EL2520 - Control Theory and Practice - Advanced Course

Solution/Answers – 210608 (mostly answers, and not motivated solutions as required on the exam)

1. (a) The determinant is $\det G = \frac{-s+1}{(s+1)^2(s+3)}$ and the denominator is the LCD for all minors. Hence there are two poles in $s=-1$, one pole in $s=-3$ and one zero at $s=1$. With 3 poles we need at least 3 states in a state-space realization.

   (b) The peak value occurs as $\omega \to \infty$ and $\|S\|_\infty = \sup_\omega \bar{\sigma}(S) = 2.5$.

   (c) The 1,1-element of the RGA is $\lambda_{11} = \frac{1}{1-\frac{G_{12}G_{21}}{G_{11}G_{22}}} = 21$. Thus, we should pair on the diagonal to avoid negative steady-state RGA. But, the RGA at the crossover is very large and decentralized control is therefore not recommended.

2. (a) (i) Laplace transform gives, with $w_1 = 0$,

$$Z(s) = 2\frac{s-1}{(s-2)(s+1)}U(s) + W_2(s)$$

   The closed-loop transfer-function from W_2 to Z is the sensitivity function S. With a RHP pole at $s=2$ and a RHP zero at $s=1$ we get a lower bound on $\|S\|_\infty$

$$\|S\|_\infty \geq \frac{|z+p|}{|z-p|} = 3$$

   Thus, the smalles achievable amplification is 3.
   (ii) The closed-loop transfer function is T and we have

$$\|W_T T\|_\infty \geq |W_T(p)\frac{z+p}{z-p}| = 7.5 > 1$$

   Thus, not feasible.

   (b) (i) G has a RHP zero at $s=1$ and this zero must be retained in T to ensure internal stability. (ii) $t(s) = (-s+1)/(s+1)$

3. (a) G has a RHP zero at $s=4$ with output direction $y_z = \sqrt{4/5}\begin{pmatrix}-1 & 1/2\end{pmatrix}^T$. The requirement to be able to achieve acceptable disturbance attenuation is $|y_z^H g_d(z)| < 1$,

$$\|y_z^H g_d(z)\| = -0.055$$

   where we have used $g_d = 5/(10s+1)\begin{pmatrix}2 & 3\end{pmatrix}^T$. Thus, feasible.

   (b) Feasible, since the open-loop is stable the controller $F_y = 0$ is sufficient for robust stability.

(c)
$$J = \left\| \begin{matrix} Sg_d \\ W_I T_I \end{matrix} \right\|_\infty$$
where g_d is as given below and T_I is the complementary sensitivity at the input. An uncertainty weight W_I covering the given input uncertainty is
$$W_T = 0.2 \frac{s+1}{0.1s+1}$$

(d) For Sg_d we use d as the input at z as the output. For $W_I T_I$ we add a disturbance at the input u_d and use $W_I(u - u_d)$ as the ouput. This choice also includes more transfer-functions in the objective function than what is given in (c), but it is difficult to find an input-output pair that has the transfer-function in J.

4. (a) We have $A = 0, B = 3, C = 1, D = 0, Q_1 = Q_2 = 1$ and the LQ riccati equation is
$$1 - 9P^2 = 0$$
and $P = 1/3$ is the positive definite solution. Hence, $L = 1$ and $u(t) = -z(t)$ is the sought controller.

(b) The Riccati equation for the Kalman filter becomes
$$-P^2 R_n^{-1} + R_w = 0$$
and $P = \sqrt{0.08}$ is the pos def solution, and the observer gain is $K_f = 1/\sqrt{2}$. The controller from (a) is still optimal if we use the observer estimate of z as input to the controller according to the separation principle.

(c) (i) $A_d = 1, B_d = 3, C_d = 1$. (ii) With $N = 1$, the objective funcgion becomes
$$V = Q_z z_k^2 + Q_z(z_k + 3u_k)^2 + u_k^2 + u_{k+1}^2 = 2Q_z z_k^2 + 6Q_z u_k z_k + (9Q_z + 1)u_k^2 + u_{k+1}^2$$
Note that z_k is the current value of z and can not be influenced by the control, so can be removed from V. Also, the optimal value for $u_{k+1} = 0$ so
$$V = (9Q_z + 1)u_k + 6Q_z z_k u_k$$
and hence $H = 9Q_z + 1, h = 6Q_z z_k$. The constraint $|u_k| < 0.2$ can be represented by $L = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $b = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$.

5. (a) To simplify we introduce $\hat{u} = bu$ and $\hat{Q}_2 = Q_2/b^2$. LQ-problem with Riccati equation is then
$$-2aP + Q_1 - \hat{Q}_2^{-1} P^2 = 0$$

and we get

$$P = \hat{Q}_2(-a + \sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}})$$

and

$$\hat{u} = (a - \sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}})x$$

(b) The closed-loop dynamics become

$$\dot{x} = -(a + \Delta_a)x + (a - \sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}})x$$

or

$$\dot{x} = (-\Delta_a - \sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}})x$$

Thus, by choosing $Q_1/\hat{Q}_2$ such that

$$\sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}} > -\Delta_a$$

we can stabilize the system for any Δ_a. Thus, stability can be guaranteed by choosing $Q_1/\hat{Q}_2$ sufficiently large.

(c) With uncertain stability of the open-loop system, we should use the uncertainty description

$$G_p = G(1 + W_{iI}\Delta_I)^{-1},\ \|\Delta_I\|_\infty < 1$$

and the robust stability condition is then

$$\|W_{iI}S\|_\infty \leq 1$$

The uncertain model is $G_p = \frac{1}{s+a+\Delta_a}$, $|\Delta_a| < \delta$ and we then get

$$W_{iI} = \frac{\delta}{s+a}$$

The nominal loop-gain is

$$L = \frac{1}{s+a}(-a + \sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}})$$

and the sensitivity function is then

$$S = \frac{s+a}{s + \sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}}}$$

and

$$W_{iI}S = \frac{\delta}{s + \sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}}}$$

The peak value of $|W_{iI}S(i\omega)|$ occurs for $\omega = 0$ and hence

$$\|W_{iI}S\|_\infty = \frac{\delta}{\sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}}}$$

and we get the criterion for robust stability

$$\sqrt{a^2 + \frac{Q_1}{\hat{Q}_2}} > \delta$$

(d) The criterions in (b) and (c) are equivalent if we consider $|\Delta_a|$ bounded. In (b) we can allow "non-symmetric" ranges of Δ_a, e.g., allowing only positive Δ_a would give stability for all Δ_a. Given that the robustness condition in (c) is conservative due to the use of the SGT and complex perturbations, while Δ_a is real, it may appear surprising that we get an equally tight condition as in (b) if we consider symmetric ranges of Δ_a. The reason is mainly that the worst frequency is $\omega = 0$ for which all values are real and allowing the Δ_a to both positive and negative implies that the SGT is not conservative.