EL2520
Control Theory and Practice

MIMO Robustness, Controller Design

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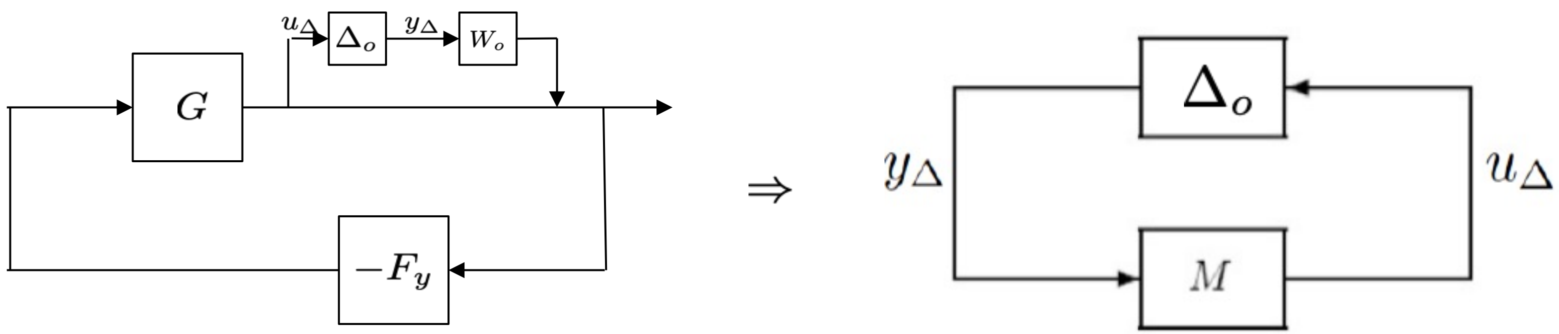
Today's Lecture

- Robust stability of MIMO feedback systems
- Controller design to meet performance specifications and robust stability
  - Loop shaping
  - $\mathcal{H}_\infty$-optimal control

Robust Stability

- Assume model uncertainty

$$G_p(s) = (I + W_o(s)\Delta_o(s))G(s), \ \|\Delta_o\|_\infty < 1 \ (\Rightarrow \bar{\sigma}(\Delta_o) < 1 \ \forall\omega)$$



- Small Gain Theorem: closed-loop stable if $M, \ \Delta_o$ stable and

$$\|M\|_\infty \|\Delta_o\|_\infty < 1 \ \ \Rightarrow \|M\|_\infty < 1$$

- Identify M from block-diagram

$$M = -TW_o$$

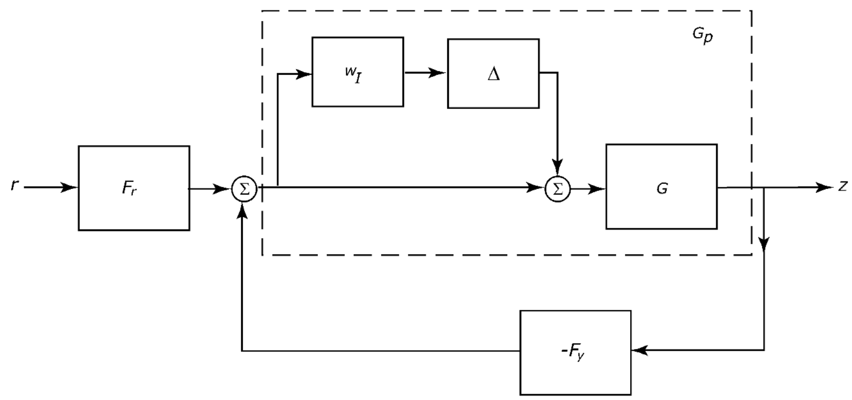
- Thus, robust stability condition

$$\|TW_o\|_\infty < 1$$

Robust stability

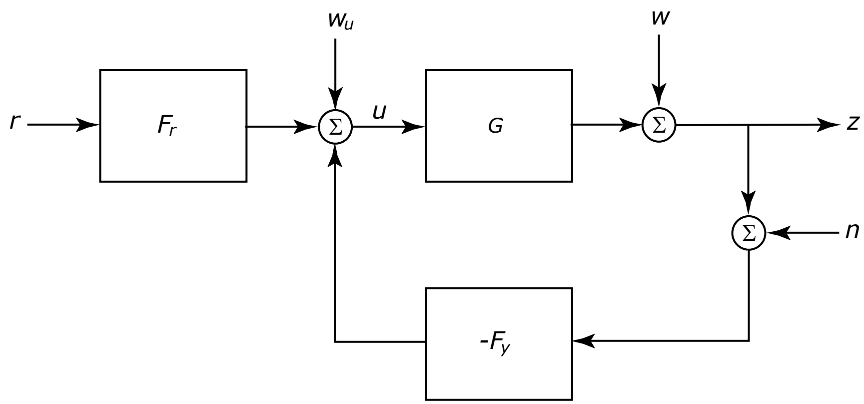
- Same result as for SISO case
- What if uncertainty is on input side of G?

$$G_p(s) = G(s)(I + W_I(s)\Delta_I(s)), \ \|\Delta_I\|_\infty < 1$$



  - Exercise: derive the robust stability condition for this case!

Controller Design



- *Aim:* determine controller $F_y(s)$ (and $F_r(s)$) to shape desired closed-loop transfer-functions, e.g., S and T
- *Example:* attenuate disturbance w and noise n

$$z = Sw - Tn \,, \quad u = -F_y S(w+n) = G_{wu}(w+n)$$

  – thus, make S, T and Gwu ''small''

  – introduce weights and design controller to achieve

$$\|W_S S\|_\infty < 1, \quad \|W_T T\|_\infty < 1, \quad \|W_u G_{wu}\|_\infty < 1$$

Controller Design

How design controller to achieve objectives?

1. *Loop-shaping:* shape open-loop $L = GF_y$, or
2. *Controller synthesis:* solve optimization problem

$$F_y = \arg\min_{F_y} \left\| \begin{matrix} W_S S \\ W_T T \\ W_u G_{wu} \end{matrix} \right\|_\infty$$

Loop Shaping (for S and T)

Translate bounds on $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ into bounds on $\sigma_i(L),\ L = GF_y$:

- We have $\bar{\sigma}(S) = \bar{\sigma}((I+L)^{-1})$. Since $\bar{\sigma}(A^{-1}) = 1/\underline{\sigma}(A)$ we get

$$\bar{\sigma}(S) = \frac{1}{\underline{\sigma}(I+L)}$$

- From Fan's Theorem (see lec notes 6)

$$\underline{\sigma}(L) - 1 \leq \underline{\sigma}(I+L) \leq \underline{\sigma}(L) + 1$$

- Thus,

$$\underline{\sigma}(L) >> 1 \quad \Rightarrow \quad \bar{\sigma}(S) \approx \frac{1}{\underline{\sigma}(L)}$$

- This gives the loop-shaping bounds

$$\bar{\sigma}(S) \leq |W_S^{-1}| \quad \Rightarrow \quad \underline{\sigma}(L) \geq |W_S|,\ |W_S| >> 1$$

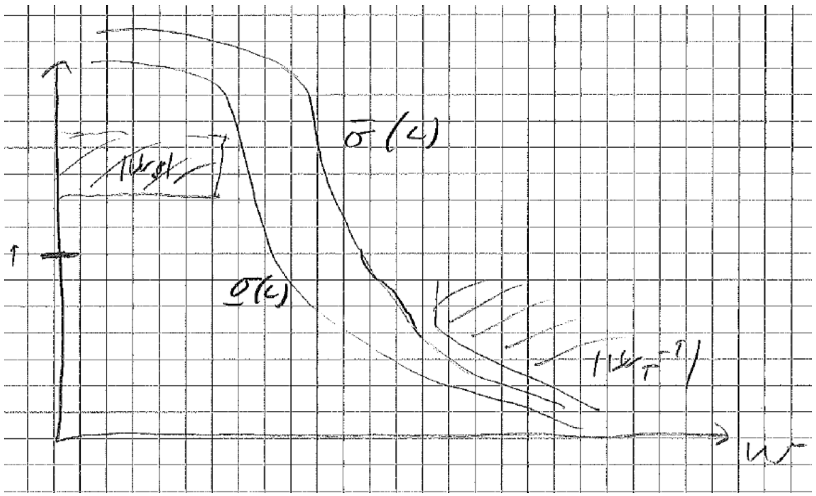
Loop Shaping (for S and T)

- Similar derivation for T (see lec notes 7):

$$\bar{\sigma}(L) << 1 \quad \Rightarrow \quad \bar{\sigma}(T) \approx \bar{\sigma}(L)$$

  and

$$\bar{\sigma}(T) \leq |W_T^{-1}| \quad \Rightarrow \quad \bar{\sigma}(L) \leq |W_T^{-1}|,\ |W_T| >> 1$$

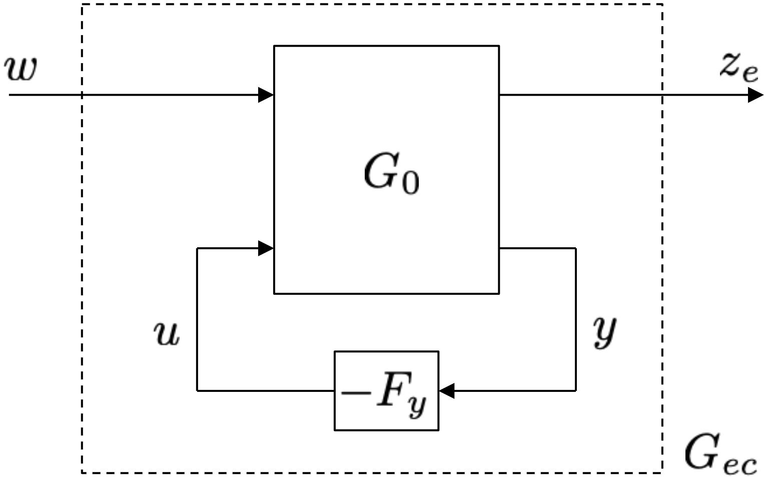


- Note: difficult to deal with stability since there is no phase defined for MIMO systems. This will be addressed when we consider *robust loopshaping* later in the course

$\mathcal{H}_\infty$-Synthesis

$$F_y = \arg\min_{F_y} \left\| \begin{matrix} W_S S \\ W_T T \\ W_u G_{wu} \end{matrix} \right\|_\infty \quad (*)$$

- Solution of (*) is based on the *extended system:*



$$z_e = G_{ec} w$$

- Equivalence between signal minimization problem and system norm minimization:

$$\min_{F_y} \sup_{w} \frac{\|z_e\|_2}{\|w\|_2} = \min_{F_y} \|G_{ec}\|_\infty \quad (**)$$

$\mathcal{H}_\infty$-Synthesis

- Choose signals w and z_e such that G_{ec} corresponds to transfer-functions we want to shape, i.e.,

$$G_{ec} = \begin{bmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{bmatrix}$$

  and determine corresponding $G_0(s)$
- Solution of (**) is based on state-space realization of $G_0(s)$:

$$\begin{aligned} \dot{x} &= Ax + Bu + Nw \\ z_e &= Mx + Du \qquad\qquad (***) \\ y &= Cx + w \end{aligned}$$

  – normalized so that $D^T M = 0, \; D^T D = I$

Solution

- No direct solution to (**)
- Indirect solution: select real number γ and determine whether controller that gives $\|G_{ec}\|_\infty = \gamma$ exists
  - Let P>0 be a solution to the algebraic *Riccati equation*

$$A^T P + P^T A + M^T M + P(\gamma^{-2} N N^T - B B^T) P = 0$$

  - Then, if $A - BB^T P$ is stable, the controller

$$\left.\begin{array}{rcl} \dot{\hat{x}} & = & A\hat{x} + Bu + N(y - C\hat{x}) \\ u & = & -L_\infty \hat{x} \ , \quad L_\infty = B^T P \end{array}\right\} F_y(s)$$

    will give $\|G_{ec}\|_\infty = \gamma$
  - State feedback + Observer !
- Optimal controller: iterate on γ until $\gamma \approx \gamma_{min}$ (known as γ-iterations)

Outline of Proof

For a proof, see Lecture notes. Main idea:

- Consider function

$$V(t) = x^T(t)Px(t) + \int_0^t (z_e^T(\tau)z_e(\tau) - \gamma^2 w_e^T(\tau)w_e(\tau))d\tau$$

- If P>0 and V(t)<0 for all t and all w_e , then the integral term must be negative and it follows that

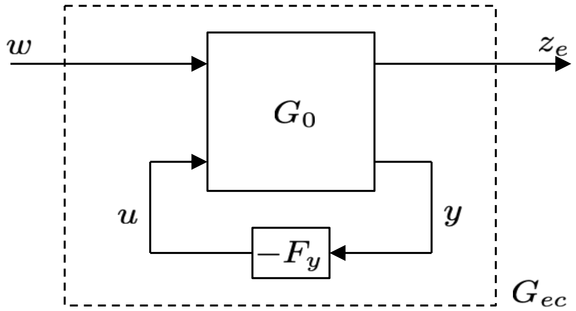
$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} < \gamma$$

- Since V(0)=0 it is sufficient to show that $\dot{V}(t) < 0 \ \forall t$

$$\begin{aligned} \dot{V} &= x^T(A^TP + PA + M^TM - P(BB^T - \gamma^{-2}NN^T)P)x + (u + B^TPx)^T(u + B^TPx) \\ &\quad -\gamma^{-2}(w - \gamma^{-2}N^TPx)^T(w - \gamma^{-2}N^TPx) \end{aligned}$$

- Thus, if P>0 solves Riccati equation and we choose $u = -B^TPx$ we get the result

Summary of $\mathcal{H}_\infty$-synthesis



1. Choose signals z_e and w such that the corresponding closed-loop transfer-function G_{ec} corresponds to the transfer-function we want to minimize the norm of
2. Solve equivalent signal minimization problem

$$\min_{F_y} \sup_{w} \frac{\|z_e\|_2}{\|w\|_2} = \min_{F_y} \|G_{ec}\|_\infty$$

3. The signal optimization problem is solved in state-space based on state-space realization of corresponding open-loop model $G_0(s)$
4. Solution based on γ-iterations and the optimal controller is on the form of state feedback combined with an observer.

Selecting signals

- Choose z_e and w such that

$$G_{ec} = \begin{bmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{bmatrix}$$

1. $W_S S$: $z_{e1} = W_S S w$
   - we have $z = Sw \;\Rightarrow\; z_{e1} = W_S z,\; w = w$ (disturbance on output)
   - for $G_0(s)$: $z = Gu + w \;\Rightarrow\; \boxed{z_{e1} = W_S(Gu + w)}$
2. $W_T T$: $z_{e2} = W_T T w$
   - we have

     $$z = Sw = (I - T)w \;\Rightarrow\; z - w = -Tw \;\Rightarrow\; z_{e2} = -W_T(z - w)$$
   - for $G_0(s)$: $z - w = Gu \;\Rightarrow\; \boxed{z_{e2} = -W_T G u}$

Selecting Signals

3. $W_u G_{wu}$: $z_{e3} = W_u G_{wu} w$
   - we have $u = G_{wu} w \;\Rightarrow\; z_{e3} = W_u u$
   - for $G_0(s)$: $u = u \;\Rightarrow\; \boxed{z_{e3} = W_u u}$

- Thus, we get

$$G_0 = \begin{bmatrix} W_S & W_S G \\ 0 & -W_T G \\ 0 & W_u \\ I & G \end{bmatrix} \qquad \left(\begin{bmatrix} z_e \\ y \end{bmatrix} = G_0 \begin{bmatrix} w \\ u \end{bmatrix} \right)$$

Next Time

- Classical optimal control LQG
- Comparison with $\mathcal{H}_\infty$- optimal control