SF2943 Time Series Analysis: Lecture 3

April 18, 2022

In this lecture, we will mention some essential features of autocovariance functions and introduce linear processes that contain significant time series models, including moving average processes as well as autoregressive processes.

1 Autocovariance Functions

We recall that a function $\kappa : \mathbb{Z} \to \mathbb{R}$ is nonnegative definite if for any $n \in \mathbb{N}$, and for any $(a_1, \ldots, a_n) \in \mathbb{R}^n$, we have

$$\sum_{i,j=1}^{n} a_i \kappa(i-j) a_j \geq 0.$$

Theorem 1.1 (Characterization of Autocovariance Functions) *A function $\gamma : \mathbb{Z} \to \mathbb{R}$ is the autocovariance function of a stationary time series if and only if it is even and nonnegative definite.*

Proof. We will only prove that given any autocovariance function of a stationary time series γ, it must be even and nonnegative definite. The other direction holds due to Kolmogorov's extension theorem (aka Kolmogorov's consistency theorem). See Theorem 1.5.1 in *Time Series: Theory and Methods* for more detail.

Let γ be the acvf of some time series $\{X_t\}$, i.e., $\gamma(h) = \mathrm{Cov}(X_{t+h}, X_t)$ for all t and h. Then for any $n \in \mathbb{N}$, and for any $(a_1, \ldots, a_n) \in \mathbb{R}^n$, we have

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) &= \mathrm{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{n} a_j X_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}\left(a_i X_i, a_j X_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{Cov}\left(X_i, X_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma(i-j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \gamma(i-j) a_j.
\end{aligned}$$

Since we know variance is always nonnegative, this implies that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \gamma(i-j) a_j = \mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) \geq 0.$$

Hence, γ is nonnegative definite.

To show that γ is also even, namely, $\gamma(h) = \gamma(-h)$ for all h. Notice that

$$\gamma_X(h) = \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(X_t, X_{t+h}) = \mathrm{Cov}(X_{(t+h)-h}, X_{t+h}) = \gamma_X(-h).$$

The proof is complete. ■

Remark 1.2 *Due to this characterization of autocovariance functions, if a function $\gamma : \mathbb{Z} \to \mathbb{R}$ is even and nonnegative definite, then we will call it an autocovariance function.*

Remark 1.3 *An autocorrelation function ρ satisfies the same properties as an autocovariance function, i.e., it is both even and nonnegative definite.*

In the following, we will show one use of the characterization of autocovariance functions in determining if a function is an acvf or not.

Example 1.1 *Consider a function $\kappa : \mathbb{Z} \to \mathbb{R}$ defined by*

$$\kappa(h) = \begin{cases} 1, & \text{if } h = 0 \\ \rho, & \text{if } h = 1, -1 \\ 0, & \text{otherwise} \end{cases}$$

for some $\rho \in \mathbb{R}$. This function κ is an acvf (i.e., even and nonnegative definite) if and only if $|\rho| \leq 1/2$.

Proof. If $\rho = 0$, then it is easy to see that κ is the acvf of a white noise $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ with $\sigma^2 = 1$.

For $\rho \neq 0$ and $|\rho| \leq 1/2$, we will show that κ is the acvf for a MA(1) process with

$$\theta = \frac{1 \pm \sqrt{1 - 4\rho^2}}{2\rho} \text{ and } \sigma^2 = \frac{\rho}{\theta}.$$

Recall that a MA(1) process $\{X_t\}$ is defined as $X_t = Z_t + \theta Z_{t-1}$ with $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. In addition, we showed in our previous lecture that its acvf γ_X is

$$\gamma_X(h) = \begin{cases} \sigma^2(1 + \theta^2), & \text{if } h = 0 \\ \sigma^2\theta, & \text{if } h = 1, -1 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, κ could be an acvf γ_X for some MA(1) process if $\sigma^2(1 + \theta^2) = 1$ and $\sigma^2\theta = \rho$.

Because $\rho \neq 0$ and $\sigma^2 = \rho/\theta$ and plug it into $\sigma^2(1 + \theta^2) = 1$ gives

$$\rho\theta^2 - \theta + \rho = 0.$$

This implies that

$$\theta = \frac{1 \pm \sqrt{1 - 4\rho^2}}{2\rho}$$

and if $|\rho| > 1/2$, then θ is not a real number and thus κ can not be an acvf for any MA(1) process.

However, in order to show that κ is not an acvf, we have to either prove it is not even or not nonnegative definite. Since κ is apparently even, we will prove it is not nonnegative definite when $|\rho| > 1/2$.

If $\rho > 1/2$, consider $(a_1, a_2, \ldots, a_n) = (-1, (-1)^2, \ldots, (-1)^n)$, we find

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\kappa(i-j)a_j &= \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^i\kappa(i-j)(-1)^j = \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}\kappa(i-j) \\
&= \sum_{i=1}^{n}(-1)^{2i}\kappa(0) + \sum_{i=1}^{n-1}(-1)^{2i+1}\kappa(-1) + \sum_{i=2}^{n}(-1)^{2i-1}\kappa(1) \\
&= \sum_{i=1}^{n}(-1)^{2i}1 + \sum_{i=1}^{n-1}(-1)^{2i+1}\rho + \sum_{i=2}^{n}(-1)^{2i-1}\rho \\
&= n + (n-1)(-\rho) + (n-1)(-\rho) = n - 2(n-1)\rho \\
&= (1-2\rho)n + 2\rho.
\end{aligned}$$

Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\kappa(i-j)a_j = (1-2\rho)n + 2\rho < 0$$

for any

$$n > \frac{2\rho}{2\rho - 1} > 0.$$

This implies that κ is not nonnegative definite when $\rho > 1/2$.

If $\rho < -1/2$, consider $(a_1, a_2, \ldots, a_n) = (1, 1, \ldots, 1)$, we find

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\kappa(i-j)a_j &= \sum_{i=1}^{n}\sum_{j=1}^{n}\kappa(i-j) = \sum_{i=1}^{n}\kappa(0) + \sum_{i=1}^{n-1}\kappa(-1) + \sum_{i=2}^{n}\kappa(1) \\
&= \sum_{i=1}^{n}1 + \sum_{i=1}^{n-1}\rho + \sum_{i=2}^{n}\rho = n + (n-1)\rho + (n-1)\rho \\
&= n + 2(n-1)\rho = (1+2\rho)n - 2\rho.
\end{aligned}$$

Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\kappa(i-j)a_j = (1+2\rho)n - 2\rho < 0$$

for any

$$n > \frac{2\rho}{2\rho + 1} > 0.$$

This implies that κ is not nonnegative definite when $\rho < -1/2$. ■

2 Stationary Time Series

In this section, we introduce some properties of strictly stationary time series and stationary time series. In particular, we introduce the notion of q-dependent and q-correlated, which describe the "relation" of a time series at different time points.

Definition 2.1 *$\{X_t\}$ is a strictly stationary time series if*

$$(X_1, \ldots, X_n) \stackrel{d}{=} (X_{1+h}, \ldots, X_{n+h})$$

for all $h \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Proposition 2.2 *Properties of a strictly stationary time series $\{X_t\}$:*

1. *$\{X_t\}$ are identically distributed.*
2. *$(X_t, X_{t+h}) \stackrel{d}{=} (X_1, X_{1+h})$ for all $t, h \in \mathbb{Z}$.*
3. *$\{X_t\}$ is stationary if $EX_t^2 < \infty$ for all t.*
4. *An iid sequence is strictly stationary.*
5. *Stationarity does not imply strict stationarity.*

A simple way to construct a strictly stationary time series is by applying a function to an iid sequence. To be more precise, let $\{Z_t\}$ be an iid sequence of random variables, consider

$$X_t \doteq g(Z_t, Z_{t+1}, \ldots, Z_{t+q}) \tag{2.1}$$

for all $t \in \mathbb{Z}$ and for some function $g : \mathbb{R}^q \to \mathbb{R}$. Then $\{X_t\}$ is strictly stationary. Moreover, notice that X_t depends on $Z_t, Z_{t+1}, \ldots, Z_{t+q}$ and X_{t+h} depends on $Z_{t+h}, Z_{t+h+1}, \ldots, Z_{t+h+q}$. Hence, X_t is independent of X_{t+h} if and only if $h > q$ or $h < -q$.

Definition 2.3 *A time series $\{X_t\}$ is q-dependent for some $q \in \mathbb{N} \cup \{0\}$ if X_t and X_s are independent when $|s - t| > q$.*

According to this definition, it is not hard to see that an iid sequence is 0-dependent. Additionally, the process $\{X_t\}$ defined in (2.1) is q-dependent. We can define an analogous concept in terms of correlation.

Definition 2.4 *A time series $\{X_t\}$ is q-correlated for some $q \in \mathbb{N} \cup \{0\}$ if X_t and X_s are uncorrelated when $|s - t| > q$.*

Remark 2.5 *For a stationary time series $\{X_t\}$, it is q-correlated for some $q \in \mathbb{N} \cup \{0\}$ if its acvf γ_X satisfying $\gamma_X(h) = 0$ for all $|h| > q$.*

Based on the definition, it is easy to check that white noise is 0-correlated and a MA(1) process is 1-correlated. We can extend the order of moving average processes to any q-th order, and the resulting process is q-correlated.

Definition 2.6 *A time series $\{X_t\}$ is a moving-average process of order q (aka MA(q) process or q-th order moving average process) for some $q \in \mathbb{N}$ if*

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\theta_1, \ldots, \theta_q \in \mathbb{R}$.

By showing the mean function and covariance function of a MA(q) process is invariant with time, one can find that a MA(q) process is stationary. Moreover, $\mathrm{Cov}(X_s, X_t) = 0$ for any s, t such that $|s - t| > q$, this means that a MA(q) process is q-correlated. Therefore, for any MA(q) process, it is stationary and q-correlated. It turns out that the reverse statement is true. We will state it without proof.

Proposition 2.7 *If $\{X_t\}$ is a stationary and q-correlated time series with mean 0, then it can be represented as a MA(q) process, i.e., there exists some $\{Z_t\} \sim WN(0, \sigma^2)$ such taht*

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

for all t.

3 Linear Processes

In this section, we will introduce an important class of time series models called linear processes. This class of models contains MA(q) processes. We will study the properties of linear processes, such as computing their acvfs.

Definition 3.1 *A time series $\{X_t\}$ is a linear process if it has the representation*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \text{ for all } t \in \mathbb{Z},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\{\psi_j\} \subset \mathbb{R}$ with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.

Remark 3.2 *The short reason for requiring $\{\psi_j\}$ satisfying $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ is to make sure $\sum_{j=-n}^{n} \psi_j Z_{t-j}$ has a well-defined limit, which makes $\{X_t\}$ a well-defined process.*

According to the definition of a MA(q) process $\{X_t\}$,

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

for all t. Therefore, $\{X_t\}$ is a linear process with

$$\psi_j = \begin{cases} 0, & \text{for } j < 0 \\ 1, & \text{for } j = 0 \\ \theta_j, & \text{for } j = 1, \ldots, q \\ 0, & \text{for} j > q. \end{cases}$$

One can define a MA(∞) process in the following way.

Definition 3.3 *A time series $\{X_t\}$ is a moving-average process of order ∞ (aka MA(∞) process or ∞-th order moving average process) if*

$$X_t = Z_t + \sum_{j=1}^{\infty} \theta_j Z_{t-j},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\{\theta_j\} \subset \mathbb{R}$ with $\sum_{j=1}^{\infty} |\theta_1| < \infty$.

As mentioned in the previous section, any MA(q) process is stationary. The following proposition says that any linear process is also stationary.

Proposition 3.4 *Let $\{Y_t\}$ be a stationary time series with mean function $\mu_Y \equiv 0$ and acvf γ_Y. If $\{\psi_j\} \subset \mathbb{R}$ satisfying $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, then the time series $\{X_t\}$ defined as*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}$$

is stationary with mean function $\mu_X \equiv 0$ and acvf

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h + k - j).$$

Proof. The following proof is not very rigorous in that we pretend the expectation and infinite sum can be exchanged. The rigorous proof requires the use of dominated convergence theorem and the assumption that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, which is beyond the scope of this course.

To find the mean function,

$$\begin{aligned} \mu_X(t) = EX_t = E\left(\sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}\right) &= \sum_{j=-\infty}^{\infty} E\left(\psi_j Y_{t-j}\right) \\ = \sum_{j=-\infty}^{\infty} \psi_j E\left(Y_{t-j}\right) &= \sum_{j=-\infty}^{\infty} \psi_j \mu_Y(t) = 0. \end{aligned}$$

As for the covariance function,

$$\begin{aligned} \gamma_X(t+h, t) = \operatorname{Cov}(X(t+h), X(t)) &= E\left[\left(\sum_{j=-\infty}^{\infty} \psi_j Y_{t+h-j}\right)\left(\sum_{k=-\infty}^{\infty} \psi_k Y_{t-k}\right)\right] \\ &= E\left[\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k Y_{t+h-j} Y_{t-k}\right] = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} E\left[\psi_j \psi_k Y_{t+h-j} Y_{t-k}\right] \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k E\left[Y_{t+h-j} Y_{t-k}\right] = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \operatorname{Cov}(Y_{t+h-j}, Y_{t-k}) \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y((t+h-j) - (t-k)) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j). \end{aligned}$$

This shows that the mean function and covariance function of $\{X_t\}$ is invariant with time; thus, it is stationary. Moreover, the acvf is

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j).$$

■

Corollary 3.5 *Every linear process $\{X_t\}$ is stationary with mean function $\mu_x(h) = 0$ for all h and acvf*

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_\ell \psi_{\ell+h} \sigma^2.$$

Proof. Since a white noise $\{Z_t\}$ is stationary with mean function $\mu_Z \equiv 0$, we know from Proposition 3.4 that $\{X_t\}$ is stationary with mean function $\mu_x(h) = 0$ for all h. Furthermore, $\gamma_Z(h) = 0$ for all $h > 0$ and $\gamma(0) = \sigma^2$, from Proposition 3.4 we know that the acvf

$$\begin{aligned}\gamma_X(h) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Z(h+k-j) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j-h} \gamma_Z(0) \\ &= \sum_{j=-\infty}^{\infty} \psi_j \psi_{j-h} \sigma^2 = \sum_{j=-\infty}^{\infty} \psi_\ell \psi_{\ell+h} \sigma^2.\end{aligned}$$

The second equality holds because $\gamma_Z(h+k-j)$ is not zero only when $k = j-h$ (i.e., $h+k-j = 0$); the last equality holds due to a change of variable. ■