EL2520
Control Theory and Practice

LQG (cont’d) and $\mathcal{H}_2$-optimal control

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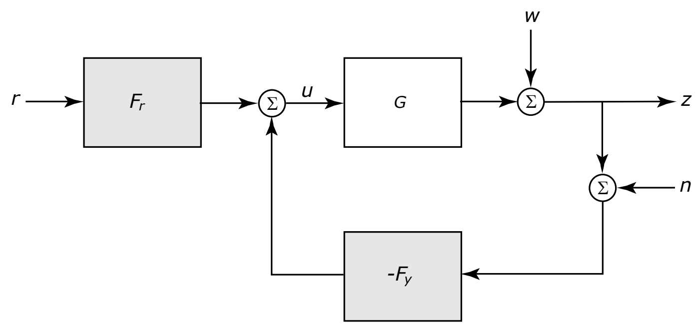
Controller Design – Signals vs Systems



$$z = Sw + Tn$$

- Design controller so that output z ‘’small’’ in the presence of disturbance w and noise n
- Corresponds to making transfer-functions S and T ‘’small’’
- Thus, we can either solve signal minimization problem, i.e., minimize some norm of z given inputs w and n, or solve corresponding transfer-function minimization problem.

H-infinity, LQG and H2

- $\mathcal{H}_\infty$:

$$\min_{F_y} \sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} = \min_{F_y} \|G_{ec}\|_\infty \; ; \quad z_e = G_{ec} w_e$$

  - minimize worst case amplification in terms of 2-norm of signals is equivalent to minimizing infinity-norm of corresponding transfer-function

- LQG:

$$J = \mathbf{E}\left\{ \lim_{T\to\infty} \int_0^T [z^T Q_1 z + u^T Q_2 u] \, dt \right\}$$

  - minimize expected 2-norm of weighted output z and weighted input u when w and n white noise
  - *will show today:* special case of LQG corresponds to minimizing 2-norm of corresponding transfer-function

Today’s lecture

- LQG recap and additional remarks
- H2-optimal control

- Lec 10: Robust Loopshaping
- Lec 11: Case study and comparison of methods

Linear Quadratic Gaussian control

Model: linear system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nv_1(t)$$
$$y(t) = Cx(t) + v_2(t)$$
$$z(t) = Mx(t)$$

where v_1, v_2 are white noise with

$$\mathrm{cov}([v_1,\ v_2]) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Objective: minimize effect of noise on output, while punishing control cost

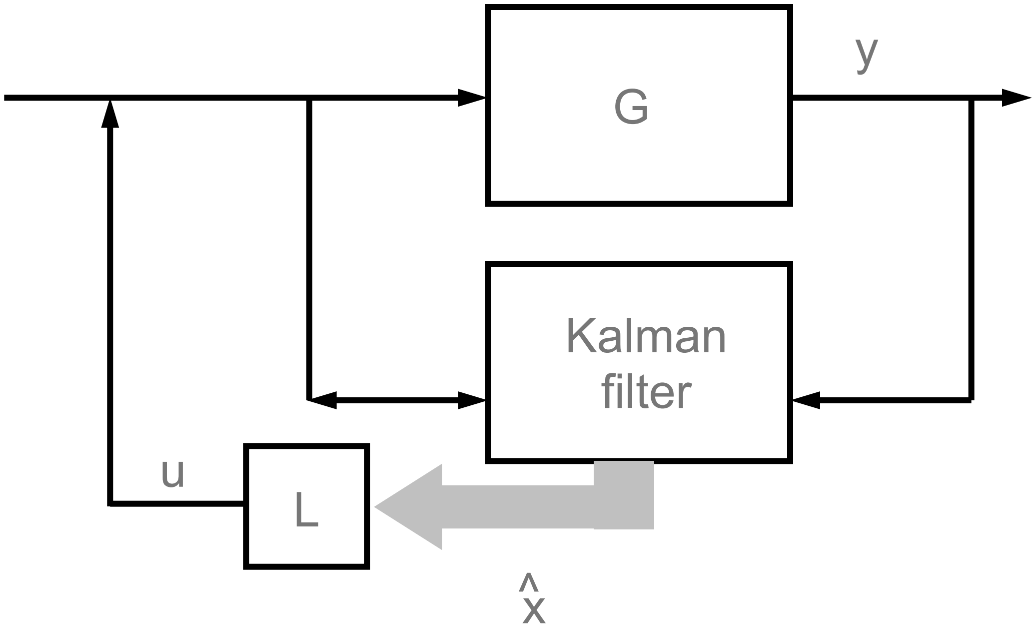
$$J = \mathbf{E}\left\{\lim_{T\to\infty}\int_0^T [z^T Q_1 z + u^T Q_2 u]\ dt\right\}$$

LQG: Linear system, Quadratic cost, Gaussian noise

Solution structure

Optimal solution satisfies *separation principle*, composed of

- Optimal linear state feedback (Linear Quadratic regulator)
- Optimal observer (Kalman filter)



The Optimal solution

State feedback

$$u(t) = -L\widehat{x}(t) = -Q_2^{-1}B^T S\widehat{x}(t)$$

where S>0 is solution to the algebraic Riccati equation

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1}B^T S = 0$$

Observer (Kalman filter)

$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where $K=(PC^T+NR_{12})R_2^{-1}$ and P>0 is solution to the algebraic Riccati equation

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

White Noise

- Inputs v_1 and v_2 are white noise signals with covariance matrices

$$E\{v_1 v_1^T\} = R_1 \ ; \quad E\{v_2 v_2^T\} = R_2$$

- The corresponding frequency spectra are

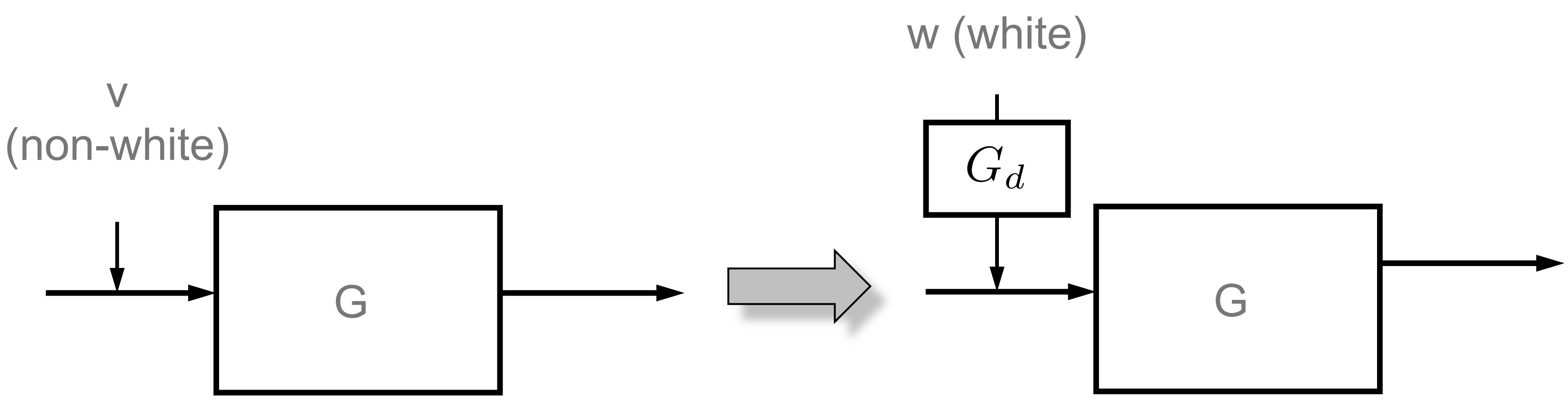
$$\Phi_{v_1}(\omega) = R_1 \ ; \quad \Phi_{v_2}(\omega) = R_2$$

  – white noise = constant spectra, i.e., same energy at all frequencies

Filtered White noise

Assumption of white noise no serious restriction:

- practically all disturbance spectra can be realized as filtered white noise
- need to augment system model with disturbance model



The servo problem

- The output z should follow a reference signal r
- Let r be the output of a linear system with white noise as input

➔ let v_1 include also the driving source of r

Extended system state:
$$\begin{cases} z = M_1 x \\ r = M_2 x \end{cases}$$

Error: $e = r - z = [-M_1 \quad M_2]x = Mx$

Note: r is known, and can be included in the measurement $\overline{y} = \begin{bmatrix} y \\ r \end{bmatrix}$

Controller: $u = -F_{\overline{y}}\overline{y} = F_r r - F_y y$

Objective: minimize
$$J = \mathbf{E}\left\{\lim_{T\to\infty}\int_0^T [e^T Q_1 e + u^T Q_2 u]\, dt\right\}$$

LQG and loop shaping

- LQG: simple to trade-off response-time vs. control effort
  - but what about sensitivity and robustness?
- These aspects can indirectly be accounted for using the noise models
  - Sensitivity function: transfer matrix w→z
  - Complementary sensitivity: transfer matrix n→z

Example: S forced to be small at low frequencies by letting (some component of) w affect the output, and let w have large energy at low frequencies,

$$W(s) = \frac{1}{s+\delta} V(s)$$

(delta small, strictly positive, to ensure stabilizability)

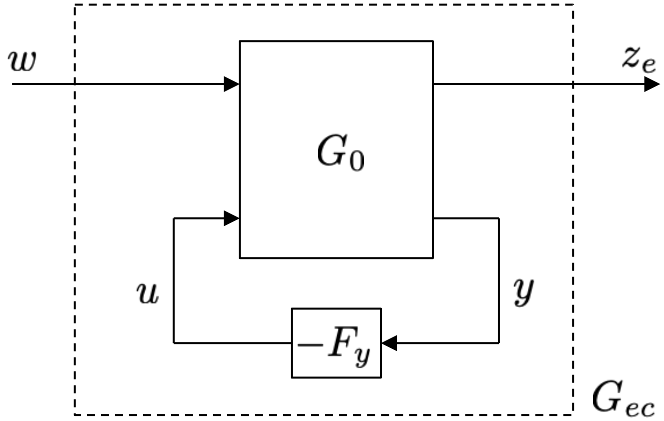
$\mathcal{H}_2$-optimal control

- Synthesize controller by solving optimization problem

$$F_y = \arg\min_{F_y} \|P(s)\|_2^2 = \arg\min_{F_y} \frac{1}{2\pi} \int_{-\infty}^{\infty} tr(P(i\omega)P^H(i\omega))d\omega \qquad (*)$$

  – P(s) is system we want to shape, e.g., $P = \begin{bmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{bmatrix}$

- Similar to in $\mathcal{H}_\infty$–optimal control, we determine an extended system



$$z_e = G_{ec} w$$

such that $G_{ec}(s) = P(s)$

A Signal Minimization Problem

- Basis for solution of (*) is state-space realization of $G_0(s)$:

$$\begin{aligned} \dot{x} &= Ax + Bu + Nw \\ z &= Mx + Du \qquad\qquad (**) \\ y &= Cx + w \end{aligned}$$

  – such that $D^T M = 0 \;\wedge\; D^T D = I$

- Assume disturbance w is white noise with unit intensity, i.e.,

$$E\{w(t)w^T(\tau)\} = I\delta(t-\tau) \quad \Rightarrow \quad \Phi_w(w) = I$$

- Then, from Parseval's Theorem

$$\begin{aligned} \|z(t)\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |Z(i\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_{ec}(i\omega)W(i\omega)|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} tr(G_{ec}(i\omega)G_{ec}^H(i\omega))d\omega = \|G_{ec}\|_2^2 = \|P\|_2^2 \end{aligned}$$

- Thus, minimizing $\|z(t)\|_2$ in presence of white noise w with unit intensity corresponds to minimizing 2-norm of system P

An LQG Problem

- Note that, with $D^T M = 0, D^T D = I$,

$$\|z(t)\|_2^2 = \|Mx + Du\|_2^2 = \|Mx\|_2^2 + \|u\|_2^2 = \int_0^\infty x^T M^T M x + u^T u dt$$

- Thus, (*) equivalent to LQG problem with $Q_1 = Q_2 = R_1 = R_2 = I$
- Solution: LQ + Kalman filter based on model (**)

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + N(y - C\hat{x}) \\ u &= -L\hat{x}, \quad L = B^T S, \ A^T S + SA + M^T M - SBB^T S = 0 \ \wedge \ S > 0 \end{aligned}$$

- Note
  - weights included in model (**)
  - Kalman gain K=N since model on innovation form (see also course book)

Summary: $\mathcal{H}_2$ and LQG

- Minimizing 2-norm of output in presence of white noise with $\Phi_w = I$ is equivalent to minimizing 2-norm of corresponding transfer-matrix
- Minimizing 2-norm of output in presence of white noise is also equivalent to an LQG problem
- Thus, minimizing 2-norm of weighted closed-loop transfer-matrices can be solved as an equivalent LQG problem

$\mathcal{H}_2$- vs $\mathcal{H}_\infty$-optimal control

H_2-optimal control

$$\min_{F_y} \|G_{ec}\|_2^2 = \min_{F_y} \int_{-\infty}^{\infty} \sum_i \sigma_i^2(G_{ec}(i\omega))\, d\omega$$

(push down all singular values at all frequencies)

H_∞-optimal control

$$\min_{F_y} \|G_{ec}\|_\infty = \min_{F_y} \sup_\omega \overline{\sigma}(G_{ec}(i\omega))$$

(push down maximum singular value at worst frequency)

Design example

DC servo:

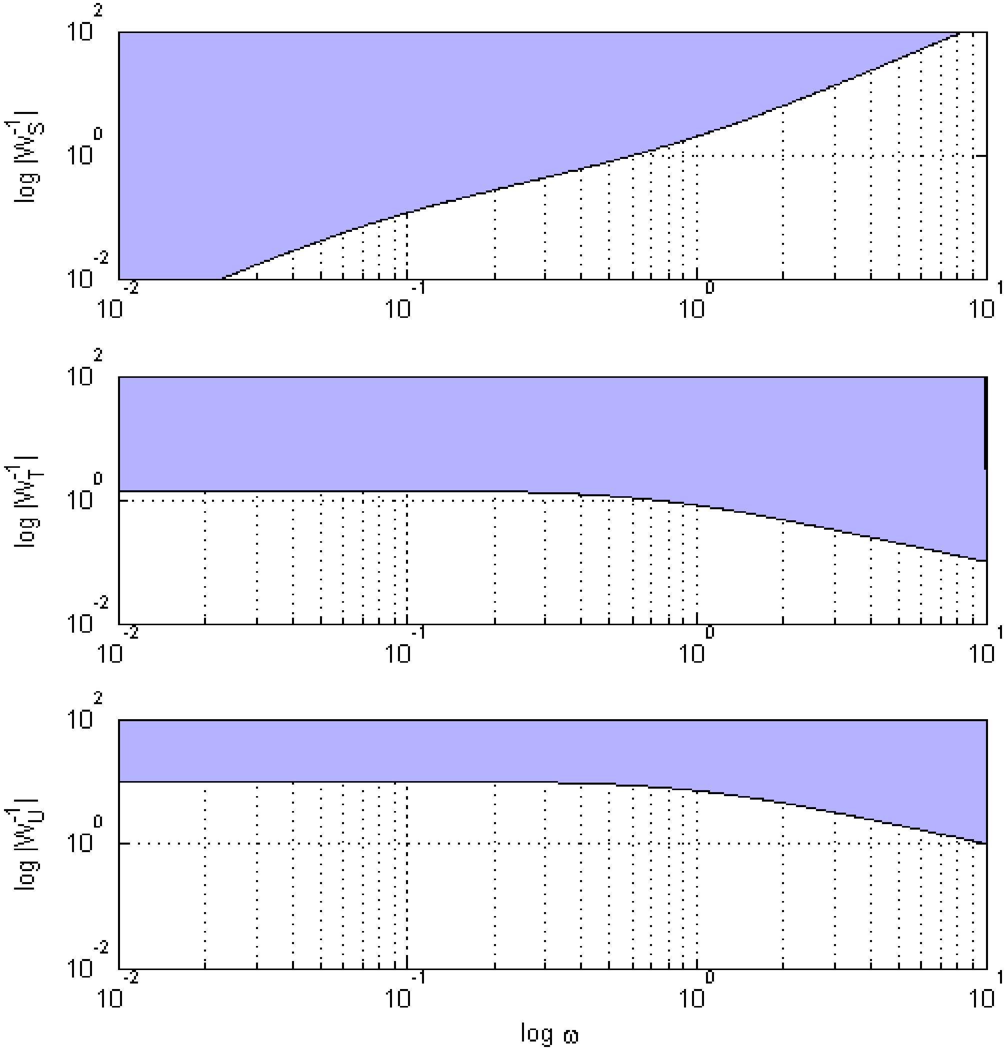
$$G(s) = \frac{1}{s(s+1)}$$

Same performance requirements as previously (see next slide)

Two key points:

- H_∞ optimal design allows to work directly with constraints
- The relation between H_2 and H_∞ optimal controllers

Weights

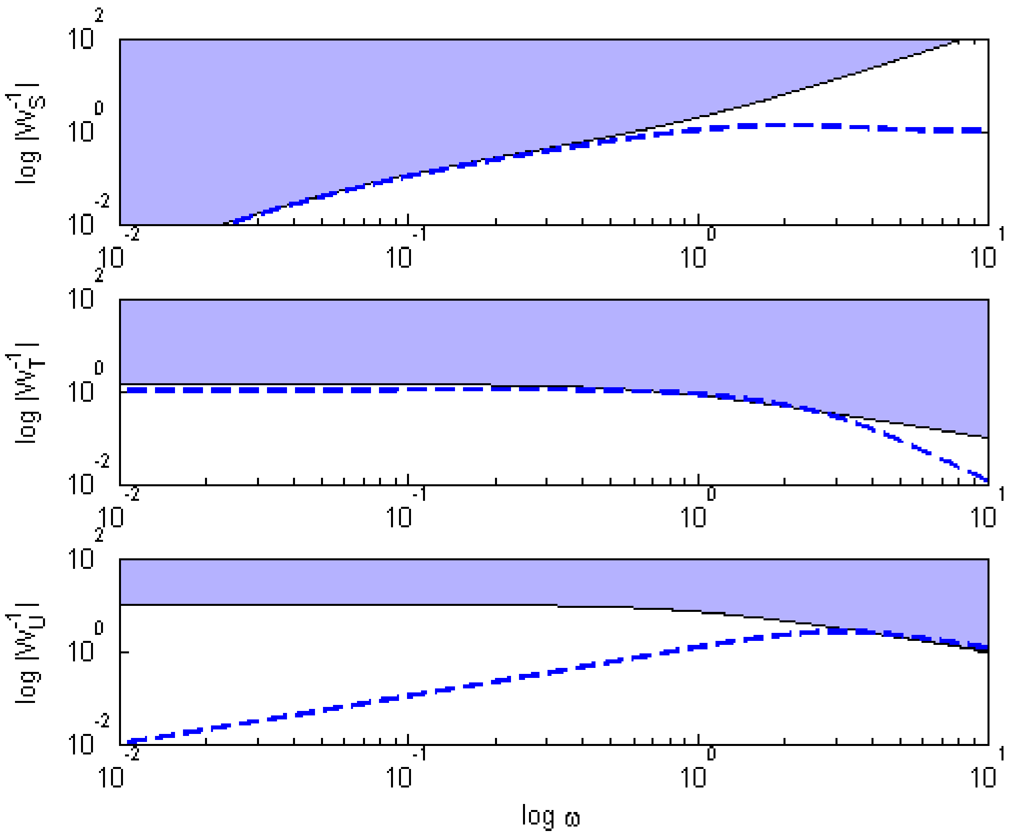


$$W_S(s) = \frac{0.71s + 0.05}{s^2(s+1)}$$

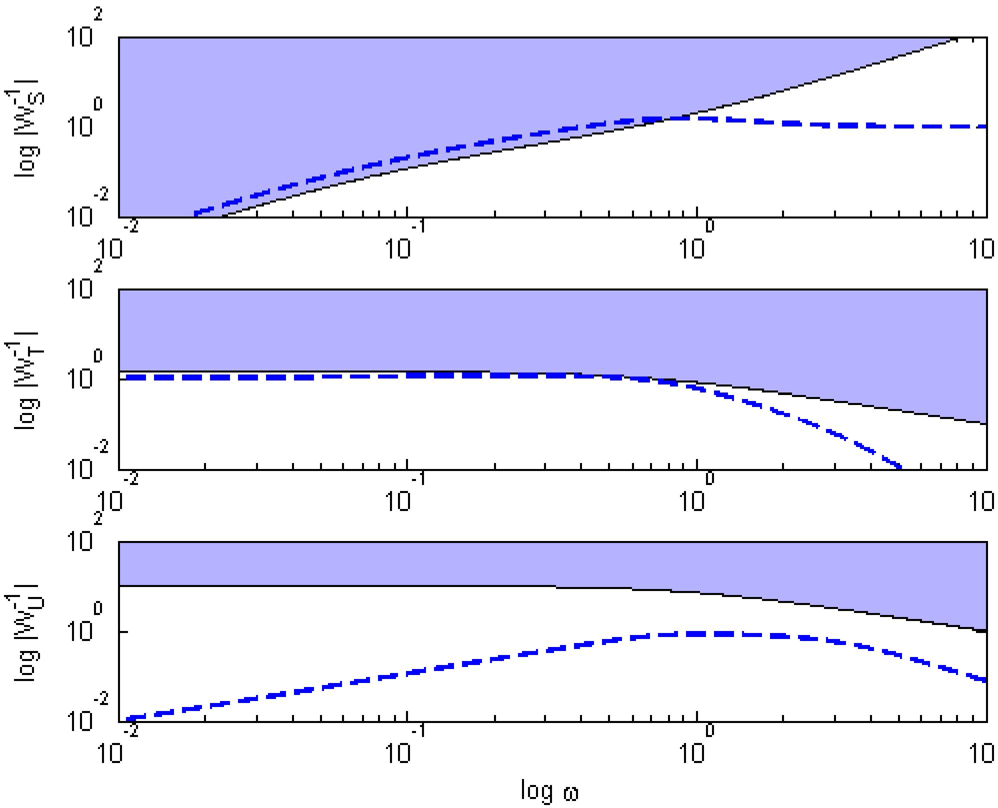
$$W_T(s) = s + 0.71$$

$$W_U(s) = \frac{10s + 10}{s + 100}$$

H_∞ optimal control



H_2-optimal controller



Quiz: why doesn't the H_2-optimal controller "meet the specs"?

Comparing the controllers