2 EL2520 Lecture notes 2: The Closed-Loop System

Consider the two-degree of freedom feedback control system in Figure 2.1. As discussed in Lecture 1, the aim of the control system is to make the output z follow the reference r, in the presence of disturbances on the input w_u, disturbances on the output w and measurement noise n (and model uncertainty Δ_G).

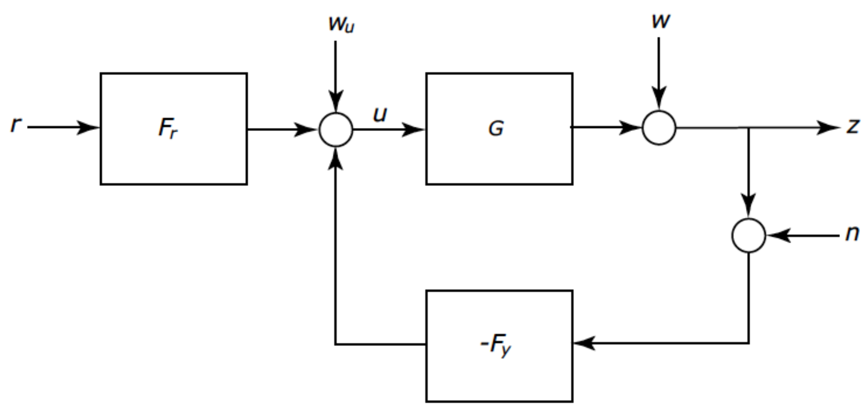


Figure 2.1: Feedback control system.

In classical control, dealing with SISO systems only, the control problem is typically solved in the frequency domain, i.e., based on the frequency response $G(i\omega)$. A standard approach to controller design is then to specify properties such as the closed-loop bandwidth and maximum peak of the closed-loop transfer-function G_c from r to z. To simplify the design, the specifications are then translated into specifications on the loop transfer-function $L = GF_y$, such as crossover frequency and phase margin, and these specifications are then met by *loop shaping* using the controller F_y (also known as lead-lag compensation). Robustness, i.e., stability in the presence of model uncertainty, is adressed through the phase- and amplitude margins in the classical framework.

In this course, we will introduce what is known as modern control[1]. This approach is also based on formulating and analyzing the control problem in the frequency domain. However, to accomodate for MIMO systems, specifications are instead based on norms of signals and/or systems. The use of norms will, as we shall see, also enable us to quantify limitations for what can achieved with feedback for a given system. For instance, if there is a time delay in the system, what is the maximum bandwidth that can be achieved even with the best possible controller? In addition, the use of norms in combination with the

[1]Sometimes modern control refers to the optimal control theory developed in the 1960s, using a state space approach to control, and then the control theory introduced here, developed mainly in the 1980s-90s, is referred to as postmodern. We will in this course also consider optimal control theory from the 60s, and then in the form of LQG optimal control in Lecture 8.

small gain theorem will enable us to quantify robust stability is a much more rigorous way than with the classical stability margins. Most importantly, we can extend the analysis of robust stability to MIMO systems for which robustness typically is a much more critical issue than for SISO systems.

We will start out by introducing the modern approach to control for SISO systems, and later extend the results to MIMO systems. Thus, in Lecture 2-4 we limit ourselves to systems with scalar inputs and outputs, i.e., SISO systems. In Lecture 5-13 we will deal with the more general MIMO problem.

The aim of control is, as stated above, to keep signals like the control error $e = r - z$ and the control input u small in some (norm) sense. This again implies making the transfer-functions from the external inputs w, w_u, r, n to the outputs e, u small in some (norm) sense. To derive the relevant closed-loop transfer-functions we use simple block-diagram algebra, i.e., we write up the relations in the block-diagram of Fig.1 and then solve for the variables of interest

$$z = w + G(w_u + F_r r - F_y(z + n))$$

Solving for z gives

$$z = \underbrace{\frac{1}{1 + GF_y}}_{S} w + \underbrace{\frac{G}{1 + GF_y}}_{SG} w_u + \underbrace{\frac{GF_r}{1 + GF_y}}_{G_c} r - \underbrace{\frac{GF_y}{1 + GF_y}}_{T} n \tag{1}$$

Here S is called the *Sensitivity function*, T is called the *Complementary Sensitivity function* and G_c is called the *Closed-Loop transfer function.* Note that

$$S + T = \frac{1}{1 + GF_y} + \frac{GF_y}{1 + GF_y} = 1$$

From (1) we see that we should make $|S(i\omega)|$ small for disturbance attenuation, $|T(i\omega)|$ small for noise attenuation and $|1 - G_c(i\omega)|$ small for setpoint following. Since $S + T = 1$, we see that we can not make both $|S|$ and $|T|$ small at the same frequency and hence we have to make a trade-off between attenuation of disturbances w, w_u on one hand and amplification of measurement noise n on the other hand.

The closed-loop transfer-functions for the control input u are derived in the same manner

$$u = w_u + F_r r - F_y(n + w + Gu) \quad \Rightarrow \quad u = \underbrace{\frac{1}{1 + GF_y}}_{S} w_u + \underbrace{\frac{F_r}{1 + GF_y}}_{SF_r} r - \underbrace{\frac{F_y}{1 + GF_y}}_{SF_y} (n + w)$$

Thus, to keep u small we need to make $|S|, |SF_r|$ and $|SF_y|$ small. We will below discuss how to define more precisely what we mean by "small" transfer-functions. First, we will introduce the concept of *internal stability* since closed-loop stability is a pre-requisite for discussing anything related to control performance.

2.1 Internal Stability

A system is said to be *internally stable* if it is input-output stable from any external input to any output. For a SISO system, the outputs are u and z and the external inputs are r, w, w_u, n and we hence require the corresponding transfer-functions $S, SG, G_c, T, SF_r, SF_y, F_r$ to all be stable, i.e., have all poles strictly in the complex LHP. Note that we include also the feedforward controller F_r since it is outside the feedback loop and hence has to be stable. Thus, in principle we must check the poles of seven different transfer-functions to analyze internal stability. However, we can reduce the number to four transfer-functions since stability of certain transfer-functions implies stability of others. In particular,

- If S is stable, then so is T since $S + T = 1$.
- If S and F_r stable, then so is SF_r.
- If SG and F_r stable, then so is G_c.

Thus, it suffices to check stability of the transfer-functions S, SG, SF_y, F_r (also known as the Gang of Four).

Example: Consider the following control system

$$G = \frac{1}{s-1}\,, \quad F_y = F_r = \frac{s-1}{s}$$

which gives

$$S = \frac{s}{s+1}\,, \quad SG = \frac{s}{(s+1)(s-1)}\,, \quad SF_y = \frac{s-1}{s+1}\,, \quad F_r = \frac{s-1}{s}$$

and the closed-loop system is unstable since SG has a pole in the complex RHP.

Note that in the example above the controller has a zero at $s = 1$ which cancels the pole in G at $s = 1$. It is this cancellation that causes instability in this case. In fact, one should never cancel RHP poles and RHP zeros between the controller and plant since there always will be signals that can enter between the controller and plant which then implies instability from these signals. In the example above we have instability in SG, i.e., from w_u to z.

Internal stability as defined above is often also termed *nominal stability* since it only concerns stability of the closed-loop with the nominal model G. This in contrast to *robust stability* which concerns stability in the presence of model uncertainty, or model errors. Nominal stability, i.e., stability without uncertainty, is obviously a prerequisite for robust stability.

2.2 The Sensitivity Function

As seen from (1), the sensitivity function S is the transfer-function from disturbances on the output w to the output z. Without any feedback, i.e., $F_y = 0$, this transfer-function is simply 1.

– without feedback $z = w$

– with feedback $z = Sw$

Thus, feedback provides disturbance attenuation at frequencies where $|S(i\omega)| < 1$, while it amplifies disturbances at frequencies where $|S(i\omega)| > 1$. Ideally, considering only disturbance attenuation, we would like to achieve $|S(i\omega)| = 0 \ \forall\omega$. However, there are several reasons why this is not possible. One simple reason, due to a result by Bode, is that any system for which the loop transfer-function $L = GF_y$ is stable and has a pole excess of a least 2 (at least two more poles than zeros, which is true for any real system) must have $|S(i\omega)| > 1$ at some frequency. This can easily be understood by considering the Nyquist plot of L shown in Figure 2.2. With a pole excess ≥ 2, L will have a phase-lag of $-\pi$ at some frequency which implies that the Nyquist curve must approach and possibly cross the negative real axis at some frequency and eventually end up in $|L| = 0$ at $\omega = \infty$. Furthermore, $L(i\omega)$ will have phase between $-\pi/2$ and $-\pi$ in some frequency range, implying that L must pass through the 3rd quadrant in the complex plane. According to the Nyquist criterion, L can not encircle -1 and hence L must pass through a unit circle centered at -1. Since $|1 + L|$ is the distance between L and the point -1 on the negative real axis, we must have $|1 + L| < 1$ at some frequency. Since $|S| = 1/|1 + L|$ it trivially follows that we must have $|S| > 1$ at some frequency. In fact, as we shall see in Lecture 4, any frequency for which $|S| < 1$ must be balanced by another frequency for which $|S| > 1$.

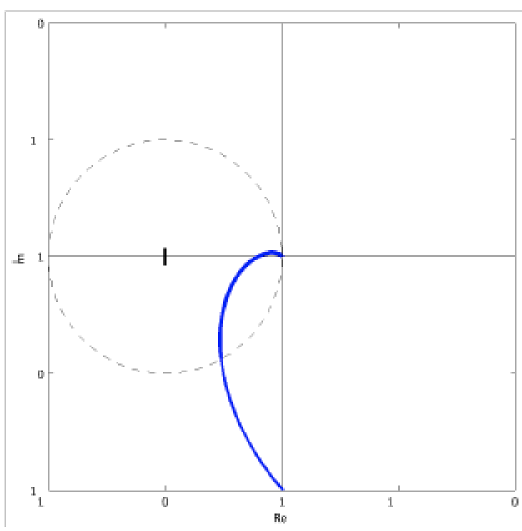


Figure 2.2: Nyquist plot, i.e., plot of loop gain $L(i\omega)$ as a function of frequency $\omega \in [0, \infty]$ in the complex plane. The dashed line is the unit circle with center in -1.

In general, to keep $|S(i\omega)| < M_S \ \ \forall\omega$ implies that the loop gain $L(i\omega)$ must be kept outside a circle centered at -1 and with radius M_S^{-1}. Typically, one allows maximum peak of the sensitivity $M_s = 2$ which implies avoiding that $L(i\omega)$ enters a circle centered at -1 and with a radius $M_S^{-1} = 0.5$

From the above, it is clear that we can not make the sensitivity small at all frequencies, and that we also need to have a peak in $|S|$ larger than one at some frequency. For control design, we then need to determine in what frequency ranges we want to make $|S|$ small and also what peak values we allow. A convenient way to do this is to define a frequency dependent bound[2] $|W_S^{-1}(i\omega)|$ that $|S|$ should stay below

$$|S(i\omega)| < |W_S^{-1}(i\omega)| \ \forall\omega \quad \Leftrightarrow \quad \|W_S S\|_\infty \leq 1$$

See also Figure 2.3. The second inequality follows from the fact the the $\|\cdot\|_\infty$-norm is

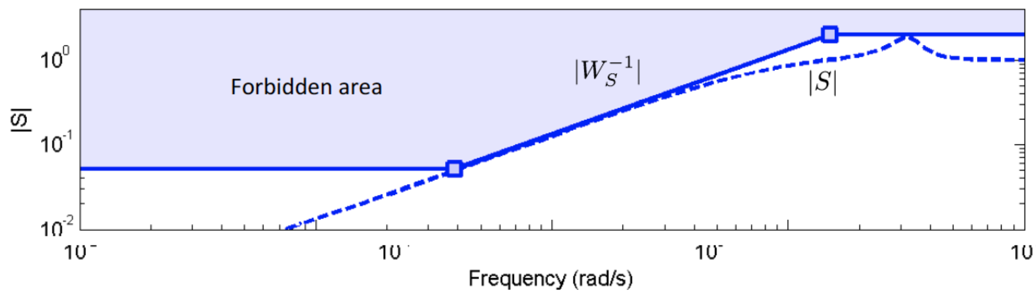


Figure 2.3: Plot of amplitude of sensitivity function S and bound W_S^{-1} as functions of frequency.

equal to the peak value of the amplitude evaluated over all frequencies (see Lecture 1). Thus, we have formulated our design objective for the sensitivity function in terms of the infinity-norm of the weighted sensitivity sensitivity $W_S S$ where W_S is the weight that we specify. Note that we now have the performance measure in the form of a single number $\|W_S S\|_\infty$ which makes it suitable for solving using optimization techniques. That is, we can determine the feedback controller F_y based on solving an optimization problem in which we minimize $\|W_S S\|_\infty$. We will return to such formulations later.

The origin of the name *sensitivity function* is in fact not related to the disturbance sensitivity discussed above. Rather, it is due to Bode and reflects the fact that the sensitivity function quantifies the relative sensitivity of the closed-loop transfer-function G_c to changes in the model G. Taking the derivative

$$\frac{dG_c}{dG} = \frac{F_r}{(1+GF_y)^2} = S\frac{F_r}{1+GF_y} = S\frac{G_c}{G}$$

and thus the relative sensitivity is

$$\frac{dG_c/G_c}{dG/G} = S$$

Considering the relative uncertainty description introduced in Lecture 1, we get

$$\tilde{G} = G(1+\Delta_G) \quad \Rightarrow \quad \tilde{G}_c = G_c(1+S\Delta_G)$$

where $\tilde{G}$ is the true system, G is the model and Δ_G is the relative model error. Thus, the feedback changes the relative error in the closed-loop relative to the open-loop by a factor equal to the sensitivity function S.

[2]We define the bound as the inverse of W_S since W_S serves as a weight on S in $\|W_S S\|_\infty$. We hence call W_S the performance weight, or weight on the sensitivity.

2.3 The Complementary Sensitivity Function

While the sensitivity function S reflects the ability of a feedback loop to attenuate disturbances and suppress the impact of model uncertainty on the closed-loop transfer-function G_c, the complementary sensitivity function T quantifies the amplification of measurement noise as well as the robust stability of a feedback loop.

From (1) we see that the complementary sensitivity function T is the closed-loop transfer-function from measurement noise n to the output z. Thus, to avoid amplifying noise in the loop we should make $|T(i\omega)|$ small at frequencies where noise is large. In fact, $T = 0$ without any feedback, so the fact that we feed measurement noise into the plant is a cost of employing feedback control. Note again that making $|T|$ small implies that we make $|S| \approx 1$, since $S + T = 1$, and hence a trade-off has to be made. More about this below.

Apart from reflecting the amplification of noise in a feedback loop, the complementary sensitivity function also has an important role in quantifying robust stability, that is, stability in the presence of model uncertainty. In particular, consider again the relative uncertainty description introduced in Lecture 1

$$\tilde{G}(s) = G(s)(1 + \Delta_G(s)) \tag{2}$$

where $\tilde{G}$ is the true system (unknown), G is the nominal model and $\Delta_G(s)$ is the relative model uncertainty. Given a controller which yields nominal stability, i.e., closed-loop stable with model G, the question of robust stability is then how large the uncertainty Δ_G can become without the closed-loop becoming unstable. In the basic control course (EL1000), this problem was solved using the Nyquist plot for the loop gain (assuming open-loop stability). Here we will instead approach the problem using the small gain theorem, mainly because we can then easily extend the results to MIMO systems and also to alternative uncertainty descriptions later (and we do not need to assume open-loop stability).

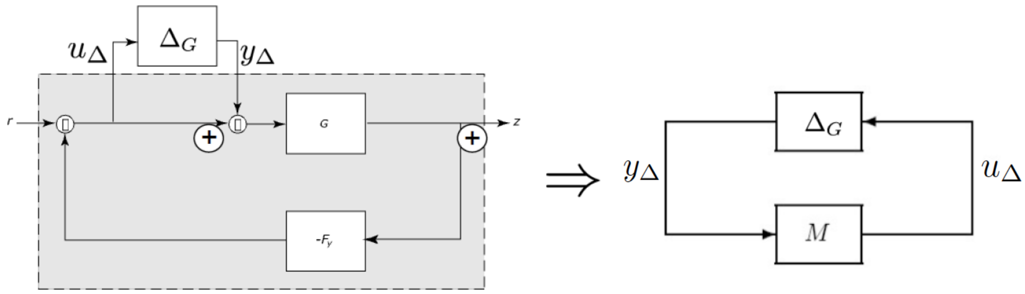
Consider the feedback loop in Figure 2.4, where we have included the uncertainty as given by (2). The question we would like to answer is what linear $\Delta_G(s)$ can be tolerated without the loop loosing stability? For this purpose we rewrite the block-diagram such that we isolate the uncertainty from the nominal system. See Figure 2.4. This then results in a feedback loop with two blocks; the uncertainty $\Delta_G(s)$ and the nominal transfer-function from the output y_Δ to the input u_Δ of Δ_G such that

$$u_\Delta = M y_\Delta$$

Using simple block-diagram algebra we identify

$$u_\Delta = -GF_y(y_\Delta + u_\Delta) \quad \Leftrightarrow \quad u_\Delta = \underbrace{\frac{GF_y}{1 + GF_y}}_{M} y_\Delta$$

That is, the block $M = T$, i.e., the complementary sensitivity function. Now, from the Small Gain Theorem we know that the M-Δ_G-loop in Figure 2.4 is stable if $M(s)$ and $\Delta_G(s)$ are both stable and the loop-gain $\|M\Delta_G\|_\infty < 1$. That $M(s)$ is stable follows from nominal closed-loop stability, i.e., stability without uncertainty, and the requirement of



$$u_\Delta = Ty_\Delta = My_\Delta$$

Figure 2.4: Block diagram with uncertainty. Corresponds to M-Δ_G-loop where M is nominal transfer-function (no uncertainty) and Δ_G is the relative model uncertainty.

$\Delta_G(s)$ stable means we can not allow unstable uncertainty $\Delta_G(s)$ when using the Small Gain Theorem for analysis. The robust stability condition is then $T(s)$ and $\Delta_G(s)$ stable and

$$\|T\Delta_G\|_\infty < 1 \quad \Leftrightarrow \quad |T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|} \quad \forall\omega \tag{3}$$

Thus, we need to make the complementary sensitivity $|T|$ small for frequencies where the uncertainty $|\Delta_G(i\omega)|$ is large.

Example: Consider the nominal system

$$G(s) = \frac{1}{\tau s + 1} \; ; \quad \tau = 3$$

and the PI-controller

$$F_y(s) = 3\frac{3s+1}{3s}$$

which yields

$$T(s) = \frac{1}{s+1}$$

Assume now that the time-constant τ of the true system can change between 3 and 1, i.e., $\tau \in [1, 3]$. We model this uncertainty as

$$\tilde{G}(s) = \frac{1}{s+1} = \frac{1}{3s+1}(1 + \Delta_G(s)) \quad \Rightarrow \quad \Delta_G(s) = \frac{2s}{s+1}$$

Figure 2.5 shows the plot of $|T(i\omega)|$ and $1/|\Delta_G(i\omega)|$, and we see that condition (3) is just satisfied, thus we have robust stability for the changes in the plant time-constant between 3 and 1. Note that in this simple example we could simply have computed the closed-loop poles for the different time-constants in G instead and would then have found that it was robustly stable with some margin. In fact, if we allowed for variations in the plant time-constant between 0 and 3, the poles would still be all in the LHP, i.e., the system would be robustly stable, but the robust stability condition (3) would not be satisfied anymore. The conservativeness would in this case partly be due to sufficiency only of the small gain theorem, and partly by the fact that the uncertainty model we use would cover

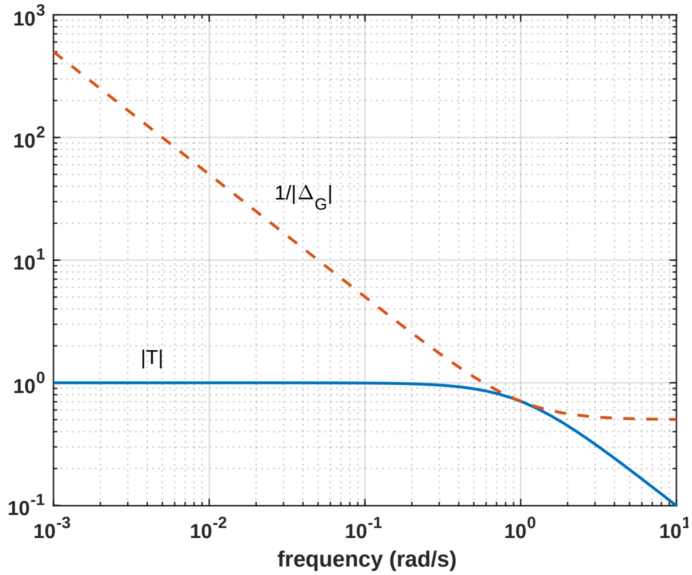


Figure 2.5: Plot of $|T(i\omega)|$ vs $1/|\Delta_G(i\omega)|$ for the example problem.

more plants than only those with time-constants between 0 and 3. More about this in Lecture 3.

Similar to what we did for the sensitivity function above, it is reasonable to define a frequency dependent bound $|W_T^{-1}(i\omega)|$ that $|T(i\omega)|$ should stay within. We then get

$$|T(i\omega)| \leq |W_T^{-1}(i\omega)| \ \forall\omega \quad \Leftrightarrow \quad \|W_T T\|_\infty \leq 1$$

Thus, by designing the weight $W_T(i\omega)$ we get a scalar measure $\|W_T T\|_\infty$ of the size of the complementary sensitivity T.

2.4 Sensitivity Shaping

Based on the above results we can see the control design problem as the problem of shaping the sensitivity functions S and T (and possibly other transfer-functions, such as those related to input usage) by designing the weights W_S and W_T and then e.g., leaving to an optimization algorithm to find a controller that minimizes the norms $\|W_S S\|_\infty$ and $\|W_T T\|_\infty$[3]. We will in the next few lectures focus on the design of the weights W_S and W_T, that is, formulating the control problem, and return to the problem of how to solve the resulting optimization problem later. Some simple observations are

- We should choose $|W_S|$ large in the frequency range where we want $|S|$ small. This is typically the frequency range where we need disturbance attenuation..

[3]Since we need a scalar objective function, we need to combine the two norms somehow.

- We should choose $|W_T|$ large in the frequency range where we want $|T|$ small. This is typically the frequency range where we have the most measurement noise and where the model uncertainty is the largest.

- Since $S + T = 1$, we can not choose $|W_S|$ and $|W_T|$ large in the same frequency range if we are to achieve the control objectives $\|W_S S\|_\infty \leq 1, \; \|W_T T\|_\infty \leq 1$.

See also Figure 2.6. In Lecture 4 we will discuss the trade-off discussed in the last point

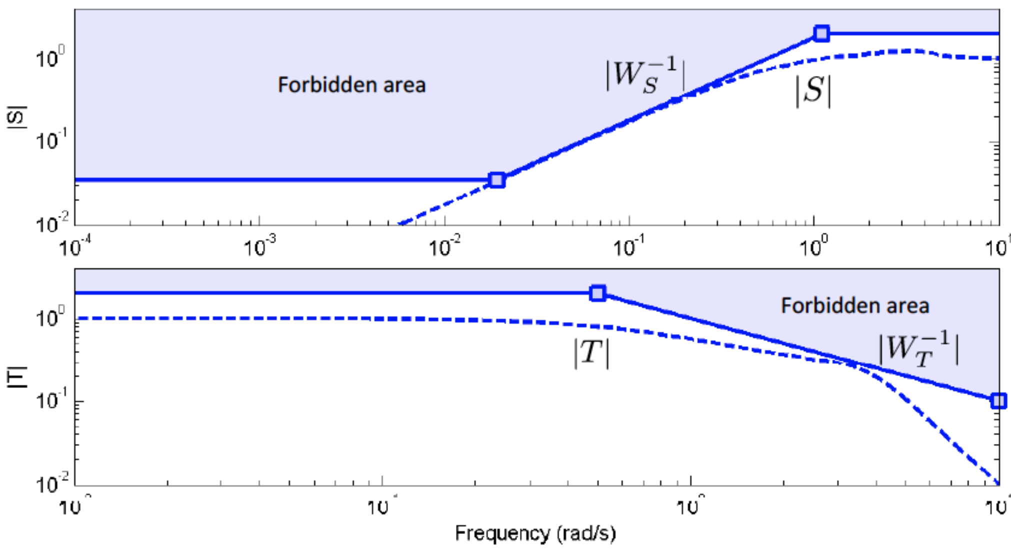


Figure 2.6: Shaping of sensitivity function S and complementary sensitivity function T.

above in more detail, and also other limitations for the design of the weights W_S and W_T. However, we will first discuss robustness in more detail in Lecture 3.