EL2520
Control Theory and Practice

Robust Loop Shaping
+ Model Reduction

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Today’s lecture

- Glover McFarlane loop shaping
  - robustifying controller ”around” nominal design
  - a design example
- Introduction to model reduction
  - balanced truncation

Loop Shaping (Lec 7)

Translate bounds on $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ into bounds on $\sigma_i(L), \;\; L = GF_y$

- From Fan's Thm:

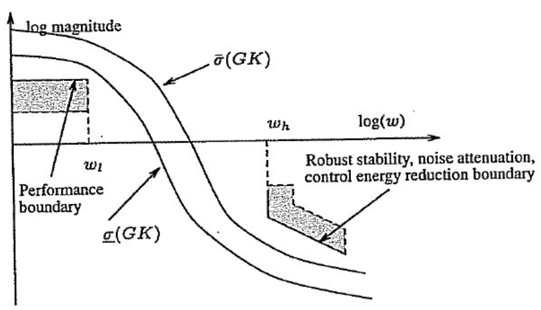
$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1$$

- Then, $\underline{\sigma}(L) >> 1 \;\; \Rightarrow \;\; \bar{\sigma}(S) \approx 1/\underline{\sigma}(L)$ and we get condition

$$\bar{\sigma}(S) < |W_S^{-1}| \;\; \Rightarrow \;\; \underline{\sigma}(L) > |W_S|, \;\; |W_S| >> 1$$

- Similarily, $\bar{\sigma}(L) << 1 \Rightarrow \bar{\sigma}(T) \approx \bar{\sigma}(L)$ and we get condition

$$\bar{\sigma}(T) < |W_T^{-1}| \;\; \Rightarrow \;\; \bar{\sigma}(L) < |W_T^{-1}|, \;\; |W_T| >> 1$$



But, difficult to address stability margins in MIMO case (no definition of phase); make robust by optimizing robustness for some generic uncertainty.

A robust stabilization problem

Write model as (normalized coprime factorization)

$$G(s) = M(s)^{-1}N(s)$$

Find a controller that stabilizes the model set

$$G(s) = (M(s) + \Delta_M(s))^{-1}(N(s) + \Delta_N(s))$$

for all uncertainties satisfying

$$\|\Delta_M(s)\ \Delta_N(s)\|_\infty \leq \epsilon$$

More general uncertainty description than input or output uncertainty, e.g., allows different number of RHP poles and zeros in model set.

Co-prime factorization

Any transfer matrix can be (left) co-prime factorized

$$G(s) = M(s)^{-1}N(s)$$

where M and N are stable and co-prime. N contains RHP zeros of G, M contains RHP poles of G as RHP zeros

M and N coprime if they satisfy Bezout identity NU+MV=I, for stable U,V

The co-prime factorization is not unique. It is *normalized* if N, M satisfy

$$M(s)M(-s)^T + N(s)N(-s)^T = I$$

Normalized co-prime factorizations are unique.

Co-prime factorization cont'd

Example: The system

$$G(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)}$$

has a coprime factorization given by

$$N(s) = \frac{s-1}{s+4}, \quad M(s) = \frac{s-3}{s+2}$$

Another factorization is

$$N(s) = \frac{(s-1)(s+2)}{s^2 + k_1 s + k_2}, \quad M(s) = \frac{(s-3)(s+4)}{s^2 + k_1 s + k_2}$$

This one is normalized for appropriate values of k_1, k_2

A robust stabilization problem

Find a controller that stabilizes

$$G(s) = (M(s) + \Delta_M(s))^{-1}(N(s) + \Delta_N(s))$$

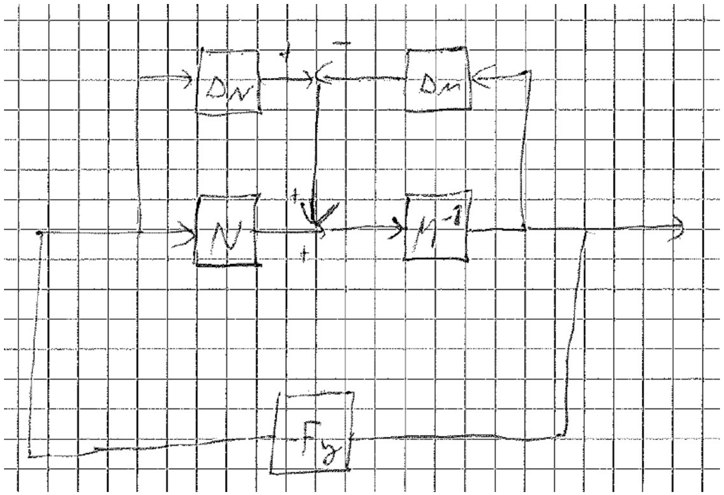
for all uncertainties satisfying $\|\Delta_M(s) \;\; \Delta_N(s)\|_\infty \leq \epsilon$

From Small Gain Thm we get the requirement (see next slide)

$$\underbrace{\left\| \begin{bmatrix} -F_y \\ I \end{bmatrix} (I + GF_y)^{-1} M^{-1} \right\|_\infty}_{\gamma} < 1/\epsilon$$

An alternative H_∞ control problem: minimize γ (to maximize robustness)

The Uncertain System



- Write it on P-Δ form with $\Delta = [\Delta_N \;\; \Delta_M]$ then we find

$$P = -\begin{pmatrix} F_y \\ I \end{pmatrix} (I + GF_y)^{-1} M^{-1}$$

- From Small Gain Theorem we have robust stability if P,Δ stable and

$$\|\Delta\|_\infty \|P\|_\infty < 1$$

Robust stabilization: the solution

Consider a state-space representation of G:

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

1. Solve the Riccati equations for Z>0, X>0

$$AZ + ZA^T - ZC^TCZ + BB^T = 0$$
$$A^TX + XA - XBB^TX + C^TC = 0$$

2. Let λ_m be the maximum eigenvalue of XZ, and introduce

$$\gamma = \alpha(1 + \lambda_m)^{1/2}, \qquad R = I - \frac{1}{\gamma^2}(I + ZX)\,, \alpha \geq 1$$
$$L = B^TX, \qquad K = R^{-1}ZC^T$$

3. Then, the following controller stabilizes all plants with $\|\Delta\|_\infty < 1/\gamma$

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K(y - C\hat{x}), \qquad u = -L\hat{x}$$

4. Minimum γ obtained for $\alpha = 1$

Robustification of control laws

- Robust stability usually of little interest on its own; must also address performance
- 3 step method:
  1. Design nominal controls to get an appropriate loop gain
  $$L(s) = W_2(s)G(s)W_1(s)$$
  2. Robust stabilization applied to W_2GW_1 yields robustly stabilizing controller $\tilde{F}_y(s)$. Recommendation is to use $\alpha = 1.1$
  3. Use the controller:
  $$F_y(s) = W_1(s)\tilde{F}_y(s)W_2(s)$$

  Rule of thumb: if minimum γ small (<4), then robustification has little impact on performance. Otherwise performance and robust stability is in conflict

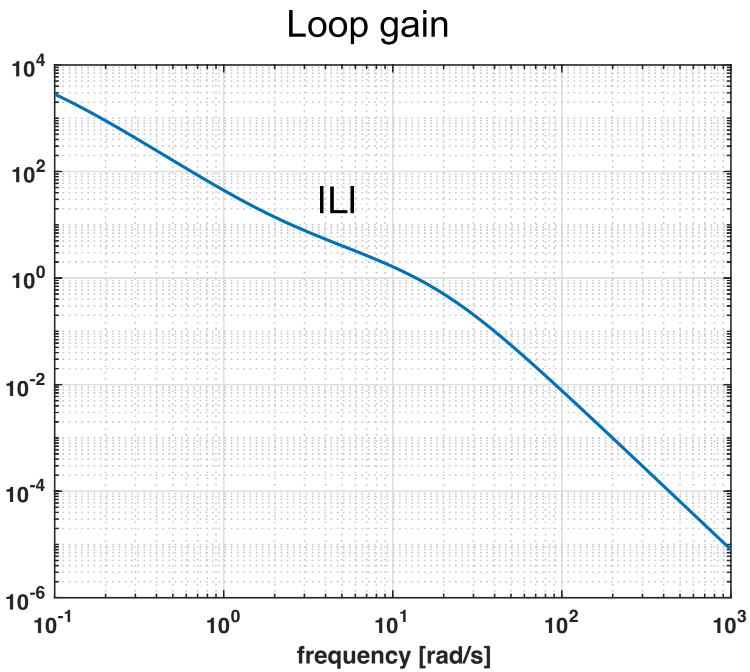
Example

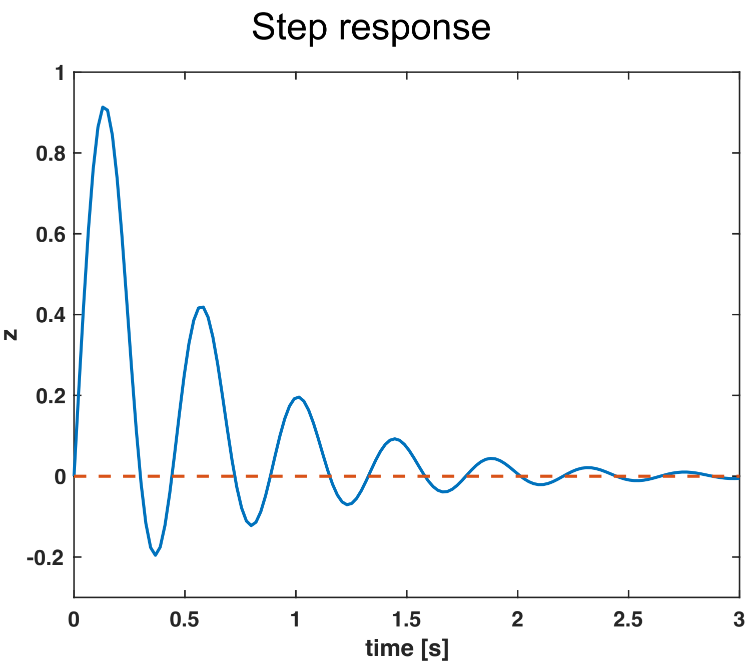
- Process with disturbance

$$z = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2}u + \frac{100}{10s+1}d$$

- Loop shaping controller so that $|L| > |G_d|$

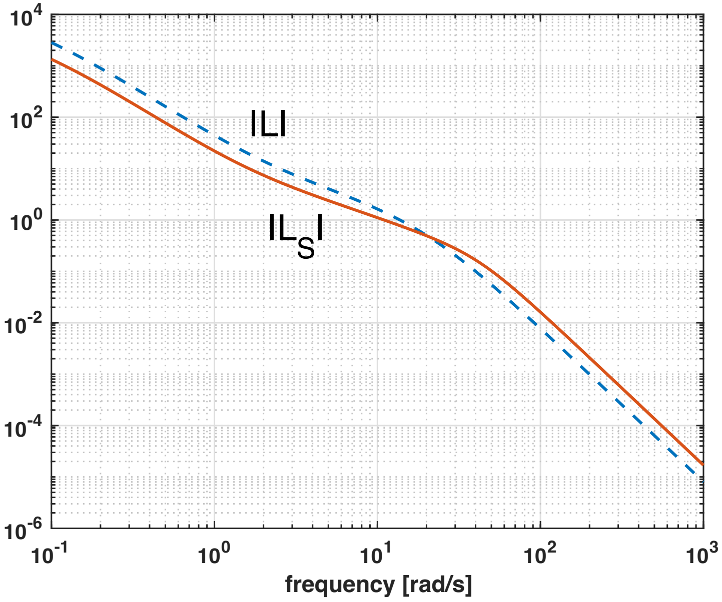
$$F_y = \frac{s+2}{s}$$

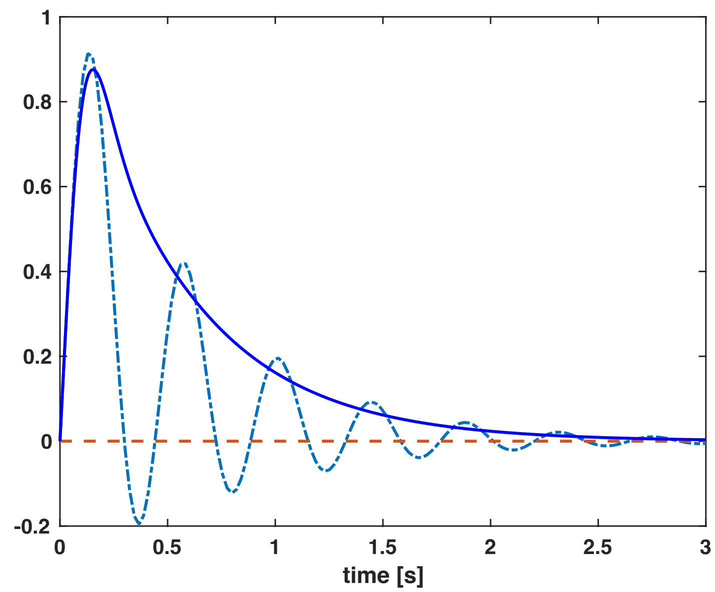




Robustification

- Matlab robust control toolbox: $[Ks, Cl, gam] = ncfsyn(L);$
  - gam=2.34 (<4, OK!)
  - robust controller: $F_{ys} = K_s F_y \quad \Rightarrow \quad L_s = G F_{ys}$





Today’s lecture

- Glover McFarlane loop shaping
  - robustifying controller ”around” nominal design
- A design example
- **Model order reduction – reducing the order of controllers (mostly for orientation)**

Controller simplification

- LQG, H_2, H_∞ and Glover-McFarlane designs typically give high-order controllers (extended systems)
- Often desirable to reduce the controller order (number of states)
- Easier implementation, reduced computational load …
- … but we need to ensure that simplified controller is "close" to original design
- Original and approximate models: $G,\ G_a$. We wish to ensure that

$$\|G - G_a\|_\infty < \epsilon$$

State-space realizations

A linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

can be represented in many ways (observable canonical form, controllable canonical form, …) via change of variables $\xi = Tx$
Which gives

$$\dot{\xi} = TAT^{-1}\xi + TBu$$
$$y = CT^{-1}\xi + Du$$

We should select a description that reveals the state variables with the largest influence the input-output relationship.

The Controllability Gramian

Measures how states are influenced by impulse inputs

- Impulse input: $u(t) = e_i \delta(t), \ x(0) = 0$

  Gives state: $x(t) = e^{At} B_i, \ B_i$ the *i*-th column of *B*
- Impulse in each input: $x(t) = e^{At} B$
- Size of the state measured through the *controllability gramian*:

$$S_x = \int_0^\infty x(t)x(t)^T dt = \int_0^\infty e^{At} B B^T e^{A^T t} dt$$

$$S_\xi = T S_x T^T$$

Diagonal Controllability Gramian

The gramian is symmetric and can be diagonalized:

There exists transformation matrix *T* such that:

$$TS_xT^T = \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$$

So if $\xi = Tx$, σ_k measures how much the state ξ_k is influenced by the input.

The Observability Gramian

Measures how different states contribute to the output energy

$$(u(t) = 0,\ x(0) = x_0) \implies y(t) = Ce^{At}x_0$$

Energy at the output:

$$\int_0^\infty y(t)^T y(t) dt = x_0^T \left[\int_0^\infty e^{A^T t} C^T C e^{At} dt \right] x_0$$

The *observability gramian:* $O_x = \int_0^\infty e^{A^T t} C^T C e^{At} dt$

Change of coordinate $O_\xi = (T^T)^{-1} O_x T^{-1}$

Diagonal Observability Gramian

The gramian is symmetric and can be diagonalized:

There exists transformation matrix *T* such that:

$$(T^T)^{-1} O_x T^{-1} = \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$$

So if $\xi = Tx$, σ_k measures how much the initial state ξ_0 influences the output:

$$\int_0^\infty y(t)^T y(t) dt = \xi_0^T \Sigma \xi_0$$

Balanced representation

Theorem. There exists *T* such that $S_\xi = O_\xi = \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$

- $\xi = Tx$ is called the balanced state representation. Essentially, all state variables ξ_k as controllable as they are observable.
- The singular values σ_k are called *Hankel singular values*
- States corresponding to small Hankel singular values may be removed without affecting the input-output behavior much.

Balanced truncation

Write the balanced representation as:

$$
\begin{aligned}
\dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\
\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2 u \\
y &= C_1 x_1 + C_2 x_2 + Du
\end{aligned}
$$

Observability gramian: $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$

$$\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_k),\ \Sigma_2 = \mathrm{diag}(\sigma_{k+1}, \ldots, \sigma_n)$$

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$$

Balanced truncation

Theorem: Replace G=(A, B, C, D) by G_a=(A_{11}, B_1, C_1, D). Then

$$\|G - G_a\|_\infty \leq 2(\sigma_{k+1} + \ldots \sigma_n)$$

Example. H_∞-optimal controller for a DC motor in Lecture 9 has Hankel singular values

$$\begin{bmatrix} 262.7436 & 1.0656 & 0.6449 & 0.0188 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

so a fourth order controller seems (and is) appropriate!

Summary

- Glover McFarlane loop shaping
  - robustifying controller ”around” nominal design
- A design example
- Simplification of control laws
  - balanced truncation