SF2943 Time Series Analysis: Lecture 9

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The analysis of a stationary time series we have introduced is based on the autocovariance function. This type of analysis is known as *time domain analysis*. We will introduce another approach for analyzing a stationary time series, known as *spectral analysis* or *frequency domain analysis*, in these two lectures.

It is well-known that Fourier analysis is a powerful tool in studying signals. In Fourier analysis, a function can be viewed as a Fourier series, a summation of several trigonometric functions with appropriate coefficients. This type of representation is especially suitable when we believe the function is like a "wave". For example, when we try to describe signals with a function. Just like in Fourier analysis, spectral analysis is a technique that allows us to discover underlying periodicities of a stationary time series. Roughly speaking, we can decompose our time series as a combination of several seasonal components. By identifying the period for each seasonal component, we can understand the behavior of the whole time series.

1 Spectral Density

A crucial quantity to capture the "underlying periodicities" is called *spectral density*. Below we will give its definition in a particula case.

Definition 1.1 *Let $\{X_t\}$ be a zero-mean stationary process with acvf γ satisfying absolute summability, i.e.,*

$$\sum_{h=\infty}^{\infty} |\gamma(h)| < \infty.$$

The spectral density of $\{X_t\}$ is defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h), \qquad \text{for all } \lambda \in \mathbb{R}. \tag{1.1}$$

Remark 1.2

- *In this notes, the notation i is only used to denote the imaginary number. Namely, $i = \sqrt{-1}$.*
- $e^{i\lambda} = \cos\lambda + i \sin\lambda$.

- *$f(\lambda)$ is well-defined since γ is absolutely summable and $|e^{ih\lambda}|^2 = |\cos^2(h\lambda)+\sin^2(h\lambda)| = 1$.*
- *Because $e^{i\lambda}$ is 2π-periodic, $f(\lambda)$ is also 2π-periodic. Therefore, it suffices to confine attention to the values of f on the interval $[-\pi,\pi]$.*

Lemma 1.3 (Basic Properties of Spectral Density)

1. *f is even, i.e., $f(\lambda) = f(-\lambda)$.*
2. *$f(\lambda) \geq 0$, for all $\lambda \in [-\pi,\pi]$.*
3. *$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda) d\lambda$.*

Proof. For the first one, using (1.1), change of variable $k = -h$, and because γ is even, we find

$$\begin{aligned} f(-\lambda) &= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih(-\lambda)}\gamma(h) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih(-\lambda)}\gamma(h) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i(-h)\lambda}\gamma(h) \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-ik\lambda}\gamma(-k) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-ik\lambda}\gamma(k) = f(\lambda). \end{aligned}$$

As for the second property, consider

$$f_N(\lambda) \doteq \frac{1}{2\pi n} E\left[\left|\sum_{t=1}^{n} X_t e^{-it\lambda}\right|^2\right] \geq 0,$$

where $\{X_t\}$ is the time series. Notice that

$$\begin{aligned} f_N(\lambda) &= \frac{1}{2\pi n} E\left[\left|\sum_{t=1}^{n} X_t e^{-it\lambda}\right|^2\right] = \frac{1}{2\pi n} E\left[\left(\sum_{t=1}^{n} X_t e^{-it\lambda}\right)\overline{\left(\sum_{s=1}^{n} X_s e^{-is\lambda}\right)}\right] \\ &= \frac{1}{2\pi n} E\left[\left(\sum_{t=1}^{n} X_t e^{-it\lambda}\right)\left(\sum_{s=1}^{n} X_s e^{is\lambda}\right)\right] = \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n} E\left[X_t X_s\right] e^{-i(t-s)\lambda} \\ &= \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n} e^{-i(t-s)\lambda}\gamma(t-s) = \frac{1}{2\pi n}\sum_{h=-n}^{n}\left(n-|h|\right) e^{-ih\lambda}\gamma(h) \\ &= \frac{1}{2\pi}\sum_{h=-n}^{n}\left(1-\frac{|h|}{n}\right) e^{-ih\lambda}\gamma(h) \to \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h) = f(\lambda). \end{aligned}$$

Since $f_N(\lambda) \geq 0$, its limit $f(\lambda)$ must be greater than or equal to zero.

Lastly, for the third property, we first prove the second equation. Using

$$e^{ik\lambda} = \cos(k\lambda) + i\sin(k\lambda)$$

gives

$$\int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda) d\lambda + i\int_{-\pi}^{\pi} \sin(k\lambda) f(\lambda) d\lambda.$$

It suffices to show that

$$\int_{-\pi}^{\pi} \sin(k\lambda) f(\lambda) d\lambda = 0.$$

This is true because sine function is odd and f is even, we have for any λ

$$\sin(-k\lambda) f(-\lambda) = -\sin(k\lambda) f(\lambda)$$

and

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(k\lambda) f(\lambda) d\lambda &= \int_{0}^{\pi} \sin(k\lambda) f(\lambda) d\lambda + \int_{-\pi}^{0} \sin(k\lambda) f(\lambda) d\lambda \\
&= \int_{0}^{\pi} \sin(k\lambda) f(\lambda) d\lambda + \int_{0}^{\pi} \sin(-k\lambda) f(-\lambda) d\lambda \\
&= \int_{0}^{\pi} \sin(k\lambda) f(\lambda) d\lambda - \int_{0}^{\pi} \sin(k\lambda) f(\lambda) d\lambda = 0.
\end{aligned}$$

For the first equality in the third property, we use (1.1) to find

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda &= \int_{-\pi}^{\pi} e^{ik\lambda} \left(\frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \right) d\lambda = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) \int_{-\pi}^{\pi} e^{i(k-h)\lambda} d\lambda \\
&= \frac{1}{2\pi} \gamma(k) \int_{-\pi}^{\pi} 1 d\lambda = \gamma(k),
\end{aligned}$$

where the second last equation holds due to the fact that for any $k \neq h$,

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{i(k-h)\lambda} d\lambda &= \int_{-\pi}^{\pi} \cos((k-h)\lambda) d\lambda + i \int_{-\pi}^{\pi} \sin((k-h)\lambda) d\lambda \\
&= \int_{-\pi}^{\pi} \cos((k-h)\lambda) d\lambda = \frac{1}{k-h} \sin((k-h)\lambda)|_{-\pi}^{\pi} = 0.
\end{aligned}$$

■

It turns out that for a not absolutely summable γ, there may exist a function f such that it is even, $f(\lambda) \geq 0$, and

$$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda.$$

For instance, consider

$$\gamma(h) = \begin{cases} 1, & \text{for } h = p \\ \frac{(-1)^{\frac{h-1}{2}}}{h}, & \text{for } h \neq 0 \text{ and } h \text{ is even} \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad f(\lambda) = \begin{cases} \frac{1}{2}, & \text{for } \lambda \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0, & \text{otherwise} \end{cases} .$$

Therefore, we can generalize the definition of spectral density to the following.

Definition 1.4 *A function $f : [-\pi, \pi] \to \mathbb{R}$ is the spectral density of a stationary time series $\{X_t\}$ with acvf γ if*

- *$f(\lambda) \geq 0$ for all $\lambda \in [-\pi, \pi]$ and*
- *$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda$ for all $h \in \mathbb{Z}$.*

Remark 1.5 *If γ is absolutely summable, then the spectral density defined in Definition 1.4 must be the same as the spectral density defined in Definition 1.1, that is,*

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h), \quad \text{for all } \lambda \in \mathbb{R}.$$

Remark 1.6 *The spectral density defined in Definition 1.4 must be unique, because if for all $h \in \mathbb{Z}$*

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda$$

and

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} g(\lambda) d\lambda,$$

then f and g have the same Fourier coefficients and hence $f(\lambda) = g(\lambda)$ for all λ.

The following proposition characterizes spectral densities.

Proposition 1.7 *A real-valued function f defined on $[-\pi, \pi]$ is the spectral density of a real-valued stationary process if and only if*

1. *$f(\lambda) = f(-\lambda)$.*
2. *$f(\lambda) \geq 0$ for all $f(\lambda)$.*
3. *$\int_{-\pi}^{\pi} f(\lambda) d\lambda < \infty$.*

Proof. We have the direction from left to right in a special case when the acvf γ is absolutely summable. For the general case, see Time Series: Theory and Methods, Section 4.3.

As for the other direction, suppose f satisfies conditions $1, 2, 3$. Consider the function

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda.$$

Using conditions 2 and 3, we know that the function γ is well-defined. If we can show that γ is an acvf, namely, it is even and nonnegative definite, then γ is the acvf of some stationary process, and according to Definition 1.4, f is the spectral density of this stationary process.

It is not hard to see that the function γ is even because f is even

$$\gamma(-h) = \int_{-\pi}^{\pi} e^{i(-h)\lambda} f(\lambda) d\lambda = \int_{-\pi}^{\pi} e^{ih\lambda} f(-\lambda) d\lambda = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda = \gamma(h).$$

Moreover, for any $a_1, \ldots, a_n \in \mathbb{R}$

$$\begin{aligned}\sum_{t=1}^{n}\sum_{s=1}^{n} a_t\gamma(t-s)a_s &= \sum_{t=1}^{n}\sum_{s=1}^{n} a_t a_s \left(\int_{-\pi}^{\pi} e^{i(t-s)\lambda} f(\lambda)d\lambda\right) = \int_{-\pi}^{\pi}\left(\sum_{t=1}^{n}\sum_{s=1}^{n} a_t a_s e^{i(t-s)\lambda}\right) f(\lambda)d\lambda \\ &= \int_{-\pi}^{\pi}\left(\sum_{t=1}^{n} a_t e^{it\lambda}\right)\left(\sum_{s=1}^{n} a_t e^{-is\lambda}\right) f(\lambda)d\lambda = \int_{-\pi}^{\pi}\left|\sum_{t=1}^{n} a_t e^{it\lambda}\right|^2 f(\lambda)d\lambda \geq 0,\end{aligned}$$

where the first equation comes from the definition of γ and the inequality holds due to the condition 2. Hence, γ is also nonnegative definite. ■

Corollary 1.8 *An absolutely summable $\gamma : \mathbb{Z} \to \mathbb{R}$ is the acvf of a stationary process if and only if it is even and*

$$f(\lambda) \doteq \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h) \geq 0 \textit{ for all } \lambda \in [-\pi, \pi],$$

i.e., the spectral density f is nonnegative everywhere.

Proof. The proof from left to right is given in Lemma 1.3. The same argument in Proposition 1.7 can prove the other direction for the nonnegative definiteness of γ. ■

Corollary 1.8 provides a useful tool to check if a function γ is the acvf for a stationary time series. In particular, it provides a more convenient way to show that a function γ is nonnegative definite. We can see the power of Corollary 1.8 from the same example we gave in Lecture 3 (also Example 1.1).

Example 1.1 *Consider a function $\kappa : \mathbb{Z} \to \mathbb{R}$ defined by*

$$\kappa(h) = \begin{cases} 1, & \textit{if } h = 0 \\ \rho, & \textit{if } h = 1, -1 \\ 0, & \textit{otherwise} \end{cases}$$

for some $\rho \in \mathbb{R}$. This function κ is an acvf (i.e., even and nonnegative definite) if and only if $|\rho| \leq 1/2$.

In Lecture 3, we proved this result by first identifying $\kappa(h)$ as the acvf for an MA(1) process when $|\rho| \leq 1/2$, and proving $\kappa(h)$ is NOT nonnegative definite when $|\rho| > 1/2$. Both steps are not obvious, especially for the second step, we need to find a certain $n \in \mathbb{N}$ and a certain vector $(a_1, a_2, \ldots, a_n)$ to make sure

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\kappa(i-j)a_j < 0.$$

On the contrary, since κ is even and absolutely summable, indeed,

$$\sum_{h=-\infty}^{\infty} |\kappa(h)| = 1 + 2|\rho| < \infty,$$

according to Corollary 1.8, it suffices to prove that $f(\lambda) \geq 0$ for all λ if and only if $|\rho| \leq 1/2$. To see the nonnegativity of f, observe that

$$\begin{aligned} f(\lambda) &\doteq \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) = \frac{1}{2\pi} \left[\kappa(0) + e^{-i\lambda}\kappa(1) + e^{i\lambda}\kappa(-1) \right] \\ &= \frac{1}{2\pi} \left[1 + (e^{-i\lambda} + e^{i\lambda})\rho \right] = \frac{1}{2\pi} \left[1 + 2\rho \cos \lambda \right] \end{aligned}$$

and $|\cos \lambda| \leq 1$, hence $f(\lambda) \geq 0$ for all λ if and only if $|\rho| \leq 1/2$.

2 Spectral Distribution Function

Although we generalized the definition of spectral density, there exists a stationary time series $\{X_t\}$ which has no spectral density, i.e., its acvf γ can not be written as

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda \tag{2.1}$$

for some f even and nonnegative. For example, the stationary time series

$$X_t = A\cos(\omega t) + B\sin(\omega t),$$

where A, B are uncorrelated random variables with mean 0 and variance 1, and ω is a positive constant. For this time series, it is not hard to find that its acvf is $\gamma(h) = \cos(\omega h)$. This formula might look quite different from (2.1), but one can rewire the cosine function to find

$$\gamma(h) = \cos(\omega h) = \frac{1}{2} e^{-i\omega h} + \frac{1}{2} e^{i\omega h}.$$

The last quantity turns out to be a *Rieman-Stieltjes integral* (defined later)

$$\frac{1}{2} e^{-i\omega h} + \frac{1}{2} e^{i\omega h} = \int_{-\pi}^{\pi} e^{ih\lambda} dF(\lambda)$$

with

$$F(\lambda) = \begin{cases} 0, & \text{if } \lambda < -\omega \\ \frac{1}{2}, & \text{if } \lambda \in [-\omega, \omega) \\ 1, & \text{if } \lambda \geq \omega \end{cases}.$$

Therefore, we have a similar expression for the acvf γ in terms of

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} dF(\lambda).$$

Definition 2.1 *Let F and g be two real-valued functions on $[a, b]$. Assume that F is right-continuous, nondecreasing, and bounded. Then the Riemann-Stieltjes integral*

$$\int_a^b g(x) dF(x)$$

is given by the limit of

$$\sum_{i=0}^{n-1} g(x_i)\left(F(x_{i+1}^n) - F(x_i^n)\right),$$

where $\{x_i^n\}$ *is a partition of* $[a,b]$ *with* $\max_i |x_{i+1}^n - x_i^n| \to 0$.

Remark 2.2 *In the case when* $F(x) = x$*, the definition of a Riemann-Stieltjes integral is the same as the definition of a Riemann integral introduced in the Calculus course.*

Remark 2.3 *Suppose* F *is not only right-continuous, nondecreasing, and bounded, but also differentiable, i.e.,* $F'(x)$ *exists for all* x*, then*

$$\int_a^b g(x)dF(x) = \int_a^b g(x)F'(x)dx.$$

In other words, the Riemann-Stieltjes integral of g *with respect to* F *is simply the Riemann integral of* gF'*.*

Remark 2.4 *If* F *is piece-wise constant, nondecreasing, and bounded with "jumps" at pints* $y_1, \ldots, y_k \in (a,b)$*, then*

$$\int_a^b g(x)dF(x) = \sum_{j=1}^{k} g(y_j)(F(y_j) - F(y_j^-)),$$

where

$$F(y^-) \doteq \lim_{\substack{z\to y\\ z<y}} F(z).$$

The following theorem says that for any stationary time series with acvf γ, there might not exist f such that

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)d\lambda$$

but there is always a function F such that

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} dF(\lambda).$$

Theorem 2.5 (Herglotz's theorem) *A function* $\gamma : \mathbb{Z} \to \mathbb{R}$ *is the acvf of a stationary time series if and only if there exists a right-continuous, nondecreasing, and bounded function* $F : [-\pi, \pi] \to \mathbb{R}$ *with* $F(-\pi) = 0$ *such that*

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} dF(\lambda)$$

for all $h \in \mathbb{Z}$*. (The function* F *is called the spectral distribution function of* γ *and if* $F(\lambda) = \int_{-\pi}^{\lambda} f(v)dv$ *for all* λ *and for some nonnegative* f*, then* f *is called the spectral density function.)*

Remark 2.6 *The spectral distribution function F is not necessary a probability distribution function since $F(\pi)$ might not be one, but the rescaled function*

$$G(\lambda) = \frac{F(\lambda)}{F(\pi)}$$

is a probability distribution function on $[-\pi, \pi]$. Furthermore, since

$$\gamma(0) = \int_{-\pi}^{\pi} e^{i0\lambda} dF(\lambda) = F(\pi) - F(-\pi) = F(\pi),$$

we find that the acf

$$\rho(h) \doteq \frac{\gamma(h)}{\gamma(0)} = \frac{\gamma(h)}{F(\pi)} = \int_{-\pi}^{\pi} e^{ih\lambda} \frac{1}{F(\pi)} dF(\lambda) = \int_{-\pi}^{\pi} e^{ih\lambda} dG(\lambda).$$

This implies that G is the spectral distribution function of acf ρ.

Remark 2.7 *If the spectral distribution function F has a density f, i.e.,*

$$F(\lambda) = \int_{-\pi}^{\lambda} f(v) dv,$$

then the time series is said to have a continuous spectrum. On the other hand, if F is a piecewise-constant function, then the time series is said to have a discrete spectrum. In addition, because F could be piece-wise continuous but not piecewise-constant, the time series could have a spectrum that is neither continuous nor discrete.