AUTOMATIC CONTROL

KTH

EL2450 Hybrid and Embedded Control Systems

Exam 09:00–14:00, June 8, 2019

Aid:

Lecture notes (slides) from the course and a mathematical handbook (e.g., "Beta Mathematics Handbook" by Råde & Westergren). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used unless previously approved by course responsible.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page and write on one side per sheet.
- Each answer has to be motivated.
- Specify the total number of handed in pages on the cover.
- Each subproblem is marked with its maximum credit.

Grading:

Grade A: ≥ 43, Grade B: ≥ 38

Grade C: ≥ 33, Grade D: ≥ 28

Grade E: ≥ 23, Grade Fx: ≥ 21

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Lycka till!

1. Suppose the tasks J_1, J_2, and J_3 are executed on a preemptive CPU. All tasks are released at time zero and have the following characteristics:

Task	T_i	D_i	C_i
J_1	3	3	1
J_2	6	6	2
J_3	10	10	τ

where τ is a positive integer representing the worst-case computational time of task J_3.

(a) [2p] What is the maximum value of τ for which this task set is schedulable under the earliest deadline first (EDF) algorithm? What is the schedule length?

(b) [4p] Let τ be equal to the maximum value obtained in (a), and consider again scheduling under EDF. Draw the corresponding schedule and determine the worst-case response time for J_1, J_2, and J_3. Write down any assumptions you make.

(c) [4p] Let τ be equal to the maximum value obtained in (a), but now consider scheduling under the rate monotonic (RM) algorithm. Are the tasks schedulable in this case and why? Please motivate and justify your answer without drawing the corresponding schedule. [Hint: $2^{1/3} = 1.26$, $\frac{29}{30} \approx 0.967$]

2. Consider the following switching system:

$$\left.\begin{aligned}\dot{x}_1 &= -x_2 - x_1(1 - x_1^2 - x_2^2)\\ \dot{x}_2 &= x_1 - x_2(1 - x_1^2 - x_2^2)\end{aligned}\right\} \text{ if } \|x\| < \alpha, \tag{1}$$

$$\left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(1 + x_1^2)x_1 - x_2 + M\cos(\omega t)\end{aligned}\right\} \text{ if } \|x\| \geq \alpha, \tag{2}$$

with $x = [x_1, x_2]^\top \in \mathbb{R}^2$, and α, M, ω are positive constants.

(a) [1p] Model the switching system as a Hybrid Automaton

$$H = (Q, X, \mathrm{Init}, f, E, D, G, R).$$

(b) [3p] Consider subsystem (1) and the Lyapunov function

$$V_1 = \frac{1}{2}(x_1^2 + x_2^2).$$

(i) Prove that $\dot{V}_1 < 0$ for $\|x\| < 1$ over the trajectories of the first subsystem (1).

(ii) Is subsystem (1) *globally asymptotically stable* ? Motivate your answer.

(c) [3p] Consider subsystem (2) and the Lyapunov function

$$V_2 = x^\top P x + 2\int_0^{x_1} (y + y^3)dy,$$

where $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$.

(i) Prove that $\dot{V}_2 < 0$ for $\|x\| > M\sqrt{5}$ over the trajectories of the second subsystem (2).
(*Hint: Use the property $y^\top x \leq \|x\|\|y\|$ with $y = [1, 2]^\top$.*)

(ii) Is subsystem (2) *asymptotically stable* ? Motivate your answer.

(d) [3p]

(i) Does there exist a common Lyapunov function for the switching system? Motivate your answer.

(ii) Provide sufficient conditions on α and M such that the switching system is asymptotically stable.

3. Consider a linear system given by

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k). \tag{3}$$

(a) [1p] Check the controllability of system (3).

(b) [2p] Show where the poles of the closed-loop system can be place by using a controller $u(k) = -Lx(k)$. Compare these with the poles of the open-loop system.

Another criterion for controllability is the Popov-Belevitch-Hautus (PBH) test, which states the following:

The system

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

is controllable if and only if

$$\operatorname{rank}(\begin{bmatrix} zI - \Phi & \Gamma \end{bmatrix}) = n$$

for all $z \in \mathbb{C}$, where $\Phi \in \mathbb{R}^{n \times n}$ and $\Gamma \in \mathbb{R}^{n \times p}$.

(c) [3p] Show that if $\operatorname{rank}([\lambda I - \Phi \quad \Gamma]) < n$ for $\lambda \in \mathbb{C}$, then $\operatorname{rank}(W_c) < n$, where

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}.$$

(Hint: Let $A \in \mathbb{R}^{n \times q}$ with $q \geq n$. Recall that there exists a $w \in \mathbb{C}^n$ and $w \neq 0$ such that $w^T A = 0$ if and only if $\operatorname{rank}(A) < n$*.)*

(d) [1p] Do we need to consider all $z \in \mathbb{C}$? If not, which z do we need to consider in the PBH test for controllability?

Let us now check the controllability of the system (3) with the PBH test.

(e) [3p] Use the PBH test to determine the controllability of (3). What more information do we get from the PBH test than from determining the rank of W_c? What does this imply for the stabilization of uncontrollable systems?

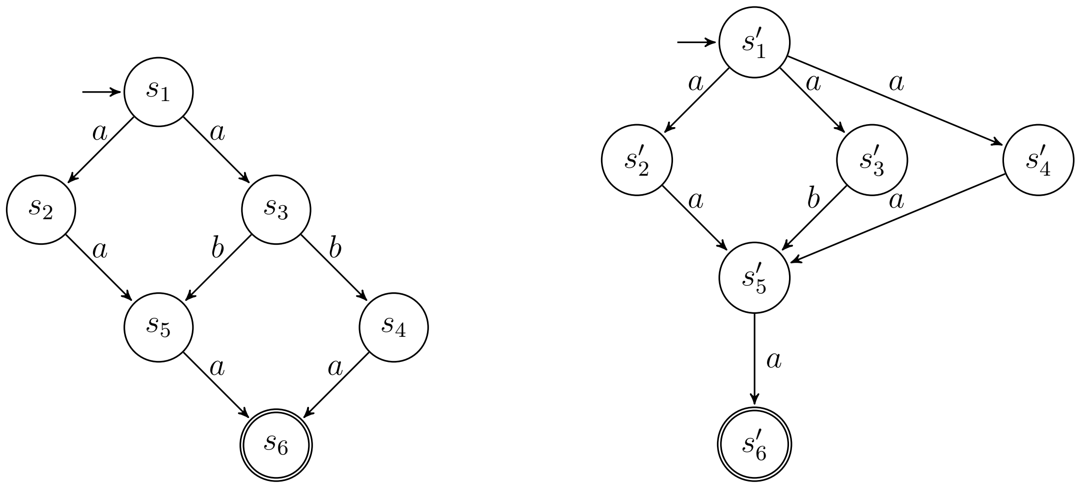

Figure 1: The transition systems $\mathcal{T}_1$ (left) and $\mathcal{T}_2$ (right).

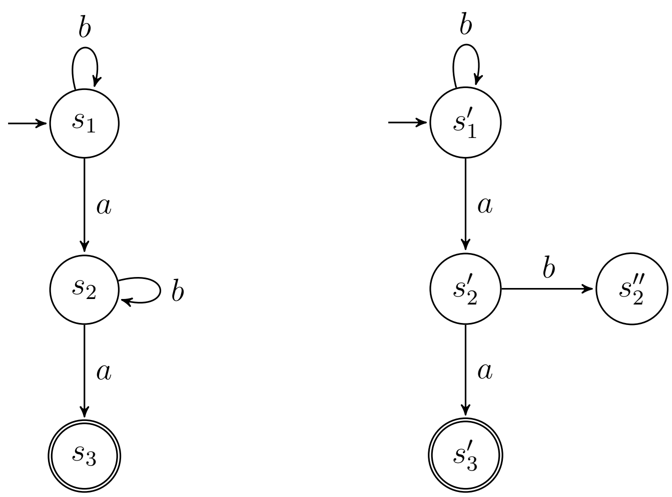

Figure 2: The transition systems $\mathcal{T}_3$ (left) and $\mathcal{T}_4$ (right).

4. (a) [3p] Consider the Transition Systems $\mathcal{T}_1$ (left), $\mathcal{T}_2$ (right) as they are depicted in Figure 1. Are the Transition Systems bisimilar? Motivate your answer. If they are, provide a bisimulation relation.

   (b) [3p] Find the coarsest quotient Transition System bisimilar to the Transition System $\mathcal{T}_1$ (left) in Figure 1.

   (c) [3p] Consider the Transition Systems $\mathcal{T}_3$ and $\mathcal{T}_4$ in Figure 2.

      (i) Consider the relation $\sim = \{(s_1, s'_1), (s_2, s'_2), (s_2, s''_2), (s_3, s'_3)\}$. Is it a bisimulation relation? Motive your answer.

      (ii) Construct a transition system that is bisimilar to $\mathcal{T}_3$ and has *no* self loops. Provide the bisimulation relation and justify your answer.

   (d) [1p] Consider the Transition System $\mathcal{T}_5$ in Figure 3. Construct a transition system with three states that simulates $\mathcal{T}_5$ and has *no* self loops. Justify your answer.

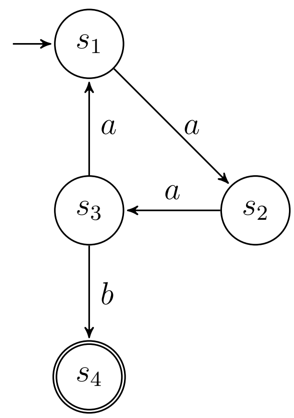

Figure 3: The transition system $\mathcal{T}_5$.

5. Consider three robots R_1, R_2 and R_3 that want to meet at the same place. Each robot is controlled as a simple integrator; i.e., if we denote as $p_i(t) \in \mathbb{R}^2$ the position of robot R_i, then the motion of the robot is described by

$$\dot{p}_i(t) = u_i(t). \tag{4}$$

In order to meet at the same place, R_1 follows R_2, R_2 follows R_3, and R_3 follows R_1, as described by the following equations:

$$\begin{aligned} u_1(t) &= p_2(t) - p_1(t), \\ u_2(t) &= p_3(t) - p_2(t), \\ u_3(t) &= p_1(t) - p_3(t). \end{aligned} \tag{5}$$

Consider the state variables $x_1(t) = p_2(t) - p_1(t)$ and $x_2(t) = p_3(t) - p_2(t)$. Note that the robots reach their goal if and only if $x_1 = x_2 = 0$.

(a) [1p] Find the state space representation

$$\dot{x}(t) = Bu(t), \tag{6}$$

where $x(t) = [x_1(t), x_2(t)]^\top$ and $u(t) = [u_1(t), u_2(t), u_3(t)]^\top$.

(b) [1p] Find the matrix K such that $u(t) = Kx(t)$.

(c) [1p] Write the closed-loop system as

$$\dot{x}(t) = BKx(t). \tag{7}$$

(d) [1p] Use the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{8}$$

to show that the robots asymptotically meet at the same place.

Now suppose that the robots measure each other's positions only on the aperiodic sampling times t_k, with $k \in \mathbb{N}$. Therefore, the control inputs become

$$\begin{aligned} u_1(t) &= p_2(t_k) - p_1(t_k), \\ u_2(t) &= p_3(t_k) - p_2(t_k), \\ u_3(t) &= p_1(t_k) - p_3(t_k), \end{aligned} \tag{9}$$

for $t \in [t_k, t_{k+1})$.

(e) [2p] Let $e(t) = x(t) - x(t_k)$, and write the closed-loop system for $t \in [t_k, t_{k+1})$ as a function of $x(t)$ and $e(t)$.

(f) [4p] Using the same Lyapunov function as in (d), to find a condition in the form $\|e(t)\| \leq \alpha \|x(t)\|$, with $\alpha > 0$, that guarantees that the robot asymptotically meet at the same place. Choose α as large as possible.
Hint: for a matrix $M \in \mathbb{R}^{n \times m}$, we have $||M|| = \sqrt{\lambda_{\max}(M^T M)}$, where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue.