EL2520
Control Theory and Practice

Lecture 13:
Dealing with Hard Constraints

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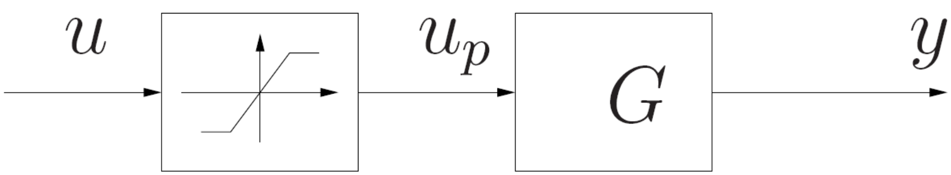
Input Constraints

Dealing with input constraints:

- Linear control design: punish large control moves, e.g.,
  - LQG: choose large input weight Q_2
  - H_∞: include e.g, $\|G_{wu}\|_\infty$ in objective function
- But, inputs often have hard constraints

$$u_{min} \leq u_p \leq u_{max}$$



Outline of Lecture

Dealing with hard constraints

- Previous lecture: Constrained Receding Horizon Control / MPC
- This lecture: Anti reset windup
  - State feedback with a nonlinear observer
  - Interpretation and extension to any controller

Model Predictive Control

- Finite-horizon discrete time LQR with hard constraints on u and y:

$$
\begin{array}{ll}
\text{minimize} & J(U) = \sum_{k=0}^{N-1} (x_k^T Q_1 x_k + u_k^T Q_2 u_k) + x_N^T Q_f x_N \\
\text{subject to} & u_{\min} \le u_k \quad \le u_{\max}, \;\; k = 0, \ldots, N-1 \\
& y_{\min} \le C x_k \le y_{\max}, \;\; k = 1, \ldots, N \\
& x_{k+1} = A x_k + B u_k
\end{array}
$$

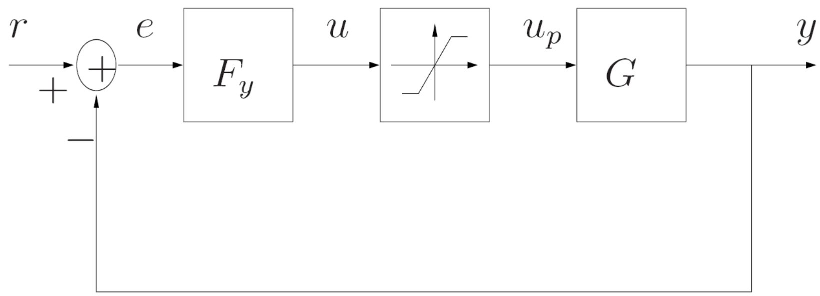
- Can be written as a quadratic programming problem in $\{u_0, \ldots, u_{N-1}\}$

- Implement only u_0, let system evolve one sample and redo optimization with new state estimate; results in receding horizon optimization

- Main advantage: constraints can be included in optimization

Anti Reset Windup

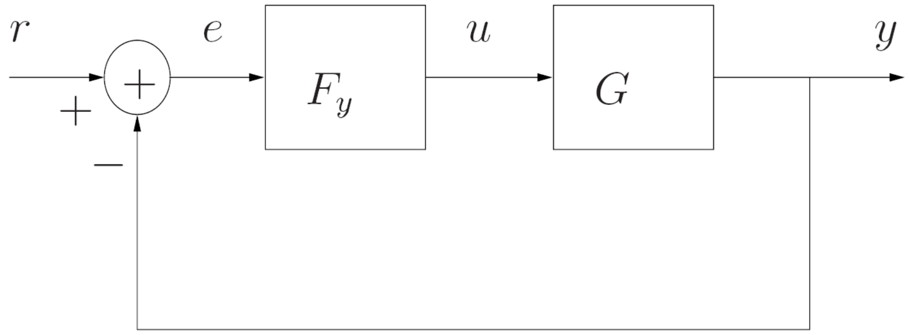
- The problem with saturating input



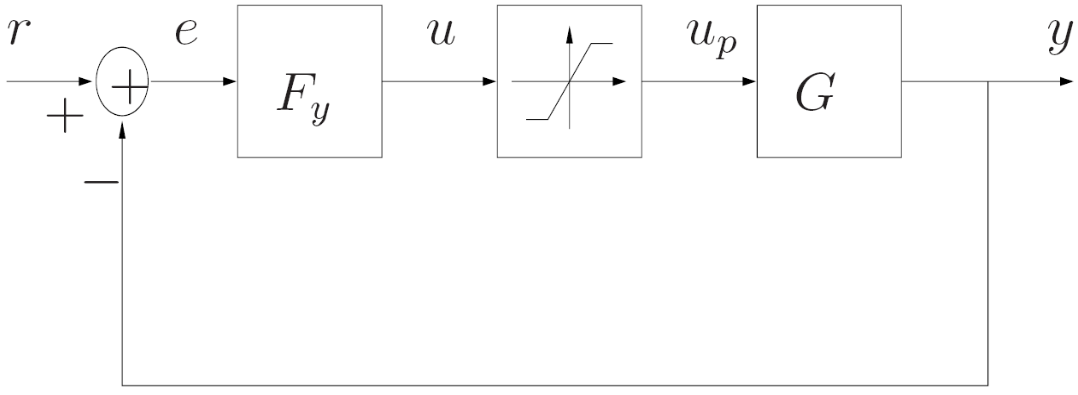
  - feedback broken, i.e., system open-loop, when u in saturation
  - problem, in particular if F or G unstable
  - F usually has integrator (unstable)
- The classical approach to deal with hard constraints on the input is called anti-reset windup

Magnitude limitations on control

Linear model



Actual implementation



Example: DC servo

Servo:

$$G(s) = \frac{1}{s(s+1)}$$

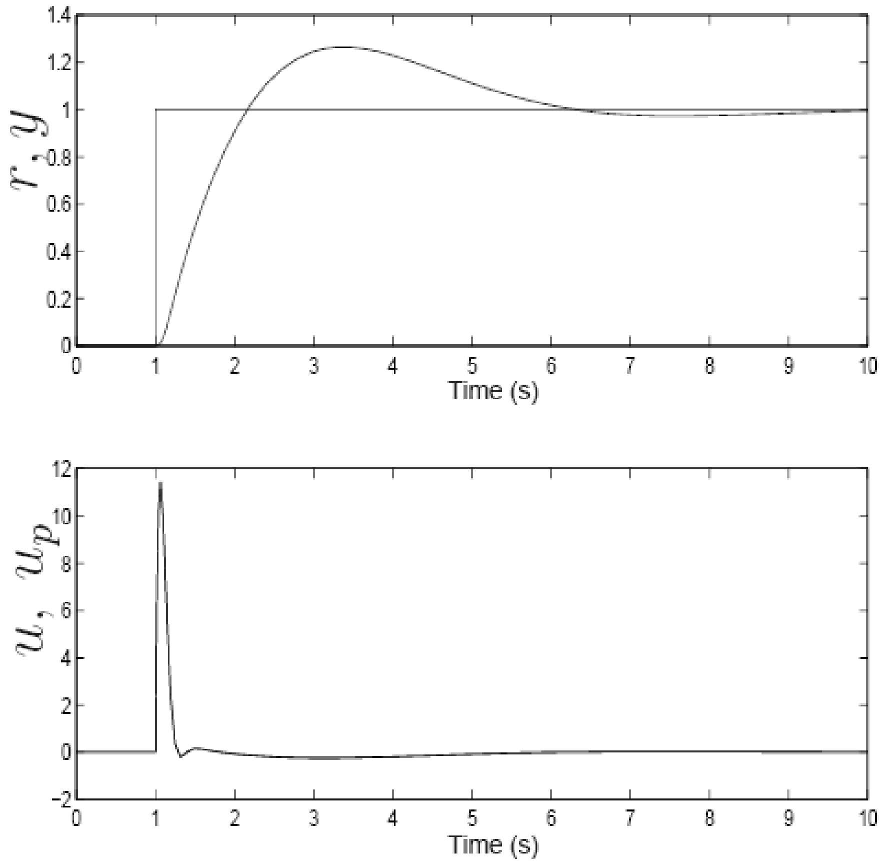
A controller designed using LQG is

$$F_y(s) = \frac{439s^2 + 710.5s + 316.2}{s^3 + 26.47s^2 + 349.8s - 7.13}$$

which has poles in -13.2444 +- 13.2255i, and 0.0204

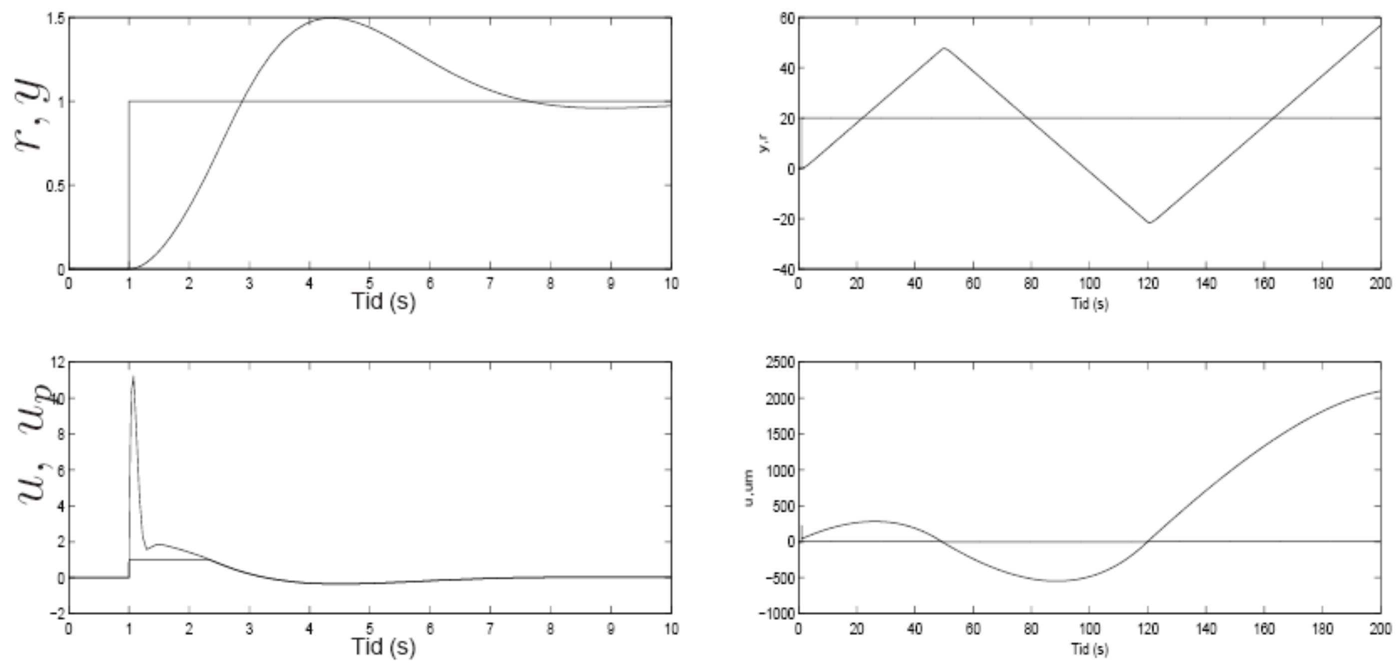
Note: controller is unstable, but closed loop is internally stable!

Step response (no constraints)



Step response with saturated input

$$-1 \leq u_p \leq 1$$



Slower, larger overshoot

unstable

Observer + State Feedback

Many controllers based on feedback from observed states

- Observer:

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

- Feedback from observed states

$$u = -L\hat{x}$$

- Controller transfer-function

$$U(s) = -F_y(s)Y(s) = -L(sI - A + BL + KC)^{-1}KY(s)$$

A solution: modified observer

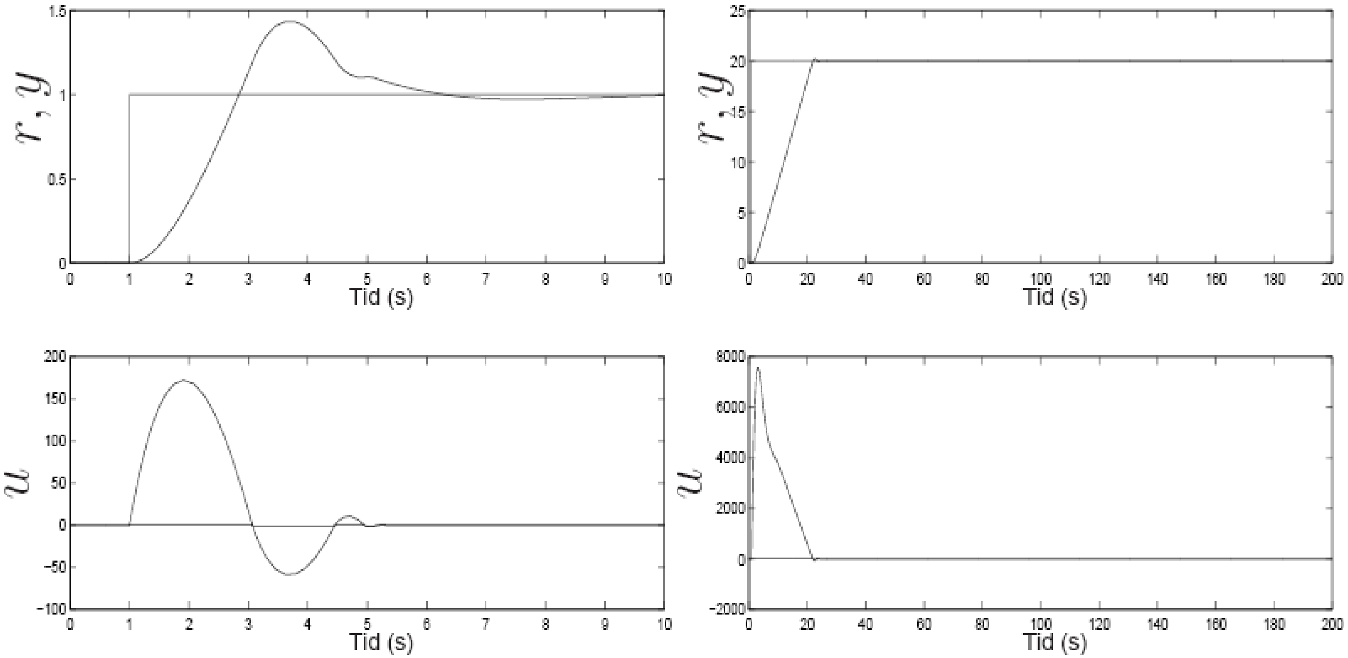
Observer should reflect true dynamics

$$\frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) + Bu_p(t) + K(y(t) - C\widehat{x}(t))$$

The constrained (actually applied) input is used in observer

- a nonlinear observer!
- based on measuring the actual input or having a model of the constraint

Step responses with modified observer



Analysis: stability also in saturation

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu_p(t) + K(y(t) - C\hat{x}(t)) =$$
$$= (A - KC)\hat{x}(t) + Bu_p(t) + Ky(t)$$

Controller transfer function

$$U(s) = -L(sI - A + KC)^{-1}KY(s) - L(sI - A + KC)^{-1}BU_p(s)$$

In saturation ($u<u_{min}$ or $u>u_{max}$), u_p is constant

Thus, in saturation, the controller dynamics is given by A-KC whose eigenvalues are -0.5446±0.7276i, -1.2106 (i.e. stable)

This modification is known as *anti-reset windup.*

Interpretation: feedback from u-u_p

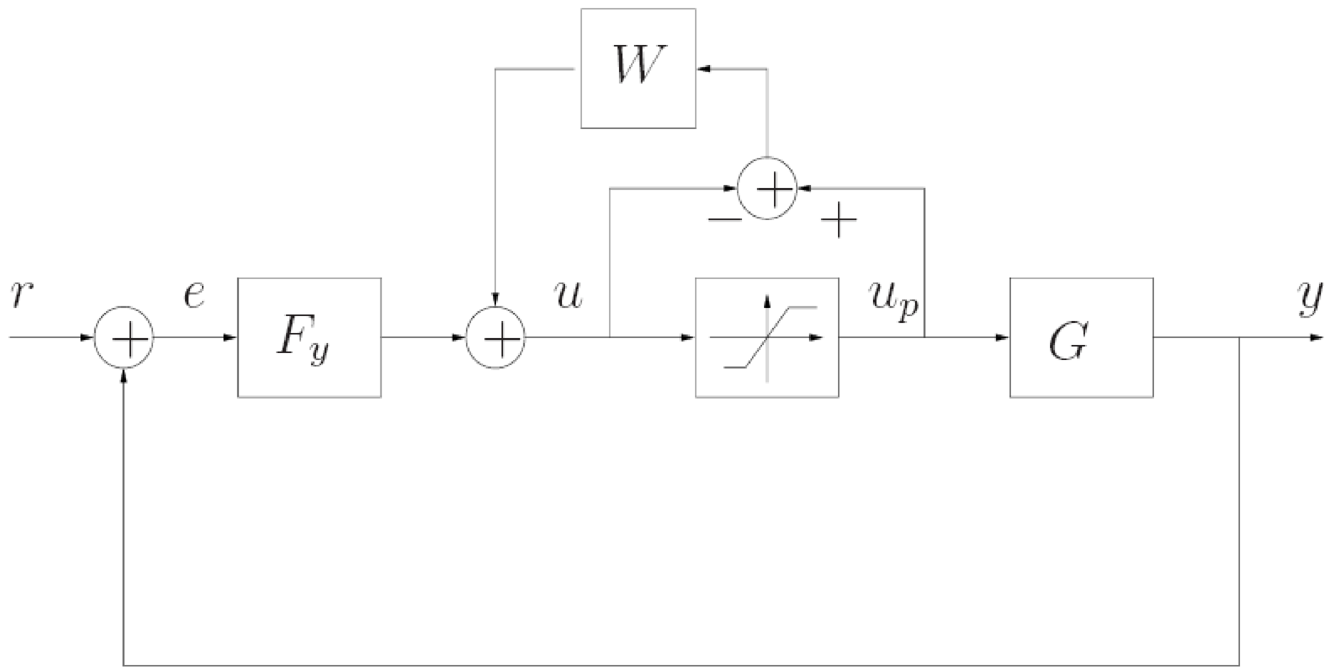
Write controller as

$$\begin{aligned}\frac{d}{dt}\widehat{x}(t) &= A\widehat{x}(t) + Bu_p(t) + K(y(t) - C\widehat{x}(t)) = \\ &= (A - KC)\widehat{x}(t) + B(u_p(t) + u(t) - u(t)) + Ky(t) = \\ &= (A - BL - KC)\widehat{x}(t) + Ky(t) + B(u_p(t) - u(t))\end{aligned}$$

Taking Laplace transforms

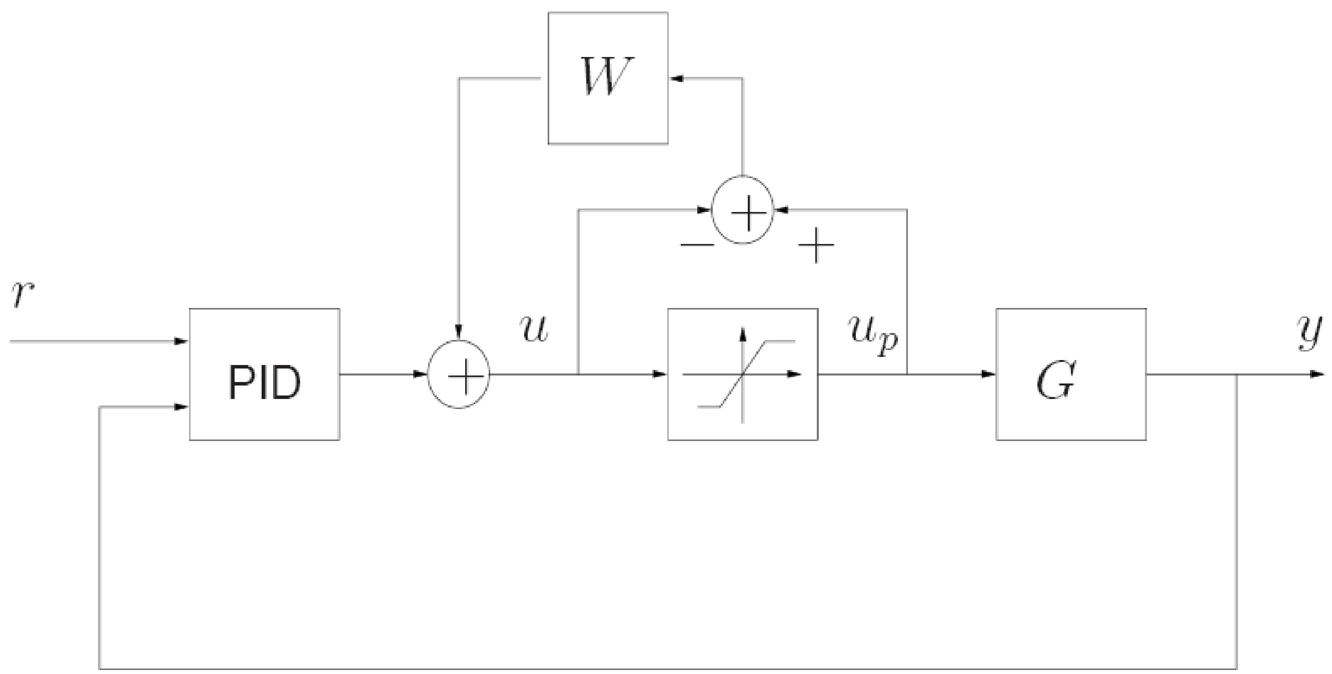
$$\begin{aligned}U(s) =& -L(sI - A + BL + KC)^{-1}KY(s) \\ & -L(sI - A + BL + KC)^{-1}B(U_p(s) - U(s)) = \\ =& -F_y(s)Y(s) + W(s)(U_p(s) - U(s))\end{aligned}$$

In block diagram



- Anti-reset windup is based on tracking input

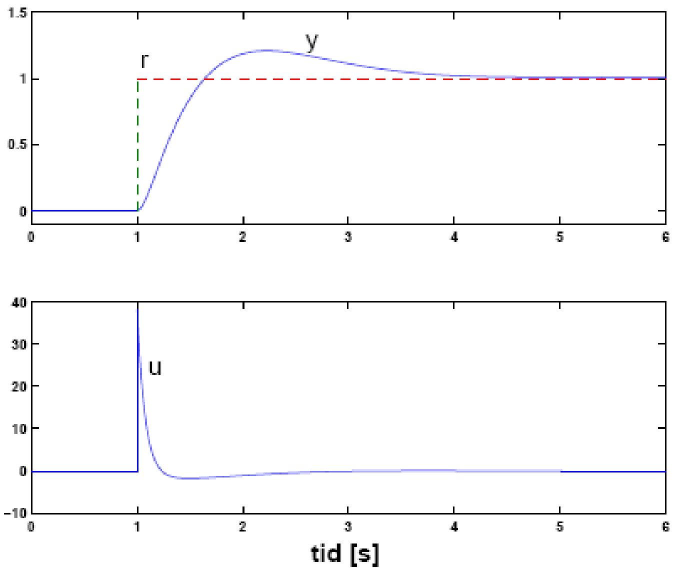
Application to PID controllers



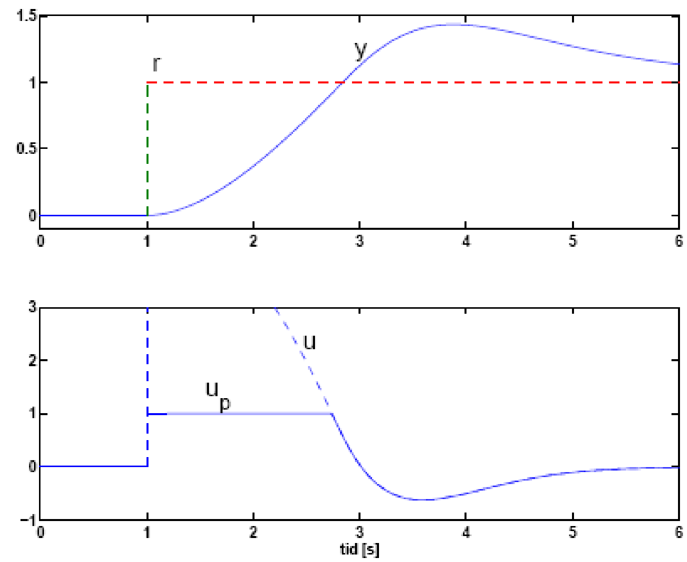
- Common choice: $W(s) = \dfrac{1}{sT_t}$

DC Servo under PID control

No constraint

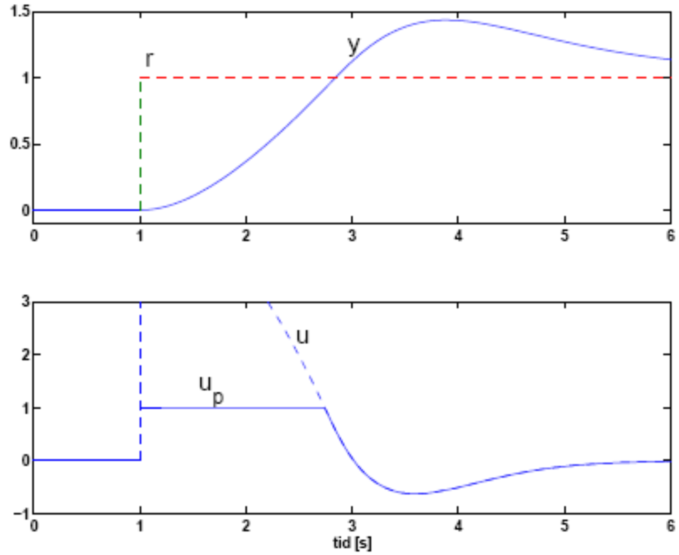


$u_p \in [-1, 1]$, no compensation

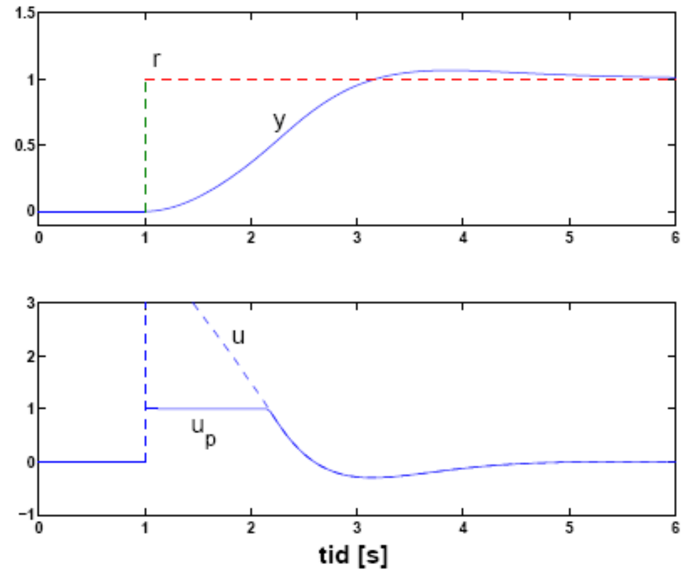


Servo: PID+Anti-reset Windup

$T_t = 1000$



$T_t = T_i = 1.9$



Summary

- Hard constraints: a nonlinearity essentially always present in real control systems
- Main problem: system drifts off when input in saturation
- Approaches to deal with hard constraints
  - constrained receding horizon LQG control (MPC)
  - anti-reset windup