



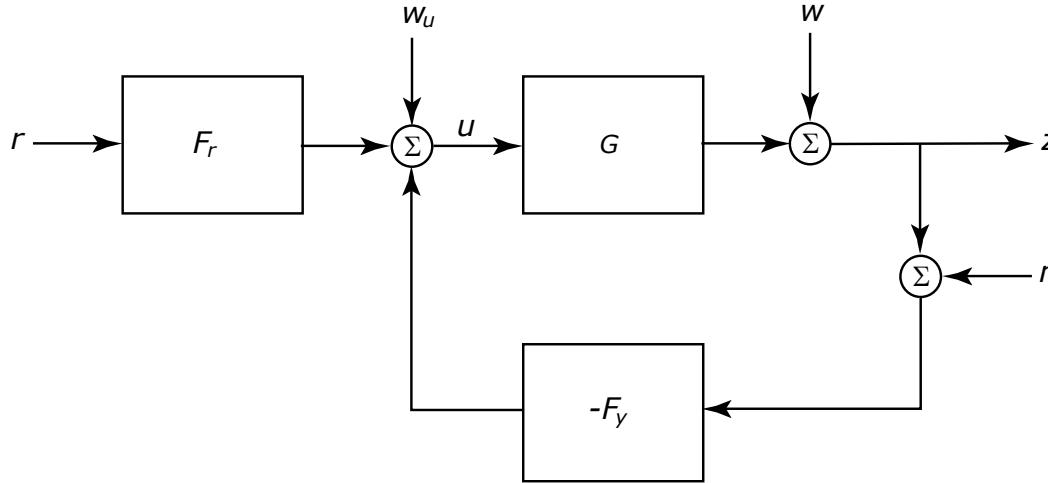
EL2520

Control Theory and Practice

Lecture 8: LQG

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Brief recap: Lecture 6 and 7



Aim: shape closed loop transfer-functions, e.g., S, T, G_{wu} to achieve desired system properties

How: introduce weights W_S, W_T, W_u and determine F_y, F_r such that

$$\|W_S S\|_\infty < 1 \quad \|W_T T\|_\infty < 1 \quad \|W_u G_{wu}\|_\infty < 1$$

where we assume $W_S = w_S I$ etc., i.e., scalar weights.

Controller Design – H_∞

Determine $F_y(s)$ to achieve $\|W_S S\|_\infty < 1$ $\|W_T T\|_\infty < 1$ $\|W_u G_{wu}\|_\infty < 1$

1. Translate closed-loop bounds into bounds on open-loop and use $F_y(s)$ to shape loop-gain $L = GF_y$, or

2. Synthesis, i.e., solve optimization problem

$$F_{y,opt}(s) = \arg \min_{F_y} \left\| \begin{array}{c} W_S S \\ W_T T \\ W_u G_{wu} \end{array} \right\|_\infty \quad (*)$$

Note that if the "stacked" objective above is less than 1, then we have achieved the three individual objectives (with some margin)

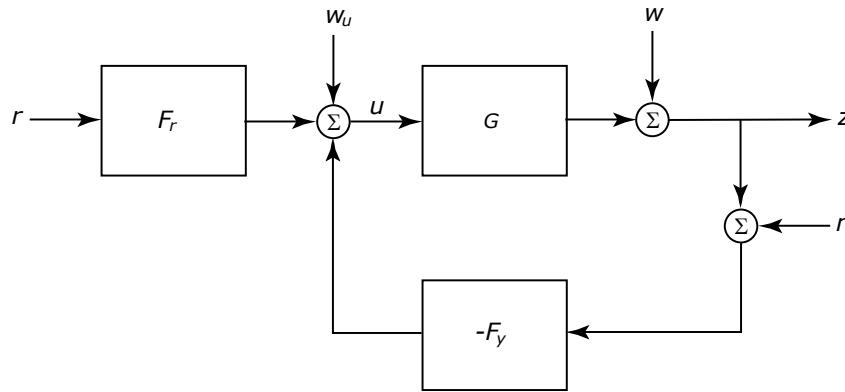
The Solution

- Observation: equivalence between signal norm minimization and system norm minimization
- Translate system norm minimization into signal norm minimization
- Solve signal norm minimization problem in state-space
- Resulting controller: state feedback + observer

Raison d'etre for H_∞

- Note that H_∞ in some sense is a worst case approach to control, i.e., we optimize with respect to worst direction and worst frequency
- The main reason is that requirements on robust stability naturally leads to H_∞ -bounds since we must guarantee stability even in the worst case
- The introduction of weights still makes it relevant to employ H_∞ also for performance

Quiz 1

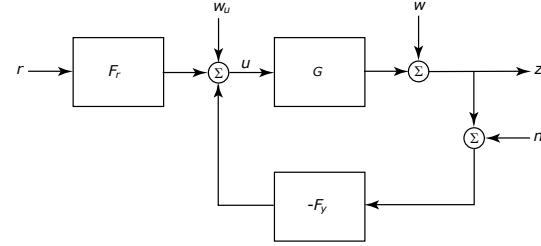


- Consider the signal minimization problem

$$\min_{F_y} \frac{\|z\|_2}{\|w\|_2}$$

- what is the corresponding system norm minimization?
- assume $G(s)$ stable, what is the resulting controller?

Quiz 2

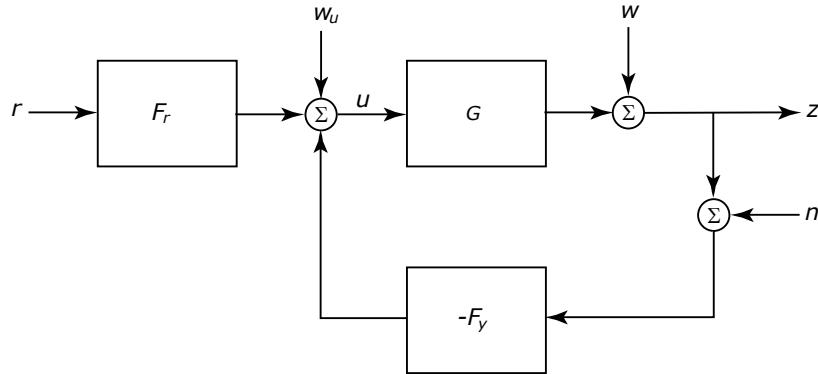


- Assume now that the disturbance at the output is $w = G_d(s)d$ and we consider the minimization

$$\min_{F_y} \frac{\|z\|_2}{\|d\|_2}$$

- what is the corresponding system norm minimization?
- if all variables are scaled to have size less than one, what is the performance requirement in terms of the system norm?

Quiz 3



- How can we include design of the prefilter F_r in \mathcal{H}_∞ -optimal design?

Today: LQG

- Formulate optimal control problem as signal minimization problem in state-space, using classical quadratic cost functions
- Result:
 - controller with similar structure as in H_∞ -optimal control, i.e., observer + state feedback
 - but, quadratic cost function does not allow explicit robustness consideration; robustness must be dealt with indirectly

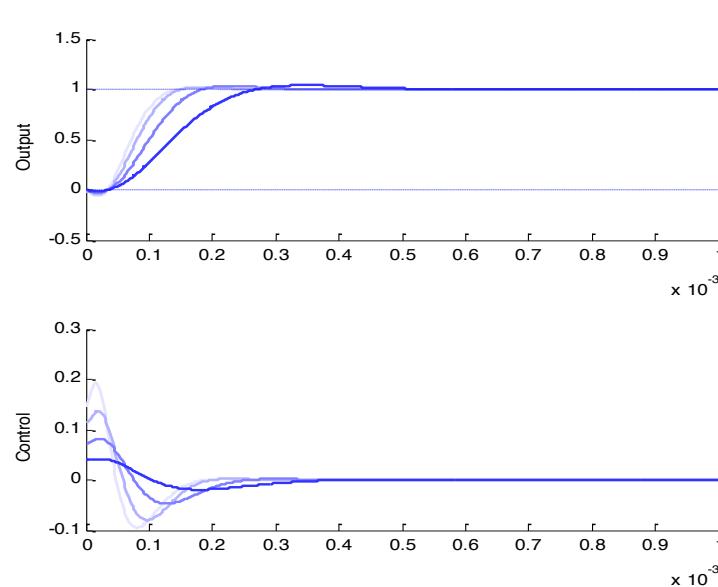
Linear quadratic control

Compute the controller $F_y(s)$ that minimizes

$$J = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [z^T Q_1 z + u^T Q_2 u] dt$$

for given (positive definite) weight matrices Q_1 and Q_2 .

Easy to influence control energy/transient performance trade-off



Linear quadratic control

Challenge: framework developed for stochastic disturbances v_1, v_2

$$\dot{x} = Ax + Bu + Nv_1$$

$$y = Cx + v_2$$

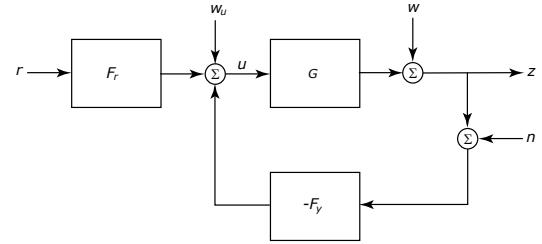
$$z = Mx$$

$$J = \mathbb{E} \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [z^T Q_1 z + u^T Q_2 u] dt \right\}$$

v_1/v_2 white noise with covariance R_1/R_2

- Need to review stochastic disturbance models
- Have to skip some details
(continuous-time stochastic processes technically tricky)

Quiz 4



- Assume the disturbance w is white noise (with covariance R) and consider the signal minimization problem

$$\min_{F_y} \|z\|_2$$

- what problem does this correspond to?
- What disturbance w is considered in \mathcal{H}_∞ -optimal control?

Learning aims

After this lecture, you should be able to

- model disturbances in terms of their spectra
- use spectral factorization to re-write disturbances as filtered white noise
- compute the LQG-optimal controller (observer/controller gains)
- describe how the LQG weights qualitatively affect the time responses

Material: Lecture notes 8, course book 5.1-5.4 + 9.1-9.3 + 9.A

Outline

- Controllability, Observability
- Modeling disturbances as filtered white noise
- Linear quadratic Gaussian (LQG) control

State-space form

State space description of multivariable linear system

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$

- $x(t)$ is called the *state vector*,
- systems on state-space form often written as (A, B, C, D)

Transfer matrix given by

$$G(s) = C(sI - A)^{-1}B + D$$

Controllability

The state \tilde{x} is *controllable* if, given $x(0)=0$, there exists $u(t)$ such that $x(t)=\tilde{x}$ for some $t < \infty$

The system is *controllable* if all \tilde{x} are controllable.

The *controllability matrix*

$$S(A, B) = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times mn}$$

- Controllable states \tilde{x} can be written as $\tilde{x} = S(A, B)\alpha$ some α in $\mathbb{R}^{mn \times 1}$
- System is controllable if $S(A, B)$ has full rank
(i.e., for each x there exists α such that $x = S(A, B)\alpha$)

Observability

The state $\tilde{x} \neq 0$ is *unobservable* if $x(0) = \tilde{x}$ and $u(t)=0$ for $t>0$ implies that $y(t)=0$ for $t \geq 0$.

The system is *observable* if no states are unobservable

The *observability matrix*

$$O(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{pm \times n}$$

- Unobservable states \tilde{x} are solutions to $O(A, C)\tilde{x} = 0$
- System is observable if $O(A, C)$ has full rank
(i.e., only $\tilde{x} = 0$ solves $O(A, C)\tilde{x} = 0$)

Modifying dynamics via state feedback

Open loop system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

can be controlled using state feedback $u(t) = -Lx(t) \rightarrow$

$$\frac{d}{dt}x(t) = (A - BL)x(t)$$

Q: can we choose L so that A-BL gets arbitrary eigenvalues?

A: if and only if the system is controllable.

Observers

The state vector of the system

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

can be estimated using an observer

$$\begin{aligned}\frac{d}{dt}\hat{x}(t) &= A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}) \Rightarrow \\ \frac{d}{dt}\tilde{x}(t) &= (A - KC)\tilde{x}(t) \quad \text{where } \tilde{x}(t) = x(t) - \hat{x}(t)\end{aligned}$$

Q: can we choose K so A-KC gets arbitrary eigenvalues?

A: if and only if system is observable

Stabilizability and detectability

Control objective concerns only output z of system, i.e., controllability and observability of all states may not be so important.

Exception: must be able to control and observe unstable modes!

A system (A, B) is *stabilizable* if there exists L so that $A - BL$ is stable

A system (A, C) is *detectable* if there exists K so that $A - KC$ is stable

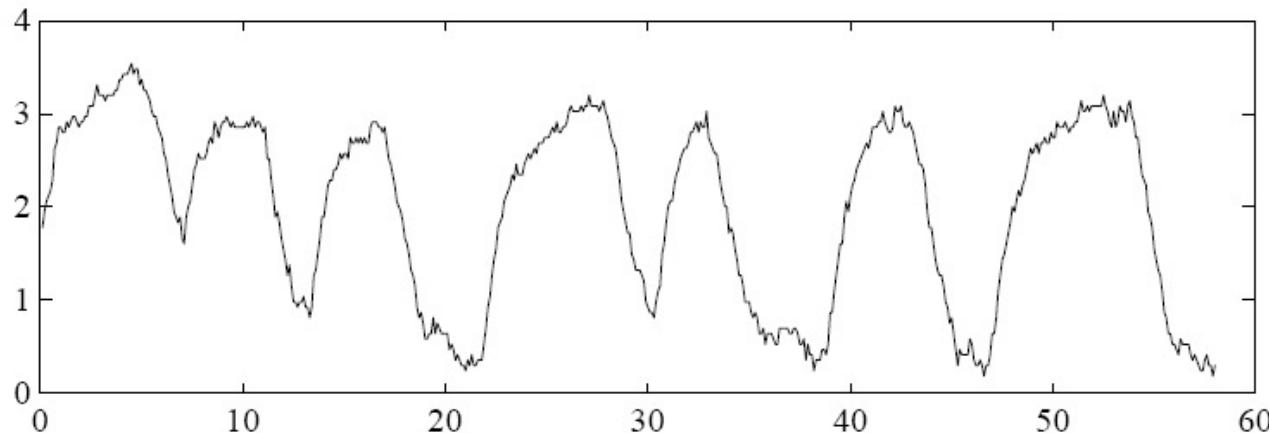
Today's lecture

- Controllability, Observability
- Modeling disturbances as filtered white noise
- Linear quadratic Gaussian (LQG) control

Disturbances

Disturbances model a wide range of phenomena that are not easily described in more detail, e.g.

- load variations, measurement noise, process variations, ...



Important to model

- size, frequency content and correlations between disturbances.

Signal sizes

So far in the course, we have used the 2-norm

$$\|z\|_2^2 = \int_0^\infty |z(t)|^2 dt$$

If the integral does not converge, we can use

$$\|z\|_e^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t=0}^T |z(t)|^2 dt$$

Only measures size of signal, i.e., not frequency etc

More informative measure

How is $z(t)$ related to $z(t-\tau)$? One measure is

$$r_z(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T z(t)z(t - \tau) dt$$

For ergodic stochastic processes, we have

$$r_z(\tau) = \mathbf{E}z(t)z(t - \tau)$$

(i.e., the covariance function of the signal)

Vector valued signals

For vector-valued z , coupling between z_i and z_j can be described by

$$r_{ij}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T z_i(t) z_j(t - \tau) dt$$

Can be combined into matrix

$$R_z(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T z(t) z^T(t - \tau) dt$$

For ergodic stochastic processes, we get

$$R_z(\tau) = \mathbf{E} z(t) z^T(t - \tau)$$

i.e., the covariance matrix for z

Signal spectra

Translating the signal measure to the frequency domain

$$\Phi_z(\omega) = \int_{-\infty}^{\infty} R_z(\tau) e^{-i\omega\tau} d\tau$$

$\Phi_z(\omega)$ is called the *spectrum* of z

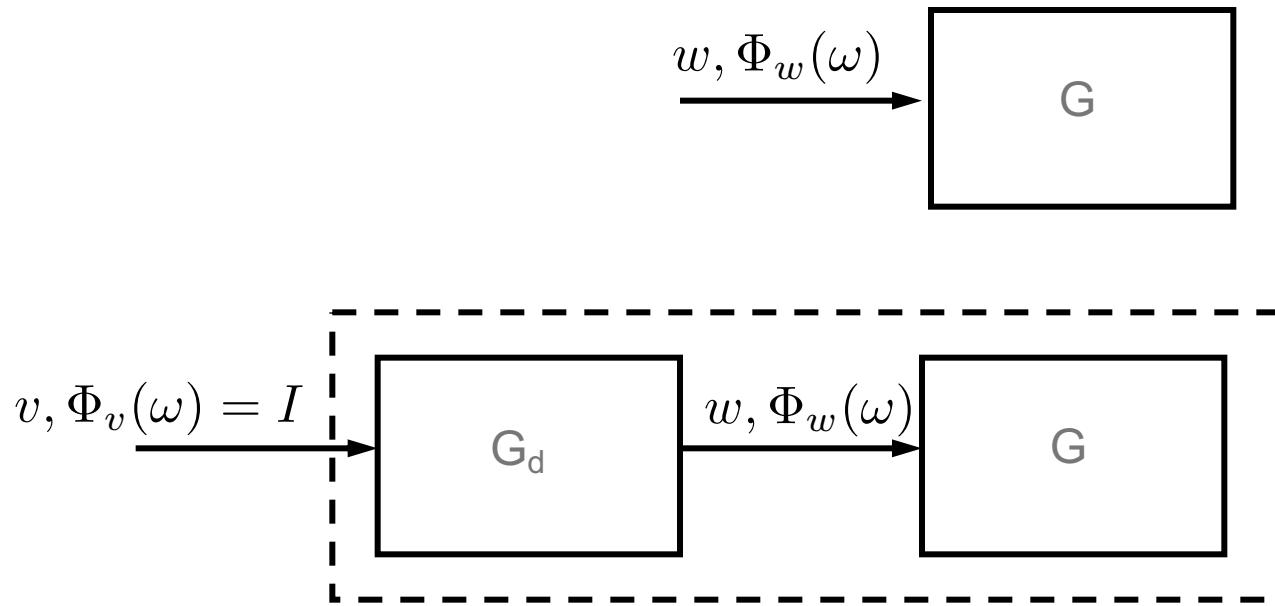
Interpretation

- $[\Phi_z(\omega)]_{ii}$ measures the energy content of z_i at frequency ω
- $[\Phi_z(\omega)]_{ij}$ measures coupling of z_i and z_j at frequency ω
- $[\Phi_z(\omega)]_{ij} = 0$ implies that z_i and z_j are uncorrelated

A signal with Φ_z constant for all ω is called *white noise*,
(in this case, we call Φ_z the *covariance matrix* of z)

Disturbances as filtered white noise

Fact (spectral factorization): any spectrum $\Phi(\omega)$ which is rational in ω^2 , can be represented as white noise filtered through a stable minimum phase linear system.



(see course book Theorem 5.1 for a precise statement)

State-space model with disturbances

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t) + Nw_1(t) \\ y(t) &= Cx(t) + Du(t) + w_2(t)\end{aligned}$$

If disturbances w_1 and w_2 are not white, but have spectra that can be obtained via $w_i = G_i v_i$ where v_i is white noise, then we can re-write system as

$$\begin{aligned}\frac{d}{dt}\bar{x}(t) &= \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{N}v_1(t) \\ y(t) &= \bar{C}\bar{x}(t) + Du(t) + v_2(t)\end{aligned}$$

Note: \bar{x} is x augmented with the states from G_1, G_2 ; \bar{A}, \bar{B}, \dots are A, B, \dots augmented with state-space descriptions of G_i

Today's lecture

- Recap: State-space representation, state feedback and observers
- Recap: Modeling disturbances as filtered white noise
- Linear quadratic Gaussian (LQG) control

Linear quadratic Gaussian control

Model: linear system with white noise

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nv_1(t)$$

$$y(t) = Cx(t) + v_2(t)$$

$$z(t) = Mx(t)$$

where v_1, v_2 are white noise with

$$\text{cov}([v_1, v_2]) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Objective: minimize effect of v on z , punish control cost

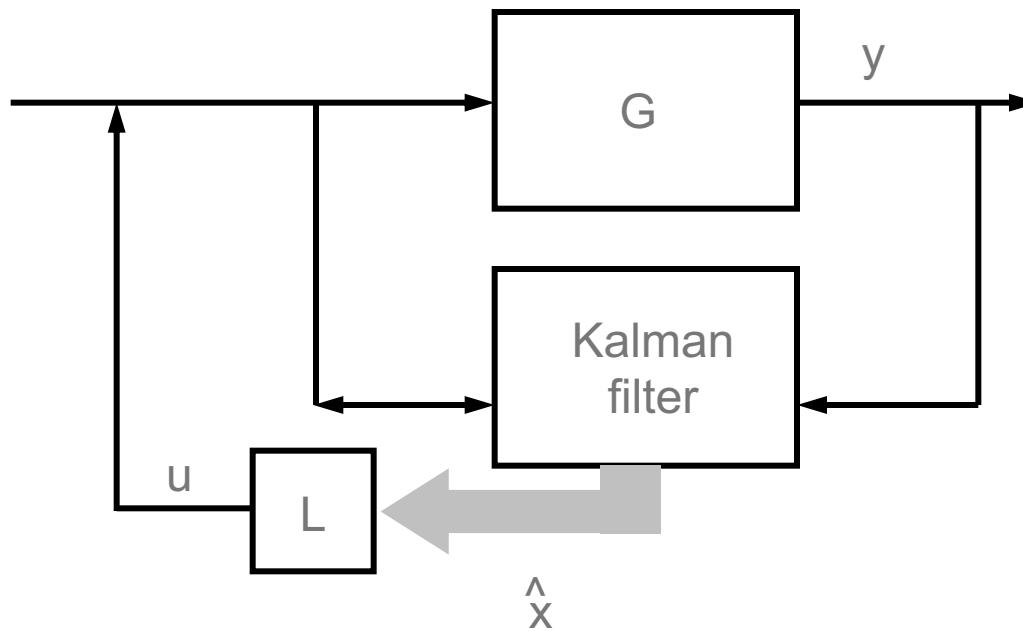
$$J = \mathbf{E} \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [z^T Q_1 z + u^T Q_2 u] dt \right\}$$

LQG: Linear system, Quadratic cost, Gaussian noise

Solution structure

Optimal solution satisfies *separation principle*, composed of

- Optimal linear state feedback (Linear-quadratic regulator)
- Optimal observer (Kalman filter)



The LQR Problem

Disturbance and noise free system

$$\begin{aligned}\dot{x} &= Ax(t) + Bu(t); \quad x(0) = x_0 \\ z(t) &= Mx(t)\end{aligned}$$

LQ problem

$$\min_u \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T z^T Q_1 z + u^T Q_2 u \, dt$$

Optimal input given by state feedback

$$u(t) = -Lx(t) = -Q_2^{-1}B^T S x(t)$$

where $S > 0$ is the solution to the algebraic Riccati equation

$$A^T S + S A + M^T Q_1 M - S B Q_2^{-1} B^T S = 0$$

Example: LQR for scalar system

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t), \quad y(t) = x(t)$$

with cost

$$J = \int_0^\infty [x^2 + \rho u^2] dt$$

Riccati equation

$$2as + 1 - s^2/\rho = 0$$

has solutions

$$s = a\rho \pm \sqrt{(\rho a)^2 + \rho}$$

so the optimal feedback law is

$$u = -(s/\rho)x = -(a + \sqrt{a^2 + 1/\rho})x$$

Example: LQR for scalar system

Closed loop system

$$\dot{x}(t) = -(\sqrt{a^2 + 1/\rho})x(t)$$

- If $\rho \rightarrow 0$, closed loop bandwidth is approx. $1/\sqrt{\rho}$
- If $\rho \rightarrow \infty$
 - and $a < 0 \Rightarrow u = 0 \cdot x$
 - and $a > 0 \Rightarrow u = -2ax$ which gives $\dot{x} = -ax$, i.e., pole is mirrored about imaginary axis

The Kalman Filter

- System with disturbance/noise

$$\dot{x} = Ax + Bu + Nv_1, \quad E\{v_1 v_1^T\} = R_1$$

$$y = Cx + v_2, \quad E\{v_2 v_2^T\} = R_2$$

- Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x})$$

- Determine the K_f that minimizes square of estimation error

$$E\{(x - \hat{x})^T(x - \hat{x})\}$$

- Solution: $K_f = PC^T R_2^{-1}$, where $P > 0$ solves Riccati eq.

$$PA^T + AP - PC^T R_2^{-1} CP + NR_1 N^T = 0$$

- The optimal observer is called the Kalman filter

Example: Scalar system Kalman filter

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t) + v_1(t), \quad y(t) = x(t) + v_2(t)$$

with covariances $E\{v_1^2\}=R_1$, $E\{v_2^2\}=R_2$, $E\{v_1v_2\}=0$.

Riccati equation

$$2ap + r_1 - p^2/r_2 = 0$$

gives

$$k = a + \sqrt{a^2 + r_1/r_2}$$

and estimation error dynamics

$$\frac{d}{dt}\tilde{x}(t) = -(\sqrt{a^2 + r_1/r_2})\tilde{x}$$

Interpretation: measurements discarded if too noisy.

The LQG Controller

- The LQG controller that minimizes the objective function

$$J = \mathbb{E} \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [z^T Q_1 z + u^T Q_2 u] dt \right\}$$

is given by the Separation Principle, i.e., LQR + Kalman filter

$$\begin{aligned} u &= -L\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + Bu + K_f(y - C\hat{x}) \end{aligned}$$

Quiz 6

- How can the sensitivity function be shaped using LQG?
- How can robust stability (indirectly) be addressed in LQG?

LQG and loop shaping

LQG: simple to trade-off response-time vs. control effort

- but what about sensitivity and robustness?

These aspects can be to some extent be accounted for using the noise models

- Sensitivity function S: transfer matrix $w_u \rightarrow z$
- Complementary sensitivity T: transfer matrix $n \rightarrow z$

Example: S forced to be small at low frequencies by letting
(some component of) w_1 affect the output of the system, and
let w_1 have large energy at low frequencies,

$$w_1(t) = \frac{1}{p + \delta} v_1(t)$$

(delta small, strictly positive, to ensure stabilizability)

LQG Control: pros and cons

Pros:

- Simple to trade off response time vs. control effort
- Applies to multivariable systems

Cons:

- Often hard to see connection between weight matrices Q_1 , Q_2 , R_1 , R_2 and desired system properties (e.g. sensitivity, robustness, etc)
- In practice, iterative process in which Q_1 and Q_2 are adjusted until closed loop system behaves as desired
- Poor robustness properties in general

Summary

- State-space theory recap:
 - Controllability, observability, stabilizability, detectability
 - State feedback and observers
- Modeling disturbances as white noise
 - Mean, covariance, spectrum
 - Spectral factorization: disturbances as filtered white noise
- Linear-quadratic controller
 - Kalman-filter + state feedback
 - Obtained by solving Riccati equations
 - Focuses on time-responses
 - Loop shaping and robustness less direct

Next time: the link between LQG and \mathcal{H}_2 -optimal control