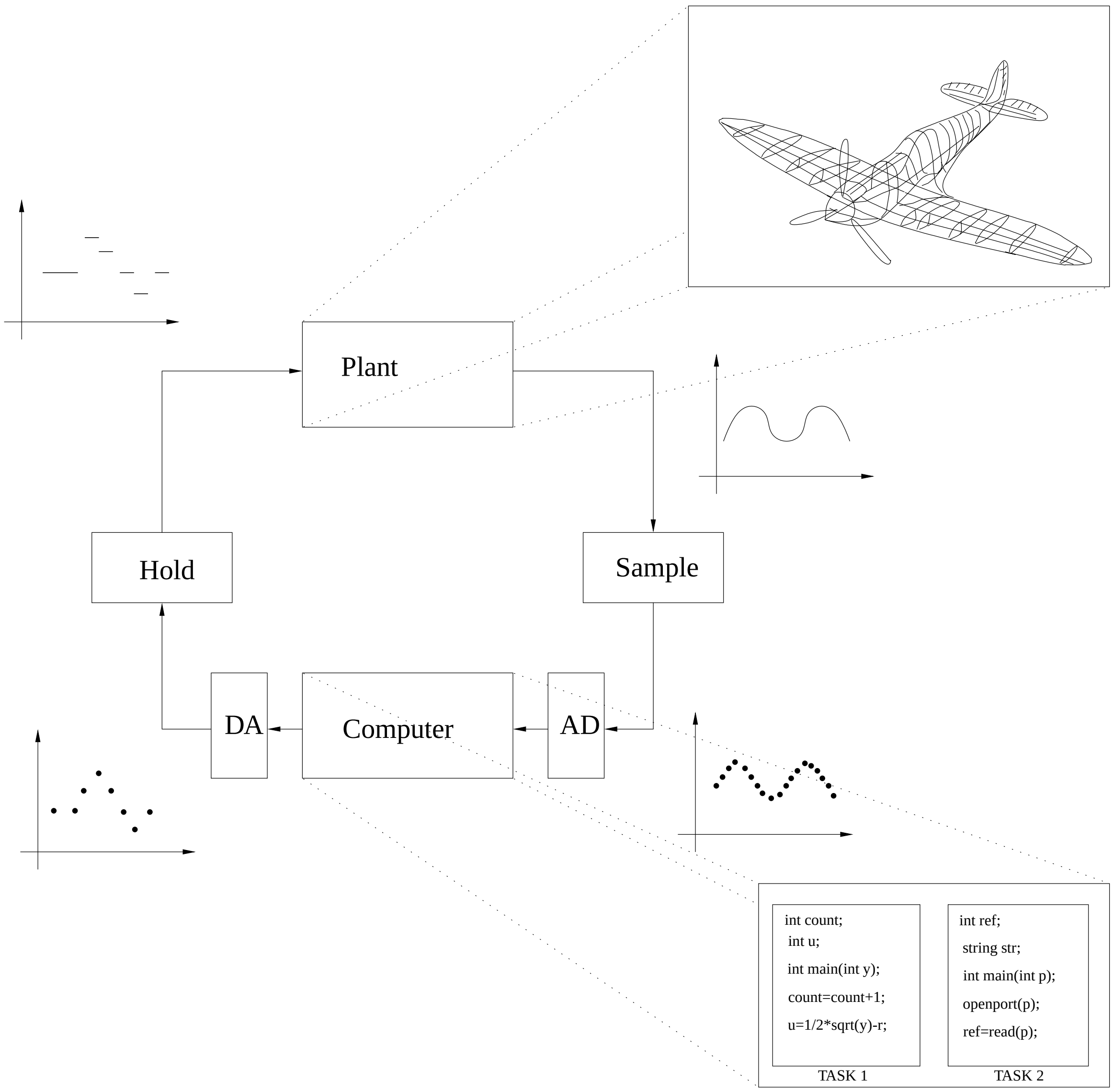
ROYAL INSTITUTE OF TECHNOLOGY

EL2450 - Hybrid and Embedded Control Systems

Exercises



Automatic Control Lab, School of Electrical Engineering
Royal Institute of Technology (KTH), Stockholm, Sweden

CONTENTS

Exercises

Part I

Time-triggered control

1 Review exercises: aliasing, z-transform, matrix exponential

EXERCISE 1.1 (Ex. 7.3 in [14])

Consider a sampling and reconstruction system as in Figure 1.1.1. The input signal is $x(t) = \cos(\omega_0 t)$. The Fourier transform of the signal is

$$X(j\omega) = \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

and the reconstruction (low-pass) filter has the transfer function

$$F(j\omega) = \begin{cases} h, & -\omega_s/2 < \omega < \omega_s/2 \\ 0, & \text{else} \end{cases}$$

where $\omega_s = \frac{2\pi}{h}$ is the sampling frequency. Find the reconstructed output signal $x_r(t)$ for the following input frequencies

(a) $\omega_0 = \omega_s/6$

(b) $\omega_0 = 2\omega_s/6$

(c) $\omega_0 = 4\omega_s/6$

(d) $\omega_0 = 5\omega_s/6$

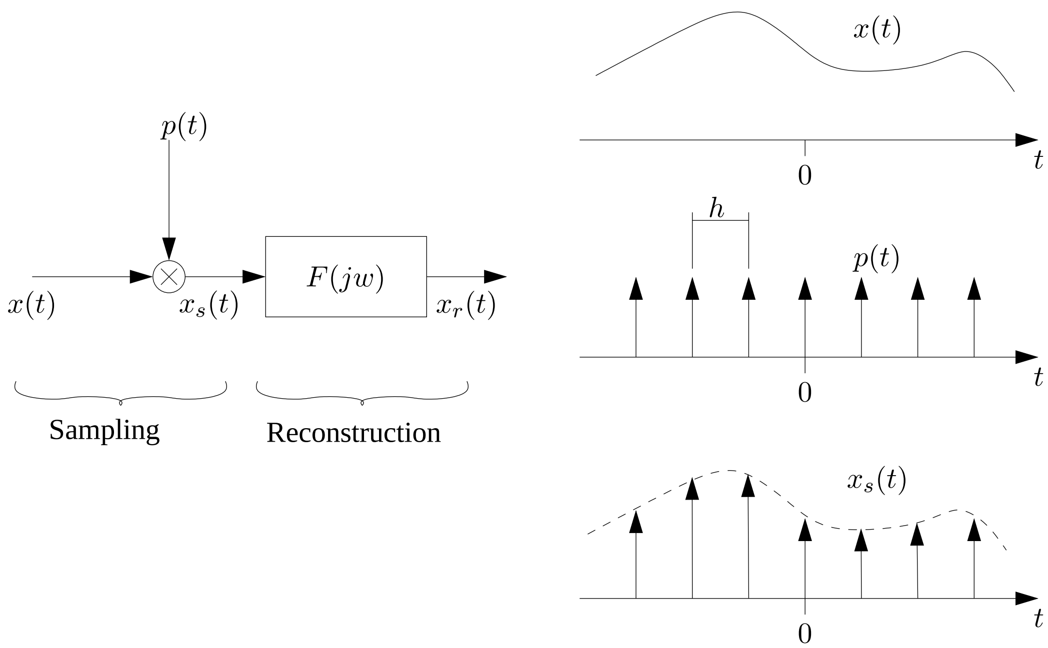


Figure 1.1.1: Sampling and reconstruction of a band-limited signal.

EXERCISE 1.2

Let the matrix A be

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Compute the matrix exponential e^A.

EXERCISE 1.3

Compute the z-transform of

$$x(kh) = e^{-kh/T} \quad T > 0.$$

EXERCISE 1.4

Compute the z-transform of

$$x(kh) = \sin(wkh)$$

EXERCISE 1.5

Given the following system described by the following difference equation

$$y(k+2) - 1.5y(k+1) + 0.5y(k) = u(k+1)$$

with initial condition $y(0) = 0.5$ and $y(1) = 1.25$, determine the output when the input $u(k)$ is a unitary step.

2 Models of sampled systems

EXERCISE 2.1 (Ex. 2.1 in [2])

Consider the scalar system

$$\begin{aligned} \frac{dx}{dt} &= -ax + bu \\ y &= cx. \end{aligned}$$

Let the input be constant over periods of length h. Sample the system and discuss how the poles of the discrete-time system vary with the sampling frequency.

EXERCISE 2.2

Consider the following continuous-time transfer function

$$G(s) = \frac{1}{(s+1)(s+2)}.$$

The system is sampled with sampling period $h = 1$.

(a) Derive a state-space representation of the sampled system.

(b) Find the pulse-transfer function corresponding to the system in (a).

EXERCISE 2.3 (Ex. 2.2 in [2])

Derive the discrete-time system corresponding to the following continuous-time systems when a zero order-hold circuit is used

(a)

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x\end{aligned}$$

(b)

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{du}{dt} + 3u$$

(c)

$$\frac{d^3y}{dt^3} = u$$

EXERCISE 2.4 (Ex. 2.3 in [2])

The following difference equations are assumed to describe continuous-time systems sampled using a zero-order-hold circuit and the sampling period h. Determine, if possible, the corresponding continuous-time systems.

(a)

$$y(kh) - 0.5y(kh - h) = 6u(kh - h)$$

(b)

$$\begin{aligned}x(kh + h) &= \begin{pmatrix} -0.5 & 1 \\ 0 & -0.3 \end{pmatrix} x(kh) + \begin{pmatrix} 0.5 \\ 0.7 \end{pmatrix} u(kh) \\ y(kh) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x(kh)\end{aligned}$$

(c)

$$y(kh) + 0.5y(kh - h) = 6u(kh - h)$$

EXERCISE 2.5 (Ex. 2.11 in [2])

The transfer function of a motor can be written as

$$G(s) = \frac{1}{s(s+1)}.$$

Determine:

(a) the sampled system

(b) the pulse-transfer function

(c) the pulse response

(d) a difference equation relating input and output

(e) the variation of the poles and zeros of the pulse-transfer function with the sampling period

EXERCISE 2.6 (Ex. 2.12 in [2])

A continuous-time system with transfer function

$$G(s) = \frac{1}{s}\, e^{-s\tau}$$

is sampled with sampling period $h = 1$, where $\tau = 0.5$.

(a) Determine a state-space representation of the sampled system. What is the order of the sampled-system?

(b) Determine the pulse-transfer function and the pulse response of the sampled system

(c) Determine the poles and zeros of the sampled system.

EXERCISE 2.7 (Ex. 2.13 in [2])

Solve Problem 2.6 with

$$G(s) = \frac{1}{s+1}\, e^{-s\tau}$$

and $h = 1$ and $\tau = 1.5$.

EXERCISE 2.8 (Ex. 2.15 in [2])

Determine the polynomials $A(q)$, $B(q)$, $A^*(q^{-1}))$ and $B^*(q^{-1})$ so that the systems

$$A(q)y(k) = B(q)u(k)$$

and

$$A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d)$$

represent the system

$$y(k) - 0.5y(k-1) = u(k-9) + 0.2u(k-10).$$

What is d? What is the order of the system?

EXERCISE 2.9 (Ex. 2.17 in [2])

Use the z-transform to determine the output sequence of the difference equation

$$y(k+2) - 1.5y(k+1) + 0.5y(k) = u(k+1)$$

when $u(k)$ is a step at $k = 0$ and when $y(0) = 0.5$ and $y(-1) = 1$.

EXERCISE 2.10

Consider the following continuous time controller

$$U(s) = -\frac{s_0 s + s_1}{s + r_1} Y(s) + \frac{t_0 s + t_1}{s + r_1} R(s)$$

where s_0, s_1, t_0, t_1 and r_1 are parameters that are chosen to obtain the desired closed-loop performance. Discretize the controller using exact sampling by means of sampled control theory. Assume that the sampling interval is h, and write the sampled controller on the form $u(kh) = -H_y(q)y(kh) + H_r(q)r(kh)$.

EXERCISE 2.11 (Ex. 2.21 in [2])

If $\beta < \alpha$, then

$$\frac{s+\beta}{s+\alpha}$$

is called a lead filter (i.e. it gives a phase advance). Consider the discrete-time system

$$\frac{z+b}{z+a}$$

(a) Determine when it is a lead filter

(b) Simulate the step response for different pole and zero locations

3 Analysis of sampled systems

EXERCISE 3.1 (Ex. 3.2 in [2])

Consider the system in Figure 3.1.1 and let

$$H(z) = \frac{K}{z(z-0.2)(z-0.4)} \qquad K > 0$$

Determine the values of K for which the closed-loop system is stable.

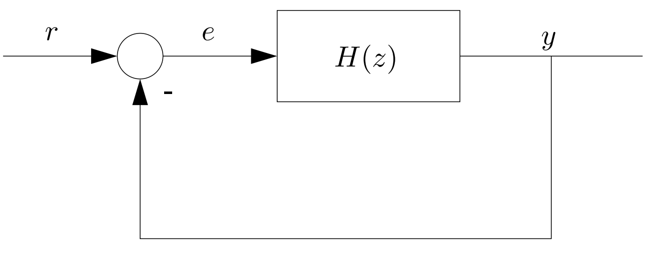


Figure 3.1.1: Closed-loop system for Problem 3.1.

EXERCISE 3.2 (Ex. 3.3 in [2])

Consider the system in Figure 3.2.1. Assume the sampling is periodic with period h, and that the D-A converter holds the control signal constant over a sampling interval. The control algorithm is assumed to be

$$u(kh) = K\big(r(kh - \tau) - y(kh - \tau)\big)$$

where $K > 0$ and τ is the computation time. The transfer function of the process is

$$G(s) = \frac{1}{s}.$$

(a) How large are the values of the regulator gain, K, for which the closed-loop system is stable when $\tau = 0$ and $\tau = h$?

(b) Compare this system with the corresponding continuous-time systems, that is, when there is a continuous-time proportional controller and a time delay in the process.

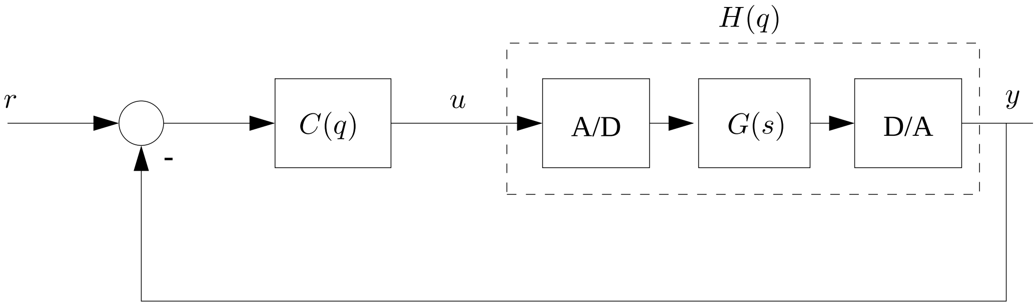


Figure 3.2.1: Closed-loop system for Problem 3.2.

EXERCISE 3.3 (Ex. 3.6 in [2])

Is the following system (a) observable, (b) reachable?

$$x(k+1) = \begin{pmatrix} 0.5 & -0.5 \\ 0 & 0.25 \end{pmatrix} x(k) + \begin{pmatrix} 6 \\ 4 \end{pmatrix} u(k)$$
$$y(k) = \begin{pmatrix} 2 & -4 \end{pmatrix} x(k)$$

EXERCISE 3.4 (Ex. 3.7 in [2])

Is the following system reachable?

$$x(k+1) = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} x(k) + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} u(k).$$

Assume that a scalar input $v(k)$ such that

$$u(k) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} v(k)$$

is introduced. Is the system reachable from $v(k)$?

EXERCISE 3.5 (Ex. 3.11 in [2])

Determine the stability and the stationary value of the output for the system described by Figure 3.2.1 with

$$H(q) = \frac{1}{q(q-0.5)}$$

where r is a step function and $C(q) = K$ (proportional controller), K>0.

EXERCISE 3.6 (Ex. 3.12 in [2])

Consider the Problem 3.5. Determine the steady-state error between the reference signal r and the output y, when r is a unit ramp, that is $r(k) = k$. Assume $C(q)$ to be a proportional controller.

EXERCISE 3.7 (Ex. 3.18 in [2])

Consider a continuous-time (CT) system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t).\end{aligned}$$

The zero-order hold sampling of CT gives the discrete-time (DT) system

$$\begin{aligned}x(kh+h) &= \Phi x(kh) + \Gamma u(kh)\\ y(kh) &= Cx(kh).\end{aligned}$$

Consider the following statements:

(a) CT stable $\Rightarrow$ DT stable

(b) CT unstable $\Rightarrow$ DT unstable

(c) CT controllable $\Rightarrow$ DT controllable

(d) CT observable $\Rightarrow$ DT observable.

Which statements are true and which are false (explain why) in the following cases:

(i) For all sampling intervals $h > 0$

(ii) For all $h > 0$ except for isolated values

(iii) Neither (i) nor (ii).

EXERCISE 3.8 (Ex. 3.20 in [2])

Given the system

$$(q^2 + 0.4q)y(k) = u(k),$$

(a) for which values of K in the proportional controller

$$u(k) = K\big(r(k) - y(k)\big)$$

is the closed-loop system stable?

(b) Determine the stationary error $r - y$ when r is a step and K=0.5 in the controller (a).

EXERCISE 3.9 (Ex. 4.1 in [2])

A general second-order discrete-time system can be written as

$$\begin{aligned} x(k+1) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} x(k) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u(k) \\ y(k) &= \begin{pmatrix} c_1 & c_2 \end{pmatrix} x(k). \end{aligned}$$

Determine a state-feedback controller in the form

$$u(k) = -Lx(k)$$

such that the characteristic equation of the closed-loop system is

$$z^2 + p_1 z + p_2 = 0.$$

Use the previous result to compute the deadbeat controller for the double integrator.

EXERCISE 3.10 (Ex. 4.2 in [2])

Given the system

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 1 & 0.1 \\ 0.5 & 0.1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) \\ y(k) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x(k). \end{aligned}$$

Determine a linear state-feedback controller

$$u(k) = -Lx(k)$$

such that the poles of the closed-loop system are placed in 0.1 and 0.25.

EXERCISE 3.11 (Ex. 4.5 in [2])

The system

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 0.78 & 0 \\ 0.22 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 0.22 \\ 0.03 \end{pmatrix} u(k) \\ y(k) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(k). \end{aligned}$$

represents the normalized motor for the sampling interval of $h = 0.25$. Determine observers for the state based on the output by using each of the following:

(a) Direct calculation.

(b) An full-state observer.

(c) The reduced-order observer.

EXERCISE 3.12 (Ex. 4.8 in [2])

Given the discrete-time system

$$x(k+1) = \begin{pmatrix} 0.5 & 1 \\ 0.5 & 0.7 \end{pmatrix} x(k) + \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} u(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v(k)$$
$$y(k) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(k).$$

where v is a constant disturbance. Determine controller such that the influence of v can be eliminated in steady state in each of the following cases:

(a) The state and v can be measured.

(b) The state can be measured.

(c) Only the output can be measured.

EXERCISE 3.13 (Ex. 4.6 in [2])

Figure 3.13.1 shows a system with two tanks, where the input signal is the flow to the first tank and the output is the level of water in the second tank. The continuous-time model of the system is

$$\dot{x} = \begin{pmatrix} -0.0197 & 1 \\ 0.0178 & -0.0129 \end{pmatrix} x + \begin{pmatrix} 0.0263 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x.$$

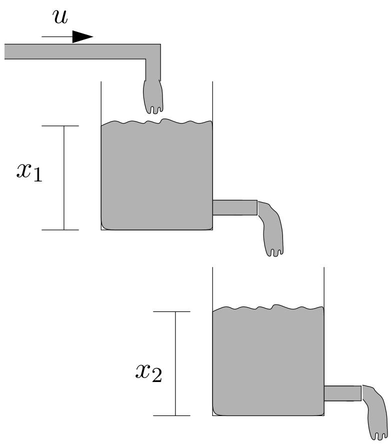


Figure 3.13.1: Closed-loop system for Problem 3.13.

(a) Sample the system with $h = 12$.

(b) Verify that the pulse-transfer operator for the system is

$$H(q) = \frac{0.030q + 0.026}{q^2 - 1.65q + 0.68}$$

(c) Determine a full-state observer. Choose the gain such that the observer is twice as fast as the open-loop system.

EXERCISE 3.14

Consider the following scalar linear system

$$\begin{aligned}\dot{x}(t) &= -5x(t) + u(t)\\ y(t) &= x(t).\end{aligned}$$

(a) Sample the system with sampling period $h = 1$,

(b) Show, using Lyapunov result, that the sampled system is stable when the input $u(kh) = 0$ for $k \geq 0$.

EXERCISE 3.15

Consider the following linear system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)\\ y(t) &= x(t).\end{aligned}$$

(a) Sample the system with sampling period $h = 1$

(b) Design a controller that place the poles in 0.1 and 0.2.

(c) Show, using Lyapunov result, that the closed loop sampled system is stable

4 Computer realization of controllers

EXERCISE 4.1

Consider the following pulse-transfer

$$H(z) = \frac{z-1}{(z-0.5)(z-2)}$$

(a) Design a digital PI controller

$$H_c(z) = \frac{(K + K_i)z - K}{z-1}$$

that places the poles of the closed-loop system in the origin.

(b) Find a state-space representation of the digital controller in (a).

EXERCISE 4.2 (Ex. 8.2 in [2])

Use different methods to make an approximation of the transfer function

$$G(s) = \frac{a}{s+a}$$

(a) Euler's method

(b) Tustin's approximation

(c) Tustin's approximation with pre-warping using $\omega_1 = a$ as warping frequency

EXERCISE 4.3 (Ex. 8.3 in [2])

The lead network with transfer function

$$G_\ell(s) = 4\frac{s+1}{s+2}$$

Give a phase advance of about 20° at $\omega_c = 1.6\text{rad}/s$. Approximate the network for $h = 0.25$ using

(a) Euler's method

(b) Backward differences

(c) Tustin's approximation

(d) Tustin's approximation with pre-warping using $\omega_1 = \omega_c$ as warping frequency

EXERCISE 4.4 (Ex. 8.7 in [2])

Consider the tank system in Problem 2.13. Assume the following specifications:

1. The steady-state error after a step in the reference value is zero
2. The crossover frequency of the compensated system is 0.025 rad/s
3. The phase margin is about 50°.

(a) Design a PI-controller such that the specifications are fulfilled.

(b) Determine the poles and the zeros of the closed-loop system. What is the damping corresponding to the complex poles?

(c) Choose a suitable sampling interval and approximate the continuous-time controller using Tustin's method with pre-warping. Use the crossover frequency as warping frequency.

EXERCISE 4.5 (Ex. 8.4 in [2])

The choice of sampling period depends on many factors. One way to determine the sampling frequency is to use continuous-time arguments. Approximate the sampled system as the hold circuit followed by the continuous-time system. Assuming that the phase margin can be decreased by 5° to 15°, verify that a rule of thumb in selecting the sampling frequency is

$$h\omega_c \approx 0.15 \text{ to } 0.5$$

where ω_c is the crossover frequency of the continuous-time system.

EXERCISE 4.6 (Ex. 8.12 in [2])

Consider the continuous-time double integrator described by

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x.
\end{aligned}
$$

Assume that a time-continuous design has been made giving the controller

$$
\begin{aligned}
u(t) &= 2r(t) - \begin{pmatrix} 12 \end{pmatrix} \hat{x}(t) \\
\frac{d\hat{x}(t)}{dt} &= A\hat{x}(t) + Bu(t) + K\big(y(t) - C\hat{x}(t)\big)
\end{aligned}
$$

with $K^T = (1, 1)$.

(a) Assume that the controller should be implemented using a computer. Modify the controller (not the observer part) for the sampling interval $h = 0.2$ using the approximation for state models.

(b) Approximate the observer using a backward-difference approximation

EXERCISE 4.7

Consider the following continuous time controller

$$
U(s) = -\frac{s_0 s + s_1}{s + r_1} Y(s) + \frac{t_0 s + t_1}{s + r_1} R(s)
$$

where s_0, s_1, t_0, t_1 and r_1 are parameters that are chosen to obtain the desired closed-loop performance.

(a) Discretize the controller using forward difference approximation. Assume that the sampling interval is h, and write the sampled controller on the form $u(kh) = -H_y(q)y(kh) + H_r(q)r(kh)$.

(b) Assume the following numerical values of the coefficients: $r_1 = 10$, $s_0 = 1$, $s_1 = 2$, $t_0 = 0.5$ and $t_1 = 10$. Compare the discretizations obtained in part (a) for the sampling intervals $h = 0.01$, $h = 0.1$ and $h = 1$. Which of those sampling intervals should be used for the forward difference approximation?

EXERCISE 4.8

Consider the following continuous-time controller in state-space form

$$
\begin{aligned}
\dot{x} &= Ax + Be \\
u &= Cx + De
\end{aligned}
$$

(a) Derive the backward-difference approximation in state-space form of the controller, i.e. derive Φ_c, Γ_c, H and J for a system

$$
\begin{aligned}
w(k+1) &= \Phi_c w(k) + \Gamma_c e(k) \\
u(k) &= Hw(k) + Je(k)
\end{aligned}
$$

(b) Prove that the Tustin's approximation of the controller is given by

$$\begin{aligned}
\Phi_c &= \left(I + \frac{A_c h}{2}\right)\left(I - \frac{A_c h}{2}\right)^{-1} \\
\Gamma_c &= \left(I - \frac{A_c h}{2}\right)^{-1} \frac{B_c h}{2} \\
H &= C_c \left(I - \frac{A_c h}{2}\right)^{-1} \\
J &= D_c + C_c \left(I - \frac{A_c h}{2}\right)^{-1} \frac{B_c h}{2}.
\end{aligned}$$

5 Implementation aspects

EXERCISE 5.1

Consider the discrete-time controller characterized by the pulse-transfer function

$$H(z) = \frac{1}{(z-1)(z-1/2)(z^2+1/2z+1/4)}.$$

Implement the controller in parallel form.

EXERCISE 5.2

(a) Given the system in Figure 5.2.1, find the controller $C_s(s)$ such that the closed loop transfer function from r to y becomes

$$H_{c\ell} = \frac{C(s)P(s)}{1+C(s)P(s)} e^{-s\tau}$$

(b) Let

$$\begin{aligned}
P(s) &= \frac{1}{s+1} \\
H_{c\ell}(s) &= \frac{8}{s^2+4s+8} e^{-s\tau}
\end{aligned}$$

find the expression for the Smith predictor $C_s(s)$.

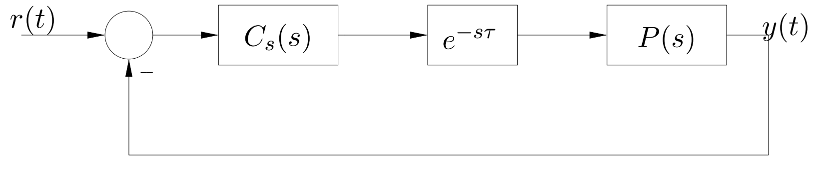


Figure 5.2.1: System of Problem 5.2.

EXERCISE 5.3

A process with transfer function

$$P(z) = \frac{z}{z-0.5}$$

is controlled by the PI-controller

$$C(z) = K_p + K_i \frac{z}{z-1}$$

where $K_p = 0.2$ and $K_i = 0.1$. The control is performed over a wireless network, as shown in Figure 5.3.1. Due to retransmission of dropped packets, the network induces time-varying delays. How large can the maximum delay be, so that the closed loop system is stable?

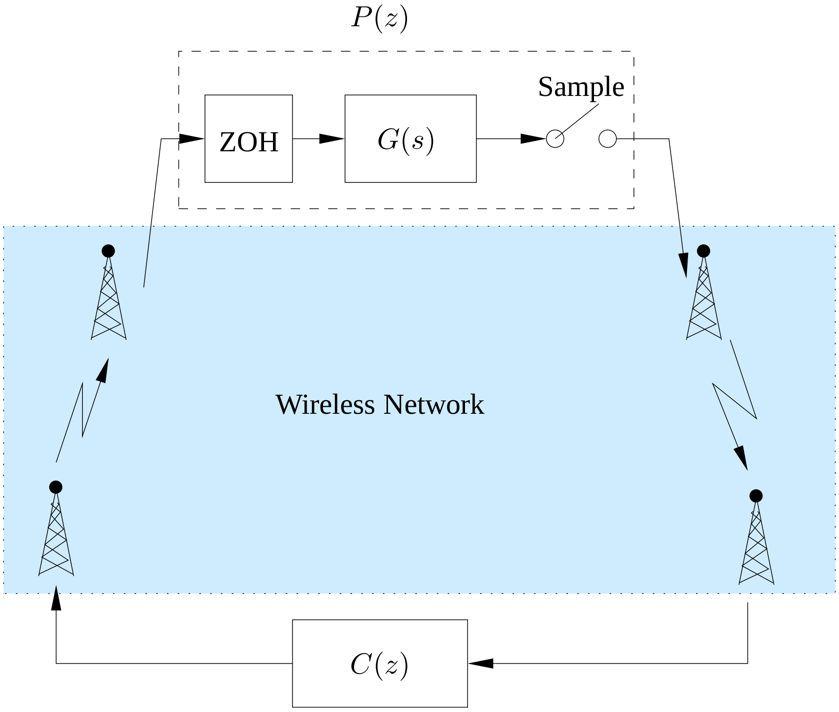


Figure 5.3.1: Closed loop system for Problem 5.3.

EXERCISE 5.4 (Inspired by Ex. 9.15 in [2])

Two different algorithms for a PI-controller are listed. Use the linear model for roundoff to analyze the sensitivity of the algorithms to roundoff in multiplications and divisions. Assume that the multiplications happen in the order as they appear in the formula and that they are executed before the division.

Algorithm 1:

```
repeat
  e:=r-y
  u:=k*(e+i)
  i:=i+e*h/ti
forever
```

Algorithm 2:

```
repeat
  e:=r-y
  u:=i+k*e
  i:=k*i+k*h*e/ti
forever
```

EXERCISE 5.5

Consider a first-order system with the discrete transfer function

$$H(z) = \frac{1}{1 - az^{-1}} \quad a = \frac{1}{8}.$$

Assume the controller is implemented using fixed point arithmetic with 8 bits word length and $h = 1$ second. Determine the system's unit step response for sufficient number of samples to reach steady-state. Assume that the data representation consists of

- 1 bit for sign
- 2 bits for the integer part
- 5 bits for the fraction part

and consider the cases of truncation and round-off.

Part II

Event-triggered control

6 Event-based control and Real-time systems

EXERCISE 6.1

Consider the following first-order system:

$$\dot{x} = x + u$$

where $x, u \in \mathbb{R}$ are the state and control input, respectively. Moreover, $x = 5$ when $t = 0$. The feedback control law is given by:

$$u = -2\,x$$

(a) Show that under this control law, the closed-loop system asymptotically converges to the origin, using a Lyapunov function.

(b) Suppose $x(t)$ is sampled periodically and the controller is followed by a Zero Order Hold, namely

$$u(t) = -2\,x(kh), \qquad t \in [kh, kh + h),$$

where $k \in \mathbb{N}$, $h > 0$ is the sampling period. What is the maximal value $h_{\max}$ of h before the closed-loop system becomes unstable?

(c) Now $x(t)$ is sampled aperiodically at the sequence $\{t_k\}$, $k \in \mathbb{N}$ and the controller is followed by a Zero Order Hold, namely

$$u(t) = -2\,x(t_k),\ t \in [t_k, t_{k+1}).$$

how to design the sequence $\{t_k\}$, $k \in \mathbb{N}$ such that the closed-loop system still converges to the origin? what is the maximal sampling interval $\max_k(t_{k+1} - t_k)$ in this case?

(d) Describe how the event-triggered controller could be implemented on a digital platform and compare it with periodic controller in (b).

EXERCISE 6.2

Consider the following unstable second-order system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u(t).$$

(a) Show that a linear feedback control law $u(t) = Kx(t)$ with

$$K = \begin{bmatrix} -3 & -1 \end{bmatrix},$$

renders the closed-loop system asymptotically stable.

(b) In order to implement the controller on a digital platform, the state of the system is sampled aperiodically at a sequence of time instants $\{t_k\}$, $k \in \mathbb{N}$, and the control signal is now given by

$$u(t) = Kx(t_k), \quad t \in [t_k, t_{k+1}).$$

Find the closed loop equation of the system in terms of the state $x(t)$ and the state error $e(t)$, where

$$e(t) = x(t_k) - x(t), \quad t \in [t_k, t_{k+1}).$$

(c) Using the quadratic Lyapunov function

$$V(x) = \frac{1}{2} x^T \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} x,$$

design the sequence $\{t_k\}$, $k \in \mathbb{N}$, such that the origin is still a stable equilibrium point of the closed-loop system.

EXERCISE 6.3

In an embedded control system the control algorithm is implemented as a task in a CPU. The control task J_c can compute the new control action only after the acquisition task J_a has acquired new sensor measurements. The two tasks are independent and they share the same CPU. Suppose the sampling period is $h = 0.4$ seconds and the tasks have the following specifications

	C_i
J_c	0.1
J_a	0.2

We assume that the period T_i and the deadline D_i are the same for the two tasks, and the release time is 0 for both tasks.

(a) Is possible to schedule the two tasks J_c and J_a? Determine the schedule length and draw the schedule.

(b) Suppose that a third task is running in the CPU. The specifications for the task are

	C_i	$T_i = D_i$	r_i
J_x	0.2	0.8	0.3

and we assume that the task J_x has higher priority than the tasks J_c and J_a. We also assume the CPU can handle preemption. Are the three tasks schedulable? Draw the schedule and determine the worst-case response time for the control task J_c.

EXERCISE 6.4

A digital PID controller is used to control the plant, which sampled with period $h = 2$ has the following transfer function

$$P(z) = \frac{1}{100} \frac{z - 0.1}{z - 0.5}.$$

The control law is

$$C(z) = 15 \left(1 + \frac{z}{z - 1} \right).$$

Assume that the control task J_c is implemented on a computer and has $C_c = 1$ as the worst case computation time. Assume that a higher priority interrupt occurs at time $t = 2$ which has a worst case computation time C_I. Determine the largest value of C_I such that the closed loop system is stable.

EXERCISE 6.5

A robot has been designed with three different tasks J_A, J_B, J_C, with increasing priority. The task J_A is a low priority thread which implements the DC-motor controller, the task J_B periodically send a "ping" through the wireless network card so that it is possible to know if the system is running. Finally the task J_C, with highest priority, is responsible to check the status of the data bus between two I/O ports, as shown in Figure 6.5.1. The control task is at low priority since the robot is moving very slowly in a cluttered environment. Since the data bus is a shared resource there is a semaphore that regulates the access to the bus. The tasks have the following characteristics

	T_i	C_i
J_A	8	4
J_B	5	2
J_C	1	0.1

Assuming the kernel can handle preemption, analyze the following possible working condition:

- at time $t = 0$, the task J_A is running and acquires the bus in order to send a new control input to the DC-motors,
- at time $t = 2$ the task J_C needs to access the bus meanwhile the control task J_A is setting the new control signal,
- at the same t J_B is ready to be executed to send the "ping" signal.

(a) Show graphically which tasks are running. What happens to the high priority task J_C? Compute the response time of J_C in this situation.

(b) Suggest a possible way to overcome the problem in (a).

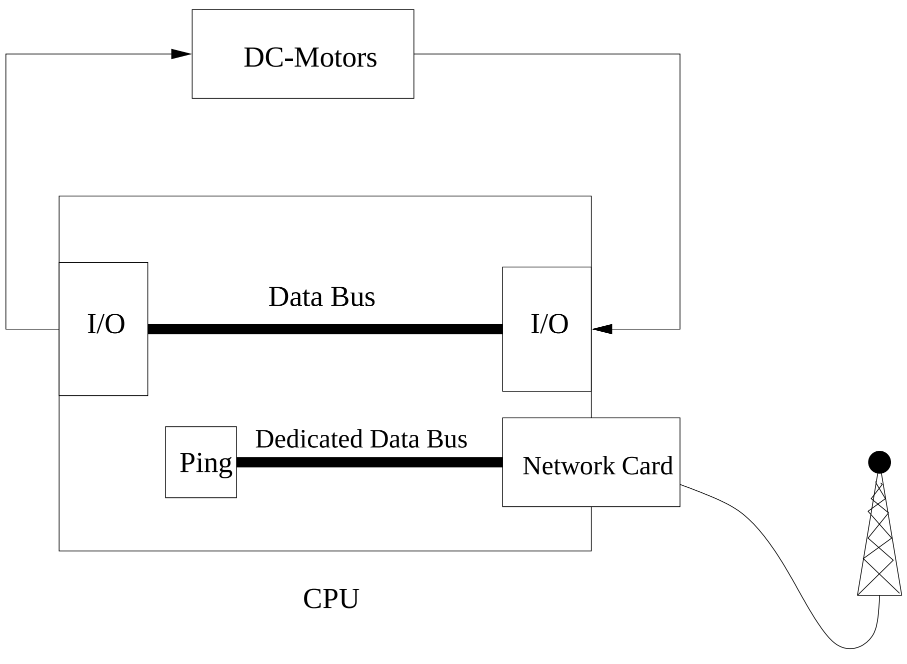


Figure 6.5.1: Schedule for the control task J_c and the task handling the interrupt, of Problem 6.4.

EXERCISE 6.6 (Jackson's algorithm, page 52 in [4])

All the tasks consist of a single job, have synchronous arrival times but have different computation times and deadlines. They are assumed to be independent. Each task can be characterized by two parameters, deadline d_i and computation time C_i

$$\mathcal{J} = \{J_i | J_i = J_i(C_i, d_i), \quad i = 1, \dots, n\}.$$

We consider here the Jackson's algorithm to schedule a set $\mathcal{J}$ of n aperiodic tasks minimizing a quantity called *maximum lateness* and defined as

$$L_{max} := \max_{i \in \mathcal{J}} \big(f_i - d_i\big).$$

The algorithm, also called *Earliest Due Date* (EDD), can be expressed by the following rule

Theorem 1. *Given a set of n independent tasks, any algorithm that executes the tasks in order of nondecreasing deadlines is optimal with respect to minimizing the maximum lateness.*

(a) Consider a set of 5 independent tasks simultaneously activated at time $t = 0$. The parameters are indicated in the following table

	J_1	J_2	J_3	J_4	J_5
C_i	1	1	1	3	2
d_i	3	10	7	8	5

Determine what is the maximum lateness using the scheduling algorithm EDD.

(b) Prove the optimality of the algorithm.

EXERCISE 6.7

Consider the following linear second-order system

$$(S) : \begin{cases} \dot{x}_1(t) = 3x_1(t) + x_2(t) + u(t) \\ \dot{x}_2(t) = 5x_1(t) - 2x_2(t) + u(t) \end{cases}$$

with $[x_1, x_2]^\top = x \in \mathbb{R}^2, u \in \mathbb{R}, t \geq 0$ and initial conditions $x_1(0) = x_2(0) = 1$.

(a) Show that the system is unstable.

(b) Determine a linear state-feedback controller $u(t) = Kx(t)$ with $K = [K_1\ K_2]$, $K_1, K_2 \in \mathbb{R}$ such that the poles of the closed loop system are placed in -2 and -4.

(c) In order to implement the controller on a digital platform, the state of the system is sampled aperiodically at a sequence of time instants $\{t_k\}, k \in \mathbb{N}$, and the control signal is now given by

$$u(t) = Kx(t_k), t \in [t_k, t_{k+1}).$$

Find the closed loop equation of the system in terms of the state $x(t)$ and the state error $e(t)$, where

$$e(t) = x(t_k) - x(t), t \in [t_k, t_{k+1}).$$

(d) By using the positive definite quadratic Lyapunov function

$$V(x) = \frac{1}{2}\, x^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x$$

find a relation between the error $e(t)$ and the state $x(t)$ such that the system is still asymptotically stable.

EXERCISE 6.8

Consider three robots R_1, R_2 and R_3 that want to meet at the same place. Each robot is controlled as a simple integrator; i.e., if we denote as $p_i(t) \in \mathbb{R}^2$ the position of robot R_i, then the motion of the robot is described by

$$\dot{p}_i(t) = u_i(t).$$

In order to meet at the same place, R_1 follows R_2, R_2 follows R_3, and R_3 follows R_1, as described by the following equations:

$$\begin{aligned} u_1(t) &= p_2(t) - p_1(t), \\ u_2(t) &= p_3(t) - p_2(t), \\ u_3(t) &= p_1(t) - p_3(t). \end{aligned}$$

Consider the state variables $x_1(t) = p_2(t) - p_1(t)$ and $x_2(t) = p_3(t) - p_2(t)$. Note that the robots reach their goal if and only if $x_1 = x_2 = 0$.

(a) Find the state space representation

$$\dot{x}(t) = Bu(t),$$

where $x(t) = [x_1(t), x_2(t)]^\top$ and $u(t) = [u_1(t), u_2(t), u_3(t)]^\top$.

(b) Find the matrix K such that $u(t) = Kx(t)$.

(c) Write the closed-loop system as

$$\dot{x}(t) = BKx(t).$$

(d) Use the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

to show that the robots asymptotically meet at the same place.

Now suppose that the robots measure each other's positions only on the aperiodic sampling times t_k, with $k \in \mathbb{N}$. Therefore, the control inputs become

$$\begin{aligned} u_1(t) &= p_2(t_k) - p_1(t_k), \\ u_2(t) &= p_3(t_k) - p_2(t_k), \\ u_3(t) &= p_1(t_k) - p_3(t_k), \end{aligned}$$

for $t \in [t_k, t_{k+1})$.

(e) Let $e(t) = x(t) - x(t_k)$, and write the closed-loop system for $t \in [t_k, t_{k+1})$ as a function of $x(t)$ and $e(t)$.

(f) Using the same Lyapunov function as in (d), to find a condition in the form $\|e(t)\| \leq \alpha\|x(t)\|$, with $\alpha > 0$, that guarantees that the robot asymptotically meet at the same place. Choose α as large as possible.
Hint: for a matrix $M \in \mathbb{R}^{n\times m}$, we have $||M|| = \sqrt{\lambda_{\max}(M^T M)}$, where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue.

EXERCISE 6.9

The motion of a car is modeled as

$$\begin{aligned}\dot{p}(t) =& v(t),\\ \dot{v}(t) =& u(t),\end{aligned}$$

where $p \in \mathbb{R}$ is the position of the car, $v \in \mathbb{R}$ is the velocity of the car, and $u \in \mathbb{R}$ is a control input. The car is controlled with an event-triggered controller such that

$$u(t) = -p(t_k) - v(t_k)$$

for all $t \in [t_k, t_{k+1})$. The goal of the controller is to bring the car to rest at the origin (i.e., $p = 0$ and $v = 0$). We consider the candidate Lyapunov function $V(t) = 3p(t)^2 + 2v(t)^2 + 2p(t)v(t)$.

(a) Letting $\tilde{p}(t) = p(t) - p(t_k)$ and $\tilde{v}(t) = v(t) - v(t_k)$, write the closed-loop dynamics of the car as a function of $p(t)$, $\tilde{p}(t)$, $v(t)$ and $\tilde{v}(t)$.

(b) Compute the time-derivative $\dot{V}(t)$ of the candidate Lyapunov function as a function of $p(t)$, $\tilde{p}(t)$, $v(t)$ and $\tilde{v}(t)$.

(c) Denoting $x(t) = [p(t), v(t)]^\intercal$ and $\tilde{x}(t) = [\tilde{p}(t), \tilde{v}(t)]^\intercal$, rewrite $V(t)$ in the form $V(t) = x(t)^\intercal P x(t)$, where $P \in \mathbb{R}^{2\times 2}$ is symmetric and positive semidefinite. Write the numerical value of P explicitly.

(d) Rewrite $\dot{V}(t)$ in the form $\dot{V}(t) = -x(t)^\intercal Q x(t) + x(t)^\intercal R \tilde{x}(t)$, where $Q \in \mathbb{R}^{2\times 2}$ is symmetric positive definite and $R \in \mathbb{R}^{2\times 2}$. Write the numerical value of Q and R explicitly.

(e) Using the results in (c) and (d), find a condition in the form $\|\tilde{x}\| < \alpha \|x\|$ that guarantees that $\dot{V}(t) < -\frac{2}{5+\sqrt{5}} V(t)$. Choose α as large as possible. Hints: $x^\intercal P x \leq \lambda_{\max}(P)\|x\|^2$ where $\lambda_{\max}$ is the largest eigenvalue, $x^\intercal Q x \geq \lambda_{\min}(Q)\|x\|^2$, where $\lambda_{\min}$ is the smallest eigenvalue, $|x^\intercal R \tilde{x}| \leq \sigma_{\max}(R)\|x\|\|\tilde{x}\|$, where $\sigma_{\max}$ denotes the largest singular value. Remember that the maximum singular value of a matrix R is computed as $\sigma_{\max}(R) = \sqrt{\lambda_{\max}(R^\intercal R)}$. Moreover,

$$\begin{aligned}\operatorname{eig}\left(\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}\right) =& \left\{\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right\},\\ \operatorname{eig}\left(\begin{bmatrix}5 & 5\\ 5 & 5\end{bmatrix}\right) =& \{0, 10\}.\end{aligned}$$

(f) Use the result in (e) to write an event-triggered law to schedule the control updates t_k that guarantees that $V(t)$ converges exponentially to zero.

7 Real-time scheduling

EXERCISE 7.1 (Ex. 4.3 in [4])

Verify the schedulability and construct the schedule according to the rate monotonic algorithm for the following set of periodic tasks

	C_i	T_i
J_1	1	4
J_2	2	6
J_3	3	10

EXERCISE 7.2 (Ex. 4.4 in [4])

Verify the schedulability under EDF of the task set given in Exercise 7.1 and then construct the corresponding schedule.

EXERCISE 7.3

Consider the following set of tasks

	C_i	T_i	D_i
J_1	1	3	3
J_2	2	4	4
J_3	1	7	7

Are the tasks schedulable with rate monotonic algorithm? Are the tasks schedulable with earliest deadline first algorithm?

EXERCISE 7.4

Consider the following set of tasks

	C_i	T_i	D_i
J_1	1	4	4
J_2	2	5	5
J_3	3	10	10

Assume that task J_1 is a control task. Every time that a measurement is acquired, task J_1 is released. When executing, it computes an updated control signal and outputs it.

(a) Which scheduling of RM or EDF is preferable if we want to minimize the delay between the acquisition and control output?

(b) Suppose that J_2 is also a control task and that we want its maximum delay between acquisition and control output to be two time steps. Suggest a schedule which guarantees a delay of maximally two time steps, and prove that all tasks will meet their deadlines.

EXERCISE 7.5

Consider the two tasks J_1 and J_2 with computation times, periods and deadlines as defined by the following table:

	C_i	T_i	D_i
J_1	1	3	3
J_2	1	4	4

(a) Suppose the tasks are scheduled using the rate monotonic algorithm. Will J_1 and J_2 meet their deadlines according to the schedulability condition on the utilization factor? What is the schedule length, i.e., the shortest time interval that is necessary to consider in order to describe the whole time evolution of the scheduler? Plot the time evolution of the scheduler when the release time for both tasks is at $t = 0$.

(b) If the two tasks implement a controller it is important to know what is the worst-case delay between the time the controller is ready to sample and the time a new input $u(kh)$ is ready to be released. Find the worst-case response time for J_1 and J_2. Compare with the result in (a).

EXERCISE 7.6

Consider the set of periodic tasks given in the table below:

	C_i	T_i	O_i
J_1	1	3	1
J_2	2	5	1
J_3	1	6	0

where for task i, C_i the worst-case execution time, T_i denotes the period, and O_i the offset for the respective tasks. Assume that the deadlines coincide with the period. The offset denotes the relative release time of the first task instance for each task. Assume that all tasks are released at time 0 with their respective offset O_i.

(a) Determine the schedule length.

Determine the worst-case response time for task J_2 for each of the following three scheduling policies:

(b) Rate-monotonic scheduling

(c) Deadline-monotonic scheduling

(d) Earliest-deadline-first scheduling

EXERCISE 7.7

A control task J_c is scheduled in a computer together with two other tasks J_1 and J_2. Assume that the three tasks are scheduled using a rate monotonic algorithm. Assume that the release time for all tasks are at zero and that the tasks have the following characteristics

	C_i	T_i	D_i
J_1	1	4	4
J_2	1	6	6
J_c	2	5	5

(a) Is the set of tasks schedulable with rate monotonic scheduling? Determine the worst-case response time for the control task J_c.

(b) Suppose the control task implements a sampled version of the continuous-time controller with delay

$$\dot{x}(t) = Ax(t) + By(t - \tau)$$
$$u(t) = Cx(t)$$

where we let τ be the worst-case response time R_c of the task J_c. Suppose that the sampling period of the controller is $h = 2$ and $R_c = 3$. Derive a state-space representation for the sampled controller. Suggest also an implementation of the controller by specifying a few lines of computer code.

(c) In order to improve performance the rate monotonic scheduling is substituted by a new scheduling algorithm that give highest priority to the control task and intermediate and lowest to the task J_1 and J_2, respectively. Are the tasks schedulable in this case?

EXERCISE 7.8

Compute the maximum processor utilization that can be assigned to a polling server to guarantee the following periodic task will meet their deadlines

	C_i	T_i
J_1	1	5
J_2	2	8

EXERCISE 7.9

Together with the periodic tasks

	C_i	T_i
J_1	1	4
J_2	1	8

we want to schedule the following aperiodic tasks with a polling server having $T_s = 5$ and $C_s = 2$. The aperiodic tasks are

	r_i	C_i
a_1	2	3
a_2	7	2
a_3	9	1
a_3	29	4

EXERCISE 7.10

Consider the set of tasks in Problem 7.5, assuming that an aperiodic task could ask for CPU time. In order to handle the aperiodic task we run a polling server J_s with computation time $C_s = 3$ and period $T_s = 6$. Assume that the aperiodic task has computation time $C_a = 3$ and asks for the CPU at time $t = 3$. Plot the time evolution when a polling server is used together with the two tasks J_1 and J_2 scheduled as in Problem 7.5 part (a). Describe the scheduling activity illustrated in the plots.

8 Models of computation: Discrete-event systems and Transition systems

EXERCISE 8.1

Consider the problem of controlling a gate which is lowered when a train is approaching and it is raised when the train has passed. We assume that the railway is unidirectional and that a train can be detected 1500m before the gate and 1000m after the gate. The sensors give binary outputs i.e., they give a '0' when the train is not over the sensor and a '1' when the train is over the sensor. The gate has a sensor which gives a binary information and in particular gives '0' if the gate is (fully) closed and '1' if the gate is (fully) opened. Figure 8.1.1 shows a schema of the system. The gate needs to be lowered as soon as a train is approaching, and raised when the train has passed. Model the system as a discrete-event system. Assume that trains, for safety reasons are distant from each other, so that no train approaches before the previous train has left.

EXERCISE 8.2

A vending machine dispenses soda for \$0.45. It accepts only dimes (\$0.10) and quarters (\$0.25). It does not give change in return if your money is not correct. The soda is dispensed only if the exact amount of money is inserted. Model the vending machine using a discrete-event system. Is it possible that the machine does not dispense soda? Prove it formally.

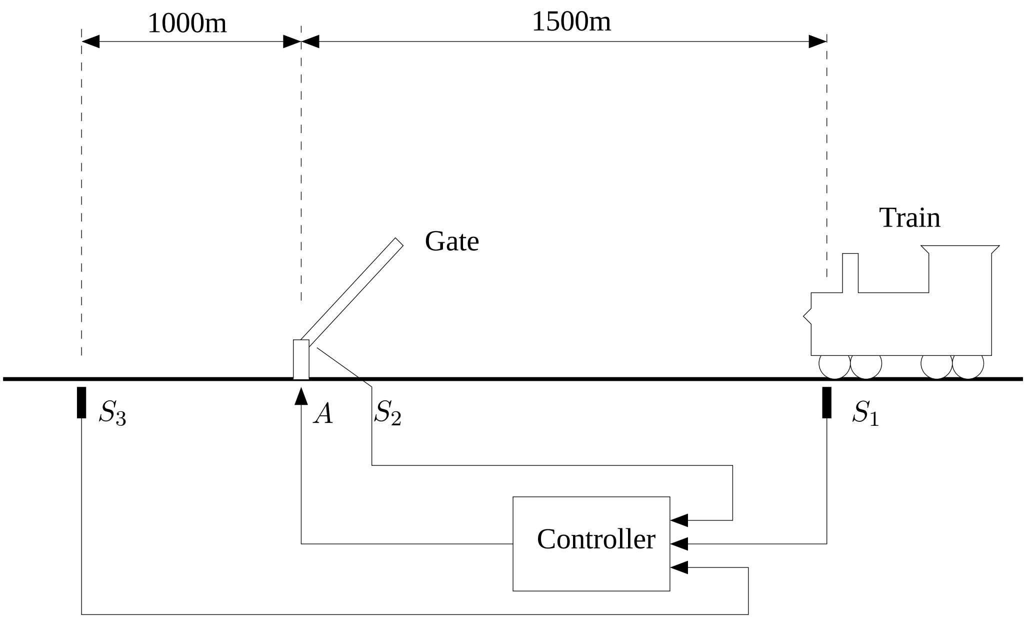


Figure 8.1.1: Control of a gate. Problem 8.1.

EXERCISE 8.3

Consider the automaton describing some discrete-event system shown in Figure 8.3.1. Describe formally the

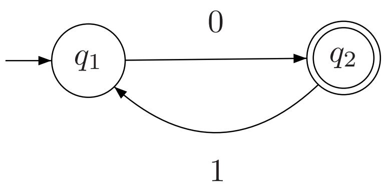


Figure 8.3.1: Automaton A of Problem 8.3.

DES. Compute the marked language L_m and the generated language L.

EXERCISE 8.4 (Example 2.13 in [6])

Consider the automaton A of Figure 8.4.1. Compute the language marked by the automaton A, $L_m(A)$ and the language generated by the automaton, $L(A)$.

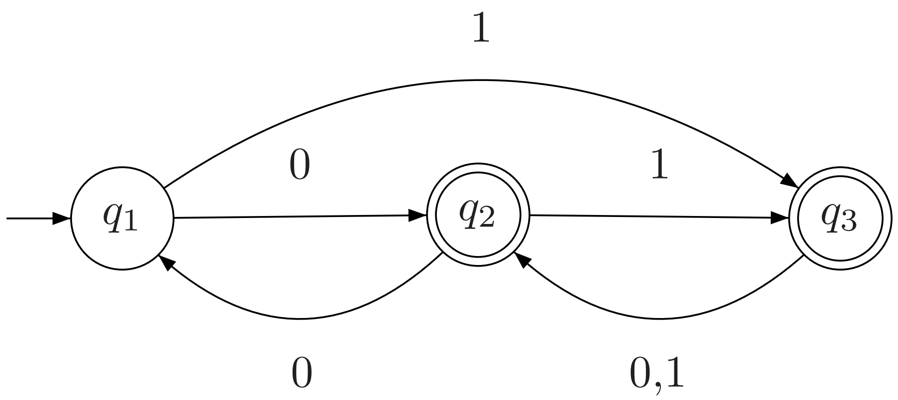


Figure 8.4.1: Automaton A of Problem 8.4.

EXERCISE 8.5 (Example 3.8 in [6])

Consider the automaton A of Figure 8.5.1. Determine the minimum state automaton.

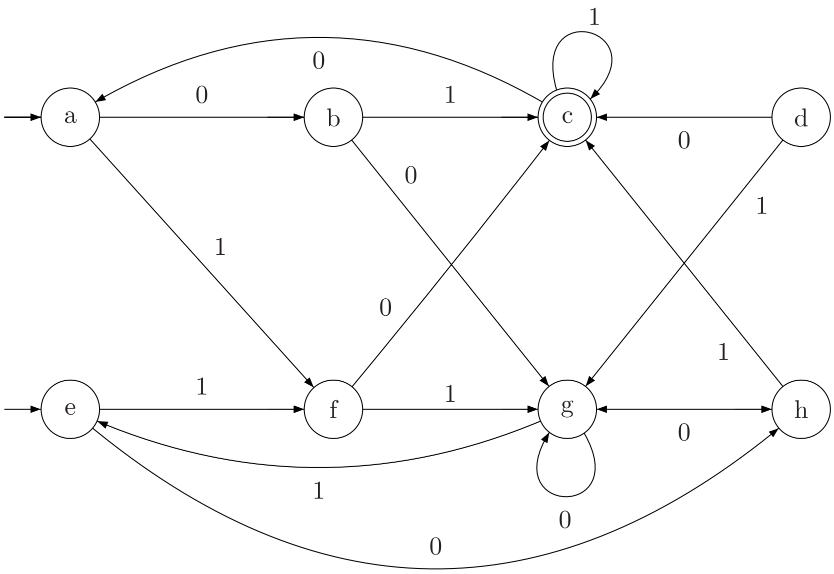


Figure 8.5.1: Automaton A of Problem 8.5.

EXERCISE 8.6 (Example 2.5 in [6])

Consider the automaton

$$A = (\{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{q_1\})$$

be a nondeterministic automaton where

$$\delta(q_0, 0) = \{q_0, q_1\} \quad \delta(q_0, 1) = \{q_1\} \quad \delta(q_1, 0) = \quad \delta(q_1, 1) = \{q_0, q_1\}.$$

Construct an deterministic automaton A' which accept the same L_m.

EXERCISE 8.7 [3]

Consider the circuit diagram of the sequential circuit with input variable x, output variable y, and register r, cf. Figure 8.7.1. The control function for output variable y is given by

$$\lambda_y = \neg(x \oplus r)$$

where $\oplus$ stands for exclusive (XOR, or parity function). The register evaluation changes according to the circuit function

$$\delta_r = x \vee r$$

where $\vee$ stands for disjunction (OR). Initially, the register evaluation is $[r = 0]$. Model the circuit behavior by a finite transition system.

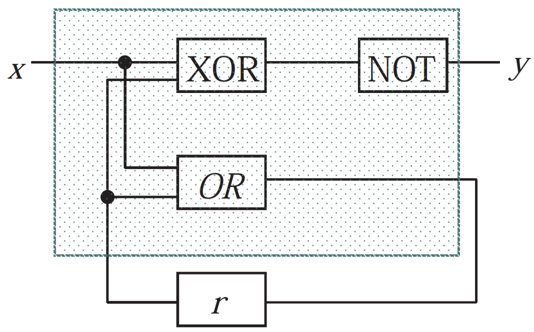


Figure 8.7.1: Diagram for a simple hardware circuit.

EXERCISE 8.8 [5]

Consider an automaton as shown in Figure. 8.8.1, where $E = \{a_1, a_2, b\}$ is the set of events; $X = \{0, 1, 2, 3, 4, 5\}$ is the set of states; note that state 0 is the initial state and state 4 is the marked state. Try to understand the automaton and explain the set of languages it marks.

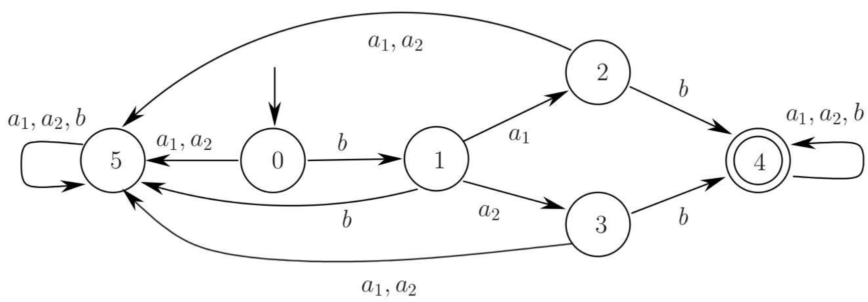


Figure 8.8.1: This automaton can be used as a model of a “status check” system when a machine is turned on, if its states are given the following interpretation: 0 = “Initializing”, 1 = “I am ON”, 2 = “My status is OK”, 3 = “I have a problem”, 4 = “Status report done”, and 5 = “Error”.

EXERCISE 8.9 [5]

Consider a simple T-intersection of the traffic system, as shown in Fig. 8.9.1. There are four types of vehicles:

- $(1, 2)$-type vehicles coming from point 1 and turning right towards 2;
- $(1, 3)$-type vehicles coming from point 1 and turning right towards 3;
- $(2, 3)$-type vehicles going straight from 2 to 3;
- $(3, 2)$-type vehicles going straight from 3 to 2.

The traffic light is set so that it either turns red for $(1, 2)$ and $(1, 3)$ vehicles (green for $(2, 3)$ and $(3, 2)$ vehicles), or it turns green for $(1, 2)$ and $(1, 3)$ vehicles (red for $(2, 3)$ and $(3, 2)$ vehicles). Model the traffic system as a transition system $\mathcal{T} = (S, Act, \rightarrow, I)$ to describe the changes to the number of the vehicles of the individual types waiting at the intersection over time. Model arrivals of new cars and departures of the waiting cars and changes of the traffic light colors. Assume that initially, there is no vehicle at the intersection.

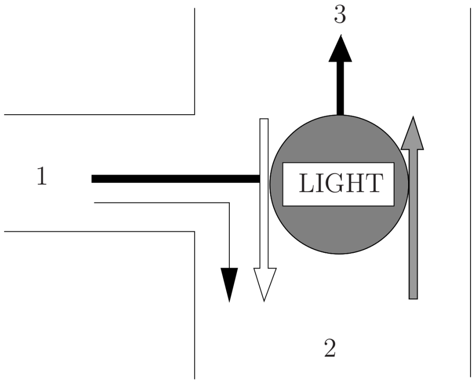


Figure 8.9.1: A simple T-intersection with four types of vehicles.

EXERCISE 8.10 [5]

Queuing systems arises in many application domain such as computer networks, manufacturing, logistics and transportation. A queuing systems is composed by three basic elements: 1) the entities, generally referred to as *customers*, that do the waiting in their request for resources, 2) the resources for which the waiting is done, which are referred to as *servers*, and 3) the space where the waiting is done, which is defined as *queue*. Typical examples of servers are communications channels, which have a finite capacity to transmit information. In such a case, the customers are the unit of information and the queue is the amount of unit of information that is waiting to be transmitted over the channel.

A basic queue system is reported in figure 8.10.1. The circle represent a server, the open box is a queue preceding the server. The slots in the queue are waiting customers. The arrival rate of customers in the queue is denoted by a, whereas the departure rate of customers is denoted by b.

Model the queue system of figure 8.10.1 by a transition system. How many states has the system?

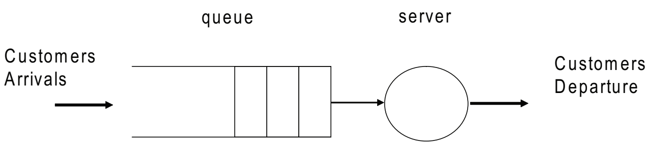


Figure 8.10.1: A basic queue system.

EXERCISE 8.11 [15]

Consider the transition system $T = \{S, \Sigma, \rightarrow, S_S\}$, where the cardinality of S is finite. The reachability algorithm is

$$\begin{aligned}
\text{Initialization}: \quad & \mathrm{Reach_1} = \emptyset; \\
& \mathrm{Reach_0} = \mathrm{S_S}; \\
& i = 0; \\
\text{Loop}: \quad & While \quad \mathrm{Reach_i} \neq \mathrm{Reach_{i-1}} \quad do \\
& \quad \mathrm{Reach_{i+1}} = \mathrm{Reach_i} \cup \{\mathrm{s'} \in \mathrm{S} : \exists\ : \mathrm{s} \in \mathrm{Reach_i}, \sigma \in \Sigma, \mathrm{s} \rightarrow^{\sigma} \mathrm{s'} \in \rightarrow\}; \\
& \quad i = i + 1;
\end{aligned}$$

Prove formally that

- the reachability algorithm finishes in a finite number of steps;
- upon exiting the algorithm, $Reach_i = Reach_T(S_S)$.

EXERCISE 8.12 [15]

Give the Transition System $T = \{S, \Sigma, \rightarrow, S_S\}$ reported in figure 8.12.1, describe the reach set when $S_S = \{3\}$ and $S_S = \{2\}$ by using the reachability algorithm.

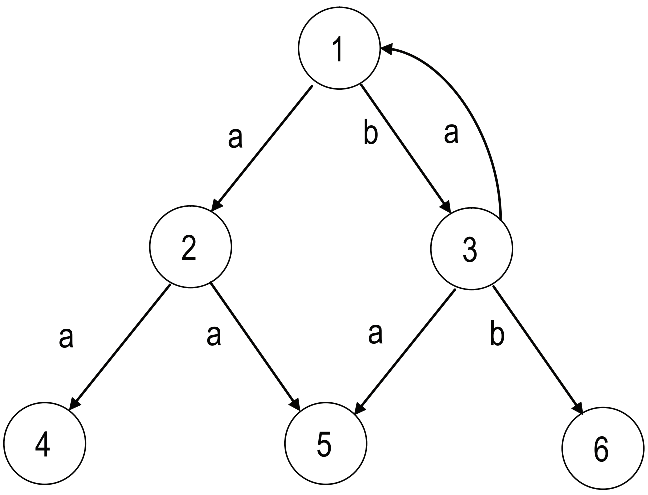


Figure 8.12.1: A Transition System.

Part III

Hybrid control

9 Modeling of hybrid systems

EXERCISE 9.1

A water level in a tank is controlled through a relay controller, which senses continuously the water level and turns a pump on or off. When the pump is off the water level decreases by 2 cm/s and when it is on, the water level increases by 1 cm/s. It takes 2 s for the control signal to reach the pump. It is required to keep the water level between 5 and 12 cm.

(a) Assuming that the controller starts the pump when the level reaches some threshold and turns it of when it reaches some other threshold, model the closed-loop system as a hybrid automaton.

(b) What thresholds should be used to fulfill the specifications?

EXERCISE 9.2

Consider the quantized control system in Figure 9.2.1. Such a system can be modeled as a hybrid automaton with continuous dynamics corresponding to $P(s)C(s)$ and discrete states corresponding to the levels of the quantizer. Suppose that each level of the quantizer can be encoded by a binary word of k bits. Then, how many discrete states N should the hybrid automaton have? Describe when discrete transitions in the hybrid automaton should take place.

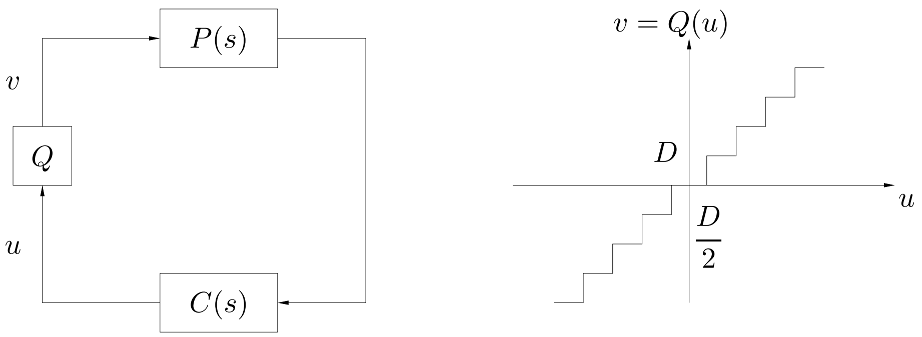


Figure 9.2.1: Quantized system in Problem 9.2.

EXERCISE 9.3

A system to cool a nuclear reactor is composed by two independently moving rods. Initially the coolant temperature x is 510 degrees and both rods are outside the reactor core. The temperature inside the reactor increases according to the following (linearized) system

$$\dot{x} = 0.1x - 50.$$

When the temperature reaches 550 degrees the reactor must be cooled down using the rods. Three things can happen

- the first rod is put into the reactor core
- the second rod is put into the reactor core
- none of the rods can be put into the reactor

For mechanical reasons a rod can be placed in the core if it has not been there for at least 20 seconds. If no rod is available the reactor should be shut down. The two rods can refrigerate the coolant according to the two following ODEs

$$\begin{aligned} \text{rod 1:} \quad & \dot{x} = 0.1x - 56 \\ \text{rod 2:} \quad & \dot{x} = 0.1x - 60 \end{aligned}$$

When the temperature is decreased to 510 degrees the rods are removed from the reactor core. Model the system, including controller, as a hybrid system.

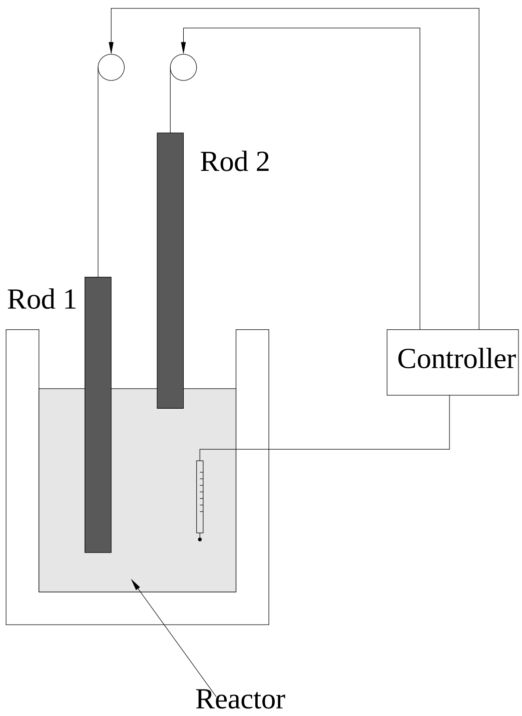


Figure 9.3.1: Nuclear reactor core with the two control rods

EXERCISE 9.4

Consider the classical sampled control system, shown in Figure 9.4.1. Model the system with a hybrid automaton. Suppose that the sampling period is k and that the hold circuit is a zero-order hold.

EXERCISE 9.5

Consider a hybrid system with two discrete states q_1 and q_2. In state q_1 the dynamics are described by the linear system

$$\dot{x} = A_1 x = \begin{pmatrix} -1 & 0 \\ p & -1 \end{pmatrix}$$

and in state q_2 by

$$\dot{x} = A_2 x = \begin{pmatrix} -1 & p \\ 0 & -1 \end{pmatrix}.$$

Assume the system is

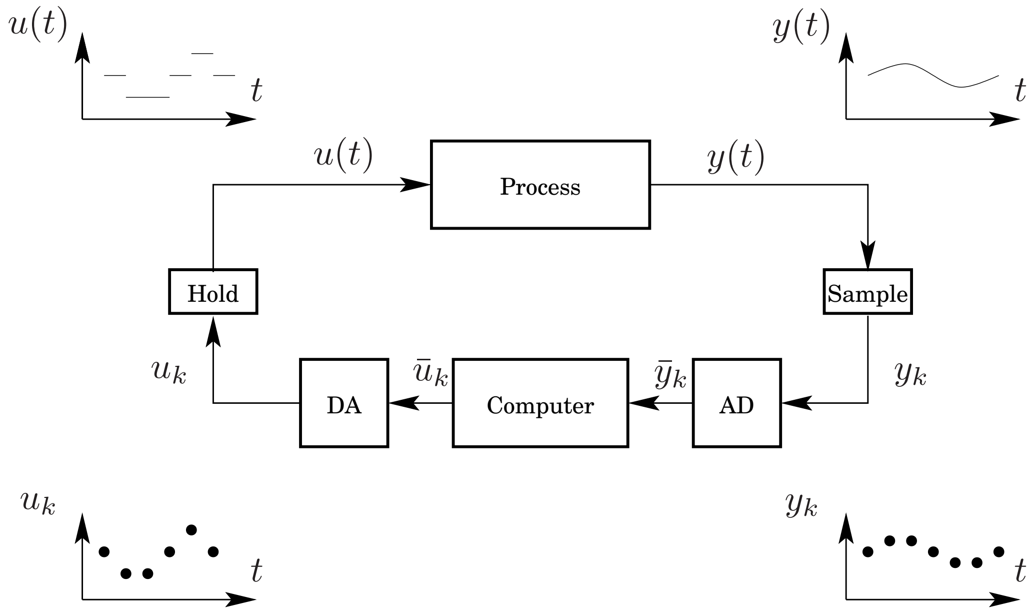


Figure 9.4.1: Sampled data control system of Problem 9.4.

in state q_1 if $2k \leq t < 2k + 1$ and

in state q_2 if $2k + 1 \leq t < 2k + 2$,

where $k = 0, 1, 2, \ldots$.

(a) Formally define a hybrid system with initial state q_1, which operates in the way described above.

(b) Starting from $x(0) = x_0$, specify the evolution of the state $x(t)$ in the interval $t \in [0, 3)$ as a function of x_0.

EXERCISE 9.6

Consider the hybrid system of Figure 9.6.1:

(a) Describe it as a hybrid automaton, $H = (Q, X, \text{Init}, f, D, E, G, R)$

(b) Find all the domains $D(q_3)$ so that the hybrid system is live?

(c) Plot the solution of the hybrid system.

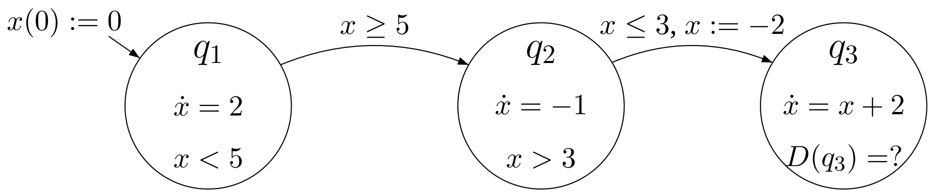


Figure 9.6.1: Hybrid system for Problem 9.6.

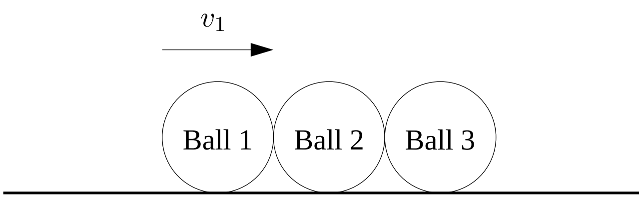


Figure 10.1.1: Three balls system. The Ball 1 has velocity v_1 at time $t = 0$.

10 Stability of hybrid systems

EXERCISE 10.1

Consider three balls with unit mass, velocities v_1, v_2, v_3, and suppose that they are touching at time $t = \tau_0$, see Figure 10.1.1. The initial velocity of Ball 1 is $v_1(\tau_0) = 1$ and Balls 2 and 3 are at rest, i.e., $v_2(\tau_0) = v_3(\tau_0) = 0$. Assume that the impact is a sequence of simple inelastic impacts occurring at $\tau_0' = \tau_1' = \tau_2' = \dots$ (using notation from hybrid time trajectory). The first inelastic collision occurs at τ_0' between balls 1 and 2, resulting in $v_1(\tau_1) = v_2(\tau_1) = 1/2$ and $v_3(\tau_1) = 0$. Since $v_2(\tau_1') > v_3(\tau_1')$, Ball 2 hits Ball 3 instantaneously giving $v_1(\tau_2) = 1/2$, and $v_2(\tau_2) = v_3(\tau_2) = 1/4$. Now $v_1(\tau_2') > v_2(\tau_2')$, so Ball 1 hits Ball 2 again resulting in a new inelastic collision. This leads to an infinite sequence of collisions.

(a) Model the inelastic collisions of the three-ball system described above as a hybrid automaton $H = (Q, X, \text{Init}, f, D, E, G, R)$ with one discrete variable $Q = \{q\}$ and three continuous variables $X = \{v_1, v_2, v_3\}$.

(b) Is the execution described above a Zeno execution? Motivate.

(c) What is the accumulation point of the infinite series of hits described above? Make a physical interpretation.

EXERCISE 10.2

Consider the following system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 3 \\ -2 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Show, using a Lyapunov function, that the system is asymptotically stable.

EXERCISE 10.3

Consider the following theorem:

Theorem 2. *A linear system*

$$\dot{x} = Ax$$

is asymptotically stable if and only if for any positive definite symmetric matrix Q the equation

$$A^T P + PA = -Q$$

in the unknown $P \in \mathbb{R}^{n\times n}$ has a solution which is positive definite and symmetric.

Show the necessary part of the previous theorem (*i.e.*, the *if* part).

EXERCISE 10.4

Consider the following system

$$\begin{aligned}\dot{x}_1 &= -x_1 + g(x_2)\\ \dot{x}_2 &= -x_2 + h(x_1)\end{aligned}$$

where the functions g and h are such that

$$|g(z)| \leq |z|/2 \qquad |h(z)| \leq |z|/2$$

Show that the system is asymptotically stable.

EXERCISE 10.5

Consider the following discontinuous differential equations

$$\begin{aligned}\dot{x}_1 &= -\text{sgn}(x_1) + 2\text{sgn}(x_2)\\ \dot{x}_2 &= -2\text{sgn}(x_1) - \text{sgn}(x_2).\end{aligned}$$

where

$$\text{sgn}(z) = \begin{cases} +1 & \text{if } z \geq 0 \\ -1 & \text{if } z < 0. \end{cases}$$

Assume $x(0) \neq 0$,

(a) define a hybrid automaton that models the discontinuous system

(b) does the hybrid automaton exhibit Zeno executions for every initial state?

EXERCISE 10.6

Consider the following switching system

$$\dot{x} = a_q x, \quad a_q < 0 \quad \forall q$$

where $q \in \{1, 2\}$ and

$$\begin{aligned}\Omega_1 &= \{x \in \mathbb{R} | x \in [2k, 2k+1), k = 0, 1, 2, \dots\}\\ \Omega_2 &= \{x \in \mathbb{R} | x \in [2k+1, 2k+2), k = 0, 1, 2, \dots\}\end{aligned}$$

Show that the system is asymptotically stable.

EXERCISE 10.7

Consider the following switching system

$$\dot{x} = A_q x$$

where $q \in \{1, 2\}$ and

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -3 & 0 \\ 0 & -5 \end{pmatrix}.$$

Let Ω_q be such that

$$\Omega_1 = \{x \in \mathbb{R}^2 | x_1 \geq 0\}$$

$$\Omega_2 = \{x \in \mathbb{R}^2 | x_1 < 0\}$$

Show that the system is asymptotically stable.

EXERCISE 10.8

Consider the following switching system

$$\dot{x} = A_q x$$

where $q \in \{1, 2\}$ and

$$A_1 = \begin{pmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{pmatrix}.$$

Assume that a_i, b_i and c_i, $i = 1, 2$ are real numbers and that $a_i, c_i > 0$. Show that the switched system is asymptotically stable.

EXERCISE 10.9

Consider a system that follows the dynamics

$$\dot{x} = A_1 x$$

for a time $\epsilon/2$ and then switches to the system

$$\dot{x} = A_2 x$$

for a time $\epsilon/2$. It then switches back to the first system, and so on.

(a) Model the system as a switched system

(b) Model the system as a hybrid automaton

(c) Let t_0 be a time instance at which the system begins a period in mode 1 (the first system) with initial condition x_0. Determine the state at $t_0 + \epsilon/2$ and $t_0 + \epsilon$.

(d) Let ϵ tend to zero (very fast switching). Determine the solution the hybrid system will tend to.

EXERCISE 10.10

Consider the following hybrid system

$$\dot{x} = A_q x$$

where

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} -3 & 0 \\ 0 & -5 \end{pmatrix}.$$

Let Ω_q be such that

$$\Omega_1 = \{x \in \mathbb{R}^2 | x_1 \geq 0\}$$
$$\Omega_2 = \{x \in \mathbb{R}^2 | x_1 < 0\}$$

show that the switched system is asymptotically stable using a common Lyapunov function.

EXERCISE 10.11 (Example 2.1.5 page 18-19 in [13])

Consider the following switched system with $q \in \{1, 2\}$

$$\dot{x} = A_q x$$

where

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}.$$

(a) Show that is impossible to find a quadratic common Lyapunov function.

(b) Show that the origin is asymptotically stable for any switching sequence.

EXERCISE 10.12

Consider the following two-dimensional state-dependent switched system

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_1 \leq 0 \\ A_2 x & \text{if } x_1 > 0 \end{cases}$$

where

$$A_1 = \begin{pmatrix} -5 & -4 \\ -1 & -2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix}.$$

(a) Prove that there is not a common quadratic Lyapunov function suitable to prove stability of the system

(b) Prove that the switched system is asymptotically stable using the multiple Lyapunov approach.

EXERCISE 10.13

Consider the following switched system:

$$\dot{x} = A_q x,$$

with $q \in \{1, 2\}, x \in \Omega_q$, where $\Omega_1 = \{x_1 \leq 0\}$, and $\Omega_2 = \{x_1 > 0\}$, and

$$A_1 = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -5 & -50 \\ 0.5 & -5 \end{bmatrix}.$$

(a) Model formally the system as a hybrid automaton.

(b) Is it possible to find a common quadratic Lyapunov function? Motivate your answer.

(c) Show that the origin is asymptotically stable.

EXERCISE 10.14

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= g\sin(x_1) - u\cos(x_1), \\ x_1(0) &= x_2(0) = u(0) = 0, \end{aligned}$$

where $x_1 \in [-\pi, \pi]$ is an angle, $x_2 \in \mathbb{R}$ is the corresponding angular velocity, u is the acceleration of the system, representing the control input, and $g > 0$ is the gravity acceleration. The goal is to drive $x = [x_1, x_2]^\top$ to the desired configuration $x_{\text{des}} = [\pi, 0]^\top$. To this end, we use a greedy controller, where we apply the maximum and minimum acceleration $u = u_{\max}$ and $u = -u_{\max}$, respectively, based on the energy of the system, which can be approximated by the function $\beta : [-\pi, \pi] \times \mathbb{R} \to \mathbb{R}$, with $\beta(x_1, x_2) = [\frac{x_2^2}{2} + g(\cos(x_1) - 1)]x_2 \cos x_1$. More specifically, we apply $u = u_{\max}$ when $\beta(x_1, x_2) \geq 0$ and $u = -u_{\max}$ when $\beta(x_1, x_2) < 0$. In order to avoid chattering, when x_1 is close to the desired configuration π (within a fixed angle θ, with $\pi > \theta > 0$), we apply a local stabilizing controller $u = \gamma_1 x_1 + \gamma_2 x_2, \gamma_1, \gamma_2 \in \mathbb{R}$, regardless of the value of β.

(a) *Using the augmented state* $z = [x^\top, u]^\top$, model the system as a hybrid automaton

$$H = (Q, X, \text{Init}, f, D, E, G, R).$$

(b) Consider that after appropriate state transformation and linearization, we obtain a switching system of the form

$$\dot{x} = A_q x,$$

with $q \in \{1, 2\}$, i.e., we only consider two of the aforementioned three states, and

$$A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$A_2 = \begin{bmatrix} a & c \\ 0 & d \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$ are scalar constants with $a < 0, d < 0$. Prove that the origin is asymptotically stable, given that the equation

$$x^2 + ((b+c)^2 - 4ad)x + a^2 + 3a + 9 = 0,$$

has two real solutions, $x_1, x_2 \in \mathbb{R}$, where at least one of them is positive.

EXERCISE 10.15

Consider the following switching system

$$\dot{x} = A_q x$$

where $x \in \mathbb{R}^2, q \in \{1, 2\}$ and

$$A_1 = \begin{bmatrix} a_1 & 0 \\ b_1 & a_2 \end{bmatrix}, A_2 = \begin{bmatrix} c_1 & d_1 \\ 0 & c_2 \end{bmatrix}$$

with $a_1, a_2, c_1, c_2 < 0$ and $0 < b_1 d_1 < 4\sqrt{a_1 a_2 c_1 c_2}$. Show that the system is asymptotically stable.

11 Control of hybrid systems

EXERCISE 11.1

Consider the system

$$\dot{x} = A_{p^\star} x + B_{p^\star} u \tag{11.1}$$

$$y = C_{p^\star} x, \tag{11.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, $\{A_{p^\star}, B_{p^\star}, C_{p^\star} : p \in \mathcal{P}\}$ is a given finite family of matrices, and $p^\star$ is unknown. The system (11.1) is considered controllable and observable. Propose a control strategy to drive the state x to zero.

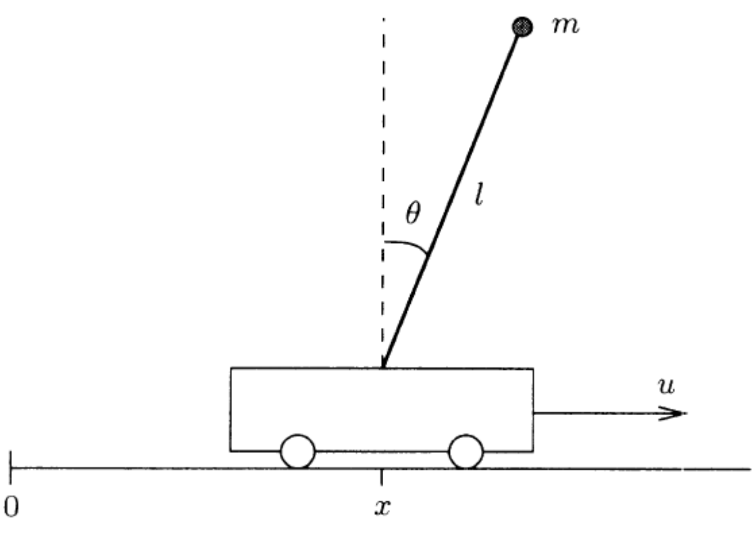


Figure 11.1: An inverted pendulum on a cart.

EXERCISE 11.2

Consider the inverted pendulum on a cart (see Fig. 11.1) described by the equations

$$\ddot{x} = u \tag{11.2a}$$

$$J\ddot{\theta} = mgl\sin\theta - ml\cos\theta u, \tag{11.2b}$$

where x is the location of the cart, θ is the angle between the pendulum and the vertical axis, the control input u is the acceleration of the cart, l is the length of the rod, m is the mass of the pendulum (concentrated at its tip), and J is the moment of inertia with respect to the pivot point.

(a) Is the upright position $\theta = 0$ globally stabilizable via continuous feedback ?

(b) Propose a feedback design that stabilizes the system to upright position $\theta = 0$.

12 Simulation and bisimulation

EXERCISE 12.1

Consider the following two bisimilar transition systems $T = (S, \Sigma, \rightarrow, S_0, S_F)$ and $T' = (S', \Sigma, \rightarrow', S'_0, S'_F)$ illustrated in Fig. 12.1.1 below. Find the bisimulation relation $\sim \in S \times S'$.

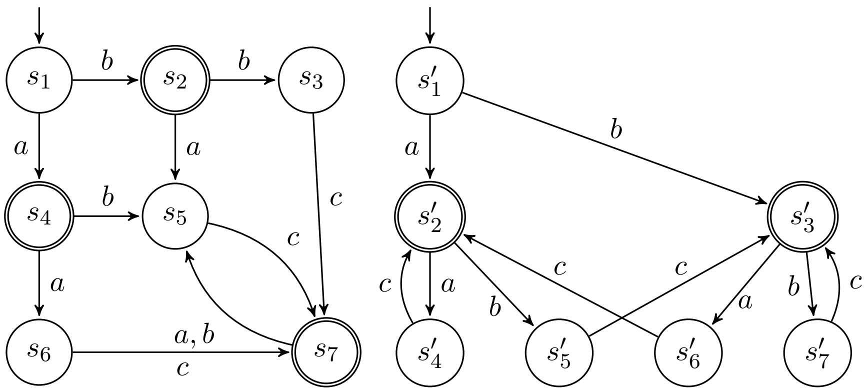


Figure 12.1.1: The transition systems T (left) and T' (right).

EXERCISE 12.2

Find the minimal quotient T that is bisimilar to the transition system T of Fig. 12.2.1.

EXERCISE 12.3

Find two transition systems $T = (S, \Sigma, \rightarrow, S_0, S_F)$ and $T' = (S', \Sigma, \rightarrow', S'_0, S'_F)$, with the property that there exist two bisimulations $\sim_1 \subseteq S \times S'$ and $\sim_2 \subseteq S \times S'$, such that $\sim_1 \neq \sim_2$.
Hint: First find a transition system T that is not equal to its coarsest bisimulation quotient. Then look for an appropriate T' that satisfies the desired condition.

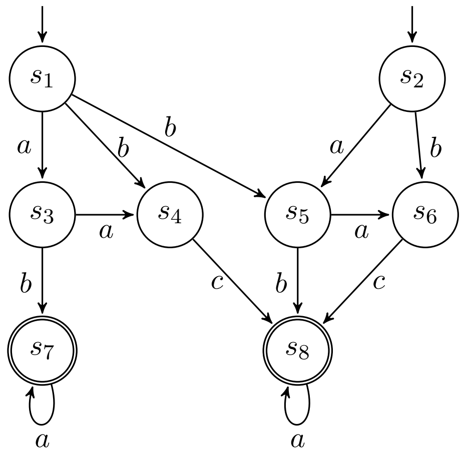


Figure 12.2.1: The transition systems T.

EXERCISE 12.4

Consider the infinite transition system $T = (S, \Sigma = \{a, b, c, d\}, \rightarrow, S_0, S_F)$ and finite transition systems $T_1 = (Q, \Sigma, \rightarrow_1, Q_0, Q_F), T_2 = (R, \Sigma, \rightarrow_2, R_0, R_F), T_3 = (U, \Sigma, \rightarrow_3, U_0, U_F), T_4 = (V, \Sigma, \rightarrow_4, V_0, V_F)$.

a) Which of T_1, T_2, T_3, T_4 are bisimilar with T? Which of them are not and why?

b) Find all simulation relations between the given transition systems.

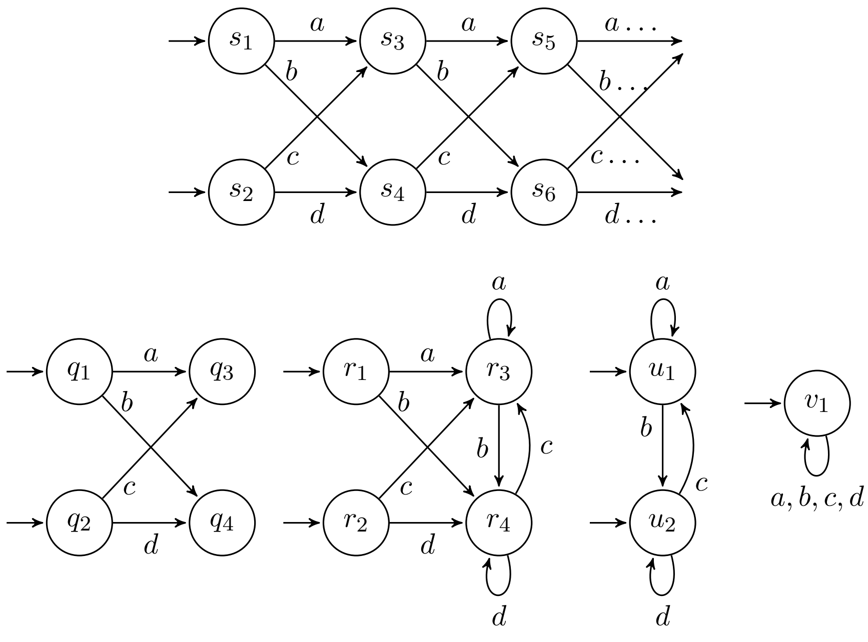


Figure 12.4.1: The transition systems T (top) and $T_1 - T_4$ (bottom, from left to right).

EXERCISE 12.5

Consider the following three transition systems with initial and final states

$$T = (S = \{s_1, s_2, s_3\}, \Sigma = \{a, b, c\}, \rightarrow, S_0 = \{s_1\}, S_F = \{s_2, s_3\}),$$

$$T' = (S' = \{s_1', s_2'\}, \Sigma = \{a, b, c\}, \rightarrow', S_0 = \{s_1'\}, S_F = \{s_2'\}), \text{ and}$$

$$T'' = (S'' = \{s_1'', s_2'', s_3''\}, \Sigma = \{a, b, c\}, \rightarrow'', S_0 = \{s_1''\}, S_F = \{s_2'', s_3''\}),$$

depicted in Fig. 12.5.1. Which of them are bisimilar? Explain.

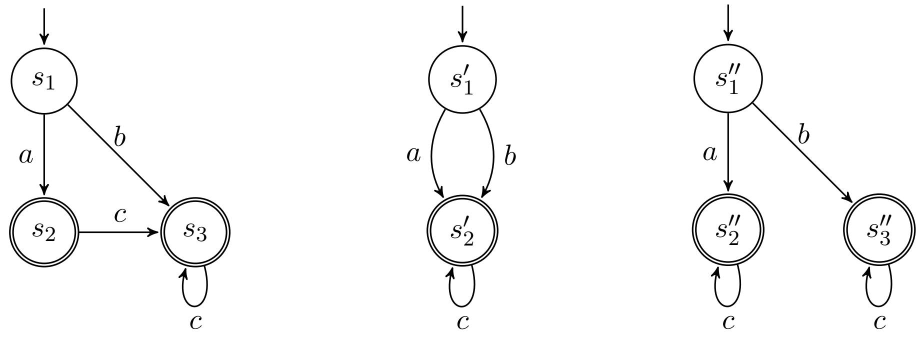


Figure 12.5.1: The transition systems T (left) and T' (middle) and T'' (right).

EXERCISE 12.6

Consider the following hybrid automaton H, with the set of initial states $Init = \{(q_1, x = 0, y = 0)\}$ and view it as a transition system with the set of generators $\Sigma = \{turn\_right, turn\_left, t\}$, where t represents the time, and $turn\_right, turn\_left$ are the events. Is it bisimilar to any of the transition systems $T_1, T_2, T_3, or T_4$?

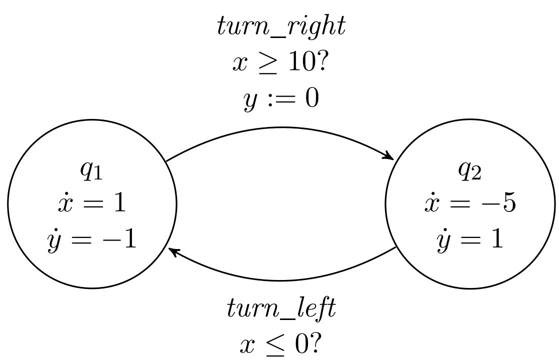


Figure 12.6.1: Hybrid automaton H.

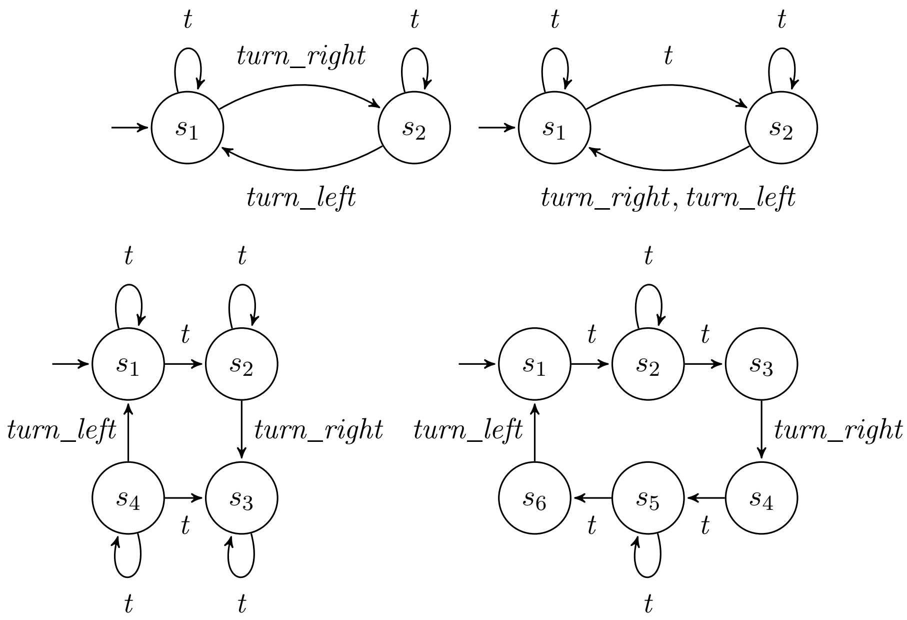


Figure 12.6.2: Transition systems T_1 (top left), T_2 (top right), T_3 (bottom left) and T_4 (bottom right).

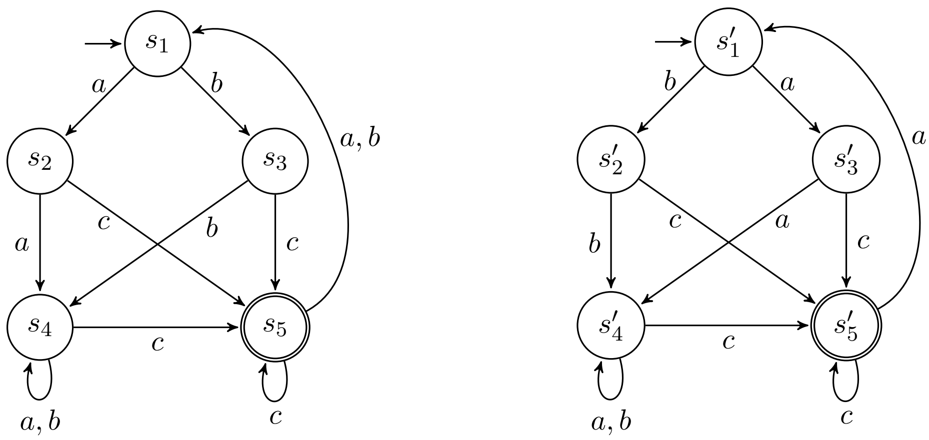


Figure 12.7.1: The transition systems $\mathcal{T}_1$ (left) and $\mathcal{T}_2$ (right).

EXERCISE 12.7 (Exam 2017)

(a) Consider the Transition Systems $\mathcal{T}_1$ (left), $\mathcal{T}_2$ (right) as they are depicted in Figure 12.7.1. Are the Transition Systems bisimilar? Motivate your answer.

(b) Find the minimal quotient Transition System bisimilar to the Transition System $\mathcal{T}_1$ (left) in Figure 12.7.1.

(c) Consider the timed automaton $\mathcal{T}$ depicted in Figure 12.7.2. Decide, whether the following states are reachable in the $\mathcal{T}$, i.e. whether they belong to the set $Reach(\{(q_1, 0, 0, 0)\})$. Motivate your answer.

- (i) $(q_3, 0, 2, 1)$
- (ii) $(q_2, 1, 0, 2)$
- (iii) $(q_2, 0, 3, 1)$

(iv) $(q_3, 0, 3, 2)$

(v) $(q_3, 3, 4, 2)$

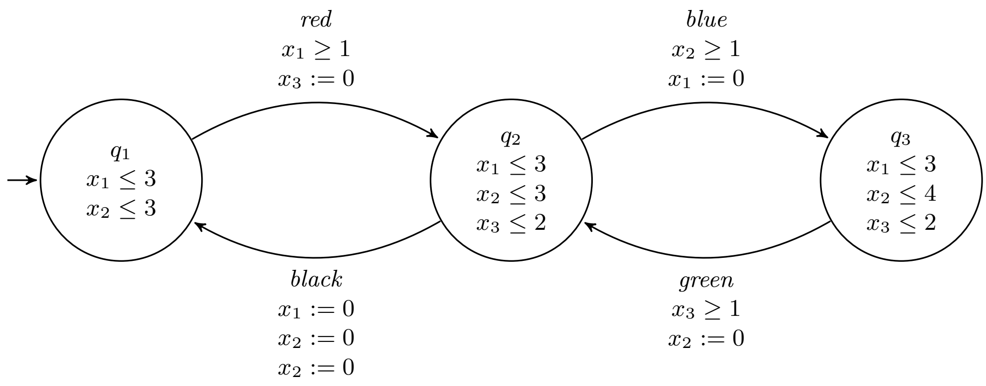


Figure 12.7.2: The timed automaton $\mathcal{T}$.

13 Reachability, timed automata and rectangular automata

EXERCISE 13.1

Consider the following two tank system in Fig. 13.1.1 modeled as a hybrid automaton A in Fig. 13.1.2.

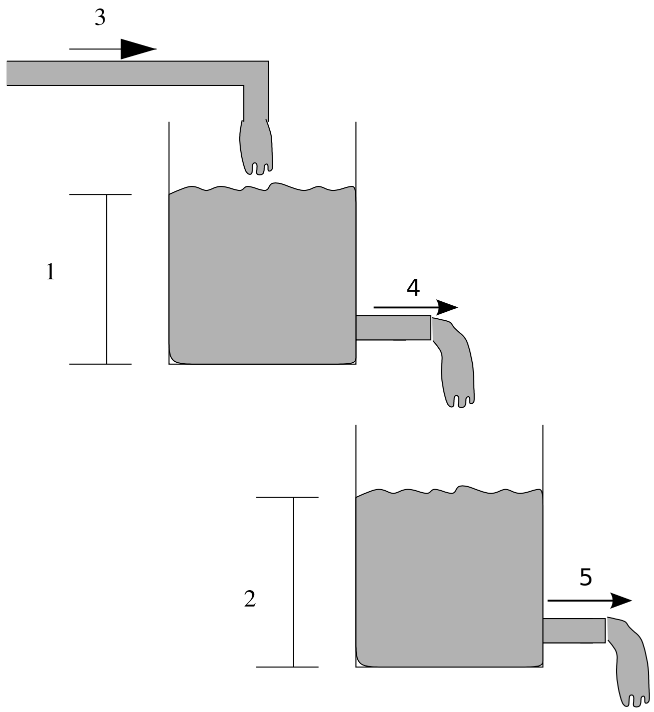


Figure 13.1.1: Two tank system, 1: x, 2: y, 3: $\alpha_1; \alpha_2$ (on or off), 4: α_3 (on or off), 5: α_4.

a) Compute $Post(q_{\text{off,off}}, x = 175, y \in [100, 200])$.

b) Compute $Post(q_{\text{off,off}}, x = 200, y = 200)$.

c) Compute $Pre(q_{\text{on,on}}, x = 150, y = 200)$.

Let the initial state be $(q_{\text{off,off}}, x = 190, y = 200)$.

d) Verify that the level of water in the second water tank always remains within safe bounds, i.e., that $y(t) \in [150, 350]$, for each $t \geq 0$.
Hint: Compute the reach set from the initial state through repeated computation of $Post$.

e) Is there a state $(q_{\text{off,off}}, x, y)$, where $x = y$ reachable from the initial state?

EXERCISE 13.2

Consider the following timed automaton A of a time code lock. Find the corresponding region automaton $Reg(A)$?

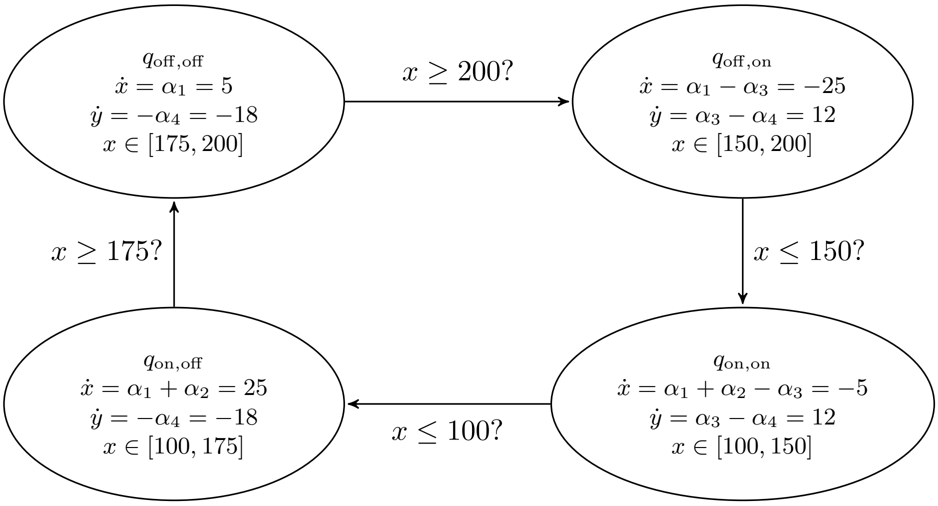


Figure 13.1.2: Two tanks hybrid automaton A.

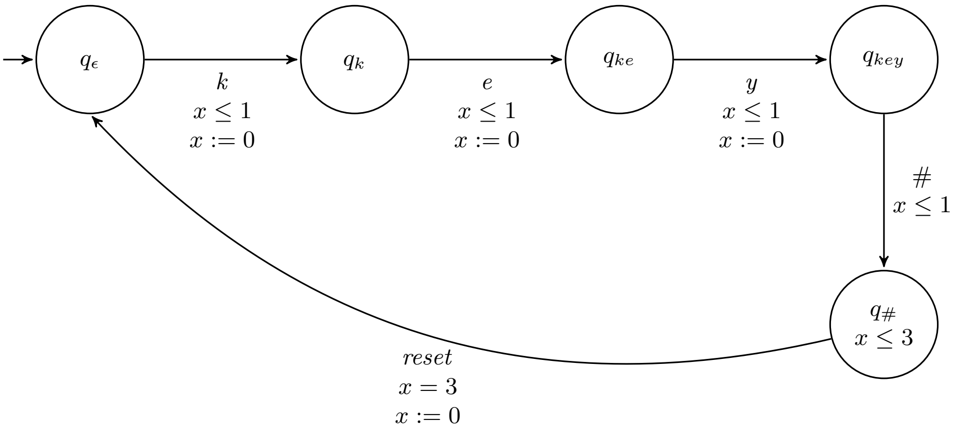


Figure 13.2.1: Timed automaton A.

EXERCISE 13.3

Consider the timed automaton TA with $Q = \{q_1, q_2\}, X = \{x\}, Act = \{tick, tock\}$, and the sets of domains, edges, guards and resets D, E, G, R given in Fig. 13.3.1. (1) Give an example of three different states of its corresponding untimed transition system $[[TA]]$ that are reachable from its initial state $(q_1, 0)$. (2) Find the region graph $[[TA]]/_\sim$ for the timed automaton TA from Fig. 13.3.1.

EXERCISE 13.4

Is the following rectangular automaton A initialized? If yes, find a bisimilar timed automaton A'.

EXERCISE 13.5

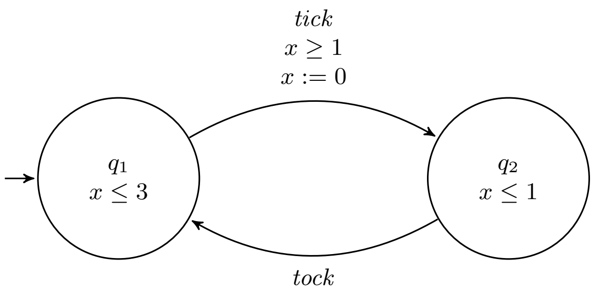


Figure 13.3.1: The timed automaton TA.

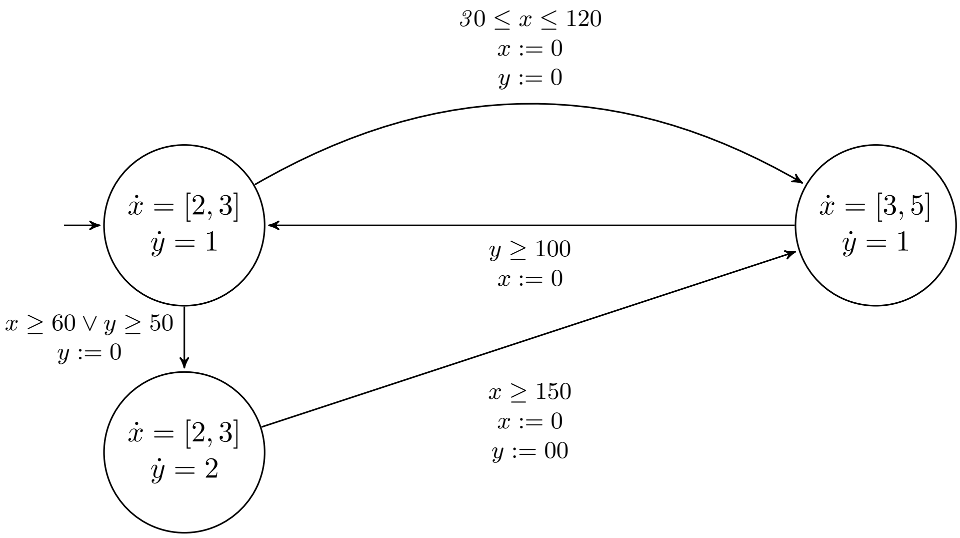


Figure 13.4.1: Rectangular automaton A.

Consider the following linear system

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix} x$$

Assume that the initial condition is defined in the following set

$$x_0 \in \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = x_2, -10 \leq x_1 \leq 10\}$$

We want to verify that no trajectories enter in a *Bad* set defined as

$$Bad = \{(x_1, x_2) \in \mathbb{R} | -8 \leq x_1 \leq 0 \wedge 2 \leq x_2 \leq 6\}.$$

EXERCISE 13.6

Consider the following linear system

$$\dot{x} = \begin{pmatrix} -5 & -5 \\ 0 & -1 \end{pmatrix} x.$$

Assume that the initial condition lies in the following set

$$x_0 \in \{(x_1, x_2) \in \mathbb{R}^2 | -2 \leq x_1 \leq 0 \wedge 2 \leq x_2 \leq 3\}.$$

Describe the system as a transition system and verify that no trajectories enter a *Bad* set defined as the triangle with vertices $v_1 = (-3, 2)$, $v_2 = (-3, -3)$ and $v_3 = (-1, 0)$.

EXERCISE 13.7

Consider the following controlled switched system

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + B_1 u \qquad \text{if} \quad \|x\| < 1 \\
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \qquad \text{if} \quad 1 \leq \|x\| \leq 3 \\
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= -\left(\begin{pmatrix} 0 & 1 \\ \frac{1}{3} & 5 \end{pmatrix} \begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \right) \qquad \text{otherwise}
\end{aligned}$$

Assume that the initial conditions are $x_0 \in \{x \in \mathbb{R}^2 | \|x\| > 3\}$,

(a) Determine a control strategy such that $\text{Reach}_q \cup \Omega_1 \neq \emptyset$, i.e. Ω_1 can be reached from any initial condition when $B_1 = 0$.

Suppose in the following $B_1 = (0, 1)^T$,

(b) Is it possible to determine a linear control input such that $(0, 0)$ is globally asymptotically stable?

(c) Construct a piecewise linear system such that $(0, 0)$ is globally asymptotically stable.

(d) Suppose now, that we do not want that the solution of the linear system would enter the *Bad* set Ω_1. Determine a controller such that $\text{Reach}_q \cap \Omega_1 = \emptyset$.

EXERCISE 13.8

A system to cool a nuclear reactor is composed by two independently moving rods. Initially the coolant temperature x is 510 degrees and both rods are outside the reactor core. The temperature inside the reactor increases accordingly to the following (linearized) system

$$\dot{x} = 0.1x - 50.$$

When the temperature reaches 550 degrees the reactor mush be cool down using the rods. Three things can happen

- the first rod is put into the reactor core
- the second rod is put into the reactor core
- none of the rods can be put into the reactor

For mechanical reasons the rods can be placed in the core if it has not been there for at least 20 seconds. The two rods can refrigerate the coolant accordingly to the two following ODEs

$$\begin{aligned} \text{rod 1:} \quad & \dot{x} = 0.1x - 56 \\ \text{rod 2:} \quad & \dot{x} = 0.1x - 60 \end{aligned}$$

When the temperature is decreased to 510 degrees the rods are removed from the reactor core.

a Model the system as a hybrid system.

b If the temperature goes above 550 degrees, but there is no rod available to put down in the reactor, there will be a meltdown. Determine if this *Bad* state can be reached.

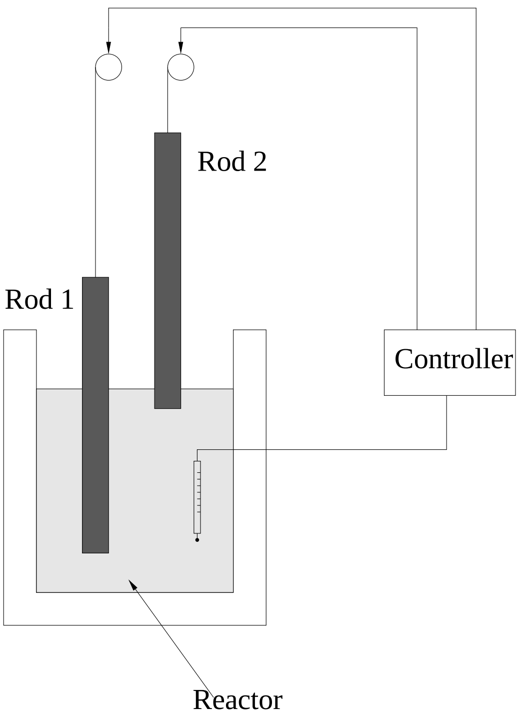


Figure 13.8.1: Nuclear reactor core with the two control rods

EXERCISE 13.9

Consider the transition system $\mathcal{TS}$ shown in Fig. 13.9.1, with atomic proposition $\Pi = \{\text{red}, \text{green}, \text{blue}\}$ and labeling function $\mathcal{L}(s_0) = \{\text{red}\}$,$\mathcal{L}(s_1) = \{\text{green}\}$, $\mathcal{L}(s_2) = \{\text{blue}\}$. Using formal verification techniques, find a path on $\mathcal{TS}$ that satisfies the specification “infinitely often green".

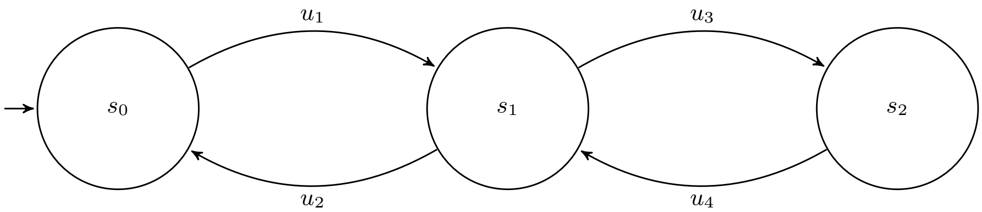


Figure 13.9.1: The transition system $\mathcal{TS}$.

EXERCISE 13.10

Consider the timed automaton with $Q = \{q_1, q_2\}$, $X = \{x_1, x_2\}$, $Act = \{tick, tock\}$ and the sets of domains, edges, guards and resets given in Fig. 13.10.1.

- Model the automaton as a Hybrid Automaton
- Model the automaton as a Transition System
- Decide whether the following states are reachable from the initial state $(q_1, 0, 0)$:

1. $(q_2, 0, 1)$
2. $(q_1, 4, 4)$
3. $(q_1, 1, 2)$
4. $(q_2, 2, 3)$
5. $(q_2, 3, 4)$
6. $(q_2, 0, 0.5)$
7. $(q_2, 0.5, 3)$

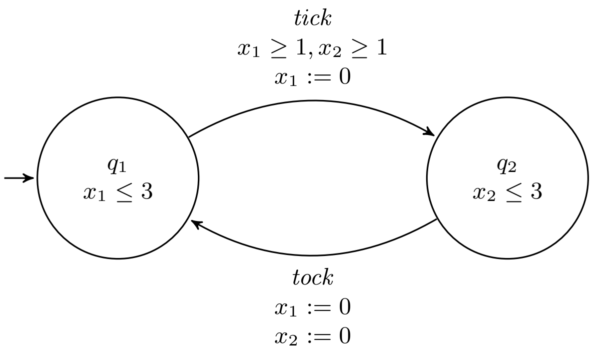


Figure 13.10.1: The timed automaton TA.

Solutions

Part I

Time-triggered control

Solutions to review exercises

SOLUTION 1.1

Before solving the exercise we review some concepts on sampling and aliasing

Shannon sampling theorem

Let $x(t)$ be band-limited signal that is, $X(j\omega) = 0$ for $|\omega| > \omega_m$. Then $x(t)$ is uniquely determined by its samples $x(kh)$, $k = 0, \pm 1, \pm 2, \dots$ if

$$\omega_s > 2\omega_m$$

where $\omega_s = 2\pi/h$ is the sampling frequency, h the sampling period. The frequency $w_s/2$ is called the Nyquist frequency.

Reconstruction

Let $x(t)$ be the signal to be sampled. The sampled signal $x_s(t)$ is obtained multiplying the input signal $x(t)$ by a period impulse train signal $p(t)$, see Figure 1.1.1. We have that

$$\begin{aligned} x_s(t) &= x(t)p(t) \\ p(t) &= \sum_{k=-\infty}^{\infty} \delta(t - kh). \end{aligned}$$

Thus the sampled signal is

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kh)\delta(t - kh).$$

If we let the signal $x_s(t)$ pass through an ideal low-pass filter (see Figure 1.1.1) with impulse response

$$f(t) = \text{sinc}\left(\frac{w_s}{2}t\right)$$

and frequency response

$$F(j\omega) = \begin{cases} h, & -\omega_s/2 < \omega < \omega_s/2; \\ 0, & \text{otherwise.} \end{cases}$$

as shown in Figure 1.1.1. The output signal is

$$\begin{aligned} x_r(t) = x_s(t) * f(t) &= \int_{-\infty}^{\infty} x_s(t-\tau)f(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x(kh)\delta(t - \tau - kh) \right) f(\tau)d\tau \\ &= \sum_{k=-\infty}^{\infty} x(kh)\text{sinc}\left(\frac{\omega_s}{2}(t - kh)\right) \\ &= \sum_{k=-\infty}^{\infty} x(kh)\text{sinc}\left(\frac{\pi}{h}(t - kh)\right). \end{aligned}$$

Notice that perfect reconstruction requires an infinite number of samples.

Returning to the solution of the exercise we have that the Fourier transform of the sampled signal is given by

$$X_s(\omega) = \frac{1}{h} \sum_{k=-\infty}^{\infty} X(\omega + k\omega_s)$$

(a) The reconstructed signal is $x_r(t) = \cos(\omega_0 t)$ since $\omega_s = 6\omega_0 > 2\omega_0$.

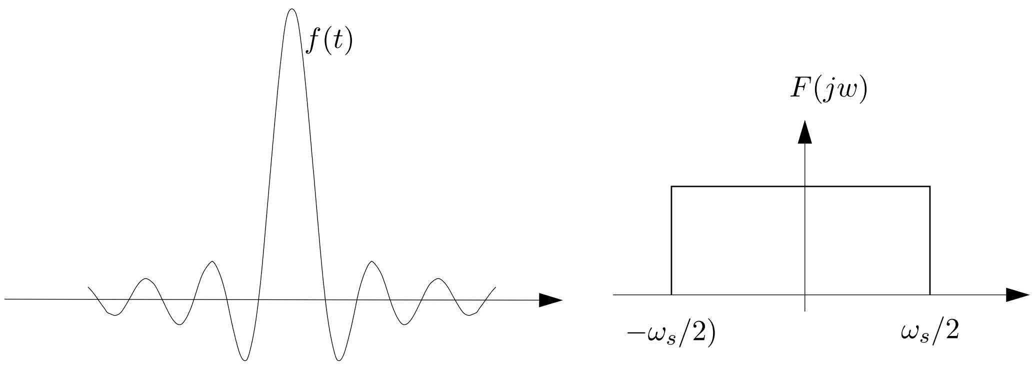


Figure 1.1.1: Impulse and frequency response of an ideal low-pass filter.

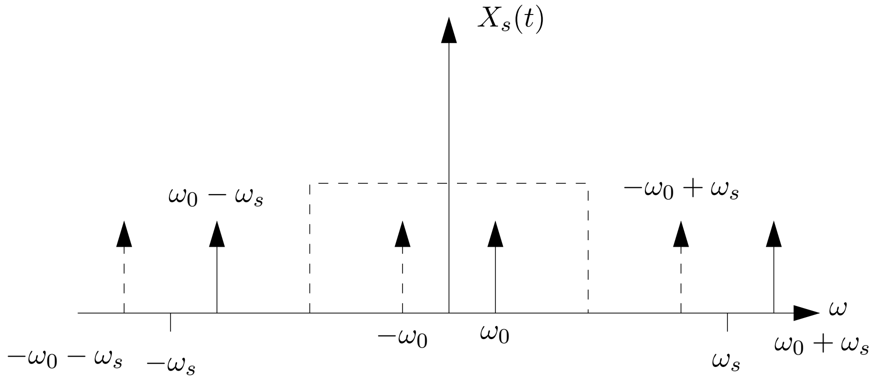


Figure 1.1.2: Frequency response of the signal with $\omega_0 = \omega_s/6$

(b) The reconstructed signal is $x_r(t) = \cos(\omega_0 t)$ since $\omega_s = 6\omega_0/2 > 2\omega_0$.

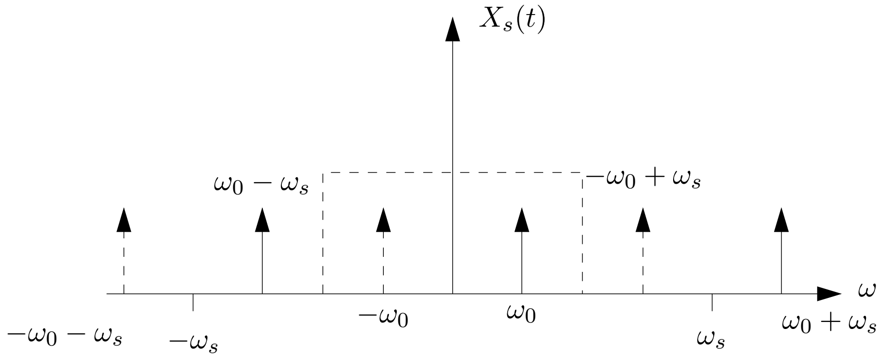


Figure 1.1.3: Frequency response of the signal with $\omega_0 = 6\omega_s/2$

(c) The reconstructed signal is $x_r(t) = \cos\big((-\omega_0 + \omega_s)t\big) = \cos(\omega_0/2t)$ since $\omega_s = 6\omega_0/4 < 2\omega_0$.

(d) The reconstructed signal is $x_r(t) = \cos\big((-\omega_0 + \omega_s)t\big) = \cos(\omega_0/5t)$ since $\omega_s = 6\omega_0/5 < 2\omega_0$.

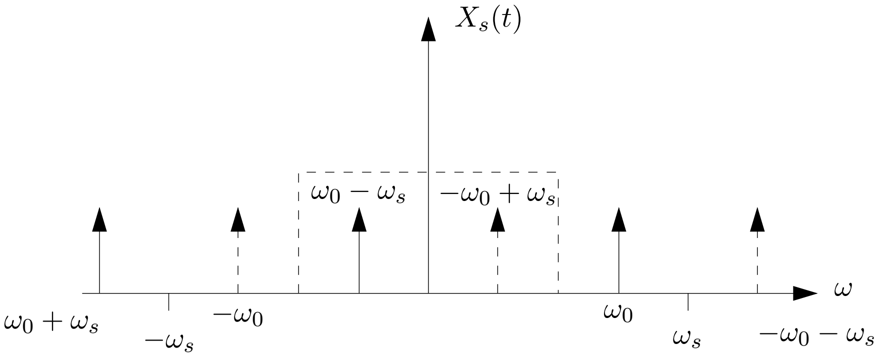


Figure 1.1.4: Frequency response of the signal with $\omega_0 = 6\omega_s/4$

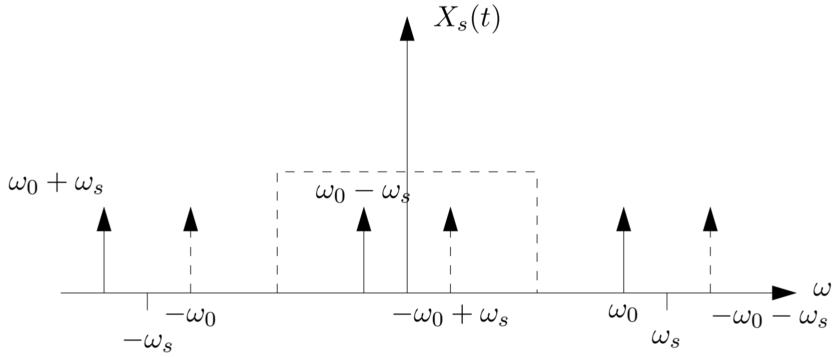


Figure 1.1.5: Frequency response of the signal with $\omega_0 = 6\omega_s/5$

FROM OPPENHEIM, WILLSKY
"SIGNALS & SYSTEMS"

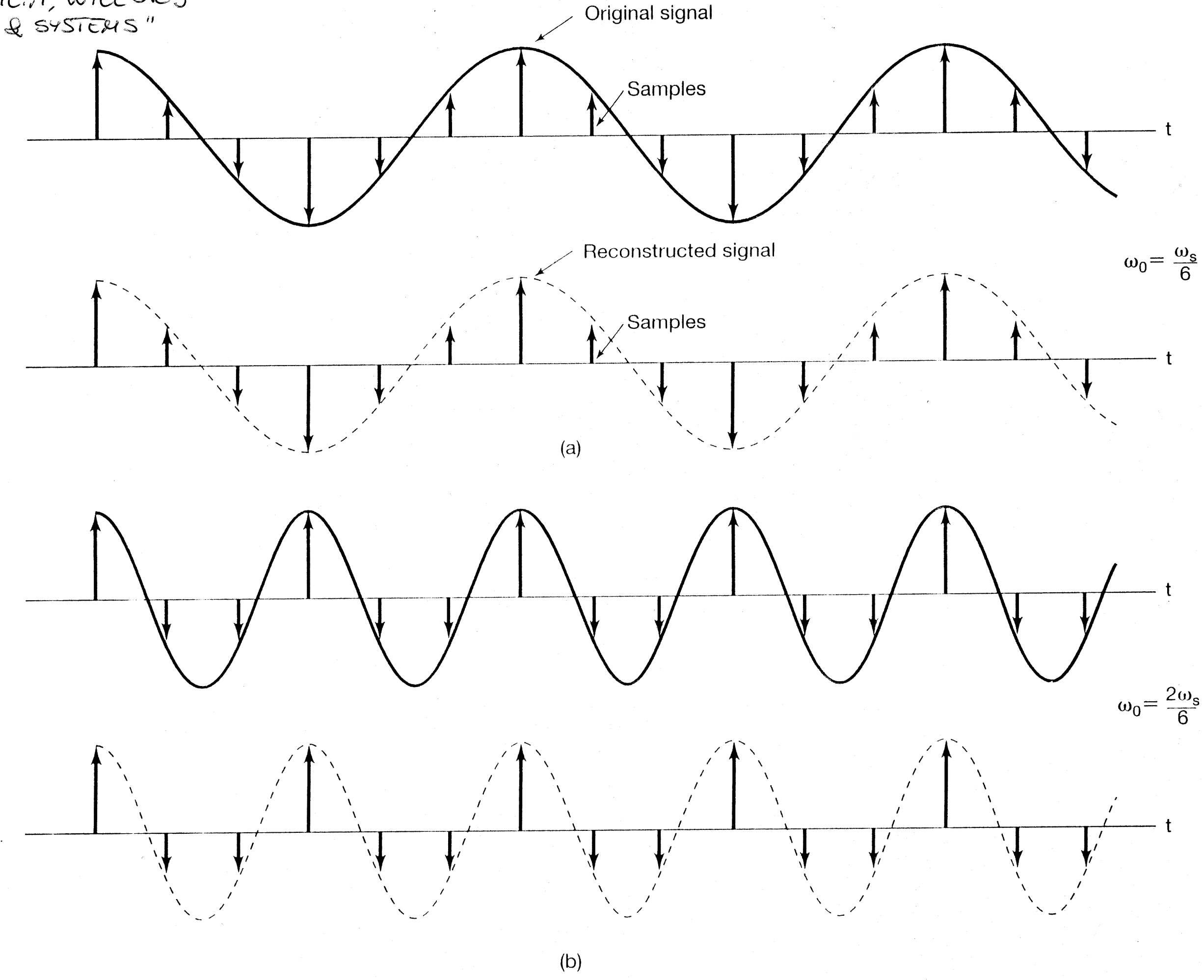


Figure 7.16 Effect of aliasing on a sinusoidal signal. For each of four values of ω_0, the original sinusoidal signal (solid curve), its samples, and the reconstructed signal (dashed curve) are illustrated: (a) $\omega_0 = \omega_s/6$; (b) $\omega_0 = 2\omega_s/6$; (c) $\omega_0 = 4\omega_s/6$; (d) $\omega_0 = 5\omega_s/6$. In (a) and (b) no aliasing occurs, whereas in (c) and (d) there is aliasing.

FROM OPPENHEIM, WILLSKY
"SIGNALS & SYSTEMS"

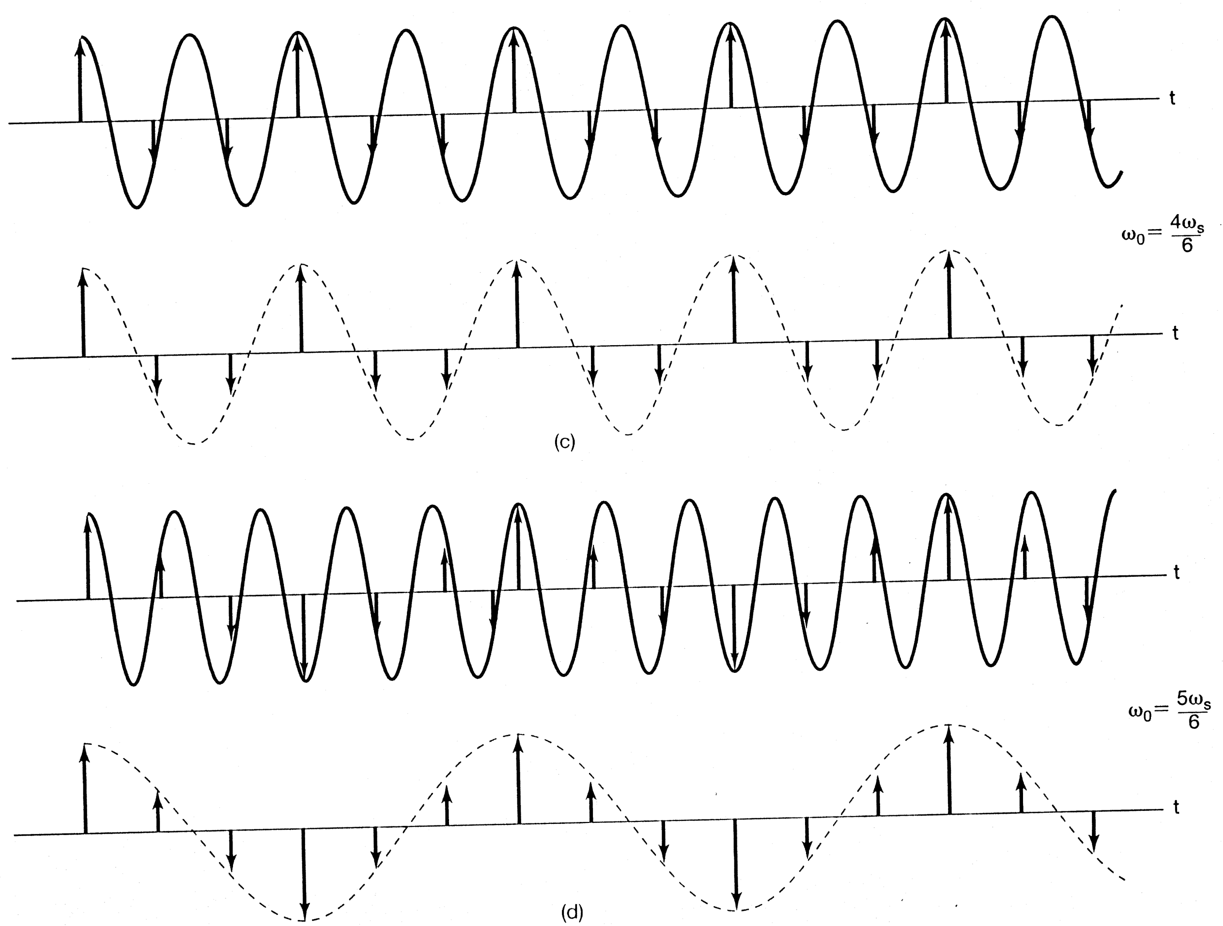


Figure 7.16 *Continued*

SOLUTION 1.2

We have that

$$e^{Ah} = \alpha_0 Ah + I\alpha_1.$$

The eigenvalues of Ah are $\pm ih$ thus we need to solve

$$\begin{aligned} e^{ih} &= \alpha_0 ih + \alpha_1 \\ e^{-ih} &= -\alpha_0 ih + \alpha_1. \end{aligned}$$

This gives

$$\begin{aligned} \alpha_0 &= \frac{e^{ih} - e^{-ih}}{2ih} = \frac{\sin h}{h} \\ \alpha_1 &= \frac{e^{ih} + e^{-ih}}{2} = \cos h. \end{aligned}$$

Thus

$$e^{Ah} = \frac{\sin h}{h} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h + \cos h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We remind here some useful way of computing the matrix exponential of a matrix $A \in \mathbb{R}^{n\times n}$. Depending on the form of the matrix A we can compute the exponential in different ways

- If A is diagonal then

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \Rightarrow \quad e^A = \begin{pmatrix} e^{a_{11}} & 0 & \dots & 0 \\ 0 & e^{a_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_{nn}} \end{pmatrix}$$

- A is nilpotent of order m. Then $A^m = 0$ and $A^{m+i} = 0$ for $i = 1, 2, \dots$. Then it is possible to use the following series expansion to calculate the exponential

$$e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{A^{m-1}}{(m-1)!}$$

- Using the inverse Laplace transform we have

$$e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)$$

- In general it is possible to compute the exponential of a matrix (or any continuous matrix function $f(A)$)) using the Cayley-Hamilton Theorem. For every function f there is a polynomial p of degree less than n such that

$$f(A) = p(A) = \alpha_0 A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-1} I.$$

If the matrix A has distinct eigenvalues, the n coefficient $\alpha_0, \dots, \alpha_{n-1}$ are computed solving the system of n equations

$$f(\lambda_i) = p(\lambda_i) \quad i = 1, \dots, n.$$

If the there is a multiple eigenvalue with multiplicitity m, then the additional conditions

$$\begin{aligned} f^{(1)}(\lambda_i) &= p^{(i)}(\lambda_i) \\ &\vdots \\ f^{(m-1)}(\lambda_i) &= p^{(m-1)}(\lambda_i) \end{aligned}$$

hold, where $f^{(i)}$ is the ith derivative with respect to λ.

SOLUTION 1.3

We recall here what is the z-transform of a signal. Consider a *discrete-time signal* $x(kh)$, $k = 0, 1, \ldots$. The z-transform of $x(kh)$ is defined as

$$\mathcal{Z}\{x(kh)\} = X(z) = \sum_{k=0}^{\infty} x(kh) z^{-k}$$

where z is a complex variable.

Using the definition

$$X(z) = \sum_{k=0}^{\infty} e^{-kh/T} z^{-k} = \sum_{k=0}^{\infty} \left\{ e^{-h/T} z^{-1} \right\}^k.$$

If $|e^{-h/T} z^{-1}| < 1$ then

$$X(z) = \frac{1}{1 - z^{-1} e^{-h/T}} = \frac{z}{z - e^{-h/T}}.$$

SOLUTION 1.4

Using the definition

$$X(z) = \sum_{k=0}^{\infty} \sin(wkh) z^{-k}.$$

Since

$$\sin wkh = \frac{e^{jwkh} - e^{-jwkh}}{2i}$$

then

$$X(z) = \frac{1}{2i} \sum_{k=0}^{\infty} \left\{ e^{iwh} z^{-1} \right\}^k - \frac{1}{2i} \sum_{k=0}^{\infty} \left\{ e^{-iwh} z^{-1} \right\}^k$$

If $|e^{\pm iwh} z^{-1}| < 1$ then

$$X(z) = \frac{1}{2i} \left(\frac{z}{z - e^{iwh}} - \frac{z}{z - e^{-iwh}} \right) = \cdots = \frac{z \sin wh}{z^2 - 2z \cos wh + 1}.$$

SOLUTION 1.5

For a discrete-time signal $x(k)$, we have the following

$$\begin{aligned}
\mathcal{Z}(x_k) &= X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} = x(0) + x(1) z^{-1} + \ldots \\
\mathcal{Z}(x_{k+1}) &= x(1) + x(2) z^{-1} + \ldots \\
&= z\, x(0) - z\, x(0) + z \left(x(1) z^{-1} + x(2) z^{-2} + \ldots \right) \\
&= z \left(x(0) + x(1) z^{-1} + x(2) z^{-2} + \ldots \right) - z\, x(0) \\
&= z\, X(z) - z\, x(0)
\end{aligned}$$

similarly,

$$\mathcal{Z}(x_{k+2}) = z^2\, X(z) - z^2\, x(0) - z\, x(1)$$

The discrete step is the following function

$$u(k) = \begin{cases} 0, & \text{if } k < 0; \\ 1, & \text{if } k \geq 0. \end{cases}$$

Thus

$$U(z) = \frac{z}{z-1}\,.$$

Using the previous z-transform we have

$$z^2\,Y(z) - z^2\,y(0) - z\,y(1) - 1.5z\,Y(z) + 1.5z\,y(0) + 0.5\,Y(z) = z\,U(z) - z\,u(0).$$

Collecting $Y(z)$ and substituting the initial conditions we get

$$Y(z) = \frac{0.5z^2 - 0.5z}{z^2 - 1.5z + 0.5} + \frac{z}{z^2 - 1.5z + 0.5}U(z)$$

Since $U(z) = z/(z-1)$ then

$$Y(z) = \frac{0.5z}{z-0.5} + \frac{z^2}{(z-1)^2(z-0.5)}.$$

Inverse transform gives

$$\begin{aligned} y(k) &= 0.5^{k+1} + \left(\frac{0.5(k+1)-1}{0.5^2} + \frac{0.5^{k+1}}{0.5^2}\right)u(k-1) \\ &= 0.5^{k+1} + \left(\frac{k-1}{0.5} + 0.5^{k-1}\right)u(k-1) \end{aligned}$$

Solutions to models of sampled systems

SOLUTION 2.1

The sampled system is given by

$$\begin{aligned} x(kh+h) &= \Phi x(kh) + \Gamma u(kh) \\ y(kh) &= Cx(kh) \end{aligned}$$

where

$$\begin{aligned} \Phi &= e^{-ah} \\ \Gamma &= \int_0^h e^{-as} ds b = \frac{b}{a}\left(1 - e^{-ah}\right). \end{aligned}$$

Thus the sampled system is

$$\begin{aligned} x(kh+h) &= e^{-ah}x(kh) + \frac{b}{a}\left(1 - e^{-ah}\right)u(kh) \\ y(kh) &= cx(kh). \end{aligned}$$

The poles of the sampled system are the eigenvalues of Φ. Thus there is a real pole at e^{-ah}. If h is small $e^{-ah} \approx 1$. If $a > 0$ then the pole moves towards the origin as h increases, if $a < 0$ it moves along the positive real axis, as shown in Figure 2.1.1.

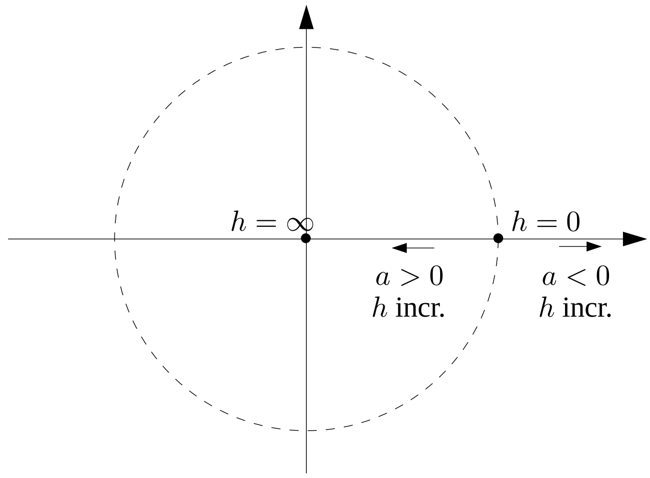


Figure 2.1.1: Closed-loop system for Problem 2.1.

SOLUTION 2.2

(a) The transfer function can be written as

$$G(s) = \frac{1}{(s+1)(s+2)} = \frac{\alpha}{s+1} + \frac{\beta}{s+2} = \frac{1}{s+1} - \frac{1}{s+2}.$$

A state-space representation (in diagonal form) is then

$$\begin{aligned}\dot{x} &= \underbrace{\begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{B} u\\ y &= \underbrace{\begin{pmatrix}1 & -1\end{pmatrix}}_{C} x.\end{aligned}$$

The state-space representation of the sampled system is

$$\begin{aligned}x(k+1) &= \Phi x(k) + \Gamma u(k)\\ y(k) &= Cx(k)\end{aligned}$$

where

$$\begin{aligned}\Phi &= e^{Ah} = \begin{pmatrix}e^{-1} & 0\\ 0 & e^{-2}\end{pmatrix}\\ \Gamma &= \int_0^h e^{As}\,ds\,B = \int_0^1 \begin{pmatrix}e^{-s}\\ e^{-2s}ds\end{pmatrix} = \begin{pmatrix}1-e^{-1}\\ \frac{1-e^{-2}}{2}\end{pmatrix}\end{aligned}$$

since A is diagonal.

(b) The pulse-transfer function is given by

$$\begin{aligned}H(z) = C(zI-\Phi)^{-1}\Gamma &= \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{z-e^{-1}} & 0\\ 0 & \frac{1}{z-e^{-2}}\end{pmatrix}\begin{pmatrix}1-e^{-1}\\ \frac{1-e^{-2}}{2}\end{pmatrix}\\ &= \frac{z(3/2+e^{-1}-1/2e^{-2}) + (3/2e^{-3}-e^{-2}-1/2e^{-1})}{(z-e^{-1})(z-e^{-2})}\end{aligned}$$

SOLUTION 2.3

(a) The sampled system is

$$\begin{aligned} x(kh+h) &= \Phi x(kh) + \Gamma u(kh) \\ y(kh) &= Cx(kh) \end{aligned}$$

where

$$\Phi = e^{Ah} \qquad\qquad \Gamma = \int_0^h e^{Ah} B ds.$$

To compute e^{Ah} we use that

$$e^{Ah} = \mathcal{L}^{-1}\left((sI-A)^{-1}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix}\right).$$

Since

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) &= \cos h \\ \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) &= \sin h \end{aligned}$$

then

$$e^{Ah} = \begin{pmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{pmatrix}.$$

Equivalently we can compute e^{Ah} using Cayley-Hamilton's theorem. The matrix e^{Ah} can be written as

$$e^{Ah} = a_0 Ah + a_1 I$$

where the constants a_0 and a_1 are computed solving the characteristic equation

$$e^{\lambda_k} = a_0\lambda_i + a_1 \qquad k = 1, \dots, n$$

where n is the dimension of the matrix A and λ_k are distinct eigenvalues of the matrix Ah. In this example the eigenvalues of Ah are $\pm hi$. Thus we need to solve the following system of equations

$$\begin{aligned} e^{ih} &= a_0 ih + a_1 \\ e^{-ih} &= -a_0 ih + a_1 \end{aligned}$$

which gives

$$\begin{aligned} a_0 &= \frac{1}{2hi}\left(e^{ih} - e^{-ih}\right) = \frac{\sin h}{h} \\ a_1 &= \frac{1}{2}\left(e^{ih} + e^{-ih}\right) = \cos h. \end{aligned}$$

Finally we have

$$e^{Ah} = \frac{\sin h}{h}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h + \cos h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{pmatrix}.$$

(b) Using Laplace transform we obtain

$$s^2Y(s) + 3sY(s) + Y(s) = sU(s) + 3U(s).$$

Thus the system has the transfer function

$$G(s) = \frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}.$$

One state-space realization of the system with transfer function $G(s)$ is

$$\begin{aligned} \dot{x} &= \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{B} u \\ y &= \underbrace{\begin{pmatrix} 2 & -1 \end{pmatrix}}_{C} x. \end{aligned}$$

Thus the sampled system is

$$\begin{aligned} x(kh+h) &= \Phi x(kh) + \Gamma u(kh) \\ y(kh) &= Cx(kh) \end{aligned}$$

with

$$\begin{aligned} \Phi &= e^{Ah} = \begin{pmatrix} e^{-h} & 0 \\ 0 & e^{-2h} \end{pmatrix} \\ \Gamma &= \int_0^h e^{As} B ds = \int_0^h \begin{pmatrix} e^{-s} \\ e^{-2s} \end{pmatrix} ds = \begin{pmatrix} 1 - e^{-h} \\ \frac{1-e^{-2h}}{2} \end{pmatrix} \end{aligned}$$

(c) One state-space realization of the system is

$$\begin{aligned} \dot{x} &= \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{B} u \\ y &= \underbrace{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}_{C} x. \end{aligned}$$

We need to compute Φ and Γ. In this case we can use the series expansion of e^{Ah}

$$e^{Ah} = I + Ah + \frac{A^2h^2}{2} + \ldots$$

since $A^3 = 0$, and thus all the successive powers of A. Thus in this case

$$\begin{aligned} \Phi &= e^{Ah} = \begin{pmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ h^2/2 & h & 1 \end{pmatrix} \\ \Gamma &= \int_0^h e^{As} B ds = \int_0^h \begin{pmatrix} 1 & s & s^2/2 \end{pmatrix}^T ds = \begin{pmatrix} h & h^2/2 & h^3/6 \end{pmatrix}^T \end{aligned}$$

SOLUTION 2.4

We will use in this exercise the following relation

$$\Phi = e^{Ah} \Rightarrow A = \frac{\ln \Phi}{h}$$

(a)

$$y(kh) - 0.5y(kh-h) = 6u(kh-h) \Rightarrow y(kh) - 0.5q^{-1}y(kh) = 6q^{-1}u(kh)$$

which can be transformed in state-space as

$$\begin{aligned} x(kh+h) &= \Phi x(kh) + \Gamma u(kh) = 0.5x(kh) + 6u(kh) \\ y(kh) &= x(kh). \end{aligned}$$

The continuous time system is then

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) \\ y(t) &= x(t), \end{aligned}$$

where, in this case since Φ and Γ are scalars, we have

$$\begin{aligned} a &= \frac{\ln \Phi}{h} = -\frac{\ln 2}{h} \\ b &= \Gamma / \int_0^h e^{as} ds = \frac{12 \ln 2}{h} \end{aligned}$$

(b)

$$\begin{aligned} x(kh+h) &= \underbrace{\begin{pmatrix} -0.5 & 1 \\ 0 & -0.3 \end{pmatrix}}_{\Phi} x(kh) + \begin{pmatrix} 0.5 \\ 0.7 \end{pmatrix} u(kh) \\ y(kh) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x(kh). \end{aligned}$$

We compute the eigenvalues of Φ

$$\begin{aligned} &\det(sI - \Phi) = \begin{pmatrix} s+0.5 & -1 \\ 0 & s+0.3 \end{pmatrix} = 0 \Leftrightarrow (s+0.5)(s+0.3) = 0 \\ &\lambda_1 = -0.5, \quad \lambda_2 = -0.3. \end{aligned}$$

Both eigenvalues of Φ are on the negative real axis, thus no corresponding continuous system exists.

(c) We can proceed as in (a). In this case $\Phi = -0.5$ which means that the sampled system has a pole on the negative real axis. Thus, as in (b), no corresponding continuous system exists.

SOLUTION 2.5

(Ex. 2.11 in [2])

(a) A state space representation of the transfer function is

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & -1 \end{pmatrix} x. \end{aligned}$$

In this case

$$\begin{aligned} \Phi &= e^{Ah} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-h} \end{pmatrix} \\ \Gamma &= \int_0^h e^{As} B ds = \begin{pmatrix} h \\ 1 - e^{-h} \end{pmatrix} \end{aligned}$$

(b) The pulse-transfer function is given by

$$H(z) = C(zI-\Phi)^{-1}\Gamma = \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}z-1 & 0\\ 0 & z-e^{-h}\end{pmatrix}^{-1}\begin{pmatrix}h\\ 1-e^{-h}\end{pmatrix}$$
$$= \frac{(h+e^{-h}-1)z + (1-e^{-h}-he^{-h})}{(z-1)(z-e^{-h})}$$

(c) The pulse response is

$$h(k) = \begin{cases} 0, & k=0;\\ C\Phi^{k-1}\Gamma, & k\geq 1.\end{cases}$$

Since

$$\Phi^k = \left(e^{Ah}\right)^k = e^{Akh}$$

then we have

$$h(k) = C\Phi^{k-1}\Gamma = \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{-(k-1)h}\end{pmatrix}\begin{pmatrix}h\\ 1-e^{-h}\end{pmatrix}$$
$$= h - e^{-(kh-h)} + e^{-hk}$$

(d) A difference equation relating input and output is obtained from $H(q)$.

$$y(kh) = H(q)u(kh) = \frac{(h+e^{-h}-1)q + (1-e^{-h}-he^{-h})}{(q-1)(q-e^{-h})}u(kh)$$

which gives

$$y(kh+2h) - (1+e^{-h})y(kh+h) + e^{-h}y(kh) = (h+e^{-h}-1)u(kh+h) + (1-e^{-h}-he^{-h})u(kh)$$

(e) The poles are in $z=1$ and $z=e^{-h}$. The second pole moves from 1 to 0 as h goes from 0 to ∞. There is a zero in

$$z = -\frac{1-e^{-h}-he^{-h}}{h+e^{-h}-1}.$$

The zero moves from -1 to 0 as h increases, see Figure 2.5.1

SOLUTION 2.6

(a) We notice that $\tau < h$. The continuous-time system in state-space is

$$\dot{x} = \underbrace{0}_{A}\cdot x + \underbrace{1}_{B}\cdot u(t-\tau)$$
$$y = \underbrace{1}_{C}\cdot x$$

Sampling the continuous-time system with sampling period $h=1$ we get

$$x(k+1) = \Phi x(k) + \Gamma_0 u(k) + \Gamma_1 u(k-1)$$
$$y(k) = x(k),$$

where

$$\Phi = e^{Ah} = e^0 = 1$$
$$\Gamma_0 = \int_0^{h-\tau} e^{As}ds B = 0.5$$
$$\Gamma_1 = e^{A(h-\tau)}\int_0^{\tau} e^{As}ds B = 0.5.$$

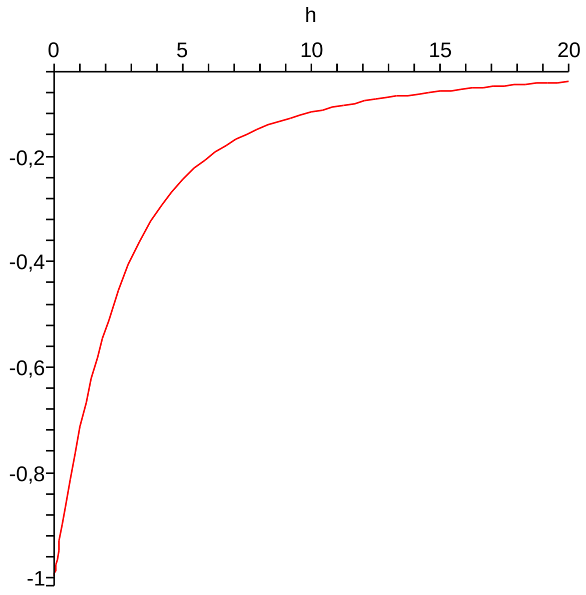


Figure 2.5.1: The zero of Problem 2.5 as function of h.

The system in state space is

$$\begin{pmatrix} x(k+1) \\ u(k) \end{pmatrix} = \begin{pmatrix} 1 & 0.5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k-1) \end{pmatrix} + \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} u(k)$$
$$y(k) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k-1) \end{pmatrix}.$$

The system is of second order.

(b) The pulse-transfer function is

$$H(z) = C(zI - \Phi)^{-1}\Gamma = \cdots = \frac{0.5(z+1)}{z(z-1)}.$$

To determine the pulse-response we inverse-transform $H(z)$.

$$H(z) = \frac{0.5(z+1)}{z(z-1)} = 0.5\left(z^{-1}\frac{z}{z-1} + z^{-2}\frac{z}{z-1}\right).$$

The inverse transform of $z/(z-1)$ is a step. Thus we have the sum of two steps delayed of 1 and 2 time-steps, thus the pulse-response is

$$h(kh) = \begin{cases} 0, & k = 0; \\ 0.5, & k = 1; \\ 1, & k > 1. \end{cases}$$

(c) We can consider $H(z)$ computed in (b). There are to poles, one in $z = 0$ and another in $z = 1$. There is a zero in $z = -1$.

SOLUTION 2.7

In this case the time delay is longer than the sampling period, $\tau > h$. The sampled system with $h = 1$ is

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma_0 u\big(k-(d-1)\big) + \Gamma_1 u(k-d) \\ y(k) &= x(k), \end{aligned}$$

where we compute d as the integer such that

$$\tau = (d-1)h + \tau', \qquad 0 < \tau' \leq h$$

and where Γ_0 and Γ_1 are computed as in the solution of exercise 2.6, where τ is replaced by τ'. In this example $d = 2$ and $\tau' = 0.5$, and where

$$\begin{aligned} \Phi &= e^{-1} \\ \Gamma_0 &= 1 - e^{-0.5} \\ \Gamma_1 &= e^{-0.5} - e^{-1}. \end{aligned}$$

A state representation is then

$$\begin{aligned} \begin{pmatrix} x(k+1) \\ u(k-1) \\ u(k) \end{pmatrix} &= \begin{pmatrix} \Phi & \Gamma_0 & \Gamma_1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k-2) \\ u(k-1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k) \\ y(k) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k-2) \\ u(k-1) \end{pmatrix}. \end{aligned}$$

We still have a finite dimensional system (third order).

SOLUTION 2.8

The system

$$y(k) - 0.5y(k-1) = u(k-9) + 0.2u(k-10)$$

can be shifted in time so that

$$y(k+10) - 0.5y(k+9) = u(k+1) + 0.2u(k)$$

which can be written as

$$(q^{10} - 0.5q^9)y(k) = (q + 0.2)u(k).$$

Thus

$$\begin{aligned} A(q) &= q^{10} - 0.5q^9 \\ B(q) &= q + 0.2 \end{aligned}$$

and the system order is $\deg A(q) = 10$. We can rewrite the given system as

$$(1 - 0.5q^{-1})y(k) = (1 + 0.2q^{-1})u(k-9).$$

where

$$\begin{aligned} A^*(q^{-1}) &= 1 - 0.5q^{-1} \\ B^*(q^{-1}) &= 1 + 0.2q^{-1} \end{aligned}$$

with $d = 9$. Notice that

$$\frac{B(q)}{A(q)} = \frac{q + 0.2}{q^{10} - 0.5q^9} = q^{-9}\frac{1 + 0.2q^{-1}}{1 - 0.5q^{-1}} = \frac{B^*(q^{-1})}{A^*(q^{-1})}$$

SOLUTION 2.9

We can rewrite the system

$$y(k+2) - 1.5y(k+1) + 0.5y(k) = u(k+1)$$

as

$$q^2 y(k) - 1.5qy(k) + 0.5y(k) = qu(k).$$

We use the z-transform to find the output sequence when the input is a step, namely

$$u(k) = \begin{cases} 0, & k < 0; \\ 1, & k \geq 0. \end{cases}$$

when $y(0) = 0.5$ and $y(-1) = 1$. We have

$$z^2(Y(z) - y(0) - y(1)z^{-1}) - 1.5z\big(Y(z) - y(0)\big) + 0.5Y(z) = z\big(U - u(0)\big).$$

We need to compute $y(1)$. From the given difference equation we have

$$y(1) = 1.5y(0) - 0.5y(-1) + u(0) = 1.25$$

Thus substituting in the z-transform and rearranging the terms, we get

$$(z^2 - 1.5z + 0.5)Y(z) - 0.5z^2 - 1.25z + 0.75z = zU(z) - z.$$

Thus we have

$$Y(z) = \frac{0.5z(z-1)}{(z-1)(z-0.5)} + \frac{z}{(z-1)(z-0.5)}U(z).$$

Now $U(z) = z/(z-1)$ this we obtain

$$Y(z) = \frac{0.5z}{z-0.5} + \frac{2}{(z-1)^2} + \frac{1}{z-0.5}.$$

Using the following inverse z-transforms

$$\begin{aligned}
\mathcal{Z}^{-1}\left(\frac{z}{z - e^{1/T}}\right) &= e^{-k/T}, \quad e^{-1/T} = 0.5 \Rightarrow T = 1/\ln 2 \\
\mathcal{Z}^{-1}\left(z^{-1}\frac{z}{z-0.5}\right) &= e^{-(k-1)\ln 2} \\
\mathcal{Z}^{-1}\left(\frac{1}{(z-1)^2}\right) &= \mathcal{Z}^{-1}\left(z^{-1}\frac{z}{(z-1)^2}\right) = k-1
\end{aligned}$$

we get

$$y(k) = 0.5e^{-k\ln 2} + 2(k-1) + e^{-(k-1)\ln 2}$$

SOLUTION 2.10

The controller can be written as

$$U(s) = -U_y(s) + U_r(s) = -G_y(s)Y(s) + G_r(s)R(s)$$

where the transfer functions $G_y(s)$ and $G_r(s)$ are given

$$\begin{aligned}
G_y(s) &= \frac{s_0 s + s_1}{s + r_1} = s_0 + \frac{s_1 - s_0 r_1}{s + r_1} \\
G_r(s) &= \frac{t_0 s + t_1}{s + r_1} = t_0 + \frac{t_1 - t_0 r_1}{s + r_1}.
\end{aligned}$$

We need to transform this two transfer functions in state-space form. We have

$$\begin{aligned}\dot{x}_y(t) &= -r_1 x_y(t) + (s_1 - s_0 r_1) y(t)\\ u_y(t) &= x_y(t) + s_0 y(t)\end{aligned}$$

and

$$\begin{aligned}\dot{x}_r(t) &= -r_1 x_r(t) + (t_1 - t_0 r_1) r(t)\\ u_r(t) &= x_r(t) + t_0 r(t).\end{aligned}$$

The sampled systems corresponding to the previous continuous time systems, when the sampling interval in h, are

$$\begin{aligned}x_y(kh+h) &= \Phi x_y(kh) + \gamma_y y(kh)\\ u_y(kh) &= x_y(kh) + s_0 y(kh)\end{aligned}$$

and

$$\begin{aligned}x_r(kh+h) &= \Phi x_r(kh) + \gamma_r r(kh)\\ u_r(kh) &= x_r(kh) + t_0 r(kh)\end{aligned}$$

where

$$\begin{aligned}\Phi &= e^{-r_1 h}\\ \gamma_y &= \int_0^h e^{-r_1 s} ds\,(s_1 - s_0 r_1) = -(e^{-r_1 h} - 1)\frac{s_1 - s_0 r_1}{r_1}\\ \gamma_r &= \int_0^h e^{-r_1 s} ds\,(t_1 - t_0 r_1) = -(e^{-r_1 h} - 1)\frac{t_1 - t_0 r_1}{r_1}.\end{aligned}$$

From the state representation we can compute the pulse transfer function as

$$\begin{aligned}u_y(kh) &= \left(\frac{\gamma_y}{q - \phi_y} + s_0\right) y(kh)\\ u_r(kh) &= \left(\frac{\gamma_r}{q - \phi_r} + t_0\right) r(kh).\end{aligned}$$

Thus the sampled controller in the form asked in the problem is

$$u(kh) = -\underbrace{\frac{s_0 q + \gamma_y - s_0 \phi_y}{q - \phi_y}}_{H_y(q)} y(kh) + \underbrace{\frac{t_0 q + \gamma_r - t_0 \phi_r}{q - \phi_r}}_{H_r(q)} r(kh).$$

SOLUTION 2.11

(Ex. 2.21 in [2]) Consider the discrete time filter

$$\frac{z+b}{z+a}$$

(a)

$$\begin{aligned}\arg\left(\frac{e^{i\omega h} + b}{e^{i\omega h} + a}\right) &= \arg\left(\frac{\cos\omega h + b + i\sin\omega h}{\cos\omega h + b + i\sin\omega h}\right)\\ &= \arctan\left(\frac{\sin\omega h}{b + \cos\omega h}\right) - \arctan\left(\frac{\sin\omega h}{a + \cos\omega h}\right).\end{aligned}$$

We have a phase lead if

$$\arctan\left(\frac{\sin\omega h}{b+\cos\omega h}\right) > \arctan\left(\frac{\sin\omega h}{a+\cos\omega h}\right), \quad 0<\omega h<\pi$$
$$\frac{\sin\omega h}{b+\cos\omega h} > \frac{\sin\omega h}{a+\cos\omega h}.$$

Thus we have lead if $b<a$.

Solutions to analysis of sampled systems

SOLUTION 3.1

The characteristic equation of the closed loop system is

$$z(z-0.2)(z-0.4)+K=0 \quad K>0.$$

The stability can be determined using the root locus. The starting points are $z=0$, $z=0.2$ and $z=0.4$. The asymptotes have the directions $\pm\pi/3$ and $-\pi$. The crossing of the asymptotes is 0.2. To find the value of K such that the root locus intersects the unit circle we let $z=a+ib$ with $a^2+b^2=1$. This gives

$$(a+ib)(a+ib-0.2)(a+ib-0.4)=-K.$$

Multiplying by $a-ib$ and since $a^2+b^2=1$ we obtain

$$a^2-0.6a-b^2+0.08+i(2ab-0.6b)=-K(a-ib).$$

Equating real parts and imaginary parts we obtain

$$a^2-0.6a-b^2+0.08=-Ka$$
$$b(2a-0.6)=Kb.$$

If $b\neq 0$ then

$$a^2-0.6a-(1-a^2)+0.087=-a(2a-0.6)$$
$$4a^2-1.2a-0.92=0.$$

Solving with respect to a we get

$$a=0.15\pm\sqrt{0.0225+0.23}=\begin{cases}0.652\\-0.352\end{cases}$$

This gives $K=0.70$ and $K=-1.30$. The root locus may also cross the unit circle if $b=0$ for $a=\pm1$. A root at $z=-1$ is obtained when

$$-1(-1-0.2)(-1-0.4)+K=0$$

namely when $K=1.68$. There is a root at $z=1$ when $K=-0.48$. The closed loop system is stable for

$$K\leq 0.70$$

SOLUTION 3.2

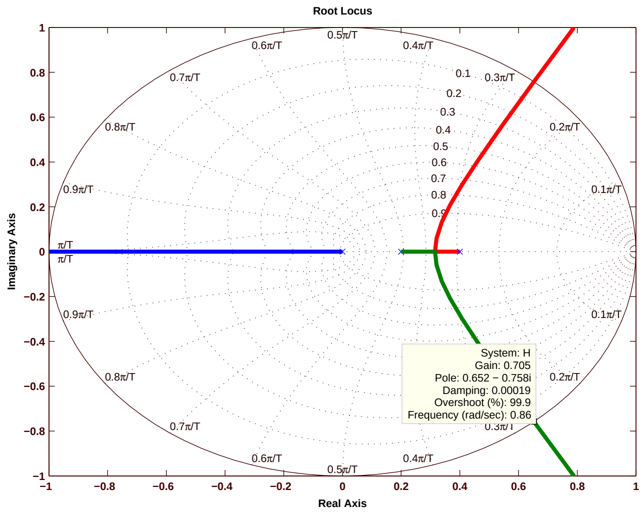


Figure 3.1.1: Root locus for the system in Problem 3.1

We sample the system $G(s)$. In order to do this we derive a state-space realization of the given system

$$\begin{aligned}\dot{x} &= u \\ y &= x\end{aligned}$$

which gives the following matrices of the sampled system

$$\begin{aligned}\Phi &= e^{-h\,0} = 1 \\ \Gamma &= \int_0^h ds = h.\end{aligned}$$

The pulse transfer operator is

$$H(q) = C(qI - \Phi)^{-1}\Gamma = \frac{h}{q-1}.$$

(a) When $\tau = 0$ the regulator is

$$u(kh) = Ke(kh)$$

and the characteristic equation of the closed loop system becomes

$$1 + C(z)H(z) = Kh + z - 1 = 0.$$

The system is stable if

$$|1 - Kh| < 1 \;\Rightarrow\; 0 < K < 2/h.$$

When there is a delay of one sample, $\tau = h$ then the characteristic equation becomes

$$z^2 - z + Kh = 0.$$

The roots of the characteristic equation (the poles of the system) must be inside the unit circle for guaranteeing stability and thus $|z_1| < 1$ and $|z_2| < 1$. Thus $|z_1||z_2| = |z_1 z_2| < 1$. Since $z_1 z_2 = Kh$ we have

$$K < 1/h.$$

(b) Consider the continuous-time system $G(s)$ in series with a time delay of τ seconds. The transfer function is then

$$\bar{G}(s) = \frac{K}{s} e^{-s\tau}.$$

The phase of the system as function of the frequency is

$$\arg \bar{G}(j\omega) = -\frac{\pi}{2} - \omega\tau$$

and the gain is

$$|\bar{G}(j\omega)| = \frac{K}{\omega}.$$

The system is stable if the gain is less than 1 at the cross over frequency, which satisfies

$$-\frac{\pi}{2} - \omega_c\tau = \pi \ \Rightarrow \ \omega_c = \frac{\pi}{2\tau}$$

The system is stable if

$$|\bar{G}(j\omega_c)| = \frac{K}{\omega_c} < 1$$

which yields

$$K < \frac{\pi}{2\tau} = \begin{cases} \infty & \tau = 0 \\ \frac{\pi}{2h} & \tau = \pi \end{cases}$$

The continuous-time system will be stable for all values of K if $\tau = 0$ and for $K < \pi/2h$ when $\tau = h$. This value is about 50% larger than the value obtained for the sampled system in (a).

SOLUTION 3.3

(a) The observability matrix is

$$W_o = \begin{pmatrix} C \\ C\Phi \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}.$$

The system is not observable since rank$(W_o) = 1$, or det$W_o = 0$.

(b) The controllability matrix is

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 1 \end{pmatrix}$$

which has full rank. Thus the system is reachable.

SOLUTION 3.4

The controllability matrix is

$$W_c = \begin{pmatrix}\Gamma & \Phi\Gamma\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 0 & 0.5 & 0\end{pmatrix}$$

which has full rank (check the first two rows of W_c), thus the system is reachable. From the input u we get the system

$$x(k+1) = \begin{pmatrix}1 & 0\\ 0 & 0.5\end{pmatrix} x(k) + \begin{pmatrix}0\\ 1\end{pmatrix} v(k).$$

In this case

$$W_c = \begin{pmatrix}0 & 0\\ 1 & 0.5\end{pmatrix}$$

which has rank one, and thus the system is not reachable from v.

SOLUTION 3.5

The closed loop system is

$$y(k) = H_{c\ell}(q)r(k) = \frac{C(q)H(q)}{1 + C(q)H(q)} r(k).$$

(a) With $C(q) = K$, $K > 0$ we get

$$y(k) = \frac{K}{q^2 - 0.5q + K} r(k).$$

The characteristic polynomial of the closed loop system is

$$z^2 - 0.5z + K = 0,$$

and stability is guaranteed if the roots of the characteristic polynomial are inside the unit circle. In general for a second order polynomial

$$z^2 + a_1 z + a_2 = 0$$

all the roots are inside the unit circle if[1]

$$\begin{aligned} a_2 &< 1\\ a_2 &> -1 + a_1\\ a_2 &> -1 - a_1. \end{aligned}$$

If we apply this result to the characteristic polynomial of the given system $H_{c\ell}$ we get

$$\begin{aligned} K &< 1\\ K &> -1.5\\ K &> -0.5 \end{aligned}$$

which together with the hypothesis that $K > 0$ gives $0 < K < 1$. The steady state gain is given by

$$\lim_{z\to 1} H_{c\ell}(z) = \frac{K}{K + 0.5}.$$

[1]The conditions come from the Jury's stability criterion applied to a second order polynomial. For details see pag. 81 in [2].

SOLUTION 3.6

The z-transform of a ramp is given in Table 2, pag. 22 in [1] and we get

$$R(z) = \frac{z}{(z-1)^2}.$$

Using the pulse transfer function from Problem 3.5 and the final value theorem we obtain

$$\lim_{k\to\infty} e(k) = \lim_{k\to\infty} (r(k) - y(k)) = \lim_{z\to 1} \left(\frac{z-1}{z} \underbrace{H_{c\ell}(z)R(z)}_{F(z)} \right)$$

if $(1-z^{-1})F(z)$ does not have any root on or outside the unit circle. For the $H_{c\ell}(z)$ as in this case the condition is not fulfilled. Let us consider the steady state of the first derivative of the error signal $e(k)$

$$\lim_{k\to\infty} = \lim_{z\to 1} \frac{z-1}{z} \frac{z^2 - 0.5z}{z^2 - 0.5z + K} \frac{z}{(z-1)^2} = \frac{0.5}{K+0.5}$$

which is positive, meaning that in steady state the reference and the output diverge.

SOLUTION 3.7

(a) (i) - Poles are mapped as $z = e^{sh}$. This mapping maps the left half plane on the unit circle

(b) (i) - The right half plane is mapped outside the unit circle

(c) (ii) - Consider the harmonic oscillator:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

which is reachable since

$$W_c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has full rank. If we sampled with sampling period h we get

$$\begin{aligned} x(kh+h) &= \begin{pmatrix} \cos\omega h & \sin\omega h \\ -\sin\omega h & \cos\omega h \end{pmatrix} x(kh) + \underbrace{\begin{pmatrix} 1-\cos\omega h \\ \sin\omega h \end{pmatrix}}_{\Gamma} u(kh) \\ y(kh) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(kh) \end{aligned}$$

If we choose $h = 2\pi/\omega$ then Γ is the 0 vector and clearly the system is not controllable.

(d) (ii) - as in (c) we can find example sampling periods that make the system not observable.

SOLUTION 3.8

The open loop system has pulse transfer operator

$$H_0 = \frac{1}{q^2 + 0.4q}$$

and the controller is proportional, thus $C(q) = K$.

(a) The closed loop system has pulse transfer operator

$$H_{c\ell} = \frac{KH_0}{1+KH_0} = \frac{K}{q^2+0.4q+K}.$$

From the solution of Problem 3.5 we know that the poles are inside the unit circle if

$$\begin{aligned} &K < 1 \\ &K > -1 + 0.4 \\ &K > -1 - 0.4 \;\Rightarrow\; -0.6 < K < 1 \end{aligned}$$

(b) Let $e(k) = r(k) - y(k)$ then

$$E(z) = (1 - H_{c\ell})\, R(z).$$

If K is chosen such that the closed loop system is stable, the final-value theorem can be used and

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1} \frac{z-1}{z} \frac{1}{1+KH_0} R(z) = \frac{z-1}{z} \frac{z^2+0.4z}{z^2+0.4z+K} \frac{z}{z-1} = \frac{1.4}{1.4+K}$$

If for example we choose $K = 0.5$ then $\lim_{k\to\infty} e(k) = 0.74$.

SOLUTION 3.9

The closed loop system has the following characteristic equation

$$\det\left(zI - (\Phi - \Gamma L)\right) = z^2 - (a_{11}+a_{22}-b_2\ell_2-b_1\ell_1)z + a_{11}a_{22} - a_{12}a_{21} + (a_{12}b_2 - a_{22}b_1)\ell_1 + (a_{21}b_1 - a_{11}b_2)\ell_2.$$

This must be equal to the given characteristic equations, thus

$$\begin{pmatrix} b_1 & b_2 \\ a_{12}b_2 - a_{22}b_1 & a_{21}b_1 - a_{11}b_2 \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} p_1 + \operatorname{tr}\Phi \\ p_2 - \det\Phi \end{pmatrix}$$

where $\operatorname{tr}\Phi = a_{11} + a_{22}$ and $\det\Phi = a_{11}a_{22} - a_{12}a_{21}$. The solution is

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} a_{21}b_1 - a_{11}b_2 & -b_2 \\ -a_{12}b_2 + a_{22}b_1 & b_1 \end{pmatrix} \begin{pmatrix} p_1 + \operatorname{tr}\Phi \\ p_2 - \det\Phi \end{pmatrix}$$

where $\Delta = a_{21}b_1^2 - a_{12}b_2^2 + b_1b_2(a_{22} - a_{11})$. To check when $\Delta = 0$ we consider the controllability matrix of the system

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma \end{pmatrix} = \begin{pmatrix} b_1 & a_{11}b_1 + a_{12}b_2 \\ b_2 & a_{21}b_1 + a_{22}b_2 \end{pmatrix}$$

and we find that $\Delta = \det W_c$. There exists a solution to the system of equation above if the system is controllable, since then the rank of the controllability matrix is full and $\det W_c \neq 0$.

For the double integrator $a_{11} = a_{12} = a_{22} = b_2 = 1$ and $a_{21} = 0$ and $b_1 = 0.5$. In order to design an dead beat controller we need to place the poles in zero, thus $p_1 = p_2 = 0$. This yields

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = -1 \begin{pmatrix} -1 & -1 \\ -0.5 & -0.5 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}.$$

SOLUTION 3.10

In this case the characteristic equation is

$$(z - 0.1)(z - 0.25) = z^2 - 0.35z + 0.025.$$

Using the result from Problem 3.9 we find that $\Delta = 0.5$ and L is obtained from

$$L^T = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} 0.5 & 0 \\ 0.1 & 1 \end{pmatrix} \begin{pmatrix} 0.75 \\ -0.025 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.1 \end{pmatrix}.$$

SOLUTION 3.11

(a) The static observer gives

$$
\hat{x}(k) = \Phi^{n-1}W_o^{-1}\begin{pmatrix} y(k-n+1) \\ \vdots \\ y(k) \end{pmatrix}
+ \underbrace{\begin{pmatrix} \Phi^{n-2}\Gamma & \Phi^{n-3}\Gamma & \dots & \Gamma \end{pmatrix} - \Phi^{n-1}W_o^{-1} \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 \\ C\Gamma & 0 & \dots & 0 \\ C\Phi\Gamma & C\Gamma & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C\Phi^{n-2}\Gamma & C\Phi^{n-3}\Gamma & \dots & C\Gamma \end{pmatrix}}_{W_u}}_{\Psi} \begin{pmatrix} u(k-n+1) \\ \vdots \\ u(k-1) \end{pmatrix}.
$$

In this case we have

$$
\begin{aligned}
W_o &= \begin{pmatrix} C \\ C\Phi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.22 & 1 \end{pmatrix} \quad W_o^{-1} = \begin{pmatrix} -4.55 & 4.55 \\ 1 & 0 \end{pmatrix} \\
W_u &= \begin{pmatrix} 0 \\ C\Gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0.03 \end{pmatrix} \\
\Psi &= \Gamma - \Phi W_o^{-1} W_u \\
&= \begin{pmatrix} 0.22 \\ 0.03 \end{pmatrix} - \begin{pmatrix} 0.78 & 0 \\ 0.22 & 1 \end{pmatrix}\begin{pmatrix} -4.55 & 4.55 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0.03 \end{pmatrix} = \begin{pmatrix} 0.114 \\ 0 \end{pmatrix}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
\hat{x}(k) &= \Phi W_o^{-1}\begin{pmatrix} y(k-1) \\ y(k) \end{pmatrix} + \Psi u(k-1) \\
&= \begin{pmatrix} -3.55 & 3.55 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y(k-1) \\ y(k) \end{pmatrix} + \begin{pmatrix} 0.114 \\ 0 \end{pmatrix} u(k-1).
\end{aligned}
$$

(b) The dynamic observer has the form

$$
\hat{x}(k+1|k) = (\Phi - KC)\hat{x}(k|k-1) + \Gamma u(k) + Ky(k).
$$

We choose K such that the the eigenvalues of $\Phi - KC$ are in the origin (very fast observer). Using the results in Problem 3.9 where we use Φ^T and C^T instead of Φ and Γ, we obtain

$$
K = \begin{pmatrix} 2.77 \\ 1.78 \end{pmatrix}.
$$

(c) The reduced observer has the form

$$
\hat{x}(k|k) = (I - KC)\left[\Phi\hat{x}(k-1|k-1) + \Gamma u(k-1)\right] + Ky(k).
$$

In this case we want to find K such that

- $CK = 1$
- $(I - KC)\Phi$ has the eigenvalues in the origin.

The first condition imposes $k_2 = 1$. Since

$$(I - KC)\Phi = \begin{pmatrix} 0.78 - 0.22k_1 & -k_1 \\ 0 & 0 \end{pmatrix}$$

in order to have eigenvalues in the origin we need to choose $k_1 = 0.78/0.22 = 3.55$. The reduced observer is then

$$\hat{x}(k|k) = \begin{pmatrix} 0 & -3.55 \\ 0 & 0 \end{pmatrix} \hat{x}(k-1|k-1) + \begin{pmatrix} 0.114 \\ 0 \end{pmatrix} u(k-1) + \begin{pmatrix} 3.55 \\ 1 \end{pmatrix} y(k).$$

Since $\hat{x}_2(k|k) = y(k)$ the we get

$$\hat{x}(k|k) = \begin{pmatrix} -3.55 & 3.55 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(k-1) \\ y(k) \end{pmatrix} + \begin{pmatrix} 0.114 \\ 0 \end{pmatrix} u(k-1)$$

which is the same as the static observer computed in (a).

SOLUTION 3.12

The constant disturbance $v(k)$, which typically has high energy at low frequencies, can be described by the dynamical system

$$\begin{aligned} w(k+1) &= A_w w(k) \\ v(k) &= C_w w(k), \end{aligned}$$

where the matrix A_w typically has eigenvalues at the origin or on the imaginary axis. We consider the augmented state vector

$$z = \begin{pmatrix} x \\ w \end{pmatrix}.$$

The augmented system ca be described by

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} A & C_w \\ 0 & A_w \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \end{aligned}$$

Sampling the system gives the following discrete-time system

$$\begin{aligned} \begin{pmatrix} x(k+1) \\ w(k+1) \end{pmatrix} &= \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k) \\ y(k) &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} w(k+1) &= \Phi_w w(k) \\ v(k) &= C_w w(k), \end{aligned}$$

and Φ_{xw} relates $x(k+1)$ and $w(k)$. In this exercise the disturbance can be modeled by the system

$$\begin{aligned} w(k+1) &= w(k) \\ v(k) &= w(k), \end{aligned}$$

and the process is described by

$$\begin{aligned} \Phi_w &= 1 \\ \Phi_{xw} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

(a) If the state x and the disturbance v can be measured then we can use the controller

$$u(k) = -Lx(k) - L_w w(k).$$

This gives the closed loop system

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Phi_{xw} w(k) - \Gamma L x(k) - \Gamma L_w w(k) \\ y(k) &= Cx(k) \end{aligned}$$

In general it is not possible to eliminate completely the influence of $w(k)$. This is possible only if $\Phi_{xw} - \Gamma L_w = 0$. We will therefore consider the situation at the output in steady state

$$y(\infty) = C\left(I - (\Phi - \Gamma L)\right)^{-1} (\Phi_{xw} - \Gamma L_w) w(\infty) = H_w(1) w(\infty).$$

The influence of w (or v) can be zero in steady state if

$$H_w(1) = 0.$$

Let ϕ_{ij} the (i,j) element of $\Phi - \Gamma L$ and γ_i the ith element of Γ. Then

$$C\left(I - (\Phi - \Gamma L)\right)^{-1} (\Phi_{xw} - \Gamma L_w) = -\frac{1 - L_w\gamma_1 - \phi_{22} + \phi_{22}L_w\gamma_1 - \phi_{12}L_w\gamma_2}{-1 + \phi_{22} + \phi_{11} - \phi_{11}\phi_{22} + \phi_{12}\phi_{21}} = 0$$

yields

$$L_w = \frac{-1 + \phi_{22}}{-\gamma_1 + \phi_{22}\gamma_1 - \phi_{12}\gamma_2}.$$

If L is the feedback matrix that gives a dead beat controller, that is

$$L = \begin{pmatrix} 3.21 & 5.57 \end{pmatrix}$$

such that

$$\Phi - \Gamma L = \begin{pmatrix} -0.142 & -0.114 \\ 0.179 & 0.142 \end{pmatrix}$$

then we have $L_w = 5.356$.

(b) In this case the state is measurable, but not the disturbance. The disturbance can be calculated from the state equation

$$\Phi_{xw} w(k-1) = x(k) - \Phi x(k-1) - \Gamma u(k-1).$$

The first element of this vector gives

$$w(k-1) = (1\ 0)\left(x(k) - \Phi x(k-1) - \Gamma u(k-1)\right).$$

Since $w(k)$ is constant and $x(k)$ is measurable, it is possible to calculate $\hat{w}(k) = w(k-1)$. The following control law can then be used

$$u(k) = -Lx(k) - L_w\hat{w}(k)$$

where L and L_w are the same as in (a). Compared with the controller in (a) there is a delay in the detection of the disturbance.

(c) If only the output is measurable then the state and the disturbance can be estimated by using the following observer:

$$\begin{pmatrix} \hat{x}(k+1) \\ \hat{w}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}(k) \\ \hat{w}(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} K \\ K_w \end{pmatrix} \epsilon(k)$$

$$\epsilon(k) = y(k) - C\hat{x}(k).$$

The gains K and K_w can be determined so that the error goes to zero, since the augmented system is observable. Let $\tilde{x}(k) = \hat{x}(k) - x(k)$ and similarly $\tilde{w}(k) = \hat{w}(k) - w(k)$. Then

$$\begin{pmatrix} \tilde{x}(k+1) \\ \tilde{w}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi - KC & \Phi_{xw} \\ -K_w C & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{w}(k) \end{pmatrix}.$$

The characteristic equation of the system matrix for the error is

$$z^3 + (k_1 - 2.2)z^2 + (1.05 - 1.7k_1 + k_2 + k_w)z + 0.7k_1 + 0.15 - 0.7k_w - k_2 = 0.$$

The eigenvalues can be placed at the origin if

$$K = \begin{pmatrix} 2.2 \\ -0.64 \end{pmatrix} K_w \qquad\qquad = 3.33.$$

The controller is

$$u(k) = -L\hat{x}(k) - L_w\hat{w}(k)$$

where L and L_w are the same as (a).

SOLUTION 3.13

(a) The eigenvalues of the matrix A are

$$\lambda_1 = -0.0197$$
$$\lambda_2 = -0.0129.$$

The matrices of the sampled system are

$$\Phi = e^{Ah} = \begin{pmatrix} 0.790 & 0 \\ 0.176 & 0.857 \end{pmatrix}$$
$$\Gamma = \begin{pmatrix} 0.281 \\ 0.0296 \end{pmatrix}$$

(b) The pulse transfer operator is given by

$$H(q) = C(qI - \Phi)^{-1}\Gamma = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} q - 0.790 & 0 \\ -0176 & q - 0.857 \end{pmatrix}^{-1} \begin{pmatrix} 0.281 & 0.0297 \end{pmatrix} = \frac{0.030q + 0.026}{q^2 - 1.65q + 0.68}$$

(c) The poles of the continuous-time system are at -0.0197 and -0.0129. The observer should be twice as fast as the fastest mode of the open-loop system, thus we choose the poles of the observer in

$$z = e^{-0.0192 \cdot 2 \cdot 12} = 0.62.$$

The desired characteristic equation of $\Phi - KC$ is then

$$z^2 - 1.24z + 0.38 = 0.$$

Using the results from Problem 3.9 we obtain

$$K = \begin{pmatrix} 0.139 & 0.407 \end{pmatrix}.$$

SOLUTION 3.14

(a) The sampled system is

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = x(k)$$

where

$$\Phi = e^{-5}$$
$$\Gamma = \frac{1-e^{-5}}{5}.$$

(b) In order to to prove stability with Lyapunov argument we need to choose a Lyapunov function V. Let $V = |x|$. The increment of the Lyapunov function is then

$$\Delta V = |x^+| - |x| = |e^{-5}x| - |x| = (e^{-5}-1)|x| \leq 0.$$

since $u(k) = 0$. Since $\Delta V \leq 0$, then the system is stable. We actually know that the system is asymptotically stable, since the pole is inside the unit circle. We can conclude on *asymptotical* stability using the Lyapunov argument, noticing that $\Delta V < 0$ if $x \neq 0$. Thus the system is asymptotically stable.

SOLUTION 3.15

(a) The sampled system is

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = x(k)$$

where

$$\Phi = \begin{pmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{pmatrix}$$
$$\Gamma = \begin{pmatrix} 1-e^{-1} \\ \dfrac{1-e^{-2}}{2} \end{pmatrix}.$$

(b) The characteristic polynomial is

$$(z-0.1)(z-0.2) = z^2 - 0.3z + 0.02.$$

Using the result from Problem 3.9 we obtain

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \frac{1}{-0.0636} \begin{pmatrix} -e^{-1}\dfrac{1-e^{-2}}{2} & -\dfrac{1-e^{-2}}{2} \\ -e^{-2}(1-e^{-1}) & 1-e^{-1} \end{pmatrix} \begin{pmatrix} -0.3+e^{-1}+e^{-2} \\ 0.02-e^{-1}e^{-2} \end{pmatrix} = \begin{pmatrix} 0.3059 \\ 0.0227 \end{pmatrix}$$

(c) The closed loop system is

$$x(k+1) = \underbrace{\begin{pmatrix} 0.1745 & -0.0143 \\ -0.1323 & 0.1255 \end{pmatrix}}_{\Phi_c} x(k).$$

We consider the following Lyapunov function

$$V(x) = x^T x$$

then

$$\Delta V(x) = x^T \Phi_c^T \Phi_c x - x^T x = x^T \underbrace{\left(\Phi_c^T \Phi_c - I\right)}_{\Omega,\ \text{symmetric}} x.$$

Since the eigenvalues of the symmetric matrix Ω are negative then $\Delta V \leq 0$. Thus the closed loop system is stable. Notice that $\Delta V < 0$ if $x \neq 0$ thus the closed loop system is asymptotically stable (the eigenvalues are placed in 0.1 and 0.2).

Solutions to computer realization of controllers

SOLUTION 4.1

(a) The characteristic polynomial of the closed loop system is equal to the numerator of $1 + HH_c$, that is,

$$z^2 + (K + K_i - 2.5)z - K + 1.$$

The poles of the closed-loop system are in the origin if

$$\begin{aligned} K + K_i - 2.5 &= 0 \\ -K + 1 &= 0 \end{aligned}$$

which yields $K = 1$ and $K_i = 1.5$.

(b) We can use the following partial fraction expansion for $H_c(z)$

$$H_c(z) = M + \frac{N}{z-1}.$$

With simple calculations we obtain $M = 2.5$ and $N = 1.5$. Thus the state-space representation of the controller is

$$\begin{aligned} x(k+1) &= x(k) + e(k) \\ u(k) &= 1.5x(k) + 2.5e(k) \end{aligned}$$

SOLUTION 4.2

(a) Using Euler's method (Forward difference) we get

$$H(z) = \frac{a}{(z-1)/h + a} = \frac{ah}{z - 1 + ah}.$$

This corresponds to the difference equation

$$y(kh + h) + (ah - 1)y(kh) = a\,h\,u(kh).$$

The difference equation is stable if

$$|ah - 1| < 1 \;\Rightarrow\; 0 < h < 2/a.$$

The approximation may be poor even if the difference equation is stable.

(b) Tustin's approximation gives

$$
\begin{aligned}
H(z) &= \frac{a}{\dfrac{2}{h}\dfrac{z-1}{z+1} + a} = \frac{(z + 1ah/2)}{(1 + ah/2)z + (ah/2 - 1)} \\
&= \frac{ah/2}{1 + ah/2} \frac{z+1}{z + \dfrac{ah/2 - 1}{ah/2 + 1}}.
\end{aligned}
$$

The pole of the discrete-time system will vary from 1 to - 1 when h varies from 0 to ∞. The discrete-time approximation is always stable if $a > 0$.

(c) Tustin's approximation with prewarping gives

$$
H(z) = \frac{a}{\alpha \dfrac{z-1}{z+1} + a} = \frac{a/\alpha}{1 + a/\alpha} \frac{z+1}{z + \dfrac{a/\alpha - 1}{a/\alpha - 1}}
$$

where

$$
\alpha = \frac{a}{\tan(ah/2)}.
$$

SOLUTION 4.3

(a) Euler's method gives

$$
H(z) = 4\frac{(z-1)/h + 1}{(z-1)/h + 2} = 4\frac{z - 1 + h}{z - 1 + 2h} = 4\frac{z - 0.75}{z - 0.5}
$$

(b) Backward differences give

$$
H(z) = 4\frac{(z-1)/(zh) + 1}{(z-1)/(zh) + 2} = 4\frac{z(1+h) - 1}{z(1+2h) + 1} = 3.33\frac{z - 0.80}{z - 0.667}
$$

(c) Tustin's approximation gives

$$
H(z) = 4\frac{\dfrac{2}{h}\dfrac{z-1}{z+1} + 1}{\dfrac{2}{h}\dfrac{z-1}{z+1} + 2} = 4\frac{z(1 + h/2) - (1 - h/2)}{z(1+h) - (1-h)} = 3.6\frac{z - 0.778}{z - 0.6}
$$

(d) Tustin's approximation with pre-warping

$$
H(z) = 4\frac{\alpha\dfrac{z-1}{z+1} + 1}{\alpha\dfrac{z-1}{z+1} + 2} = 4\frac{z(1 + 1/\alpha) - (1 - 1\alpha)}{z(1 + 2/\alpha) - (1 - 2/\alpha)} = 3.596\frac{z - 0.775}{z - 0.596}
$$

All the four approximation has the form

$$
H(z) = K\frac{z + a}{z + b}.
$$

The gain and phase at $\omega = 1.6$rad/s are obtained from

$$H(e^{i\omega h}) = K\frac{e^{i\omega h}+a}{e^{i\omega h}+b} = K\frac{(e^{i\omega h}+a)(e^{-i\omega h}+b)}{(e^{i\omega h}+b)(e^{-i\omega h}+b)}$$
$$= K\frac{1+ab+(a+b)\cos(\omega h)+i(b-a)\sin(\omega h)}{1+b^2+2b\cos(\omega h)}$$

$$\arg H(e^{i\omega h}) = \arctan\frac{(b-a)\sin\omega h}{1+ab+(a+b)\cos(\omega h)}$$

$$|H(e^{i\omega h})| = K\sqrt{\frac{1+a^2+2a\cos(\omega h)}{1+b^2+2b\cos(\omega h)}}.$$

The four different approximations give at $\omega = 1.6$rad/s the following results

	$\lvert H(.)\rvert$	arg$H(.)$	Rel. err. |.|	Rel. err. arg(.)
Continuous-time (no approx.)	2.946680646	0.3374560692		
Euler	2.966414263	0.4105225474	0.67%	21.65%
Backward	2.922378065	0.2772636846	-0.82%	-17.83%
Tustin	2.959732059	0.3369122161	0.44%	-0.16%
Tustin with prewarping	2.946680646	0.3374560692	0.0%	0.0%

SOLUTION 4.4

The tank process in Problem 2.13 has the transfer function

$$G(s) = \frac{0.000468}{(s+0.0197)(s+0.0129)}$$

(a) At the desired cross-over frequency we have

$$|G(i\omega_c)| = 0.525$$
$$\arg G(i\omega_c) = -115^\circ.$$

We use a PI controller in the form

$$F(s) = \frac{K(T\,s+1)}{T\,s}$$

and we have the following specifications atω_c

- gain 1/0.525
- phase -15°.

This gives $K = 1.85$ and $T = 149$.

(b) The characteristic equation of the closed loop system is

$$s^2 + 0.0326s^2 + 0.00112s + 5.91\ 10^{-6} = 0$$

which has roots $s_{1,2} = -0.0135 \pm i0.0281$ and $s_3 = -0.006$. The complex poles have a damping of $\zeta = 0.43$. The zero of the closed loop system is -0.0062.

(c) Tustin's approximation with given warping is

$$H_c(z) = \frac{1.85\left(\alpha\dfrac{z-1}{z+1} + 0.0067\right)}{\alpha\dfrac{z-1}{z+1}} = \frac{1.85(\alpha+0.0067)}{\alpha}\left(1 + \frac{0.0134}{(\alpha+0.0067)(z-1)}\right).$$

The rule of thumb for the selection of the sampling period gives

$$h \approx 6 \ - \ 20\text{sec}.$$

The choice of $h = 12$ seems to be reasonable. This gives $\alpha = 0.165$ and

$$H_c(z) = 1.925\left(1 + \frac{0.0778}{z-1}\right).$$

SOLUTION 4.5

The sample and zero-order hold circuit can be approximate by a delay of $h/2$ seconds. Indeed the output of the zero-order hold is

$$u_h(t) = u(kh), \qquad kh < t < kh + h$$

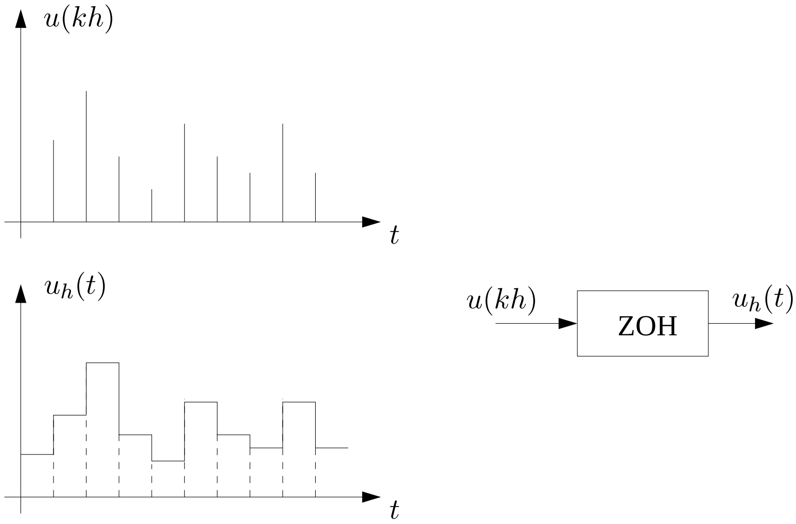


Figure 4.5.1: The zero-order hold.

If $u(kh) = \delta(kh)$ then $u_h(t)$ is the impulse response of the ZOH filter. In this case the output, let us call it, $u_h^\delta(t)$ is a pulse of height 1 and duration h i.e.,

$$u_h^\delta(t) = (1(t) - 1(t-h)).$$

The Laplace transform of the impulse response is the transfer function of the ZOH filter, which is

$$ZOH(s) = \mathcal{L}\{u_h^\delta(t)\} = \int_0^\infty \frac{1}{h}\left(1(t) - 1(t-h)\right)e^{-st}dt = (1 - e^{-sh})/s.$$

When we sample a signal we have a scaling factor of $1/h$, as

$$X_s(j\omega) = \frac{1}{h}\sum_{k=-\infty}^{\infty} X(j(\omega + k\omega_s)).$$

Thus for small h we have

$$SAMPLE(s) \cdot ZOH(s) = \frac{1}{h}\frac{(1-e^{-sh})}{s} = \frac{1-1+sh-(sh)^2/2+\dots}{sh} = 1-\frac{sh}{2}+\dots \approx e^{-sh/2}$$

which is approximately a delay of $h/2$. If we assume a decrease of the phase margin of $5^\circ - 15^\circ$, then

$$\Delta\phi_{ZOH} = \omega_c h/2 = \frac{180^\circ \omega_c h}{2\pi} = \frac{\omega_c h}{0.035} = 5^\circ - 15^\circ,$$

which then gives

$$\omega_c h = 0.17 - 0.52$$

or

$$\omega_c h \approx 0.15 - 0.5.$$

SOLUTION 4.6

Consider the general problem of controlling a continuous-time system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

with the continuous-time controller

$$u(t) = Mr(t) - Lx(t).$$

The closed-loop system is

$$\begin{aligned} \dot{x}(t) &= (A-BL)x(t) + BMr(t) = A_c x(t) + BMr(t) \\ y(t) &= Cx(t). \end{aligned}$$

If $r(t)$ is constant over one sampling period, then the previous equation can be sampled, giving

$$\begin{aligned} x(kh+h) &= \Phi_c x(kh) + \Gamma_c M r(kh) \\ y(kh) &= Cx(kh), \end{aligned}$$

where

$$\begin{aligned} \Phi_c &= e^{A_c h} \\ \Gamma_c &= \int_0^h e^{A_c s} ds B. \end{aligned}$$

Let us assume that the controller

$$u(kh) = \tilde{M}r(kh) - \tilde{L}x(kh)$$

is used to control the sampled system

$$\begin{aligned} x(kh+h) &= \Phi x(kh) + \Gamma u(kh) \\ y(kh) &= Cx(kh), \end{aligned}$$

where

$$\begin{aligned} \Phi &= e^{Ah} \\ \Gamma &= \int_0^h e^{Ah} ds B. \end{aligned}$$

In this case the closed-loop system is

$$\begin{aligned}x(kh+h) &= (\Phi - \Gamma\tilde{L})x(kh) + \Gamma\tilde{M}r(kh)\\ y(kh) &= Cx(kh).\end{aligned}$$

It is in general not possible to choose $\tilde{L}$ such that

$$\Phi_c = \Phi - \Gamma\tilde{L}.$$

However, we can make a series expansion and equate terms. Assume

$$\tilde{L} = L_0 + L_1 h/2$$

then

$$\Phi_c \approx I - (A - BL)h + \left(A^2 - BLA - ABL - (BL)^2\right)h^2/2 + \ldots$$

and

$$\Phi - \Gamma\tilde{L} \approx I + (A - BL_0)h + \left(A^2 - ABL_0 - (BL_1)^2\right)h^2/2 + \ldots$$

The systems (**??**) and (**??**) have the same poles up to and including order h^2 if

$$\tilde{L} = L\left(I + (A - BL)h/2\right).$$

The modification on $\tilde{M}$ is determined by assuming the steady-state values of (**??**) and (**??**) are the same. Let the reference be constant and assume that the steady-state value of the state is x^0. This gives the relations

$$(I - \Phi_c)x^0 = \Gamma_c M r$$

and

$$\left(I - (\Phi - \Gamma\tilde{L})\right)x^0 = \Gamma\tilde{M}r$$

The series expansion of the left-hand sides of these two relations are equal for power of h up to and including h^2. Then, we need to determine $\tilde{M}$ so that the expansions of the right-hands are the same for h and h^2. Assuming

$$\tilde{M} = M_0 + M_1 h/2$$

then

$$\Gamma_c M \approx BMh + (A - BL)BMh^2/2 + \ldots$$

and

$$\Gamma\tilde{M} \approx BM_0 h + (BM_1 - ABM_0)h^2/2 + \ldots,$$

which gives

$$\tilde{M} = (I - LBh/2)M.$$

(a) We need to compute $\tilde{L}$ and $\tilde{M}$, which are

$$\begin{aligned}\tilde{L} &= L\begin{pmatrix}1 & h/2\\ -h/2 & 1-h\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & h/2\\ -h/2 & 1-h\end{pmatrix} = \begin{pmatrix}0.8 & 1.7\end{pmatrix}\\ \tilde{M} &= 2 - 2h = 1.6\end{aligned}$$

(b) The backward difference approximation gives

$$\begin{aligned}\frac{1-q^{-1}}{h}\hat{x}(kh) &= (A - KC)\hat{x}(kh) + Bu(kh) + Ky(kh)\\ (I - Ah - KCh)\hat{x}(kh) &= q^{-1}\hat{x}(kh) + Bu(kh) + Ky(kh).\end{aligned}$$

Let

$$\Phi_0 = (I - Ah - KCh)^{-1} = \frac{1}{h^2 + h + 1}\begin{pmatrix} 1 & h \\ -h & 1+h \end{pmatrix}.$$

This gives,

$$\begin{aligned} \hat{x}(kh) &= (\Phi_0 \hat{x}(kh-h) + \Phi_0 Bu(kh) + \Phi_0 Ky(kh) \\ &= \begin{pmatrix} 0.81 & 0.16 \\ -0.16 & 0.97 \end{pmatrix}\hat{x}(kh-h) + \begin{pmatrix} 0.03 \\ 0.19 \end{pmatrix}u(kh) + \begin{pmatrix} 0.19 \\ 0.16 \end{pmatrix}y(kh). \end{aligned}$$

(c) Simulate the continuous-time controller and the discrete-time approximation. Let $x(0) = (1,1)^T$ and $\hat{x}(t) = (0,0)^T$.

SOLUTION 4.7

(a) To obtain the forward difference approximation we substitute $s = \frac{q-1}{h}$ in the continuous-time expression. We obtain

$$u(kh) = -\frac{s_0 q - s_0 + hs_1}{q + r_1 h - 1}y(kh) + \frac{t_0 q - t_0 + ht_1}{q + r_1 h - 1}r(kh)$$

(b) In order to compare the discretizations we examine the location of the controller pole. We use the graph available in order to compute the poles for the exact discretization. We have the following results:

Discretization	$h = 0.01$	$h = 0.1$	$h = 1$
Exact	0.9048	0.3679	0.00004540
Forward difference	0.9	0	-9

We notice that the forward difference approximation yields an unstable system for $h = 1$ (pole in -9). For $h = 0.1$, the forward difference approximation gives a pole at the origin, which corresponds to a one step delay. Thus this approximations of the continuous-time controller cannot be considered satisfactory. For $h = 0.01$ the pole of the forward difference approximation is very closed to the exact discretization which means that this is a better choice.

We could also consider how the sampling interval relates to the dominating time constant of the system. We know that the relation between this two parameters is given by the following rule-of-thumb

$$N_r = \frac{T_r}{h} \approx 4 - 10.$$

In this case only the controller dynamics are available, but using the time constant of the controller that is $T_c = 1/r_1 = 0.1$ we see that only for $h = 0.01$ the rule-of-thumb is fulfilled.

SOLUTION 4.8

(a) The backward differences approximation substitutes s with $(z-1)/zh$. We can consider the Laplace transform of the controller's equation yields

$$\begin{aligned} sX(s) &= AX(s) + BE(s) \\ U(s) &= CX(s) + DE(s) \end{aligned}$$

Substituting s with $(z-1)/zh$ we get the following equations

$$\begin{aligned} x(k+1) - x(k) &= hAx(k+1) + hBe(k+1) \\ u(k) &= Cx(k) + De(k). \end{aligned}$$

We consider first the state update equation. Dividing the signals at time $k+1$ and k we get the following

$$x(k+1) - hAx(k+1) - hBe(k+1) = x(k) =: w(k+1).$$

Solving this equation for x in terms of w and e we get

$$x(k+1) = (I - hA)^{-1}w(k+1) + (I - hA)^{-1}Bhe(k+1)$$

thus, since $w(k+1) = x(k)$ we get

$$w(k+1) = (I - hA)^{-1}w(k) + (I - hA)^{-1}Bhe(k).$$

This gives that

$$\begin{aligned}\Phi_c &= (I - hA)^{-1} \\ \Gamma_c &= (I - hA)^{-1}Bh.\end{aligned}$$

From the output equation substituting $x(k)$ we get directly

$$u(k) = C(I - hA)^{-1} + \{D + C(I - hA)^{-1}Bh\}e(k).$$

Thus

$$\begin{aligned}H &= C(I - hA)^{-1} \\ J &= D + C(I - hA)^{-1}Bh.\end{aligned}$$

(b) To compute the Tustin's approximation we proceed in the same way, and we substitute s with $2(z-1)/(h(z+1))$. The state equation then becomes

$$x(k+1) - x(k) = \frac{A_c h}{2}\big(x(k+1) - x(k)\big) + \frac{B_c h}{2}(e(k+1) + e(k)).$$

Again collecting the $k+1$ terms on one side we get

$$\begin{aligned}x(k+1) - \frac{Ah}{2}x(k) - \frac{Bh}{2}e(k+1) &= x(k) + \frac{Ah}{2}x(k) + \frac{Bh}{2}e(k) \\ &=: w(k+1).\end{aligned}$$

Thus we can derive $x(k+1)$ from the previous equation as function of $w(k+1)$ and $e(k+1)$

$$x(k+1) = (I - \frac{Ah}{2})^{-1}w(k+1) - (I - \frac{Ah}{2})^{-1}\frac{Bh}{2}e(k+1).$$

The state equation becomes

$$w(k+1) = \underbrace{(I + \frac{Ah}{2})(I - \frac{Ah}{2})^{-1}}_{\Phi_c} w(k) + \underbrace{\big((I + \frac{Ah}{2})(I - \frac{Ah}{2})^{-1} + I\big)\frac{Bh}{2}}_{\Gamma_c} e(k).$$

The output equation becomes

$$u(k) = \underbrace{C(I - \frac{Ah}{2})^{-1}}_{H_c} w(k) + \underbrace{\{D + C(I - \frac{Ah}{2})^{-1}\frac{Bh}{2}\}}_{J_c} e(k).$$

Notice that is possible to write Γ_c in a more compact way as follows

$$\begin{aligned}\big((I + \frac{Ah}{2})(I - \frac{Ah}{2})^{-1} + I\big)\frac{Bh}{2} &= \big((I + \frac{Ah}{2}) + (I - \frac{Ah}{2})\big)(I - \frac{Ah}{2})^{-1}\frac{Bh}{2} \\ &= (I - \frac{Ah}{2})^{-1}\frac{Bh}{2}.\end{aligned}$$

Solutions to implementation aspects

SOLUTION 5.1

The poles of the controller are

$$z_1 = 1; \quad z_2 = \frac{1}{2}; \quad z_{3,4} = -1/4 \pm i\frac{\sqrt{3}}{4}$$

We can represent the controller in Jordan form as

$$\Sigma_1 : s_1(k+1) = 1\, s_1(k) + 2.286 y(k)$$
$$\Sigma_2 : s_2(k+1) = \frac{1}{2}\, s_2(k) - 3.073 y(k)$$
$$\Sigma_3 : \begin{pmatrix} s_3(k+1) \\ s_4(k+1) \end{pmatrix} = \begin{pmatrix} \frac{-1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{-1}{4} \end{pmatrix} \begin{pmatrix} s_3(k) \\ s_4(k) \end{pmatrix} + \begin{pmatrix} 1.756 \\ 1.521 \end{pmatrix} y(k)$$

The control input $u(k)$ is then

$$u(k) = 0.5 s_1(k) + 0.8677 s_2(k) + 0.8677 s_3(k)$$

Another way to get the parallel form is the following. We write the given transfer function as

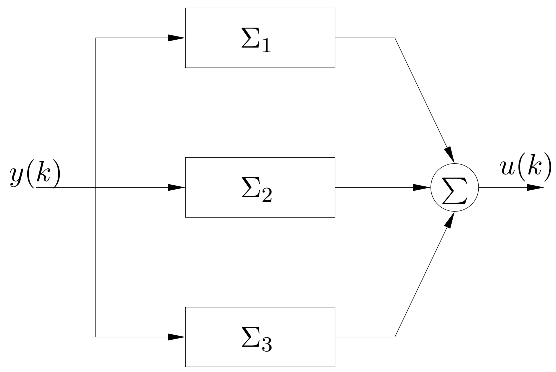


Figure 5.1.1: Parallel form for the controller of Problem 5.1

$$H(z) = \frac{1}{(z-1)(z-1/2)(z^2+1/2z+1/4)} = \frac{A}{z-1} + \frac{B}{z-1/2} + \frac{Cz+D}{z^2+1/2z+1/4}$$

where the constants A, B, C, D are to be determined. It easy to find that the two polynomial in z are equal if and only if

$$A = 1.14, \quad B = -2.67, \quad D = 0.95, \quad C = 1.52.$$

Then the system is the parallel of

$$\Sigma_1 : \quad H_{\Sigma_1} = \frac{1.14}{z-1}$$
$$\Sigma_2 : \quad H_{\Sigma_2} = \frac{-2.67}{z-1/2}$$
$$\Sigma_3 : \quad H_{\Sigma_3} = \frac{1.52z+0.95}{z^2+1/2z+1/4}.$$

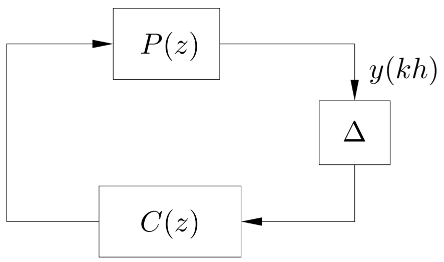


Figure 5.3.1: Closed loop system for Problem 5.3.

SOLUTION 5.2

(a) The closed loop system from r to y shown in Figure has the following transfer function

$$H(s) = \frac{C_s(s)P(s)e^{-s\tau}}{1 + C_s(s)P(s)e^{-s\tau}}.$$

In order to find $C_s(s)$ we need to solve $H(s) = H_{c\ell}(s)$ which gives

$$\begin{aligned}
\frac{C_s(s)P(s)e^{-s\tau}}{1 + C_s(s)P(s)e^{-s\tau}} &= \frac{C(s)P(s)}{1 + C(s)P(s)}e^{-s\tau} \\
C_s(s)\big(1 + C(s)P(s)\big) &= C(s)\big(1 + C_s(s)P(s)e^{-s\tau}\big) \\
C_s(s) &= \frac{C(s)}{1 + C(s)P(s) - C(s)P(s)e^{-s\tau}}
\end{aligned}$$

(b) We first find $C(s)$:

$$\frac{C(s)\dfrac{1}{s+1}}{1 + C(s)\dfrac{1}{s+1}}e^{-s\tau} = \frac{8}{s^2 + 4s + 8} \qquad \Rightarrow \qquad C(s) = \frac{8s + 8}{s^2 + 4s}.$$

Using the result in (a) we get

$$C_s(s) = \frac{8(s+1)}{s^2 + 4s + 8(1 - e^{-s\tau})}$$

SOLUTION 5.3

The control problem over wireless network can be represented as shown Figure 5.3.1. The delay Δ is such that

$$\Delta(y(kh)) = y(kh - d(k)) \qquad d(k) \in \{0, \ldots, N\}$$

The closed loop system is stable if

$$\left| \frac{P(e^{i\omega})C(e^{i\omega})}{1 + P(e^{i\omega})C(e^{i\omega})} \right| < \frac{1}{N|e^{i\omega} - 1|} \qquad \omega \in [0, \pi]$$

where N is the number of samples that the control signal is delayed. Notice that the previous result is valid if the closed loop transfer function is stable. In this case the closed loop transfer function is stable with poles

$$z_1 = 0.861, \quad z_2 = 0.447.$$

If we plot the bode diagram of the closed loop system without delay versus the function $1/(N|e^{i\omega} - 1|)$ for different values of N we obtain the results shown in Figure 5.3.2. It can be seen that the closed loop system is stable if $N \leq 3$. Thus the maximum delay is 3 samples. Notice that the result is only a sufficient condition. This means that it might be possible that the system is stable for larger delays than 3 samples.

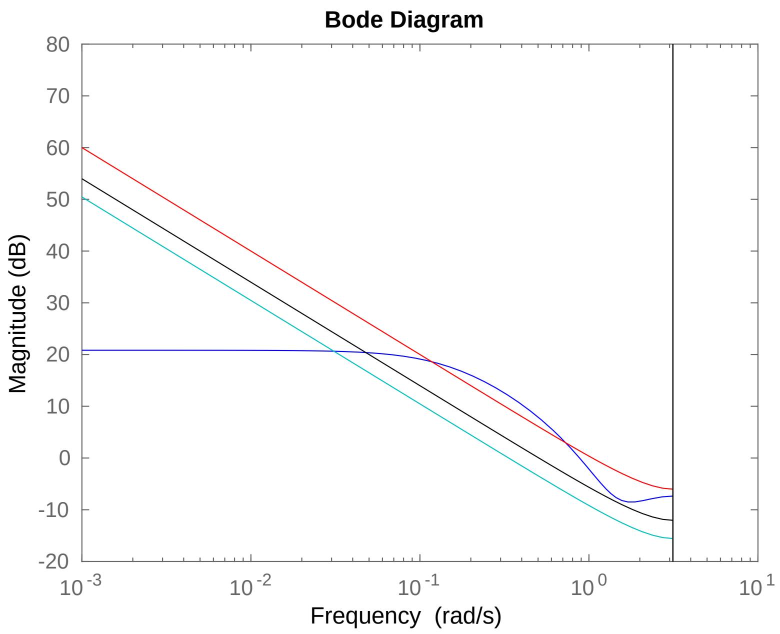


Figure 5.3.2: Bode diagram of the closed loop system of Problem 5.3 for different values of N.

SOLUTION 5.4

The linear model for multiplication with roundoff is shown in Figure 5.4.1. The signal ϵ is a stochastic variable representing the roundoff, and it is uniformly distributed in the interval $(-\delta/2, \delta/2)$. It thus has variance $\delta^2/12$. Before computing the variance of u as function of the variance of i we show that the two algorithms compute

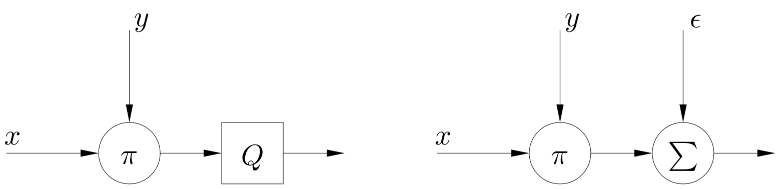


Figure 5.4.1: Linear model for multiplication with roundoff.

the same u. For Algorithm 1 we have

$$u = k(e + i) \overset{i^+ = i + eh/t_i}{\Longrightarrow} u = ke + k(i - 1) + keh/t_i$$

whereas for algorithm 2 we have

$$u = i + ke \overset{i^+ = ki + keh/t_i}{\Longrightarrow} u = ke + k(i - 1) + keh/t_i\,.$$

Alg 1: We have that

$$Var\{i^+\} = Var\{i\} + Var\left\{(eh + \epsilon)\frac{1}{t_i} + \epsilon\right\} = Var\{i\} + \left(1 + \frac{1}{t_i}\right)\frac{\delta^2}{12}\,.$$

Using the approximation of roundoff we have that $u \approx k(e + i) + \epsilon$ and thus

$$Var\{u\} = kVar\{i\} + \frac{\delta^2}{12}\,.$$

Alg 2: We have that

$$Var\{i^+\} = kVar\{i\} + \frac{\delta^2}{12} + Var\left\{((ke+\epsilon)\,h+\epsilon)\,\frac{1}{t_i} + \epsilon\right\} = kVar\{i\} + \left(2 + \frac{h}{t_i} + \frac{1}{t_i}\right)\frac{\delta^2}{12}\,.$$

Using the approximation of roundoff we have that $u \approx i + ke + \epsilon$ and thus

$$Var\{u\} = Var\{i\} + \frac{\delta^2}{12}\,.$$

If we consider

$$Var\{i^+\} = \alpha Var\{i\} + f(\delta^2)$$

then after n iteration we have that

$$Var\{i\}_n = \left(\sum_{i=0}^{n-1}\alpha^i\right) f(\delta^2)\,,$$

where we have considered $Var\{i\}_0 = 0$. Thus we have that

Alg 1: In this case

$$Var\{i\}_n = n\left(1 + \frac{1}{t_i}\right)\frac{\delta^2}{12}$$

and thus

$$Var\{u\}_n = \left(kn + \frac{kn}{t_i} + 1\right)\frac{\delta^2}{12}$$

Alg 2: In this case

$$Var\{i\}_n = \left(\sum_{i=0}^{n-1}k^i\right)\left(\frac{h}{t_i} + \frac{1}{t_i} + 2\right)\frac{\delta^2}{12}$$

and thus

$$Var\{u\}_n = \left(\frac{k^n-1}{k-1}\right)\left(\frac{h}{t_i} + \frac{1}{t_i} + 2\right)\frac{\delta^2}{12} + \frac{\delta^2}{12}$$

In order to compare the two algorithms, we assume $h = t_i = 1$. (This is just for comparison, since multiplication with 1 normally does not add any roundoff error.) We then have for the two algorithms, that

$$Var\{u\}_n^{(1)} = (2kn+1)\frac{\delta^2}{12}$$
$$Var\{u\}_n^{(2)} = \left(4\frac{k^n-1}{k-1} + 1\right)\frac{\delta^2}{12}\,.$$

If we assume $k < 1$ and n large then

$$4\frac{k^n-1}{k-1} + 1 \approx \frac{4}{1-k} + 1$$

and thus

$$2kn + 1 > \frac{4}{1-k} + 1$$

which allows to conclude that the first algorithm is worst than the second. If $k > 1$ and n large then

$$4\frac{k^n-1}{k-1} + 1 \approx \frac{k^n}{k-1}$$

and thus

$$2kn(k-1) + 1 < k^n$$

which allows to conclude that in this case the second algorithm is worst than the first.

SOLUTION 5.5

For $a = 1/8$ the transfer function is

$$H(z) = \frac{1}{1 - \frac{1}{8}z^{-1}} = \frac{y(z)}{r(z)}$$

thus in time we have

$$y(k) = \frac{1}{8}y(k-1) + r(k).$$

For a step $r(k) = 1$ for $k \geq 0$ thus

$$y(k) = \frac{1}{8}y(k-1) + 1 \quad k \geq 0.$$

The data representation is the following

$\pm$	2	1		1/2	1/4	1/8	1/16	1/32
$\square$	$\square$	$\square$	$\bullet$	$\square$	$\square$	$\square$	$\square$	$\square$
Sign	Integer			$\longleftarrow$		Fraction		$\longrightarrow$

The steady state value is

$$\lim_{z\to 1} \frac{z-1}{z} \frac{1}{1 - \frac{1}{8}z^{-1}} \frac{z}{z-1} = \frac{1}{1 - 0.125} = 1.14285714.$$

If we compute the steady-state value when truncation and roundoff are considered, and we have limited word-length we obtain

iteration	$y(k)$ with truncation	$y(k)$ with roundoff	exact
0	1	1	1.0
1	$001.00100b$	$001.00100b$	$1 + \frac{1}{8}$
2	$001.00100b$	$001.00101b$	$1 + \frac{1}{8}(1 + \frac{1}{8}) = 1 + \frac{1}{8} + \frac{1}{64} = 1 + \frac{9}{64}$
3	$001.00100b$	$001.00101b$	$1 + \frac{1}{8} + \frac{1}{64} + \frac{1}{512} = 1 + \frac{73}{512}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	$001.00100b = 1.125$	$001.00101b = 1.15625$	1.14285714

which then gives the following steady-state values

$$\begin{aligned} y_{ss}^T &= 1.125 \quad \text{Truncation} \\ y_{ss}^R &= 1.15625 \quad \text{Roundoff} \\ y_{ss}^E &= 1.14285714 \quad \text{Exact.} \end{aligned}$$

If we compute the relative error for the truncation an round-off we have

- Truncation: $\dfrac{y_{ss}^T - y_{ss}^E}{y_{ss}^E} \approx -1.6\%$
- Roundoff: $\dfrac{y_{ss}^R - y_{ss}^E}{y_{ss}^E} \approx +1.2\%$

Part II

Event-triggered control

Solutions to event-based control and real-time systems

SOLUTION 6.1

(a) The closed-loop system is given by

$$\dot{x} = x + u = x - 2x = -x.$$

Consider the Lyapunov function candidate $V(x) = x^2$. Clearly $V(x) \geq 0$ and its time derivative along the closed-loop system is

$$\begin{aligned}\dot{V} &= 2x\,\dot{x} \\ &= 2x\,(-2x) = -4x^2 \leq 0,\end{aligned}$$

which means that $\dot{V} < 0$ when $x \neq 0$ and $\dot{V} = 0$ when $x = 0$. By Lyapunov's stability theorem, the closed-loop systemconverges to the origin asymptotically.

(b) Given the periodic control input, for $t \in [kh, kh+h)$, system evolves by

$$\begin{aligned}x(t) &= e^{t-kh}x(kh) + (\int_0^{t-kh} e^s ds)\,u(kh) \\ &= e^{t-kh}x(kh) + (e^{t-kh} - 1)\,u(kh)\end{aligned}$$

When $t = kh + h$, we get

$$\begin{aligned}x(kh+h) &= e^h x(kh) + (e^h - 1)\,u(kh) \\ &= e^h x(kh) + (e^h - 1)(-2\,x(kh)) \\ &= (2 - e^h)\,x(kh).\end{aligned}$$

Thus $x(kh) = (2 - e^h)^k\,x(0)$, $k \in \mathbb{N}$. It means that for the closed-loop system to converge to the origin, the following condition should hold

$$|2 - e^h| < 1\,.$$

Thus $-1 < 2 - e^h < 1 \Rightarrow 1 < e^h < 3 \Rightarrow 0 < h < \ln 3$, i.e., $h_{\max} = \ln 3 \approx 1.1s$.

(c) Denote by $e(t) = x(t_k) - x(t)$, $t \in [t_k, t_{k+1})$. Then $u(t) = -2(x(t) + e(t))$, $\forall t \geq 0$. Consider the same Lyapunov function candidate as before $V(x) = x^2$. Its time derivative along the closed-loop system is

$$\begin{aligned}\dot{V} &= 2x\,\dot{x} \\ &= 2x\,(-x - 2e)) = -2(x^2 + 2x\,e).\end{aligned}$$

Since $x, e \in \mathbb{R}$, $x^2 = |x|^2$ and $xe \geq -|x||e|$, it gives that

$$\begin{aligned}\dot{V} &\leq -2\,(|x|^2 - 2|x||e|) \\ &= -2|x|\,(|x| - 2|e|).\end{aligned}$$

By enforcing $|e(t)| < 0.5|x(t)|$, $\forall t > 0$, $\dot{V} \leq 0$ and $\dot{V} = 0$ only when $x = 0$. The same convergence result can be obtained as in Problem 1.

For $t \in [t_k, t_{k+1})$ the closed-loop system evolves by

$$\begin{aligned}x(t) &= e^{t-t_k}x(t_k) + (\int_0^{t-t_k} e^s ds)\,u(t_k) \\ &= e^{t-t_k}x(t_k) + (e^{t-t_k} - 1)\,u(t_k) \\ &= e^{t-t_k}x(t_k) + (e^{t-t_k} - 1)\,(-2x(t_k)) \\ &= (2 - e^{t-t_k})x(t_k).\end{aligned}$$

Thus for $t \in [t_k, t_{k+1})$, the state error is given by

$$\begin{aligned} e(t) &= x(t_k) - x(t) \\ &= \frac{1}{2 - e^{t-t_k}} x(t) - x(t) \\ &= \frac{e^{t-t_k} - 1}{2 - e^{t-t_k}} x(t). \end{aligned}$$

To enforce $|e(t)| < 0.5|x(t)|$, $\forall t > 0$, we get $\forall t \in [t_k, t_{k+1})$

$$\begin{aligned} |e(t)| &= |\frac{e^{t-t_k} - 1}{2 - e^{t-t_k}} x(t)| < 0.5|x(t)| \\ &\Rightarrow |\frac{e^{t-t_k} - 1}{2 - e^{t-t_k}}| < 0.5 \\ &\Rightarrow 2|e^{t-t_k} - 1| < |2 - e^{t-t_k}| \\ &\Rightarrow |e^{t-t_k}| < 4/3 \\ &\Rightarrow |e^{t_{k+1}-t_k}| < 4/3 \\ &\Rightarrow t_{k+1} - t_k < \ln 4/3. \end{aligned}$$

That is to say, the maximal sampling interval is upper-bounded by $\ln 4/3 \approx 0.3s$.

(d) In order to implement the controller with event-based sampling and ZOH, the following steps should be followed:

(a) **Monitor**: keep monitoring $x(t)$, and check if $|e(t)| < |x(t)|$ holds, where $e(t) = x(t) - x(t_k)$. If so, go to 'sample' step.

(b) **Sample**: sample $x(t)$ to obtain $x(t_k)$. Save $x(t_k)$, go to 'update' step.

(c) **Update**: set $u = -2x(t_k)$, go back to 'Monitor'.

(d) **ZOH**: hold u.

While to implement the periodic controller, only time t needs to be monitored and the following steps should be followed:

(a) **Monitor**: keep monitoring t, and check if $t = kh$ holds. If so, go to 'sample' step.

(b) **Sample**: sample $x(t)$ to obtain $x(kh)$, go to 'update' step.

(c) **Update**: set $u = -2x(kh)$, go back to 'Monitor'.

(d) **ZOH**: hold u.

SOLUTION 6.2

(a) The closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= (A + BK)\, x(t) \\ &= \underbrace{\begin{bmatrix} -1 & 0 \\ -2 & -3 \end{bmatrix}}_{A_{cl}} x(t). \end{aligned}$$

The eigenvalues of the matrix A_{cl} are $\lambda_1 = -1$ and $\lambda_2 = -3$, which implies that the closed-loop system is asymptotically stable.

(b) According to the definition of the state error $e(t)$, the control signal can be rewritten as

$$u(t) = K\big(x(t) + e(t)\big),$$

so the closed loop equation of the system becomes

$$\begin{aligned}\dot{x}(t) &= (A + BK)\, x(t) + BKe(t) \\ &= \begin{bmatrix} -1 & 0 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} -3 & -1 \\ -3 & -1 \end{bmatrix} e(t).\end{aligned}$$

(c) The time-derivative of the candidate Lyapunov function along the closed-loop system is given by

$$\begin{aligned}\dot{V} &= 3x_1\dot{x_1} + x_2\dot{x_2} \\ &= -3x_1^2 - 2x_1x_2 - 3x_2^2 - 9x_1e_1 - 3x_1e_2 - 3x_2e_1 - x_2e_2 \\ &\leq -2(x_1^2 + x_2^2) + 9|x_1||e_1| + 3|x_1||e_2| + 3|x_2||e_1| + |x_2||e_2| \\ &\leq -2\|x\|^2 + 16\|x\|\|e\|.\end{aligned}$$

By choosing the event triggering condition

$$\|e\| < \frac{1}{8}\|x\|,$$

we have $\dot{V} < 0$, which implies that the closed-loop system still converges to the origin.

SOLUTION 6.3

(a) The two tasks are not independent in this particular case however. We first compute the CPU utilization

$$U = \sum_{i=1}^{n} \frac{C_i}{T_i} = \frac{C_c}{T_c} + \frac{C_a}{T_a}$$

where the period $T_c = T_a = h$. The two tasks needs to be finished within the new sampling interval. Thus we have

$$U = \frac{0.1}{0.4} + \frac{0.2}{0.4} = 0.75$$

thus the CPU is utilized 75% of the time. We still do not know if it is possible to schedule the tasks so that they meet their deadlines. In order to do this we calculate the schedule length and we draw the schedule. The schedule length is

$$\text{lcm}\{T_a, T_c\} = 0.4$$

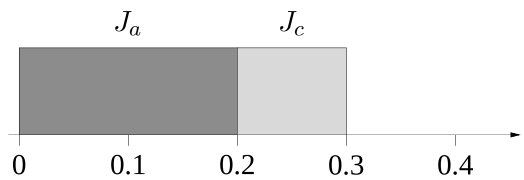


Figure 6.3.1: Schedule for the two tasks J_a and J_c of Problem 6.3.

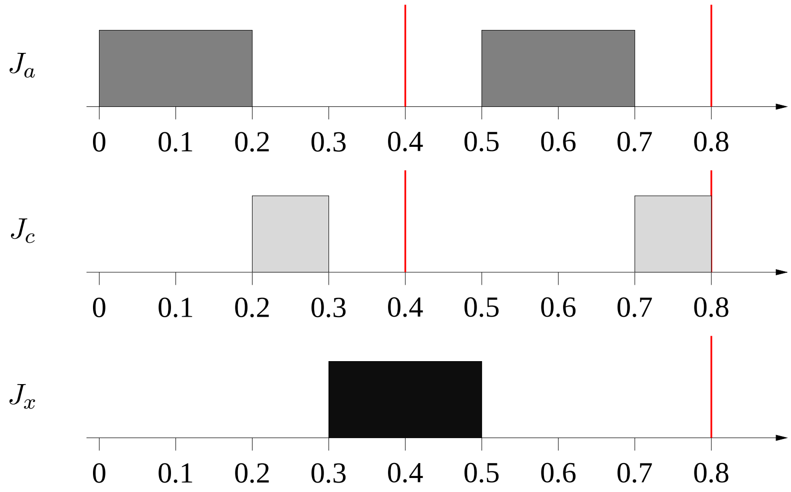


Figure 6.3.2: Schedule for the two tasks J_a, J_c and J_x of Problem 6.3.

(b) The utilization factor in this case is

$$U = \frac{0.1}{0.4} + \frac{0.2}{0.4} + \frac{0.2}{0.8} = 1$$

Thus the CPU will be fully occupied. In order to see if the tasks are schedulable we draw the schedule. In this case the schedule length is

$$lcm\{T_a, T_c, T_x\} = 0.8.$$

Notice that J_x starts after 3 time steps since the release time is $r_x = 0.3$. The worst case response time for the control task J_c is 0.4 as is can be seen from Figure 6.3.2.

SOLUTION 6.4

In a closed loop with a delay we know that the stability is guaranteed if

$$\left| \frac{P(e^{i\omega})C(e^{i\omega})}{1 + P(e^{i\omega})C(e^{i\omega})} \right| < \frac{1}{N|e^{i\omega} - 1|} \qquad \omega \in [0, \pi]$$

where $N = \lceil R_c/h \rceil$, with R_c the worst case response time of the control task J_c caused by the high priority task.

From the previous equation we have immediately that

$$N < \frac{|1 + P(e^{i\omega})C(e^{i\omega})|}{|e^{i\omega} - 1||P(e^{i\omega})C(e^{i\omega})|}$$

and this should hold for all $\omega \in [0, \pi]$. We have that the magnitude of the closed loop transfer function is

$$\left| \frac{P(e^{i\omega})C(e^{i\omega})}{1 + P(e^{i\omega})C(e^{i\omega})} \right| = 3\frac{|10e^{i\omega} - 1|}{|130e^{i\omega} - 103|}$$

and thus we have that

$$N < \frac{1}{3}\frac{|130e^{i\omega} - 103|}{|(e^{i\omega} - 1)(10e^{i\omega} - 1)|}$$

for all $\omega \in [0, \pi]$. This means that

$$N < \min_{\omega \in [0,\pi]} \frac{1}{3}\frac{|130e^{i\omega} - 103|}{|(e^{i\omega} - 1)(10e^{i\omega} - 1)|}$$

Since the right hand side of the previous inequality can be regarded as the magnitude of a transfer function with one zero and two poles on the positive real axis, it is clear the minimum is at $\omega = \pi$.

Thus we have that

$$N < \frac{1}{3} \frac{|130e^{i\omega} - 103|}{|(e^{i\omega} - 1)(10e^{i\omega} - 1)|}\bigg|_{\omega=\pi} \approx 3.53\,.$$

Since N is an integer then we have that $N = 3$. Thus we have that $R_c = 2N = 6$.

We know that the worst case computation time for the control task is $C_c = 1$ with period $T_c = h = 2$. and that the high priority task is not released until time $t = 2$. The worst case response time for the control task is given by

$$R_c = C_c + C_I$$

which than gives $C_I = 5$. The schedule in this case looks as in Figure 6.4.1. As we can notice the control task

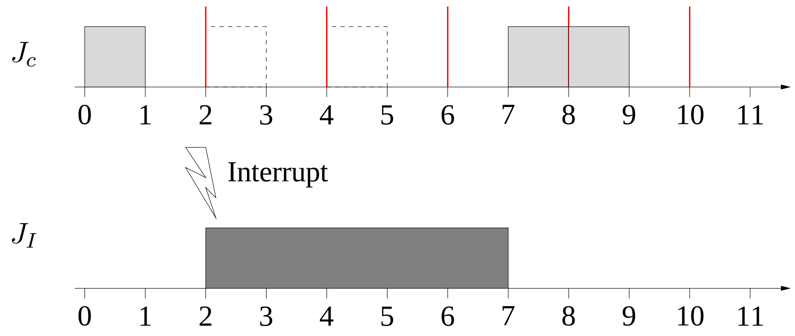


Figure 6.4.1: Schedule for the control task J_c and the task handling the interrupt, of Problem 6.4.

misses its deadlines at $t = 4$ and $t = 6$. The control task is then delayed and the worst case response time is exactly $R_c = 6$, which represents the maximum delay of the control task.

SOLUTION 6.5

(a) The situation is depicted in Figure 6.5.1. Let us consider the single instances.

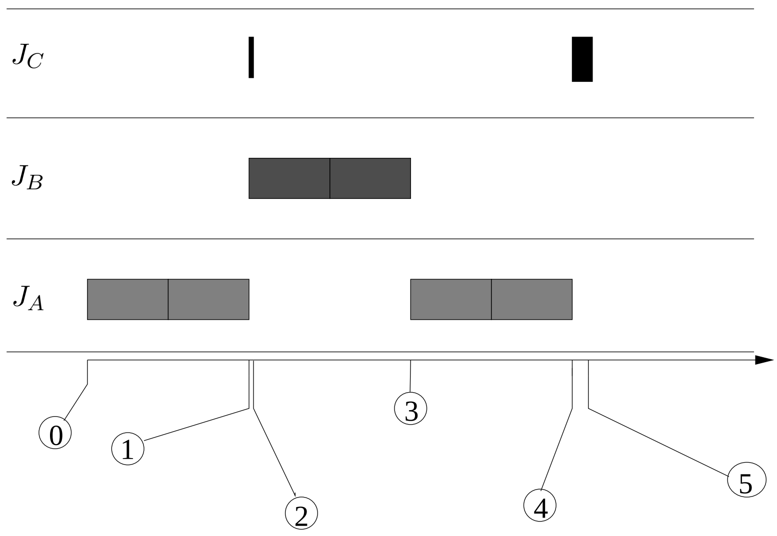


Figure 6.5.1: Priority inversion. Problem 6.5

0) The task J_A is released and its starts its computations. The bus is locked.

1) The task J_C requires to be executed. Since it has high priority the CPU preempts the task J_A.

2) Since J_C needs to access the bus, and this resource is occupied by the task J_A the task J_C is stopped. At the same time task J_B asks for CPU time, and since J_C is stopped and it has higher priority of J_A, J_B can be executed. The execution of J_B prevents J_A to release the bus.
3) The task J_B with medium priority has finished to be executed and the CPU is given to the task J_A.
4) Task J_A finishes to write on the bus, and it releases it.
5) Task J_C finally can use the bus and it is scheduled by the CPU.

What we can see from this particular situation is that a high priority task as J_C is blocked by a low priority task. We have a situation called *priority inversion*. The response time for the high priority task J_C in this case is $R_C = 4.1$.

(b) A possible way to overcome the problem is to use the *priority inheritance protocol*. In this case the task J_A inherits the priority of the task which has higher priority and needs to use the blocked resource. The task J_A, thus acquires high priority and is able to release the resource as soon as possible, so that the high priority task J_C can be served as soon as possible. The priorities are set to the default one, once the bus is released.

SOLUTION 6.6

(a) Using the EDD algorithm the tasks are scheduled considering the deadlines. Those that have early deadline are scheduled first. From the table given in the problem we see that the order of scheduling is

$$J_1 \rightarrow J_5 \rightarrow J_3 \rightarrow J_4 \rightarrow J_2.$$

The schedule is shown in Figure 6.6.1. The maximum lateness is equal to -1 due to task J_4. The fact

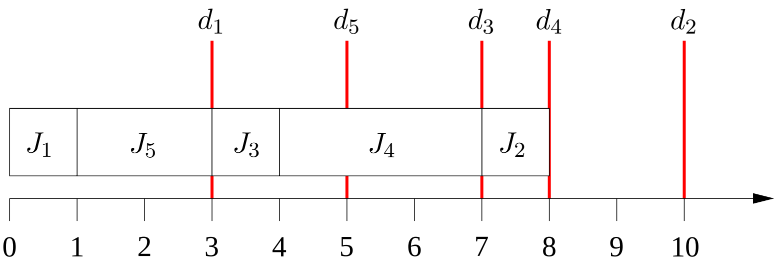


Figure 6.6.1: Jackson's scheduler

that the lateness is negative means that all the tasks meet their deadlines.

(b) Let σ be a schedule produced by any algorithm A. If A is different from EDD, then there exist two tasks J_a and J_b with $d_a < d_b$ with J_b scheduled before the task J_a. Now, let σ' be a schedule obtained from σ by exchanging J_a with J_b in the schedule σ, so that J_a is scheduled before J_b in σ'. If we exchange J_a with J_b in σ the maximum lateness cannot increase. In fact for the schedule σ we have that the maximum lateness is

$$L_{\max_{ab}} = f_a - d_a$$

as can be seen from Figure 6.6.2.

In σ' we have that

$$L'_{\max_{ab}} = \max(L'_a, L'_b).$$

We can have two possible cases

– $L'_a \geq L'_b$, then $L'_{\max_{ab}} = L'_a = f'_a - d_a$, and since $f'_a < f_a$ then we have $L'_{\max_{ab}} < L_{max_{ab}}$,

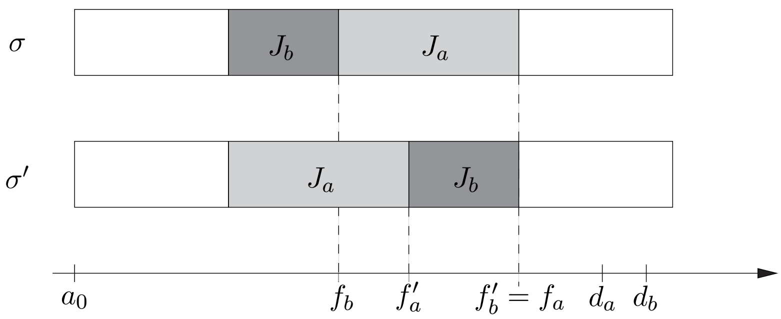


Figure 6.6.2: Jackson's scheduling - proof

– $L'_a \leq L'_b$, then $L'_{\max_{ab}} = L'_b = f'_b - d_b = f_a - d_b$, and since $d_a < d_b$ we have that $L'_{\max_{ab}} < L_{\max_{ab}}$.

Since in both cases $L'_{\max_{ab}} < L_{\max_{ab}}$, then we can conclude that interchanging the tasks J_a and J_b in σ we cannot increase the lateness of a set of tasks. By a finite number of such transpositions, σ can be transformed in σ_{EDD}. Since at each step the lateness cannot increase, σ_{EDD} is optimal.

SOLUTION 6.7

(a) By writing the system in vector form we have that $\dot{x} = Ax + Bu$ where

$$A = \begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The characteristic equation is given by

$$\begin{aligned} \psi_o(\lambda) &= \det\left[\lambda I_2 - A\right] \\ &= \det\begin{bmatrix} \lambda - 3 & -1 \\ -5 & \lambda + 2 \end{bmatrix} \\ &= \lambda^2 - \lambda - 11. \end{aligned}$$

which has one root to the RHP, hence the system is *unstable*.

(b) By substituting the given controller in the system (S) we have the closed loop form

$$\dot{x}(t) = Ax(t) + B\left[Kx(t)\right] = (A + BK)x(t).$$

The closed loop characteristic polynomial is given by

$$\begin{aligned} \psi_c(\lambda) &= \det\left[\lambda I_2 - A - BK\right] \\ &= \det\left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix} - \begin{bmatrix} K_1 & K_2 \\ K_1 & K_2 \end{bmatrix}\right] \\ &= \det\begin{bmatrix} \lambda - K_1 - 3 & -K_2 - 1 \\ -K_1 - 5 & \lambda - K_2 + 2 \end{bmatrix} \\ &= \lambda^2 + \left(-K_1 - K_2 - 1\right)\lambda + \left(-3K_1 - 2K_2 - 11\right). \end{aligned}$$

The desired characteristic polynomial is given by $\psi_{\text{des}}(\lambda) = \lambda^2 + 6\lambda + 8 = 0$. Hence, the following holds

$$\begin{cases} -K_1 - K_2 - 1 = 6 \\ -3K_1 - 2K_2 - 11 = 8 \end{cases} \iff \boxed{\begin{cases} K_1 = -5 \\ K_2 = -2 \end{cases}}.$$

(c) We have that $u(t) = Kx(t_k) = K\left(e(t) + x(t)\right)$. By substituting it to the clossed loop system we get

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + BK\left[e(t) + x(t)\right] \\
&= (A + BK)\, x(t) + (BK)e(t) \\
&= \begin{bmatrix} -2 & -1 \\ 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} e(t)
\end{aligned}$$

or

$$\boxed{(S') : \begin{cases} \dot{x}_1(t) = -2x_1(t) - x_2(t) - 5e_1(t) - 2e_2(t) \\ \dot{x}_2(t) = -4x_2(t) - 5e_1(t) - 2e_2(t) \end{cases} \quad ,\ t \geq 0}.$$

(d) By taking the time derivative of the given Lyapunov function we have that

$$\begin{aligned}
\dot{V}(x) &= 2x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= 2x_1\left\{-2x_1 - x_2 - 5e_1 - 2e_2\right\} + x_2\left\{-4x_2 - 5e_1 - 2e_2\right\} \\
&= -4x_1^2 - 2x_1x_2 - 10x_1e_1 - 4x_1e_2 - 4x_2^2 - 5x_2e_1 - 2x_2e_2 \\
&= -4\left(x_1^2 + x_2^2\right) - 2x_1x_2 - 10x_1e_1 - 4x_1e_2 - 5x_2e_1 - 2x_2e_2 \\
&= -4\|x\|^2 - 2x_1x_2 - 10x_1e_1 - 4x_1e_2 - 5x_2e_1 - 2x_2e_2.
\end{aligned}$$

By using the elementary inequalities

$$\begin{gathered}
-2ab \leq a^2 + b^2, -ab \leq |a||b|,\ \forall\, a, b \in \mathbb{R} \\
|x_1| \leq \|x\|, |x_2| \leq \|x\|, |e_1| \leq \|e\|, |e_2| \leq \|e\|,\ \forall\, x, e \in \mathbb{R}^2.
\end{gathered}$$

we get

$$\begin{aligned}
\dot{V}(x) &\leq -4\|x\|^2 + x_1^2 + x_2^2 + 10|x_1||e_1| + 4|x_1||e_2| + 5|x_2||e_1| + 2|x_2||e_2| \\
&= -4\|x\|^2 + \|x\|^2 + 10|x_1||e_1| + 4|x_1||e_2| + 5|x_2||e_1| + 2|x_2||e_2| \\
&\leq -3\|x\|^2 + 10\|x\|\|e\| + 4\|x\|\|e\| + 5\|x\|\|e\| + 2\|x\|\|e\| \\
&= -3\|x\|^2 + 21\|x\|\|e\| \\
&= -3\|x\|\left(\|x\| - 7\|e\|\right).
\end{aligned}$$

Thus, for $\|x\| - 7\|e\| > 0 \Leftrightarrow \boxed{\|e\| < \frac{1}{7}\|x\|}$ the system remains asymptotically stable.

SOLUTION 6.8

(Exam March 2017, Problem 4)

(a) We have

$$\begin{aligned}
\dot{x}_1 &= \dot{p}_2 - \dot{p}_1 = u_2 - u_1 \\
\dot{x}_2 &= \dot{p}_3 - \dot{p}_2 = u_3 - u_2.
\end{aligned}$$

Therefore,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

(b) We have

$$\begin{aligned}u_1 =& p_2 - p_1 = x_1,\\ u_2 =& p_3 - p_2 = x_2,\\ u_3 =& p_1 - p_3 = -x_1 - x_2.\end{aligned}$$

Therefore

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

(c) Using the results in (a) and (b), we have

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

(d) Using the suggested Lyapunov function, we find

$$\dot{V}(x) = -x_1^2 - 2x_2^2,$$

therefore, $x(t)$ asymptotically converges to zero, which implies that the robots asymptotically meet at the same place.

(e) With the sampled measurements, we have $\dot{x} = BK(x - e)$, or equivalently

$$\begin{aligned}\dot{x}_1 =& -x_1 + x_2 + e_1 - e_2,\\ \dot{x}_2 =& -x_1 - 2x_2 + e_1 + 2e_2.\end{aligned}$$

(f) Using the same Lyapunov function, we have

$$\dot{V}(t) = -x(t)^T Q x(t) + x(t)^T R e(t),$$

where

$$Q = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}, \quad R = \begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix}.$$

Therefore, $\dot{V}(t)$ is guaranteed if

$$-\lambda_Q \|x(t)\|^2 + \|x(t)\|\lambda_R\|e(t)\| \le 0,$$

where λ_Q is the smallest eigenvalue of Q and λ_R is the 2-norm of R (maximum singular value). Solving the inequality for $\|e(t)\|$ gives

$$\|e(t)\| \le \frac{\lambda_Q}{\lambda_R}\|x(t)\|.$$

As for numerical values, we have immediately $\lambda_Q = 1$, while a quick calculation gives $\lambda_R = 2.3028$. Therefore, we conclude

$$\alpha = 0.4343.$$

SOLUTION 6.9

(Exam June 2017, Problem 1)

(a) Substituting $u(t) = -p(t) + \tilde{p}(t) - v(t) + \tilde{v}(t)$ in the open-loop dynamics, we obtain

$$\begin{aligned}\dot{p}(t) =& v(t),\\ \dot{v}(t) =& -p(t) + \tilde{p}(t) - v(t) + \tilde{v}(t).\end{aligned}$$

(b) Taking the derivatives and collecting similar terms, we obtain

$$\dot{V}(t) = -2v(t)^2 - 2p(t)^2 + 4v(t)\tilde{p}(t) + 4v(t)\tilde{v}(t) + 2p(t)\tilde{p}(t) + 2p(t)\tilde{v}(t).$$

(c) We have

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

(d) We have

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$
$$R = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}.$$

(e) We have $\dot{V}(t) = -x(t)^\mathsf{T} Q x(t) + x(t)^\mathsf{T} R \tilde{x}(t) \leq -\lambda_{\min}(Q)\|x\|^2 + \sigma_{\max}(R)\|x\|\|\tilde{x}\| = -2\|x\|^2 + 2\sqrt{10}\|x\|\|\tilde{x}\|$. At the same time, $V(t) = x(t)^\mathsf{T} P x(t) \leq \lambda_{\max}(P)\|x\|^2 = \frac{5+\sqrt{5}}{2}\|x\|^2$. Therefore, the condition $\dot{V}(t) \leq -\frac{2}{5+\sqrt{5}} V(t)$ is satisfied if $-2\|x(t)\|^2 + 2\sqrt{10}\|x(t)\|\|\tilde{x}(t)\| \leq -\|x(t)\|^2$, which is equivalent to $\|\tilde{x}(t)\|^2 \leq \frac{1}{2\sqrt{10}}\|x(t)\|$.

(f) From the previous points, we know that if we enforce the updates as

$$t_{k+1} = \inf\{t > t_k : \ \|\tilde{x}(t)\| \geq \alpha\|x(t)\|\},$$

then $V(t)$ converges exponentially to zero, with convergence rate at least as large as $\frac{2}{5+\sqrt{5}}$.

Solutions to real-time scheduling

SOLUTION 7.1

Notice that the four assumptions under which the schedulability test can be use are satisfied. We can remember here the assumptions

A1 the instances of the task J_i are regularly activated at constant rate,

A2 all instances of the periodic task J_i have the same worst case execution time C_i,

A3 all instances of a periodic task J_i have the same relative deadline, which is equal to the period,

A4 all tasks are independent i.e., there are no precedence relations and no resource constraints.

The schedulability test gives

$$U = \frac{1}{4} + \frac{2}{6} + \frac{3}{10} = 0.8833 \nleq 0.7798 = 3(2^{1/3} - 1)$$

thus we cannot conclude if it is possible to schedule the three tasks. We need to use the necessary condition, which ensures that the tasks are schedulable if the worst-case response time is less than the deadline. The priority is assigned so that J_1 has highest priority, J_2 medium priority and J_3 lowest priority. The analysis yields

$$R_1 = C_1 = 1 \leq D_1 = 4$$

for task J_2 we have

$$\begin{aligned}
R_2^0 &= C_2 = 2 \\
R_2^1 &= C_2 + \left\lceil \frac{R_2^0}{T_1} \right\rceil C_1 = 2 + 1 = 3 \\
R_2^2 &= C_2 + \left\lceil \frac{R_2^1}{T_1} \right\rceil C_1 = 2 + 1 = 3 \leq D_2 = 6
\end{aligned}$$

for task J_3 we have

$$\begin{aligned}
R_3^0 &= C_3 = 3 \\
R_3^1 &= C_3 + \left\lceil \frac{R_3^0}{T_1} \right\rceil C_1 + \left\lceil \frac{R_3^0}{T_2} \right\rceil C_2 = 3 + 2 + 1 = 6 \\
R_3^2 &= C_3 + \left\lceil \frac{R_3^1}{T_1} \right\rceil C_1 + \left\lceil \frac{R_3^1}{T_2} \right\rceil C_2 = 3 + 2 + 2 = 7 \\
R_3^3 &= C_3 + \left\lceil \frac{R_3^2}{T_1} \right\rceil C_1 + \left\lceil \frac{R_3^2}{T_2} \right\rceil C_2 = 3 + 2 + 2 = 9 \\
R_3^4 &= C_3 + \left\lceil \frac{R_3^3}{T_1} \right\rceil C_1 + \left\lceil \frac{R_3^3}{T_2} \right\rceil C_2 = 3 + 3 + 4 = 10 \\
R_4^5 &= C_3 + \left\lceil \frac{R_3^4}{T_1} \right\rceil C_1 + \left\lceil \frac{R_3^4}{T_2} \right\rceil C_2 = 3 + 3 + 4 = 10 \leq D_3 = 10
\end{aligned}$$

Thus the three tasks are schedulable with rate monotonic. The schedule length is given by

$$\text{lcm}(T_1, T_2, T_{,3}) = 60$$

The schedule is shown in figure 7.1.1.

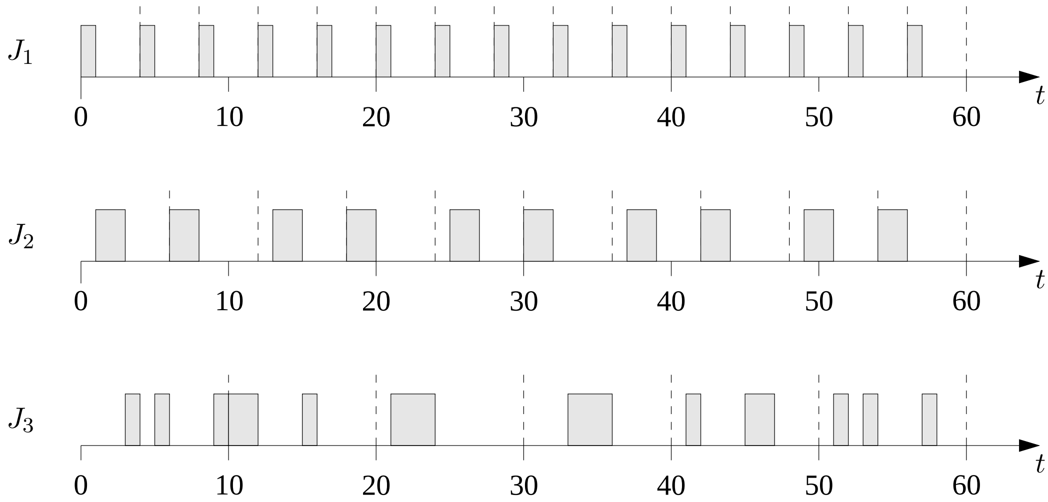


Figure 7.1.1: Rate monotonic scheduling for the tasks J_1, J_2 and J_3, in Problem 7.1

SOLUTION 7.2

The CPU utilization is

$$U = \frac{1}{4} + \frac{2}{6} + \frac{3}{10} = 0.8833 \leq 1$$

thus the schedulability test for EDF ensures that the three tasks are schedulable. The schedule length, as in Problem 7.1, is

$$\text{lcm}(T_1, T_2, T_{,3}) = 60.$$

The schedule is shown in figure 7.2.1.

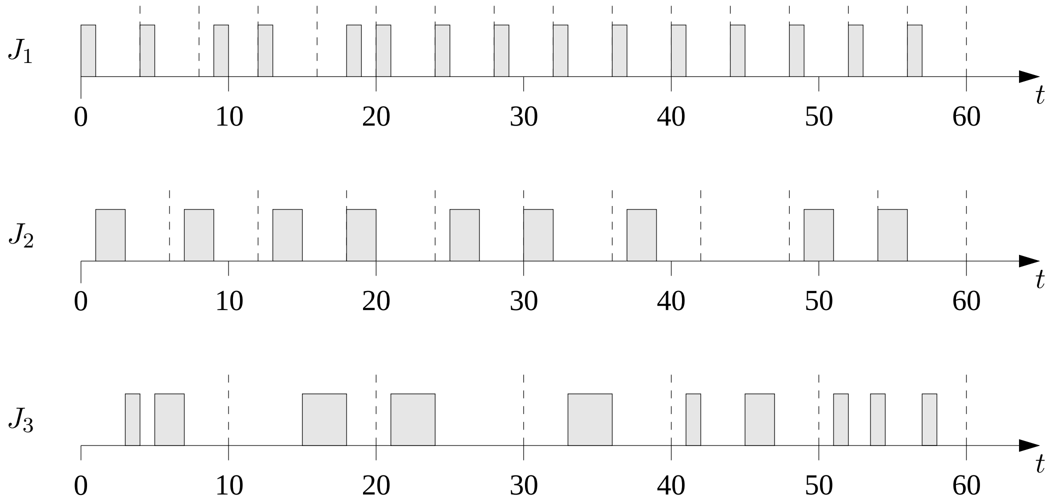


Figure 7.2.1: EDF scheduling for the tasks J_1, J_2 and J_3, in Problem 7.2

SOLUTION 7.3

To decide if the set of tasks are schedulable we consider the utilization factor:

$$U = \frac{1}{3} + \frac{2}{4} + \frac{1}{7} = \frac{14 + 21 + 6}{42} = \frac{41}{42} \approx 0.98$$

The set is schedulable with RM if $U \leq n(2^{1/n} - 1)$ where n is the number of tasks. In this case, $n = 3 \rightarrow n(2^{1/n} - 1) \approx 0.78$. The condition is not satisfied and hence we can not guarantee that the set of tasks is schedulable with RM. However, it may still be! The condition is sufficient not necessary. To make sure we have to try it! By doing so we conclude that it is schedulable with RM.

The set is schedulable with EDF if and only if $U \leq 1$. This condition is satisfied and hence it is schedulable.

SOLUTION 7.4

(a) The CPU utilization is

$$U = \frac{1}{4} + \frac{2}{5} + \frac{3}{10} = 0.95$$

The RM algorithm assigns the highest priority to J_1, middle to J_2 and the lowest to J_3. The schedule length is given by

$$\text{lcm}(4, 5, 10) = 20.$$

The RM scheduling algorithm assigns the highest priority to the control task J_1 which then has worst-case response time of $R_1 = 1$. In the case of the EDF we need to draw the schedule in order to compute R_1. The schedule is shown in Figure 7.4.1. The EDF gives a worst-case response time of $R_1 = 2$ as

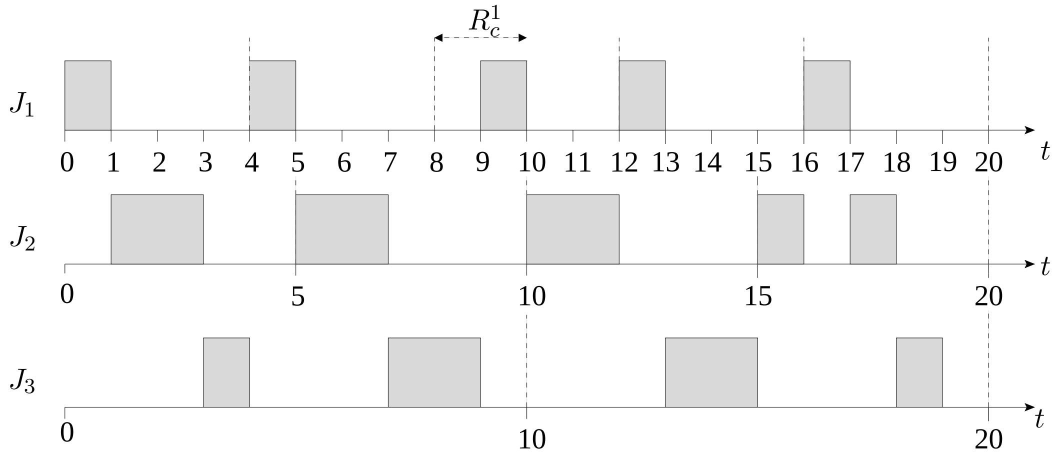


Figure 7.4.1: EDF scheduling for the tasks J_1, J_2 and J_3 in Problem 7.4.

is shown in Figure 7.4.1. With the EDF algorithm is possible to schedule tasks that are not schedulable with RM, but we do not have anymore "control" on the response time of some tasks.

(b) In order to have $R_2 = 2$ we need to assign the highest priority to the task 2. We then give to J_1 middle priority and J_3 the lowest. The tasks meet their deadline if the worst-case response time is less than the deadline. In this case we have

$$R_2 = C_2 = 2 \leq D_2 = 5$$

for task J_1 we have

$$\begin{aligned}
R_1^0 &= C_1 = 1 \\
R_1^1 &= C_1 + \left\lceil \frac{R_1^0}{T_2} \right\rceil C_2 = 1 + 2 = 3 \\
R_1^2 &= C_1 + \left\lceil \frac{R_1^1}{T_2} \right\rceil C_2 = 1 + 2 = 3 \leq D_1 = 4
\end{aligned}$$

for task J_3 we have

$$\begin{aligned}
R_3^0 &= C_3 = 3\\
R_3^1 &= C_3 + \left\lceil \frac{R_3^0}{T_1} \right\rceil C_1 + \left\lceil \frac{R_3^0}{T_2} \right\rceil C_2 = 3+1+2 = 6\\
R_3^2 &= C_3 + \left\lceil \frac{R_3^1}{T_1} \right\rceil C_1 + \left\lceil \frac{R_3^1}{T_2} \right\rceil C_2 = 3+2+4 = 9\\
R_3^3 &= C_3 + \left\lceil \frac{R_3^2}{T_1} \right\rceil C_1 + \left\lceil \frac{R_3^2}{T_2} \right\rceil C_2 = 3+3+4 = 10\\
R_3^4 &= C_3 + \left\lceil \frac{R_3^3}{T_1} \right\rceil C_1 + \left\lceil \frac{R_3^3}{T_2} \right\rceil C_2 = 3+3+4 = 10 \leq D_3 = 10
\end{aligned}$$

Thus the three tasks are schedulable. The schedule is shown in Figure 7.4.2

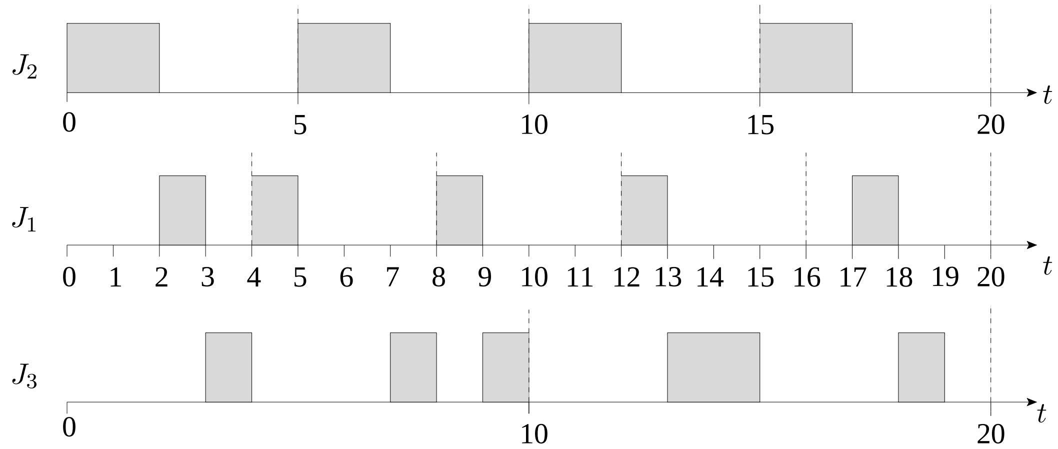


Figure 7.4.2: Fixed-priority scheduling for the tasks J_1, J_2 and J_3, in Problem 7.4. J_2 in this case has the highest priority.

SOLUTION 7.5

(a) The tasks will meet their deadlines since the schedulability condition for RM scheduling is fulfilled:

$$U = \sum_{i=1}^{2} \frac{C_i}{T_i} = \frac{C_1}{T_1} + \frac{C_2}{T_2} = \frac{7}{12} < 2\big(2^{1/2} - 1\big) \approx 0.8$$

The schedule length is equal to lcm$(T1, T2) = 12$. The time evolution is shown in Figure 7.5.1.

(b) The worst-case response time for the higher priority tasks is given by the worst-case computation time. Thus $R_1 = C_1 = 1$. For the second task we the worst-case response time is computed by

$$R_2 = C_2 + \left\lceil \frac{R2}{T1} \right\rceil C_1.$$

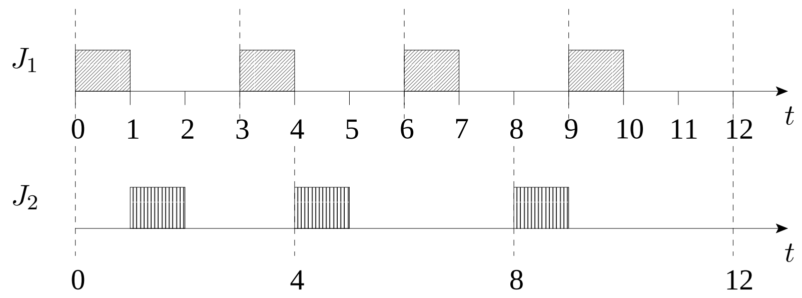


Figure 7.5.1: Rate monotonic scheduling for the tasks J_1 and J_2, in Problem 7.5

In order to solve the equation we need to use an iterative procedure. Let $R_2^0 = 0$, then

$$R_2^1 = C_2 + \left\lceil \frac{R_2^0}{T1} \right\rceil C_1 = C_2 = 1$$
$$R_2^2 = C_2 + \left\lceil \frac{R_2^1}{T1} \right\rceil C_1 = C_2 + C_1 = 2$$
$$R_3^2 = C_2 + \left\lceil \frac{R_2^2}{T1} \right\rceil = C_2 + C_1 = 2.$$

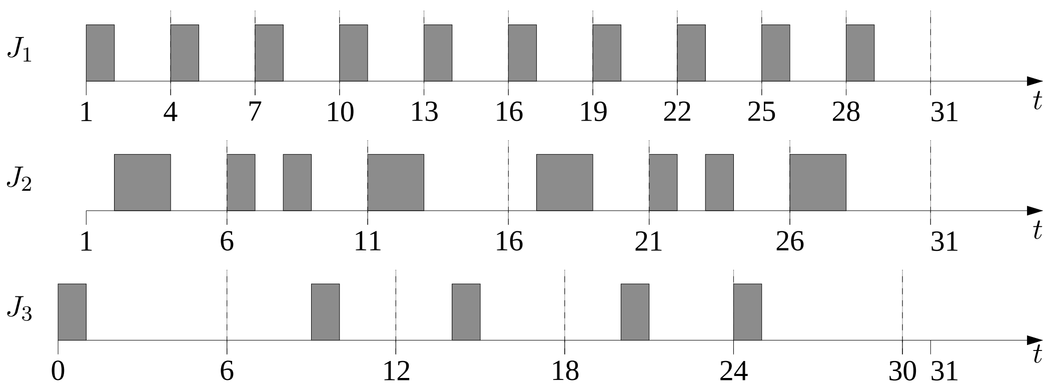
Thus $R_2 = 2$. This agrees with the time evolution plotted in (a).

SOLUTION 7.6

Since the tasks have an offset O_i we cannot use the response time formula for rate-monotonic since it assumes that all tasks are released simultaneously at time 0. We need to solve the problem drawing the schedule.

(a) In order to draw the schedule for the three tasks we need to compute the schedule length. The schedule length is the least common multiple (l.c.m.) of the task periods plus the offset. The l.c.m. is 30, thus the schedule length is 30+1=31.

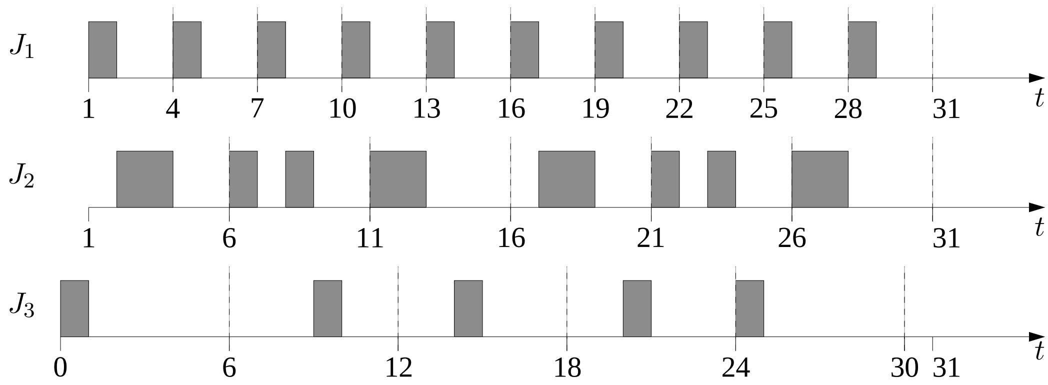
(b) *Rate-monotonic*
The rate-monotonic assigns the priority so that J_1 has the highest priority, J_2 medium priority and J_3 the lowest.



Thus the worst-case response time for task J_2 is $R_2 = 3$.

(c) *Deadline-monotonic*
The deadlines agree by assumption with the periods, so deadline-monotonic scheduling is identical to rate-monotonic scheduling. Thus the worst-case response time for task J_2 is $R_2 = 3$.

(d) *Earliest-deadline-first*
The schedule is shown below.



The worst-case response time for J_2 is $R_2 = 3$.

SOLUTION 7.7

(a) The utilization factor

$$U = \sum_{i=1}^{3} \frac{C_i}{T_i} = \frac{C_1}{T_1} + \frac{C_2}{T_2} + \frac{C_3}{T_3} = \frac{1}{4} + \frac{2}{5} + \frac{1}{6} \approx 0.82 \nleq 3(2^{1/3} - 1) \approx 0.76.$$

so we derive the worst-case response times for all tasks. The priority assignment is such that J_1 has the highest priority, J_c intermediate priority and J_2 the lowest priority. Thus the analysis yields

$$\begin{aligned}
\Rightarrow\ & R_1 = C_1 = 1 \\
& R_c^0 = C_c = 2 \\
& R_c^1 = C_c + \left\lceil \frac{R_c^0}{T_1} \right\rceil C_1 = 3 \\
& R_c^2 = C_c + \left\lceil \frac{R_c^1}{T_1} \right\rceil C_1 = 3 \\
\Rightarrow\ & R_c = 3 \leq D_c = 5 \\
& R_2^0 = C_2 = 2 \\
& R_2^1 = C_2 + \left\lceil \frac{R_2^0}{T_1} \right\rceil C_1 + \left\lceil \frac{R_2^0}{T_c} \right\rceil C_c = 4 \\
& R_2^2 = C_2 + \left\lceil \frac{R_2^1}{T_1} \right\rceil C_1 + \left\lceil \frac{R_2^1}{T_c} \right\rceil C_c = 4 \\
\Rightarrow\ & R_2 = 4 \leq D_2 = 6.
\end{aligned}$$

Thus the set of task is schedulable with the ate monotonic algorithm, with the worst-case response time for the control task equal to $R_c = 3$.

(b) Since $\tau = R_c = 3 > h = 2$ then the sampled version controller in state space from is

$$\begin{aligned}
x(kh + h) &= \Phi x(kh) + \Gamma_0 y(kh - (d-1)h) + \Gamma_1 y(kh - dh) \\
u(kh) &= Cx(kh)
\end{aligned}$$

where $\tau = (d-1)h + \tau'$ with $\tau' \leq h$ and $d \in \mathbb{N}$. Thus if we choose $d = 2$ we get $\tau' = 1$. The state space description becomes

$$\begin{pmatrix} x(kh+h) \\ y(kh-h) \\ y(kh) \end{pmatrix} = \begin{pmatrix} \Phi & \Gamma_1 & \Gamma_0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(kh) \\ y(kh-2h) \\ y(kh-h) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} y(kh)$$

$$u(kh) = \begin{pmatrix} C & 0 & 0 \end{pmatrix} \begin{pmatrix} x(kh) \\ y(kh-2h) \\ y(kh-h) \end{pmatrix}$$

A sketch of a computer program is given below. Notice that we assume that initial values of $x(0)$, $y(-2h)$, $y(-h)$ are known.

Computer Code

```
y=array{y(-1),y(-2)};
nexttime = getCurrentTime();
k = 0;
while true do
   new_y=AD_conversion();
   x(k) = Φx(k − 1) + Γ1y(k − 3) + Γ0y(k − 2);
   u(k) = Cx(k);
   DA_conversion();
   y(k) =new_y;
   k = k + 1;
   nexttime = nexttime + h;
   sleepUntil(nexttime);
end while
```

(c) The set of tasks will be schedulable if the worst-case response time, R_i is less than the D_i for $i \in 1, 2, c$ In this case the control task has the highest priority, J_1 intermediate priority and J_2 the lowest priority. The analysis yields

$$\begin{aligned}
\Rightarrow R_c &= C_c = 2 \\
R_1^0 &= C_1 = 1 \\
R_1^1 &= C_1 + \left\lceil \frac{R_1^0}{T_c} \right\rceil C_c = 3 \\
R_1^2 &= C_1 + \left\lceil \frac{R_1^1}{T_c} \right\rceil C_c = 3 \\
\Rightarrow R_1 &= 3 \leq D_1 = 4 \\
R_2^0 &= C_2 = 1 \\
R_2^1 &= C_2 + \left\lceil \frac{R_2^0}{T_c} \right\rceil C_c + \left\lceil \frac{R_2^0}{T_1} \right\rceil C_1 = 4 \\
R_2^2 &= C_2 + \left\lceil \frac{R_2^1}{T_c} \right\rceil C_c + \left\lceil \frac{R_2^1}{T_1} \right\rceil C_1 = 4 \\
\Rightarrow R_2 &= 4 \leq D_2 = 6.
\end{aligned}$$

Thus the tasks are schedulable even if the task J_c is a high priority task. In doing this the delay τ is then equal to $R_c = 2$.

SOLUTION 7.8

The schedulability of periodic tasks can be guaranteed by evaluating the interference by the polling server on periodic execution. In the worst case, such an interference is the same as the one introduced by an equivalent periodic task having a period equal to T_s and computation time equal to C_s. Independently of the number of aperiodic tasks handled by the server, a maximum time equal to C_s is dedicated to aperiodic requests at each server period. The utilization factor of the polling server is $U_s = C_s/T_s$ and hence the schedulability of a periodic set with n tasks and utilization U_p can be guaranteed if

$$U_p + U_s \leq (n+1)(2^{1/(n+1)} - 1)$$

Thus we need to have

$$U_s \leq (n+1)(2^{1/(n+1)} - 1) - U_p = (n+1)(2^{1/(n+1)} - 1) - \sum_{i=1}^{n} \frac{C_i}{T_i}$$

which in this case is

$$U_s \leq 3(2^{1/3} - 1) - \frac{1}{5} - \frac{2}{8} \approx 0.3298 \quad \Rightarrow \quad U_s^{max} = 0.3298$$

SOLUTION 7.9

The tasks and the polling server can be scheduled since we computed the maximum utilization as

$$U = \sum_{i=1}^{2} \frac{C_i}{T_i} + \frac{C_s}{T_s} = \frac{1}{4} + \frac{2}{8} + \frac{1}{5} = 0.77 \leq 3(2^{1/3} - 1) = 0.78$$

The schedule is shown is Figure 7.9.1

SOLUTION 7.10

The polling server has period $T_s = 6$ thus has lower priority with respect to J_1 and J_2. The time evolution is given in the following Figure 7.10.1.
Note that the server is preempted both by J_1 and J_2, so the aperiodic task is finished to be served only after one period of the server i.e., $T_s = 6$. The server has lower priority, so it runs only after J_1 and J_2 have terminated. At time $t = 0$ the server is ready to be executed, but since no aperiodic tasks are waiting the server is suspended. At $t = 3$ the aperiodic task is requiring the CPU but it will need to wait until the polling server is reactivated i.e., until $t = T_s = 6$. At this time the server is ready to handle the request of the aperiodic task, but the J_1 and J_2 with higher priority preempt the server.

Solutions to models of computation: Discrete-event systems and Transition systems

SOLUTION 8.1

Let us define the following event alphabet

$$E = \{s_1, \bar{s}_1, s_2, \bar{s}_2, s_3, \bar{s_3}\}$$

where

$$\begin{aligned} &s_i \text{ is triggered when } S_i \text{ becomes } 1 \\ &\bar{s}_i \text{ is triggered when } S_i \text{ becomes } 0 \qquad i = 1, 2, 3. \end{aligned}$$

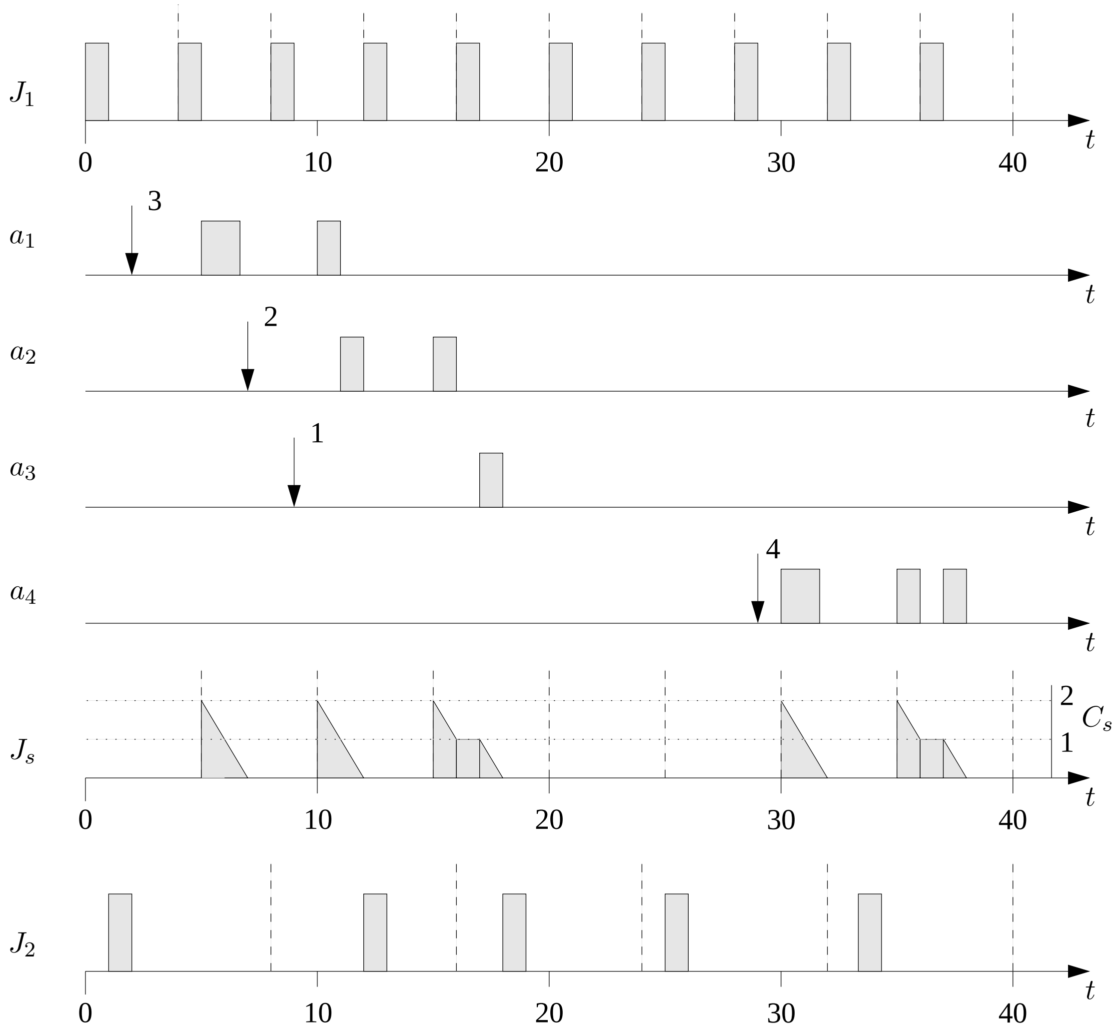


Figure 7.9.1: Polling server of Problem 7.9

We consider the following four possible states for the gate

$$Q = \{O, C, R, L\}$$

where O stands for ’opened’, C for ’closed’, R for ’raising’ and L for ’lowering’. The transition function is defined as following

$$\begin{aligned}
\delta(O, s_1) &= L \\
\delta(O, \bar{s}_1) &= O \\
\delta(L, \bar{s}_2) &= C \\
\delta(L, s_2) &= L \\
\delta(C, \bar{s}_3) &= C \\
\delta(C, s_3) &= R \\
\delta(R, s_2) &= O \\
\delta(R, \bar{s}_2) &= R \\
\delta(R, s_1) &= L.
\end{aligned}$$

We defined as initial state the state O, and as set of final states

$$Q_m = \{O\}.$$

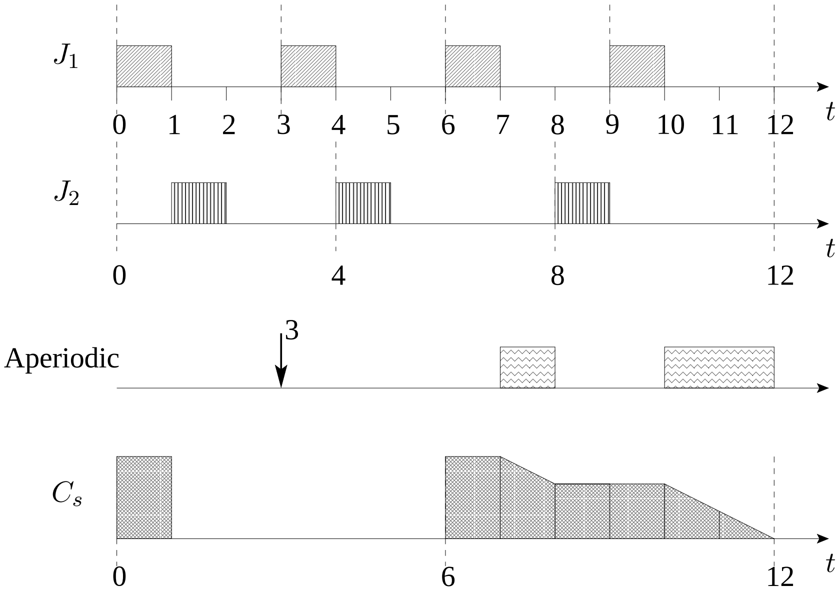


Figure 7.10.1: Polling server of Problem 7.5

Thus the automaton is

$$A = (Q, E, \delta, O, O)$$

where Q, E and δ are defined as before. In Figure 8.1.1 is shown the automaton.

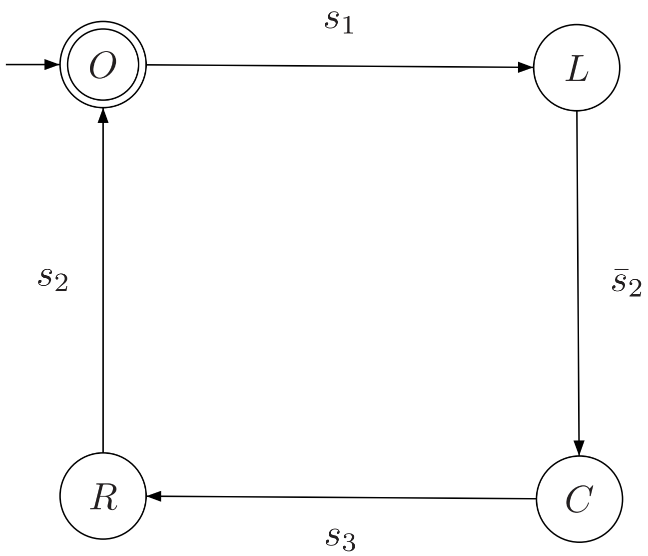


Figure 8.1.1: Control of a gate. Problem 8.1.

SOLUTION 8.2

We consider the following automaton as model of the vending machine

$$A = (Q, E, \delta, q_0, Q_m)$$

where we have the state-space

$$Q = \{0, 10, 20, 30, 40, 25, 35, 45, O\}$$

where the state O denote 'overfull'. The event alphabet is

$$E = \{\mathbb{D}, \mathbb{Q}\}.$$

The initial state is $q_0 = 0$ and the set of final states is $Q_m = \{45\}$. The transition map is

$$\begin{aligned}
\delta(0, \mathbb{D}) &= 10 \\
\delta(0, \mathbb{Q}) &= 25 \\
\delta(10, \mathbb{D}) &= 20 \\
\delta(10, \mathbb{Q}) &= 35 \\
\delta(20, \mathbb{D}) &= 30 \\
\delta(20, \mathbb{Q}) &= 45 \\
\delta(25, \mathbb{D}) &= 35 \\
\delta(25, \mathbb{Q}) &= 0 \\
\delta(30, \mathbb{D}) &= 40 \\
\delta(30, \mathbb{Q}) &= 0 \\
\delta(35, \mathbb{D}) &= 45 \\
\delta(35, \mathbb{Q}) &= 0 \\
\delta(40, \mathbb{Q}) &= 0 \\
\delta(40, \mathbb{D}) &= 0.
\end{aligned}$$

The automaton is shown in Figure 8.2.1. We can compute the marked language by inspection

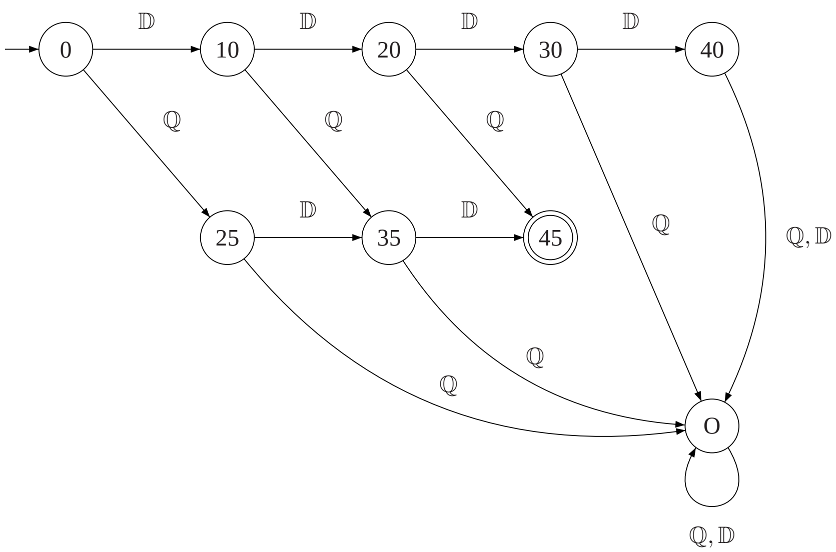


Figure 8.2.1: Automaton A of Problem 8.2.

$$L_m(A) = \{\mathbb{D}\mathbb{D}\mathbb{Q}, \mathbb{D}\mathbb{Q}\mathbb{D}, \mathbb{Q}\mathbb{D}\mathbb{D}\}.$$

We notice directly form Figure 8.2.1 that the machine will not dispense soda if the wrong amount of money is inserted. In that case we can reach the states 30 or 40 or O from where is not possible to reach the state 45 or if we reach the states 25 or 35 and we overpay we will not get a soda. Notice that we can continue to insert coins and the soda will never be dispensed i.e., we have a livelock.

In order to prove it formally we can notice that

$$\mathbb{D}\mathbb{D}\mathbb{D} \in L(A)$$

but clearly

$$\mathbb{DDD} \notin L_m(A).$$

Thus the automaton blocks.

SOLUTION 8.3

In order to compute the marked language and generated language we consider the following notation. Let R_{ij}^k be the set of all strings x such that $\delta(q_i, x) = q_j$, and if $\delta(q_i, y) = q_\ell$, for any y that is prefix (initial segment) of x or ϵ, then $\ell \leq k$. That is, R_{ij}^k is the set of all strings that take the finite automaton from state q_i to state q_j without going through any state numbered higher than k. Note that by "going through a state" we mean both entering and leaving that state. Thus i or j may be greater than k. It is possible to define R_{ij}^k recursively

$$R_{ij}^k = R_{ik}^{k-1}\big(R_{kk}^{k-1}\big)R_{kj}^{k-1} \cup R_{ij}^{k-1}$$

and

$$R_{ij}^0 = \begin{cases} \{a|\delta(q_i, a) = q_i\} & \text{if } i = j \\ \{a|\delta(q_i, a) = q_j\} & \text{if } i \neq j \end{cases}$$

We then have that

$$L_m(A) = \bigcup_{q_j \in Q_m} R_{0j}^n$$

and

$$L(A) = \bigcup_{q_i \in Q} R_{0i}^n.$$

A set of strings R_{ij}^k can be in general represented by the *regular expressions* r_{ij}^k, which allows a more compact notation.

The automaton is formally defined by the following five-tuple

$$A = (Q, E, \delta, q_0, Q_m)$$

where

$$\begin{aligned} &Q = \{q_1, q_2\} \\ &E = \{0, 1\} \\ &\delta(q_1, 0) = q_2 \\ &\delta(q_1, 1) = q_1 \\ &\delta(q_2, 0) = q_2 \\ &\delta(q_2, 1) = q_1 \\ &q_0 = q_1 \\ &Q_m = \{q_2\} \end{aligned}$$

The first column of the table is computed by inspection can compute the r_{ij}^0,

	k=0	k=1
r_{11}^k	ϵ	ϵ
r_{12}^k	0	0
r_{21}^k	1	1
r_{22}^k	ϵ	10

The second column is computed using the recursive formula introduced before. In particular we have

$$\begin{aligned}
r_{11}^1 &= r_{11}^0 \left(r_{11}^0\right)^* r_{11}^0 + r_{11}^0 = \epsilon(\epsilon)^*\epsilon + \epsilon = \epsilon \\
r_{12}^1 &= r_{11}^0 \left(r_{11}^0\right)^* r_{12}^0 + r_{12}^0 = \epsilon(\epsilon)^* 0 + 0 = 0 \\
r_{21}^1 &= r_{21}^0 \left(r_{11}^0\right)^* r_{11}^0 + r_{21}^0 = 1(\epsilon)^*\epsilon + 1 = 1 \\
r_{22}^1 &= r_{21}^0 \left(r_{11}^0\right)^* r_{12}^0 + r_{22}^0 = 1(\epsilon)^* 0 + \epsilon = 10
\end{aligned}$$

The marked language is then

$$L_m(A) = r_{12}^2 = r_{12}^1 \left(r_{22}^1\right)^* r_{22}^1 + r_{12}^1 = 0(10)^* 10 + 0 = 0(10)^*$$

In this case the generated language is

$$L(A) = r_{11}^2 + r_{12}^2 = 0(10)^* 1 + \epsilon + 0(10)^*$$

The prefix closure $\overline{L_m(A)}$ is

$$\overline{L_m(A)} = \{s \in E^* | \exists t \in E^*, st \in L_m(A)\}.$$

We prove that $\overline{L_m(A)} = L(A)$.

Prove $\overline{L_m(A)} \subseteq L(A)$: Let $x \in \overline{L_m(A)}$ then is trivial to see that $x \in L(A)$.

Prove $\overline{L_m(A)} \supseteq L(A)$: Let now $x \in L(A)$. Then $x = \{\epsilon, 0, 010, 01010, 0101010, \dots, 01, 0101, 010101, 01010101, \dots$
The strings in $L_m(A)$ are $0, 010, 01010, 0101010, \dots$ thus prefixes are $\{\epsilon, 0, 01, 0101, 01010, 010101, \dots\}$ and as we can see then $x \in \overline{L_m(A)}$.

We can conclude that $\overline{L_m(A)} = L(A)$, thus the DES is nonblocking.

SOLUTION 8.4

In this case the automaton is

$$A = (Q, E, \delta, q_0, Q_m)$$

where

$$\begin{aligned}
&Q = \{q_1, q_2, q_3\} \\
&E = \{0, 1\} \\
&\delta(q_1, 0) = q_2 \\
&\delta(q_1, 1) = q_3 \\
&\delta(q_2, 0) = q_1 \\
&\delta(q_2, 1) = q_3 \\
&\delta(q_3, 0) = q_2 \\
&\delta(q_3, 1) = q_2 \\
&q_0 = q_1 \\
&Q_m = \{q_2, q_3\}
\end{aligned}$$

We apply the recursive formula introduced in the solution of Problem 8.3 to compute the marked language and the generated language.

	k=0	k=1	k=2
r_{11}^k	ϵ	ϵ	$(00)^*$
r_{12}^k	0	0	$0(00)^*$
r_{13}^k	1	1	0^*1
r_{21}^k	0	0	$0(00)^*$
r_{22}^k	ϵ	ϵ+00	$(00)^*$
r_{23}^k	1	1+01	0^*1
r_{31}^k	$\emptyset$	$\emptyset$	$(0+1)(00)^*0$
r_{32}^k	0+1	0+1	$(0+1)(00)^*$
r_{33}^k	ϵ	ϵ	$\epsilon+(0+1)0^*1$

Certain equivalences among regular expressions has been used to simplify the expressions. For example

$$r_{22}^1 = r_{21}^0\big(r_{11}^0\big)^* r_{12}^0 + r_{22}^0 = 0(\epsilon)^*0 + \epsilon = \epsilon + 00$$

or, for example for r_{13}^2 we have

$$r_{13}^2 = r_{12}^1\big(r_{22}^1\big)^* r_{23}^1 + r_{13}^1 = 0(\epsilon+00)^*(1+01)+1$$

Since $(\epsilon+00)^* = (00)^*$ and $(1+01) = (\epsilon+0)1$, the expression becomes

$$r_{13}^2 = 0(00)^*(\epsilon+0)+1+1$$

and since $(00)^*(\epsilon+0) = 0^*$ the expression reduces to

$$r_{13}^2 = 00^*1+1 = 0^*1.$$

We have that

$$L_m(A) = r_{12}^3 + r_{13}^3$$

where

$$\begin{aligned} r_{12}^3 &= r_{13}^2\big(r_{33}^2\big)^* r_{32}^2 + r_{12}^2 = 0^*1\big(\epsilon+(0+1)0^*1\big)^*(0+1)(00)^* + 0(00)^* \\ &= 0^*1\big((0+1)0^*1^*\big)^*(0+1)(00)^* + 0(00)^* \end{aligned}$$

and

$$\begin{aligned} r_{13}^3 &= r_{13}^2\big(r_{33}^2\big)^* r_{31}^2 + r_{13}^2 = 0^*1\big(\epsilon+(0+1)0^*1\big)^*\big(\epsilon+(0+1)0^*1\big) + 0^*1 \\ &= 0^*1\big((0+1)0^*1^*\big)^* \end{aligned}$$

Thus we have

$$L_m(A) = r_{12}^3 + r_{13}^3 = 0^*1\big((0+1)0^*1\big)^*\big(\epsilon+(0+1)(00)^*\big) + 0(00)^*.$$

The generated language $L(A)$ is given by

$$L(A) = r_{11}^3 + r_{12}^3 + r_{13}^3$$

where

$$r_{11}^3 = r_{13}^2\big(r_{33}^2\big)^* r_{31}^2 + r_{11}^2 = \big(0^*1((0+1)0^*1)^*(0+1)0+\epsilon\big)(00)^*$$

SOLUTION 8.5

Two states p and q are said to be *equivalent*, $p \equiv q$ if and only if for each input string x, $\delta(p,x)$ is a marked state if an only if $\delta(q,x)$ is a marked place. Two states p and q are said to be distinguishable if there exists an x such that $\delta(p,x)$ is in Q_m and $\delta(q,x)$ is not.

We build a table as follows

b	X						
c	X	X					
d	X	X	X				
e		X	X	X			
f	X	X	X		X		
g	X	X	X	X	X	X	
h	X		X	X	X	X	X
	a	b	c	d	e	f	g

where an 'X' is placed in the table each time a pari of states cannot be equivalent. Initially an 'X' is placed in each entry corresponding to one marked state and one non-marked state. It is in fact impossible that a marked state and a non-marked state are equivalent. In the example, place an 'X' in the entries (a,c), (b,c), (c,d), (c,e), (c,f), (c,g) and (c,h). Next for each pair of states p and q that are not already known to be distinguishable, we consider the pairs of states $r = \delta(p,a)$ and $s = \delta(q,a)$ for a input symbol a. If states r adn s have been shown to be distinguishable by some string x then p and q are distinguishable by the string ax. Thus if the entry (r,s) n the table has an 'X', an 'X' is also placed at the entry (p,q). If the entry (r,s) does not yet have an 'X', then the pair (p,q) is placed on a list associated with the (r,s)-entry. At some future time, if (r,s) entry receives and 'X', then each pair on the list associated with the (r,s)-entry also receives and 'X'.

In the example , we place and 'X' in the entry (a,b), since the entry $(\delta(b,1), \delta(a,1)) = (c,f)$ is already with 'X'. Similarly, the (a,d)-entry receives an 'X' since the entry $(\delta(a,0), \delta(b,0)) = (b,c)$ has an 'X'. Consideration of the (a,e)-entry on input 0 results in the pair (a,e) being placed on the list associated with (b,h). Observe that on input 1, both a and e go to the same state f and hence no string starting with 1 can distinguish a from e. Because of the 0-input, the pair (a,g) is placed on the list associated with (b,g). When the (b,g)-entry is considered, it receives an 'X' on account of a 1-input and (a,g) receives a 'X'. The string 01 distinguishes a from g.

At the end of the algorithm $a \equiv e$, $b \equiv h$ and $d \equiv f$. The minimum automaton is shown in Figure 8.5.1.

SOLUTION 8.6

We can construct a deterministic automaton

$$A' = (Q', \{0,1\}, \delta', [q_0], Q_m)$$

accepting $L_m(A)$ as follows. The set Q consists of all subsets of Q. These we denote with $[q_0]$, $[q_1]$, $[q_0,q_1]$, and . Since $\delta(q_0,0) = \{q_0,q_1\}$ then we have

$$\delta'([q_0],0) = [q_0,q_1]$$

and similarly

$$\delta'([q_0],1) = [q_1] \quad \delta'([q_1],0) = \quad \delta'([q_1],1) = [q_0,q_1].$$

Naturally, $\delta'(,0) = \delta'(,1) =$. Finally

$$\delta'([q_0,q_1],0) = [q_0,q_1]$$

since

$$\delta(\{q_0,q_1\},0) = \delta(q_0,0) \cup \delta(q_1,0) = \{q_0,q_1\} \cup = \{q_0,q_1\},$$

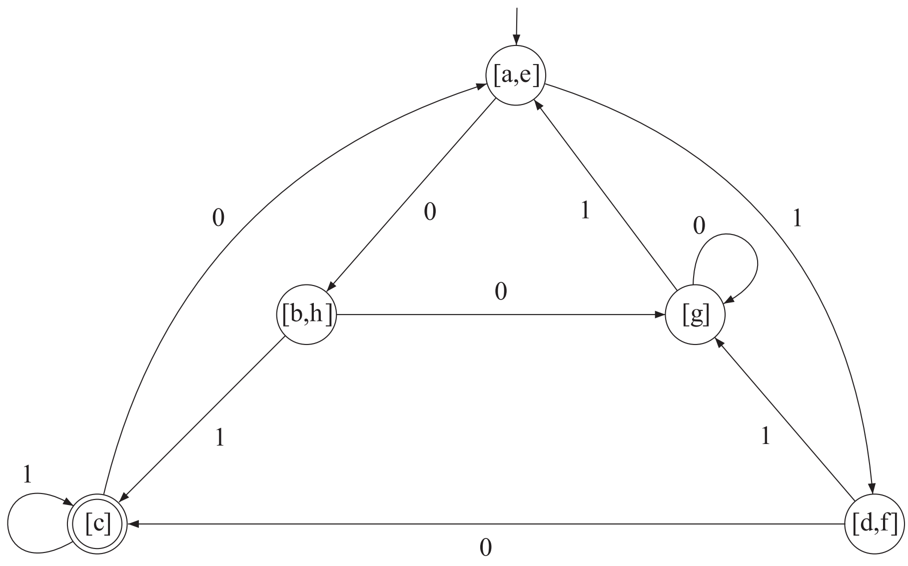


Figure 8.5.1: Minimum automaton of Problem 8.5.

and similarly we get

$$\delta([q_0, q_1], 1) = [q_0, q_1].$$

The final states are $Q'_m = \{[q_1], [q_0, q_1]\}$.

SOLUTION 8.7

The FTS is defined by $\mathcal{T} = (S, Act, \rightarrow, I)$, where the set of states is given by $S = \{\langle x = 0, r = 0\rangle, \langle x = 1, r = 0\rangle, \langle x = 0, r = 1\rangle, \langle x = 1, r = 1\rangle\}$; the set of actions is irrelevant and omitted here; the transition relation results directly from the evaluation function δ_r. For example, $\langle x = 0, r = 1\rangle \rightarrow \langle x = 0, r = 1\rangle$ if the next input bit x is 0, and $\langle x = 0, r = 1\rangle \rightarrow \langle x = 1, r = 1\rangle$ if the next input bit x is 1; the set of initial sets is $\{\langle x = 0, r = 1\rangle, \langle x = 0, r = 0\rangle\}$. You can also model y as the output of the FTS, as shown in Figure 8.7.1.

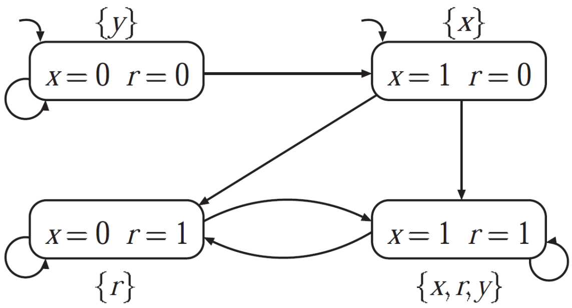


Figure 8.7.1: FTS for a simple hardware circuit.

SOLUTION 8.8

This automaton recognizes event sequences of length 3 or more, which begin with b and contain a_1 or a_2 in the second position, followed by b in their third position. In other words, each such string must begin with b and include a_1 or a_2 in the second position, followed by b in the third position. What follows after the third position does not matter in this example.

SOLUTION 8.9

Denote by the queue lengths formed by four vehicles types and the state of the traffic light, i.e.,

$$S = \{(x_{12}, x_{13}, x_{23}, x_{32}, y) : x_{12}, x_{13}, x_{23}, x_{32} \in \mathbb{Z}, y \in \{G, R\}\},$$

where G denotes green for $(1,2)$ and $(1,3)$ while R denotes red for $(1,2)$ and $(1,3)$. The event set is given by

$$Act = \{a_{12}, a_{13}, a_{23}, a_{32}, d_{12}, d_{13}, d_{23}, d_{32}, g, r\},$$

where $a_{12}, a_{13}, a_{23}, a_{32}$ is a vehicle arrival for each of the four types; $d_{12}, d_{13}, d_{23}, d_{32}$ is a vehicle departure upon clearing the intersection for each of the four types; g indicates the light turns green for $(1,2)$ and $(1,3)$ type vehicles; r indicates the light turns red for $(1,2)$ and $(1,3)$ type vehicles. The initial state can be arbitrarily set. The transition relation can be derived easily following the rules described above. For example, $(x_{12}, x_{13}, x_{23}, x_{32}, R) \xrightarrow{g} (x_{12}, x_{13}, x_{23}, x_{32}, G)$, $(x_{12}, x_{13}, x_{23}, x_{32}, G) \xrightarrow{d_{12}} (x_{12} - 1, x_{13}, x_{23}, x_{32}, G)$, $(x_{12}, x_{13}, x_{23}, x_{32}, R) \xrightarrow{a_{13}} (x_{12}, x_{13} + 1, x_{23}, x_{32}, R)$.

SOLUTION 8.10

The queuing system of figure 8.10.1 has a generator set $\Sigma = \{a, d\}$.

A natural state variable is the number of customers in queue, thus the state-space is the set of non-negative integers $S = \{0, 1, 2, \ldots\}$.

The transition relation is

$$\begin{aligned} f(x, a) &= x + 1 \qquad \forall x \geq 0 \\ f(x, d) &= x - 1 \qquad \forall x > 0\,. \end{aligned}$$

The starting state q_0 is chosen to be the initial number of customers in the system.

In figure 8.10.1, the transition system representing the basic queue system is reported. It is evident that the cardinality of the state is infinite, but it is also countable.

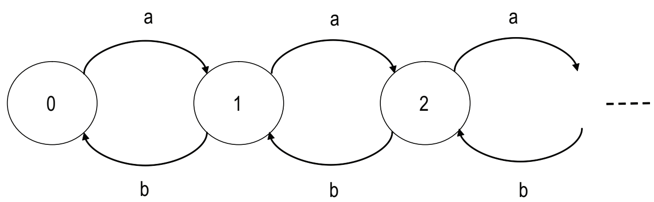


Figure 8.10.1: A basic queue system.

SOLUTION 8.11

The proof is as follows:

1. In each iteration the number of elements in $\mathrm{Reach_i}$ increases by at least 1. Since it can have, at most, as many elements as S, there can only be as many iterations as the number of elements in S (minus the number of elements in S_S).

2. $\mathrm{Reach_i}$ is the set of states that can be reached in i steps, thus any state that can be reached in a finite number of steps must be in one of the $\mathrm{Reach_i}$.

SOLUTION 8.12

Let $S_S = \{3\}$. By applying the reachability algorithm we have

$$\begin{aligned}
&S_S = 3\\
&\mathrm{Reach}_0 = \{3\}\\
&\mathrm{Reach}_1 = \{1,3,5,6\}\\
&\mathrm{Reach}_2 = \{1,2,3,5,6\}\\
&\mathrm{Reach}_3 = S\\
&\mathrm{Reach}_4 = S\\
&\mathrm{Reach}_{\mathrm{T}}(\{3\}) = S
\end{aligned}$$

Let $S_S = \{2\}$. By applying the reachability algorithm we have

$$\begin{aligned}
&S_S = 2\\
&\mathrm{Reach}_0 = \{2\}\\
&\mathrm{Reach}_1 = \{2,4,5\}\\
&\mathrm{Reach}_2 = \{2,4,5\}\\
&\mathrm{Reach}_{\mathrm{T}}(\{2\}) = \{2,4,5\}
\end{aligned}$$

Part III

Hybrid control

Solution to modeling of hybrid systems

SOLUTION 9.1

(a) Let the water level be $x \geq 0$ m and use four discrete states. Let the thresholds be x_{on} and x_{off}. The suggested model is illustrated in Figure 9.1.1.

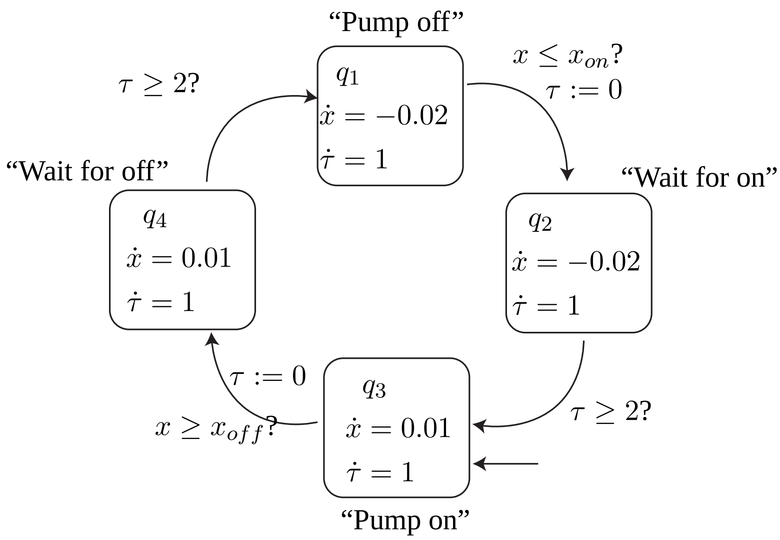


Figure 9.1.1: Hybrid model of the tank and relay controller in Problem 9.1.

(The variable τ is included as a "timer", used to keep track of how long the system stays in the waiting states.)

(b) The water level decreases $2 \cdot 0.02$ m=0.04 m while waiting for the pump to turn on. Thus we need to choose x_{on}=0.09 to prevent underfilling. Similarly, choose

$$x_{off} = 0.12 - 2 \cdot 0.01 = 0.10.$$

SOLUTION 9.2

We see that v can assume 2^k values, called $v_i, i \in \{0, \dots, 2^k - 1\}$. So we introduce $N = 2^k$ discrete states q_i. In state q_i, the system has v_i as constant control signal, illustrated in Figure 9.2.1.

Figure 9.2.1: Dynamics of state q_i in Problem 9.2.

When do we switch states? We define the edges as

$$E = \{(q_i, q_{i+1}), (q_{i+1}, q_i) | i = 0, \dots, N-2\},$$

i.e., the system can only switch between adjacent states. The switchings are controlled by guards and domains. The guards define when switching is *allowed*, and when the system is outside the domain, it *must* switch:

$$\begin{aligned} G(q_i, q_{i+1}) &= \{u \geq v_i + D/2\} \\ G(q_{i+1}, q_i) &= \{u \leq v_{i+1} - D/2\} \\ D(q_i) &= \{v_i - D/2 \leq u \leq v_i + D/2\} \end{aligned}$$

So the hybrid system can be illustrated as in Figure 9.2.2.

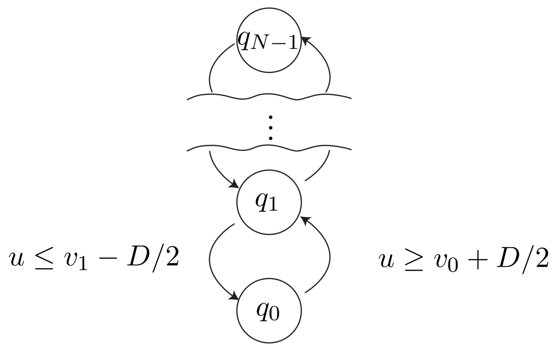


Figure 9.2.2: Control system with quantizer, modeled as a hybrid system. (Problem 9.2)

SOLUTION 9.3

A hybrid system system modeling the nuclear reactor is shown in Figure 9.3.1.

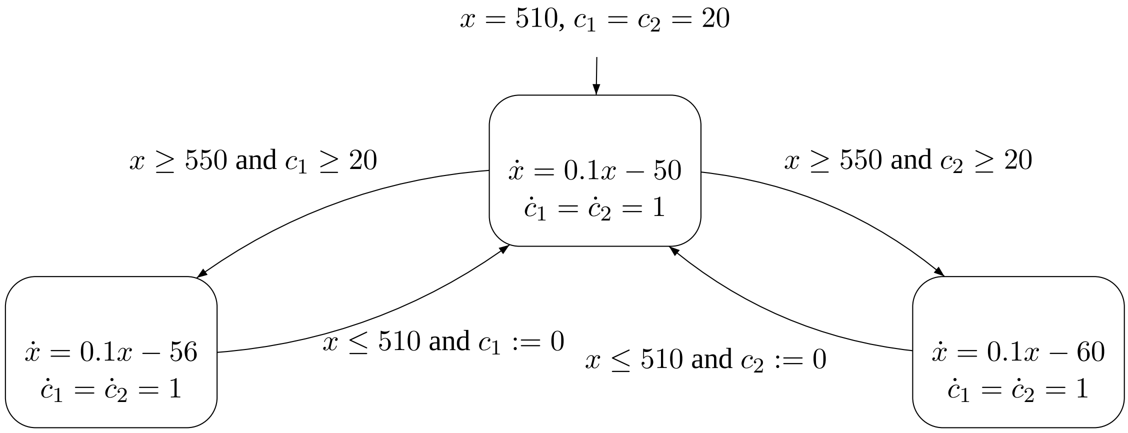


Figure 9.3.1: The hybrid system for the nuclear reactor

SOLUTION 9.4

We model the system on linear state space form with a controller $u = f(y)$ and neglect the time for computing the control signal. (The computing time could otherwise be modeled as an extra waiting state between sampling $y(t)$ and applying the new $u(t)$.) The suggested hybrid model is illustrated in Figure 9.4.1.

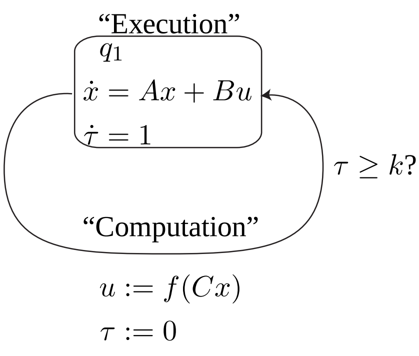


Figure 9.4.1: The sampled control system in Problem 9.4.

The control law, as well as any quantization effects in the D/A or A/D converters are incorporated into $f(y)$.

SOLUTION 9.5

(a) We need a timer variable t to control the switching, so the full state is $(x,t)^T$. Now the hybrid automaton can be described as $H = (Q, X, \text{Init}, f, D, E, G, R)$, with

$$
\begin{aligned}
&Q = \{q_1, q_2\} \\
&X = \mathbb{R}^2 \times \mathbb{R}^+ \\
&\text{Init} = q_1 \times (0,0) \\
&f(q_i, x, t) = \begin{bmatrix} A_i x & 1 \end{bmatrix}^T \\
&D(q_i) = \{(x,t)^T | t < 1\} \\
&E = \{(q_1, q_2), (q_2, q_1)\} \\
&G(q_1, q_2) = G(q_2, q_1) = \{(x,t)^T | t \geq 1\} \\
&R(q_1, q_2, x, t) = R(q_2, q_1, x, t) = \begin{bmatrix} x & 0 \end{bmatrix}^T.
\end{aligned}
$$

(b) The trajectory evolves as follows:

$$
x(t) = \begin{cases} e^{A_1 t} x_0, & t \in [0,1) \\ e^{A_2(t-1)} x(1), & t \in [1,2) \\ e^{A_1(t-2)} x(2), & t \in [2,3) \end{cases}
$$

In order to express $x(t)$ as a function of x_0 we need to derive $x(1)$ and $x(2)$ as functions of x_0. We have

$$
x(1) = e^{A_1(1-0)} x_0 = e^{A_1} x_0
$$

and

$$
x(2) = e^{A_2(2-1)} x(1) = e^{A_2} x(1).
$$

Combining the two previous equations we have

$$
x(t) = \begin{cases} e^{A_1 t} x_0, & t \in [0,1) \\ e^{A_2(t-1)} e^{A_1} x_0, & t \in [1,2) \\ e^{A_1(t-2)} e^{A_2} e^{A_1} x_0, & t \in [2,3). \end{cases}
$$

SOLUTION 9.6

(a) The hybrid automaton is defined as $H = (Q, X, \text{Init}, f, D, E, G, R)$ with

$$
\begin{aligned}
&Q = \{q_1, q_2, q_3\} \\
&X = \mathbb{R} \\
&\text{Init} = q_1 \times 0 \\
&f(q_1, x) = 2 \\
&f(q_2, x) = -1 \\
&f(q_3, x) = x + 2 \\
&D(q_1) = \{x \in \mathbb{R} | x < 5\} \\
&D(q_2) = \{x \in \mathbb{R} | x > 3\} \\
&D(q_3) \text{ to be defined} \\
&E = \{(q_1, q_2), (q_2, q_3)\} \\
&G(q_1, q_2) = \{x \in \mathbb{R} | x \geq 5\} \\
&G(q_2, q_3) = \{x \in \mathbb{R} | x \leq 3\} \\
&R(q_1, q_2, x) = x \\
&R(q_2, q_3, x) = -2
\end{aligned}
$$

(b) In order for the hybrid automaton to be live we need to ensure that the domain $D(q_3)$ contains the trajectory of the system $\dot{x} = x + 2$ with initial condition $x_0 = -2$. But

$$x_0 = -2 \Rightarrow \dot{x} = -2 + 2 = 0,$$

so the state will stay constant. Thus all domains that fulfill $\{x = 2\} \in D(q_3)$ guarantee liveness of the system.

(c) The state-trajectory of the hybrid system is shown in Figure 9.6.1, where $\tau_0 = 0$, $\tau_1 = 2.5$ and $\tau_2 = 4.5$.

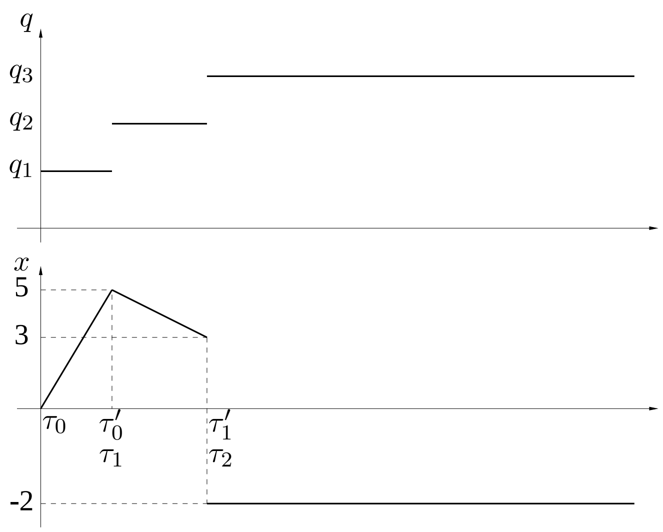


Figure 9.6.1: Trajectory of the hybrid system in Problem 9.6

Solutions to stability of hybrid systems

SOLUTION 10.1

(a) A possible solution is $H = (Q, X, \text{Init}, f, D, E, G, R)$ where

$$\begin{aligned}
&Q = \{q\} \\
&X = \mathbb{R}^3 \text{ and } x = (v_1, v_2, v_3)^T \\
&\text{Init} = q \times (1, 0, 0) \\
&f(q, x) = (0, 0, 0)^T \\
&D(q) = \{x \in \mathbb{R}^3 | v_1 \le v_2 \le v_3\} \\
&E = \{(q, q)\} \\
&G(q, q) = \{x \in \mathbb{R}^3 | v_1 > v_2\} \cup \{x \in \mathbb{R}^3 | v_2 > v_3\} \\
&R(q, q, x) = \begin{cases} \left(\dfrac{v_1 + v_2}{2}, \dfrac{v_1 + v_2}{2}, v_3\right) & \text{if } v_1 > v_2 \\ \left(v_1, \dfrac{v_2 + v_3}{2}, \dfrac{v_2 + v_3}{2}\right) & \text{if } v_2 > v_3 \end{cases}
\end{aligned}$$

(b) The collisions result in the following evolution for the continuous state:

$$(1, 0, 0) \to (\frac{1}{2}, \frac{1}{2}, 0) \to (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) \to (\frac{3}{8}, \frac{3}{8}, \frac{1}{4}) \to \dots$$

which constitute a infinite sequence. We have that the solution is Zeno since it has an infinite number of discrete transitions over a finite time interval, which in this case is a zero time interval.

(b) Since $v_1(\tau_i) - v_3(\tau_i) = 2^{-i} \to 0$ as $i \to \infty$ and $v_1(\tau_\infty) = v_2(\tau_\infty) = v_3(\tau_\infty)$, it follows that the accumulation point is (1/3,1/3,1/3). The physical interpretation is that after an infinite number of hits, the balls will have the same velocity (equal to one third of the initial velocity of ball 1).

SOLUTION 10.2

We can consider the following Lyapunov function:

$$V(x) = \frac{1}{2} x^T x$$

It is an acceptable candidate, since

$$V(x) \ge 0,$$

with equality only at $x = 0$. We then have

$$\dot{V} = x^T \dot{x} = x^T A x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 3 \\ -2 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

which gives

$$\dot{V}(x) = -x_1^2 - x_2^2 - 2x_3^2 < 0 \qquad \forall x \neq 0$$

Thus we can conclude that the system is asymptotically stable.

SOLUTION 10.3

Let P be a positive definite symmetric matrix which satisfies

$$A^T P + PA = -Q.$$

Then the function $V(x) = x^T P x$ is a positive definite function. Let us compute $\dot{V}(x)$

$$\dot{V}(x) = \frac{d}{dt}(x^T P x) = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x = x^T (A^T P + PA) x = -x^T Q x.$$

Thus $\dot{V}(x)$ is negative definite. For Lyapunov's stability criteria we can conclude that the system is asymptotically stable.

SOLUTION 10.4

Let us consider the following Lyapunov function

$$V(x) = \frac{x^T x}{2} = \frac{x_1^2 + x_2^2}{2}$$

Let us compute the time derivative of $V(x)$. We have

$$\begin{aligned}
\dot{V}(x) &= x_1 \dot{x}_1 + x_2 \dot{x}_2 = -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1) \\
&\leq -x_1^2 - x_2^2 + \frac{1}{2} x_1 |x_2| + \frac{1}{2} x_2 |x_1| \\
&\leq -x_1^2 - x_2^2 + |x_1 x_2| \\
&\leq -\frac{1}{2}(x_1^2 + x_2^2)
\end{aligned}$$

where we used $(|x_1| - |x_2|)^2 \geq 0$, which gives $|x_1 x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$. Thus

$$\dot{V} < 0 \quad \forall x \neq 0$$

thus the system is asymptotically stable. Notice that we do not need to know the expressions for $g(.)$ and $h(.)$ to show asymptotic stability.

SOLUTION 10.5

(a) Depending on the sign of x_1 or x_2 we have four systems. If $x_1 > 0$ and $x_2 > 0$ then we have

$$\begin{aligned}
\dot{x}_1 &= -1 + 2 = 1 \\
\dot{x}_2 &= -2 - 1 = -3
\end{aligned}$$

If $x_1 < 0$ and $x_2 > 0$ then we have

$$\begin{aligned}
\dot{x}_1 &= 1 + 2 = 3 \\
\dot{x}_2 &= 2 - 1 = 1
\end{aligned}$$

If $x_1 < 0$ and $x_2 < 0$ then we have

$$\begin{aligned}
\dot{x}_1 &= 1 - 2 = -1 \\
\dot{x}_2 &= 2 + 1 = 3
\end{aligned}$$

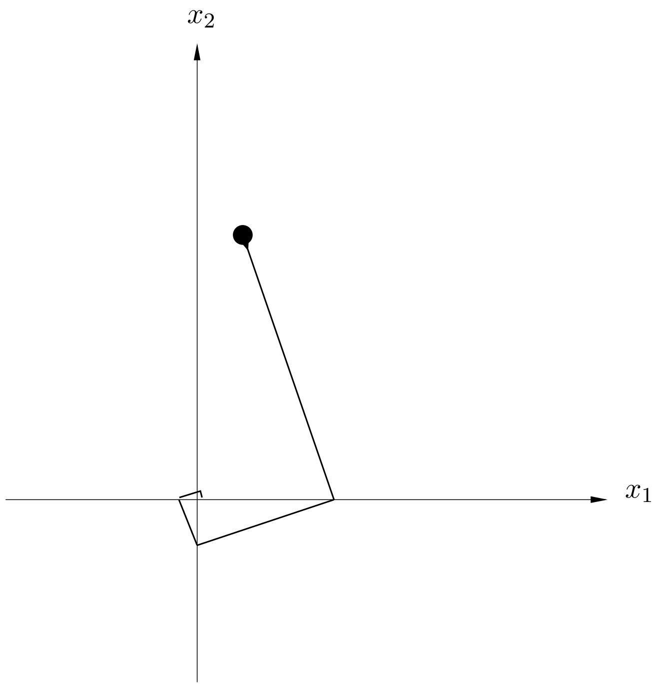


Figure 10.5.1: A trajectory of the system in Problem 10.5. The dot represents the initial state.

If $x_1 > 0$ and $x_2 < 0$ then we have

$$\begin{aligned}\dot{x}_1 &= -1 - 2 = -3\\ \dot{x}_2 &= -2 + 1 = -1\end{aligned}$$

A trajectory is represented in Figure 10.5.1. We can then represent the discontinuous system as a hybrid system with four states. Formally we have $H = (Q, X, \text{Init}, f, D, E, G, R)$ with

$$\begin{aligned}&Q = \{q_1, q_2, q_2, q_4\}\\ &X = \mathbb{R}^2\\ &\text{Init} = q_0 \times (x_{10}, x_{20}),\end{aligned}$$

where q_0 depends on the state x_0.

$$\begin{aligned}&f(q_1, x) = (1, -3)^T\\ &f(q_2, x) = (3, 1)^T\\ &f(q_3, x) = (-1, 3)^T\\ &f(q_4, x) = (-3, -1)^T\\ &D(q_1) = \{x \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0\}\\ &D(q_2) = \{x \in \mathbb{R}^2 | x_1 < 0, x_2 \geq 0\}\\ &D(q_3) = \{x \in \mathbb{R}^2 | x_1 < 0, x_2 < 0\}\\ &D(q_4) = \{x \in \mathbb{R}^2 | x_1 \geq 0, x_2 < 0\}\\ &R(q_i, q_j, x) = x\end{aligned}$$

We still need to define the set of edges E and the guards G. Since a trajectory of the given discontinuous system is always such that from the 1st quadrant it moves to the 4th and then 3rd and 2nd (see

Figure 10.5.1), the set of edges is then

$$E = \{(q_1, q_4), (q_4, q_3), (q_3, q_2), (q_2, q_1)\}.$$

The guards are

$$\begin{aligned} G(q_1, q_4) &= \{x \in \mathbb{R}^2 | x_2 < 0\} \\ G(q_4, q_3) &= \{x \in \mathbb{R}^2 | x_1 < 0\} \\ G(q_3, q_2) &= \{x \in \mathbb{R}^2 | x_2 \geq 0\} \\ G(q_2, q_1) &= \{x \in \mathbb{R}^2 | x_1 \geq 0\}. \end{aligned}$$

(b) We can notice that for any initial condition ($x(0) \neq 0$) a solution of the hybrid system will look like a 'spiral' which is moving towards the origin. As we approach the origin the number of switches increases, until we have an infinite number of switches in finite time. Thus we have Zeno behavior.

SOLUTION 10.6

Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}x^2.$$

For both $q = 1$ and $q = 2$ it holds that

$$\dot{V}(x) = x\dot{x} = a_q x^2 \leq 0,$$

with equality only for $x = 0$. Then $V(x)$ is a common Lyapunov function for the switched system, and it is asymptotically stable.

SOLUTION 10.7

The matrices A_1 and A_2 commute, *i.e.*

$$A_1 A_2 = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix} = A_2 A_1.$$

Further, they are both diagonal, which means that the eigenvalues are the diagonal elements. All eigenvalues are negative, so each matrix is asymptotically stable.

If the system matrices commute and are asymptotically stable, then the switched system is also asymptotically stable.

SOLUTION 10.8

1st method The second component of the state x satisfies

$$\dot{x}_2 = -c_q x_2$$

where $q = \{1, 2\}$. Therefore $x_2(t)$ decays to 0 asymptotically at the rate corresponding to $\min\{c_1, c_2\}$. The first component of x satisfies the equation

$$\dot{x}_1 = -a_q x_1 + b_q x_2.$$

This can be viewed as the asymptotically stable system $\dot{x}_1 = -a_q x_1$ excited by a asymptotically decaying input $b_q x_2$. Thus x_1 also converges to zero asymptotically.

2nd method A second way to show that the switched system is asymptotically stable consists in constructing a common Lyapunov function for the family of linear system A_q. In this case it is possible to find a quadratic common Lyapunov function

$$V(x) = x^T P x$$

with $P = \text{diag}(d_1, d_2)$ with $d_1, d_2 > 0$, since P must be positive definite. We then have

$$-\left(A_q^T P + P A_q\right) = \begin{pmatrix} 2d_1 a_q & -d_1 b_q \\ -d_1 b_q & 2d_2 c_q \end{pmatrix} \qquad q = 1, 2$$

To ensure that this matrix is positive definite we can fix $d_1 > 0$ and then choose $d_2 > 0$ large enough so that

$$4d_2 d_1 a_q c_q - d_1^2 b_q^2 > 0, \qquad q = 1, 2.$$

Thus this concludes the proof.

SOLUTION 10.9

(a) To model the system as a switching system, we introduce a timer variable τ:

$$\begin{aligned} \dot{x} &= A_q x \\ \dot{\tau} &= 1, \end{aligned}$$

where $q \in \{1, 2\}$ and

$$\begin{aligned} \Omega_1 &= \{x, \tau | k\epsilon \leq \tau < (k + \frac{1}{2})\epsilon, k = 0, 1, \ldots\} \\ \Omega_2 &= \{x, \tau | (k + \frac{1}{2})\epsilon \leq \tau < (k+1)\epsilon, k = 0, 1, \ldots\}. \end{aligned}$$

(b) A corresponding hybrid automaton is $H = (Q, X, \text{Init}, f, D, E, G, R)$, with

$$\begin{aligned} &Q = \{q_1, q_2\} \\ &X = \mathbb{R}^n \times \mathbb{R}^+ \text{ (state vector: } (x, \tau)^T) \\ &\text{Init} = q_1 \times (x_0, t_0)^T \\ &f(q_1, x, \tau) = (A_1 x, 1)^T \\ &f(q_2, x, \tau) = (A_2 x, 1)^T \\ &D(q) = \{x, \tau | \tau < \epsilon/2\} \; \forall q \in Q \\ &E = \{(q_1, q_2), (q_2, q_1)\} \\ &G(e) = \{x, \tau | \tau \geq \epsilon/2\} \; \forall e \in E \\ &R(e, x, \tau) = (x, 0)^T \; \forall e \in E. \end{aligned}$$

Note that here we let τ be reset at every switching instant, to simplify the system.

(c) If the system starts in $q = q_1$, with $x = x_0$ at $t = t_0$, we get

$$x(t_0 + \epsilon/2) = e^{A_1 \epsilon/2} x_0$$

and

$$x(t_0 + \epsilon) = e^{A_2 \epsilon/2} e^{A_1 \epsilon/2} x_0.$$

(d) We use the definition of the matrix exponential:

$$e^{A\epsilon} = I + A\epsilon + A^2\epsilon^2 + \mathcal{O}(\epsilon^3)$$

This allows us to write

$$\begin{aligned} e^{A_2\epsilon/2}e^{A_1\epsilon/2} &= (I + A_2\epsilon/2 + \ldots)(I + A_1\epsilon/2 + \ldots) \\ &= I + (A_1 + A_2)\epsilon/2 + A_2A_1\epsilon^2/4 + \ldots \approx \{ \text{ as } \epsilon \to 0 \} \approx e^{\frac{A_1+A_2}{2}\epsilon} \end{aligned}$$

So for fast switching (small ϵ), the system behaves like the average of the two subsystems.

SOLUTION 10.10

We choose $Q = I$ and test if

$$P = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$$

solves the Lyapunov equation

$$PA_1 + A_1^T P = -I \Leftrightarrow \begin{bmatrix} -2p_1 & 0 \\ 0 & -4p_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Apparently it does, if we choose $p_1 = 1/2$ and $p_2 = 1/4$. So $V(x) = x^T Px$ is a Lyapunov function for system 1. Does it also work for system 2?

$$\begin{aligned} \dot V(x) &= \dot x^T Px + x^T P\dot x = x^T A_2 Px + x^T PA_2 x \\ &= x^T \begin{bmatrix} -3 & 0 \\ 0 & -5/2 \end{bmatrix} x \le 0 \, \forall x, \end{aligned}$$

with equality only for $x = 0$. So $V(x)$ is a common Lyapunov function for systems 1 and 2 and the switched system is asymptotically stable.

SOLUTION 10.11

(a) Without loss of generality we can consider a positive definite matrix P in the form

$$P = \begin{pmatrix} 1 & q \\ q & r \end{pmatrix}.$$

Assume the matrix P satisfies

$$A_q^T P + PA_q < 0 \qquad \forall q$$

then for $q = 1$ we have

$$-A_1^T P - PA_1 = \begin{pmatrix} 2-2q & 2q+1-r \\ 2q+1-r & 2q+2r \end{pmatrix}$$

and this matrix is positive definite only if

$$\begin{aligned} &q < 1 \\ &8q^2 + 1 + r^2 - 6r < 0 \Rightarrow q^2 + \frac{(r-3)^2}{8} < 1 \end{aligned}$$

Similarly for the other matrix we have

$$-A_2^T P - PA_2 = \begin{pmatrix} 2 - \frac{q}{5} & 2q + 10 - \frac{r}{10} \\ 2q + 10 - \frac{r}{10} & 20q + 2r \end{pmatrix}$$

which is a positive definite matrix if

$$q < 10$$
$$600r - 800q^2 - 1000 - r^2 \Rightarrow q^2 + \frac{(r-300)^2}{800} < 100.$$

The aforementioned inequalities represent the interiors of two ellipsoids As can be seen in Figure 10.11.1 the two ellipsoids do not intersect.

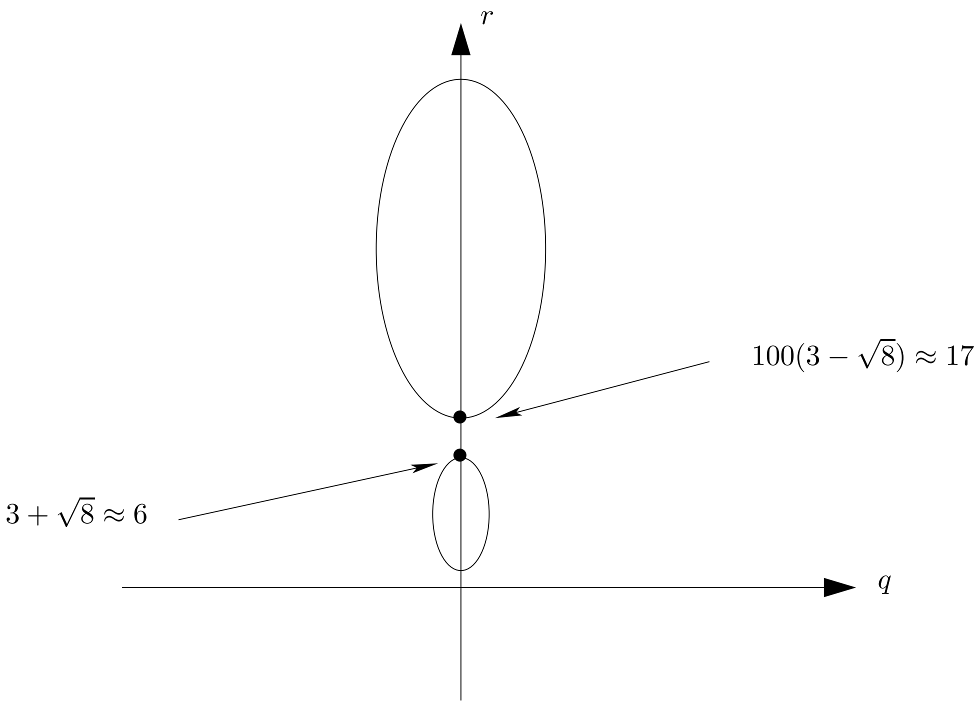


Figure 10.11.1: Ellipsoids representing the aforementioned inequalities.

(b) We would like to show that the system is however asymptotically stable. In order to do that, since the switching is arbitrary, we consider a 'worst-case-switching', which we define as follows. The vectors A_1x and A_2x are parallel on two lines which pass through the origin. If such lines define a switching surface, we then choose to follow the trajectory of the vector field which is pointing outward, so that we have a 'worst-case-switching', in the sense that it should move away from the origin. The situation is shown in Figure 10.11.2. The switching surface can be found solving the equation which describes collinearity

$$(A_1x) \times (A_2x) = 0$$

which gives the equation

$$(-x_1 - x_2)(0.1x_1 - x_2) + (x_1 - x_2)(x_1 + 10x_2) = 0.$$

The equations of the two lines are

$$x_2 = 1.18x_1$$
$$x_2 = -0.08x_1.$$

In order know which of the two vectors is pointing outward we can determine when $(A_1x) \times (A_2x) > 0$ which gives that between the line $-0.08x_1$ and $1.18x_1$ the vectors A_2x are pointing outward. We now

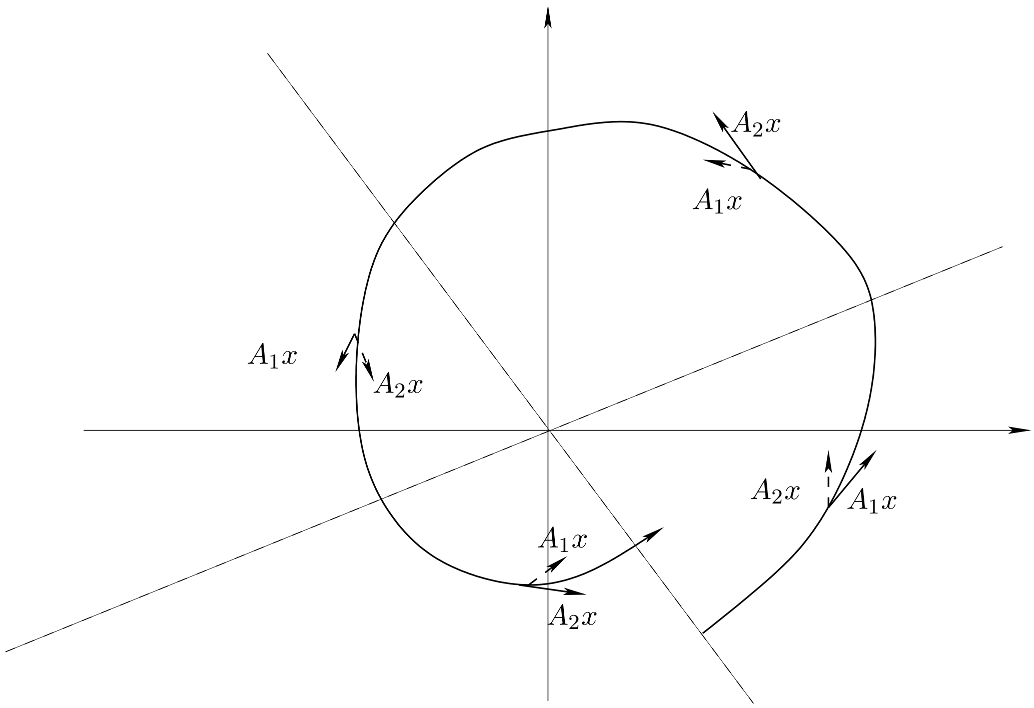


Figure 10.11.2: Worst case switching in Problem 10.11

need to show that the trajectory is converging toward the origin. Let us consider, for example, the following initial condition

$$x_0 = (100, -8)$$

which is a point on the switching surface $x_2 = -0.08x_1$. The trajectory follows the vector field A_1x and intersects the next switching surface at the point

$$\begin{pmatrix} \bar{x}_1 \\ 1.18\bar{x}_1 \end{pmatrix} = e^{A_1\bar{t}}x_0$$

which in our case is the point with coordinates $(25, 29.5)$. Then the trajectory is given by the vector field A_2x and the intersection point with the switching surface is $(-86.3, 6.8)$. The next point is $(-21.7, -25)$. The next point is the intersection of the trajectory with the switching surface that contained the initial condition x_0. We have the intersection at $(25, -2)$. Thus after one rotation we have that the distance of the trajectory to the origin is decreased from 100.1 to 25.1. Thus the system converges to the origin considering the 'worst-case-switching'.

SOLUTION 10.12

(a) Use the same ideas as in Problem 10.11.

(b) Let us consider the following two positive definite symmetric matrices

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

If we consider the following Lyapunov equations

$$A_q^T P_q + P_q A_q \quad q = 1, 2$$

we have

$$A_1^T P_1 + P_1 A_1 = -\begin{pmatrix} 10 & 7 \\ 7 & 12 \end{pmatrix} = Q_1$$
$$A_2^T P_2 + P_2 A_2 = \begin{pmatrix} -40 & 20 \\ 20 & 12 \end{pmatrix} = Q_2$$

Since the matrices Q_1 and Q_2 are symmetric matrices and the eigenvalues are $\sigma(Q_1) = \{-3.9, -18.1\}$ and $\sigma(Q_2) = \{-1.6, -50.1\}$, the two matrices are negative definite. Thus the two linear systems are asymptotically stable.

Let us consider the two Lyapunov functions

$$V_1 = x^T P_1 x$$
$$V_2 = x^T P_2 x.$$

In order to prove stability using the Lyapunov function approach we need to prove that the sequence

$$\{V_q(x(\tau_{i_q}))\}$$

is non-decreasing. In this case we can notice that we have the switching when x_1 changes sign. We can notice that

$$\lim_{x_1 \to 0^+} V_2(x) = 3x_2^2$$

and

$$\lim_{x_1 \to 0^-} V_1(x) = 3x_2^2$$

Thus the two Lyapunov function form a continuous non-increasing function, see Figure 10.12.1.

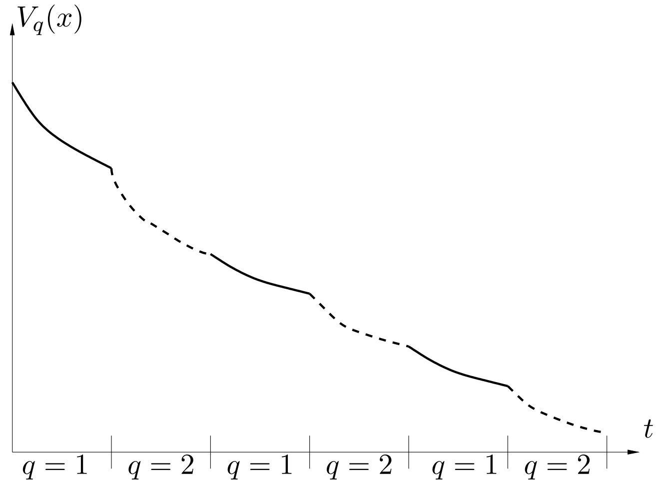


Figure 10.12.1: Multiple Lyapunov functions for Problem 10.12

SOLUTION 10.13

Use the ideas from the previous exercises.

SOLUTION 10.14

(a)
- We have three states $Q = \{q_1, q_2, q_3\}$. In q_1 we have $|x_1 - \pi| \geq \theta$, $\beta(x_1, x_2) \geq 0$ and $u = u_{\max}$. In q_2 we have $|x_1 - \pi| \geq \theta$, $\beta(x_1, x_2) < 0$ and $u = -u_{\max}$. In q_3 we have $|x_1 - \pi| < \theta$ and $u = \gamma_1 x_1 + \gamma_2 x_2$.
- The domain is $z \in X = [-\pi, \pi] \times \mathbb{R} \times [-u_{\max}, u_{\max}]$.
- The initial condition is $z(0) = [0, 0, 0]$, so $|x_1(0) - \pi| = \pi > \theta$ and $\beta(x_1(0), x_2(0)) = 0$. Hence, the initial state is q_1 and Init $= \{q_1, 0, 0, 0\}$.
- The vector field corresponding to the three states is $f = [f_1^\top, f_2^\top, f_3^\top]^\top$, with

$$f_1(q_1, z) = \begin{bmatrix} x_2 \\ g\sin(x_1) - u_{\max}\cos(x_1) \\ 0 \end{bmatrix},$$
$$f_2(q_2, z) = \begin{bmatrix} x_2 \\ g\sin(x_1) + u_{\max}\cos(x_1) \\ 0 \end{bmatrix},$$
$$f_3(q_3, z) = \begin{bmatrix} x_2 \\ g\sin(x_1) - (\gamma_1 x_1 + \gamma_2 x_2)\cos(x_1) \\ \gamma_1 x_2 + \gamma_2(g\sin(x_1) - (\gamma_1 x_1 + \gamma_2 x_2)\cos(x_1)) \end{bmatrix}.$$

- The domains are

$$\begin{aligned} D_1(q_1) =& \{z \in X : |x_1 - \pi| \geq \theta, \beta(x_1, x_2) \geq 0, u = u_{\max}\}, \\ D_2(q_2) =& \{z \in X : |x_1 - \pi| \geq \theta, \beta(x_1, x_2) < 0, u = -u_{\max}\}, \\ D_3(q_3) =& \{z \in X : |x_1 - \pi| < \theta, u = \gamma_1 x_1 + \gamma_2 x_2\}. \end{aligned}$$

- The possible edges are $E = \{(q_1, q_2), (q_1, q_3), (q_2, q_3), (q_2, q_1), (q_3, q_1), (q_3, q_2)\}$.
- The guards are

$$\begin{aligned} G(q_1, q_2) &= \{z \in D_1 : \beta(x_1, x_2) < 0\}, \\ G(q_2, q_1) &= \{z \in D_2 : \beta(x_1, x_2) = 0\}, \\ G(q_1, q_3) &= \{z \in D_1 : |x_1 - \pi| < \theta\}, \\ G(q_3, q_1) &= \{z \in D_3 : |x_1 - \pi| = \theta \wedge \beta(x_1, x_2) \geq 0\}, \\ G(q_2, q_3) &= \{z \in D_2 : |x_1 - \pi| < \theta\}, \\ G(q_3, q_2) &= \{z \in D_3 : |x_1 - \pi| = \theta \wedge \beta(x_1, x_2) < 0\}, \end{aligned}$$

- The reset maps are

$$\begin{aligned} R((q_1, q_2), z) &= \{u = -u_{\max}\}, \\ R((q_2, q_1), z) &= \{u = u_{\max}\}, \\ R((q_1, q_3), z) &= \{u = \gamma_1 x_1 + \gamma_2 x_2\}, \\ R((q_2, q_3), z) &= \{u = \gamma_1 x_1 + \gamma_2 x_2\}, \\ R((q_3, q_1), z) &= \{u = u_{\max}\}, \\ R((q_3, q_2), z) &= \{u = -u_{\max}\}, \end{aligned}$$

(b) We use the positive definite indentity matrix $P = I$ and we calculate the matrices

$$\begin{aligned}
Q_1 &= -(A_1^\top P + PA_1), \\
Q_2 &= -(A_2^\top P + PA_2), \\
Q_1 &= -\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}, \\
Q_2 &= -\begin{bmatrix} 2a & c \\ c & 2d \end{bmatrix},
\end{aligned}$$

We need to prove that $\det(Q_1), \det(Q_2) > 0$, i.e.,

$$\begin{aligned}
&-b^2 - 2bc - c^2 + 4ad > 0, \Rightarrow 4ad > (b+c)^2, \\
&-c^2 + 4ad > 0 \Rightarrow 4ad > c^2
\end{aligned}$$

The given equation

$$x^2 + ((b+c)^2 - 4ad)x + a^2 + 3a + 9 = 0,$$

can be written in the form $x^2 - (x_1 + x_2)x + x_1 * x_2 = 0$, where x_1, x_2 are the solutions. The last coefficient, corresponding to the product can be shown to be positive

$$x_1 * x_2 = a^2 + 3a + 9 = \frac{1}{2} 2 * (a^2 + 3a + 9) = \frac{1}{2}((a+3)^2 + 9 + a^2) > 0.$$

Therefore, since one solution is positive, then the other must be positive as well, i.e., $x_1 > 0, x_2 > 0$. That means that the sum of the solutions is also positive, $x_1 + x_2 > 0$, i.e., $-(b+c)^2 + 4ad > 0$ which means that $\det(Q_1) > 0$. Note that this also implies that $\det(Q_2) > 0$.

SOLUTION 10.15

Let a common lyapunov function for the family of the linear systems $A_q, q \in \{1, 2\}$ be $V(x) = x^\top Px$ with $P = \text{diag}(p_1, p_2) > 0$ and $p_1, p_2 > 0$. Then we have

$$\dot{V}(x) = \dot{x}^\top Px + x^\top Px = x^\top A_q^\top Px + x^\top PA_q x = x^\top \{A_q^\top P + PA_q\} x = -x^\top Q_q x,$$

where $Q_q = -(A_q^\top P + PA_q)$ and $q \in \{1, 2\}$. By doing the calculations we have that

$$Q_1 = \begin{bmatrix} -2a_1p_1 & -b_1p_2 \\ -b_1p_2 & -2a_2p_2 \end{bmatrix}, Q_2 = \begin{bmatrix} -2c_1p_1 & -d_1p_1 \\ -d_1p_1 & -2c_2p_2 \end{bmatrix}.$$

In order to have $Q_1, Q_2 > 0$ we need $-2a_1p_1 > 0, -2c_1p_1 > 0$ (which obviously hold since $a_1, a_2 < 0$ and $p_1, p_2 > 0$) and

$$\begin{aligned}
\det(Q_1) &= p_2(4a_1a_2p_1 - b_1^2p_2) > 0, \\
\det(Q_2) &= p_1(4c_1c_2p_2 - d_1^2p_1) > 0.
\end{aligned}$$

It can be observed that the latter equations become

$$\begin{aligned}
4a_1a_2p_1 - b_1^2p_2 > 0 &\Longleftrightarrow \frac{p_1}{p_2} > \frac{b_1^2}{4a_1a_2}, \\
4c_1c_2p_2 - d_1^2p_1 > 0 &\Longleftrightarrow \frac{p_1}{p_2} < \frac{4c_1c_2}{d_1^2},
\end{aligned}$$

respectively, or

$$\boxed{\frac{b_1^2}{4a_1a_2} < \frac{p_1}{p_2} < \frac{4c_1c_2}{d_1^2}.}$$

The last holds since we have

$$0 < b_1 d_1 < 4\sqrt{a_1 a_2 c_1 c_2} \Longleftrightarrow b_1^2 d_1^2 < 4a_1 a_2\, 4c_1 c_2 \Longleftrightarrow \frac{b_1^2}{4a_1 a_2} < \frac{4c_1 c_2}{d_1^2}.$$

Thus, by chosing p_1, p_2 such the last inequality holds, we have that $\dot{V}(x) < 0$ and the switching system is asymptotically stable.

Solutions to control of hybrid systems

SOLUTION 11.1

Since p^* is unknown, we adopt the supervisory control approach. Consider a family of observer-based estimators parameterized by $\mathcal{P}$ of the form

$$\dot{x}_p = (A_p + K_p C_p)x_p + B_p u - K_p y \tag{0.0}$$
$$y_p = C_p x_p \tag{0.0}$$

and the corresponding control laws

$$u_p = F_p x_p, \;\; p \in \mathcal{P}, \tag{0.0}$$

where we choose the matrices K_p and F_p such that $A_p + K_p C_p$ and $A_p + B_p F_p$ are Hurwitz, $\forall p \in \mathcal{P}$. Consider now the output estimation errors

$$e_p = y_p - y, \;\; \forall p \in \mathcal{P}, \tag{0.0}$$

and in particular, $e_{p^*} = y_{p^*} - y$, i.e., the $p^\star$- output element of the aforementioned designed system minus the actual measured output of the system. Then we prove that e_{p^*} converges to zero exponentially, regardless of the control input u. Consider first the respective state estimation error $e_{p_x^*} = x_{p^*} - x$, with

$$\dot{e}_{p_x^*} = (A_{p^*} + K_{p^*} C_{p^*})x_{p^*} + B_{p^*} u - K_{p^*} y - A_{p^*} x - B_{p^*} u \tag{0.0}$$
$$= K_{p^*} C_{p^*} x_{p^*} - K_{p^*} C_{p^*} x + A_{p^*}(x_{p^*} - x) \tag{0.0}$$
$$= (A_{p^*} + K_{p^*} C_{p^*})x_{p^*} - (A_{p^*} + K_{p^*} C_{p^*})x \tag{0.0}$$
$$= (A_{p^*} + K_{p^*} C_{p^*})e_{p_x^*} \tag{0.0}$$

which implies that $e_{p_x^\star}$ converges to zero exponentially fast, since $A_{p^\star} + K_{p^\star} C_{p^\star}$ is Hurwitz. Therefore $e_{p^*} = C_{p^*} e_{p_x^\star}$ also converges to zero exponentially fast.

We will design the control law as $u = u_\sigma$, where $\sigma : [0, \infty) \to \mathcal{P}$ is a switching signal, i.e., $u(t) = F_{\sigma(t)} x_{\sigma(t)}$. Intuition suggests to pick, as an estimate of p^*, the index of the smallest estimation error e_p. However, we would like to consider also the past behavior of e_p and therefore we consider the monitoring signals

$$\mu_p(t) = \int_0^t |e_p(t)|^2 d\tau, \;\; \forall p \in \mathcal{P}, \tag{0.0}$$

which can be generated by the dynamics $\dot{\mu}_p = |e_p|^2$, $\mu_p(0) = 0$, $\forall p \in \mathcal{P}$. We can generate the switching signal σ by the so-called *hysteresis switching logic* (Fig. 11.1.1). Fix a positive number h called the *hysteresis constant*. Set $\sigma(0) = \arg\min_{p\in\mathcal{P}} \mu_p(0)$. Suppose now that at a certain time σ has switched to some $q \in \mathcal{P}$. The value of σ is held fixed until we have $\min_{p\in\mathcal{P}} \mu_p(t) + h \le \mu_q(t)$. If and when that happens, we set σ equal to $\arg\min_{p\in\mathcal{P}} \mu_p(t)$. When the indicated $\arg\min$ is not unique, a particular value for σ among those that achieve the minimum can be chosen arbitrarily. Repeating this procedure, we obtain a piecewise constant signal which is continuous from the right everywhere.

Since e_{p^*} converges to zero exponentially fast, $\mu_{p^*}(t)$ is bounded from above by some number K for all $t \ge 0$. Moreover, all signals μ_p are nondecreasing by construction (sumation of positive quadratic terms).

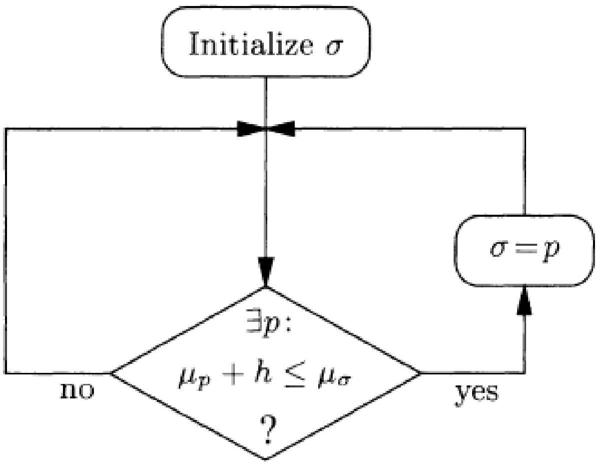


Figure 11.1.1: Hysteresis switching logic.

Hence, each μ_p has a limit (possibly ∞) as $t \to \infty$. Since $\mathcal{P}$ is finite, there exists a time T such that for each $p \in \mathcal{P}$ we either have $\mu_p(T) > K$ or $\mu_p(t_2) - \mu_p(t_1) < h$ for all $t_2 > t_1 \geq T$. Then for $t \geq T$ at most one more switch can occur. We conclude that there exists a time T^* such that $\sigma(t) = q^* \in \mathcal{P}$ for all $t \geq T^*$. Moreover, μ_{q^*} is bounded because μ_{p^*} is, hence e_{q^*} is also bounded. After the switching stops, the closed-loop system can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{q^*} \end{bmatrix} = \bar{A} \begin{bmatrix} x \\ x_{q^*} \end{bmatrix}, \tag{0.0}$$

where

$$\bar{A} = \begin{bmatrix} A_{p^*} & B_{p^*}F_{q^*} \\ -K_{q^*}C_{p^*} & A_{q^*} + K_{q^*}C_{q^*} + B_{q^*}F_{q^*} \end{bmatrix} \tag{0.0}$$

$$\bar{C} = \begin{bmatrix} -C_{p^*} & C_{q^*} \end{bmatrix} \tag{0.0}$$

If we let

$$\bar{K} = \begin{bmatrix} K_{p^*} \\ K_{q^*} \end{bmatrix} \tag{0.0}$$

then it is easy to check that

$$\bar{A} - \bar{K}\bar{C} = \begin{bmatrix} A_{p^*} + K_{p^*}C_{p^*} & B_{p^*}F_{p^*} - K_{p^*}C_{q^*} \\ 0 & A_{q^*} + B_{q^*}F_{q^*} \end{bmatrix} \tag{0.0}$$

is Hurwitz, and therefore the system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{q^*} \end{bmatrix} = (\bar{A} - \bar{K}\bar{C}) \begin{bmatrix} x \\ x_{q^*} \end{bmatrix} + \bar{K}e_{q^*} \tag{0.0}$$

converges to zero if $q^* = p^*$, and bounded close to zero if $q^* \neq p^*$.

The result of a simulation example with $x \in \mathbb{R}^2$, $p \in \{1, \ldots, 10\}$, $p^* = 5$, and random controllable/observable triplets A_p, B_p, C_p is shown in Fig. 11.1.2.

SOLUTION 11.2

(a) Since we are only concerned with the pendulum position, we can focus only on (11.2b). A possible state space for $(\theta, \dot{\theta})$ is $\mathbb{S} \times \mathbb{R}$, where $\mathbb{S}$ denotes the unit circle. This space is not contractible, therefore

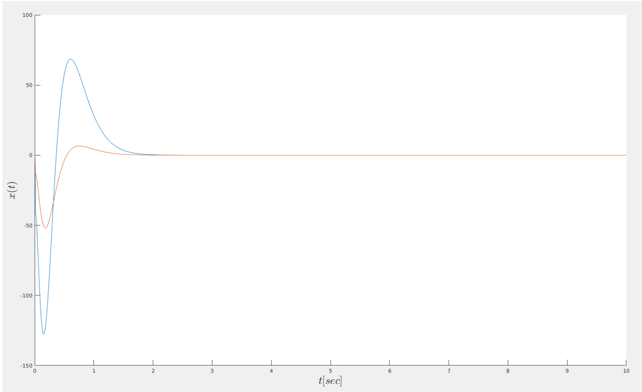


Figure 11.1.2: Simulation results for supervisory control.

global continuous feedback stabilization is not possible. There is also a more direct way to see this. Let a continuous feedback control law $u = k(\theta, \dot{\theta})$. The point $(\theta, \dot{\theta})$ is an equilibrium of the closed-loop system

$$J\ddot{\theta} = mgl\sin\theta - ml\cos\theta k(\theta, \dot{\theta})$$

if and only if $\dot{\theta} = 0$ and

$$mgl\sin\theta - ml\cos\theta k(\theta, \dot{\theta}) = 0,$$

whose left-hand side takes the positive value mgl at $\theta = \frac{\pi}{2}$ and the negative value $-mgl$ at $\theta = \frac{3\pi}{2}$. Thus if the feedback law k is continuous, then there will necessarily be an undesired equilibrium for some $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$. This means that the system (11.2b) cannot be globally asymptotically stabilized by continuous feedback.

(b) We need therefore to design a switching controller to stabilized the pendulum to the upright position. We will first use a bang-bang controller based on the phase plot of the system. The phase plot of the unactuated system ($u = 0$) is pictured in Fig. 11.2.1 for $m = l = J = 1$, $g = 9.81$ and it's basically the contours of the system energy:

$$E = \frac{1}{2}J\dot{\theta}^2 + mgl(1 + \cos\theta),$$

which is depicted in Fig. 11.2.2 for $m = l = J = 1$, $g = 9.81$. For the unactuated system, the equilibrium $(0, 0)$ is a saddle point (the initial conditions that end up there is a set of measure zero). The key observation here is that E is preserved for the unactuated system. In the desired equilibrium $\theta = \dot{\theta} = 0$, the energy is $E = 2mgl$. Therefore we could drive the system to this energy value by using bang-bang control inputs and then when the energy reaches this value, we could turn off the control.

The bang-bang control protocol is implemented as follows. Depending on the initial energy of the system $E(0)$, we set appropriately the control input $u \in [-u_{\max}, u_{\max}]$, where $u_{\max}$ is the maximum admissible input, in order to reach $E = 2mgl$. This is done based on the derivative

$$\dot{E} = -ml\dot{\theta}\cos\theta u,$$

i.e., at time t, choose $u(t) = u_{\max}$ or $u(t) = -u_{\max}$ such that $\dot{E}(t) > 0$ if $E(t) < 2mgl$ and $\dot{E}(t) < 0$ if $E(t) > 2mgl$. Once $E(t^\star) = 2mgl$ for a given $t^\star$, turn off the control, i.e., $u(t) = 0$, $\forall t \geq t^\star$. Since the energy of the unactuated system is preserved, it holds that $E(t^\star) = 2mgl$, $\forall t \geq t^\star$, and therefore, the system will converge asymptotically to the desired equilibrium $(0, 0)$. This is illustrated in Figs. 11.2.3-11.2.5 for $m = l = J = 1$, $g = 9.81$ and $u_{\max} = 1$. This, however, is not an efficient strategy, for two main reasons. Firstly, the real dynamical system of the pendulum includes dissipative terms of the form $-b\dot{\theta}$, which means that the energy of the unactuated system is no longer preserved. Secondly,

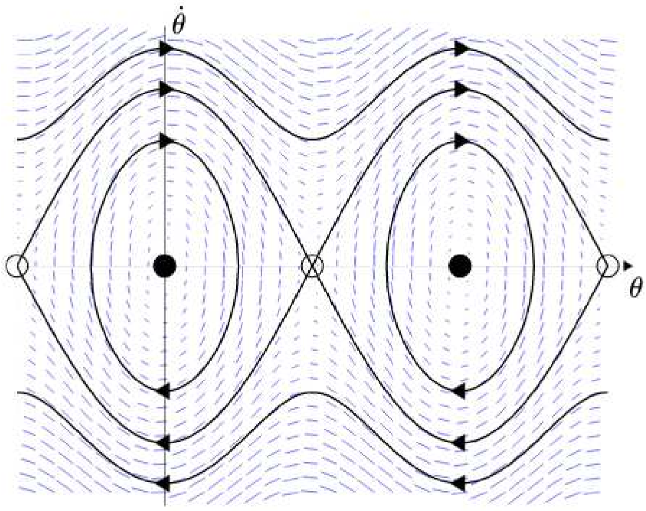


Figure 11.2.1: The phase plot of the unactuated (and undamped) pendulum dynamics.

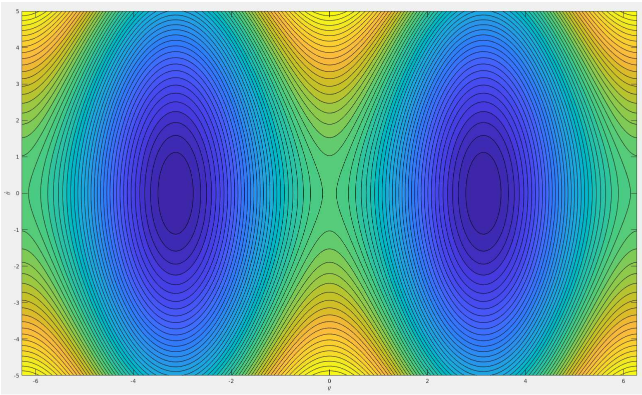


Figure 11.2.2: The energy levels of the unactuated (and undamped) inverted pendulum.

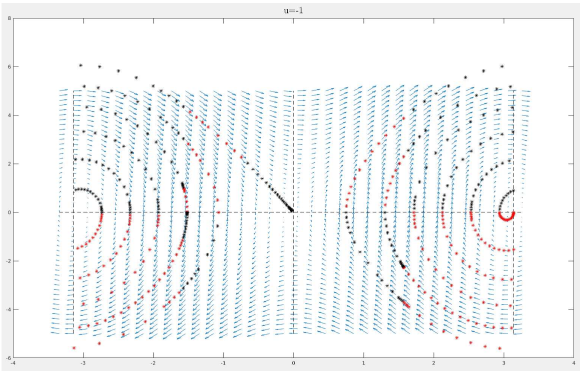


Figure 11.2.3: Phase and traj. plot for $u = -1$

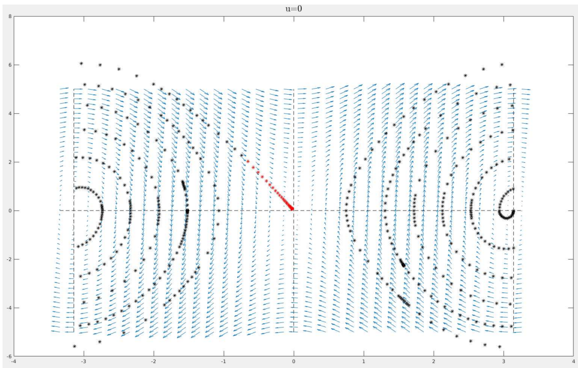


Figure 11.2.4: Phase and traj. plot for $u = 0$

any arbitrarily small disturbance will kick the system out of the energy contour of $2mgl$, and the overall protocol might lead to Zeno behavior.

We discuss now a more efficient control protocol. The idea now is to use the energy-shaping controller as before to swing up the pendulum and switch to a locally stabilizing smooth controller when it is in the respective domain of attraction. For instance, we could switch to a locally stabilizing $u = k_1\theta + k_2\dot{\theta}$ with

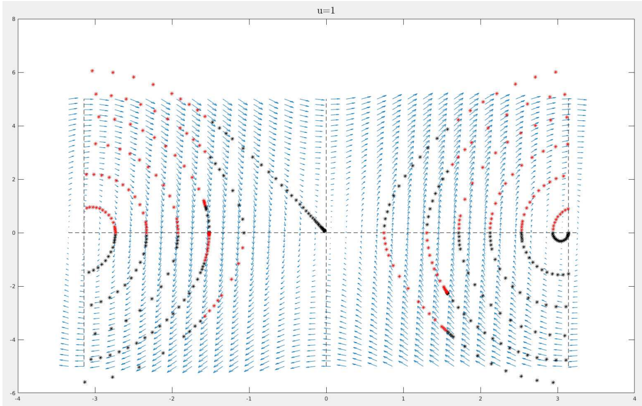


Figure 11.2.5: Phase and traj. plot for $u = 1$

appropriate gains k_1, k_2, which stabilizing the linearized system with respect to the upright configuration ($J\ddot{\theta} = mgl\theta - mlu$). To avoid the linearization around the upright configuration, we can use the locally stabilizing smooth feedback law,

$$k(\theta, \dot{\theta}) = \frac{J}{ml\cos\theta}\left(\frac{mgl}{ml\cos\theta}\sin\theta + k_g\tan\theta + \arctan\dot{\theta}\right),$$

with k_g a positive gain, whose domain of attraction is $\{(\theta, \dot{\theta})^\top : -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$. It is straightforward to verify the asymptotic stability of the pendulum by using the total energy as a candidate Lyapunov function. Hence, we employ the swing-up energy-shaping strategy as before, until the $|\theta| \leq \pi/2 - \varepsilon$, where ε is a small positive constant. Then we switch the controller to the aforementioned one. A resulting trajectory for is shown in Fig. 11.2.6. Note, however, that one needs to deal with the input saturation on the smooth feedback as well.

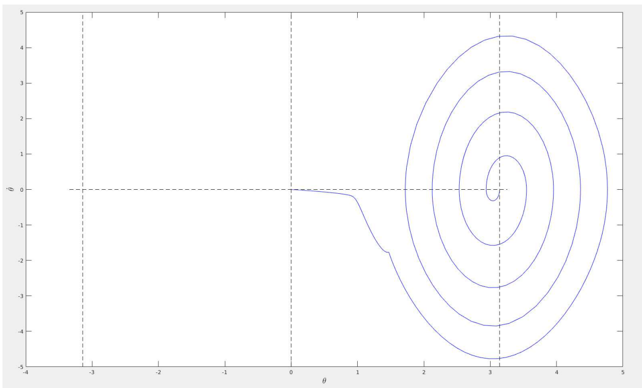


Figure 11.2.6: Phase and traj. plot including the locally stabilizing controller

Solutions to simulation and bisimulation

SOLUTION 12.1

First of all, we notice that as there is only one initial state in T and one in T', it has to hold that $s_1 \sim s'_1$, $s_1 \not\sim s'_i$, and $s'_1 \not\sim s_i$, for all $i \in \{2, \dots 7\}$.

Because $s_1 \sim s'_1$ and $s_1 \xrightarrow{a} s_4$, there must exist s'_i, such that $s'_1 \xrightarrow{a} s'_i$, with the property that $s_4 \sim s'_i$. Because s_4 is accepting, s'_i has to be accepting, too. From $Post_a(s'_1) = \{s'_2\}$, we obtain directly that $s_4 \sim s'_2$.

By similar reasoning from $s_1 \sim s'_1$, $Post_b(s_1) = \{s_2\}$ and $Post_b(s_1) = \{s'_3\}$, we learn that necessarily $s_2 \sim s'_3$.

Furthermore,

- $s_4 \sim s'_2$, $Post_a(s_4) = \{s_6\}$, and $Post_a(s'_2) = \{s'_4\}$ implies $s_6 \sim s'_4$;
- $s_4 \sim s'_2$, $Post_b(s_4) = \{s_5\}$, and $Post_b(s'_2) = \{s'_5\}$ implies $s_5 \sim s'_5$;
- $s_2 \sim s'_3$, $Post_a(s_2) = \{s_5\}$, and $Post_a(s'_3) = \{s'_6\}$ implies $s_5 \sim s'_6$;
- $s_2 \sim s'_3$, $Post_b(s_2) = \{s_3\}$, and $Post_b(s'_3) = \{s'_7\}$ implies $s_3 \sim s'_7$;
- $s_6 \sim s'_4$, $Post_c(s_6) = \{s_7\}$, and $Post_c(s'_4) = \{s'_2\}$ implies $s_7 \sim s'_2$;
- $s_5 \sim s'_5$, $s_5 \sim s'_6$, $Post_c(s_5) = \{s_7\}$, $Post_c(s'_5) = \{s'_3\}$, and $Post_c(s'_6) = \{s'_2\}$ implies $s_7 \sim s'_3$, and $s_7 \sim s'_2$;
- $s_3 \sim s'_7$, $Post_c(s_3) = \{s_7\}$, and $Post_c(s'_7) = \{s'_3\}$ implies $s_7 \sim s'_3$;
- $s_7 \sim s'_2$, $s_7 \sim s'_3$, $Post_a(s_7) = Post_b(s_7) = \{s_5\}$, and $Post_a(s'_2) = \{s'_4\}$, $Post_b(s'_2) = \{s'_5\}$, $Post_a(s'_3) = \{s'_6\}$, and $Post_b(s'_3) = \{s'_7\}$, we obtain $s_5 \sim s'_4$, $s_5 \sim s'_5$, $s_5 \sim s'_6$, and $s_5 \sim s'_7$.
- $s_5 \sim s'_4$, $s_5 \sim s'_7$, $Post_c(s_5) = \{s_7\}$, $Post_c(s'_4) = \{s'_2\}$, and $Post_c(s'_7) = \{s'_3\}$ implies $s_7 \sim s'_2$, and $s_7 \sim s'_3$;

Altogether, we obtain $S \cup S'/_\sim = \big\{\{s_1, s'_1\}, \{s_2, s_4, s_7, s'_2, s'_3\}, \{s_3, s_5, s_6, s'_4, s'_5, s'_6, s'_7\}\big\}$. It is now straightforward to verify that $\sim$ is indeed a bisimulation.

SOLUTION 12.2

We start with the initial partition $S/_\sim = \{P_1 = \{s_1, s_2, s_3, s_4, s_5, s_6\}, P_2 = \{s_7, s_8\}\}$.

Following the bisimulation algorithm, we repeatedly and systematically look for $\sigma \in \Sigma$, and $P, P' \in S/_\sim$, such that $\emptyset \neq P \cap Pre_\sigma(P') \neq P$

- We easily realize that for the action a, and sets $P = P' = P_1$, such a condition holds: $Pre_a(P_1) = \{s_1, s_3, s_2, s_5\}$, and $\emptyset \neq Pre_a(P_1) \neq P_1$. Thus, P_1 is split into $P_{11} = P_1 \cap Pre_a(P_1) = \{s_1, s_2, s_3, s_5\}$, and $P_{12} = P_1 \setminus Pre_a(P_1) = \{s_4, s_6\}$. The partition is now $S/_\sim = \{P_{11} = \{s_1, s_2, s_3, s_5\}, P_{12} = \{s_4, s_6\}, P_2 = \{s_7, s_8\}\}$.
- Now, for the action b, and sets $P = P_{11}$ and $P' = P_{11}$, we have $Pre_b(P_{11}) = \{s_1\} \neq P_1$, and therefore P_{11} has to be split into $P_{111} = \{s_1\}$, and $P_{112} = \{s_2, s_3, s_5\}$. The new partition is $S/_\sim = \{P_{111} = \{s_1\}, P_{112} = \{s_2, s_3, s_5\}, P_{12} = \{s_4, s_6\}, P_2 = \{s_7, s_8\}\}$.
- For the action a, and sets $P = P_{112}$ and $P' = P_{112}$, we have $Pre_a(P_{112}) = \{s_1, s_2\}$, $Pre_a(P_{112}) \cap P_{112} \neq P_{112}$, and therefore P_{112} has to be split into $P_{1121} = \{s_2\}$, and $P_{1122} = \{s_3, s_5\}$. The new partition is $S/_\sim = \{P_{111} = \{s_1\}, P_{1121} = \{s_2\}, P_{1122} = \{s_3, s_5\}, P_{12} = \{s_4, s_6\}, P_2 = \{s_7, s_8\}\}$. Note, that the same splitting would be induced if we consider the action c and sets $P = P_{112}$, and $P' = P_2$, because $Pre_c(P_2) = \{s_3, s_5\}$, and $P_{112} \cap Pre_c(P_2) \neq P_{112}$.
- At this point it looks like there is no other action $\sigma \in \Sigma$, and $P, P' \in S/_\sim$, such that $\emptyset \neq P \cap Pre_\sigma(P') \neq P$. Let us verify this is indeed true:
  - $Pre_a(P_{111}) = Pre_b(P_{111}) = Pre_c(P_{111}) = Pre_a(P_{1121}) = Pre_b(P_{1121}) = Pre_c(P_{1121}) = \emptyset$
  - $Pre_a(P_{1122}) = \{s_1, s_2\} = P_{111} \cup P_{1121}$, which means that $Pre_a(P_{1122}) \cap P_{111} = P_{111}$, $Pre_a(P_{1122}) \cap P_{1121} = P_{1121}$, and $Pre_a(P_{1122}) \cap P_{1122} = Pre_a(P_{1122}) \cap P_{12} = Pre_a(P_{1122}) \cap P_2 = \emptyset$.

- $Pre_b(P_{1122}) = \{s_1\} = P_{111}$, which means that $Pre_b(P_{1122}) \cap P_{111} = P_{111}$, and $Pre_b(P_{1122}) \cap P_{1121} = Pre_b(P_{1122}) \cap P_{1122} = Pre_b(P_{1122}) \cap P_{12} = Pre_a(P_{21}) \cap P_2 = \emptyset$.
- $Pre_c(P_{1122}) = \emptyset$
- $Pre_a(P_{12}) = \{s_3, s_5\} = P_{1122}$
- $Pre_b(P_{12}) = \{s_1, s_2\} = P_{111} \cup P_{1121}$
- $Pre_c(P_{12}) = \emptyset$
- $Pre_a(P_2) = \{s_7, s_8\} = P_2$
- $Pre_b(P_2) = \{s_3, s_5\} = P_{1122}$
- $Pre_c(P_2) = \{s_4, s_6\} = P_{12}$

Thus, we can conclude that $S/_\sim = \{P_{111} = \{s_1\}, P_{1121} = \{s_2\}, P_{1122} = \{s_3, s_5\}, P_{12} = \{s_4, s_6\}, P_2 = \{s_7, s_8\}\}$ is the final partition of the set of states S and the coarsest quotient transition system $T/_\sim = (\{P_{111}, P_{1121}, P_{1122}, P_{12}, P_2\}, \Sigma, \to /_\sim, \{P_{111}, P_{1121}\}, \{P_2\})$, where $\to /_\sim$ is defined by the following figure

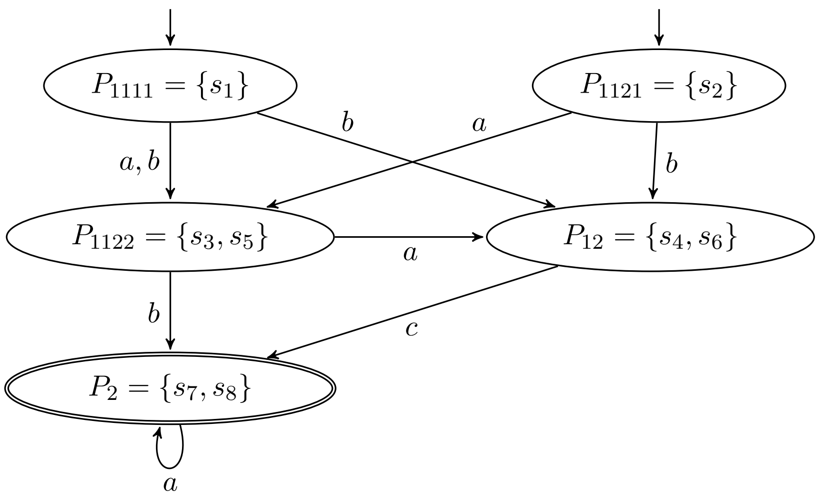


Figure 12.2.1: The transition systems $T/_\sim$.

SOLUTION 12.3

Let us follow the hint and find a transition system T that is not equal to its coarsest bisimulation quotient. For that, it is actually enough to put in T two different initial states s and s', such that $Post_\sigma(s) = Post_\sigma(s')$, for all $\sigma \in \Sigma$. A simple example of such system together with its bisimulation quotient is given in the following figure.

Now, consider the following T', which is in fact the same as T except for the state names.

The first bisimulation straightforwardly maps the states of T to the states of T':

$S \cup S'/_{\sim_1} = \{\{s_1, s'_1\}, \{s_2, s'_2\}, \{s_3, s'_3\}\}$. The second bisimulations builds on the bisimulation quotients of T and T':

$S \cup S'/_{\sim_2} = \{\{s_1, s'_1, s_2, s'_2\}, \{s_3, s'_3\}\}$.

SOLUTION 12.4

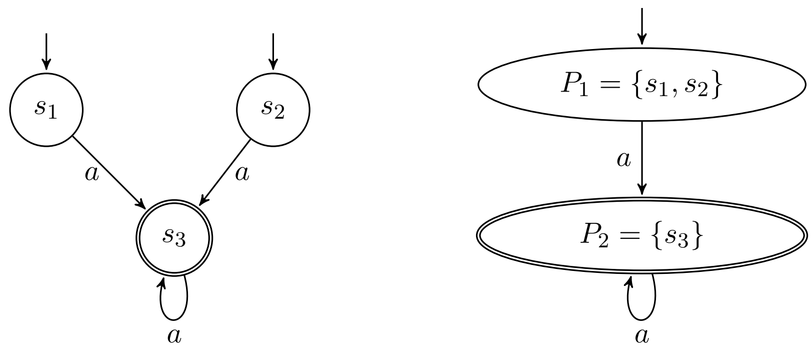


Figure 12.3.1: The transition system T and its coarsest bisimulation quotient $T/_\sim$.

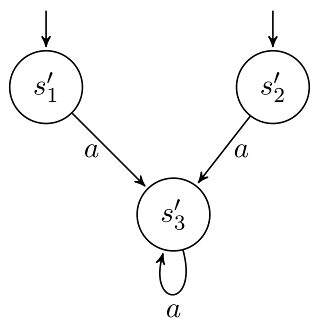


Figure 12.3.2: The transition system T'.

a)
- T is not bisimilar with T_1. The proof is lead by contradiction: Let us assume that there exists a bisimulation relation $\sim \subseteq S \times Q$. For each initial state s in T there must exist an initial state q in T_1, such that $s \sim q$. Furthermore, because $Post_c(s_1) = \emptyset$ and $Post_c(q_2) \neq \emptyset$, it must be that $s_1 \sim q_1$. Similarly, we obtain that $s_2 \sim q_2$. From these, we deduce that $s_3 \sim q_3$ and $s_4 \sim q_4$, however $Post_a(s_3) \neq \emptyset$, while $Post_a(q_3) = \emptyset$. Thus $s_3 \not\sim q_3$, which is a contradiction.
- T is bisimilar with T_2. The bisimulation relation is defined as follows: $S \cup R/_\sim = \{\{s_1, r_1\}, \{s_2, r_2\}, \{s_{2i+1}, r_3 \mid i \geq 1\}, \{s_{2i}, r_4 \mid i \geq 2\}\}$.
- T is bisimilar with T_3. The bisimulation relation is defined as follows: $S \cup U/_\sim = \{\{s_{2i+1}, u_1 \mid i \geq 0\}, \{s_{2i}, u_2 \mid i \geq 1\}\}$.
- T is not bisimilar with T_4. Let us assume that there exists a bisimulation relation $\sim \subseteq S \times V$. Necessarily, $s_i \sim v_1$, for all $i \geq 1$. However $Post_c(s_1) = \emptyset$, while $Post_c(v_1) = \{v_1\}$, and therefore $s_1 \not\sim v_1$. Contradiction.

b) From the above, we have that T simulates T_2, T_3 and that T_2, T_3 simulates T. Intuitively, T_1 allows for strictly less behavior than T, however, T covers all behaviors that are allowed in T_1, meaning that T simulates T_1. The simulation relation is $\sim \subseteq Q \times S$, where $q_1 \sim s_1, q_2 \sim s_2, q_3 \sim s_3$, and $q_4 \sim s_4$. Because T, T_2, T_3 are bisimilar, then also T_2, T_3 simulate T_1.

Dually, T_4 simulates T with simulation relation $\sim \subseteq S \times V$, where $s_i \sim v_1$, for all $i \geq 1$. T_4 thus simulates also T_1, T_2, and T_3.

SOLUTION 12.5

- Transition systems T and T' are bisimilar. To prove that, we define a relation

$$\sim \subseteq S \times S' = \{(s_1, s_1'), (s_2, s_2'), (s_3, s_2')\},$$

which is a bisimulation relation because $\sim$ is a simulation relation from T to T' and $\sim^{-1} = \{(s_1', s_1), (s_2', s_2), (s_2', s_3)\} \subseteq S' \times S$ is a simulation relation from T' to T.

- Furthermore, transition systems T' and T'' are bisimilar, too. To prove that, we define a relation

$$\sim' \subseteq S \times S' = \{(s_1', s_1''), (s_2', s_2''), (s_2', s_3'')\},$$

which is a bisimulation relation because $\sim' \subseteq S' \times S''$ is a simulation relation from T' to T'' and $sim'^{-1} = \{(s_1'', s_1'), (s_2'', s_2'), (s_3'', s_2')\} \subseteq S'' \times S'$ is a simulation relation from T'' to T'.

- Finally, since T and T' are bisimilar as well as T' and T'', we immediately obtain from the properties of bisimulation, that T and T'' are also bisimilar through the bisimulation relation

$$\sim'' \subseteq S'' \times S = \{(s_1'', s_1), (s_2'', s_2), (s_2'', s_3), (s_3'', s_3)\}.$$

To properly prove that $\sim$ is a bisimulation relation, i.e. (i) that $\sim \subseteq S \times S'$ is a simulation relation, and (ii) that $\sim^{-1} \subseteq S' \times S$ is also a simulation relation we need to check the three conditions that define the simulation relation:

(i)
1. Condition 1 ($\forall s \in S_0\ (\exists s' \in S_0'.\ s \sim s')$) holds, because $S_0 = \{s_1\}$, $s_1 \sim s_1'$, and $s_1' \in S_0'$.
2. Condition 2 ($s \sim s' \wedge s \in S_F \Rightarrow s' \in S_F'$) holds, because $S_F = \{s_2, s_3\}$, $\sim \cap S_F \times S' = \{(s_2, s_2'), (s_3, s_2')\}$ and $s_2' \in S_F'$.
3. Condition 3 ($\forall \sigma \in \Sigma\ (s \sim s' \wedge s \xrightarrow{\sigma} r \Rightarrow \exists r' \in S'$ such that $s' \xrightarrow{\sigma}{}' r'$ and $r \sim r'$)) holds, because:
   - For $s_1 \sim s_1'$, $s_1 \xrightarrow{a} s_2$, there is $s_1' \xrightarrow{a}{}' s_2'$, $s_2 \sim s_2'$
   - For $s_1 \sim s_1'$, $s_1 \xrightarrow{b} s_3$, there is $s_1' \xrightarrow{b}{}' s_2'$, $s_3 \sim s_2'$
   - For $s_2 \sim s_2'$, $s_2 \xrightarrow{c} s_3$, there is $s_2' \xrightarrow{c}{}' s_2'$, $s_3 \sim s_2'$
   - For $s_3 \sim s_2'$, $s_3 \xrightarrow{c} s_3$, there is $s_2' \xrightarrow{c}{}' s_2'$, $s_3 \sim s_2'$

(ii)
1. Condition 1 ($\forall s' \in S_0'\ (\exists s \in S_0.\ s' \sim^{-1} s)$) holds, because $S_0' = \{s_1'\}$, $s_1' \sim^{-1} s_1$, and $s_1 \in S_0$.
2. Condition 2 ($s' \sim^{-1} s \wedge s' \in S_F' \Rightarrow s \in S_F$) holds, because $S_F' = \{s_2'\}$, $\sim^{-1} \cap S_F' \times S = \{(s_2', s_2), (s_2', s_3)\}$ and $s_2, s_3 \in S_F$.
3. Condition 3 ($\forall \sigma \in \Sigma\ (s' \sim^{-1} s \wedge s' \xrightarrow{\sigma}{}' r' \Rightarrow \exists r \in S$ such that $s \xrightarrow{\sigma} r$ and $r' \sim^{-1} r$)) holds, because:
   - For $s_1' \sim^{-1} s_1$, $s_1' \xrightarrow{a}{}' s_2'$, there is $s_1 \xrightarrow{a} s_2$, $s_2' \sim^{-1} s_2$
   - For $s_1' \sim^{-1} s_1$, $s_1' \xrightarrow{b}{}' s_2'$, there is $s_1 \xrightarrow{b} s_3$, $s_2' \sim^{-1} s_3$
   - For $s_2' \sim^{-1} s_2$, $s_2' \xrightarrow{c}{}' s_2'$, there is $s_2 \xrightarrow{c} s_3$, $s_2' \sim^{-1} s_3$
   - For $s_2' \sim^{-1} s_3$, $s_2' \xrightarrow{c}{}' s_2'$, there is $s_3 \xrightarrow{c} s_3$, $s_2' \sim^{-1} s_3$

Thereby, $\sim$ is a bisimulation relation and T and T' is bisimilar.

To properly prove that $\sim'$ is a bisimulation relation, i.e. (i) that $\sim' \subseteq S' \times S''$ is a simulation relation, and (ii) that $\sim'^{-1} \subseteq S'' \times S'$ is also a simulation relation similarly as above we need to check the three conditions that define the simulation relation:

(i)
1. Condition 1 ($\forall s' \in S_0'\ (\exists s'' \in S_0''.\ s' \sim' s'')$) holds, because $S_0' = \{s_1'\}$, $s_1' \sim' s_1''$, and $s_1'' \in S_0''$.

2. Condition 2 ($s' \sim' s'' \wedge s' \in S'_F \Rightarrow s'' \in S''_F$) holds, because $S'_F = \{s'_2\}$, $\sim' \cap S'_F \times S'' = \{(s'_2, s''_2), (s'_2, s''_3)\}$ and $s''_2, s''_3 \in S''_F$.

3. Condition 3 ($\forall \sigma \in \Sigma$ ($s' \sim' s'' \wedge s' \xrightarrow{\sigma}{}' r' \Rightarrow \exists r'' \in S''$ such that $s'' \xrightarrow{\sigma}{}'' r''$ and $r' \sim' r''$)) holds, because:
   * For $s'_1 \sim' s''_1$, $s'_1 \xrightarrow{a}{}' s'_2$, there is $s''_1 \xrightarrow{a}{}'' s''_2$, $s'_2 \sim' s''_2$
   * For $s'_1 \sim' s''_1$, $s'_1 \xrightarrow{b}{}' s'_2$, there is $s''_1 \xrightarrow{b}{}'' s''_3$, $s'_2 \sim' s''_3$
   * For $s'_2 \sim' s''_2$, $s'_2 \xrightarrow{c}{}' s'_2$, there is $s''_2 \xrightarrow{c}{}'' s''_2$, $s'_2 \sim' s''_2$
   * For $s'_2 \sim' s''_3$, $s'_2 \xrightarrow{c}{}' s'_2$, there is $s''_3 \xrightarrow{c}{}'' s''_3$, $s'_2 \sim' s''_3$

(ii)
1. Condition 1 ($\forall s'' \in S''_0$ ($\exists s' \in S'_0$. $s'' \sim'^{-1} s'$)) holds, because $S''_0 = \{s''_1\}$, $s''_1 \sim'^{-1} s'_1$, and $s'_1 \in S'_0$.

2. Condition 2 ($s'' \sim'^{-1} s' \wedge s'' \in S''_F \Rightarrow s' \in S'_F$) holds, because $S''_F = \{s''_2, s''_3\}$, $\sim \cap S''_F \times S' = \{(s''_2, s'_2), (s''_3, s'_2)\}$ and $s'_2 \in S'_F$.

3. Condition 3 ($\forall \sigma \in \Sigma$ ($s'' \sim s' \wedge s'' \xrightarrow{\sigma}{}'' r'' \Rightarrow \exists r' \in S'$ such that $s' \xrightarrow{\sigma}{}' r'$ and $r \sim r'$)) holds, because:
   * For $s''_1 \sim'^{-1} s'_1$, $s''_1 \xrightarrow{a}{}'' s''_2$, there is $s'_1 \xrightarrow{a}{}' s'_2$, $s''_2 \sim'^{-1} s'_2$
   * For $s''_1 \sim'^{-1} s'_1$, $s''_1 \xrightarrow{b}{}'' s''_3$, there is $s'_1 \xrightarrow{b}{}' s'_2$, $s''_3 \sim'^{-1} s'_2$
   * For $s''_2 \sim'^{-1} s'_2$, $s''_2 \xrightarrow{c}{}'' s''_2$, there is $s'_2 \xrightarrow{c}{}' s'_2$, $s''_2 \sim'^{-1} s'_2$
   * For $s'_3 \sim'^{-1} s'_2$, $s''_3 \xrightarrow{c}{}'' s''_3$, there is $s'_2 \xrightarrow{c}{}' s'_2$, $s''_3 \sim'^{-1} s'_2$

Thereby, $\sim'$ is a bisimulation relation and T' and T'' is bisimilar.

SOLUTION 12.6

The transition system $T_H = (S = Q \times X \times Y, \Sigma, \rightarrow, S_0)$ looks as follows:

It can be seen that while in T_H, there has to be a time-driven transition labeled with t between successive event-driven transitions $turn\_right$ and $turn\_left$, in T_1, $turn\_left$ may follow $turn\_right$ immediately. Therefore, they are not bisimilar.

In T_2, we can again find behavior that is not admitted in T_H. Namely, in T_H, it is not possible to perform sequence of transitions generated by the sequence $t, turn\_right, t, turn\_right$.

T_3 is bisimilar with T_H. The bisimulation relation is $s_1 \sim (q_1, x, y)$, where $x < 10$, $s_2 \sim (q_1, x, y)$, where $x \geq 10$, $s_3 \sim (q_2, x, y)$, where $x > 0$, and finally $s_3 \sim (q_2, x, y)$, where $x \leq 0$.

T_4 clearly allows only for behavior that is allowed in T_H. However, it does not allow for all of the behavior. Namely, at least two t-transitions are required before $turn\_right$ can happen, whereas in T_H only one t-transition has to happen.

SOLUTION 12.7

(a) Consider the relations:

$$\sim = \{(s_1, s'_1), (s_2, s'_3), (s_3, s'_2), (s_4, s'_4), (s_5, s'_5)\} \subseteq S \times S',$$
$$\sim^{-1} = \{(s'_1, s_1), (s'_3, s_2), (s'_2, s_3), (s'_4, s_4), (s'_5, s_5)\} \subseteq S' \times S.$$

The relation $\sim$ is a simulation relation from T to T' since the following hold:

1. For the state $s_1 \in S$, there exists the state $s'_1 \in S'$ such that $(s_1, s'_1) \in\sim$.
2. We have that $(s_5, s'_5) \in\sim$ with $s_5 \in S_F$ and $s'_5 \in S'_F$.
3. – $(s_1, s'_1) \in\sim$, $(s_1, a, s_2) \in\rightarrow$, and at the same time $(s'_1, a, s'_3) \in\rightarrow'$, and $(s_2, s'_3) \in\sim$.

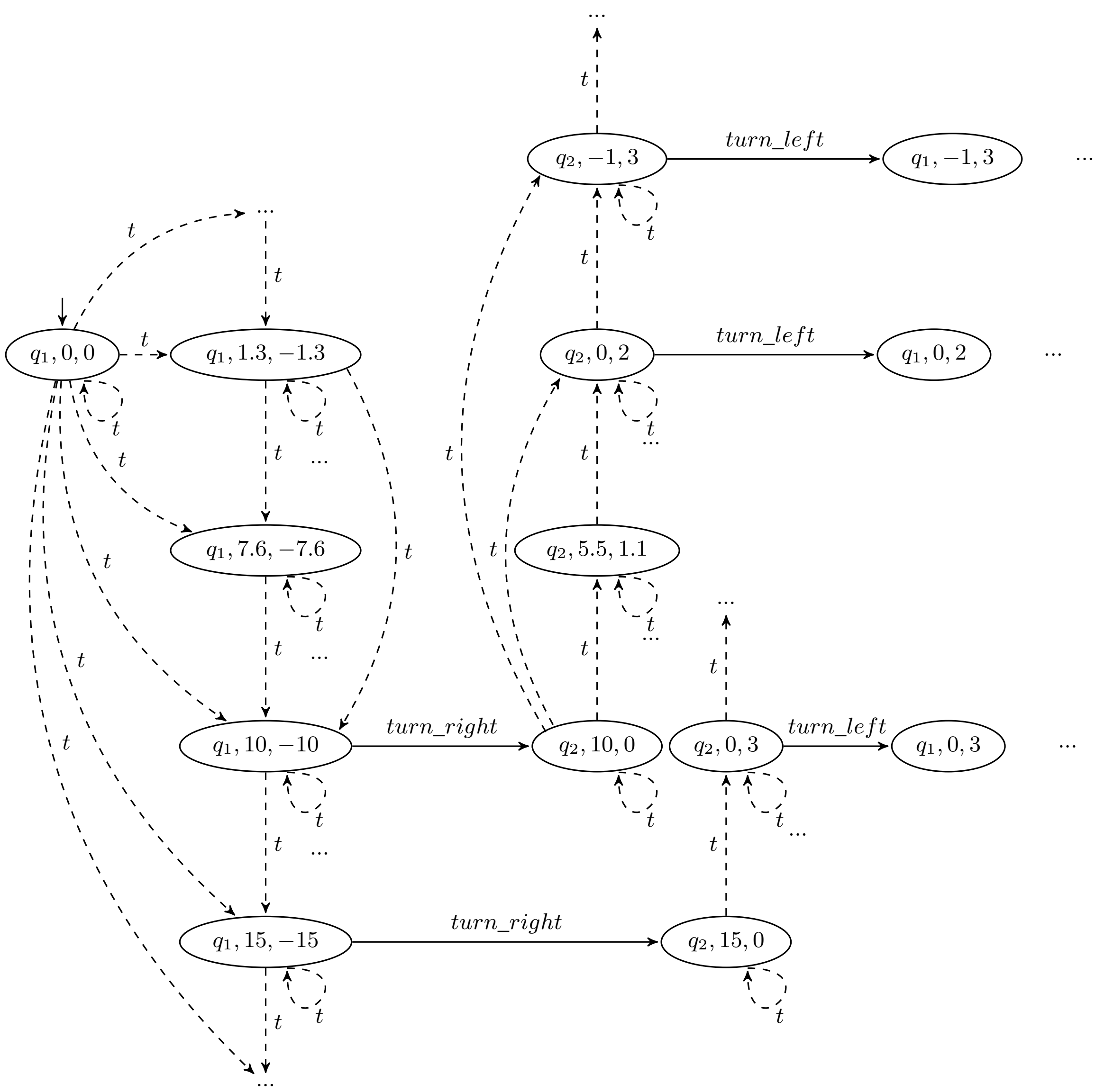


Figure 12.6.1: Informal illustration of transition system T_H. The time-driven transitions are illustrated in dashed lines and the even-driven in full lines.

- $(s_1, s_1') \in\sim, (s_1, b, s_3) \in\rightarrow$, and at the same time $(s_1', b, s_2') \in\rightarrow'$, and $(s_3, s_2') \in\sim$.
- $(s_2, s_3') \in\sim, (s_2, a, s_4) \in\rightarrow$, and at the same time $(s_3', a, s_4') \in\rightarrow'$, and $(s_4, s_4') \in\sim$.
- $(s_2, s_3') \in\sim, (s_2, c, s_5) \in\rightarrow$, and at the same time $(s_3', c, s_5') \in\rightarrow'$, and $(s_5, s_5') \in\sim$.
- $(s_3, s_2') \in\sim, (s_3, b, s_4) \in\rightarrow$, and at the same time $(s_2', b, s_4') \in\rightarrow'$, and $(s_4, s_4') \in\sim$.
- $(s_3, s_2') \in\sim, (s_3, c, s_5) \in\rightarrow$, and at the same time $(s_2', c, s_5') \in\rightarrow'$, and $(s_5, s_5') \in\sim$.
- $(s_4, s_4') \in\sim, (s_4, a, s_4) \in\rightarrow$, and at the same time $(s_4', a, s_4') \in\rightarrow'$, and $(s_4, s_4') \in\sim$.
- $(s_4, s_4') \in\sim, (s_4, b, s_4) \in\rightarrow$, and at the same time $(s_4', b, s_4') \in\rightarrow'$, and $(s_4, s_4') \in\sim$.
- $(s_4, s_4') \in\sim, (s_4, c, s_5) \in\rightarrow$, and at the same time $(s_4', c, s_5') \in\rightarrow'$, and $(s_5, s_5') \in\sim$.
- $(s_5, s_5') \in\sim, (s_5, a, s_1) \in\rightarrow$, and at the same time $(s_5', a, s_1') \in\rightarrow'$, and $(s_1, s_1') \in\sim$.
- $(s_5, s_5') \in\sim, (s_5, b, s_1) \in\rightarrow$, and at the same time $(s_5', b, s_1') \in\rightarrow'$, and $(s_1, s_1') \in\sim$.
- $(s_5, s_5') \in\sim, (s_5, c, s_5) \in\rightarrow$, and at the same time $(s_5', c, s_5') \in\rightarrow'$, and $(s_5, s_5') \in\sim$.

It can be shown analogously, that the three conditions for the relation $\sim^{-1}$ hold. Thus, $\sim^{-1}$ is a relation from $\mathcal{T}'$ to $\mathcal{T}$. Therefore, $\sim$ is a *bisimulation relation* and the Transition Systems $\mathcal{T}, \mathcal{T}'$ are *bisimilar*.

(b) We start with the partition $S_{/\sim} = \{\{s_1, s_2, s_3, s_4\}, \{s_5\}\}$.

Step 1: Set $P = \{s_1, s_2, s_3, s_4\}$ and $P' = \{s_5\}$. We have that

$$\emptyset \neq P \cap \text{Pre}_c(P') = \{s_1, s_2, s_3, s_4\} \cap \{s_2, s_3, s_4, s_5\} = \{s_2, s_3, s_4\} \neq P.$$

Then, set:

$$\begin{aligned} P_1 &= P \cap \text{Pre}_c(P') = \{s_2, s_3, s_4\}, \\ P_2 &= P \backslash \text{Pre}_c(P') = \{s_1\}, \\ S_{/\sim} &= \{\{s_5\}, \{s_1\}, \{s_2, s_3, s_4\}\}. \end{aligned}$$

Step 2: Set $P = P' = \{s_2, s_3, s_4\}$. We have that

$$\emptyset \neq P \cap \text{Pre}_b(P') = \{s_2, s_3, s_4\} \cap \{s_1, s_3, s_4\} = \{s_3, s_4\} \neq P.$$

Then, set:

$$\begin{aligned} P_1 &= P \cap \text{Pre}_b(P') = \{s_3, s_4\}, \\ P_2 &= P \backslash \text{Pre}_b(P') = \{s_2, s_3, s_4\} \backslash \{s_1, s_3, s_4\} = \{s_2\}, \\ S_{/\sim} &= \{\{s_1\}, \{s_2\}, \{s_3, s_4\}, \{s_5\}\}. \end{aligned}$$

Step 3: Set $P = P' = \{s_3, s_4\}$. We have that

$$\emptyset \neq P \cap \text{Pre}_a(P') = \{s_3, s_4\} \cap \{s_4\} = \{s_4\} \neq P.$$

Then, set:

$$\begin{aligned} P_1 &= P \cap \text{Pre}_a(P') = \{s_4\}, \\ P_2 &= P \backslash \text{Pre}_a(P') = \{s_3, s_4\} \backslash \{s_4\} = \{s_3\}, \\ S_{/\sim} &= \{\{s_1\}, \{s_2\}, \{s_3\}, \{s_4\}, \{s_5\}\}. \end{aligned}$$

Thus, the given Transition System is the minimal quotient.

(d) (i) The state is reachable with the following sequence of transitions:

$$\begin{aligned} (q_1, 0, 0, 0) &\overset{Time=1}{\longrightarrow} (q_1, 1, 1, 1) \\ &\overset{Action=red}{\longrightarrow} (q_2, 1, 1, 0) \\ &\overset{Time=1}{\longrightarrow} (q_1, 2, 2, 1) \\ &\overset{Action=blue}{\longrightarrow} (q_3, 0, 2, 1) \end{aligned}$$

(ii) The state is reachable with the following sequence of transitions:

$$
\begin{aligned}
(q_1, 0, 0, 0) &\overset{Time=2}{\longrightarrow}(q_1, 2, 2, 2)\\
&\overset{Action=red}{\longrightarrow}(q_2, 2, 2, 0)\\
&\overset{Time=1}{\longrightarrow}(q_3, 3, 3, 1)\\
&\overset{Action=blue}{\longrightarrow}(q_3, 0, 3, 1)\\
&\overset{Time=1}{\longrightarrow}(q_3, 1, 4, 2)\\
&\overset{Action=green}{\longrightarrow}(q_2, 1, 0, 2)
\end{aligned}
$$

(iii) The state is reachable with the following sequence of transitions:

$$
\begin{aligned}
(q_1, 0, 0, 0) &\overset{Time=2}{\longrightarrow}(q_1, 2, 2, 2)\\
&\overset{Action=red}{\longrightarrow}(q_2, 2, 2, 0)\\
&\overset{Time=1}{\longrightarrow}(q_3, 3, 3, 1)\\
&\overset{Action=blue}{\longrightarrow}(q_3, 0, 3, 1)\\
&\overset{Time=1}{\longrightarrow}(q_3, 1, 4, 2)
\end{aligned}
$$

(iv) The state is reachable with the following sequence of transitions:

$$
\begin{aligned}
(q_1, 0, 0, 0) &\overset{Time=1}{\longrightarrow}(q_1, 1, 1, 1)\\
&\overset{Action=red}{\longrightarrow}(q_2, 1, 1, 0)\\
&\overset{Time=2}{\longrightarrow}(q_3, 3, 3, 2)\\
&\overset{Action=blue}{\longrightarrow}(q_3, 0, 3, 2)
\end{aligned}
$$

(iv) The state is not reachable since then a run reaches state q_3, clock x_1 is reset, and then in order clock x_1 to be added by 3 time units, will make clock x_3 to go out of the domain limits.

Solutions to reachability, timed automata and rectangular automata

SOLUTION 13.1

a) In state $(q_{\text{off,off}}, x = 175, y \in [100, 200])$, only time-driven transitions are available and no event-driven ones. Furthermore, in state $q_{\text{off,off}}$, the value of x cannot exceed 200. Assuming that the value of x is fixed, the value of y can be computed as follows: $y_{min} = 100 - 18 \cdot (x - 175)/5$, and $x_{max} = 200 - 18 \cdot (x - 175)/5$. Thus, $Post(q_{\text{off,off}}, x = 175, y \in [100, 200]) =$

$$\{(q_{\text{off,off}}, x, y \mid x \in [175, 200], y \in [100 - 18 \cdot (x - 175)/5, 200 - 18 \cdot (x - 175)/5].$$

b) In state $(q_{\text{off,off}}, x = 200, y = 200)$, the only time-driven transition enabled is with $time = 0$, as x cannot exceed 200 in $q_{\text{off,off}}$. There is however an event-driven transition enabled, leading into the state $q_{\text{off,on}}, x = 200, y = 200$. Thus, $Post(q_{\text{off,off}}, x = 200, y = 200) =$

$$\{(q_{\text{off,off}}, x = 200, y = 200), (q_{\text{off,on}}, x = 200, y = 200)\}.$$

c) The only time-driven transition that leads to $(q_{\text{on,on}}, x = 150, y = 200)$ is with $t = 0$ as x cannot exceed 150 in $q_{\text{on,on}}$ while at the same time the value of x is decreasing in there. Any event-driven transition from $(q_{\text{off,on}}, x = 150, y)$, where y is arbitrary leads to $(q_{\text{on,on}}, x = 150, y = 200)$. Thus, $Pre(q_{\text{on,on}}, x = 150, y = 200) =$

$$\{(q_{\text{on,on}}, x = 150, y = 200)\} \cup \{(q_{\text{off,on}}, x = 150, y) \mid y \in \mathbb{R}\}.$$

d) Let us follow the hint:

$Post(q_{\text{off,off}}, x = 190, y = 200) = \{(q_{\text{off,off}}, x \in [190, 200], y = 200 - 18 \cdot (x - 190)/5\}$. Note that $y \in [200 - 18 * 2, 200]$, i.e. $y \in [164, 200]$ and is within safe bounds.

$Post(q_{\text{off,off}}, x = 200, y = 164) = \{q_{\text{off,on}}, x = 200, y = 164)\}$

$Post(q_{\text{off,on}}, x = 200, y = 164) = \{(q_{\text{off,on}}, x \in [150, 200], y = 164 + 12 \cdot (x - 200)/-25\}$. Note that $y \in [164, 164 + 12 \cdot 2]$, i.e. $y \in [164, 188]$ and is within safe bounds.

$Post(q_{\text{off,on}}, x = 150, y = 188) = \{q_{\text{on,on}}, x = 150, y = 188)\}$

$Post(q_{\text{on,on}}, x = 150, y = 188) = \{(q_{\text{on,on}}, x \in [100, 150], y = 188 + 12 \cdot (x - 150)/-5\}$. Note that $y \in [188, 188 + 12 \cdot 10]$, i.e. $y \in [164, 308]$ and is within safe bounds.

$Post(q_{\text{on,on}}, x = 100, y = 308) = \{q_{\text{on,off}}, x = 100, y = 308)\}$

$Post(q_{\text{on,off}}, x = 100, y = 308) = \{(q_{\text{off,on}}, x \in [100, 175], y = 308 - 18 \cdot (x - 100)/25\}$. Note that $y \in [308 - 18 \cdot 3, 308] = [308 - 54, 308]$, i.e. $y \in [254, 308]$ and is within safe bounds.

$Post(q_{\text{on,off}}, x = 175, y = 254) = \{q_{\text{off,off}}, x = 175, y = 254)\}$

$Post(q_{\text{on,off}}, x = 175, y = 182) = \{(q_{\text{off,on}}, x \in [175, 200], y = 254 - 18 \cdot (x - 175)/5\}$. Note that $y \in [254, 254 - 18 \cdot 5] = [254, 254 - 90]$, i.e. $y \in [164, 254]$ and is within safe bounds. Now, we realize that the system reached once again the state $(q_{\text{off,off}}, x = 200, y = 164)$. As the system is deterministic, we conclude that we have found all reachable states.

e) No, such state is not reachable as we see from the above reachability analysis.

SOLUTION 13.2

For simplicity, we show only the states reachable from the initial state.

SOLUTION 13.3

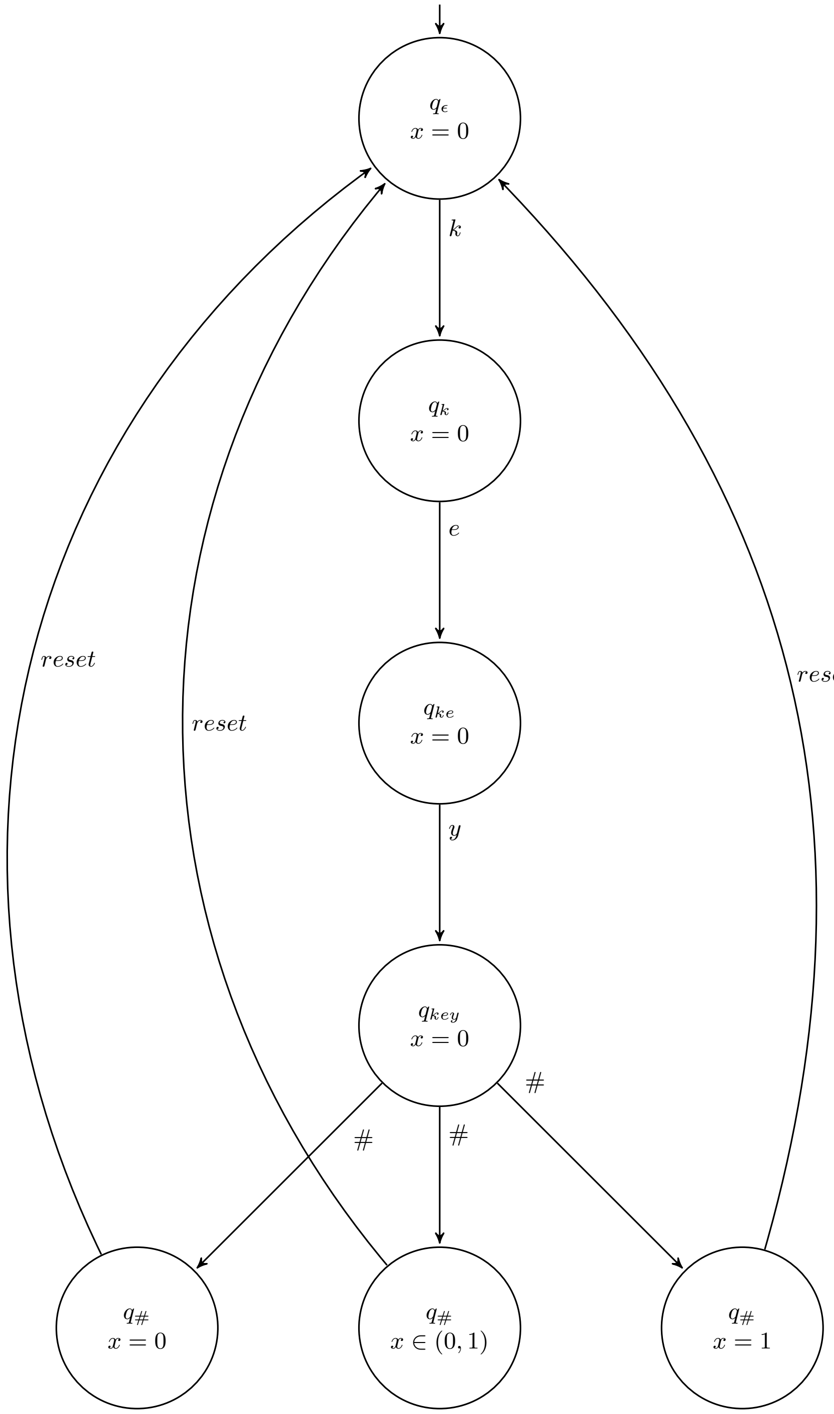


Figure 13.2.1: Region automaton $Reg(A)$.

(1) for the first part,

- The initial state $(q_1, 0)$ is trivially reachable from itself.
- The state $(q_2, 0)$ is reachable in $[TA]]/_\sim$ from $(q_1, 0)$ through a transition that corresponds to a sequence of transitions $(q_1, 0) \xrightarrow{time} (q_1, 1) \xrightarrow{tick} (q_2, 0)$ in TA.
- The state $(q_1, 0.5)$ is reachable in $[TA]]/_\sim$ from $(q_2, 0)$ through a transition that corresponds to a sequence

of transitions $(q_2, 0) \xrightarrow{time} (q_2, 0.5) \xrightarrow{tock} (q_1, 0.5)$ in TA.

Following similar reasoning as in the third point, we could come up with numerous other examples of reachable states, for instance $(q_1, 0)$, $(q_1, 0.0001)$, $(q_1, 0.99)$, or $(q_1, 1)$.

(2) The previous exercise help us to construct the region graph. Informally, we first notice, that in $[[TA]]$, the only admissible value of the clock in state q_2 is 0 due to the reset present on the only event-type transition $tick$ that leads to q_2. When leaving from q_2, the value of x belongs to $[0, 1]$ due to the domain of state q_2 and this value remains unchanged during the execution of $tock$. Hence, upon arrival to q_1, the value of x is within $[0, 1]$. Now, we can repeat the above reasoning to show that these are in fact all of the states reachable from $(q_1, 0)$. Applying the method of getting $[[TA]]/_\sim$ from $[[TA]]$, we obtain the following solution in Fig. 13.3.1.

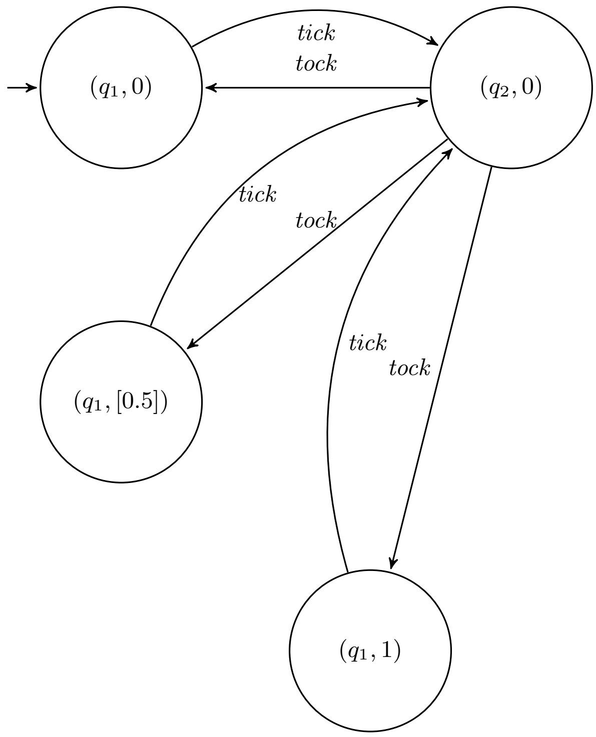


Figure 13.3.1: The region automaton $[[TA]]/_\sim$.

SOLUTION 13.4

Yes, it is initialized. We first make the translation to a multi-rate automaton and from then we continue with the translation to a timed automaton.

SOLUTION 13.5

The trajectories of the linear system are shown in Figure 13.5.1. It easy to prove that no trajectories intersects the *Bad* set.

SOLUTION 13.7

The partition of the state-space is shown in Figure 13.7.1.

(a) Let us consider the system in the region Ω_3. It is easy to determine a linear state feedback $g_3(x) = K_3 x$ which places the poles in $p_1 = p_2 = -1$, since the system is controllable. The feedback control gain is in this case

$$K_3 = \begin{pmatrix} -\frac{4}{3} & -3 \end{pmatrix}.$$

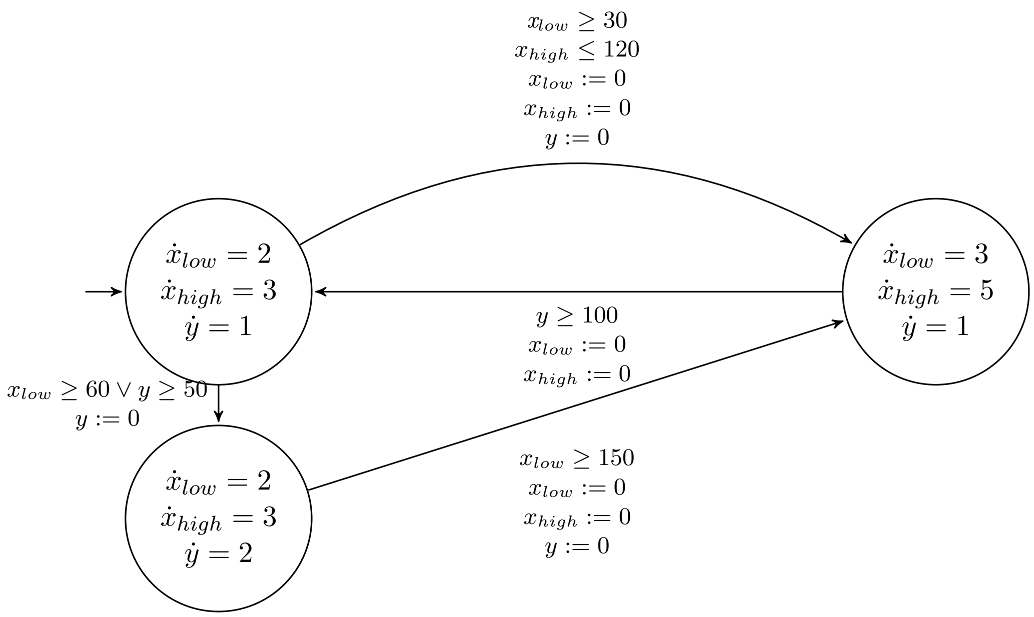


Figure 13.4.1: Multi-rate automaton $\hat{A}$.

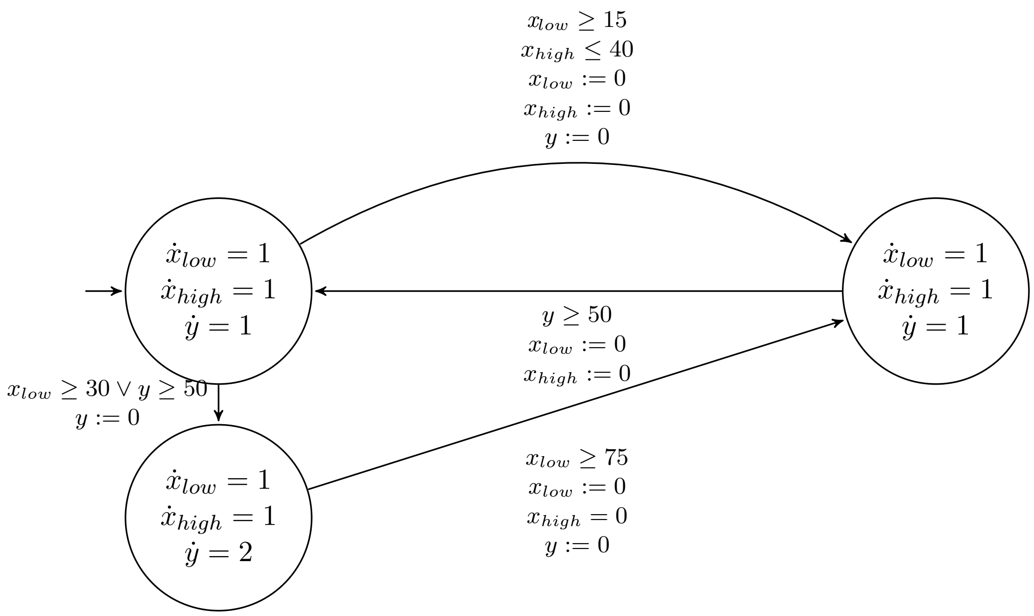


Figure 13.4.2: Timed automaton A'.

Notice that if we use g_3 as controller for the system 2 we have a closed loop stable system with poles

$$p_1 = -3 \qquad p_2 = -\frac{1}{3}.$$

Notice moreover that the closed loop system for system 3 has the equilibrium point in $(-1/3, 0)$ since the closed loop system is

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{3} \end{pmatrix}$$

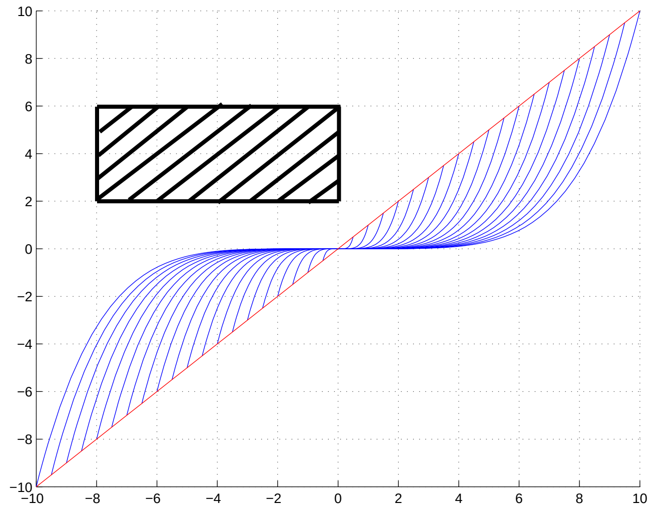


Figure 13.5.1: Trajectories of the linear system in Problem 13.5.

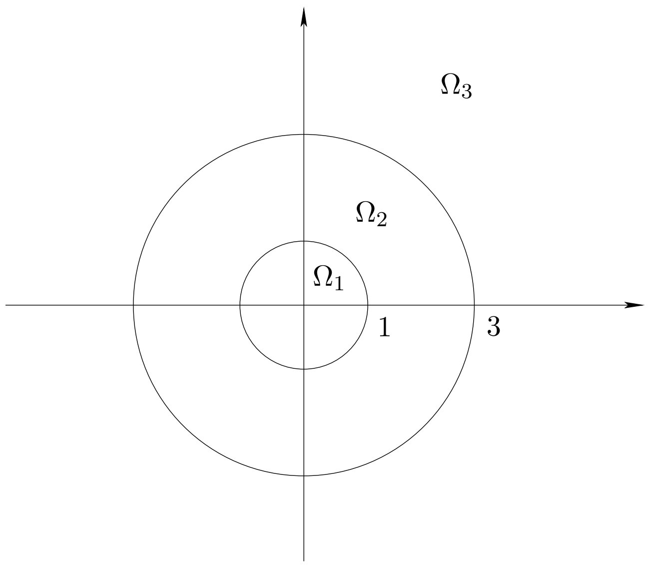


Figure 13.7.1: State-space partition for the system in Problem 13.7

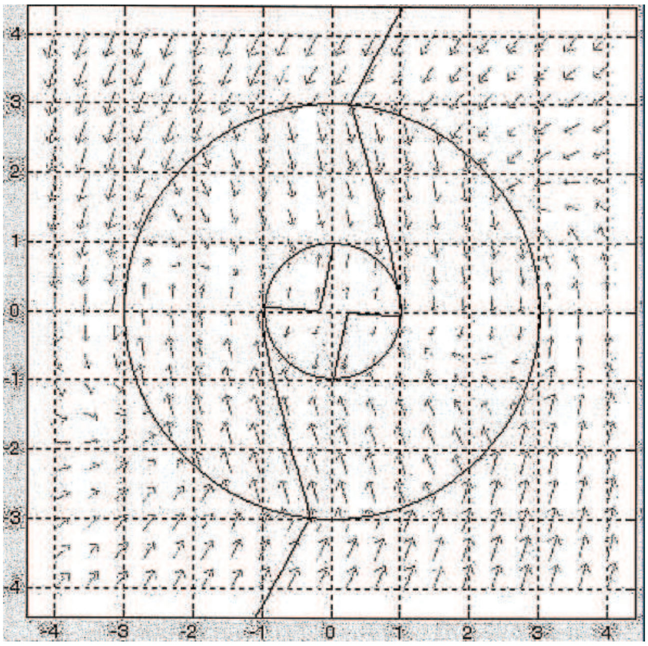


Figure 13.7.2: Trajectories of the linear system in Problem 13.7.

This means that all trajectories starting in Ω_3 will go towards $(-1/3, 0)$, which is in Ω_1. Hence they will enter Ω_2, which also is stable with $(0,0)$ as equilibrium. Thus the trajectory will enter Ω_1. An example of trajectories is shown in Figure 13.7.2.

(b) If $B_1 = (0,1)^T$ then as you can notice the dynamics in the set $\Omega_3 = \{\|x\| > 3\}$ is the negative of the dynamics in Ω_1 (modulo a bias term that only makes the equilibrium different). Therefore is we use the linear controller g_3 computed in (a), this will not make the system 1 stable. Thus, we need to use a different controller for the two regions.

(c) A simple choice to stabilize the system 1 is a feedback controller that places the poles in $p_1 = p_2 = -1$. Such controller for the system 1 is

$$g_1(x_1, x_2) = \begin{pmatrix} -\frac{4}{3} & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Notice that that controllers g_1 and g_3 place the poles of the systems 1 and 3 in $p_1 = p_2 = -1$. Both controllers stabilizes the system in Ω_2, since the closed loop for the system 2, when g_3 is used, has poles in

$$p_1 = -3 \qquad p_2 = -1/3$$

and using g_1 the poles are

$$p_{1,2} = \frac{-11 \pm 4\sqrt{7}}{3}.$$

thus a possible control strategy is

$$g(x) = \begin{cases} g_1(x) & \text{if } \|x\| \le \alpha \\ g_3(x) & \text{otherwise} \end{cases}$$

with $1 \le \alpha \le 3$, which means that we can use either g_1 or g_3 in any subset of the region Ω_2.

(d) If the bad region corresponds to Ω_1 then we need to avoid the trajectory to go enter in Ω_1 at all. One possible solution is to design a control law g_1 such that the eigenvalues of the closed loop system for

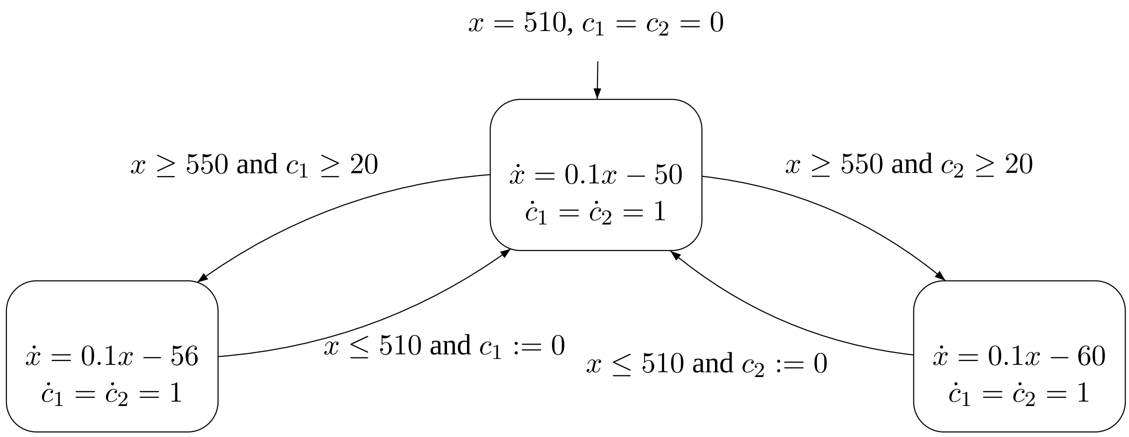


Figure 13.8.1: The hybrid system for the nuclear reactor

system 1 are placed in $p_{1,2} = \pm i$. In this way we have that the trajectories are confined in a circle with radius 1.

SOLUTION 13.8

A hybrid system system modeling the nuclear reactor is shown in Figure 13.8.1.

SOLUTION 13.9

The transition system is modeled explicitly as $\mathcal{TS} = (\mathcal{S}, \Sigma, \Pi, \rightarrow, \mathcal{L}, \mathcal{S}_0)$, where

- $\mathcal{S} = \{s_0, s_1, s_2\}$ is the set of states
- $\mathcal{S}_0 = \{s_0\}$ is the set of initial states
- $\Pi = \{\text{red}, \text{green}, \text{blue}\}$ is the set of atomic propositions
- $\Sigma = \{u_1, \ldots, u_4\}$ is the set of actions/control inputs
- $\rightarrow = \{(s_0, u_1, s_1), (s_1, u_2, s_0), (s_1, u_3, s_2), (s_2, u_4, s_1)\}$ is the transition relation
- $\mathcal{L}(s_0) = \{\text{red}\}, \mathcal{L}(s_1) = \{\text{green}\}, \mathcal{L}(s_2) = \{\text{blue}\}$ is the labeling function

We next convert the specification to an LTL formula, i.e. $\varphi = \Box\Diamond$"green", and we build the Büchi automaton that accepts it as language $\mathcal{A} = (\mathcal{Q}, 2^{\Pi}, \delta, \mathcal{Q}_0, \mathcal{F})$ (see Fig. 13.9.1), with

- $Q = \{q_0, q_1\}$ the number of states
- $Q_0 = \{q_0\}$ the initial state
- $F = \{q_1\}$ is the accepting state
- 2^{Π} is the inputs
- $\delta\left(q_0, \{\neg\text{“green”}\}\right) = \{q_0\}$, $\delta\left(q_0, \{\text{“green”}\}\right) = \{q_1\}$, $\delta\left(q_1, \{\neg\text{“green”}\}\right) = \{q_0\}$, $\delta\left(q_1, \{\text{“green”}\}\right) = \{q_1\}$ is a transition relation.

Next, we build the product Automaton $\widetilde{\mathcal{A}} = \mathcal{TS} \otimes \mathcal{A} = (\mathcal{Q}_p, \mathcal{Q}_{p,0}, \Sigma, \mathcal{F}_p, \delta_p)$ (see Fig. 13.9.2), with

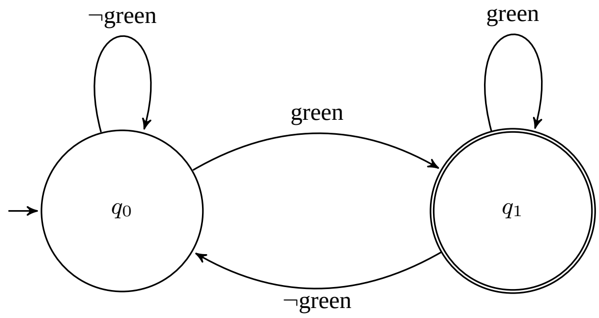


Figure 13.9.1: The Büchi Automaton $\mathcal{A}$.

- $\mathcal{Q}_p = \{(s_0, q_0), (s_1, q_0), (s_2, q_0), (s_0, q_1), (s_1, q_1), (s_2, q_1)\}$ is the set of states
- $\mathcal{Q}_{p,0} = \{(s_0, q_0)\}$ is the set of initial states
- $\mathcal{F}_p = \{(s_0, q_1), (s_1, q_1), (s_2, q_1)\}$ is the set of accepting states
- $\Sigma = \{u_1, u_2, u_3, u_4\}$ is the set of actions/inputs
- δ_p is the transition relation, according to the rule $((s, q), \sigma, (s', q') \in \delta_p)$ **if** $(s, \sigma, s') \in \rightarrow$ **and** $q' \in \delta(q, \mathcal{L}(s))$, so
  - $\big((s_0, q_0), u_1, (s_1, q_0)\big) \in \delta_p$, since $(s_0, u_1, s_1) \in \rightarrow$ and $\delta(q_0, \mathcal{L}(s_0)) = q_0 \in \delta(q_0, \{\text{“red”}\}) = \{q_0\}$
  - $\big((s_0, q_1), u_1, (s_1, q_0)\big) \in \delta_p$ since $(s_0, u_1, s_1) \in \rightarrow$ and $q_0 \in \delta(q_1, \mathcal{L}(s_0)) = \delta(q_1, \{\text{“red”}\}) = \{q_0\}$
  - $\big((s_1, q_0), u_2, (s_0, q_1)\big) \in \delta_p$ since $(s_1, u_2, s_0) \in \rightarrow$ and $q_1 \in \delta(q_0, \mathcal{L}(s_1)) = \delta(q_0, \{\text{“green”}\}) = \{q_1\}$
  - $\big((s_1, q_0), u_3, (s_2, q_1)\big) \in \delta_p$ since $(s_1, u_3, s_2) \in \rightarrow$ and $q_1 \in \delta(q_0, \mathcal{L}(s_1)) = \delta(q_0, \{\text{“green”}\}) = \{q_1\}$
  - $\big((s_1, q_1), u_2, (s_0, q_1)\big) \in \delta_p$ since $(s_1, u_2, s_0) \in \rightarrow$ and $q_1 \in \delta(q_1, \mathcal{L}(s_1)) = \delta(q_1, \{\text{“green”}\}) = \{q_1\}$
  - $\big((s_1, q_1), u_3, (s_2, q_1)\big) \in \delta_p$ since $(s_1, u_3, s_2) \in \rightarrow$ and $q_1 \in \delta(q_1, \mathcal{L}(s_1)) = \delta(q_1, \{\text{“green”}\}) = \{q_1\}$
  - $\big((s_2, q_0), u_4, (s_1, q_0)\big) \in \delta_p$ since $(s_2, u_4, s_1) \in \rightarrow$ and $q_0 \in \delta(q_0, \mathcal{L}(s_2)) = \delta(q_0, \{\text{“blue”}\}) = \{q_0\}$
  - $\big((s_2, q_1), u_4, (s_1, q_0)\big) \in \delta_p$ since $(s_2, u_4, s_1) \in \rightarrow$ and $q_0 \in \delta(q_1, \mathcal{L}(s_2)) = \delta(q_1, \{\text{“blue”}\}) = \{q_0\}$

The accepting run consists of a prefix, i.e., a path from the initial to the accepted state, and a suffix, i.e., a cycle containing at least an accepting state. In this case, $r^* = (s_0, q_0)(s_1, q_0)(s_2, q_1)\Big((s_1, q_0)(s_2, q_1)\Big)^{\omega}$.

SOLUTION 13.10

- The respective Hybrid Automaton is

$$A = (Q, X, \text{Iniit}, Act, f, D, E, G, R),$$

where

  - $Q = \{q_1, q_2\}$ is the set of states

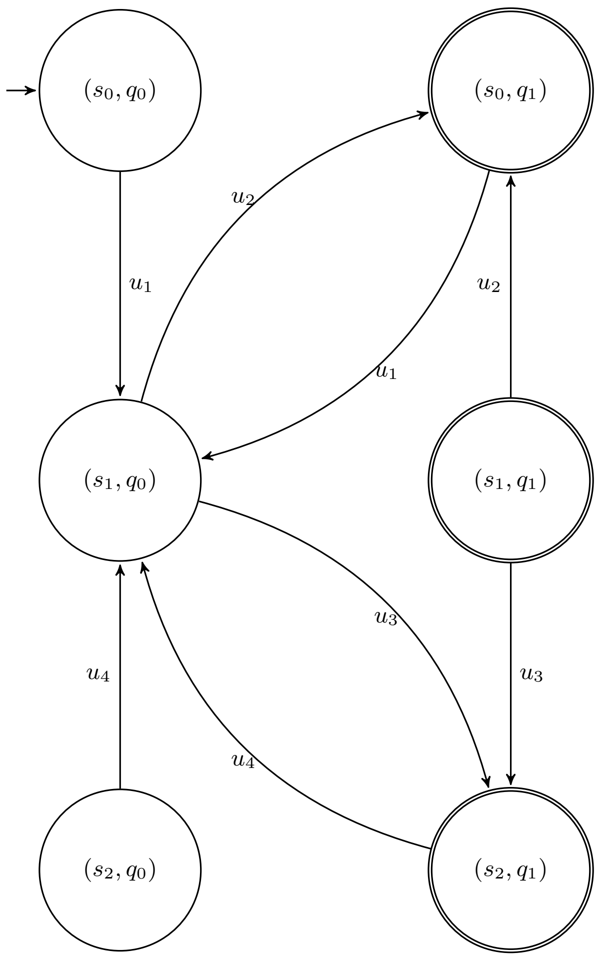


Figure 13.9.2: The product Automaton $\widetilde{\mathcal{A}}$.

- $X = \mathbb{R}^2$ is the continuous state space
- Init $= (q_1, 0, 0) \subset Q \times \{(0, 0)\}$ is the initial state
- $Act = \{tick, tock\}$ is the set of actions
- f is the vector field $\dot{x} = f(q, x)$, with $f(q, x) = (1, 1), \forall q, x$
- $E \subset S \times Act \times S = \{(q_1, tick, q_2), (q_2, tock, q_1)\}$ is the set of edges
- $D : Q \to 2^X$ is the domain, with $D(q_1) = \{x_1 \leq 1\}$, $D(q_2) = \{1 \leq x_2 \leq 3\}$
- $G : E \to 2^X$ is the guard set of the edges, with $G((q_1, tick, q_2)) = \{x_1 \geq 1, x_2 \geq 1\}$, $G((q_2, tock, q_1)) = \{x \in \mathbb{R}^2\}$
- $R : E \times X \to 2^X$ is the reset map, with $R((q_1, tick, q_2), x_1) = \{0\}$, $R((q_1, tick, q_2), x_x) = \{x_2\}$, $R((q_2, tock, q_1), x_i) = \{x_i\}$, $i = 1, 2$.

• The respective transition system is

$$T = (S, \Sigma, \to, S_0),$$

where

- $S = Q \times X$ is the set of states
- $S_0 = (q_1, 0, 0)$ is the initial state
- $\Sigma = Act \cup t$ is the set of actions (time included)

- $\rightarrow$ is a transition relation, with
  1. State transition $(q, x) \xrightarrow{\sigma} (q', y)$ if
     * $\sigma \in Act$,
     * $(q, \sigma, q') \in E$
     * $x \models G((q, \sigma, q'))$
     * $\{y\} = R((q, \sigma, q'), x)$
     * $y \models D(q')$
  2. Time transition $(q, x) \xrightarrow{t} (q, x')$ if
     * $x' = x + t$,
     * $x' \models D(q)$

- We check the reachability of the given states

1. $\rightarrow (q_1, 0, 0) \xrightarrow{t=1} (q_1, 1, 1) \xrightarrow{Act=tick} (q_2, 0, 1)$ - Accepted
2. $(q_1, 4, 4)$ cannot be reached because of the domain of q_1
3. $(q_1, 1, 2)$ cannot be reached because in q_1 we must have $x_1 = x_2$
4. $\rightarrow (q_1, 0, 0) \xrightarrow{t=1} (q_1, 1, 1) \xrightarrow{Act=Tick} (q_2, 0, 1) \xrightarrow{t=1} (q_2, 1, 2) \xrightarrow{t=1} (q_2, 2, 3)$ - Accepted
5. $(q_2, 3, 4)$ cannot be reached because of the domain of q_2
6. Same for $(q_2, 0, 0.5)$
7. $\rightarrow (q_1, 0, 0) \xrightarrow{t=2.5} (q_1, 2.5, 2.5) \xrightarrow{Act=tick} (q_2, 0, 2.5) \xrightarrow{t=0.5} (q_2, 0.5, 3)$ - Accepted

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