8 EL2520 Lecture notes 8: LQG - Linear Quadratic Optimal Control

In lecture 7, we showed that we could solve the problem of minimizing the $\mathcal{H}_\infty$-norm of weighted transfer-functions, e.g., $\min_{F_y} \|W_S S\|_\infty$, by solving a corresponding signal minimization problem

$$\min_{Fy} \sup_{w} \frac{\|z_e\|_2}{\|w\|_2}$$

in state space, where z_e and w are the output and input, respectively, of the system we want to minimize the $\mathcal{H}_\infty$-norm of. The resulting controller was on the form of state feedback combined with an observer. The classical optimal control theory from the 1960's is also based on solving a signal minimization problem in state space, but then with an objective function formulated in the time-domain and using a stochastic framework for the input signals. Below we present the main results related to this theory, the LQG-controller which also can be written as state feedback combined with an observer (the Kalman filter).

8.1 The LQG control problem

We consider a system on state space form

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) + Nv_1(t) \\ y(t) &= Cx(t) + v_2(t) \\ z(t) &= Mx(t) \end{aligned} \tag{1}$$

The aim of the control is, similar to before, to keep $z(t)$ small in the presence of the disturbance v_1 and measurement noise v_2. Recall that in H_∞-optimal control we considered the worst-case disturbances. In the LQG framework, we rather consider stochastic disturbances and noise. Assume that v_1 and v_2 are Gaussian (normally distributed) white noise with covariance matrices[1]

$$E\{v_1 v_1^T\} = R_1 \ , \quad E\{v_2 v_2^T\} = R_2$$

The optimal control problem we aim to solve, given these disturbances[2], is

$$\min_u E\{\lim_{T\to\infty} \frac{1}{T} \int_0^T z^T Q_1 z + u^T Q_2 u dt\} \tag{2}$$

The name LQG comes from Linear system, Quadratic cost and Gaussian disturbances.

[1] For a discussion on stochastic descriptions and modeling of disturbances, see course book and lecture slides for Lecture 8.

[2] A more general problem formulation allows for cross correlation between v_1 and v_2, but that is not considered here.

An important property of the solution to (2) is the *Separation Principle* which states that, provided (A, B) in (1) is stabilizable[3], (A, C) is detectable[4], and (A, R_1) stabilizable and $(A, M^T Q_1 M)$ detectable[5], the optimal control problem can be split into two subproblems

1. Optimal state feedback (LQ)
2. Optimal observer (Kalman filter)

and that these two sub-problems can be solved independently. The combined solutions is then the optimal solution to the overall LQG problem.

8.2 The LQ Problem

The LQ-problem is a deterministic control problem with no stochastic disturbances and assuming all states of the system are known (measured without noise)

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \ ; \quad x(0) = x_0 \\ z(t) &= Mx(t) \end{aligned} \tag{3}$$

The control problem we aim to solve is

$$\min_u \lim_{T \to \infty} \frac{1}{T} \int_0^T z^T Q_1 z + u^T Q_2 z dt$$

Thus, minimizing the square objective function for any initial condition x_0. The solution, which we state without proof here, is

$$u(t) = -Lx(t) \ , \quad L = Q_2^{-1} B^T P$$

and where $P \geq 0$ solves the algebraic Riccati equation

$$A^T P + PA + M^T Q_1 M - PBQ_2^{-1} B^T P = 0$$

Note that the design variables are the weights Q_1 and Q_2 that can be used to trade-off between outputs and inputs.

Example: Consider a simple first-order system

$$\dot{x} = ax(t) + u(t) \ ; \quad z(t) = x(t)$$

and performance objective

$$J = \int_0^\infty x^2 + \rho u^2 dt$$

The Riccati equation is

$$2ap + 1 - \frac{1}{\rho} p^2 = 0 \quad \Rightarrow \quad p = a\rho \pm \sqrt{(\rho a)^2 + \rho}$$

[3] All unstable states controllable

[4] All unstable states observable

[5] The last two conditions ensure a unique positive definite solution to the Riccati equations below

Thus, the optimal feedback is

$$u(t) = -\frac{p}{\rho}x(t) = -(a + \sqrt{a^2 + \frac{1}{\rho}})x(t)$$

and the closed-loop becomes

$$\dot{x} = -\sqrt{a^2 + \frac{1}{\rho}}x(t)$$

Note that if the weight $\rho \to \infty$, i.e., all weight is put on the control input u, then $u(t) = 0 \cdot x(t)$ if the system is open-loop stable ($a < 0$) while $u(t) = -2ax(t)$ if the system is open-loop unstable ($a > 0$), in which case the pole of the system is mirrored into the LHP by the feedback.

8.3 The Optimal Observer

Given that we in general do not measure all states, and furthemore, that the measurements are contaminated by measurement noise, we need to estimate the states with an observer. The system considered is

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) + Nv_1(t)\ , \quad E\{v_1 v_1^T\} = R_1 \\ y(t) &= Cx(t) + v_2(t)\ , \quad E\{v_2 v_2^T\} = R_2 \end{aligned} \tag{4}$$

To estimate the state $x(t)$ we construct an observer

$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + K_f(y(t) - C\hat{x}(t)) \tag{5}$$

where $\hat{x}$ denotes the estimated state. The design parameter is the observer gain K_f. The optimal observer is the observer which minimizes the variance of the estimation error

$$\min_{K_f} E\{(x - \hat{x})^T(x - \hat{x})\}$$

The optimal observer gain is then given by

$$K_f = PC^T R_2^{-1}$$

where $P \geq 0$ solves the Riccati equation

$$PA^T + AP - PC^T R_2^{-1} CP + NR_1N^T = 0$$

The optimal observer is called *the Kalman filter.*

Example: Consider the system

$$\dot{x} = ax(t) + u(t) + v_1(t)\ , \quad y(t) = x(t) + v_2(t)$$

where the disturbance and measurement noise variances are given by

$$E\{v_1^2\} = R_1\ ; \quad E\{v_2^2\} = R_2$$

The Riccati equation and corresponding gain for the optimal observer is

$$2ap - p^2/R_2 + R_1 = 0 \quad \Rightarrow \quad k_f = a + \sqrt{a^2 + R_1/R_2}$$

Thus, if $a < 0$ and $R_2 >> R_1$ then $k_f = 0$, i.e., disregard measurement if there is large variance in the noise v_2 relative to the variance of the process disturbance v_1. The equation for the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$ is

$$\dot{\tilde{x}} = -\sqrt{a^2 + R_1/R_2}\tilde{x}(t)$$

Thus, we get faster convergence with increasing R_1/R_2.

8.4 The Optimal LQG Controller

As stated above, the solution to the LQG problem fulfills the separation principle which means that the overall solution is a combination of the two separate problems of optimal control without any disturbance or noise (LQ) and the optimal observer without any control (Kalman). Thus, the LQG controller is given by

$$\begin{aligned} u(t) &= -L\hat{x}(t) \\ \dot{\hat{x}} &= A\hat{x}(t) + Bu(t) + K_f(y - C\hat{x}(t)) \end{aligned} \tag{6}$$

where the state controller feedback gain L is the LQ-optimal gain and the observer gain K_f is the Kalman gain. The structure of the LQG controller is illustrated in Figure 8.1. The tuning parameters of the LQG controller are the control weights Q_1, Q_2 and the

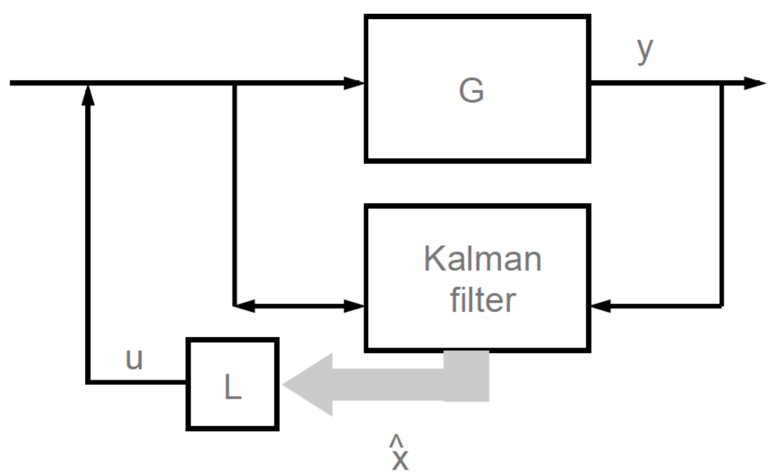

Figure 8.1: Structure of the LQG controller.

disturbance covariances R_1, R_2.

Advantages with LQG is partly that it is directly applicable to MIMO systems (which was one of the main reasons for its development in the 1950s-1960s), partly that it is relatively easy to make a trade-off between keeping the control error small on one hand and keeping input usage small on the other hand by choosing appropriate relative values of the weights Q_1 and Q_2. However, it may in general be difficult to see a direct connection between the choice of the tuning parameters Q_1, Q_2, R_1, R_2 and the real control objectives. In practice, this implies that one usually has to iterate untill finding an acceptable solution. A systematic method known as loop transfer recovery exists, but is not covered in this course. We will rather, in Lecture 9, show that LQG can be used as a machinery to shape closed-loop transfer-functions using H_2-optimal control, in which case Q_1, Q_2, R_1, R_2 are given by the objective function formulated in the input-output space.

One main disadvantage with LQG is that the only uncertainty that can be modelled is signal uncertainty, in the form of v_1 and v_2, and signals do not affect system properties like stability. Hence, it is not possible to explicitly adress robust stability with LQG. In fact, it has been shown that the LQG controller typically will have poor robustness margins and this was one of the main motivations for the introduction of robust control, or $\mathcal{H}_\infty$-optimal control, in the 1980s and 90s. We will compare the different control design methods in Lecture 11.