



# EL2520

# Control Theory and Practice

## Lecture 14: Summary

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# Outline

- Course objectives
- Checklist
- The exam and beyond
- Brief review of lectures

# Course Objectives

The course aims to provide the participants basic knowledge of approaches and methods in advanced control, in particular linear multivariable feedback systems.

## Key ingredients:

- A modern view of control
  - enabling systematic trade-off between various performance goals, respecting fundamental limitations and ensuring robust stability
- Multivariable control
  - multivariable systems (poles, zeros, gains and directions), decentralized control (RGA) and decoupling, several design methods based on controller synthesis through optimization, robust loop shaping
- Dealing with hard constraints
  - anti-windup and model predictive control

# Checklist

The basics

- Matrix manipulations, eigenvalue computations, singular values
- Complex numbers
- Differential equations, state-space models, transfer functions and frequency response

A modern perspective on SISO control:

- Signal norms, system gain and the Small Gain Theorem
- The closed loop system and the central transfer functions
- Internal stability
- Fundamental limitations due to RHP poles/zeros, time delays
- Reasonable design goals and mapping to loop gain specifications
- Uncertainty sets and robust stability / performance

# Checklist

## Multivariable linear systems

- Transfer matrices and block diagram algebra
- Multivariable poles and zeros, directions
- Amplification, gain and directions
- Extending SISO results to MIMO

## Multivariable control design techniques:

- $H_2$  and  $H_\infty$ -optimal control: weighting functions and extended system, controller structure
- LQG: design, optimal control structure and disturbance models
- Loop shaping and Glover-McFarlane robustification
- The relative gain array for decentralized control structure design

# Checklist

Dealing with hard constraints:

- Sampling of linear systems (continuous → discrete time)
- Model Predictive Control – MPC
  - finite horizon LQR w constraints → Quadratic Program
- Anti-windup (tracking) to deal with actuator saturation

# About the exam

*When?* Tue May 31 at 08.00-13.00

*How?* Written on-campus exam (remember to register)

*What?* 5 problems, 5 hours. Grading criteria on homepage

*Allowed aids?*

Course book (Glad&Ljung or Skogestad&Postlethwaite)

Basic control book (Glad&Ljung)

Lecture notes and slides from this years course (printed!)

Mathematical handbook

Pocket calculator (not symbolic)

**Note: all aids must be physical, i.e., not allowed to use any aids on computer or mobile phone**

# What is on the exam

- Signal and system norms, Small Gain Theorem
- Poles, zeros, directionality, singular values
- Closed-loop transfer functions
- Internal stability
- Performance limitations
- Input-output controllability analysis (requirements vs limitations)
- Robust stability
- RGA, decentralized control and decoupling
- LQG, H<sub>2</sub>- and H<sub>inf</sub>-optimal control
- Glover-McFarlane robust loop shaping
- Discrete time systems, finite horizon LQR and MPC
- Anti reset windup

# Beyond the exam

Learn more in our other advanced courses

- EL2820 Modeling of Dynamical Systems, per 1
- EL2700 Model Predictive Control, per 1
- EL2805 Reinforcement Learning, per 2
- EL2620 Nonlinear Control, per 2
- EL2450 Hybrid and Embedded Systems, per 3
- EL2810 Machine Learning Theory, per 3
- EL2425 Automatic Control Project Course, per 1-2

Put your skills to the test: do your master thesis at Decision and Control Systems

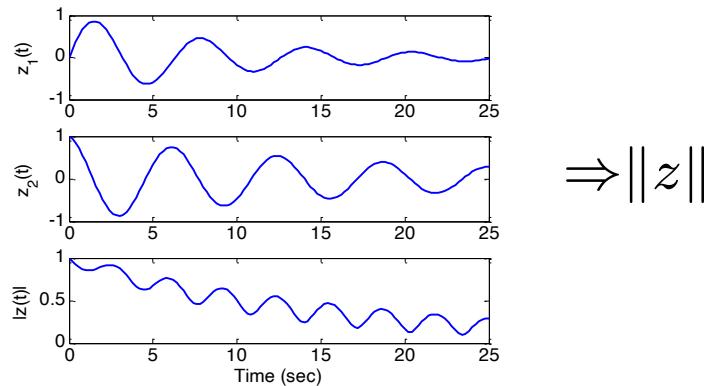
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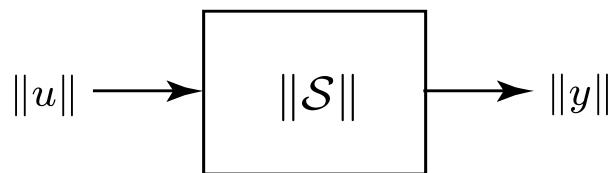
# Lecture 1

- *Signal norms:* measure signal size across space (channels) and time



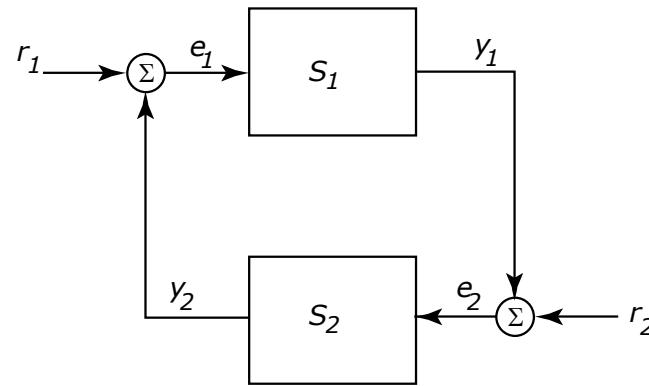
$$\Rightarrow \|z\|$$

- *System gain:* bounds signal amplification



# Small Gain Theorem

**Theorem.** Consider the interconnection



If  $S_1$  and  $S_2$  are input-output stable and

$$\|S_1\| \cdot \|S_2\| < 1$$

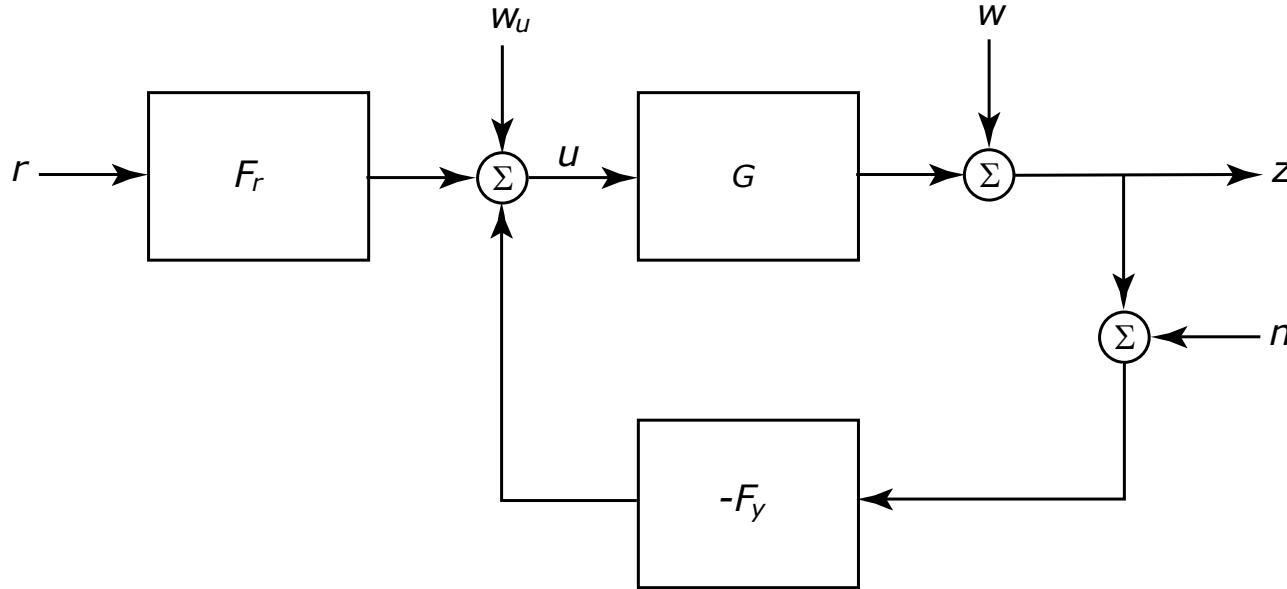
Then, the closed-loop system with  $r_1, r_2$  as inputs and  $e_1, e_2, y_1, y_2$  outputs is input-output stable.

Note: can use any norm that satisfies multiplicative property

$$\|AB\| \leq \|A\|\|B\|$$

e.g., inf-norm but not 2-norm

# Lecture 2 - The closed-loop system



Controller: feedback  $F_y$  and feedforward  $F_r$

Disturbances:  $w, w_u$  drive system from desired state

Measurement noise: corrupts information about  $z$

Aim: find controller such that  $z$  follows  $r$ .

# Transfer functions and observations

$$S = \frac{1}{1 + GF_y} \quad (w \rightarrow z, w_u \rightarrow u) \text{ sensitivity function}$$

$$T = \frac{GF_y}{1 + GF_y} \quad (n \rightarrow z) \text{ complementary sensitivity}$$

$$G_c = \frac{GF_r}{1 + GF_y} \quad (r \rightarrow z) \text{ closed loop system}$$

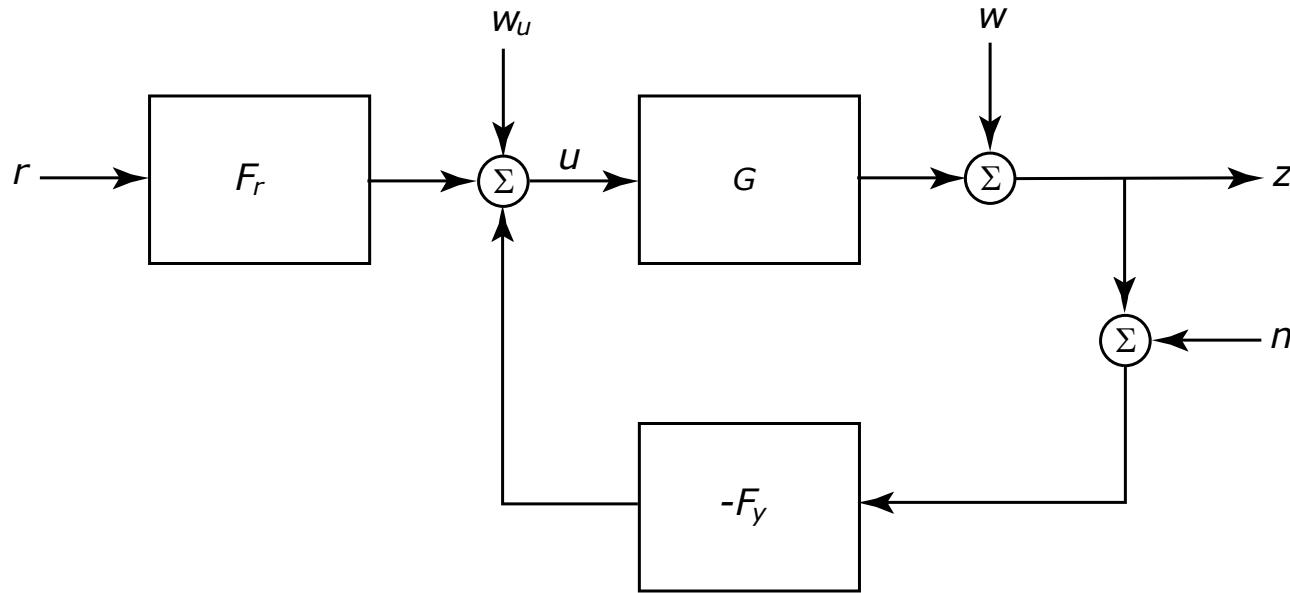
$$SG = \frac{G}{1 + GF_y} \quad (w_u \rightarrow z)$$

$$SF_y = \frac{F_y}{1 + GF_y} \quad (n \rightarrow u)$$

$$SF_r = \frac{F_r}{1 + GF_y} \quad (r \rightarrow u)$$

**Observations:** need to look at all! Many tradeoffs (e.g. S+T=1)

# Internal Stability



**Definition.** The closed loop system above is *internally stable* iff it is input-output stable from all inputs  $r, w_u, w, n$  to all outputs  $u, z, y$

**Theorem.** The closed-loop system is stable if and only if

$$S, SG, SF_y, F_r$$

are stable

# Example

**Example** Let

$$G(s) = \frac{1}{s} \left( \frac{-s + 2}{s + 2} \right)$$

Determine a controller  $U(s) = F(s)[R(s) - Y(s)]$  that achieves

$$G_c(s) = \frac{1}{1 + sT} \left( \frac{-s + 2}{s + 2} \right)$$

and show that the closed-loop system is internally stable

# The Sensitivity Functions

The sensitivity function ( $S$ ):

- Quantifies disturbance attenuation due to feedback

The complementary sensitivity function ( $T$ )

- Equals the closed-loop system  $G_c$  with 1-DOF control
- Quantifies the amplification of noise at the output
- Determines robust stability properties

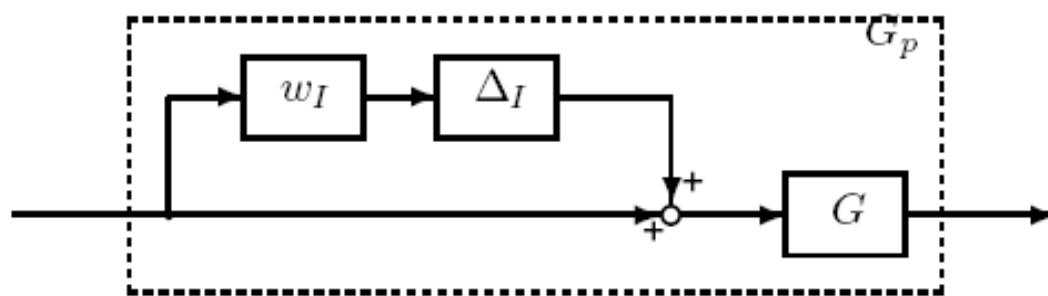
A first trade-off:  $S+T=1$

# Lecture 3 - Robustness

Robustness = property maintained under uncertainty

Idea: specify uncertainty set and guarantee stability  
and performance for all possible models within set

We have focused on multiplicative (relative) uncertainty



# SISO Robustness

Robust stability via small-gain theorem

$$|T(i\omega)| \leq |w_I^{-1}(i\omega)| \quad \forall \omega$$

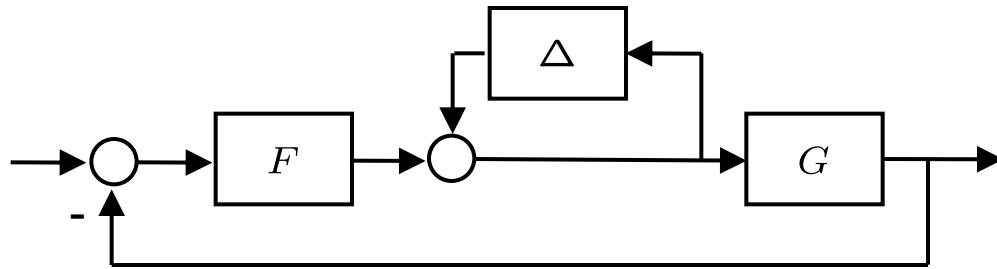
where

$$|w_I(i\omega)| \geq \left| \frac{G_p(i\omega) - G(i\omega)}{G(i\omega)} \right| \quad \forall G_p \in \Pi_i$$

Robust performance puts simultaneous bound on S and T.

# SISO Robustness

**Example:** Consider the uncertain system



Use the small-gain theorem to derive a robustness criterion

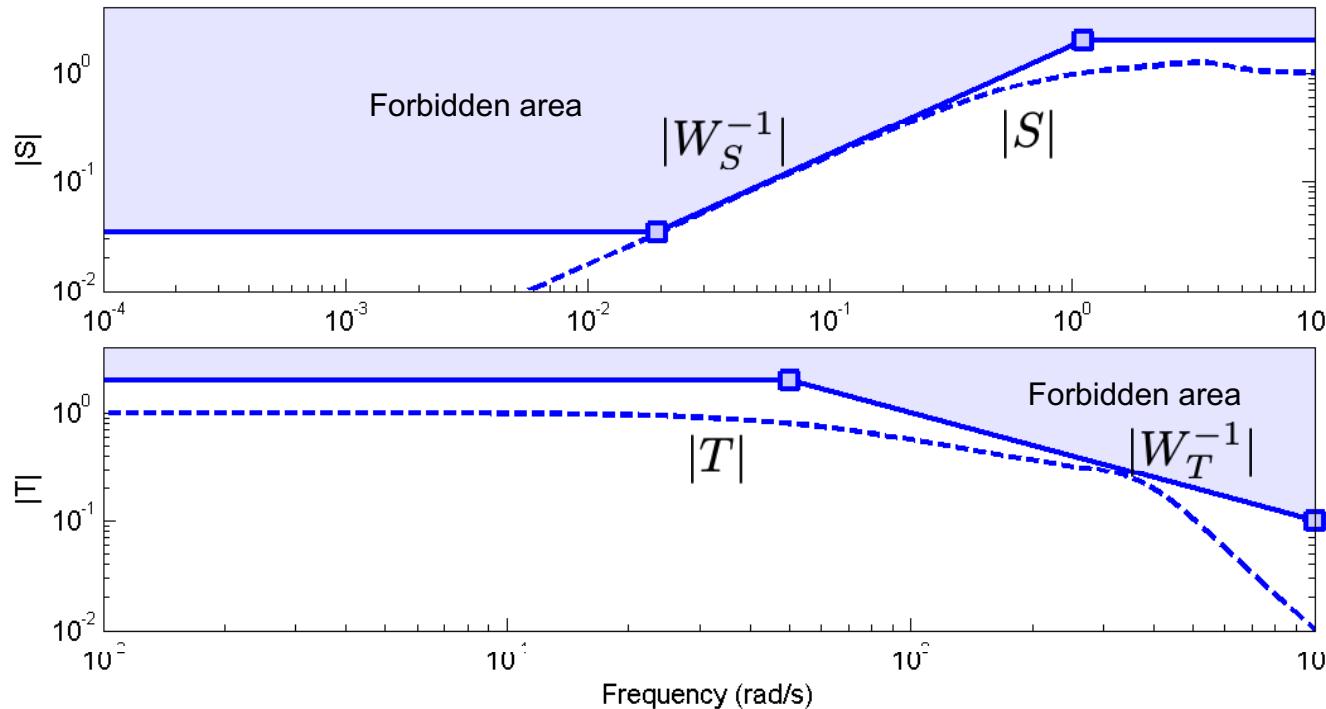
$$\|P\Delta\|_\infty \leq 1$$

for some transfer function  $P$  independent of  $\Delta$

# Question

- Why is robustness important, because we can not operate at nominal operating where model is obtained?
  - partly, this usually gives rise to a model error
  - but, we always have uncertainty in actuators (inputs) and measurements, and model uncertainty even at nominal operating point
- How is uncertainty modeled in practice
  - from system identification, e.g., parametric uncertainty. See e.g., example in Lecture notes 3
  - from knowledge of uncertainty in actuators and sensors
  - by adding some generic uncertainty to achieve a reasonable robustness (similar to AM and PM in SISO control), e.g., robust loopshaping

# Lecture 4– Limitations and Conflicts



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)| \quad \forall \omega \Leftrightarrow \|W_S S\|_\infty < 1$$

$$|T(i\omega)| \leq |W_T^{-1}(i\omega)| \quad \forall \omega \Leftrightarrow \|W_T T\|_\infty < 1$$

Can we choose weights  $w_S, w_T$  (“forbidden areas”) freely?  
– No, there are many constraints and limitations!

# Limitations and conflicts

- Fundamental trade-offs in control systems design
  - $S+T=1$  (both cannot be small at the same frequency)
  - Cannot attenuate disturbances at all frequencies (Bode Sensitivity Integral)
- Fundamental limitations:
  - Unstable poles
  - Non-minimum phase zeros
  - Time delays
- Practical limitation:
  - Control input constraints

# Rules of thumb

RHP zeros limit bandwidth (of S)

$$\omega_{BS} \leq \frac{z}{2}$$

Time-delays impose a similar bound

$$\omega_{BS} \leq \frac{1}{T}$$

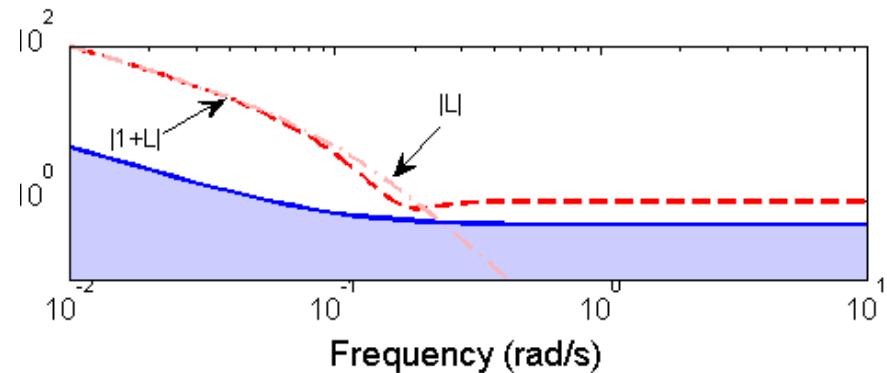
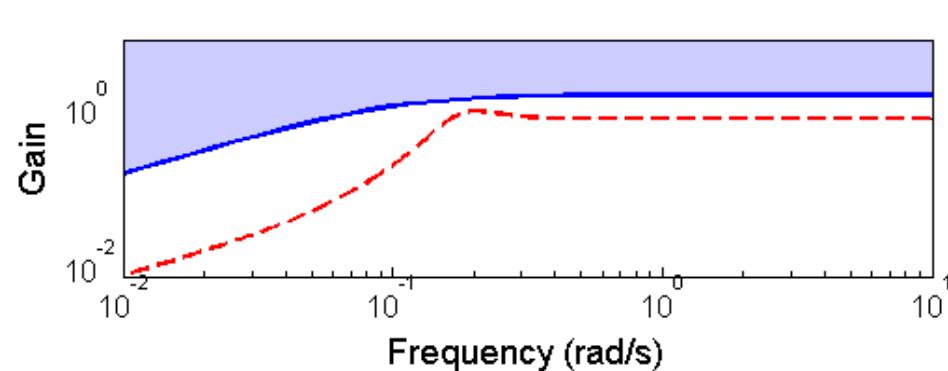
RHP poles require a minimum bandwidth (for T)

$$\omega_{BT} \geq 2p$$

# Classical Loop Shaping

In certain frequency ranges, there is a reasonable approximate mapping between constraints on S and T into requirements on the loop gain L.

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)| \Leftrightarrow |1 + L(i\omega)| \geq |W_S(i\omega)|$$



Problematic area is around cross-over frequency

- Put requirements on phase and amplitude margin

# Lecture 5 – Multivariable linear systems

Poles, zeros and gains.

**Theorem.** The *pole polynomial* of a system with transfer matrix  $G(s)$  is the least common denominator of all minors of  $G(s)$ . The poles of  $G(s)$  are the roots of the pole polynomial.

**Theorem.** The *zero polynomial* of  $G(s)$  is the greatest common divisor of the maximal minors of  $G(s)$ , normed so that they have the pole polynomial of  $G(s)$  as denominator. The zeros of  $G(s)$  are the roots of its zero polynomial.

**Theorem.** The gain of a linear system  $G(s)$  is given by

$$\|G\|_\infty = \sup_{\omega} |G(i\omega)| = \sup_{\omega} \bar{\sigma}(G(i\omega))$$

# Question

Please repeat MIMO zeros and zero directions

- A zero  $z$  is a value of  $s$  where  $G(s)$  has less rank than normal.
  - Example:

$$G(s) = \frac{1}{s+1} \begin{pmatrix} s+1 & 1 \\ 3 & 1 \end{pmatrix} \Rightarrow G(2) = \frac{1}{3} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

- When a matrix is rank deficient it has a left and right null space; the output zero direction  $y_z$  is the left nullspace and the input zero direction  $u_z$  is the right nullspace for the zero

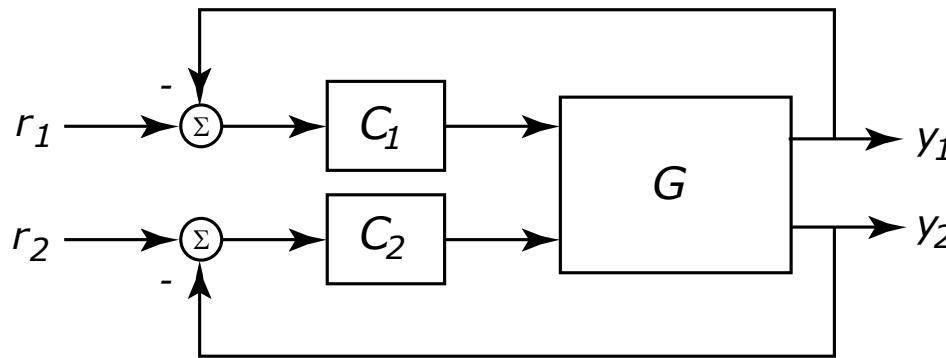
$$G(z)u_z = 0 \cdot y_z$$

- for the example:

$$u_z^H = \left( \frac{1}{3} \quad -1 \right) \quad y_z^H = (-1 \quad 1)$$

# Decentralized Control

Decentralized control:



Interactions: each input affects multiple outputs

Qualitatively: the more interactions, the harder to control

- The relative gain array tries to quantify the degree of interactions

# The Relative Gain Array

**Definition.** the *relative gain array*,  $RGA$ , of a square system is defined as

$$\Lambda(G) = G \times (G^{-1})^T$$

or, in Matlab notation,  $RGA(G) = G.*\text{inv}(G)'$

Rules of thumb:

1. Avoid pairings with  $\lambda_{ij}(0) < 0$  (why?)
2. Prefer pairings with  $|\lambda_{ij}(i\omega_c)| \approx 1$

# Decoupling

- Full decoupling

$$D(s) = d(s)G^{-1}(s) \quad \Rightarrow \quad G(s)D(s) = d(s)I$$

- Static decoupling

$$D = G^{-1}(0)$$

# Question

- How does decoupling affect robustness when the inputs are uncertain?
  - if we have input uncertainty then  $u_p = (I + \Delta)u$  and we get for the compensated plant

$$G(s)(I + \Delta)d(s)G^{-1}(s) = d(s)(I + G(s)\Delta G^{-1}(s))$$

- the term  $G(s)\Delta G^{-1}(s)$  can become very large for ill-conditioned  $G(s)$
- Example with 10% input uncertainty

$$G = \frac{1}{s+1} \begin{pmatrix} 1 & -1 \\ 1.1 & -1 \end{pmatrix} \quad \Delta = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.1 \end{pmatrix}$$

$$\Rightarrow G(s)\Delta G^{-1}(s) = \begin{pmatrix} -2.1 & 2 \\ -2.2 & 2.1 \end{pmatrix}$$

# Lecture 6 – MIMO Limitations

- Extend SISO limitations to MIMO case
- Essentially, we get interpolation constraints in certain directions (zero and pole directions)
- Otherwise derivations and limitations as for SISO case

# Question

- How to get the MIMO limitations in Lecture 6 from math?
  - Essentially combine interpolation constraints, e.g.,

$$y_z^H S(z) = y_z^H$$

with the Maximum Modulus Thm

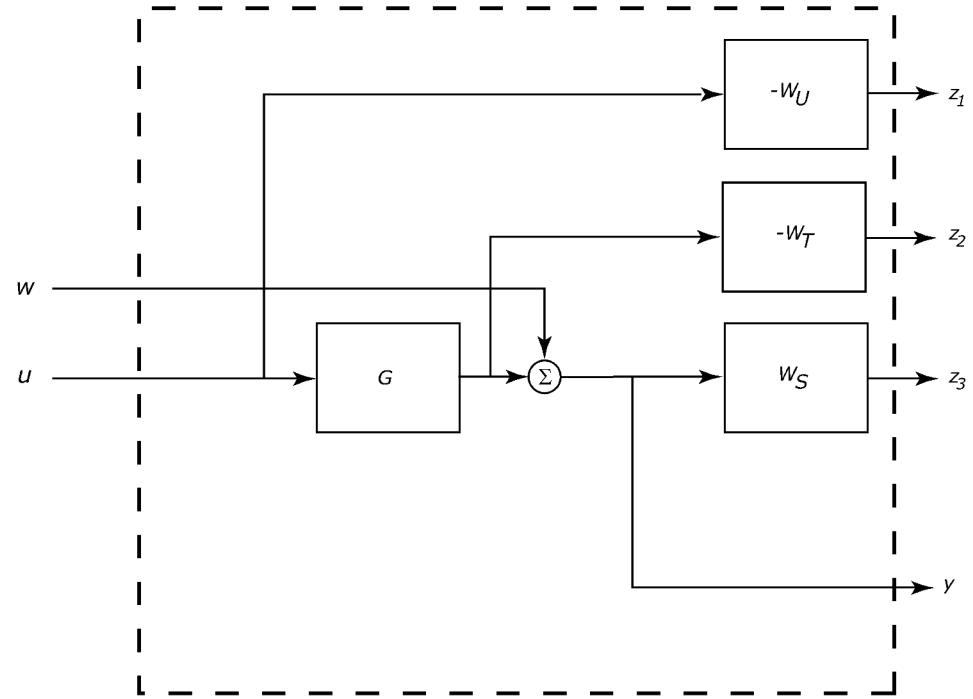
See also notes and slides from Lecture 6

# Lecture 7/9 – $H_\infty$ / $H_2$ -optimal control

Specification: optimize over all stabilizing controllers to achieve

$$\left\| \begin{pmatrix} W_{SS} \\ W_{TT} \\ W_U S F_y \end{pmatrix} \right\| \leq 1$$

Based on ‘extended system’



Optimal controller:

- observer+linear feedback  
from estimated states

# Lecture 7/9 – $H_\infty$ / $H_2$ -optimal control

$H_2$ -optimal

$$\min_{F_y} \|G_{ec}\|_2^2 = \min_{F_y} \int_{-\infty}^{\infty} \sum_i \sigma_j^2(G_{ec}(i\omega)) d\omega$$

$H_\infty$ -optimal

$$\min_{F_y} \|G_{ec}\|_\infty = \min_{F_y} \sup_{\omega} \bar{\sigma}(G_{ec}(i\omega))$$

Computed from state-space description of  $G_{ec}$

- Solution is observer + static feedback from observed states

# Question

How choose weight  $W_u$  in objective  $\|W_u G_{wu}\|_\infty < 1$

- We have

$$u = G_{wu}(s)w$$

- If system properly scaled so that  $|w| < 1$ ,  $|u| < 1 \forall \omega$ , then we require

$$\bar{\sigma}(G_{wu}(i\omega)) < 1 \quad \forall \omega \quad \Rightarrow \quad \|G_{wu}\|_\infty < 1$$

- i.e., weight is  $W_u = I$
- More generally
  - increase  $|W_u(i\omega)|$  at frequencies where you want to use less input
  - e.g., if initial  $u(t)$  too large, increase the weight at higher frequencies
  - to limit derivative of  $u(t)$ , use  $W'_u = sW_u$

# Question

- How do we choose suitable weights in  $\mathcal{H}_2$ -optimal design?
  - in the same way as in  $\mathcal{H}_\infty$ -optimal design?
  - is aim still to achieve e.g.,  $\|W_T T\|_2 \leq 1$  ?

The main difference between  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  is that, in the latter case, the aim is to push down the peak value of  $\bar{\sigma}(W_T T(i\omega))$  while in the former case the aim is to minimize the area under  $\bar{\sigma}(W_T T(i\omega))$

Thus,  $\|W_T T\|_2 \leq 1$  does not have a simple interpretation. Rather

- choose weight large at frequencies where you want T small
- if resulting T is too large in some frequency range, increase weight in that range and redo design

# Lecture 8 – Linear Quadratic Control

Objective: minimize the cost

$$J = \mathbb{E} \left\{ \lim_{T \rightarrow \infty} \int_0^T [z^T Q_1 z + u^T Q_2 u] dt \right\}$$

Optimal solution:

- State feedback and linear observer (separation principle)
- Feedback and observer gains by solving Riccati equations
- LQG controller can have very poor robustness margins
- Some equivalence to  $\mathcal{H}_2$ - optimal control

# Lecture 10 – Robust loop shaping

Glover Mc-Farlane: first perform classic loop shaping and then add robustifying controller in a second step

Solves the problem: find controller that stabilizes

$$G(s) = (M(s) + \Delta_M(s))^{-1}(N(s) + \Delta_N(s))$$

for all uncertainties

$$\|\Delta_M(s) \Delta_N(s)\|_\infty \leq \epsilon$$

Dynamic controller of high order (plant+nominal controller): model reduction based on balanced realization

# Lecture 11 – Comparing Design Methods, Case Study

# Lecture 12 – Model Predictive Control

Structured way of dealing with control and state constraints

All based on discrete time models!

1. Predict how state evolves (as function of future controls)

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u_j$$

2. Determine optimal control by minimizing LQ criterion w constraints

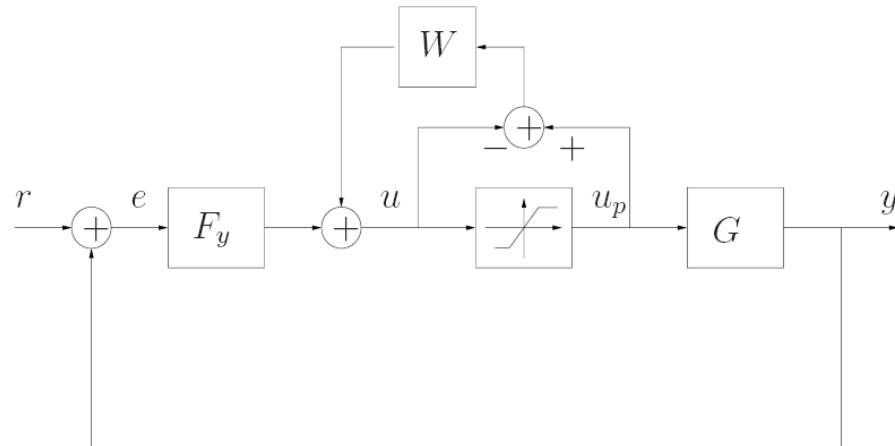
$$\begin{aligned} \text{minimize} \quad & \sum_{i=0}^{N_p-1} [(x_i - x_i^{\text{ref}})^T Q_1 (x_i - x_i^{\text{ref}}) + (u_i - u_i^{\text{ref}})^T Q_2 (u_i - u_i^{\text{ref}})] \\ & + (x_{N_p} - x_{N_p}^{\text{ref}})^T S (x_{N_p} - x_{N_p}^{\text{ref}}) \\ \text{subject to} \quad & u_{\min} \leq u_i \leq u_{\max} \\ & y_{\min} \leq C x_i \leq y_{\max} \end{aligned}$$

3. Implement first control, return to 1 at next sampling instant

Can be solved via efficient optimization (quadratic programming)

# Lecture 13 – Dealing with hard constraints

Classical approach to deal with input constraints: input tracking, i.e., use feedback from difference between computed control and saturated control (Anti Reset Windup)



# Preparations for Exam

- Old exams with answers (usually not complete solutions) in Canvas
- An old exam will be covered in the final exercise (tomorrow Friday)
- A discussion group will be opened in Canvas to answer questions that may come up before the exam. I will do my best to answer all questions, but all of you feel free to post answers!
- Remember to motivate all your answers on the exam!