AUTOMATIC CONTROL

KTH

EL2450 Hybrid and Embedded Control Systems

Exam –, March , 2017

Aid:

Lecture notes (slides) from the course, compendium ("reading material") and textbook from basic course ("Reglerteknik" by Glad & Ljung or similar text approved by course responsible). Mathematical handbook (e.g., "Beta Mathematics Handbook" by Råde & Westergren). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used unless previously approved by course responsible.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page and write on one side per sheet.
- Each answer has to be motivated.
- Specify the total number of handed in pages on the cover.
- Each subproblem is marked with its maximum credit.

Grading:

Grade A: ≥ 43, Grade B: ≥ 38

Grade C: ≥ 33, Grade D: ≥ 28

Grade E: ≥ 23, Grade Fx: ≥ 21

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Lycka till!

1. The motion of a car is modeled as

$$\begin{aligned}\dot{p}(t) =& v(t),\\ \dot{v}(t) =& u(t),\end{aligned} \tag{1}$$

where $p \in \mathbb{R}$ is the position of the car, $v \in \mathbb{R}$ is the velocity of the car, and $u \in \mathbb{R}$ is a control input. The car is controlled with an event-triggered controller such that

$$u(t) = -p(t_k) - v(t_k) \tag{2}$$

for all $t \in [t_k, t_{k+1})$. The goal of the controller is to bring the car to rest at the origin (i.e., $p = 0$ and $v = 0$). We consider the candidate Lyapunov function $V(t) = 3p(t)^2 + 2v(t)^2 + 2p(t)v(t)$.

(a) [1p] Letting $\tilde{p}(t) = p(t) - p(t_k)$ and $\tilde{v}(t) = v(t) - v(t_k)$, write the closed-loop dynamics of the car as a function of $p(t)$, $\tilde{p}(t)$, $v(t)$ and $\tilde{v}(t)$.

(b) [2p] Compute the time-derivative $\dot{V}(t)$ of the candidate Lyapunov function as a function of $p(t)$, $\tilde{p}(t)$, $v(t)$ and $\tilde{v}(t)$.

(c) [1p] Denoting $x(t) = [p(t), v(t)]^\intercal$ and $\tilde{x}(t) = [\tilde{p}(t), \tilde{v}(t)]^\intercal$, rewrite $V(t)$ in the form $V(t) = x(t)^\intercal P x(t)$, where $P \in \mathbb{R}^{2\times 2}$ is symmetric and positive semidefinite. Write the numerical value of P explicitly.

(d) [2p] Rewrite $\dot{V}(t)$ in the form $\dot{V}(t) = -x(t)^\intercal Q x(t) + x(t)^\intercal R \tilde{x}(t)$, where $Q \in \mathbb{R}^{2\times 2}$ is symmetric positive definite and $R \in \mathbb{R}^{2\times 2}$. Write the numerical value of Q and R explicitly.

(e) [3p] Using the results in (c) and (d), find a condition in the form $\|\tilde{x}\| < \alpha \|x\|$ that guarantees that $\dot{V}(t) < -\frac{2}{5+\sqrt{5}} V(t)$. Choose α as large as possible. Hints: $x^\intercal P x \leq \lambda_{\max}(P)\|x\|^2$ where $\lambda_{\max}$ is the largest eigenvalue, $x^\intercal Q x \geq \lambda_{\min}(Q)\|x\|^2$, where $\lambda_{\min}$ is the smallest eigenvalue, $|x^\intercal R \tilde{x}| \leq \sigma_{\max}(R)\|x\|\|\tilde{x}\|$, where $\sigma_{\max}$ denotes the largest singular value. Remember that the maximum singular value of a matrix R is computed as $\sigma_{\max}(R) = \sqrt{\lambda_{\max}(R^\intercal R)}$. Moreover,

$$\begin{aligned}\operatorname{eig}\left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\right) =& \left\{ \frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2} \right\},\\ \operatorname{eig}\left(\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}\right) =& \{0, 10\}.\end{aligned} \tag{3}$$

(f) [1p] Use the result in (e) to write an event-triggered law to schedule the control updates t_k that guarantees that $V(t)$ converges exponentially to zero.

Solution

(a) Substituting $u(t) = -p(t) + \tilde{p}(t) - v(t) + \tilde{v}(t)$ in the open-loop dynamics, we obtain

$$\begin{aligned} \dot{p}(t) =& v(t), \\ \dot{v}(t) =& -p(t) + \tilde{p}(t) - v(t) + \tilde{v}(t). \end{aligned} \tag{4}$$

(b) Taking the derivatives and collecting similar terms, we obtain

$$\dot{V}(t) = -2v(t)^2 - 2p(t)^2 + 4v(t)\tilde{p}(t) + 4v(t)\tilde{v}(t) + 2p(t)\tilde{p}(t) + 2p(t)\tilde{v}(t). \tag{5}$$

(c) We have

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \tag{6}$$

(d) We have

$$\begin{aligned} Q &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\ R &= \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}. \end{aligned} \tag{7}$$

(e) We have $\dot{V}(t) = -x(t)^\intercal Q x(t) + x(t)^\intercal R\tilde{x}(t) \leq -\lambda_{\min}(Q)\|x\|^2 + \sigma_{\max}(R)\|x\|\|\tilde{x}\| = -2\|x\|^2 + 2\sqrt{10}\|x\|\|\tilde{x}\|$. At the same time, $V(t) = x(t)^\intercal P x(t) \leq \lambda_{\max}(P)\|x\|^2 = \frac{5+\sqrt{5}}{2}\|x\|^2$. Therefore, the condition $\dot{V}(t) \leq -\frac{2}{5+\sqrt{5}}V(t)$ is satisfied if $-2\|x(t)\|^2 + 2\sqrt{10}\|x(t)\|\|\tilde{x}(t)\| \leq -\|x(t)\|^2$, which is equivalent to $\|\tilde{x}(t)\|^2 \leq \frac{1}{2\sqrt{10}}\|x(t)\|$.

(f) From the previous points, we know that if we enforce the updates as

$$t_{k+1} = \inf\{t > t_k : \ \|\tilde{x}(t)\| \geq \alpha\|x(t)\|\}, \tag{8}$$

then $V(t)$ converges exponentially to zero, with convergence rate at least as large as $\frac{2}{5+\sqrt{5}}$.