EL2520 - Control Theory and Practice - Advanced Course
Solution/Answers – 2020-05-25

1. (a) The LCD of all minors are $(s+1)^2(s+2)$, hence two poles in $s=-1$ and one pole in $s=-2$. The zero polynomial is $2(1-s)$ and hence a zero at $s=1$. For the zero we get from the right and left nullspace of $G(1)$
$$u_z^H = \frac{1}{\sqrt{5}}\begin{pmatrix}-1 & 2\end{pmatrix} \;;\quad y_z^H = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\end{pmatrix}$$

   (b) $\|G\|_\infty = \sup_\omega \bar{\sigma}(G(i\omega))$. Here the maximum over all frequencies occurs at $\omega = 0$ (since G is low-pass), and
$$\bar{\sigma}(G(0))^2 = \lambda_{max}(G(0)^T G(0)) = 9$$
and hence $\|G\|_\infty = \sqrt{9} = 3$.

   (c) We use the RGA. At $\omega = 0$ we get
$$\Lambda(0) = \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$$
and at the desired bandwidth, which is close to the crossover frequency,
$$|\Lambda(i1)| = \begin{pmatrix}0.2 & 1\\ 1 & 0.2\end{pmatrix}$$
where we somewhat carelessly use $|\Lambda|$ to denote the absolute values of the elements. Since we should never pair on negative RGA elements at steady-state we only have the pairing on the diagonal as the option. The RGA for this pairing is 0.2 around the crossover which is rather small (less than the recommended in the lecture notes) and it is therefore not recommended to use decentralized control for this system.

2. (a) The sensitivity function is always semi-proper and the $\mathcal{H}_2$-norm is only defined for stable and strictly proper systems.

   (b,i) The system G has a zero at $s=1$, and hence the complementary sensitivity $T(s)$ should also have a zero at $s=1$ (otherwise it has been canceled resulting in internal instability). From $T = I - S$ we get
$$T(s) = \frac{1}{s+1}\begin{pmatrix}1 & -0.1s\\ 0.1s & 1\end{pmatrix}$$
which does not have a zero at $s=1$ ($T(1)$ is full rank), and hence the closed-loop is not internally stable.

   (b,ii) We require $|W_p(z)| < 1$. Here $W_p(1) = (5\tau+10)/(10\tau+1)$ and hence we must choose $\tau > 9/5$.

(c) The system is unstable if some eigenvalue has magnitude larger than 1. The eigenvalues are 1 and $1-\gamma+\frac{\beta}{N}S_e$. Thus, unstable if

$$1-\gamma+\frac{\beta}{N}S_e > 1 \quad \Rightarrow \quad \frac{\beta}{\gamma} > \frac{N}{S_e}$$

So, with $S_e = N$ we get instability if $R_0 > 1$.

3. (a) Derive the transfer-function from u to z

$$G_{zw}(s) = 2\frac{s-2}{(s-1)(s+1)}$$

Since we have both a RHP zero and a RHP pole, we have

$$\|S\|_\infty > \frac{|z+p|}{|z-p|} = 3$$

Hence, the worst case amplification of disturbances at the output will be at least 3 with any linear controller.

(b) The transfer-function $G(s)$ has a zero at $s=1$ with output direction $y_z^H = \frac{1}{\sqrt{5}}\begin{pmatrix}-2 & 1\end{pmatrix}$. In order to keep $|z| < 1$ for $|d| < 1$ at all frequencies, we require $|y_z^H g_d(z)| < 1$

$$|y_z^H g_d(z)| = 0.89 < 1$$

and hence it should be feasible to obtain acceptable disturbance attenuation.

(c) Here $|S(i\omega)| < 1\ \forall\omega$ which is not possible if the loop gain has relative order ≥ 2 according to the Bode sensitivity integral. Since G has relative order 3, we would need an improper controller (more zeros than poles) to get a loop-gain with relative degree less than 2 and such controllers can not be realized.

4. (a) We set

$$1+W_I(s)\Delta(s) = e^{-\theta s}$$

and hence we should choose the weight

$$W_I(i\omega) = e^{-\theta i\omega} - 1$$

and the RS-condition is

$$|T(i\omega)| < \frac{1}{|e^{-\theta i\omega}-1|}\ \forall\omega$$

Consider the bandwidth ω_{BT} where $|T(i\omega_{BT}| = 1/\sqrt{2}$. At this frequency $|W_I(i\omega_{BT})| < \sqrt{2}$ for robust stability, and we find $\omega_{BT}\theta <$ 1.58 or

$$\omega_{BT} \leq \frac{1.58}{\theta}$$

(b) The first condition corresponds to requiring $|SG| < 0.1\ \omega < 0.5$. A possible weight is

$$W_1 = \frac{10}{2s+1}$$

and then requiring

$$\|W_1 SG\|_\infty < 1$$

This fulfills the requirement asymptotically, but strictly the weight is only $10/\sqrt{2}$ at the frequency 0.5. One possibility is to multiply W_1 by the factor $\sqrt{2}$, or move the pole to high frequencies, i.e., $W_1 = \frac{10}{20s+1}$. The second condition is fulfilled if the robust stability condition

$$\|W_I T_I\|_\infty < 1$$

where $T_I = (1 + F_y G)^{-1} F_y G$ is the complimentary sensitivity at the input. With 30% uncertainty we have $W_I = 0.3$. Stacking the two objectives into one matrix we get

$$F_y = \arg\min_{F_y} \left\| \begin{matrix} W_1 SG \\ 0.3 T_I \end{matrix} \right\|_\infty$$

If the resulting norm is less than 1, then we have achieved the two objectives with some margin.

5. (a) The system has a zero at $s = 0.2$ and $|W_P(0.2)| = 2.2/0.41 = 5.37 > 1$ and hence the objective $\|W_P S\|_\infty < 1$ is not feasible.

(b) The transfer-matrix has rank 1 for all s and hence there are no zeros whatsoever. However, since we only have one control input we can only specify the sensitivity for the system with the output $\hat{z} = \begin{pmatrix} z \\ y \end{pmatrix}$ and hence we can not specify the sensitivity for the output z. (We also say that the system is not functionally controllable when there are less inputs than outputs).

(c) Consider first closing the loop between v and y using a controller $v = -K_p y$. The transfer-function from the inputs to the outputs is then

$$\begin{pmatrix} z \\ y \end{pmatrix} = \underbrace{G(I + F_y G)^{-1}}_{G_c} \begin{pmatrix} u \\ v \end{pmatrix}$$

where

$$F_y = \begin{pmatrix} 0 & 0 \\ 0 & K_p \end{pmatrix}$$

We get for element $1,1$, i.e., the transfer-function from u to z

$$G_{c,11} = \frac{s^2 + (K_p + 0.8)s - 0.2 + 0.1K_p}{(s+1)(s+1+2K_p)}$$

and we see that the zero of $G_{c,11}$ depends on the controller gain K_p, and is in the LHP if $K_p > 2$. To make the control objective feasible it suffices to have the zero $z > 1.99$ or $z < 0$. Making $z > 1.99$ is not feasible with a stabilizing K_p, and hence we should choose $K_p > 2$ to move the zero into the LHP.

If we instead closed the loop between u and y with a P-controller, then we get for the transfer-function from v to z

$$G_{c,12} = \frac{s^2 + (0.5 - K_p)s - 0.5 - 0.1K_p}{(s+1)(s+1+K_p)}$$

Since $K_p > -1$ for stability, it is not possible to move the zero into the LHP. However, we can move the zero such that $z > 1.99$ by using the gain $K_p > 2.14$.

Using v to control y has the advantage that the zero can be moved into the LHP altogether, while the other pairing has the advantage that the required loop-gain for controlling y is significantly less.