EL2520
Control Theory and Practice

Signal Norms and System Gain
The Small Gain Theorem

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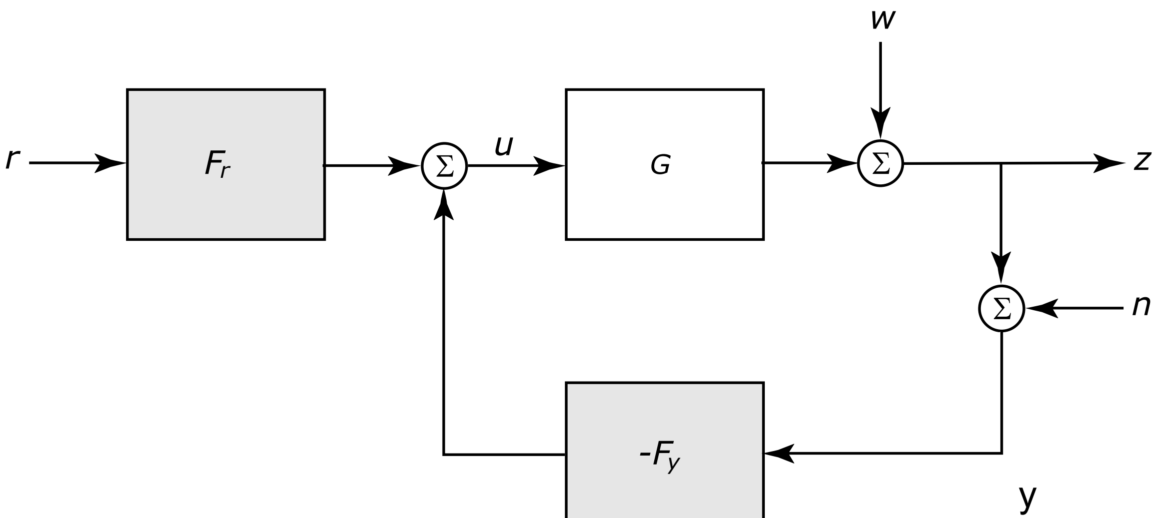
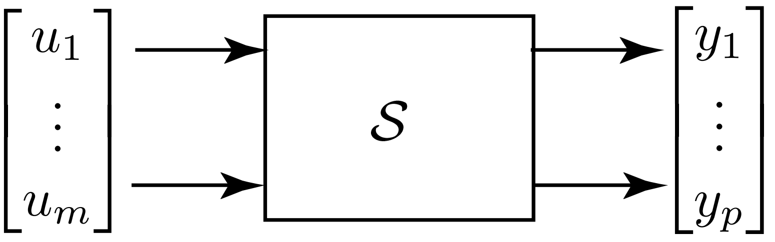
Multivariable feedback control

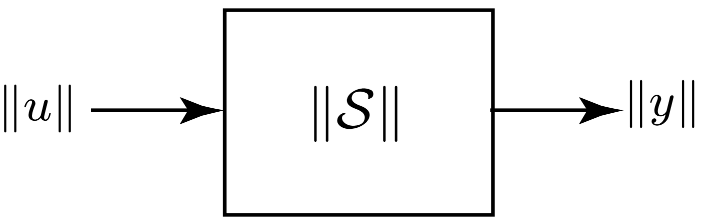


Multi-Input-Multi-Output (MIMO):

- all signals are vectors, all transfer-functions are matrices
- need scalar measures of signal size and system amplification

Systems as ”mapping of signals”





Key components:

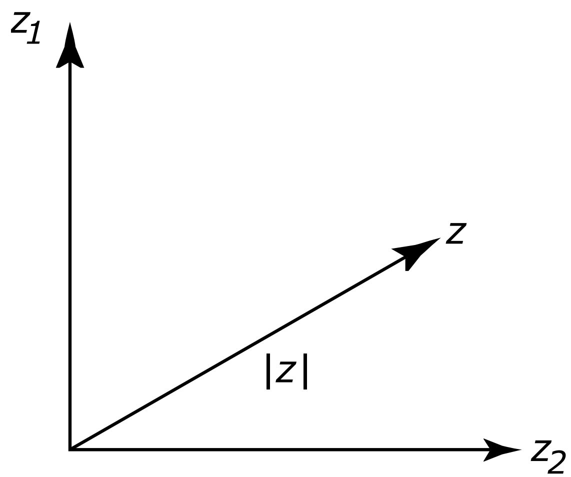
- Signal norm: quantifies ”size” of signals
- System gain: quantifies system’s (maximum) amplification

- Admits natural extensions from scalar to multivariable systems
  - frequency domain analysis and design for MIMO systems
- Enables analysis of fundamental performance limitations
- Enables analysis of robustness

Vector norms

Vector (spatial) norms measure the "size" of vectors.

– common choice: Euclidian norm (also known as 2-norm)

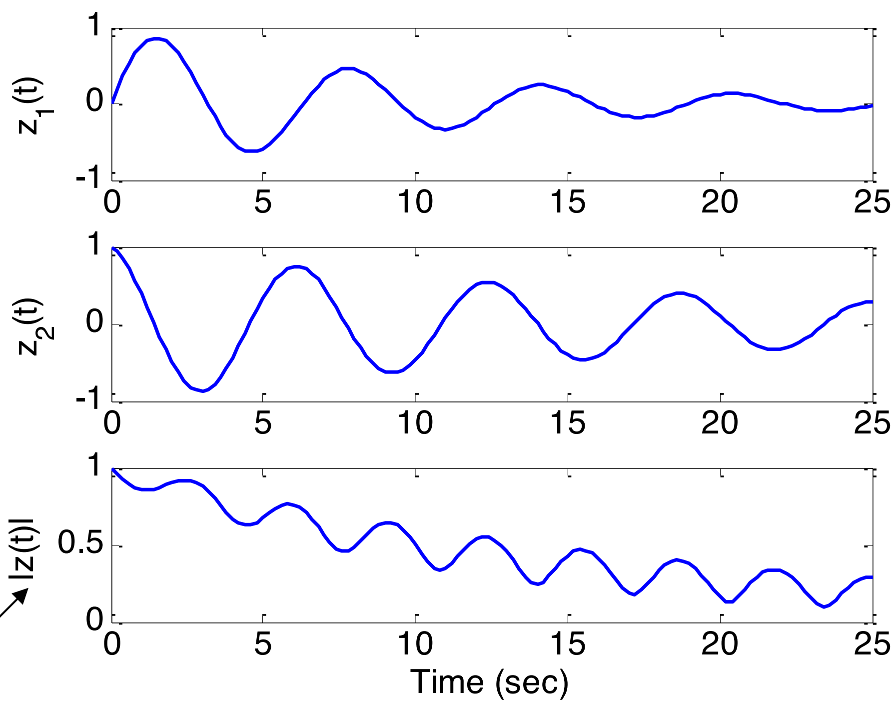


$$|z|^2 = \sum_i z_i^2 = z^T z$$

Signal norms

Signals are functions of time (or frequency)

– signal norms measure size across both space and time.



$\Rightarrow ||z||$

summing up over channels

summing up over time

Signal norms

The *peak-norm*, or L_∞ -norm, of a signal is defined as

$$\|z\|_\infty = \sup_{t\geq 0} |z(t)|$$

A signal is *bounded* if its peak-norm is bounded ($\|z\|_\infty < \infty$)

The *energy-norm*, or L_2-norm, of a signal is defined as

$$\|z\|_2 = \sqrt{\int_{-\infty}^{\infty} |z(t)|^2 \, dt}$$

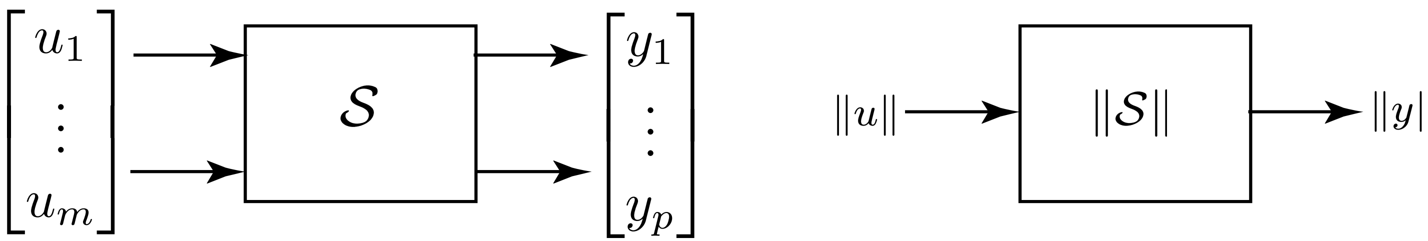
A signal is *finite-energy* if $\|z\|_2 < \infty$

Note: - bounded signals may have infinite energy (and vice versa)

- we will only work with the 2-norm in this course

The energy-gain of a system

Measures maximum "energy amplification" of system



The amplification for a specific signal $u \neq 0$ is given by

$$\frac{\|y\|_2}{\|u\|_2} = \frac{\|\mathcal{S}u\|_2}{\|u\|_2}$$

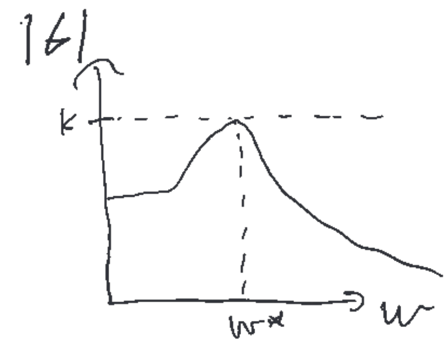
The system gain is the maximum amplification (over all finite-energy signals)

$$\|\mathcal{S}\| = \sup_{0<\|u\|_2<\infty} \frac{\|\mathcal{S}u\|_2}{\|u\|_2}$$

Energy gain of scalar linear systems

Stable scalar linear time-invariant system $\mathcal{S} : Y(s) = G(s)U(s)$

Assume that $|G(i\omega)| \leq K$ with equality for $\omega = \omega^\star$



Then, Parseval's theorem yields

$$\|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|^2 |U(i\omega)|^2 d\omega \leq K^2 \|u\|_2^2$$

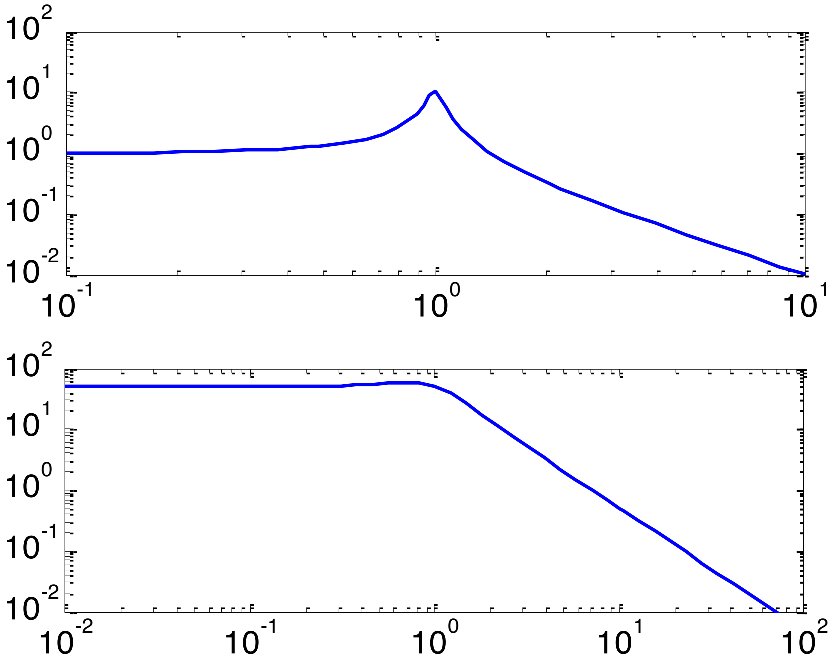
Since equality holds for $u(t) = \sin(\omega^\star t)$, we have

$$\|\mathcal{S}\| = \sup_{\omega} |G(i\omega)| = \|G\|_\infty$$

The energy-gain of scalar linear system is the $\mathcal{H}_\infty$-norm of the transfer-function G

Quiz 1: energy gains and Bode diagrams

Quiz: the Bode amplitude diagrams below represent two different linear time-invariant systems. Which one has the largest energy-gain?



Quiz 2: H_infinity norm

- Determine the $\mathcal{H}_\infty$- norm of the systems

$$G_1(s) = \frac{2}{2s+1} \qquad G_2(s) = \frac{2s}{s+1}$$

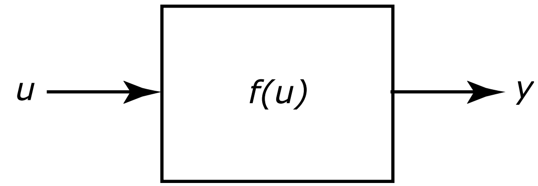
Example: gain of nonlinear system

Static nonlinear system $\mathcal{S} : y(t) = f(u(t))$

where

$$|f(x)| \leq K|x|$$

with equality for $x = x^\star$



Since

$$\|y\|_2^2 = \int_{-\infty}^{\infty} |f(u(t))|^2 \, dt \leq \int_{-\infty}^{\infty} K^2 |u(t)|^2 \, dt = K^2 \|u\|_2^2$$

the energy gain is

$$\|\mathcal{S}\| = \sup_u \frac{\|y\|_2}{\|u\|_2} = K$$

Example: gain of static linear systems

Consider the static linear system $y = Au$ with gain

$$\|A\| = \sup_{u \neq 0} \frac{|Au|}{|u|}$$

Since

$$\|A\|_2^2 = \sup_{u \neq 0} \frac{|Au|^2}{|u|^2} = \sup_{u \neq 0} \frac{u^T A^T A u}{u^T u} = \lambda_{\max}(A^T A)$$

Thus, the gain is the square root of the maximum eigenvalue of $A^T A$.

- The square roots of eig($A^T A$) are called the *singular values* of A; more about this in Video 5

Quiz: a flavour of multivariable systems

Quiz: What is the gain of the following (static) systems?

a) $$y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u$$

b) $$y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u$$

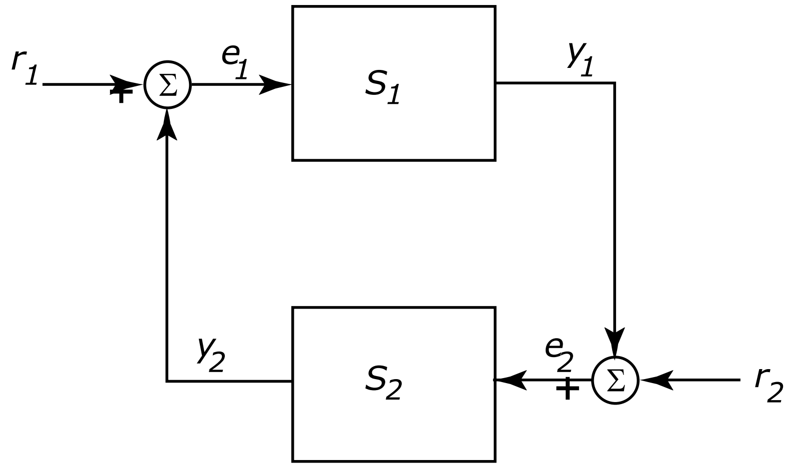
What are the corresponding "worst-case" input vectors?
(vectors u with |u|=1 that give the maximum value of |y|)

Input-output stability

Definition. A system $\mathcal{S}$ is *input-output stable* if $\|\mathcal{S}\| < \infty$

The Small Gain Theorem

Theorem. Consider the interconnection



If $\mathcal{S}_1$ and $\mathcal{S}_2$ are input-output stable and the loop gain

$$\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$$

Then, the closed-loop system with r_1, r_2 as inputs and e_1, e_2, y_1, y_2 as outputs is input-output stable.

Proof sketch

$$e_1 = r_1 + \mathcal{S}_2(r_2 + y_1)$$

$$y_1 = \mathcal{S}_1 e_1$$

$$\|e_1\| \leq \|r_1\| + \|\mathcal{S}_2\|(\|r_2\| + \|\mathcal{S}_1\| \cdot \|e_1\|)$$

$$\|e_1\| \leq \frac{\|r_1\| + \|\mathcal{S}_2\| \cdot \|r_2\|}{1 - \|\mathcal{S}_2\| \cdot \|\mathcal{S}_1\|}$$

Hence, the gain from r_1, r_2 to e_1 is finite.

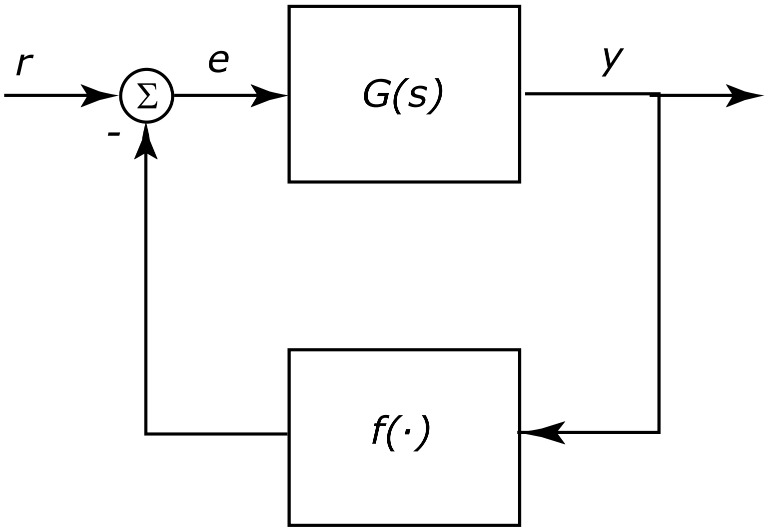
A similar argument proves that the gain from r_1, r_2 to e_2 is finite

We will present a rigorous proof later in the course (based on Nyquist)

Note: for linear system, it is sufficient that $\|\mathcal{S}_1 \mathcal{S}_2\| \leq 1$

Quiz: a nonlinear interconnection

Is the feedback interconnection



with $G(s) = \dfrac{0.4}{s+1}$ and $|f(y)| \leq 2|y|$ input-output stable?

Summary

- Systems as mappings of signals
- Norm
  - vector norms: measure size of vector “across channels”
  - signal norms: measure size of signal across time and space
- Gain
  - the maximum amplification of bounded energy signals
  - for stable linear systems, gain is the infinity norm of G
- Input-output stability and the small gain theorem