



EL2520

Control Theory and Practice

Lecture 11:

Classical and Modern Optimal Control revisited & Case Study

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Today's lecture

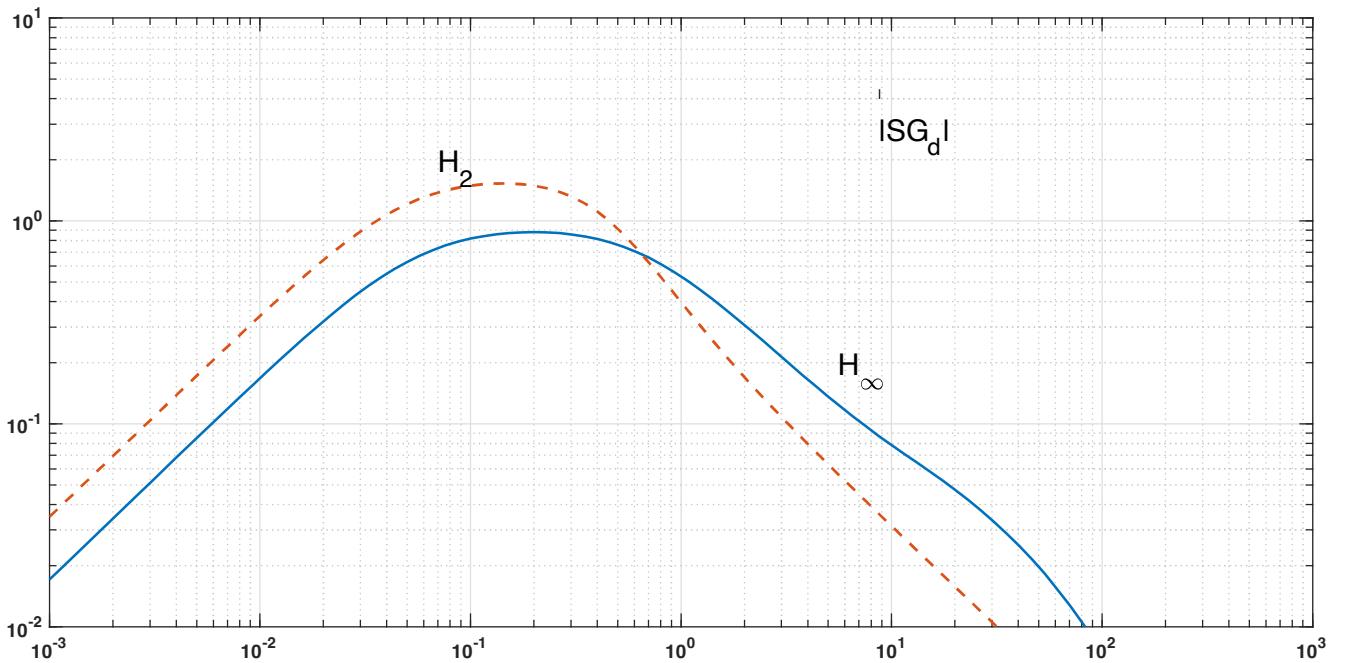
- Brief review of design methods covered so far, including interpretations and comparisons
- Case study: disturbance attenuation in a chemical reactor

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Summary

- All synthesis methods considered in this course can be used for signal minimization, or for shaping closed-loop transfer-functions (equivalence exist in all cases)
- Although possible, LQG not well suited for shaping closed-loop transfer-functions. Then, H₂-optimal control more direct.
- H _{∞} -optimal control can be used for explicit design for robust stability, and can provide performance guarantees

Disturbance sensitivity



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Optimal Control

- Classical optimal control LQG
 - originally motivated by the need for methods that could deal with MIMO systems
 - formulated as optimization problem in state-space (time)
- Modern optimal control H_∞ and H_2
 - originally motivated by need to explicitly address robustness
 - formulated as optimization problem in input-output space (frequency domain)
 - solved in state-space
- All formulations result in controllers on the form observer + state feedback

Robust Loop shaping

- Combines classical loop shaping with H_∞ -optimal control where the optimization step only addresses robust stability

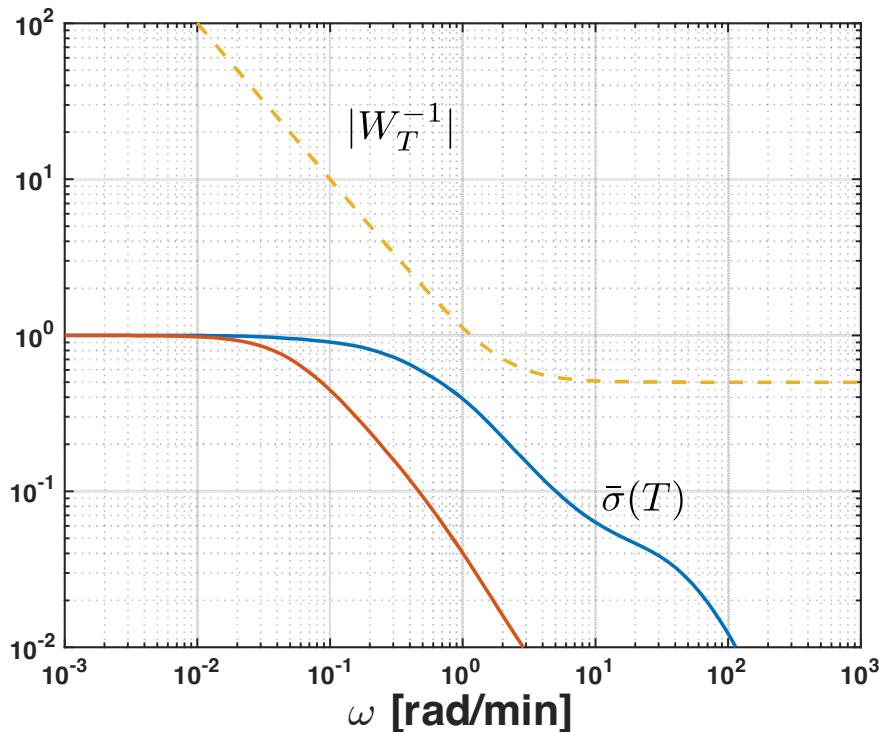
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Design III: H_2 signal based control

```
WS=((s+.05)/(s+1.d-4)); % Adds integral action  
WU=1;
```

```
d=icsignal(1);  
u=icsignal(2);  
y=icsignal(2);  
Wy=icsignal(2);  
Wu=icsignal(2);  
P=iconnect;  
P.Input=[d;u];  
P.Output=[Wy;Wu;-y];  
P.equation{1}=equate(y,G*u+Gd*d);  
P.equation{2}=equate(Wy,WS*y);  
P.equation{3}=equate(Wu,WU*u);  
  
[C,CL,gam]=h2syn(P.System,2,2);  
  
>> gam  
gam =  
0.720
```

Robustness



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Linear Quadratic Gaussian control

Model: linear system with white noise

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t) + Nv_1(t) \\ y(t) &= Cx(t) + v_2(t) \\ z(t) &= Mx(t)\end{aligned}$$

where v_1, v_2 are white noise with

$$\text{cov}([v_1, v_2]) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Objective: minimize effect of v on z , punish control cost

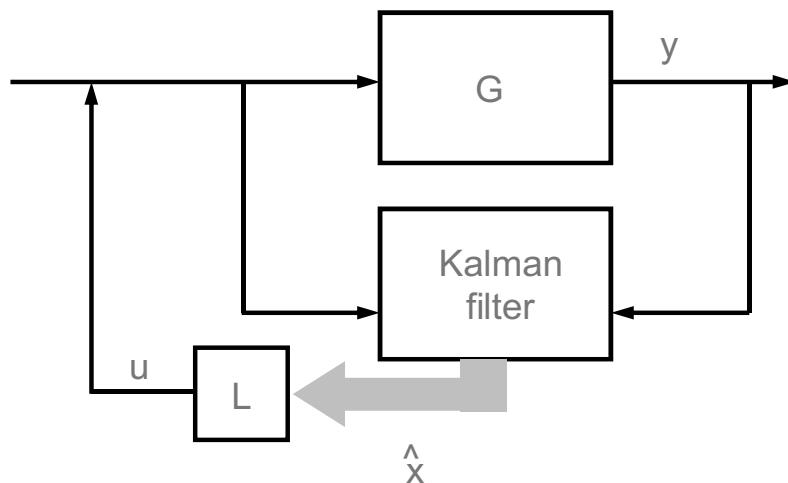
$$J = \mathbb{E} \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [z^T Q_1 z + u^T Q_2 u] dt \right\}$$

LQG: Linear system, Quadratic cost, Gaussian noise

Solution structure

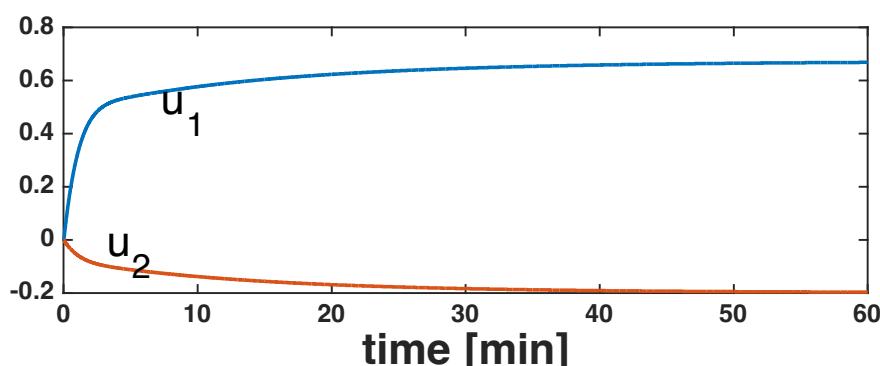
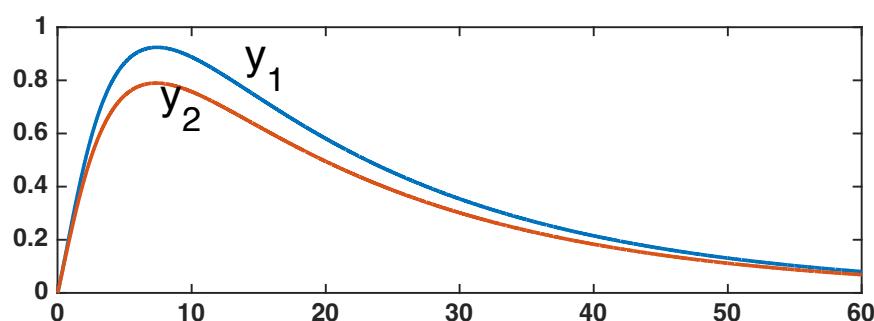
Optimal solution satisfies *separation principle*, composed of

- Optimal linear state feedback (LQ-problem, no noise)
- Optimal observer (Kalman filter, no control)



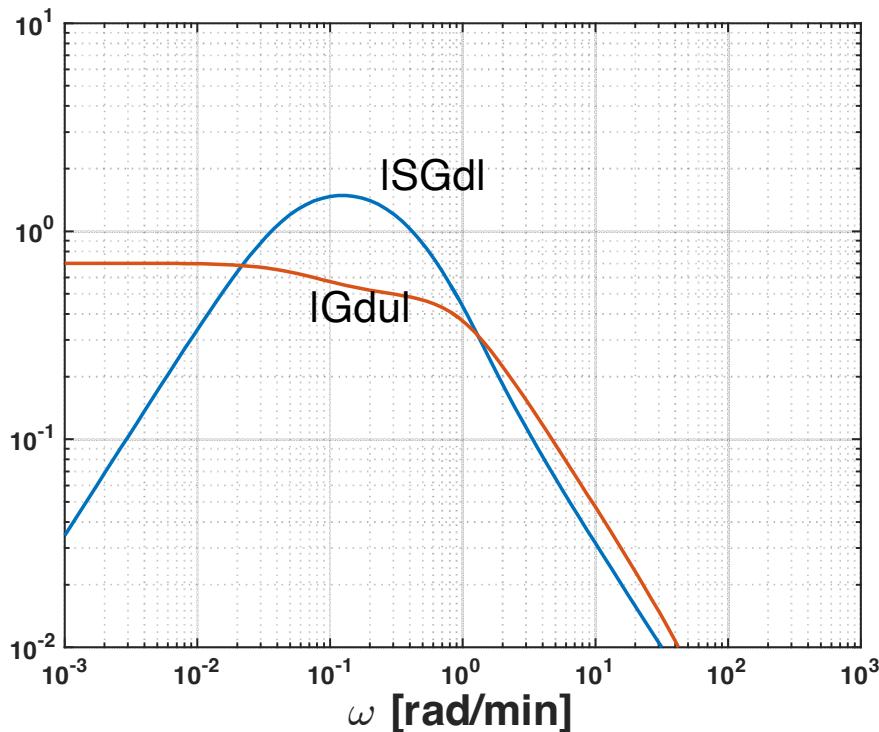
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Simulation



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Performance



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Optimal solution

State feedback (LQ problem)

$$u(t) = -L\hat{x}(t) = -Q_2^{-1}B^T S \hat{x}(t)$$

where $S > 0$ is the solution to the algebraic Riccati equation

$$A^T S + S A + M^T Q_1 M - S B Q_2^{-1} B^T S = 0$$

Kalman filter

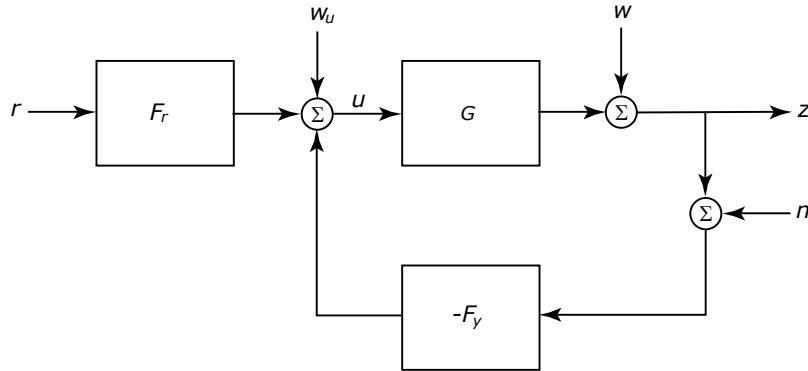
$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)); \quad K = (PC^T + NR_{12})R_2^{-1}$$

where $P > 0$ is the solution to algebraic Riccati equation

$$AP + PA^T + NR_1 N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

- Combination of optimal state feedback, with no noise, and optimal observer, with no control, is the optimal controller to the combined problem
- Tuning parameters: Q_1, Q_2, R_1, R_2 (usually assume $R_{12} = R_{21} = 0$)

H_∞ -optimal control



Aim: shape closed loop transfer-functions, e.g., S, T, G_{wu} , to achieve desired system properties

How: introduce weights W_S, W_T, W_u and determine F_y, F_r such that

$$\|W_S S\|_\infty < 1 \quad \|W_T T\|_\infty < 1 \quad \|W_u G_{wu}\|_\infty < 1$$

↓

$$\bar{\sigma}(S(i\omega)) < |w_S^{-1}(i\omega)| \quad \bar{\sigma}(T(i\omega)) < |w_T^{-1}(i\omega)| \quad \bar{\sigma}(G_{uw}(i\omega)) < |w_u^{-1}(i\omega)| \quad \forall \omega$$

where we assume $W_S = w_S I$ etc.

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Design II: add robustness criterion

```

WS=((s+.05)/(s+1.d-4)); % Adds integral action
WU=(s+0.5)/(s+1); % to put relatively smaller weight on low frequency inputs
WT=2*s/(s+2); % For robust stability in presence of output uncertainty

```

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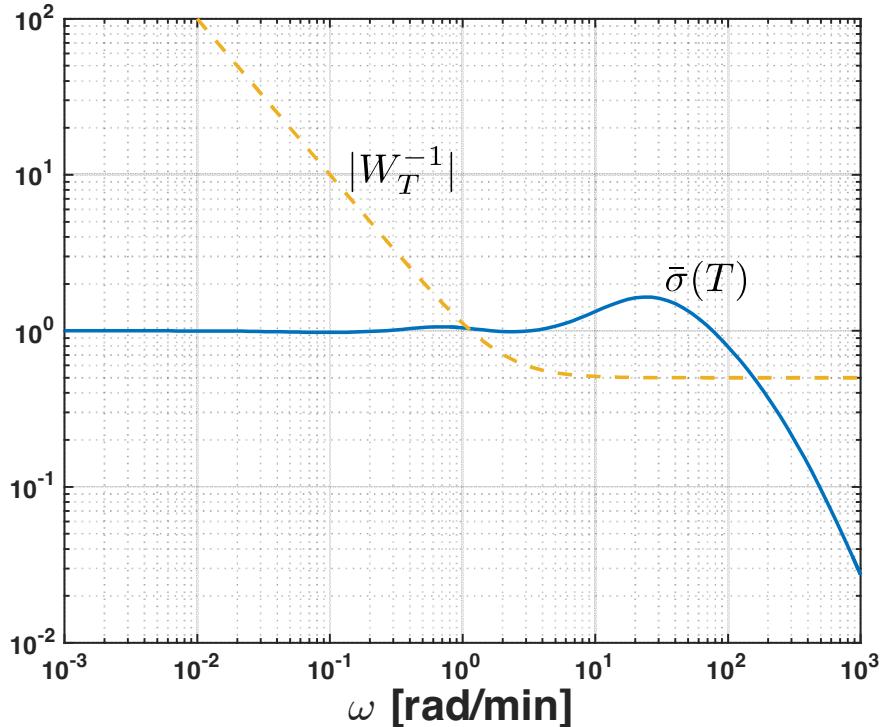
d=icsignal(1);
r=icsignal(2);
u=icsignal(2);
y=icsignal(2);
Wy=icsignal(2);
Wu=icsignal(2);
Wt=icsignal(2);
P=iconnect;
P.Input=[d;r;u];
P.Output=[Wy;Wt;Wu;r-y];
P.equation{1}=equate(y,G*u+Gd*d);
P.equation{2}=equate(Wy,WS*(r-y));
P.equation{3}=equate(Wu,WU*u);
P.equation{4}=equate(Wt,WT*y);

```

```
[C,CL,gam]=hinfsyn(P.System,2,2);
```

```
>>gam
gam = 1.796
```

Robustness



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Selecting Weights

Weights W_S, W_T, W_u should

- reflect our requirements on performance and robustness, e.g., W_S large for frequencies where we need disturbance attenuation, W_T large where we want noise attenuation and where model uncertainty is large.
- take into account trade-offs and limitations, e.g., $S+T=I$, RHP poles, RHP zeros and time delays, such that $\|\cdot\|_\infty < 1$ is feasible, i.e., there exists stabilizing controller that meets specifications.

Usually a good idea to scale all signals, such that their expected / allowed magnitude is less than 1, prior to designing weights.

Controller Design – H_∞

How determine $F_y(s)$ to achieve $\|W_S S\|_\infty < 1$ $\|W_T T\|_\infty < 1$ $\|W_u G_{wu}\|_\infty < 1$?

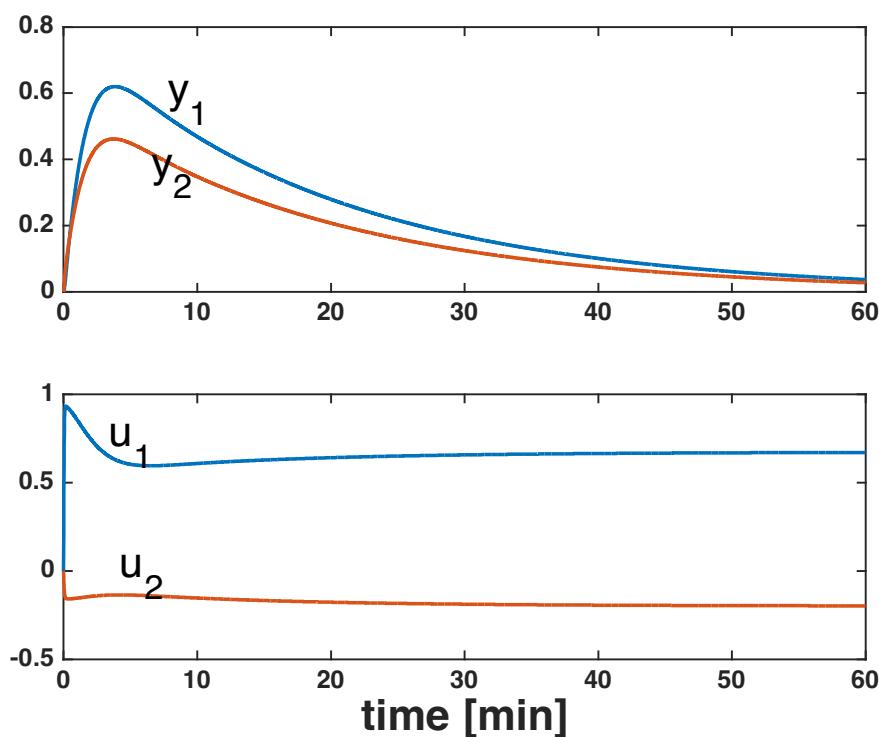
- *Loop shaping*, i.e., shape loop gain $L = GF_y$
or
- *Synthesis*, i.e., solve optimization problem

$$F_{y,opt}(s) = \arg \min_{F_y} \left\| \begin{array}{c} W_S S \\ W_T T \\ W_u G_{wu} \end{array} \right\|_\infty \quad (*)$$

Note that if the "stacked" objective above is less than 1, then we have achieved the three individual objectives

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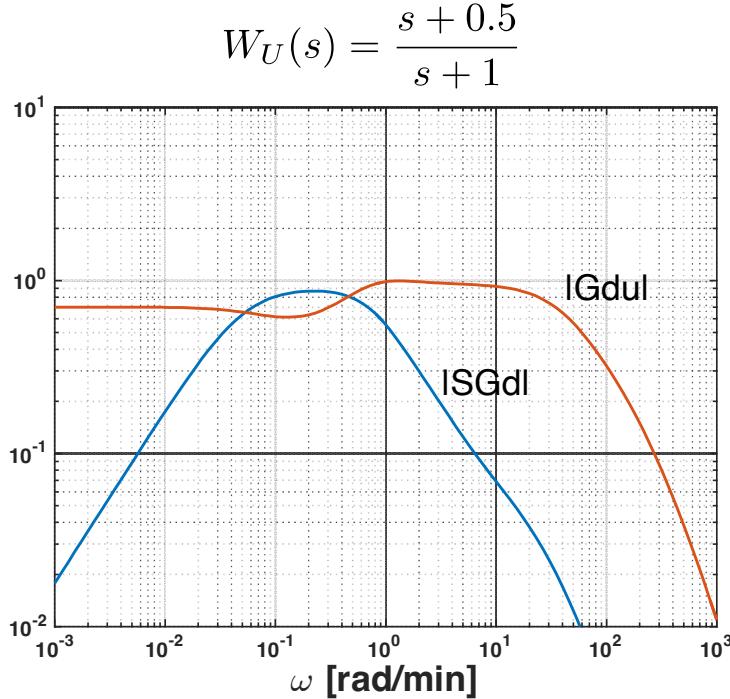
Simulation



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Add weight to input

- Try to push down input usage at high frequencies by introducing

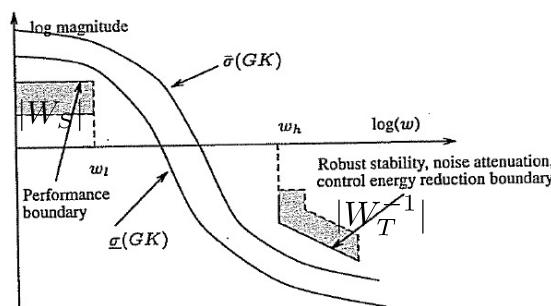


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Loop Shaping

Need to translate bounds on $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ into bounds on $\sigma_i(L)$, $L = GF_y$

- $\underline{\sigma}(L) \gg 1 \Rightarrow \bar{\sigma}(S) \approx 1/\underline{\sigma}(L)$ and hence $\underline{\sigma}(L) > |w_S|$ where $|w_S| \gg 1$
- $\bar{\sigma}(L) \ll 1 \Rightarrow \bar{\sigma}(T) \approx \bar{\sigma}(L)$ and hence $\bar{\sigma}(L) < |w_T^{-1}|$ where $|w_T| \gg 1$



- Robust loop shaping; robustify the shaped plant by maximizing robustness margin for coprime uncertainty

H_∞ Synthesis

- Given a state space system G_0 on the form

$$\begin{aligned}\dot{x} &= Ax + Bu + Nw_e \\ z_e &= Mx + Du \quad (*) \\ y &= Cx + w_e\end{aligned}$$

- Determine if a controller $u = -F_y(s)y$ exists such that for the resulting closed-loop system G_{ec} and given γ

$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} < \gamma \quad (1)$$

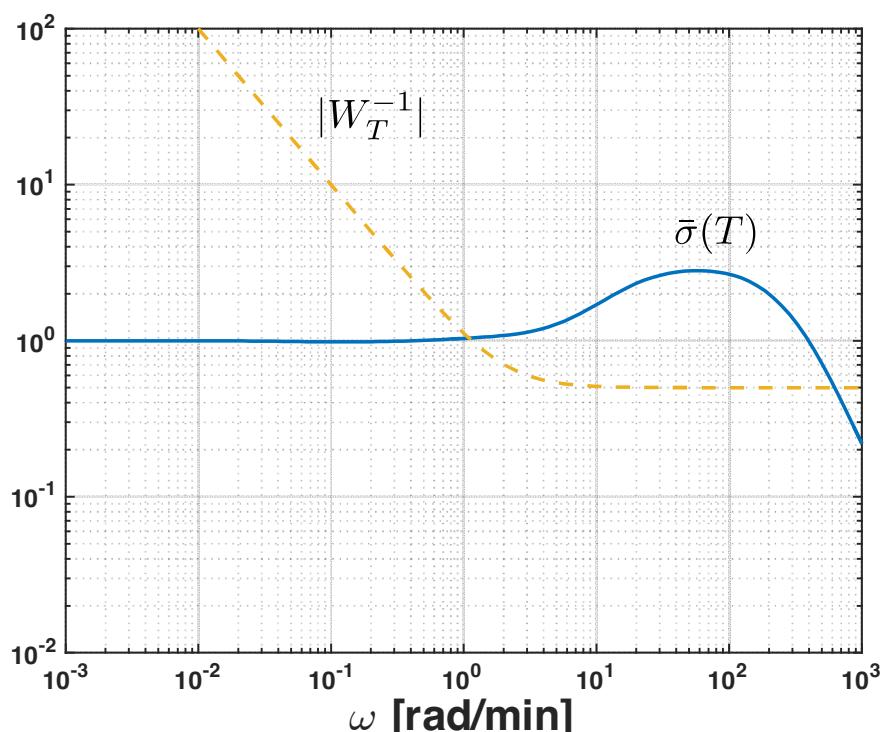
- Let $P > 0$ be a solution to the algebraic Riccati equation

$$A^T P + PA + M^T M + P(\gamma^{-2} NN^T - BB^T)P = 0$$

if $A - BB^T P$ is stable then the controller exists, otherwise not

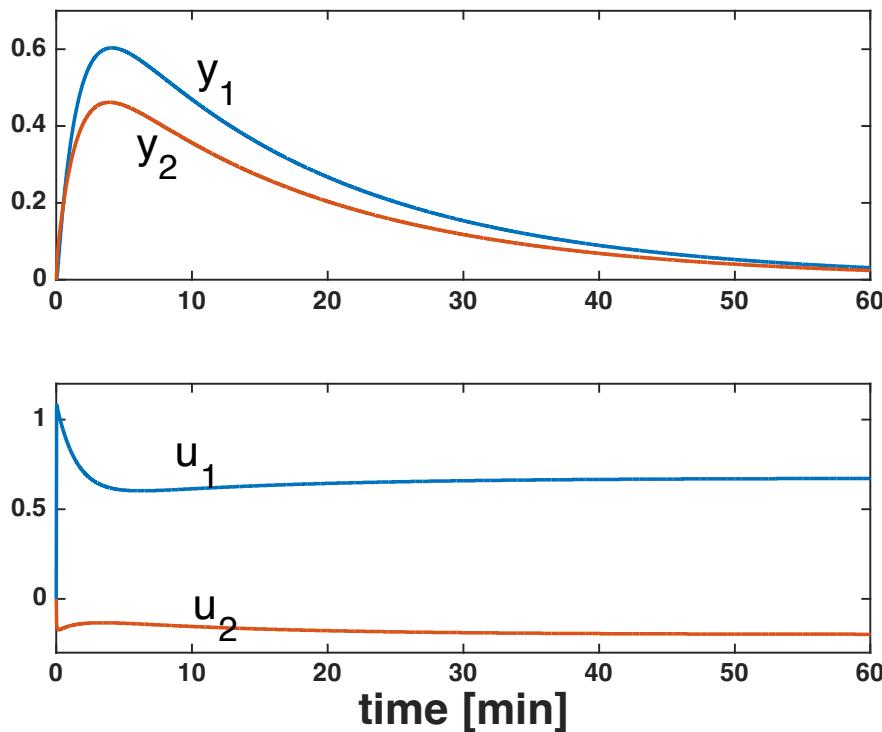
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Robust Stability



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Simulation unit step disturbance



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The H_∞ -optimal controller

- A controller satisfying requirement (1) is then given by
$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + N(y - C\hat{x}) \\ u &= -L_\infty \hat{x}; \quad L_\infty = B^T P\end{aligned}$$
i.e., state observer combined with state feedback
- The optimal controller can be found by iterating on γ until $\gamma \approx \gamma_{min}$
- To solve the original problem (*) we note that with $z_e = G_{ec}w_e$

$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} < \gamma \Leftrightarrow \|G_{ec}\|_\infty < \gamma$$

- Thus, select the output z_e and input w_e such that

$$z_e = G_{ec}w_e = \begin{bmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{bmatrix} w_e$$

and determine corresponding open-loop system G_0

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H₂-optimal control

- Similar to in H _{∞} -synthesis, define the extended system G_0

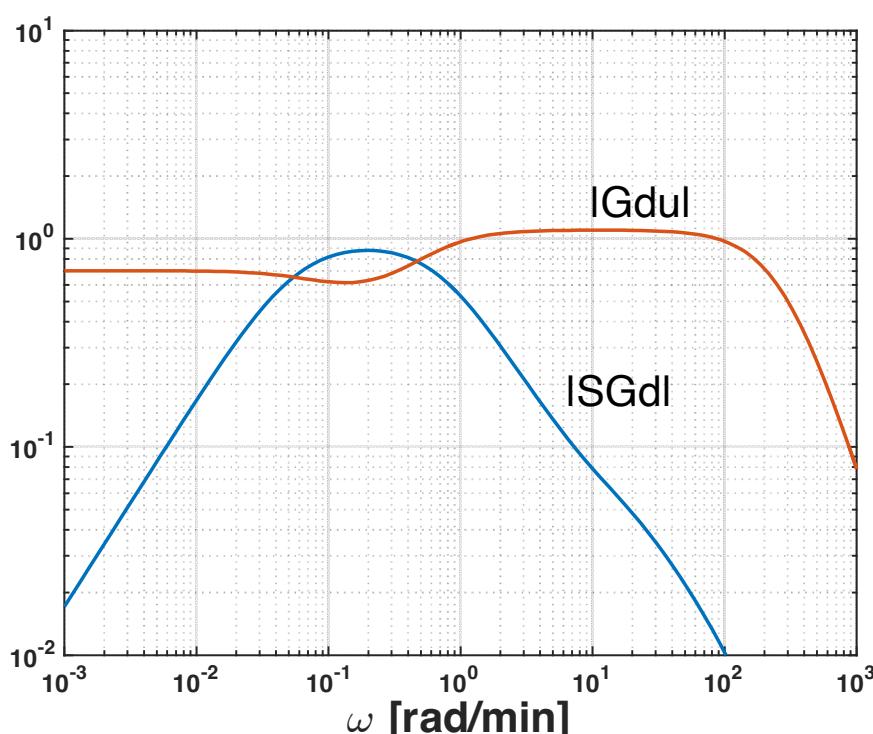
$$\begin{aligned}\dot{x} &= Ax + Bu + Nw_e \\ z_e &= Mx + Du \\ y &= Cx + w_e\end{aligned}$$

such that that the closed-loop transfer-matrix, with $u = -F_y y$, is the one we want to minimize

- If w_e is white noise with intensity $\Phi_{w_e} = I$, then minimizing $\|z_e\|_2$ corresponds to minimizing $\|G_{ec}\|_2$ where G_{ec} is closed-loop transfer-matrix from w_e to z_e
- Corresponds to LQG problem with $Q_1 = Q_2 = R_1 = R_2 = I$

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Performance



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Design I: H_∞ signal based, no robustness

```
WS=((s+.05)/(s+1.d-4)); % Adds integral action  
WU=1;  
  
d=icsignal(1);  
u=icsignal(2);  
y=icsignal(2);  
Wy=icsignal(2);  
Wu=icsignal(2);  
P=iconnect;  
P.Input=[d;u];  
P.Output=[Wy;Wu;-y];  
P.equation{1}=equate(y,G*u+Gd*d);  
P.equation{2}=equate(Wy,WS*y);  
P.equation{3}=equate(Wu,WU*u);  
  
[C,CL,gam]=hinfsyn(P.System,2,2);  
  
=> gam  
gam =  
    1.104
```

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H_2 and H_∞ optimal control

H_2 -optimal control

$$\min_{F_y} \|G_{ec}\|_2^2 = \min_{F_y} \int_{-\infty}^{\infty} \sum_i \sigma_i^2(G_{ec}(i\omega)) d\omega$$

(reduce all singular values at all frequencies)

H_∞ -optimal control

$$\min_{F_y} \|G_{ec}\|_\infty = \min_{F_y} \sup_{\omega} \bar{\sigma}(G_{ec}(i\omega))$$

(reduce maximum singular value at worst frequency)

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Design example

DC servo from Lecture 5:

$$G(s) = \frac{1}{s(s+1)}$$

Same performance requirements as in Lec 9

Two key points:

- H_∞ optimal design allows to work directly with constraints
- The relation between H_2 and H_∞ optimal controllers

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Controllability Analysis (see lec 6)

- Is control needed?

Yes, since $\|G_d\|_\infty > 1$ and requirement is $\|SG_d\|_\infty < 1$

- What is required bandwidth?

$$\|G_d(i\omega)\|_2 > 1, \omega < 0.35$$

Thus, need to attenuate disturbance up to $\omega \approx 0.35$

- Any fundamental limitations?

Yes, there is a RHP zero $z = 0.62$ which gives bandwidth limitation for S $\omega_{BS} \lesssim 0.31$

- Conflict? Need to attenuate disturbances only in disturbance direction
 - zero direction $y_z = [0.816 \ 0.578]^T$
 - with $G_d(z) = [0.313 \ 0.340]^T$ we get $|y_z^H G_d(z)| = 0.45 < 1$ OK!

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Chemical Reactor Model

- Scaled model (so that all signals should have magnitude <1 at all frequencies)

$$G(s) = \frac{1}{(15.2s + 1)(3.1s + 1)} \begin{pmatrix} 22.4(3.1s + 1) & 59.4(8.3s + 1) \\ -12.6(10.2s + 1) & -60.6(12.1s + 1) \end{pmatrix}$$

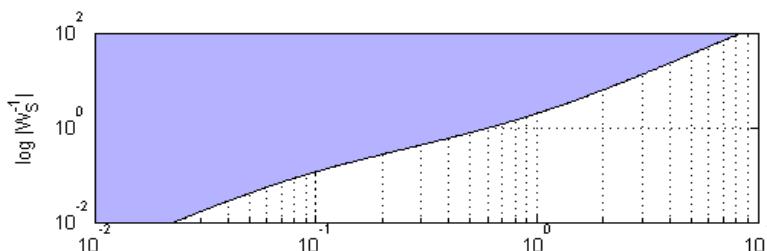
$$G_d(s) = \frac{1}{(15.2s + 1)} \begin{pmatrix} 3.28 \\ 3.56 \end{pmatrix}$$

- Aim: keep output deviations less than 1 in presence of disturbances with magnitude up to 1.
- uncertainty: measurement uncertainty exceeds 100% for frequencies above 1 rad/min. Use uncertainty weight

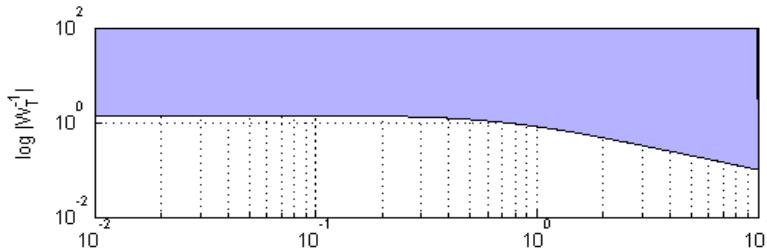
$$W_T(s) = \frac{2s}{s + 2} I$$

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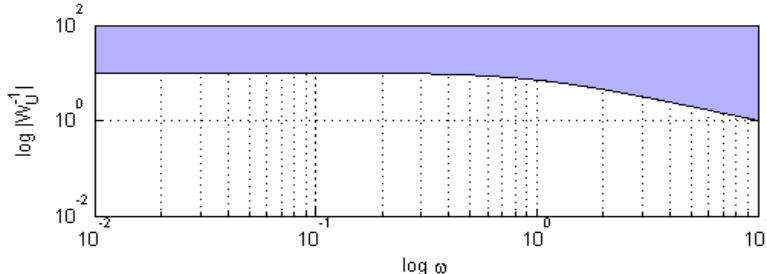
Weights



$$W_S(s) = \frac{0.71s + 0.05}{s^2(s + 1)}$$



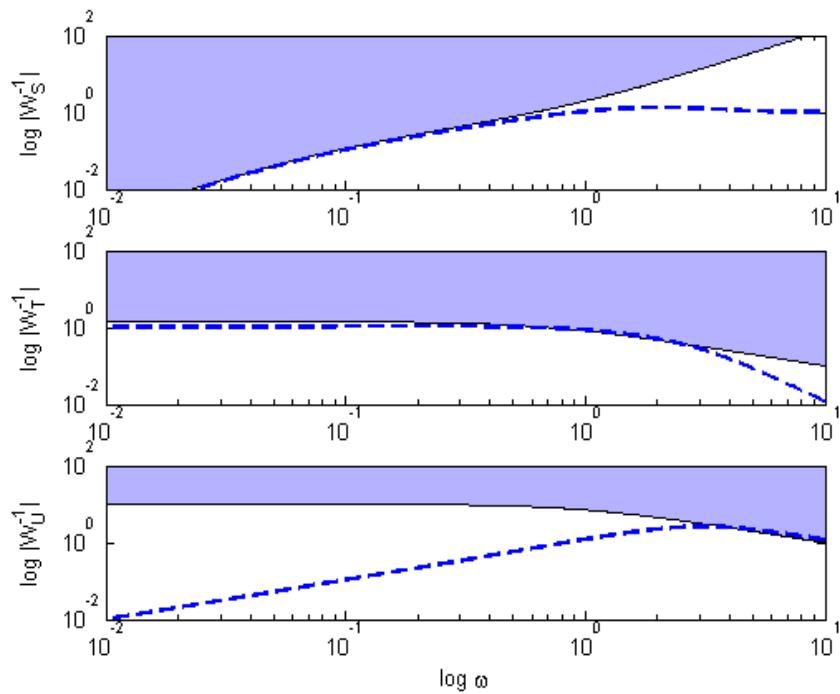
$$W_T(s) = s + 0.71$$



$$W_U(s) = \frac{10s + 10}{s + 100}$$

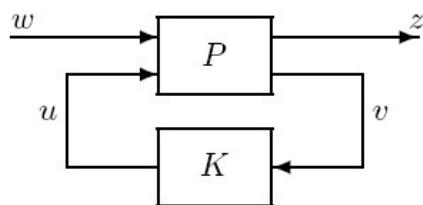
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H_∞ optimal control



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H_2/H_∞ optimal control for signal minimization



- So far: select input w and output z to reflect system transfer-function G_{ec} that we want to shape
- But: in many cases specifications may be directly on signals, e.g., keep an output less than some constraint in the presence of a bounded disturbance. Then w and z given directly by problem formulation.
- Note that weightings still will be in the frequency domain
- Case study: controller design for chemical reactor using signal minimization and sensitivity shaping, respectively

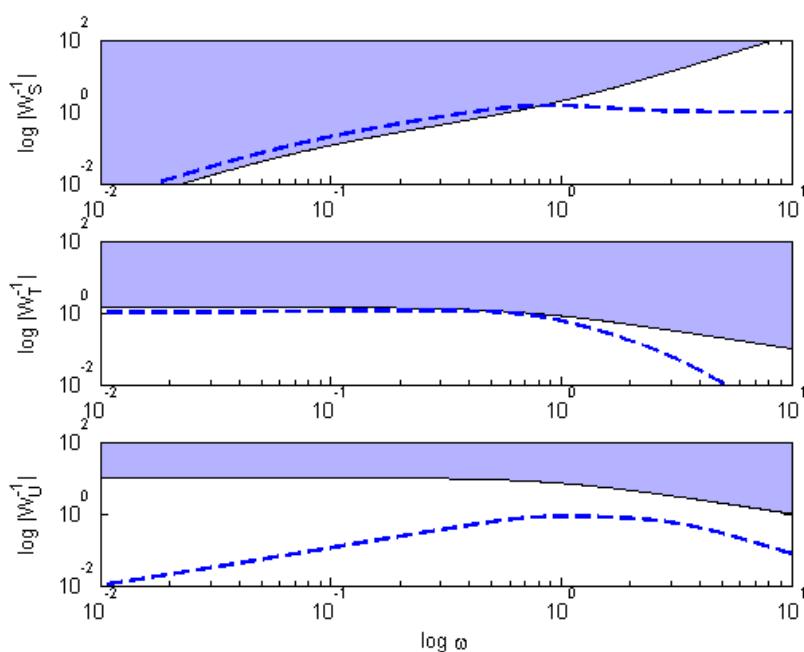
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LQG for System Optimization

- LQG may be used to obtain desired system properties, e.g., sensitivity and complementary sensitivity
- Systematic method, called Loop Transfer Recovery (LTR)
 - Main drawback: based on cancellations, and will even cancel RHP zeros and hence loss of internal stability
 - Not treated in this course
- Better to use H_2 -optimization which is a direct method, also based on using the LQG machinery

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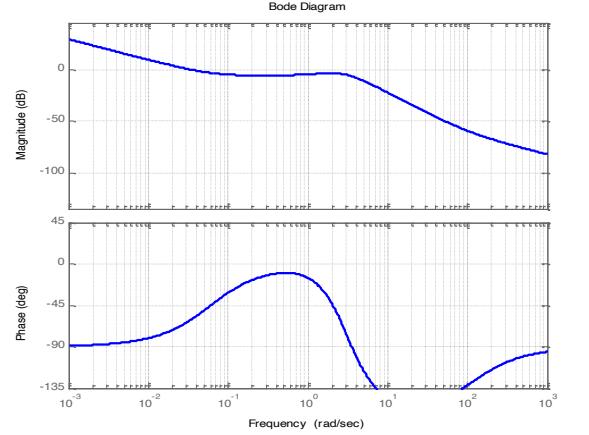
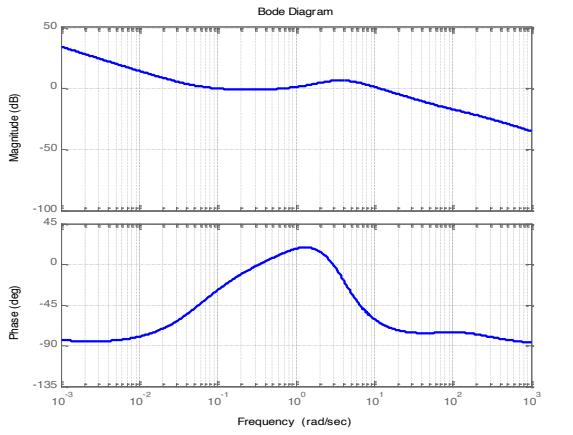
H_2 -optimal controller



Quiz: why doesn't the H_2 -optimal controller “meet the specs”?

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Comparing the controllers



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Signal vs System Optimization

- LQG is formulated as signal minimization problem, i.e., minimize weighted control error and weighted control input in the presence of (filtered) white noise disturbances
- H_2 - and H_∞ -optimal control usually considered as system norm minimization, e.g., minimize weighted sensitivity and weighted complementary sensitivity
- But, equivalence exists between system properties and signal properties, e.g.,

$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} = \|G_{ec}\|_\infty$$

- Also, equivalence exist between LQG and H_2 -optimal control; minimizing 2-norm of output in presence of white noise equals minimizing 2-norm of corresponding transfer-function

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