EL2520
Control Theory and Practice

Lecture 4: Fundamental Limitations and Conflicts

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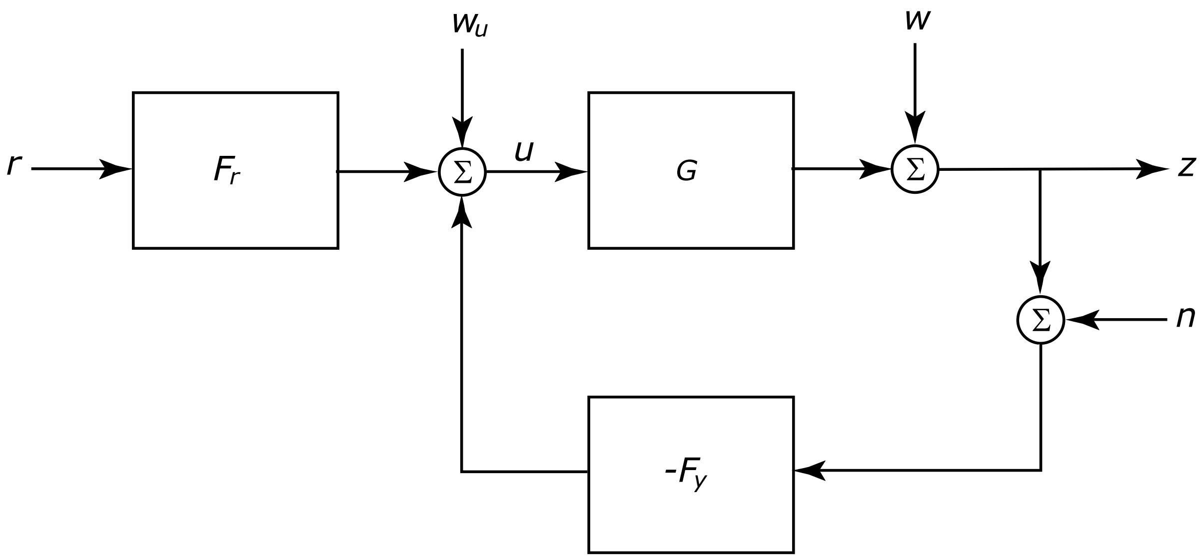
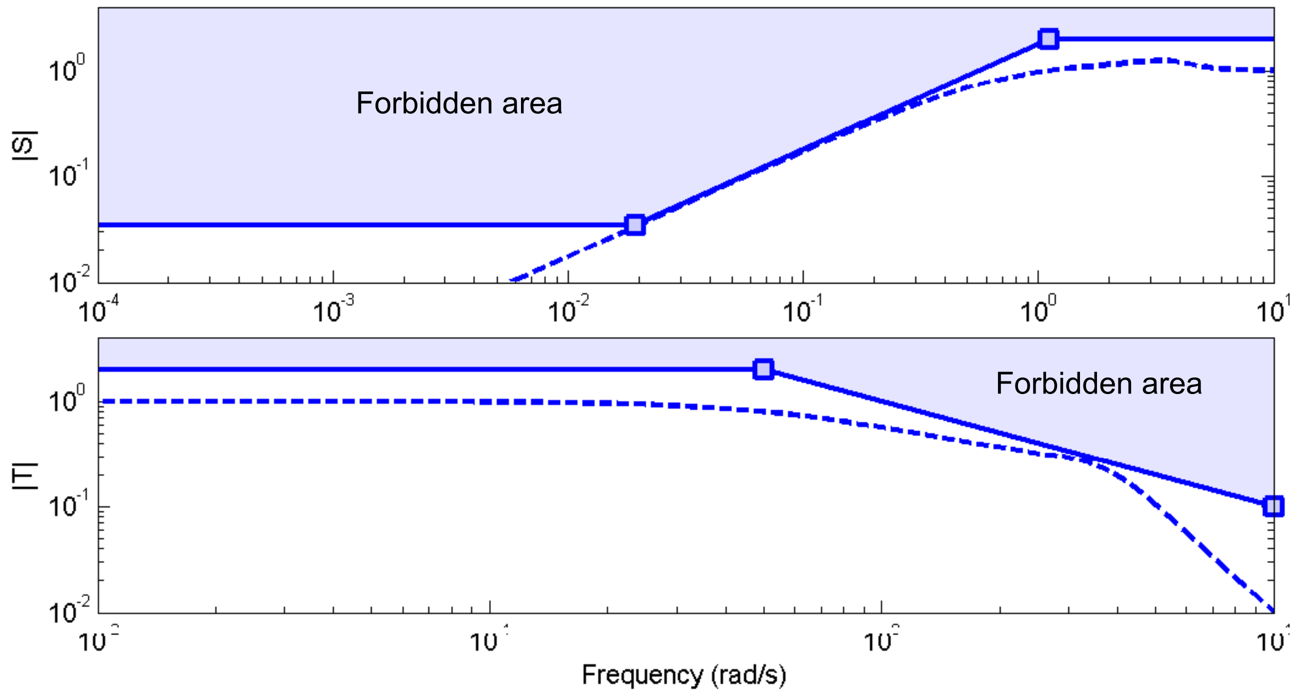
So far...



- Aim: shape closed-loop transfer-functions to achieve Nominal Stability (NS), Nominal Performance (NP), Robust Stability (RS) and Robust Performance (RP)
- Typical: shape sensitivity function S and complementary sensitivity function T

Frequency domain specifications



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)| \qquad \text{(disturbance rejection)}$$

$$|T(i\omega)| \leq |W_T^{-1}(i\omega)| \qquad \text{(noise suppression, robust stability)}$$

- Can we choose weights W_S, W_T ("forbidden areas") freely?
  - No, there are many constraints and limitations!

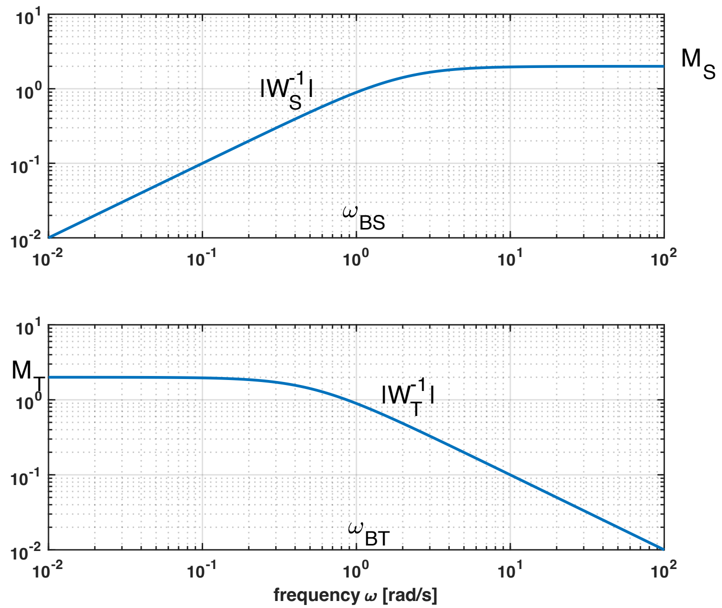
Standard Weights

- For maximum peak M_S and bandwidth ω_{BS} (where $|S| \approx 1$), we can employ

$$W_S = \frac{1}{M_S} + \frac{\omega_{BS}}{s}$$

- Similarly, for the complementary sensitivity we employ

$$W_T = \frac{1}{M_T} + \frac{s}{\omega_{BT}}$$



Introductory Example

- You are given the task of designing the control system for the cooling of an engine. A model of the system is

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_2 - u \\ \dot{x}_2 &= -0.5x_2 + 2u \\ y &= x_1 \end{aligned}$$

  y - engine temperature, u - coolant flow
- The task is to attenuate disturbances acting on the engine temperature up to a frequency 5 rad/s
- *Traditional approach:* choose controller, e.g., PID, and tune to achieve specifications
- *This course:* analyze problem to determine if task is feasible, that is, does there exist any controller that meets the specifications?

Outline of today's lecture

- Trade-off S+T=1
- Bode's Integral Theorem – the waterbed effect
- Limitations from RHP zeros and time delays
- Limitations from RHP poles
- Limitations from combined RHP poles and zeros

Sensitivity Trade-Off

Recall

$$S = \frac{1}{1+L} \;; \quad T = \frac{L}{1+L}$$

Hence

$$S(i\omega) + T(i\omega) = 1 \; \forall \omega$$

It follows that

- Either $|S(i\omega)| > 0.5$ or $|T(i\omega)| > 0.5$ at any frequency, i.e., cannot deal effectively with both disturbances and measurement noise at the same frequency
- Cannot choose $|W_S| > 2 \;\&\; |W_T| > 2$ at the same frequency
- Since |S+T|=1, the distance between S and T is always 1 in the complex plan and hence $|S| >> 1 \;\Leftrightarrow\; |T| >> 1$, i.e., large peak in S implies large peak in T and vice versa

Trade-off between frequencies - Bode’s integral theorem

Theorem. Suppose that $L(s)=F_y(s)G(s)$ has relative degree ≥2, and that $L(s)$ has N_p RHP poles located at $s=p_i$. Then the sensitivity function must satisfy

$$\int_0^\infty \log |S(i\omega)| \, d\omega = \pi \sum_{i=1}^{N_p} \mathrm{Re}(p_i)$$

Proof: based on Cauchy Integral Theorem (complex analysis)

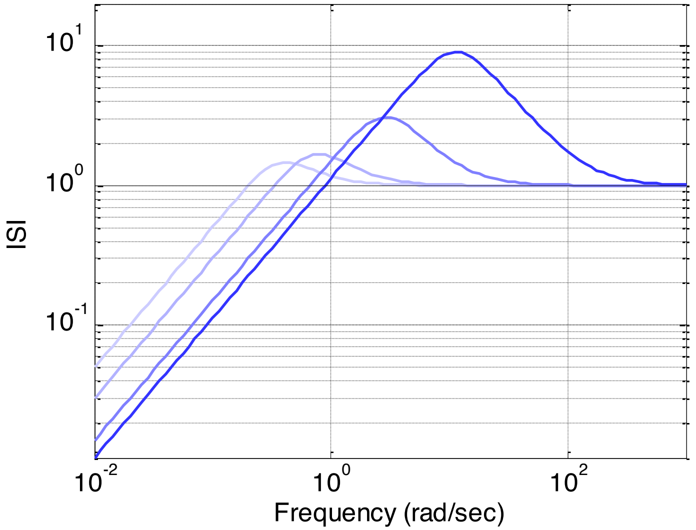
Intepretation of Bode's integral

All stable controllers give the same value of

$$\int \log |S(i\omega)|\, d\omega$$

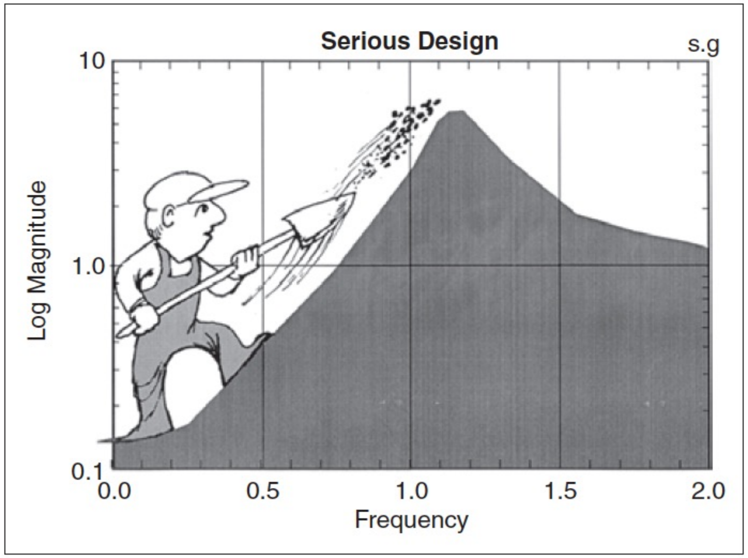
If L(s) is stable, then area for |S| above and below 1 is equal

- Sensitivity reduction in one frequency range comes at expense of sensitivity increase in another ("**waterbed effect**")

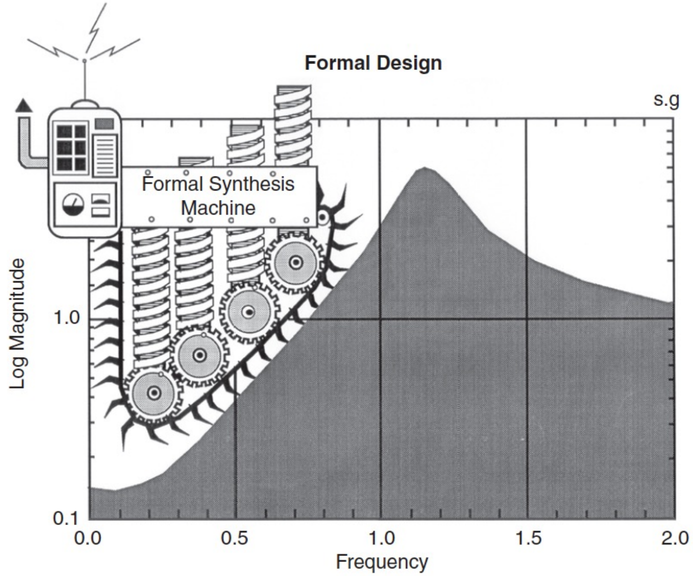


Making the Trade-Off

Manual Loop Shaping:



Optimization:



(From Stein, IEEE Control Systems Magazine, 2003)

Interpolation Constraints from RHP Zeros and Poles

$$S(s) = \frac{1}{1+L(s)} \; ; \quad T(s) = \frac{L(s)}{1+L(s)} \; ; \quad L(s) = G(s)F_y(s)$$

- Let z be a RHP zero of G(s). Then

$$S(z) = 1 \; ; \quad T(z) = 0$$

  – follows since internal stability implies L(z)=0

- Let p be a RHP pole of G(s). Then

$$S(p) = 0 \; ; \quad T(p) = 1$$

  – follows since internal stability implies L(p)=∞

These are both *interpolation constraints* that S and T must satisfy

Maximum Modulus Theorem

Maximum Modulus Thm: Suppose that Ω is a region in the complex plane and F is an analytic function on Ω and, furthermore, that F is not equal to a constant. Then **|F|** attains its maximum value at the boundary of Ω

Proof: see course on complex analysis

- S and T are stable transfer functions and hence analytic in the complex RHP, for which the boundary is the $i\omega$-axis
- A trivial consequence is then

$$\|S\|_\infty \geq S(z) = 1 \; ; \quad \|T\|_\infty \geq T(p) = 1$$

  – however, not too useful bounds. Need to add weights to get more informative constraints

Limitations from RHP Zeros

- From Maximum Modulus Thm with RHP zero z

$$\|W_S S\|_\infty \geq |W_S(z)S(z)| = |W_S(z)|$$

Thus, to achieve $\|W_S S\|_\infty < 1$ we require

$$|W_S(z)| < 1$$

Bandwidth limitation from RHP zero

Consider the weight

$$W_S(s) = \frac{1}{M_s} + \frac{\omega_{BS}}{s}$$

then

$$|W_S(z)| \leq 1 \Rightarrow \frac{1}{M_s} + \frac{\omega_{BS}}{z} \leq 1$$

So

$$\omega_{BS} \leq (1 - M_S^{-1})z$$

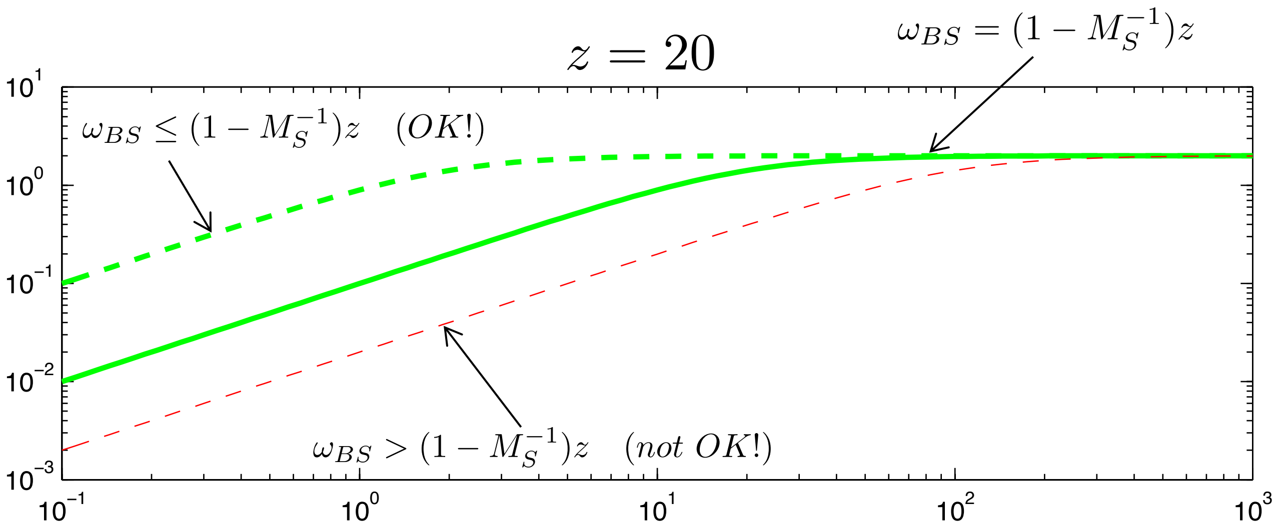
- If we allow $M_s = \infty$ we get $\omega_{BS} < z$
- The more reasonable value $M_S = 2$ gives the rule of thumb:

$$\boxed{\omega_{BS} \leq \frac{z}{2}}$$

Interpretation

To ensure that $$|S(i\omega)| \leq |W_S^{-1}(i\omega)| = M_S \left| \frac{i\omega}{i\omega + \omega_{BS} M_S} \right| \quad \forall\omega$$

one needs $\omega_{BS} \leq (1 - M_S^{-1})z$ for every RHP zero of G(s)

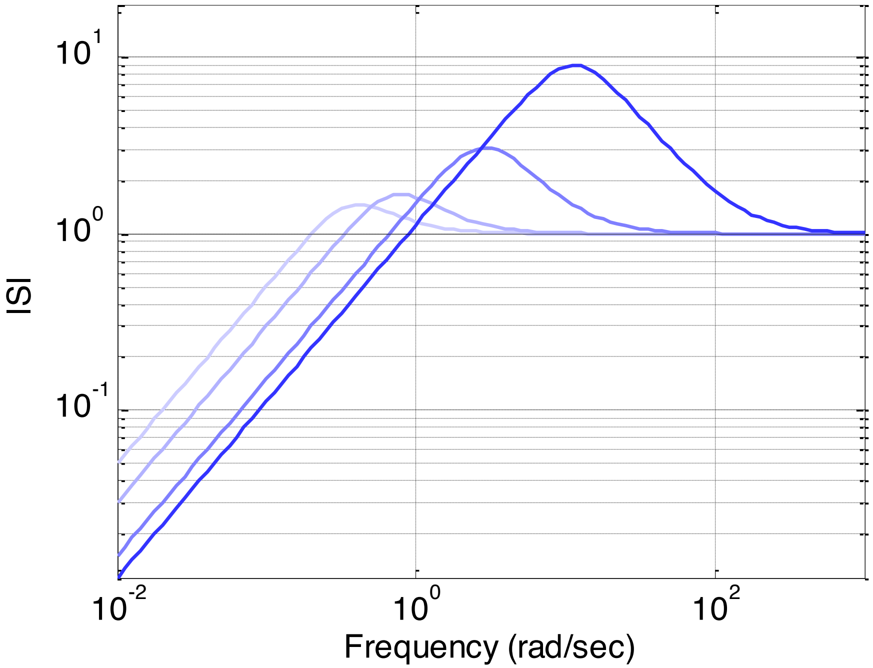


Example

Let $G(s) = \dfrac{1-s}{s(s+1)}, \quad F_y(s) = \dfrac{s+1}{a_0 s + a_1}$

If $a_0 = 1/(2\omega_0^2),\ a_1 = (\omega_0 + 1)/\omega_0$ then S has poles in $\omega_0(-1 \pm i)$

S for ω_0 =0.25, 0.5, 2, 8 – pushing bandwidth results in peaking



Bandwidth limitations from time delays

Since

$$e^{-Ts} \approx \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s} \quad \text{(Pade approximation)}$$

a system with time delay θ

$$G(s) = e^{-\theta s} G_0(s)$$

can be seen as a system with a RHP zero $z = 2/\theta$

Then, M_s=2 suggests

$$\boxed{\omega_{BS} \leq \frac{1}{\theta}}$$

Limitations from RHP Poles

- From Maximum Modulus Thm with RHP pole p

$$\|W_T T\|_\infty \geq |W_T(p)T(p)| = |W_T(p)|$$

- Thus, to achieve $\|W_T T\|_\infty < 1$ we require

$$|W_T(p)| < 1$$

Bandwidth limitation from RHP pole

Consider the weight

$$W_T(s) = \frac{s}{\omega_{BT}} + \frac{1}{M_T}$$

then

$$|W_T(p)| \leq 1 \;\; \Rightarrow \;\; \frac{p}{\omega_{BT}} + \frac{1}{M_T} \leq 1$$

So

$$\omega_{BT} \geq \frac{p}{1-1/M_T} \geq p$$

The more reasonable value M_T=2 gives the rule of thumb

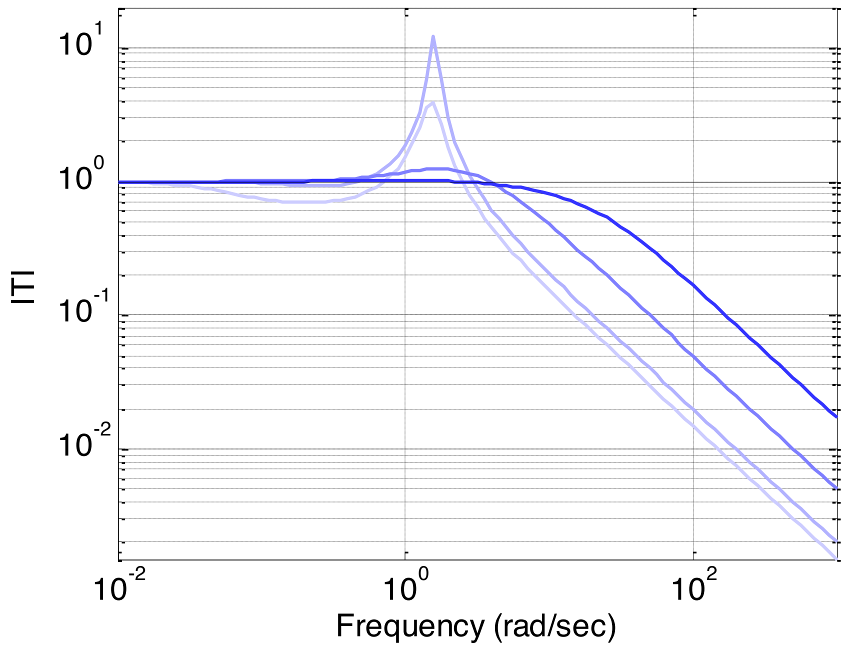
$$\boxed{\omega_{BT} \geq 2p}$$

Example

Let $G(s) = \frac{s+1}{s(s-1)}, \quad F_y(s) = \frac{b_0 s + b_1}{s+1}$

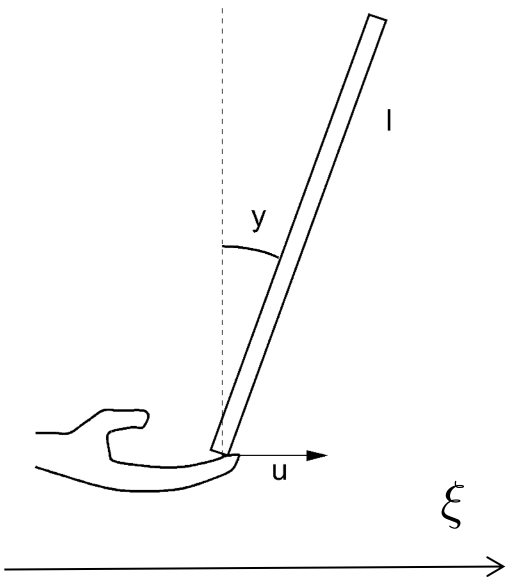
If $b_0 = 1 + 2\omega_0, \; b_1 = 2\omega_0^2$ then T has poles in $\omega_0(-1 \pm i)$

T for ω_0 =0.25, 0.5, 2, 8 – too low bandwidth forces T to peak



Example: balancing act

Balancing a stick. Input: acceleration of the finger, output: angle of the stick



Input:

$$u = \frac{d^2\xi}{dt^2}$$

Output:

$$\frac{\ell}{2}\frac{d^2y}{dt^2} - g\sin(y) = -u\cos(y)$$

Transfer function:

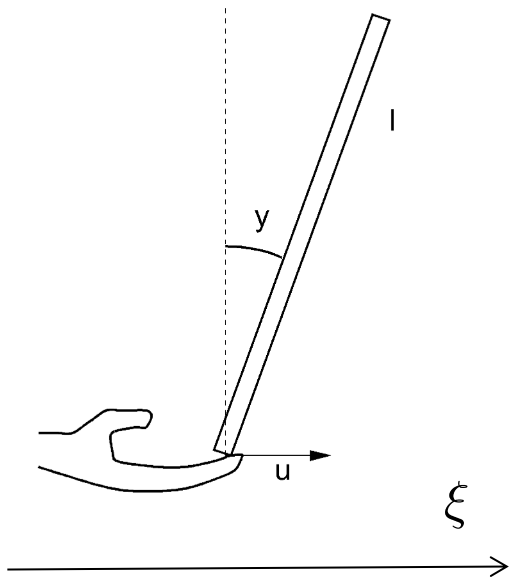
$$G(s) = \frac{-2/\ell}{s^2 - 2g/\ell}$$

Pole at

$$p = \pm\sqrt{\frac{2g}{\ell}}$$

Example: balancing act

Balancing a stick. Input: acceleration of the finger, output: angle of the stick



Pole at

$$p = \pm\sqrt{\frac{2g}{\ell}}$$

Bandwidth constraint: $\omega_{BT} \geq 2\sqrt{\frac{2g}{l}}$

When $l = 0.2m$, need to react faster than approx. 0.05s

By increasing the length of the stick we reduce bandwidth requirement

Example: X-29

Under one flying condition, the X-29 can be modeled by

$$G(s) = \hat{G}(s)\frac{s-26}{s-6}$$

RHP pole at s=6 $\Rightarrow \quad w_{BT} \geq 2 \cdot 6 = 12$

RHP zero at s=26 $\Rightarrow \quad w_{BS} \leq 26/2 = 13$

Difficult to design a controller that satisfies these requirements!

Combined RHP pole p and zero z

- Recall S(p)=0. Factor sensitivity function S as

$$S = S_{mp} \underbrace{\frac{s-p}{s+p}}_{S_{ap}}$$

- It follows that, since S(z)=1

$$S_{mp}(z) = S_{ap}^{-1}(z) = \tfrac{z+p}{z-p}$$

- Thus,

$$\|W_S S\|_\infty = \|W_S S_{mp}\|_\infty \geq |W_S(z) S_{mp}(z)| = |W_S(z) \tfrac{z+p}{z-p}|$$

- For instance, with $W_S = 1$

$$\|S\|_\infty \geq \tfrac{|z+p|}{|z-p|}$$

  - large peaks in S (and T) unavoidable with RHP zero close to RHP pole

Combined RHP poles and zeros

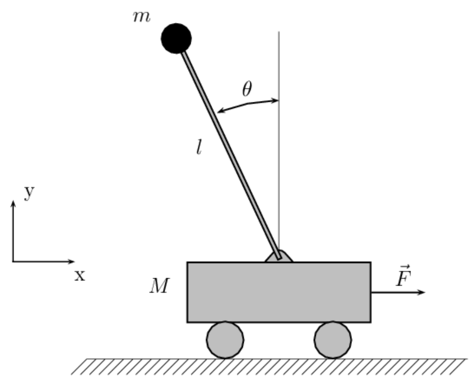
- Similarly, T(z)=0 and we get

$$\|W_T T\|_\infty \geq |W_T(p)\tfrac{p+z}{p-z}|$$

- With $W_T = 1$

$$\|T\|_\infty \geq \frac{|z+p|}{|z-p|}$$

Ex: stabilization of cart pendulum



$$X(s) = \frac{ls^2 - g}{s^2(Mls^2 - (M+m)g)} F(s)$$

$$z = \sqrt{\frac{g}{l}} \; ; \quad p = z\sqrt{1 + m/M}$$

- With l=1 and m=M we get $z = \sqrt{10}, \; p = \sqrt{20} \quad \Rightarrow$

$$\|S\|_\infty > 5.8, \; \|T\|_\infty > 5.8$$

- With l=1 and m= 0.1M we get $z = \sqrt{10}, \; p = \sqrt{11} \quad \Rightarrow$

$$\|S\|_\infty > 42, \; \|T\|_\infty > 42$$

This is essentially the rocket stabilization problem; see paper by G. Stein, IEEE Control Systems Mag. (course homepage)

Summary

S+T=1: must make trade-off between attenuation of disturbances and measurement noise at every frequency

Bode's integral theorem: must trade-off sensitivity reduction at one frequency by sensitivity increase at another frequency (waterbed)

System dynamics impose fundamental limitations on achievable control performance

- RHP zero at z: $\omega_{BS} \leq z/2$
- Time delay θ : $\omega_{BS} \leq 1/\theta$
- RHP pole at p: $\omega_{BT} \geq 2p$

Homework: does there exist a controller that meets the specifications of the introductory example?