AUTOMATIC CONTROL

KTH

EL2450 Hybrid and Embedded Control Systems

Exam 08:00–13:00, June 7, 2017

Aid:

Lecture notes (slides) from the course and textbook from basic course ("Reglerteknik" by Glad & Ljung or similar text approved by course responsible). Mathematical handbook (e.g., "Beta Mathematics Handbook" by Råde & Westergren). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used unless previously approved by course responsible.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page and write on one side per sheet.
- Each answer has to be motivated.
- Specify the total number of handed in pages on the cover.
- Each subproblem is marked with its maximum credit.

Grading:

Grade A: ≥ 43, Grade B: ≥ 38

Grade C: ≥ 33, Grade D: ≥ 28

Grade E: ≥ 23, Grade Fx: ≥ 21

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Lycka till!

1. The motion of a car is modeled as

$$\begin{aligned}\dot{p}(t) =& v(t),\\ \dot{v}(t) =& u(t),\end{aligned} \tag{1}$$

where $p \in \mathbb{R}$ is the position of the car, $v \in \mathbb{R}$ is the velocity of the car, and $u \in \mathbb{R}$ is a control input. The car is controlled with an event-triggered controller such that

$$u(t) = -p(t_k) - v(t_k) \tag{2}$$

for all $t \in [t_k, t_{k+1})$. The goal of the controller is to bring the car to rest at the origin (i.e., $p = 0$ and $v = 0$). We consider the candidate Lyapunov function $V(t) = 3p(t)^2 + 2v(t)^2 + 2p(t)v(t)$.

(a) [1p] Letting $\tilde{p}(t) = p(t) - p(t_k)$ and $\tilde{v}(t) = v(t) - v(t_k)$, write the closed-loop dynamics of the car as a function of $p(t)$, $\tilde{p}(t)$, $v(t)$ and $\tilde{v}(t)$.

(b) [2p] Compute the time-derivative $\dot{V}(t)$ of the candidate Lyapunov function as a function of $p(t)$, $\tilde{p}(t)$, $v(t)$ and $\tilde{v}(t)$.

(c) [1p] Denoting $x(t) = [p(t), v(t)]^\top$ and $\tilde{x}(t) = [\tilde{p}(t), \tilde{v}(t)]^\top$, rewrite $V(t)$ in the form $V(t) = x(t)^\top P x(t)$, where $P \in \mathbb{R}^{2\times 2}$ is symmetric and positive semidefinite. Write the numerical value of P explicitly.

(d) [2p] Rewrite $\dot{V}(t)$ in the form $\dot{V}(t) = -x(t)^\top Q x(t) + x(t)^\top R \tilde{x}(t)$, where $Q \in \mathbb{R}^{2\times 2}$ is symmetric positive definite and $R \in \mathbb{R}^{2\times 2}$. Write the numerical value of Q and R explicitly.

(e) [3p] Using the results in (c) and (d), find a condition in the form $\|\tilde{x}\| < \alpha\|x\|$ that guarantees that $\dot{V}(t) < -\frac{2}{5+\sqrt{5}}V(t)$. Choose α as large as possible. Hints: $x^\top P x \leq \lambda_{\max}(P)\|x\|^2$ where $\lambda_{\max}$ is the largest eigenvalue, $x^\top Q x \geq \lambda_{\min}(Q)\|x\|^2$, where $\lambda_{\min}$ is the smallest eigenvalue, $|x^\top R\tilde{x}| \leq \sigma_{\max}(R)\|x\|\|\tilde{x}\|$, where $\sigma_{\max}$ denotes the largest singular value. Remember that the maximum singular value of a matrix R is computed as $\sigma_{\max}(R) = \sqrt{\lambda_{\max}(R^\top R)}$. Moreover,

$$\begin{aligned}\operatorname{eig}\left(\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}\right) =& \left\{\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right\},\\ \operatorname{eig}\left(\begin{bmatrix}5 & 5\\ 5 & 5\end{bmatrix}\right) =& \{0, 10\}.\end{aligned} \tag{3}$$

where $\operatorname{eig}(M)$ denotes the eigenvalues of a matrix M.

(f) [1p] Use the result in (e) to write an event-triggered law to schedule the control updates t_k that guarantee that $V(t)$ converges exponentially to zero.

2. Consider the system

$$\begin{aligned}\dot{x}_1 =& x_2,\\ \dot{x}_2 =& g\sin(x_1) - u\cos(x_1),\\ x_1(0) = x_2(0) &= u(0) = 0,\end{aligned}$$

where $x_1 \in [-\pi, \pi]$ is an angle, $x_2 \in \mathbb{R}$ is the corresponding angular velocity, u is the acceleration of the system, representing the control input, and $g > 0$ is the gravity acceleration. The goal is to drive $x = [x_1, x_2]^\top$ to the desired configuration $x_{\text{des}} = [\pi, 0]^\top$. To this end, we use a greedy controller, where we apply the maximum and minimum acceleration $u = u_{\max}$ and $u = -u_{\max}$, respectively, based on the energy of the system, which can be approximated by the function $\beta : [-\pi, \pi] \times \mathbb{R} \to \mathbb{R}$, with $\beta(x_1, x_2) = [\frac{x_2^2}{2} + g(\cos(x_1) - 1)]x_2 \cos x_1$. More specifically, we apply $u = u_{\max}$ when $\beta(x_1, x_2) \geq 0$ and $u = -u_{\max}$ when $\beta(x_1, x_2) < 0$. In order to avoid chattering, when x_1 is close to the desired configuration π (within a fixed angle θ, with $\pi > \theta > 0$), we apply a local stabilizing controller $u = \gamma_1 x_1 + \gamma_2 x_2, \gamma_1, \gamma_2 \in \mathbb{R}$, regardless of the value of β.

(a) [5p] *Using the augmented state* $z = [x^\top, u]^\top$, model the system as a hybrid automaton $H = (Q, X, \text{Init}, f, D, E, G, R)$.

(b) [5p] Consider that after appropriate state transformation and linearization, we obtain a switching system of the form

$$\dot{x} = A_q x, \tag{4}$$

with $q \in \{1, 2\}$, i.e., we only consider two of the aforementioned three states, and

$$A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
$$A_2 = \begin{bmatrix} a & c \\ 0 & d \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$ are scalar constants with $a < 0, d < 0$. Prove that the origin is asymptotically stable, given that the equation

$$x^2 + ((b + c)^2 - 4ad)x + a^2 + 3a + 9 = 0,$$

has two real solutions, $x_1, x_2 \in \mathbb{R}$, where at least one of them is positive. Note that you do not have to calculate a, b, c, d.

3. One control task J_c and two tasks J_1 and J_2 are executed on a preemptive CPU. The tasks have the release time zero and the following characteristics:

Task	T_i	D_i	C_i
J_1	3	3	1
J_2	4	4	1
J_c	8	8	3

(a) Determine the utilization factor and the schedule length? **[2p]**

(b) Are the tasks J_1, J_2 and J_c schedulable under earliest deadline first (EDF) scheduling algorithm? What is the worst case response time for J_1, J_2 and J_c **[4p]**

(c) Are the tasks J_1, J_2 and J_c schedulable under rate monotonic (RM) scheduling? What is the worst case response time for J_1, J_2 and J_c. **[4p]**

Hint: $2^{1/3} = 1.26$.

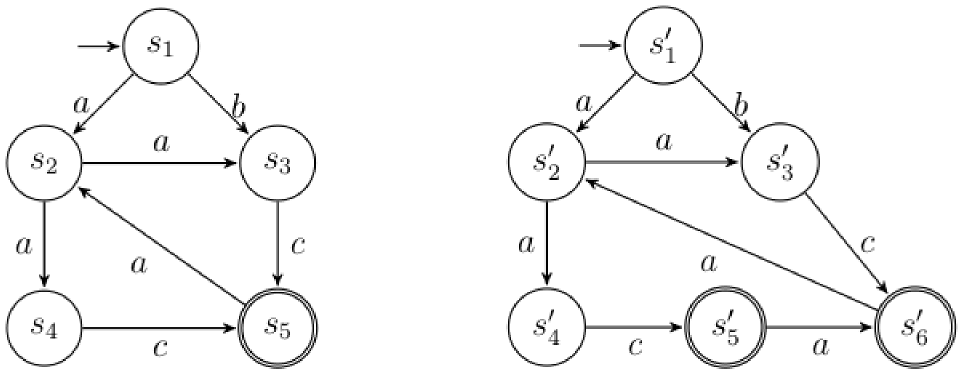

Figure 1: The transition systems $\mathcal{T}_1$ (left) and $\mathcal{T}_2$ (right).

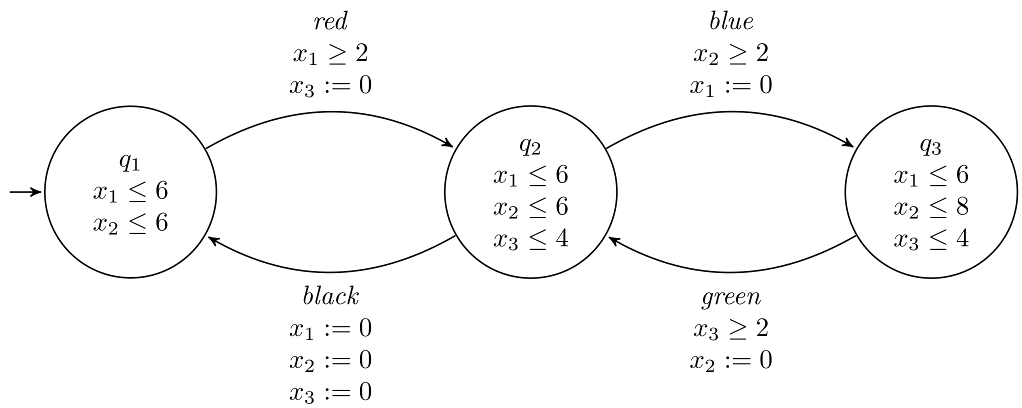

Figure 2: The timed automaton $\mathcal{T}$.

4. (a) [3p] Consider the Transition Systems $\mathcal{T}_1$ (left), $\mathcal{T}_2$ (right) as they are depicted in Figure 1. Are the Transition Systems bisimilar? Motivate your answer by following all the steps of the proof.

   (b) [3p] Consider the timed automaton $\mathcal{T}$ depicted in Figure 2. Model the timed automaton as a *hybrid automaton* $TA = (Q, X, Init, f, Act, D, E, G, R)$ and as a *Transition System* $T_{\text{TA}} = (S, \Sigma, \rightarrow, \Sigma_0)$ by writing down all the sets of the corresponding definitions.

   (c) [4p] Consider the timed automaton $\mathcal{T}$ depicted in Figure 2. Decide, whether the following states are reachable in the $\mathcal{T}$, i.e. whether they belong to the set $Reach((q_1, 0, 0, 0))$. Motivate your answer by showing the corresponding transitions.

      (i) $(q_3, 0, 4, 2)$
      (ii) $(q_2, 2, 0, 4)$
      (iii) $(q_4, 0, 6, 2)$
      (iv) $(q_3, 0, 6, 4)$
      (v) $(q_3, 6, 8, 4)$

5. Consider the hybrid automaton $H = (Q, X, Init, f, D, E, G, R)$ displayed in Figure 3 and defined by:

   - discrete state space $Q = \{q_1, q_2\}$
   - continuous state space $X = [0, 3]$
   - initial states $Init = \{(q_1, 0)\}$
   - domains $D(q_1) = D(q_2) = X$
   - edges $E = \{(q_1, q_2), (q_2, q_1)\}$
   - guards $G((q_1, q_2)) = [1, 3]$ and $G((q_2, q_1)) = [0, 1)$
   - resets $R((q_1, q_2), x) = R((q_2, q_1), x) = \emptyset$

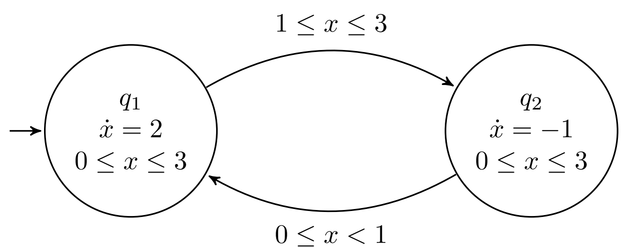

Figure 3: Hybrid automaton H

The goal of this problem is to do a safety verification on the hybrid automaton H with respect to the bad set $B = \{q_2\} \times [2, 3]$ by using the bisimulation quotient algorithm.

(a) [3p] Provide the equivalent representation of the hybrid automaton H as a transition system $T_H = (S, \Sigma, \rightarrow, S_0, S_F)$ by defining its elements S, Σ, S_0, S_F and sketching the transition system T_H.

(b) [5p] Obtain the quotient transition system $\hat{T} = (\hat{S}, \Sigma, \hat{\rightarrow}, \hat{S}_0, \hat{S}_F)$ of T_H in the following steps:

   - apply the bisimulation quotient algorithm to T_H to obtain the set of equivalence classes $\hat{S}$
   - prove that the algorithm has terminated
   - define the elements $\hat{S}$, $\hat{S}_0$, $\hat{S}_F$ of $\hat{T}$ and draw the resulting transition system $\hat{T}$.

(c) [2p] Compute the reach set $Reach(\hat{S}_0)$ of $\hat{T}$. Use this result to conclude on the safety verification of the hybrid automaton H with respect to the bad set $B = \{q_2\} \times [2, 3]$.