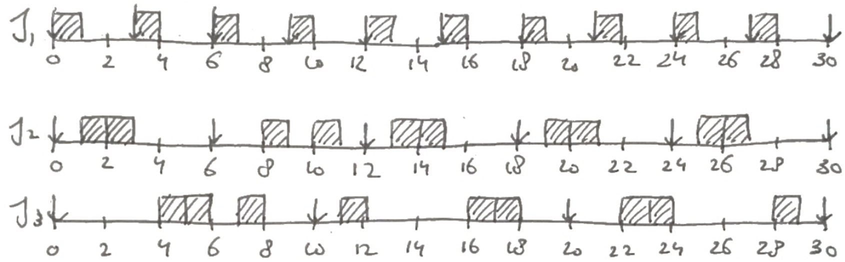

Figure 1: Schedule using EDF.

1. **Solution**

   (a) The utilization factor is given by

   $$U = \sum_{i=1}^{3} \frac{C_i}{T_i} = \frac{2}{3} + \frac{\tau}{10},$$

   and for the tasks to be schedulable using EDF it needs to hold that $U \leq 1$, which gives the condition

   $$\tau \leq \frac{10}{3}. \tag{1}$$

   So the maximum value for τ is 3. The schedule length is given by $\text{lcm}(3, 6, 10) = 30$.

   (b) The schedule is shown in Figure 1. Task J_2 is assumed to have priority when tasks have the same time until deadline. The worst-case response times can be read from the schedule and are given by $R_1 = 1$, $R_2 = 5$, and $R_3 = 9$.

   (c) The utilization factor with $\tau = 3$ is given by $U = \frac{2}{3} + \frac{3}{10} \approx 0.967$. Since U is higher than $3(2^{1/3} - 1) = 0.78$ we cannot determine if the tasks are schedulable by RM. We therefore compute the worst-case response times to determine schedulability. Let R_i denote the worst-case response time for task J_i. The fixed priority is assigned such that J_1 has the highest priority, J_2 has the middle priority, and J_3 has the lowest priority.
   For task J_1:

   $$R_1 = C_1 = 1 \leq D_1 = 3$$

   For task J_2:

   $$\begin{aligned}
   R_2^0 &= C_2 = 2 \\
   R_2^1 &= C_2 + \lceil \frac{R_2^0}{T_1} \rceil C_1 = 2 + \lceil \frac{2}{3} \rceil = 3 \\
   R_2^2 &= C_2 + \lceil \frac{R_2^1}{T_1} \rceil C_1 = 2 + \lceil \frac{3}{3} \rceil = 3 \leq D_2 = 6
   \end{aligned}$$

For task J_3:

$$R_3^0 = C_3 = 3$$
$$R_3^1 = C_3 + \lceil \frac{R_3^0}{T_1} \rceil C_1 + \lceil \frac{R_3^0}{T_2} \rceil C_2 = 3 + \lceil \frac{3}{3} \rceil + 2 \lceil \frac{3}{6} \rceil = 6$$
$$R_3^2 = C_3 + \lceil \frac{R_3^1}{T_1} \rceil C_1 + \lceil \frac{R_3^1}{T_2} \rceil C_2 = 3 + \lceil \frac{6}{3} \rceil + 2 \lceil \frac{6}{6} \rceil = 7$$
$$R_3^3 = C_3 + \lceil \frac{R_3^2}{T_1} \rceil C_1 + \lceil \frac{R_3^2}{T_2} \rceil C_2 = 3 + \lceil \frac{7}{3} \rceil + 2 \lceil \frac{7}{6} \rceil = 10$$
$$R_3^4 = C_3 + \lceil \frac{R_3^3}{T_1} \rceil C_1 + \lceil \frac{R_3^3}{T_2} \rceil C_2 = 3 + \lceil \frac{10}{3} \rceil + 2 \lceil \frac{10}{6} \rceil = 11 > D_3 = 10$$
$$R_3^5 = C_3 + \lceil \frac{R_3^4}{T_1} \rceil C_1 + \lceil \frac{R_3^4}{T_2} \rceil C_2 = 3 + \lceil \frac{11}{3} \rceil + 2 \lceil \frac{11}{6} \rceil = 11 > D_3 = 10$$

Thus the three tasks are not schedulable by RM, since task J_3 does not meet its deadline.

2. **Solution**

(a) (1) The switching system can be modeled as a Hybrid Automaton

$$H = (Q, X, \mathrm{Init}, f, E, D, G, R),$$

where

- $Q = \{q_1, q_2\}$,
- $X = \mathbb{R}^2$,
- $\mathrm{Init} \in Q \times X$,
- $f(q_1, x) = \begin{bmatrix} -x_2 - x_1(1 - x_1^2 - x_2^2) \\ x_1 - x_2(1 - x_1^2 - x_2^2) \end{bmatrix}$, $f(q_2, x) = \begin{bmatrix} x_2 \\ -(1 + x_1^2)x_1 - x_2 + M\cos(\omega t) \end{bmatrix}$,
- $E = \{(q_1, q_2), (q_2, q_1)\}$,
- $D(q_1) = \{x \in \mathbb{R}^2 : \|x\| < \alpha\}$, $D(q_2) = \{x \in \mathbb{R}^2 : \|x\| \geq \alpha\}$,
- $G((q_1, q_2)) = \{x \in \mathbb{R}^2 : \|x\| = \alpha\}$, $G((q_2, q_1)) = \{x \in \mathbb{R}^2 : \|x\| < \alpha\}$,
- $R((q_1, q_2), x) = R((q_2, q_1), x) = x$

(b) (i) By differentiating V_1 over the traj. of the first subsystem, we obtain

$$\begin{aligned} \dot{V}_1 &= -x_1x_2 - x_1^2(1 - x_1^2 - x_2^2) + x_1x_2 - x_2^2(1 - x_1^2 - x_2^2) \\ &= -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) = -\|x\|(1 - \|x\|), \end{aligned}$$

which is negative for $\|x\| < 1$.

(ii) By considering $x(0)$ such that $\|x(0)\| > 1$, $\dot{V}_1$ is positive and hence V_1, and consequently $\|x\|$, blow to infinity. Hence, it is not globally AS.

(c) (i) V_2 becomes

$$V_2 = \frac{1}{2}x_1^2 + x_2^2 + x_1x_2 + x_1^2 + \frac{x_1^4}{2},$$

and after differentiation over the traj. of the second subsystem, we obtain

$$\begin{aligned} \dot{V}_2 =& x_1x_2 - 2x_1x_2(1 + x_1^2) - 2x_2^2 + 2Mx_2\cos(\omega t) + x_2^2 - x_1^2(1 + x_1^2) - x_1x_2 + \\ & Mx_1\cos(\omega t) + 2x_1x_2 + 2x_1^3x_2 \\ =& -x_1^2 - x_2^2 + (x_1 + 2x_2)M\cos(\omega t) - x_1^4 = -\|x\|^2 - x_1^4 + (x_1 + 2x_2)\cos(\omega t) \\ \leq& -\|x\|^2 - x_1^4 + \sqrt{5}\|x\|M\cos(\omega t) \\ =& -\|x\|(\|x\| - M\sqrt{5}) - x_1^4, \end{aligned}$$

which is negative when $\|x\| > M\sqrt{5}$.

(ii) For a sufficiently large value of M, it can be shown that $\dot{x}_1 > 0$ and $\dot{x}_2 > 0$ arbitrarily close to zero, hence the origin is not asymptotically stable.

(d) The condition is

$$M\sqrt{5} < \alpha < 1.$$

3. **Solution:**

(a) The system is not controllable, because

$$W_c = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

and $\det(W_c) = 0$.

(b) With $L = [l_1 \quad l_2]$ the characteristic polynomial of the closed-loop system is

$$\begin{aligned} \det(zI - A + BL) &= \det\left(\begin{bmatrix} z - 1 + l_1 & l_2 - 1 \\ 2(l_1 - 1) & z - 2(1 - l_2) \end{bmatrix}\right) \\ &= (z - 1 + l_1)(z - 2(1 - l_2)) - 2(1 - l_1)(1 - l_2) \\ &= z\big(z - 3 + l_1 + 2l_2\big). \end{aligned}$$

One pole is located at $z_1 = 0$, while the other is located at $z_2 = 3 - l_1 - 2l_2$. The characteristic polynomial of the open-loop system is $\det(zI - A) = z(z - 3)$ and we observe that one pole cannot be moved by state-feedback. This is due to the uncontrollability of the system.

(c) If $\operatorname{rank}([\lambda I - \Phi \quad \Gamma]) < n$ for a $\lambda \in \mathbb{C}$, then there exists a w, such that $w^T[\lambda I - \Phi \quad \Gamma] = 0$. Therefore, $w^T\Phi = \lambda w^T$ and $w^T\Gamma = 0$. Note that in this case λ is an eigenvalue of Φ and w is the corresponding left eigenvector. Multiplying W_c with w^T from the left gives us

$$\begin{aligned} w^T W_c &= \begin{bmatrix} w^T\Gamma & w^T\Phi\Gamma & w^T\Phi^2\Gamma & \cdots & w^T\Phi^{n-1}\Gamma \end{bmatrix} \\ &= \begin{bmatrix} w^T\Gamma & \lambda w^T\Gamma & \lambda^2 w^T\Gamma & \cdots & \lambda^{n-1} w^T\Gamma \end{bmatrix} = 0 \end{aligned}$$

which shows us that $\operatorname{rank}(W_c) < n$. Hence, the system is uncontrollable.

(d) We only need to test z's that are eigenvalues of Φ because only then Φ will lose rank. Otherwise, $zI - \Phi$ has full rank and therefore $[zI - \Phi \quad \Gamma]$ has full rank.

(e) As shown in (b) the poles of the system are at $z_1 = 0$ and $z_2 = 3$. Applying the PBH test to the given system show us that

$$\operatorname{rank}([z_1 I - \Phi \quad \Gamma]) = \operatorname{rank}\left(\begin{bmatrix} -1 & -1 & 1 \\ -2 & -2 & 2 \end{bmatrix}\right) = 1$$

and

$$\operatorname{rank}([z_2 I - \Phi \quad \Gamma]) = \operatorname{rank}\left(\begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & 2 \end{bmatrix}\right) = 2.$$

Hence, the system is uncontrollable. The PBH test does not only show when a system is controllable but also which poles can be moved by a feedback controller. This implies that we can still stabilize an uncontrollable system if all unstable poles are controllable.

4. **Solution**

(a) Consider the relation:

$$\sim = \{(s_1, s_1'), (s_2, s_2'), (s_3, s_3'), (s_4, s_5'), (s_5, s_5'), (s_2, s_4'), (s_6, s_6')\} \subseteq S \times S',$$

The relation $\sim$ is a simulation relation from $\mathcal{T}_1$ to $\mathcal{T}_2$ since the following hold:

(a) For the state $s_1 \in S$, there exists the state $s_1' \in S'$ such that $(s_1, s_1') \in \sim$.

(b) We have that $(s_6, s_6') \in \sim$ with $s_6 \in S_F$ and $s_6' \in S_F'$.

(c)
- $(s_1, s_1') \in \sim$, $(s_1, a, s_2) \in \rightarrow$, and at the same time $(s_1', a, s_2') \in \rightarrow'$, and $(s_2, s_2') \in \sim$.
- $(s_1, s_1') \in \sim$, $(s_1, a, s_3) \in \rightarrow$, and at the same time $(s_1', a, s_3') \in \rightarrow'$, and $(s_3, s_3') \in \sim$.
- $(s_2, s_2') \in \sim$, $(s_2, a, s_5) \in \rightarrow$, and at the same time $(s_2', a, s_5') \in \rightarrow'$, and $(s_5, s_5') \in \sim$.
- $(s_2, s_4') \in \sim$, $(s_2, a, s_5) \in \rightarrow$, and at the same time $(s_4', a, s_5') \in \rightarrow'$, and $(s_5, s_5') \in \sim$.
- $(s_3, s_3') \in \sim$, $(s_3, b, s_5) \in \rightarrow$, and at the same time $(s_3', b, s_5') \in \rightarrow'$, and $(s_5, s_5') \in \sim$.
- $(s_4, s_5') \in \sim$, $(s_4, a, s_6) \in \rightarrow$, and at the same time $(s_5', a, s_6') \in \rightarrow'$, and $(s_6, s_6') \in \sim$.
- $(s_5, s_5') \in \sim$, $(s_5, a, s_6) \in \rightarrow$, and at the same time $(s_5', a, s_6') \in \rightarrow'$, and $(s_6, s_6') \in \sim$.

It can be shown analogously, that the three conditions for the relation $\sim^{-1}$ hold. Therefore, $\sim$ is a *bisimulation relation* and the Transition Systems $\mathcal{T}_1, \mathcal{T}_2$ are *bisimilar*.

(b) We start with the partition $S_{/\sim} = \{\{s_1, s_2, s_3, s_4, s_5\}, \{s_6\}\}$.

Step 1: Set $P = \{s_1, s_2, s_3, s_4, s_5\}$ and $P' = \{s_6\}$. We have that $\emptyset \neq P \cap \mathrm{Pre}_a(P') = \{s_1, s_2, s_3, s_4, s_5\} \cap \{s_4, s_5\} = \{s_4, s_5\} \neq P$. Then, set:

$$\begin{aligned} P_1 &= P \cap \mathrm{Pre}_a(P') = \{s_4, s_5\}, \\ P_2 &= P \backslash \mathrm{Pre}_a(P') = \{s_1, s_2, s_3\}, \\ S_{/\sim} &= \{\{s_6\}, \{s_4, s_5\}, \{s_1, s_2, s_3\}\}. \end{aligned}$$

Step 2: Set $P = P' = \{s_1, s_2, s_3\}$. We have that $\emptyset \neq P \cap \mathrm{Pre}_a(P) = \{s_1, s_2, s_3\} \cap \{s_1\} = \{s_1\} \neq P$. Then, set:

$$\begin{aligned} P_1 &= P \cap \mathrm{Pre}_a(P') = \{s_1\}, \\ P_2 &= P \backslash \mathrm{Pre}_a(P') = \{s_1, s_2, s_3\} \backslash \{s_1\} = \{s_2, s_3\}, \\ S_{/\sim} &= \{\{s_1\}, \{s_6\}, \{s_2, s_3\}, \{s_4, s_5\}, \}. \end{aligned}$$

Step 3: Set $P = \{s_2, s_3\}, P' = \{s_4, s_5\}$. We have that

$$\emptyset \neq P \cap \mathrm{Pre}_b(P') = \{s_2, s_3\} \cap \{s_3\} = \{s_3\} \neq P.$$

Then, set:

$$\begin{aligned} P_1 &= P \cap \mathrm{Pre}_b(P') = \{s_3\}. \\ P_2 &= P \backslash \mathrm{Pre}_b(P') = \{s_2, s_3\} \backslash \{s_3\} = \{s_2\} \\ S_{/\sim} &= \{\{s_1\}, \{s_2\}, \{s_3\}, \{s_4, s_5\}, \{s_6\}\}. \end{aligned}$$

We next check if the algorithm is finished, i.e., that we cannot refine more. The only element which is not a singleton is $\{s_4, s_5\}$. Thus, it suffices to show that for all $P' \in S_{/\sim}$ and $\sigma \in \{a, b\}$ it holds that $\{s_4, s_5\} \cap \mathrm{Pre}_\sigma(P') = \emptyset$ or $\{s_4, s_5\}$, which holds, since the only P' for which $\{s_4, s_5\} \cap \mathrm{Pre}_\sigma(P') \neq \emptyset$ is $P' = \{s_6\}$ and also $\{s_4, s_5\} \cap \mathrm{Pre}_\sigma(P') = \{s_4, s_5\}$.

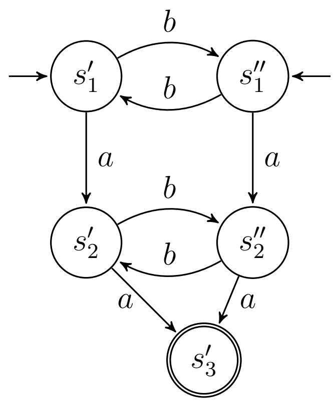

Figure 2: The transition system $\mathcal{T}_3'$.

(c) (i) No, it is not a bisimulation relation because it is not a relation from $\mathcal{T}_3$ to $\mathcal{T}_4$. Indeed, since $(s_2, s_2'') \in\sim$ and $s_2 \xrightarrow{b} s_2$, there must be a $r \in S'$ such that $s_2'' \xrightarrow{b} r$ and $(s_2, r) \in\sim$, which is not the case.

(ii) The transition system T_3' is depicted in Fig. 2. The relation

$$\sim = \{(s_1, s_1'), (s_1, s_1''), (s_2, s_2'), (s_2, s_2''), (s_3, s_3')\} \subseteq S \times S',$$

is a simulation relation from $\mathcal{T}_3$ to $\mathcal{T}_3'$ since the following hold:

i. For the state $s_1 \in S$, there exists a state $s_1' \in S'$ such that $(s_1, s_1') \in\sim$.

ii. We have that $(s_3, s_3') \in\sim$ with $s_3 \in S_F$ and $s_3' \in S_F'$.

iii.
- $(s_1, s_1') \in\sim, (s_1, a, s_2) \in\rightarrow$, and at the same time $(s_1', a, s_2') \in\rightarrow'$, and $(s_2, s_2') \in\sim$.
- $(s_1, s_1') \in\sim, (s_1, b, s_1) \in\rightarrow$, and at the same time $(s_1', b, s_1'') \in\rightarrow'$, and $(s_1, s_1'') \in\sim$.
- $(s_1, s_1'') \in\sim, (s_1, a, s_2) \in\rightarrow$, and at the same time $(s_1'', a, s_2'') \in\rightarrow'$, and $(s_2, s_2'') \in\sim$.
- $(s_1, s_1'') \in\sim, (s_1, b, s_1) \in\rightarrow$, and at the same time $(s_1'', b, s_1') \in\rightarrow'$, and $(s_1, s_1') \in\sim$.
- $(s_2, s_2') \in\sim, (s_2, a, s_3) \in\rightarrow$, and at the same time $(s_2', a, s_3') \in\rightarrow'$, and $(s_3, s_3') \in\sim$.
- $(s_2, s_2') \in\sim, (s_2, b, s_2) \in\rightarrow$, and at the same time $(s_2', b, s_2'') \in\rightarrow'$, and $(s_2, s_2'') \in\sim$.
- $(s_2, s_2'') \in\sim, (s_2, a, s_3) \in\rightarrow$, and at the same time $(s_2'', a, s_3') \in\rightarrow'$, and $(s_3, s_3'') \in\sim$.
- $(s_2, s_2'') \in\sim, (s_2, b, s_2) \in\rightarrow$, and at the same time $(s_2'', b, s_2') \in\rightarrow'$, and $(s_2, s_2') \in\sim$.

It can be shown analogously, that the three conditions for the relation $\sim^{-1}$ hold. Therefore, $\sim$ is a *bisimulation relation* and the Transition Systems $\mathcal{T}_3, \mathcal{T}_3'$ are *bisimilar*.

(d) Such a Transition system is T_5' in Fig. 3. We can check the simulation by relating $s_1, s_2, s_3 \sim s_1', s_1''$ and $s_4 \sim s_2'$.

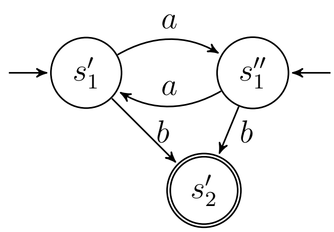

Figure 3: The transition system $\mathcal{T}_5'$.

5. **Solution**

   (a) We have

$$\begin{aligned}\dot{x}_1 =& \dot{p}_2 - \dot{p}_1 = u_2 - u_1\\ \dot{x}_2 =& \dot{p}_3 - \dot{p}_2 = u_3 - u_2.\end{aligned} \tag{2}$$

   Therefore,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}. \tag{3}$$

   (b) We have

$$\begin{aligned}u_1 =& p_2 - p_1 = x_1,\\ u_2 =& p_3 - p_2 = x_2,\\ u_3 =& p_1 - p_3 = -x_1 - x_2.\end{aligned} \tag{4}$$

   Therefore

$$\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{5}$$

   (c) Using the results in (a) and (b), we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{6}$$

   (d) Using the suggested Lyapunov function, we find

$$\dot{V}(x) = -x_1^2 - 2x_2^2, \tag{7}$$

   therefore, $x(t)$ asymptotically converges to zero, which implies that the robots asymptotically meet at the same place.

   (e) With the sampled measurements, we have $\dot{x} = BK(x+e)$, or equivalently

$$\begin{aligned}\dot{x}_1 =& -x_1 + x_2 - e_1 + e_2,\\ \dot{x}_2 =& -x_1 - 2x_2 - e_1 - 2e_2.\end{aligned} \tag{8}$$

   (f) Using the same Lyapunov function, we have

$$\dot{V}(t) = -x_1^2 - x_1 e_1 + x_1 e_2 - 2x_2^2 - e_1 x_2 - 2e_2 x_2, \tag{9}$$

which can be written as

$$\dot{V}(t) = -x(t)^T Q x(t) + x(t)^T R e(t), \tag{10}$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix}. \tag{11}$$

Therefore, $\dot{V}(t)$ is guaranteed if

$$-\lambda_Q \|x(t)\|^2 + \|x(t)\| \lambda_R \|e(t)\| \leq 0, \tag{12}$$

where λ_Q is the smallest eigenvalue of Q and λ_R is the 2-norm of R. Solving the inequality for $\|e(t)\|$ gives

$$\|e(t)\| \leq \frac{\lambda_Q}{\lambda_R} \|x(t)\|. \tag{13}$$

As for numerical values, we have immediately $\lambda_Q = 1$, while a quick calculation gives $\lambda_R = \sqrt{(7+\sqrt{13})/2}$. Therefore, we conclude

$$\alpha = \sqrt{\frac{2}{7+\sqrt{13}}}. \tag{14}$$