AUTOMATIC CONTROL

KTH

EL2450 Hybrid and Embedded Control Systems

Exam 08:00–13:00, March 13, 2019

Aid:

Lecture notes (slides) from the course and a mathematical handbook (e.g., "Beta Mathematics Handbook" by Råde & Westergren). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used unless previously approved by course responsible.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page and write on one side per sheet.
- Each answer has to be motivated.
- Specify the total number of handed in pages on the cover.
- Each subproblem is marked with its maximum credit.

Grading:

Grade A: ≥ 43, Grade B: ≥ 38

Grade C: ≥ 33, Grade D: ≥ 28

Grade E: ≥ 23, Grade Fx: ≥ 21

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Lycka till!

1. Suppose the tasks J_1, J_2, and J_3 are scheduled on a preemptive CPU. All tasks are released at time zero and have the following characteristics:

Task	T_i	D_i	C_i
J_1	4	4	1
J_2	8	8	2
J_3	12	12	4

(a) [1p] Determine the utilization factor and the schedule length.

(b) [3p] Are the tasks schedulable under the earliest deadline first (EDF) scheduling algorithm? Determine the worst-case response times of all tasks. Write down any assumptions that you make.

(c) [3p] Are the tasks schedulable under the rate monotonic (RM) scheduling algorithm? Motivate your answer by computing the worst-case response times of all tasks.

(d) [3p] Assume now that C_3 is a design parameter. Find a condition on C_3 such that all tasks are guaranteed to be schedulable under the RM scheduling algorithm. *Hint:* $36 \cdot 2^{1/3} = 45.36$

2. Consider the following switching system:

$$\ddot{y} = 2y + \dot{y} + u, \qquad \text{if } \ f(y, \dot{y}) \geq 1,$$
$$\ddot{y} = -4y - \frac{1}{2}\dot{y} - u, \quad \text{if } \ f(y, \dot{y}) < 1,$$

with $y(0) = 2$, $\dot{y}(0) = 0$, where $y, u \in \mathbb{R}$, and f is a function to be defined.

(a) [2p] Set $x_1 = y$, $x_2 = \dot{y}$, $x = [x_1, x_2]^\top \in \mathbb{R}^2$, as well as $f(y, \dot{y}) = \|x\| = \sqrt{x_1^2 + x_2^2}$.

(1) Write the system in the form

$$\dot{x} = A_q x + B_q u, \quad q \in \{1, 2\}.$$

(2) Model the switching system as a Hybrid Automaton

$$H = (Q, X, \text{Init}, f, E, D, G, R).$$

(b) [3p] Consider the control input $u = -Kx$, where $K = [k_1, k_2]$.

(1) Derive conditions on k_1 and k_2 such that the first subsystem is asymptotically stable.

(2) Derive conditions on k_1 and k_2 such that the second subsystem is asymptotically stable.

(3) Can you find values for k_1 and k_2 such that both subsystems are asymptotically stable ?

Hint: Use the Routh-Hurwitz criterion, according to which, the roots of a 2nd-order polynomial $\lambda^2 + a_1\lambda + a_0 = 0$ have negative real part if the (real) coefficients a_0, a_1 are positive.

(c) [3p]

(1) Choose appropriate values for k_1, k_2 such that the closed loop poles of the *first subsystem* are both in -1.

(2) By using these values for k_1, k_2, write the closed loop system dynamics. Does there exist a common Lyapunov function for the resulting closed loop switching system? Motivate your answer.

(3) Does the system exhibit Zeno behavior with these values for k_1, k_2 ? Motivate your answer.

(d) [2p] Design a *switching* linear feedback control law for u that renders the system asymptotically stable.

3. Let a discrete-time system be given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}$, $y(k) \in \mathbb{R}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}.$$

Definition 1 *A linear system* (A, B, C) *is said to be* positive *if and only if for every nonnegative initial state, i.e. each element of* $x(0)$ *is nonnegative, and for every nonnegative input, i.e.* $u(k) \geq 0$ *for all* $k \geq 0$*, its state and output are nonnegative.*

(a) [2p] Show that if all elements of A, B, and C are nonnegative, then system (1) is positive according to Definition 1.

(b) [3p] Here, we assume that $u(k) = 0$ for all $k \geq 0$. Show that if there exists a diagonal element a_{ii} of A such that $a_{ii} > 1$, then system (1) is unstable.

Imagine an invasive species has come to an environment in which it thrives. This species has a lifespan of two years. Let the first element of $x(k)$ model the amount of specimen that are younger than one year, while the second element of $x(k)$ models the number of specimen between one and two years old.

How the population evolves is given by the following model

$$x(k+1) = \begin{bmatrix} 2 & \alpha \\ 1 & 0 \end{bmatrix} x(k),$$

where $\alpha \geq 0$. Biologists can only measure the total population of the invasive species, i.e.

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k).$$

(c) [1p] Determine for which α the system is observable.

(d) [3p] Now assume that $\alpha = 1$. Design an observer that places the poles of the error dynamics at zero and state the observer dynamics $\hat{x}(k)$. Do the dynamics represent a positive system?

After realizing that the population of the invasive species will grow exponentially, biologists try to take countermeasures to contain the species. The model is now

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(k), \end{aligned}$$

where $u(k)$ determines the countermeasure taken by the biologists, which only influences the younger specimen. The invasive species should be removed as fast as possible from the environment.

(e) [1p] Design a deadbeat controller $u(k) = -Lx(k)$ that removes the invasive species in two time steps.

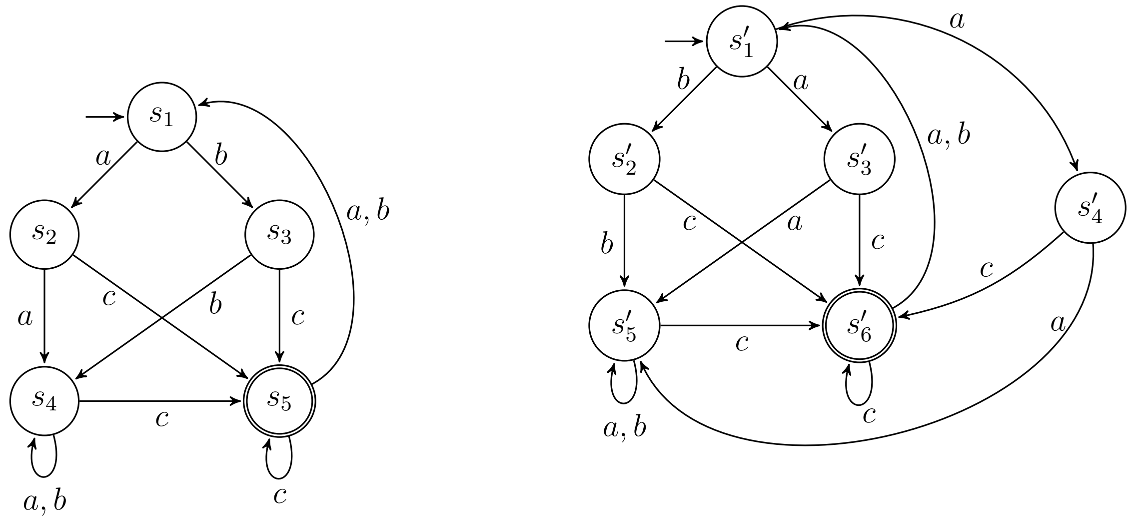

Figure 1: The transition systems $\mathcal{T}_1$ (left) and $\mathcal{T}_2$ (right).

4. (a) [4p] Consider the Transition Systems $\mathcal{T}_1$ (left), $\mathcal{T}_2$ (right) as they are depicted in Figure 1. Are the Transition Systems bisimilar? Motivate your answer.

   (b) [3p] Find the coarsest quotient Transition System bisimilar to the Transition System $\mathcal{T}_1$ (left) in Figure 1.

   (c) [3p] Consider the timed automaton $\mathcal{T}$ depicted in Fig. 2. Decide, whether the following states are reachable in the $\mathcal{T}$, i.e. whether they belong to the set $Reach(\{(q_1, 0, 0, 0)\})$. Motivate your answer.

      (i) $(q_3, 0, 2, 1)$
      (ii) $(q_2, 0, 0, 1)$
      (iii) $(q_2, 0, 2, 1)$
      (iv) $(q_1, 2, 2, 3)$
      (v) $(q_3, 3, 3, 1)$
      (vi) $(q_3, 0, 0, 1)$

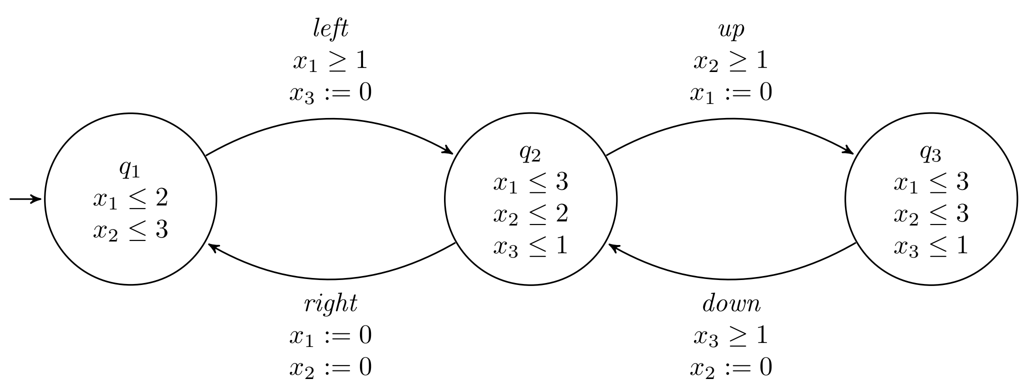

Figure 2: The timed automaton $\mathcal{T}$.

5. Consider the following linear second-order system

$$(S): \begin{cases} \dot{x}_1(t) = x_1(t) - x_2(t) + u(t) \\ \dot{x}_2(t) = 3x_1(t) + x_2(t) + u(t) \end{cases}$$

with $[x_1,\ x_2]^\top = x \in \mathbb{R}^2, u \in \mathbb{R}, t \geq 0$.

(a) [1p] Show that the open-loop system (with $u(t) = 0$) is unstable.

(b) [2p] Determine a linear state-feedback controller $u(t) = Kx(t)$ with $K = [k_1,\ k_2]$, $k_1, k_2 \in \mathbb{R}$ such that the poles of the closed-loop system are placed in -1 and -2.

(c) [2p] In order to implement the controller on a digital platform, the state of the system is sampled aperiodically at a sequence of time instants $\{t_k\}, k \in \mathbb{N}$, and the control signal is now given by

$$u(t) = Kx(t_k), t \in [t_k, t_{k+1}).$$

Find the closed-loop equation of the system dynamics in terms of the state $x(t)$ and the state error $e(t)$, where

$$e(t) = x(t_k) - x(t), t \in [t_k, t_{k+1}).$$

(d) [5p] Using the positive definite quadratic Lyapunov function

$$V(x) = \frac{1}{2}\, x^\top \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x$$

find a relation between the error $e(t)$ and the state $x(t)$ such that the system is still asymptotically stable. Determine an event-triggered rule to choose the sampling instants $\{t_k\}, k \in \mathbb{N}$. Note that the values of k_1, k_2 that have already been found can be used.

Hint: You might find the following elementary inequalities useful,

$$\begin{aligned} -ab &\leq 0.5\big(a^2 + b^2\big), \\ -ab &\leq |a||b|, \\ |x_i| &\leq ||x||, \end{aligned}$$

where $a, b, x_i \in \mathbb{R}$, $x \in \mathbb{R}^2$ and x_i is the ith element of x.