EL2520
Control Theory and Practice

Lecture 12:
Model predictive control

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Background

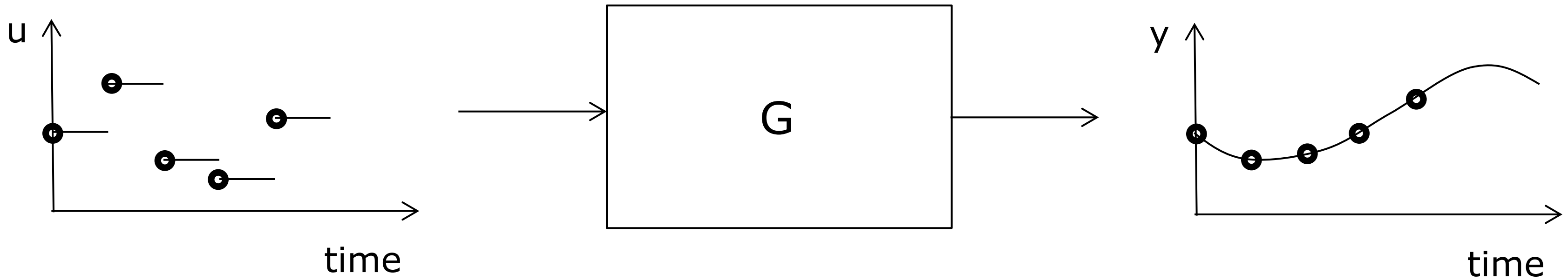
- Models have predictive power, i.e., may be used to predict future
- Idea: optimize future behavior using control input, e.g., using quadratic objective function for future outputs and inputs
- Main problem: model uncertain and future disturbances unknown $\Rightarrow$ introduce feedback by regularly updating model states based on measurements and then repeating the optimization
- Method is known as **receding horizon control**, or more commonly as **Model Predictive Control (MPC)**
- A key point is that hard constraints can be included in the optimization
- MPC is based on discrete time models

Outline

- Sampling and discrete time systems
- The Finite horizon LQR problem
- Adding constraints: the MPC controller
- Comments on tuning and an example

See also lecture notes for Lecture 12!

Computer-controlled systems



- Input to continuous time system G changed at discrete times, kept constant between time instants
- Continuous output sampled every h seconds

How does state evolve between sampling instances?

Plant dynamics at sampling instants

Recall that

$$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow x(t+h) = e^{Ah}x(t) + \int_{s=0}^{h} e^{As}Bu(t+s)\,ds$$

so if u is held constant during sample interval $u(t) = u_t,\ t \in [t, t+h)$

$$x(t+h) = A_D x(t) + B_D u_t \qquad \left(A_D = e^{Ah},\ B_D = \int_{s=0}^{h} e^{As}B\,ds \right)$$

$$y(t) = Cx(t) + Du_t$$

A discrete-time linear system!

Discrete-time linear systems

For notational convenience, we drop reference to physical time and write

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Du_k
\end{aligned}
$$

where

- $\{u_0, u_1, \ldots\}$ is an **input sequence**
- $\{y_0, y_1, \ldots\}$ is the **output sequence**
- $\{x_0, x_1, \ldots\}$ is the **state evolution**

System is stable if all eigenvalues of A are less than one in magnitude

Discrete-time linear systems

Some system theory for discrete-time linear systems (Book Ch. 2.6, 3.7, 4)

System is controllable if $S(A,B) = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$ is full rank.

System is observable if

$$O(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank

Observer-based controllers have the form

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + Bu_t + K(y_t - C\hat{x}_t) \\ u_t &= -L\hat{x}_t \end{aligned}$$

Finite-horizon LQR problem

Find control sequence

$$U = \{u_0, \ldots, u_{N-1}\}$$

that minimizes the quadratic cost function

$$J(U) = \sum_{k=0}^{N-1} (x_k^T Q_1 x_k + u_k^T Q_2 u_k) + x_N^T Q_f x_N$$

for given state cost, control cost, and final cost matrices

$$Q_1 = Q_1^T \geq 0, \quad Q_2 = Q_2^T > 0, \quad Q_f = Q_f^T \geq 0,$$

N is called the **horizon** of the problem.

Note the final state cost: mainly used to ensure stability

Finite-time LQR via least-squares

Note that $X = (x_0, \ldots, x_N)$ is a linear function of x_0 and $U = (u_0, \ldots, u_{N-1})$

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & 0 & \cdots \\ \vdots & \vdots & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} + \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x_0$$

Can express as

$$X = GU + Hx_0$$

where $G \in \mathbb{R}^{Nn \times Nm}$, $H \in \mathbb{R}^{Nn \times n}$

Finite-time LQR via least-squares

Can express finite-horizon cost as

$$J(U) = X^T \underbrace{\begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & Q_1 & 0 \\ 0 & \cdots & 0 & Q_f \end{bmatrix}}_{\overline{Q}_1} X + U^T \underbrace{\begin{bmatrix} Q_2 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & Q_2 & 0 \\ 0 & \cdots & 0 & Q_2 \end{bmatrix}}_{\overline{Q}_2} U =$$

$$= (GU + Hx_0)^T \overline{Q}_1 (GU + Hx_0) + U^T \overline{Q}_2 U =$$

$$= U^T (G^T \overline{Q}_1 G + \overline{Q}_2) U + 2x_0^T H^T \overline{Q}_1 GU + x_0^T H^T \overline{Q}_1 H x_0 =$$

$$:= U^T P_{LQ} U + 2q_{LQ}^T U + r_{LQ}$$

so optimal control is

$$U^\star = -P_{LQ}^{-1} q_{LQ}$$

for which

$$J(U^\star) = r_{LQ} - q_{LQ}^T P_{LQ}^{-1} q_{LQ}$$

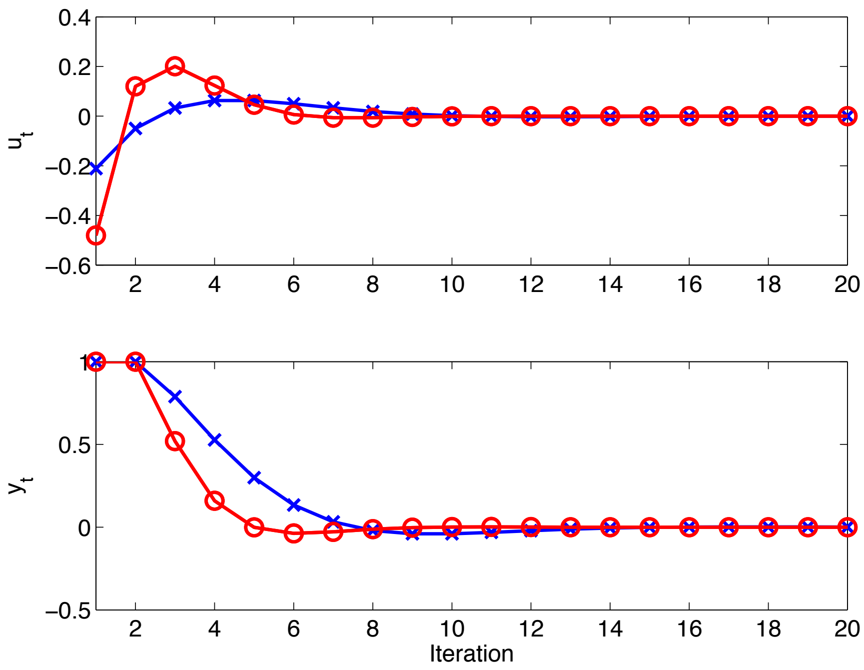
Example

LQR problem for system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t$$

$$Q_1 = Q_f = C^T C, \ \ Q_2 = \rho$$

with horizon length 20. Results for $\rho = 10$ (blue) and $\rho = 1$ (red)



Constrained Predictive Control

Finite-horizon LQR with hard constraints on u and y:

$$\begin{array}{ll}
\text{minimize} & J(U) = \sum_{k=0}^{N-1} (x_k^T Q_1 x_k + u_k^T Q_2 u_k) + x_N^T Q_f x_N \\
\text{subject to} & u_{\min} \leq u_k \quad \leq u_{\max}, \quad k = 0, \ldots, N-1 \\
& y_{\min} \leq C x_k \leq y_{\max}, \quad k = 1, \ldots, N \\
& x_{k+1} = A x_k + B u_k
\end{array}$$

Can be simplified by eliminating $\{x_1, \ldots, x_N\}$ (as above)

- results in a quadratic programming problem in $\{u_0, \ldots, u_{N-1}\}$

Quadratic programming (QP)

Minimizing a quadratic objective function subject to linear constraints

$$\begin{array}{ll} \text{minimize} & u^T P u + 2q^T u + r \\ \text{subject to} & Au \leq b \end{array}$$

Any u satisfying $Au \leq b$ is said to be **feasible.**

- clearly, not all quadratic programs are feasible (depends on A, b; more about this later…)

"Easy" to solve when objective function is **convex** (P positive semidefinite)

- optimal solution found in polynomial time
- commercial solvers deal with 10,000's of variables in a few seconds

Constrained control via QP

$$\begin{array}{ll} \text{minimize} & J(U) = \sum_{k=0}^{N-1} (x_k^T Q_1 x_k + u_k^T Q_2 u_k) + x_N^T Q_f x_N \\ \text{subject to} & u_{\min} \leq u_k \quad \leq u_{\max}, \quad k = 0, \ldots, N-1 \\ & y_{\min} \leq C x_k \leq y_{\max}, \quad k = 1, \ldots, N \\ & x_{k+1} = A x_k + B u_k \end{array}$$

As above, introducing $X = (x_0, \ldots, x_N),\ U = (u_0, \ldots, u_{N-1})$,

$$X = GU + Hx_0$$

and the objective function can be written as

$$J(U) = U^T P_{LQ} U + 2q_{LQ}^T U + r_{LQ}$$

Convex if $Q_2 \succ 0$ (implies that P_{LQ} is positive semi-definite)

What about the constraints?

Quadratic programming tricks

Example. The double inequality $u_{\min} \leq u \leq u_{\max}$ can be written as

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} u \leq \begin{bmatrix} u_{\max} \\ -u_{\min} \end{bmatrix}$$

Example. The equality $u = u_{\mathrm{tgt}}$ can be written as $u_{\mathrm{tgt}} \leq u \leq u_{\mathrm{tgt}}$, hence

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} u \leq \begin{bmatrix} u_{\mathrm{tgt}} \\ -u_{\mathrm{tgt}} \end{bmatrix}$$

Predictive control with constraints

The constraints $y_{\min} \le y_k \le y_{\max}, \;\; k = 0, \dots, N$ can be written as

$$Y \ge y_{\min}\mathbf{1}, \quad Y \le y_{\max}\mathbf{1}$$

where $Y = (y_0, \dots, y_N)$. Introducing

$$\overline{C} = \begin{bmatrix} C & 0 & \dots & 0 \\ 0 & C & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & C \end{bmatrix}$$

we can re-write these inequalities in terms of U via

$$Y \ge y_{\min}\mathbf{1} \Leftrightarrow \overline{C}(GU + Hx_0) \ge y_{\min}\mathbf{1} \Leftrightarrow \underbrace{\overline{C}G}_{A_{\underline{Y}}} U \ge \underbrace{y_{\min}\mathbf{1} - \overline{C}Hx_0}_{b_{\underline{Y}}}$$

$$Y \le y_{\max}\mathbf{1} \Leftrightarrow \overline{C}(GU + Hx_0) \le y_{\max}\mathbf{1} \Leftrightarrow \underbrace{\overline{C}G}_{A_{\overline{Y}}} U \le \underbrace{y_{\max}\mathbf{1} - \overline{C}Hx_0}_{b_{\overline{Y}}}$$

Predictive control with constraints

Hence, the constrained predictive control problem can be cast as a QP

$$
\begin{array}{ll}
\text{minimize} & U^T P_{LQ} U + 2 q_{LQ}^T U + r_{LQ} \\
\text{subject to} & \begin{bmatrix} A_{\overline{Y}} \\ -A_{\underline{Y}} \\ I \\ -I \end{bmatrix} U \leq \begin{bmatrix} b_{\overline{Y}} \\ -b_{\underline{Y}} \\ u_{\max}\mathbf{1} \\ -u_{\min}\mathbf{1} \end{bmatrix}
\end{array}
$$

Solution gives optimal finite-horizon control subject to constraints

Model predictive control:

- apply constrained optimal control in receding horizon fashion

Model predictive control algorithm

1. Given state at time t compute ("predict") future states

   $$x_{t+k}, \qquad k = 0, 1, \ldots, N$$

   as function of future control inputs

   $$u_{t+k}, \qquad k = 0, 1, \ldots, N-1$$

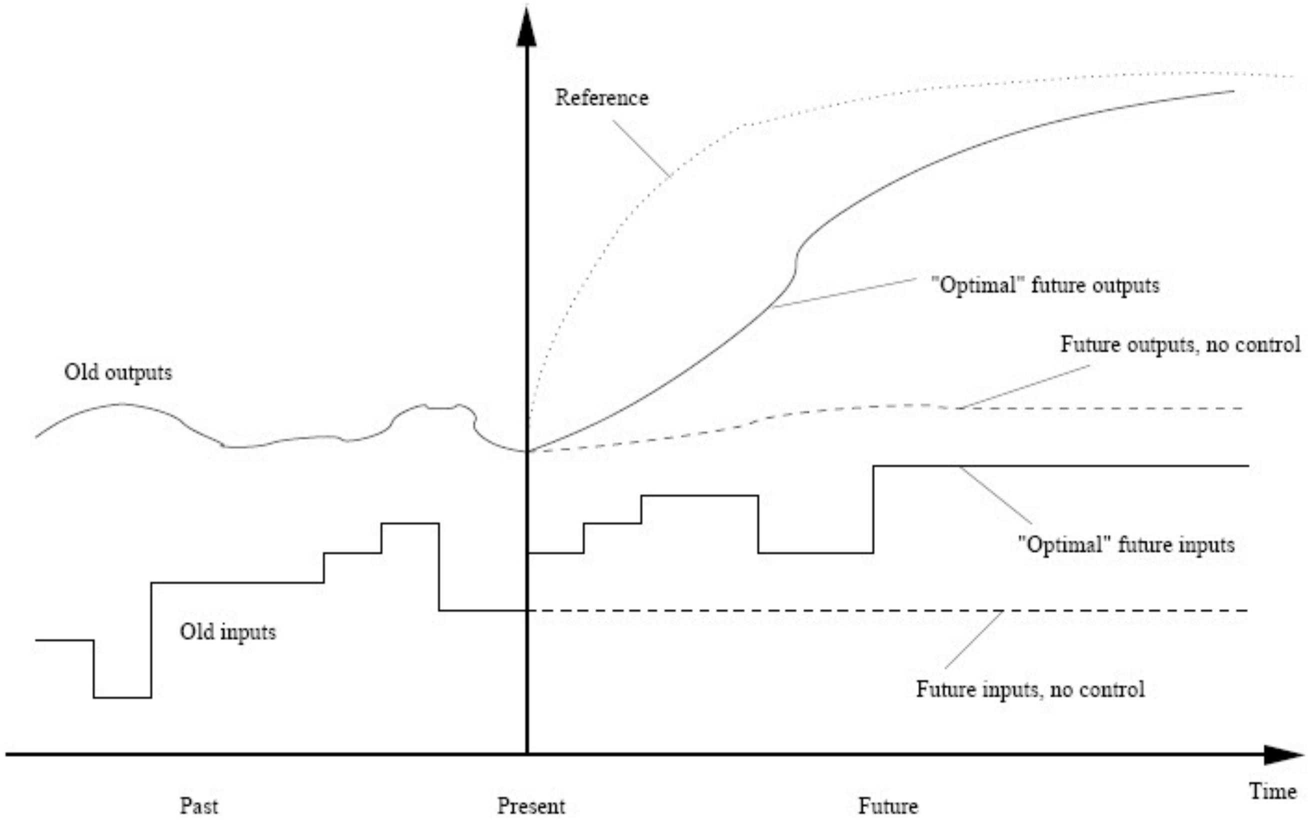
2. Find "optimal" input by minimizing constrained cost function
   - a quadratic program, efficiently solved

3. Implement u(t)

4. A next sample (t+1), return to 1.

A key is that the initial state is updated by an observer (Kalman filter) at each time step, thereby providing feedback from measurements

MPC trajectories



Example: the DC servo

Discrete-time model (sampling time 0.05 sec)

$$A = \begin{bmatrix} 1 & 0.139 \\ 0 & 0.861 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0107 \\ 0.139 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Constrained input voltage

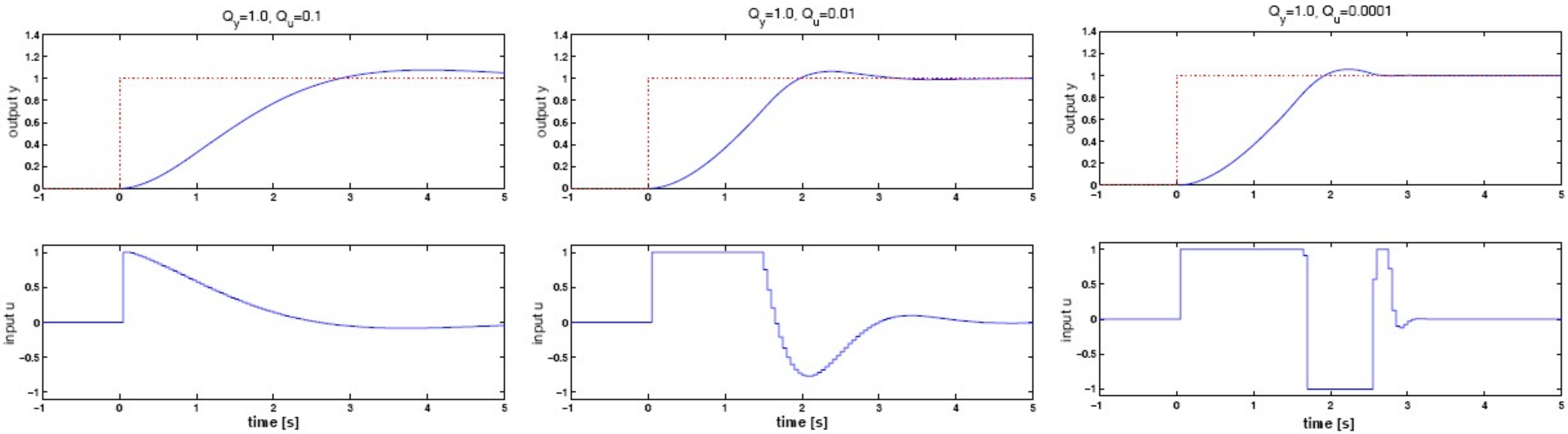
$$-1 \leq u \leq 1$$

Constrained position

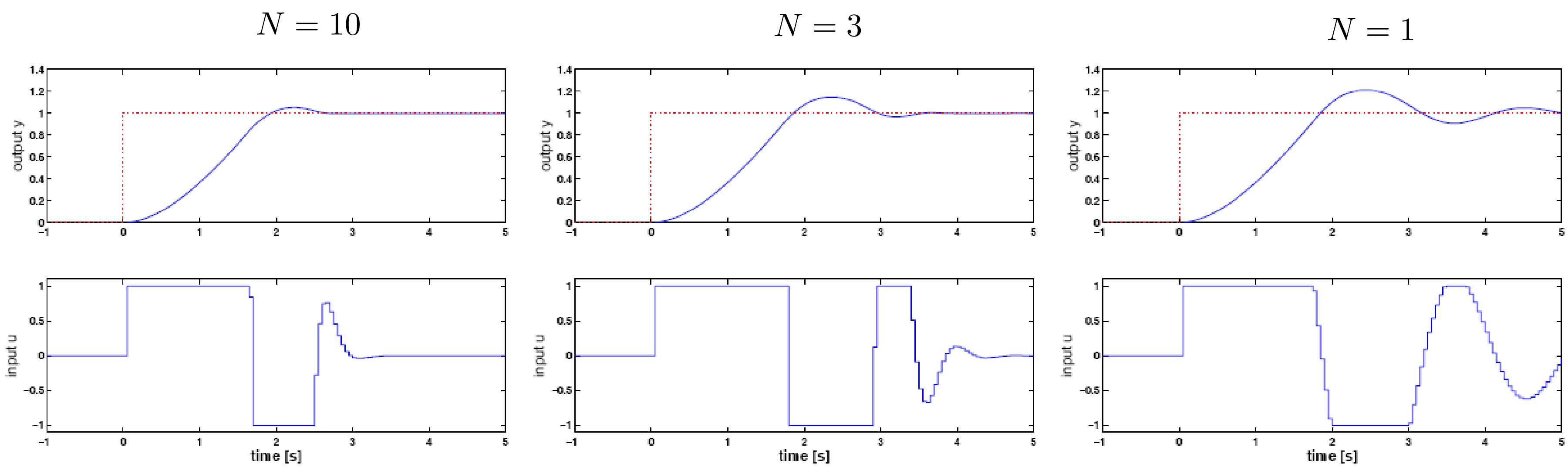
$$y_{\text{min}} \leq y_k \leq y_{\text{max}}$$

Impact of state and control weights

Prediction horizon N=10.

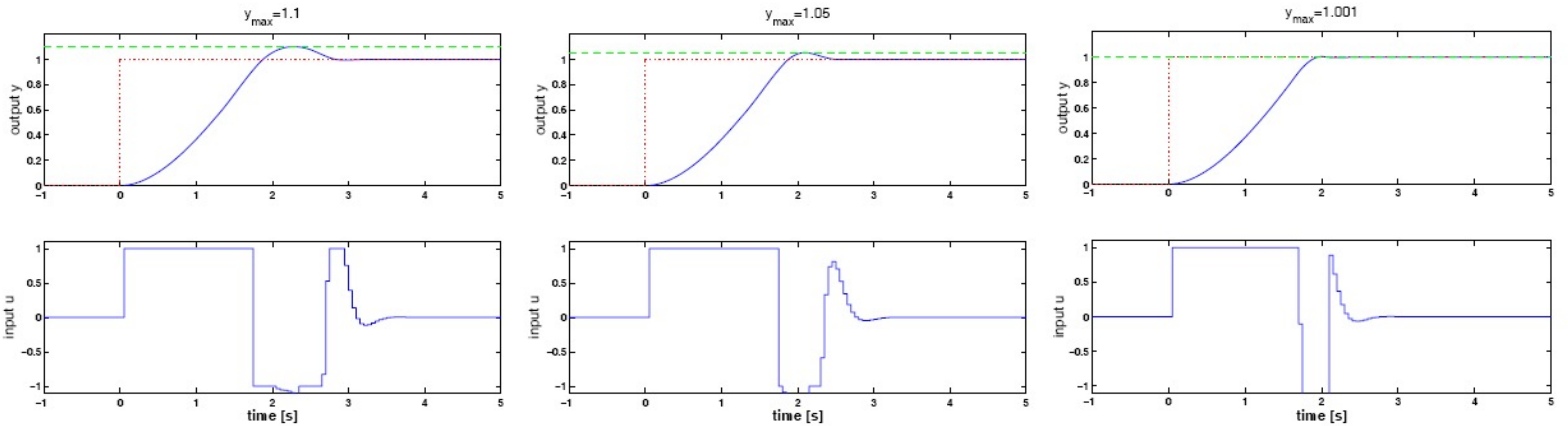


Impact of horizon



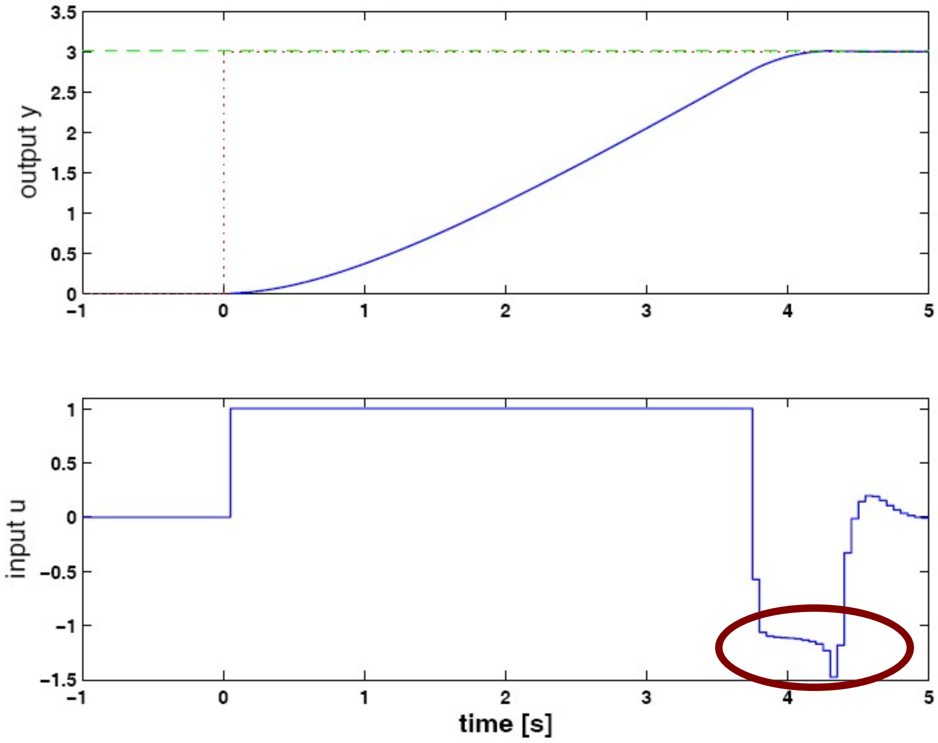
Too short horizon ➔ inaccurate predictions ➔ poor performance

Adding output constraints



Infeasibility

What happens when there is no solution to the QP?



Not clear what control to apply!

Ensuring feasibility

One way to ensure feasibility:

– introduce slack variables $s_{ck} \geq 0$

– "soften" constraints

$$u_k \leq u_{\max} \Rightarrow u_k \leq u_{\max} + s_{ck}$$

– add term in quadratic programming objective to minimize slacks

$$\underset{U}{\text{minimize}} \quad U^T P_{LQ} U + 2q_{LQ}^T U + r_{LQ}$$

$$\Downarrow$$

$$\underset{U,S}{\text{minimize}} \quad U^T P_{LQ} U + 2q_{LQ}^T U + r_{LQ} + \kappa S^T S$$

Notes:

– still QP, but more variables; can also use penalty κS (also QP)

– better to soften "physically soft" constraints (e.g. output constraints)

Reference tracking

Would like z to track a reference sequence $\{r_1, \ldots, r_N\}$, i.e. to keep

$$\sum_{k=0}^{N-1} \left((z_k - r_k)^T Q_1 (z_k - r_k) + u_k^T Q_2 u_k\right) + (z_N - r_N)^T Q_f (z_N - r_N)$$

small.

Problem: making $z_k = r_k$ typically requires $u_k \neq 0$

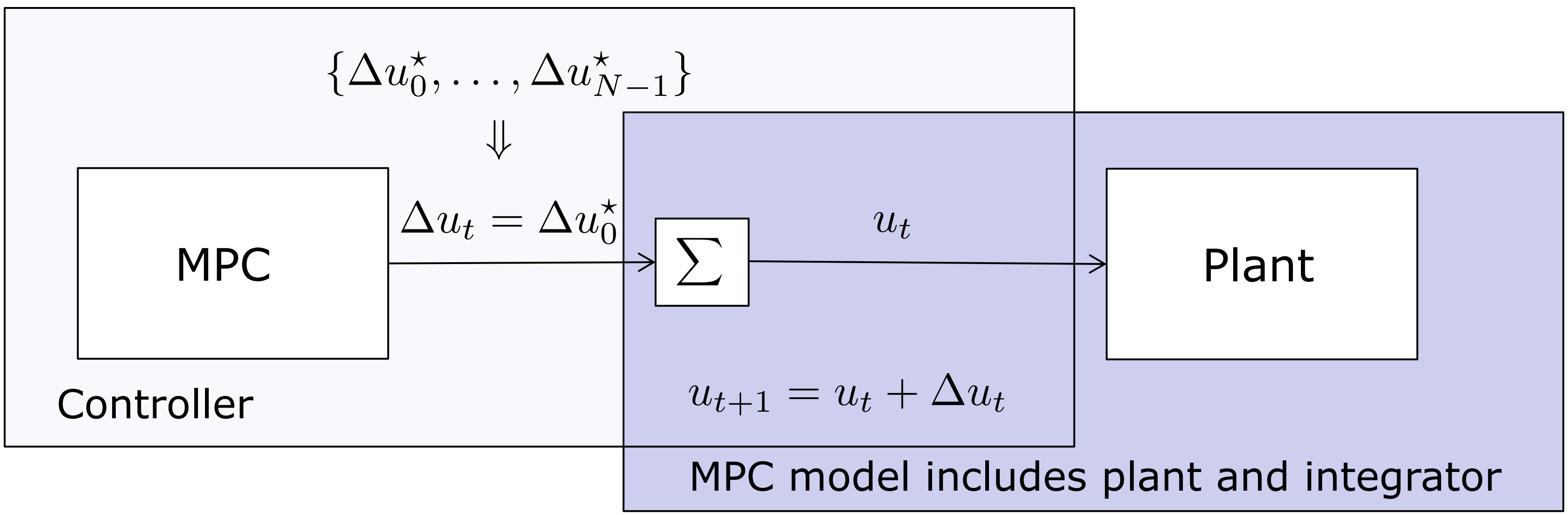
- a trade-off between zero tracking errors and using zero control
- often results in steady-state tracking error

DC motor simulations used MPC with integral action.

Including integral action

Integral action often included by a change in free variables

- Use $\Delta u_i = u_i - u_{i-1}$ as variables in the optimization
- Actual input obtained by summing up MPC outputs



Including integral action cont'd

Form augmented model with state $\bar{x}_k = (x_k,\ u_k)$ and input Δu_k:

$$\begin{bmatrix} x_{k+1} \\ u_{k+1} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta u_k$$

$$y_k = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

Consider finite-horizon cost

$$\sum_{k=0}^{N-1} \left((z_k - r_k)^T Q_1 (z_k - r_k) + \Delta u_k^T Q_2 \Delta u_k \right) + (z_N - r_N)^T Q_f (z_N - r_N)$$

Now, all terms can go to zero (at least when unconstrained, infinite horizon)

Apply control $u_t = u_{t-1} + \Delta u_{t-1}$

MPC controller tuning

MPC has a large number of “tuning” parameters.

The prediction model:

- we need to decide sampling interval
  (rule of thumb: sample 10 times desired closed-loop bandwidth)
- obtain discrete-time state-space model

Finite-horizon optimal control:

- set prediction horizon
  (rule of thumb: equal to closed-loop rise time; could be smaller)
- decide weight matrices (as for continuous-time LQG)
- decide final state penalty

MPC controller tuning

Finite-horizon optimal control, advanced:

- control horizon
  (try to set small, rule-of-thumb: use 1-10)
- inner-loop control
  (guideline: stationary LQR controller for given weight matrices)

Constraints and feasibility

- specify control and state constraints (problem dependent)
- introduce slacks to “soften” constraints
- choose constraint penalty (large value on kappa)

Integral action (almost always a good idea to include).

Advanced issues: stability

Receding horizon control might yield unstable closed-loop

Stability can be guaranteed:

- for infinite-horizon unconstrained case (this is LQR)
- for finite-horizon unconstrained case
  - if final state is penalized correctly
  - if final state is enforced to lie in a given set
- for constrained finite-horizon
  - if final state enforced to lie in a sufficiently small set **and**
  - initial QP (solved at time zero) is feasible

Hard to verify for sure in advance…

Advanced issues: robustness

Consider the unconstrained quadratic program

$$\text{minimize} \quad u^T Q u + 2q^T u$$

has optimal solution $u = -Q^{-1}q$

In the MPC setting, Q and q depend on the system model (matrices A, B, C), weights Q_1, Q_2, and also horizons.

Solution is sensitive to uncertainties if Q is ill-conditioned

- try scaling inputs and outputs in the model
- modify weight matrices Q_1 and Q_2
- almost always a good idea to include integral action

Advanced issues: observers

MPC, as presented here, assumes full state feedback.

In many cases, we will need to use an observer,

- to reconstruct states, and/or
- to filter out noise

Limited theory, but separation principle holds in some cases.

Suggests guideline

- design observer as for (unconstrained, infinite-horizon) LQG
- use estimated state in MPC calculations as if it was true state

Course on MPC

EL2700 Model Predictive Control, 7.5cr, is given in period 1.

Summary

Model predictive control (MPC)

- can handle state and control constraints
- predictive control computed via quadratic programming

Many parameters and their influence on the control

- System model, weights, horizons, constraints, …

Advanced issues:

- Feasibility and slacks to “soften” constraints
- Integral action
- Different prediction and control horizons
- Stability and the terminal weight
- The need for a state observer