Homework 1 in EL2450 Hybrid and Embedded Control Systems

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January 31, 2022

Task 1

The gain Tap refers to the state of the Tap at the bottom of the Tank 1. The value zero means that the valve is closed in this simulation. We can also simulate other states of the valve by changing this gain Tap between 0 and 1, where 1 refers to full open valve and the other value means partial open valve.

Task 2

According to the mathematical model of the coupled tank, the MATLAB codes of the transfer functions are as follows

```
uppertank = tf(k_tank,[Tau,1]);
lowertank = tf(gamma_tank,[gamma_tank*Tau, 1]);
```

Task 3

As shown in the reference block, the reference signal is a delayed step-function signal, with the step time being 25, the initial value being 0 and the final value being 10. Note that `uss` and `yss` are used as an offset for the reference value. Since we design the controller to control the level of the lower tank at 40 meter, the control signal, which design at zero, needed to be shifted by steady state value (`uss`). `yss` is also needed to subtract of the output in order to calibrate back to zero. Besides, they also represent the uncertainty of dynamic system, which will enhance the robustness of the system.

Task 4

The MATLAB codes are as follows

```
[K, Ti, Td, N] = polePlacePID(chi,omega0,zeta,Tau,gamma_tank,
    k_tank);
F = K*(tf(1)+tf(1,[Ti 0])+tf([Td*N 0],[1 N]));
```

Task 5

Add the continuous controller to the model `controllers` and connect them as shown in Fig.1.

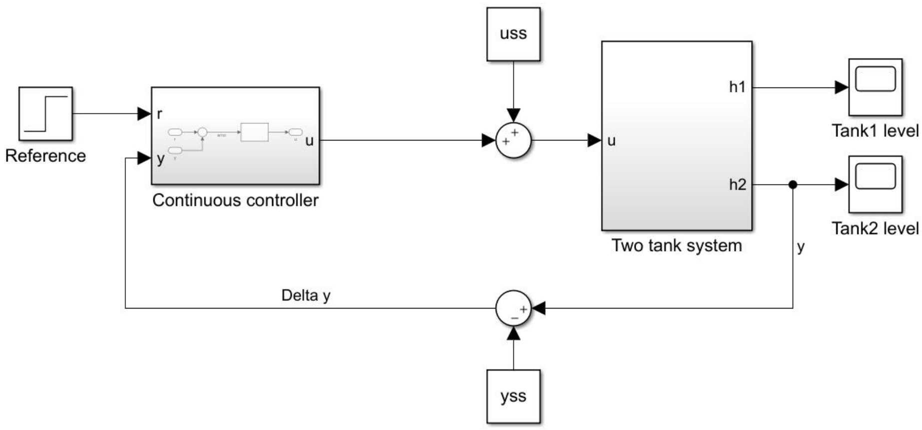


Figure 1: simulink model

Now we set a group of different values and use MATLAB to simulate the figure result of step response. According to the figures below, we could easily draw a conclusion into a table.

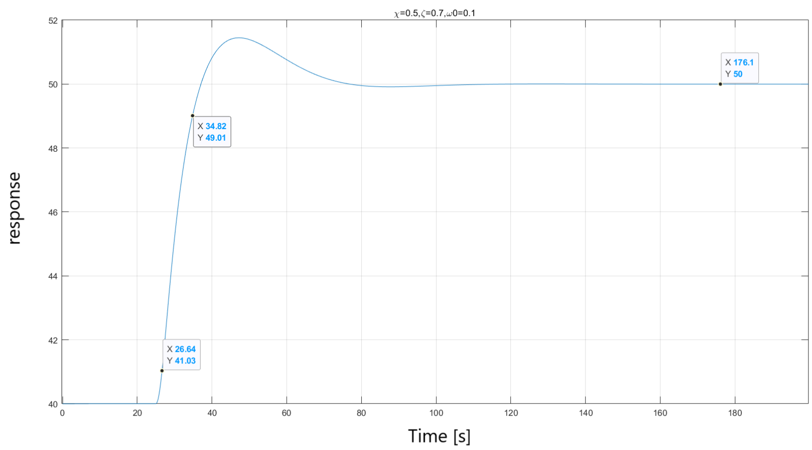


Figure 2: case 1

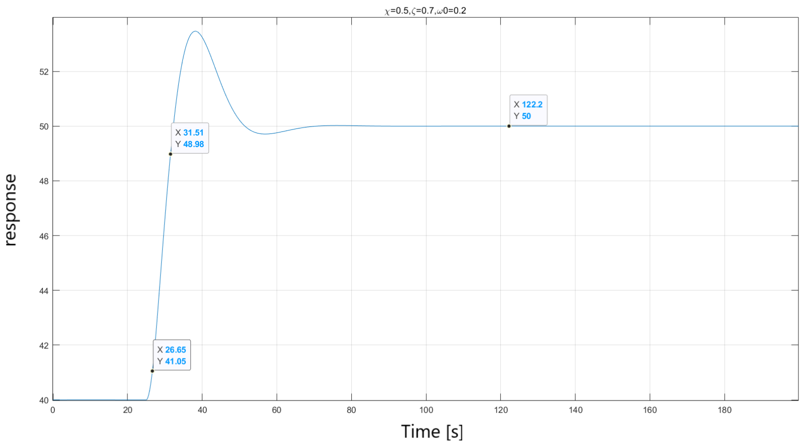


Figure 3: case 2

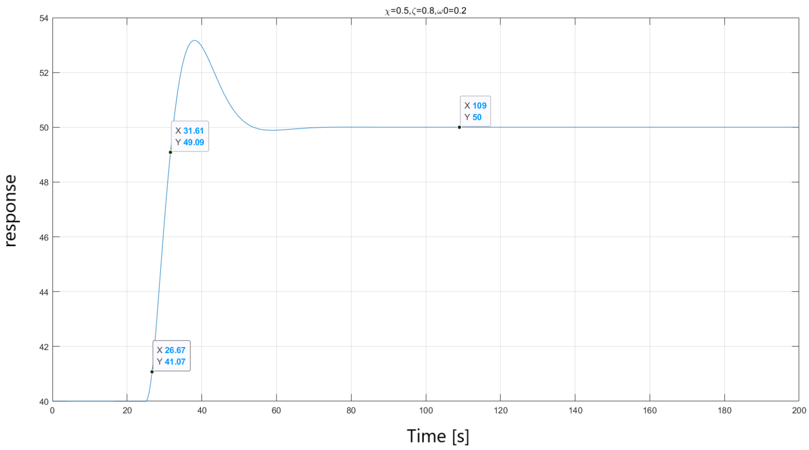


Figure 4: case 3

From Tab.1, the controller with $\chi = 0.5, \zeta = 0.8$ and $\omega_0 = 0.2$ is the only controller that fulfills the requirements, with the rise time (Tr) less than 6 s, overshoot (M) less than 35% and settling-time $(Tset)$ with a 2% error band less than 30 s. Therefore, this controller will be used in the next tasks.

Table 1: Controller performance

χ	ζ	ω_0	T_r	M	T_{set}
0.5	0.7	0.1	$8.20s$	14.50%	$44.97s$
0.5	0.7	0.2	$4.92s$	34.66%	$36.61s$
0.5	0.8	0.2	$4.94s$	31.71%	$26.38s$

Task 6

To find the cross over frequency, we can computed from the frequency where the magnitude of the open loop system equals to zero. We can also get frequency response by setting $s = j\omega$. The magnitude of the open loop system can be derived as following:

$$\begin{aligned}
Magnitude &= 20log(|F(s)G(s)|) \\
0 &= 20log(|F(s)G(s)|) \\
1 &= |F(s)G(s)| \\
1 &= \left| \frac{669.5s^2 + 180.6s + 16.05}{1605s^4 + 2119s^3 + 309.7s^2 + 12.04s} \right| \\
1 &= \left| \frac{669.5(j\omega)^2 + 180.6(j\omega) + 16.05}{1605(j\omega)^4 + 2119(j\omega)^3 + 309.7(j\omega)^2 + 12.04(j\omega)} \right| \\
\omega_{cross_over} &= 0.3619
\end{aligned}$$

Therefore, the cross over frequency equals to 0.3619 rad/s, which can also be shown in bode plot as shown in Fig.5.

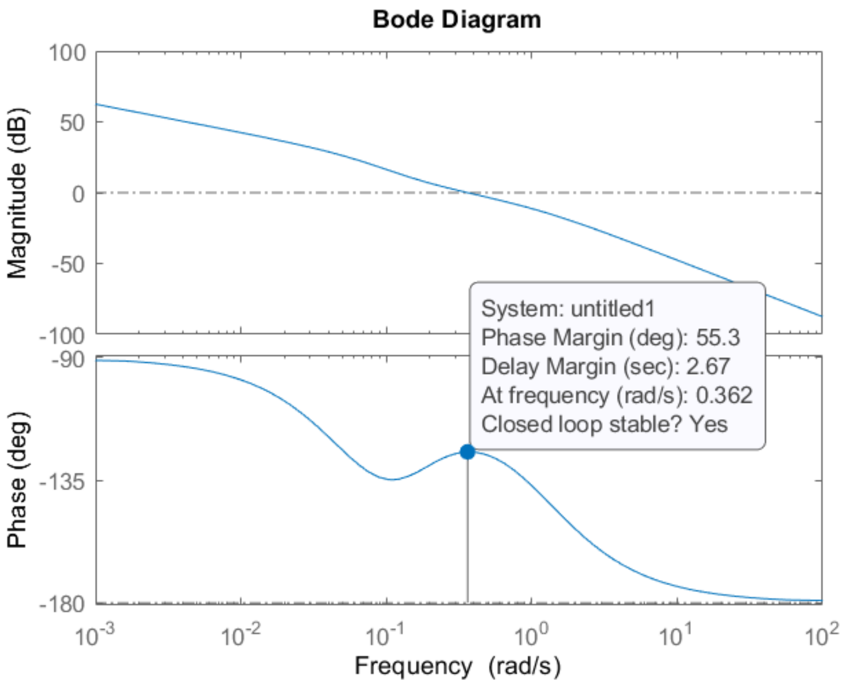


Figure 5: Bode plot of open loop system

Task 7

In order to achieve the digital control design, first we have to sample and reconstruct the input and output signals. As shown in Fig.6, we connect a Zero-Order Hold block after the continuous controller and before the double-tank process.

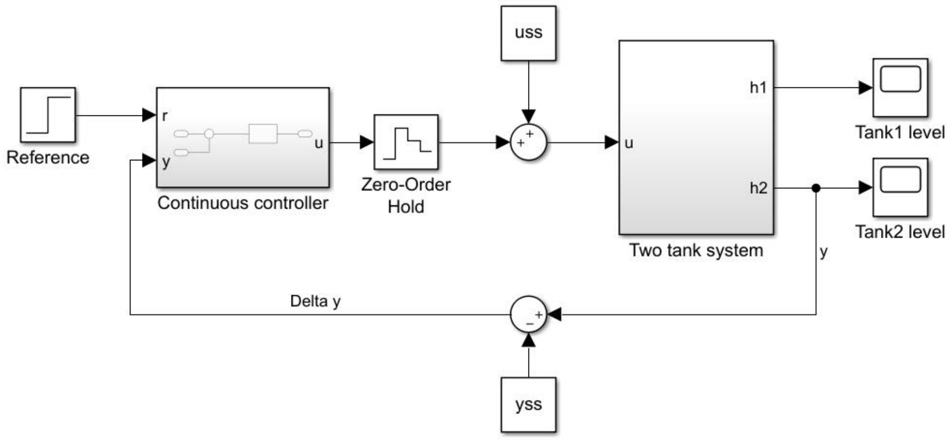


Figure 6: Simulink model

To compare the differences between the continuous case and the one with a ZOH case, we first keep the sampling time T_s a constant 1. In the meanwhile, we focus on the same values according to **Task 5**. Three comparison results are as follows.

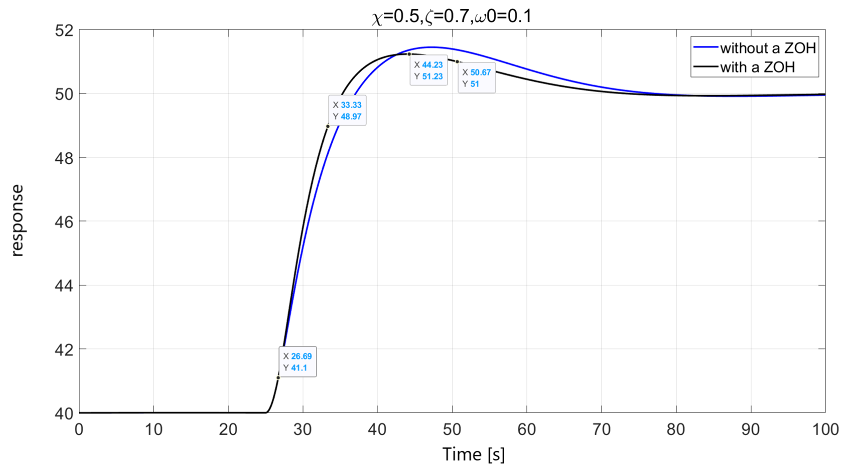


Figure 7: case 1

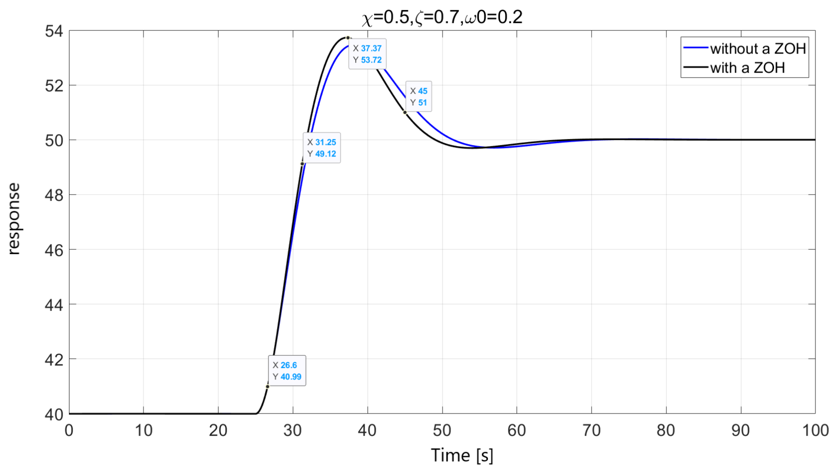


Figure 8: case 2

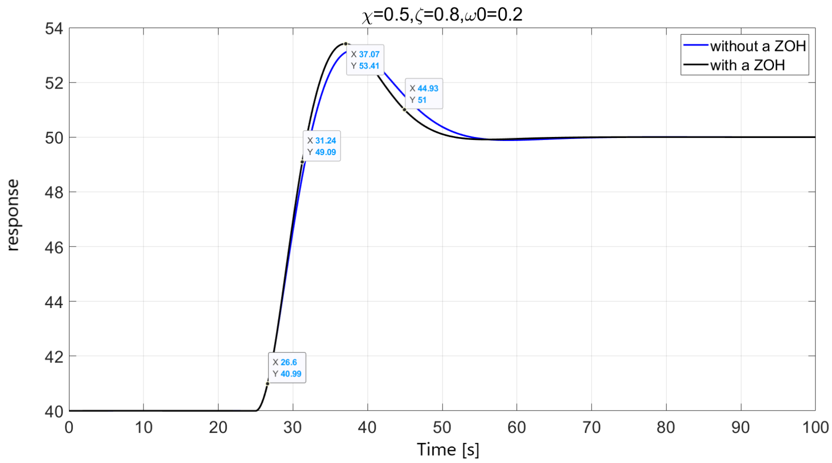


Figure 9: case 3

According to Fig.7 to Fig.9, Tab.2 is drawn below so as to further compare the three dynamic performance indexes.

Table 2: Controller performance(with the ZOH)

χ	ζ	ω_0	T_r	M	T_{set}
0.5	0.7	0.1	$6.85s$	12.30%	$48.00s$
0.5	0.7	0.2	$4.52s$	37.3%	$42.67s$
0.5	0.8	0.2	$4.52s$	34.1%	$45.33s$

Comparing Tab.1 and Tab.2, we can find that

- The rise time (T_r) decreases.
- Changes of the overshoot (M) are uncertain.
- The settiling time (T_{set}) increases.

That is because the ZOH has a low pass filter characteristic with high frequency attenuation, and its phase has a late characteristic, which has a certain negative effect on the dynamic performances of the closed-loop system.

To find out the impact of the sampling times, we now change the value of T_s and check the three dynamic performance indexes. Set the indexes of the continuous controller as $\chi = 0.5, \zeta = 0.8$ and $\omega_0 = 0.2$ and then simulate the step responses under different sampling times.

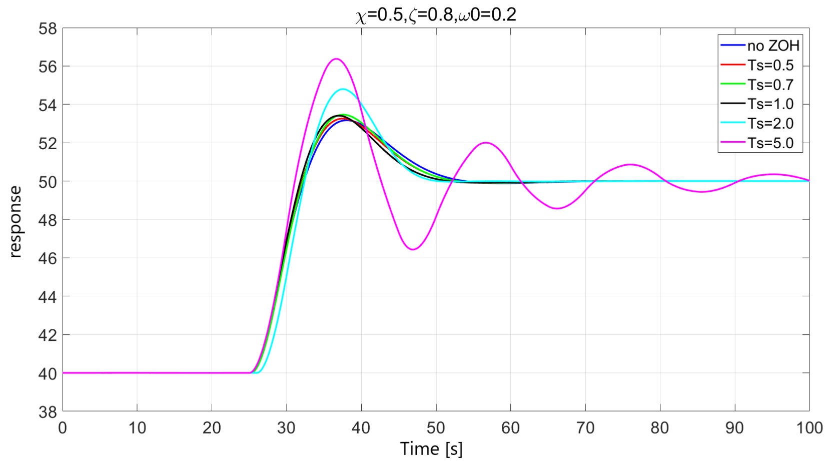


Figure 10: step response under different sampling times

And as shown in Fig.10, when the maximum sampling time is around $0.7s$, the requirements are not fulfilled anymore.

Task 8

Here we start to discretize the continuous controller into state space form using MATLAB function, shown as follows.

```
G = c2d(F,Ts,'ZOH');
[A_discretized,B_discretized,C_discretized,D_discretized] =
   tf2ss(G.num{1},G.den{1});
```

As shown in Fig.11, now we move the discretized controller and reconstruct a simulink model.

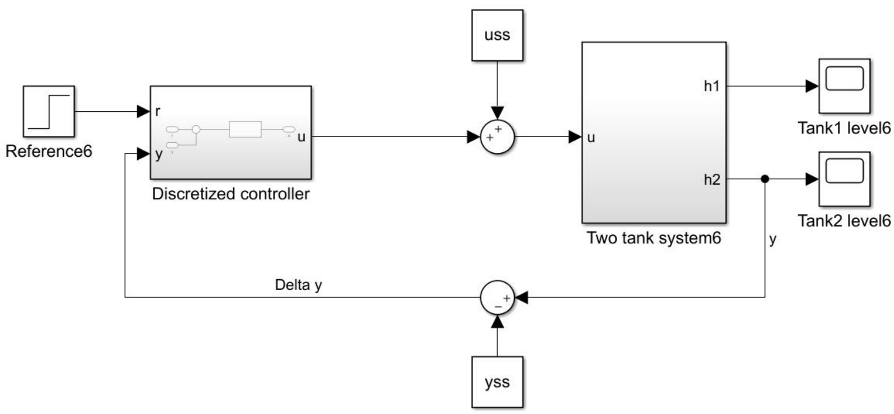


Figure 11: simulink model

Comparing the simulation results with the previous case in **Task 7** under different sampling times, we conclude Fig.12. We found that when discretizing the continuous controller, if the sampling time is too large, the step response became oscillating. When the sampling time is up to 5.0s, the system will even no longer converge to the steady state value according to our experiment.

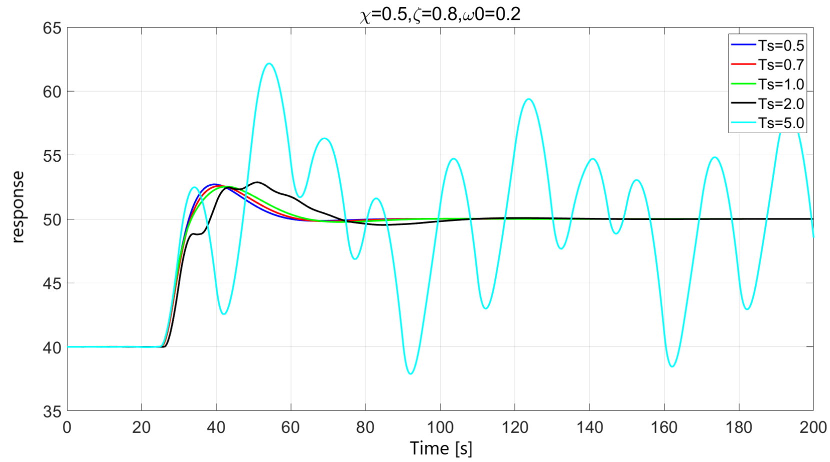


Figure 12: step response under different sampling times

In orer to find the maximum possible sampling time without affecting control performance, we now do the further experiments with the sampling time lower than 2.0s.

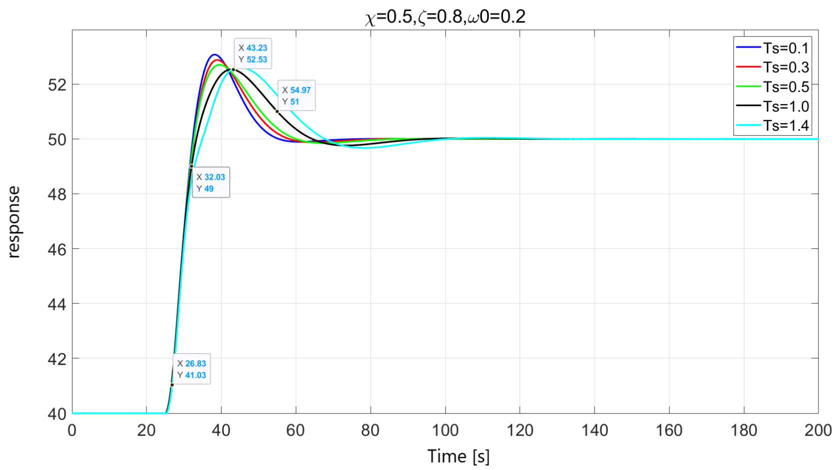


Figure 13: step response under different sampling times

According to Fig.13, we found that the maximum possible sampling time without affecting control performance is around 1.0s.

Task 9

Sampling time is always a key factor to consider when implementing a continuous controller digitally, since it might heavily influence the closed-loop performance. According to the rule of thumb which is built on extensive experience, we choose the interval of the sampling time as

$$T_s = h\omega_c \approx 0.05 \quad to \quad 0.14$$

Since the cross over frequency is 0.3619 rad/s, the sampling time should be between 0.138 and 0.387 second. We now use MATLAB to simulate the step response under the sampling time mentioned above and check the control performance, as shown in Fig.14.

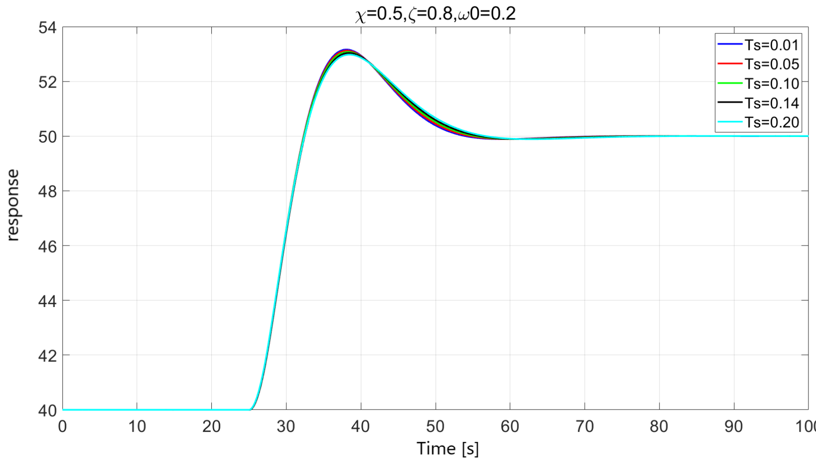


Figure 14: step response under different sampling times

Task 10

We proved **Task 10** that the maximum possible sampling time without affecting control performance is around $1.0s$. However, if compared with the caculated sampling time under the rule of thumb, the results are a bit different according to our experiment. From Fig.14, we found that when the sampling time was lower than $0.14s$ and larger than $0.05s$, the dynamic performance of the system would be roughly the same.

Task 11

Simulate the closed-loop system in **Task 8** when the sampling time $T_s = 4s$, the result of response is shown in Fig.15. Here the system will no longer converge to a steady value, instead it will oscillate randomly.

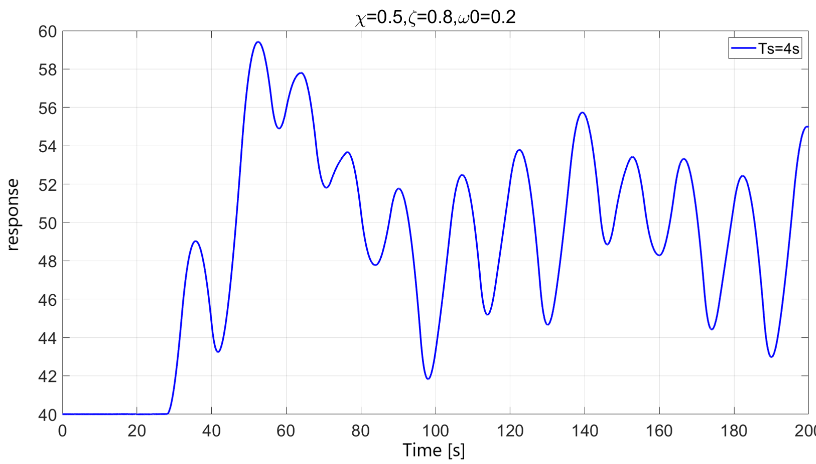


Figure 15: step response when $T_s = 4$

Task 12

First we need to transform the transfer function (Equation(1)) of this continuous-time linearized system into the standard form of state space model.

From Equation(1), we yields

$$\Delta X_1(s)(1+\tau s) = k\Delta U(s)$$
$$\Delta X_2(s)(1+\tau\gamma s) = \gamma\Delta X_1(s)$$

thus

$$\Delta X_1(s) + \Delta X_1(s)\tau s = k\Delta U(s)$$
$$\Delta X_2(s) + \Delta X_2(s)\tau\gamma s = \gamma\Delta X_1(s)$$

Take the inverse Laplace transform of both sides, we yield

$$\tau\Delta\dot{x_1}(t) + \Delta x_1(1) = k\Delta u(t)$$
$$\Delta x_2(t) + \gamma\tau\Delta\dot{x_2}(t) = \gamma\Delta x_1(t)$$

thus

$$\Delta\dot{x_1}(t) = -\frac{1}{T}\Delta x_1(t) + \frac{k}{\tau}\Delta u(t)$$
$$\Delta\dot{x_2}(t) = -\frac{1}{\gamma\tau}\Delta x_2(t) + \frac{1}{T}\Delta x_1(t)$$

Get it in matrix form

$$\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}\frac{-1}{\tau} & 0\\ \frac{1}{\tau} & \frac{-1}{\gamma\tau}\end{bmatrix}\Delta x(t) + \begin{bmatrix}\frac{k}{\tau}\\ 0\end{bmatrix}\Delta u(t)$$

where

$$A = \begin{bmatrix}\frac{-1}{\tau} & 0\\ \frac{1}{\tau} & \frac{-1}{\gamma\tau}\end{bmatrix}, B = \begin{bmatrix}\frac{k}{\tau}\\ 0\end{bmatrix}$$

thus

$$A = \begin{bmatrix}-0.08044 & 0\\ 0.08044 & 0. - 0.08044\end{bmatrix}, B = \begin{bmatrix}0.17593\\ 0\end{bmatrix}$$

Discreting this system with $T_s = 4s$ using MATLAB function `c2d`, and determining the discrete-time system of the form

$$\Delta x(k+1) = \Phi\Delta x(k) + \Gamma\Delta u(k)$$
$$\Delta y(k) = C\Delta x(k) = [0\ 1]\Delta x(k)$$

where

$$\Phi = \begin{bmatrix}0.72487 & 0\\ 0.23323 & 0.72487\end{bmatrix}, \Gamma = \begin{bmatrix}0.60172\\ 0.09162\end{bmatrix}$$

Task 13

Observability

For this Linear systems, we have

$$
\begin{aligned}
x(k+1) = \Phi x(k) = &\begin{bmatrix} 0.72487 & 0 \\ 0.23323 & 0.72487 \end{bmatrix} x(k) \\
y(k) = &Cx(k) = [0\ 1]x(k)
\end{aligned}
$$

Hence,

$$
W_o = \begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.23323 & 0.72487 \end{bmatrix}
$$

Since the rank of the $W_0 = 2$ and $det(W_0) = -0.23323 \neq 0$, this discrete-time plant model is observable.

Reachability

For this Linear systems, we have

$$
\begin{aligned}
\Delta x(k+1) = &\Phi \Delta x(k) + \Gamma \Delta u(k) \\
= &\begin{bmatrix} 0.72487 & 0 \\ 0.23323 & 0.72487 \end{bmatrix} \Delta x(k) + \begin{bmatrix} 0.60172 \\ 0.09162 \end{bmatrix} \Delta u(k)
\end{aligned}
$$

thus

$$
\begin{aligned}
x(n) = &\Phi^n x(0) + \Phi^{n-1}\Gamma u(0) + \cdots + \Gamma u(n-1) \\
= &\Phi^n x(0) + \underbrace{\begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}}_{W_c} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
W_c = \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix} = &\begin{bmatrix} 0.60172 & 0.43617 \\ 0.09162 & 0.20675 \end{bmatrix} \\
det(W_c) = &0.08445 \neq 0
\end{aligned}
$$

i.e. the discrete-time plant model is reachable.

Task 14

In order to design a state-feedback controller such that the closed-loop system has the same poles as the continuous-time closed-loop system.

Use the control law

$$\Delta u(k) = -L\Delta\hat{x}(k) + l_r r(k)$$

In order to let $\Delta\tilde{x}(k) \to 0$, here the reference gain l_r is necessary.

Task 15

The system and dynamic observer are defined as follows

$$\begin{aligned} \Delta x(k+1) &= \Phi\Delta x(k) + \Gamma\Delta u(k) \\ \Delta\hat{x}(k+1|k) &= \Phi\Delta\hat{x}(k|k-1) + \Gamma\Delta u(k) + K[y(k) - C\Delta\hat{x}(k|k-1)] \end{aligned}$$

In this experiment, we use a control law of the form

$$\Delta u(k) = -L\Delta\hat{x}(k) + l_r r(k)$$

By substituting a control law in the state space equations, we will get

$$\begin{aligned} \Delta x(k+1) &= \Phi\Delta x(k) + \Gamma(-L\Delta\hat{x}(k) + l_r r(k)) \\ \Delta\hat{x}(k+1|k) &= \Phi\Delta\hat{x}(k|k-1) + \Gamma(-L\Delta\hat{x}(k) + l_r r(k)) + K[y(k) - C\Delta\hat{x}(k|k-1)] \end{aligned}$$

We can define an augmented system as

$$\begin{aligned} x_a(k+1) &= A_a x_a(k) = B_a r(k) \\ x_a(k) &= \begin{bmatrix} \Delta x(k)^T & \Delta\hat{x}(k)^T \end{bmatrix}^T \end{aligned}$$

Therefore, we can derive the new system in the form of an augmented system as

$$x_a(k+1) = \begin{bmatrix} \Delta x(k+1) \\ \Delta\hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & -\Gamma L \\ KC & \Phi - \Gamma L - KC \end{bmatrix} \begin{bmatrix} \Delta x(k) \\ \Delta\hat{x}(k) \end{bmatrix} + \begin{bmatrix} \Gamma l_r \\ \Gamma l_r \end{bmatrix} r(k) \tag{1}$$

Thus, we get matrices A_a and B_a as

$$A_a = \begin{bmatrix} \Phi & -\Gamma L \\ KC & \Phi - \Gamma L - KC \end{bmatrix}, B_a = \begin{bmatrix} \Gamma l_r \\ \Gamma l_r \end{bmatrix}$$

Task 16

To give the controller the separation principle, we introduce new state vector

$$z(k) = \begin{bmatrix} \Delta x(k)^T & \Delta\tilde{x}(k)^T \end{bmatrix}^T, \tilde{x}(k) = \Delta x(k) - \Delta\hat{x}(k)$$

Thus, we can derive new state space equations by substituting $\Delta\hat{x}(k) = \Delta x(k) - \Delta\tilde{x}(k)$ to the Eq.1 as following

$$\begin{aligned} \Delta x(k+1) &= (\Phi - \Gamma L)\Delta x(k) + \Gamma L \Delta\tilde{x}(k) \\ \Delta\tilde{x}(k+1) &= (\Phi - KC)\Delta\tilde{x}(k) \end{aligned}$$

Therefore, we can write the equations in state space form as

$$z(k+1) = \begin{bmatrix} \Delta x(k+1) \\ \Delta\tilde{x}(k+1) \end{bmatrix} = \begin{bmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{bmatrix} z(k) + \begin{bmatrix} \Gamma l_r \\ 0 \end{bmatrix} r(k)$$

Define A_z as

$$A_z = \begin{bmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{bmatrix}$$

From the equation above, obviously, the poles of the system can be drived by caculating the equation $|zI - A_z| = 0$.

i.e.

$$(zI - \Phi + \Gamma L)(zI - \Phi + KC) = 0$$

Hence, From the equation above, the separation principle holds.

Task 17

From continuous-time closed-loop system, we can derived the transfer function as following

$$H(s) = \frac{669.5s^2 + 180.6s + 16.05}{1605s^4 + 2119s^3 + 309.7s^2 + 12.04s}$$

Therefore, we can compute continuous-time poles as following

$$poles = -0.5, -0.5, -0.16 + 0.12i, -0.16 - 0.12i$$

Using $z = e^{sh}$, poles in discrete system are

$$poles = 0.1353, 0.1353, 0.4677 - 0.2435i, 0.4677 + 0.2435i$$

From Ackermann's formula, a controller and observer gain can be computed. We design the observer to be faster than the controller, therefore the first two poles are used for the observer gain and the rest are for the controller gain. As a result, the controller gain and observer gain from `L = acker(Phi,Gamma,Pc(3:4))` and `K = acker(Phi',C',Pc(1:2))'` are

$$L = \begin{bmatrix} 0.7187 & 0.8937 \end{bmatrix}, K = \begin{bmatrix} 1.4902 \\ 1.1791 \end{bmatrix}$$

Desired poles can be verified by `eig(Aa)`, which yields the similar poles.

Task 18

As shown in Fig.16, the discrete designed controller can performed better than discretized controller with the same sampling time. For discrete designed controller, the rise time is $13.56s$, overshoot 3.08 % and settling-time $39.47s$. However, there is a steady state error occur in the response. The reason is that discrete design controller will increase the system type from holding, resulting in a steady state error.

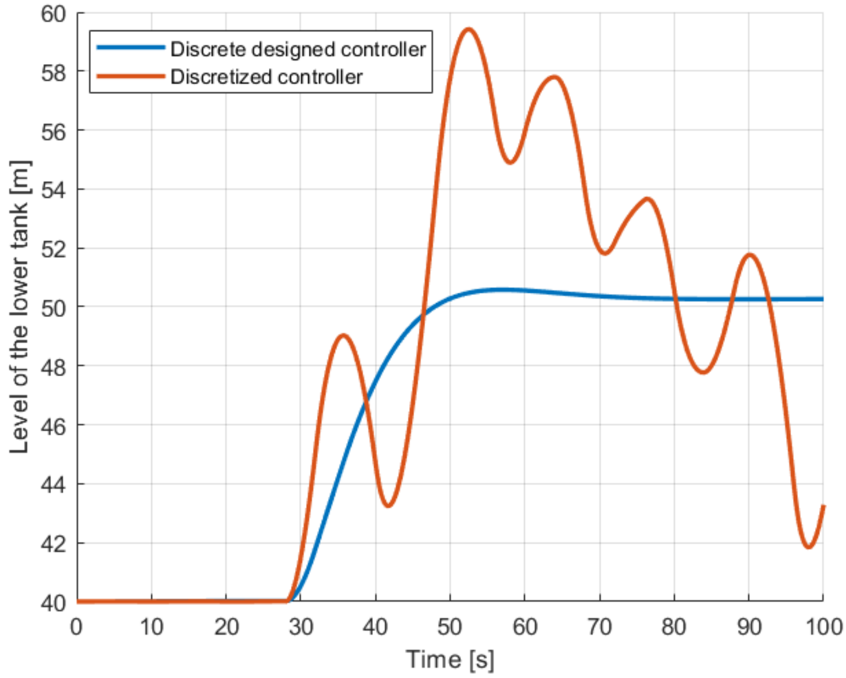


Figure 16: System response with the discrete and discretized controller

Table 3: Controller performance

controller type	T_r	M	T_{set}
discrete designed controller	$13.56s$	3.08%	$39.47s$
discretized controller	steady state error		

Task 19

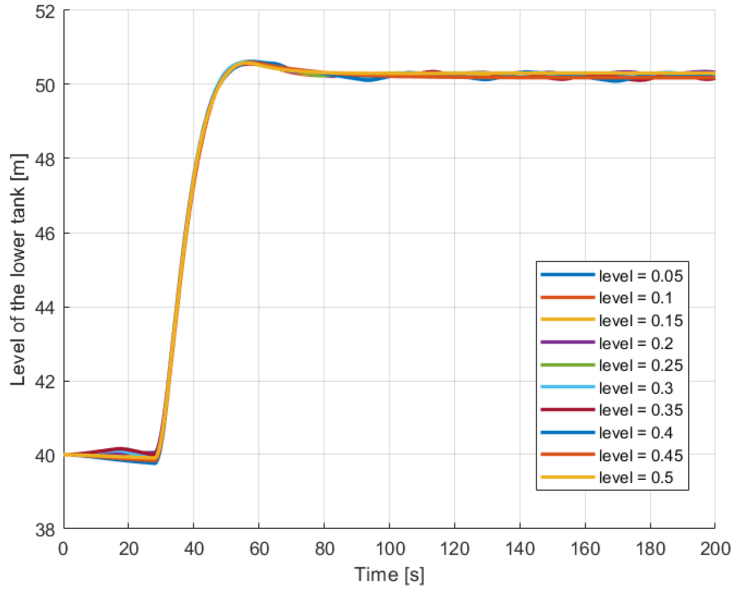
For A/D converter with the number of bits equal to 10 bits, there are $2^{10} = 1024$ quantization levels. Therefore, each interval will be $\frac{100-0}{1023} \approx 0.098$.

Task 20

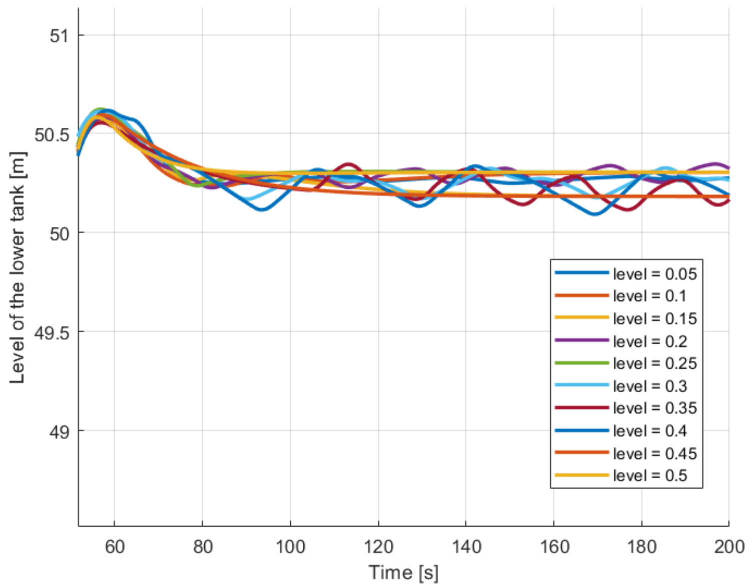
Since the quantization block can only specify the interval size and A/D and D/A can store finite set of number, we need to indicate the bound of the number. The quantization block can only specify the interval size, so the saturation block is needed to set the boundary.

Task 21

As shown in Fig.17, the performance of controller, especially the settling-time, start to degrade at quantization level equals to 0.3. However, other properties, including overshoot and rise time, do not change that much. Bits for the converter can be computed from $\text{bits} = \log 2(\frac{100-0}{0.3}) \approx 8.38$. Therefore, if the converter have accuracy 9 bits or more, the performance will be acceptable; however, if the accuracy is 8 bits or less, the degradation will occur.



(a) System response



(b) System response (enlarge to show settling-time performance)

Figure 17: The system response of different quantization level