should reflect $W_T T$, i.e.,

  $$z_{e2} = W_T T w$$

  In the original system we have $z = Sw = (I - T)w$ and hence $z - w = -Tw$ which leads to

  $$z_{e2} = -W_T(z - w)$$

  In open-loop we have $z - w = Gu$ and hence for G_0 we get

  $$z_{e2} = -W_T G u$$

- The final output z_{e3} should reflect $W_u G_{wu}$, i.e.,

  $$z_{e3} = W_u G_{wu} u$$

  In the original system we have $u = G_{wu} w$ and hence

  $$z_{e3} = W_u u$$

  In open-loop we have $u = u$ and hence for G_0 we get

  $$z_{e3} = W_u u$$

In summary, we choose the output

$$z_e = \begin{pmatrix} W_S z \\ -W_T(z - w) \\ W_u u \end{pmatrix}$$

which corresponds to the open-loop transfer-matrix

$$\begin{pmatrix} z_e \\ y \end{pmatrix} = G_0(s) \begin{pmatrix} w \\ u \end{pmatrix} \quad ; \quad G_0(s) = \begin{pmatrix} W_S & W_S G \\ 0 & -W_T G \\ 0 & W_u \\ I & G \end{pmatrix} \tag{15}$$

and the corresponding closed-loop is the desired (13). Note that the open-loop extended system contains several instances of the model $G(s)$ as well as the weights W_S, W_T and W_u.

The solution of the signal minimization problem (10) is based on the state-space realization of the open-loop model $G_0(s)$ (15)

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) + Nw(t) \\ z_e(t) &= Mx(t) + Du(t) \\ y(t) &= Cx(t) + w(t) \end{aligned} \tag{16}$$

where the realization is normalized so that $D^TM = 0$ and $D^TD = I$ (for reasons that become obvious when proving the main result below). Solving the optimal control problem in (10) is hard, and it is therefore usually preferred to solve the sub-optimal control problem. The sub-optimal problem is, given a positive number γ, determine a stabilizing controller $F_y(s)$ which provides

$$\sup_w \frac{\|z_e\|_2}{\|w\|_2} = \|G_{ec}\|_\infty = \gamma$$

If no such controller exist, then the minimum achievable value of $\|G_{ec}\|_\infty$ is larger than γ. To determine if a controller $F_y(s)$ exists that gives $\|G_{ec}\|_\infty = \gamma$ one can solve the algebraic Riccati equation

$$A^TP + PA + M^TM + P(\gamma^{-2}NN^T - BB^T)P = 0 \tag{17}$$

where A, B, M, N are all from the state-space description of $G_0(s)$ (16). If a positive definite solution $P > 0$ to (17) exists and if $A - BB^TP$ is stable, then the controller

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x}(t) + Bu(t) + N(y(t) - C\hat{x}(t)) \; ; \quad \hat{x}(0) = 0 \\ u(t) &= -L_\infty \hat{x}(t), \quad L_\infty = B^TP \end{aligned} \tag{18}$$

will give $\|G_{ec}\|_\infty = \gamma$. The Laplace transform of (18) is $u = -F_y(s)y$. Note that the feedback controller (18) is on the form of an observer combined with state feedback. As we shall see in the next lecture, this control structure also results when solving the classical optimal control problems based on objective functions defined in the time domain. Also note that the controller will have the same number as states as the open-loop model $G_0(s)$, which implies that the number of states in the controller will be at least the number of states in $G(s)$ plus the number of states in the weights W_S, W_T and W_u.

To arrive at the $\mathcal{H}_\infty$-optimal controller, i.e., the controller that solves (10) and (12), we can solve the sub-optimal problem iteratively, i.e., by iterating on γ untill $\gamma = \gamma_{min}$. The basis for this is that for every $\gamma > \gamma_{min}$, there is a solution $P > 0$ to the Riccati equation (17) which stabilizes the loop with $\|G_{ec}\|_\infty = \gamma$, while there exists no stabilizing solution $P > 0$ if $\gamma < \gamma_{min}$.

To prove that the controller (18) solves the sub-optimal control problem $\|G_{ec}\|_\infty = \gamma$, consider the function

$$V(t) = x^T(t)Px(t) + \int_0^t (z_e^T(\tau)z_e(\tau) - \gamma^2 w^T(\tau)w(\tau)d\tau$$

Now, if P is a positive definite matrix, i.e., $x^T P x > 0$, and we can show that $V(t) \leq 0 \ \forall t$, then the integral term must be negative for all t and hence it follows that $\|z_e\|_2^2 < \gamma^2 \|w\|_2^2$ for any disturbance w. Note that $V(0) = 0$ and hence it is sufficient to show that $\dot{V}(t) \leq 0 \ \forall t$. Taking the time derivative of $V(t)$ we get

$$\begin{aligned}
\dot{V} &= \dot{x}^T P x + x^T P \dot{x} + z_e^T z_e - \gamma^2 w^T w \\
&= x^T A^T P x + u^t B^T P x + w^T N^T P X + x^T P A X + x^T P B u + x^T P N w + x^T M M x + u^T u - \gamma^2 w^T w \\
&= x^T (A^T P + P A + M^T M) x + u^T B^T P + x^T P B u + u^T u + w^T N^T P x + x^T P N w - \gamma^2 w^T w \qquad (19)
\end{aligned}$$

We complete the square for $u^T B^T P + x^T P B u + u^T u$ by adding $x^T P B B^T P x$ and complete the square for $w^T N^T P x + x^T P N w - \gamma^2 w^T w$ by adding $-\gamma^{-2} x^T P N N^T P x$ (and substracting the same terms from the overall expression for $\dot{V}$) to obtain

$$\begin{aligned}
\dot{V} &= x^T (A^T P + P A + M^T M - P(B B^T - \gamma^{-2} N N^T) P) x + (u + B^T P x)^T (u + B^T P x) \\
&\quad - \gamma^{-2} (w - \gamma^{-2} N^T P x)^T (w - \gamma^{-2} N^T P x) \qquad (20)
\end{aligned}$$

Now it is easily seen that if P is a positive definite solution to (17) and we choose $u = -B^T P x$, then $\dot{V} \leq 0 \ \forall t$ and the proof is complete.