Division of Mathematical Statistics

KTH Matematik

EXAM IN SF2943 TIME SERIES ANALYSIS
MONDAY MAY 29 2017 KL 14:00–19:00.

Teacher: Pierre Nyquist, 08 – 790 6633.

Examiner: Henrik Hult, 08 – 790 6911.

Allowed aids: Pocket calculator, “Formulas and survey, Time series analysis” by Jan Grandell, without notes.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Problem 1

Consider the two ARMA models

$$X_t - X_{t-1} = Z_t - \frac{1}{2}Z_{t-1} - \frac{1}{2}Z_{t-2}, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2).$$
$$Y_t - 3Y_{t-1} = W_t + 2W_{t-1} - 8W_{t-2}, \quad \{W_t\} \sim \mathrm{WN}(0, \sigma^2).$$

Study the AR and MA polynomials and identify the values p and q for which the two processes are ARMA(p, q). Determine whether they are causal and/or invertible. If any of them is causal, compute the first five terms in the associated causal linear process representation and the (stationary) variance of the time series.

Problem 2

From a sample of size $n = 500$ from a stationary time series the corresponding ACVF γ and PACF were estimated. The estimates for the lags 0 to 7 are given in Table 1.

h	0	1	2	3	4	5	6	7
$\hat{\gamma}(h)$	18.54	12.57	8.93	6.89	5.41	4.18	3.58	2.97
$\widehat{\mathrm{PACF}}(h)$	-	0.68	0.040	0.058	0.019	0.0012	0.037	0.0023

Table 1: ACVF and PACF estimates for Problem 2

a) Based on the data of Table 1, choose between an AR(p) and a MA(q) model and suggest an appropriate order p or q. Motivate your answers.

b) For the model suggested in part (a), estimate the relevant model parameters $\phi_1, \ldots, \phi_p$, for an AR(p), or $\theta_1, \ldots, \theta_q$ for an MA(q).

Problem 3

The six figures A-F (in Figure 1) show a realisation, the sample autocorrelation function (based on a sample size of 200), and the spectral density for two AR(2) processes.

a) Which figures are realisations, sample ACFs and spectral densities, respectively? Motivate your answer.

b) Group the figures into two triplets, each containing a realisation, a sample ACF and a spectral density, such that the figures in each triplet correspond to the same time series. What can you say about the AR coefficients ϕ_1, ϕ_2 for each triplet? Motivate your answers.

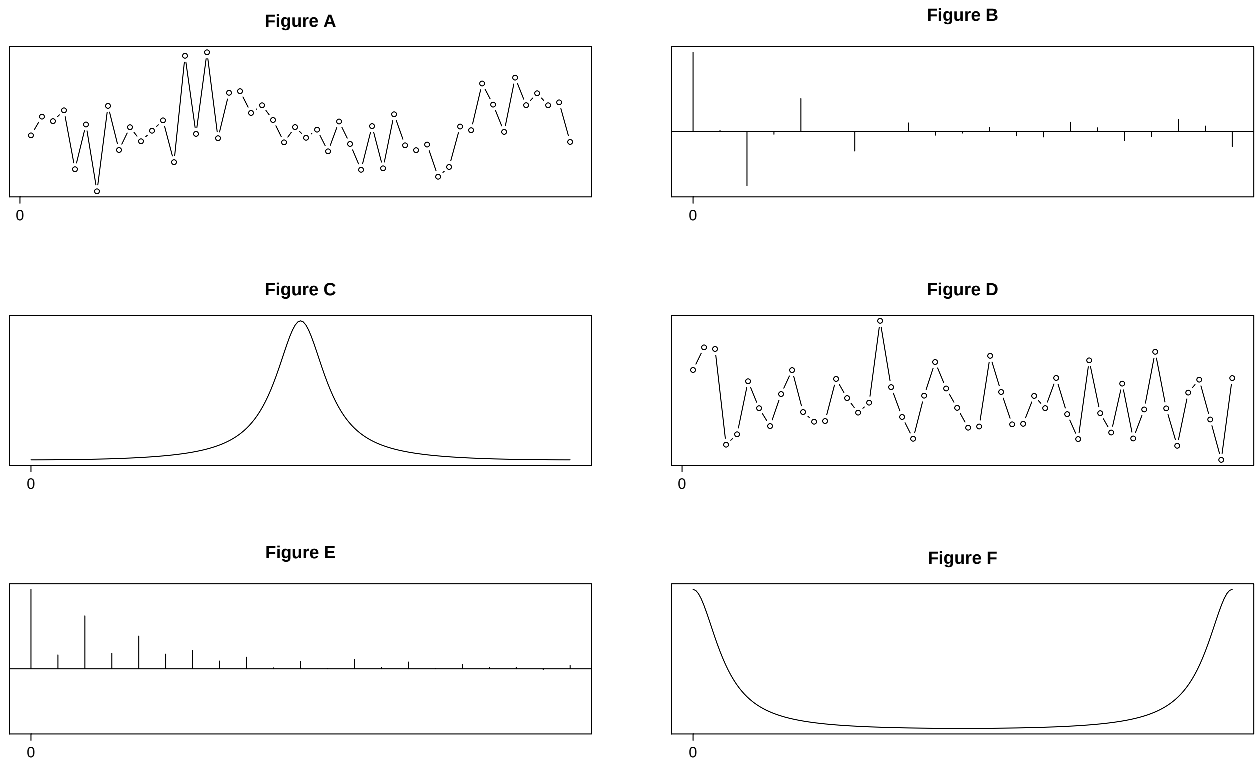

Figure 1: Figures for Problem 3

Problem 4

Consider a time series $\{Y_t\}$ expressed as

$$Y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + g(t) + X_t,$$

where $\{X_t\}$ is a causal and invertible ARMA(p, q) process, g is a function (not constant) with period 12 and $\alpha_0, \alpha_1, \alpha_2 \neq 0$.

a) Show that $\{Y_t\}$ is not stationary.

b) Suggest a linear transform ψ such that $\psi(B)Y_t$ is a stationary time series, where B is the backward shift operator.

c) Determine whether or not the resulting time series, i.e., after applying the transform $\psi(B)$ from (b), is a causal and/or invertible ARMA(p', q') process and if so determine the orders p', q'.

Problem 5

Suppose that $\{Z_t\}$ is a stationary, causal GARCH$(1,1)$ process:

$$\begin{cases} Z_t = \sqrt{h_t} e_t, \quad e_t \sim \text{IID}(0,1), \\ h_t = \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}, \end{cases}$$

for $\alpha_0 > 0$, $\alpha_1, \beta_1 \geq 0$, $\alpha_1 + \beta_1 < 1$, and such that $E[Z_t^4] < \infty$ for all t; the conditions on α_1, β_1 ensure that h_t is stationary.

a) Compute $E[Z_t]$, $E[Z_t^2]$ and $E[Z_t^2 | Z_s, s < t]$.

b) Show that $\{Z_t\}$ can be expressed as an ARCH(∞) process.

c) Compute the best, in terms of mean-square error, predictor $\hat{h}_{t+l}$ of h_{t+l} given $Z_t, Z_{t-1}, \ldots$, $l \geq 2$. How does this predictor compare to the unconditional variance $\sigma^2 = \text{Var}(Z_t)$ (the "naive" predictor) when $l \to \infty$?
Hint: Recall that the best predictor, in mean-square-sense, is $E[h_{t+l} | Z_s, s \leq t]$. Also, You may use the result of (b) and its implications even if you did not solve that part.

Good luck!

Solutions

Problem 1

Start by considering the process $\{X_t\}$. The AR and MA polynomials are given by

$$\begin{aligned} \phi(z) &= 1 - z, \\ \theta(z) &= 1 - \frac{1}{2}z - \frac{1}{2}z^2. \end{aligned}$$

The AR polynomial thus has a root at $z = 1$ and the MA polynomial factorizes as

$$\theta(z) = (1 - z)\left(1 + \frac{1}{2}z\right),$$

which shows that the roots are at $z = 1$ and $z = -2$. The AR and MA polynomials thus have a common root, $z = 1$, and there is redundancy in the description as an ARMA$(1,2)$ process. Factoring out the common factor $1 - z$, the remaining parts of the polynomials are

$$\begin{aligned} \tilde{\phi}(z) &= 1, \\ \tilde{\theta}(z) &= 1 + \frac{1}{2}z. \end{aligned}$$

This shows that the process is in fact an ARMA$(0,1)$, that is an MA$(0,1)$ process. As such it is both causal (by definition) and invertible; root $z = -2$ is outside the unit disc.

For the $\{Y_t\}$ process, the AR and MA polynomials are

$$\phi(z) = 1 - 3z,$$
$$\theta(z) = 1 + 2z - 8z^2.$$

The AR polynomial has a root at $z = 1/3$ and the MA polynomial factorizes as

$$\theta(z) = 1 + 2z - 8z^2 = (1 - 2z)(1 + 4z),$$

with has roots at $z = 1/2$ and $z = -1/4$. There are no common roots and thus $\{Y_t\}$ is an ARMA$(1, 2)$ process that, since both polynomials have roots inside the unit disc, is neither causal nor invertible.
Since only $\{X_t\}$ is causal, there is no linear process representation for $\{Y_t\}$ to consider. In order to compute the first five terms in the representation, $X_t = \sum_{i=0}^{\infty} \psi_j Z_{t-j}$, we can use the relation

$$\psi(z) = \frac{\theta(z)}{\phi(z)} = \frac{1 - \frac{1}{2}z - \frac{1}{2}z^2}{1 - z} = 1 + \frac{z}{2},$$

that is

$$\psi_0 = 1,\ \psi_1 = \frac{1}{2},\ \psi_j = 0, j > 1.$$

This is of course also readily apparent from the MA(1) representation and the associated polynomial $\tilde{\theta}$. Since $\{X_t\}$ is an MA(1) process, the stationary variance is given by

$$\text{Var}(X_t) = E[X_t^2] = \sigma^2\left(1 + \frac{1}{4}\right) = \frac{5\sigma^2}{4}$$

Problem 2

a) The estimates for the PACF at different lags seems to suggest that that PACF is zero for lags $h \geq 2$, which in turn suggests that an AR(1) model would be appropriate. Indeed, for an AR(p) process the sample PACF for lags $h > p$ are approximately independent $N(0, 1/\sqrt{n})$ random variables. As such, approximately 95% of the observations should be within $\pm 1.96/\sqrt{n} \approx 0.0877$. In Table 1 it is only for lag 1 that $\widehat{\text{PACS}}$ exceeds this value, suggesting that an AR(1) is a reasonable choice. Moreover, $\hat{\gamma}(h)$ seems to exhibit an exponential decay, also consistent with an AR model.
Answer: An AR(1) is a suitable choice.

b) In order to estimate the AR coefficient ϕ_1 we can use the equations determining the autocovariance of the process:

$$\gamma(0) = \frac{\sigma^2}{1 - \phi_1^2},$$
$$\gamma(h) = \phi_1^{|h|}\gamma(0).$$

For $h = 1$ the latter amounts to

$$\phi = \frac{\gamma(1)}{\gamma(0)}.$$

Inserting the estimates from the Table,

$$\hat{\phi}_1 = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0) \approx \frac{12.57}{18.54}} = 0.68.$$

Furthermore, one can also estimate the variance σ^2. Perhaps the easiest way to obtain such an estimate is by means of $\hat{\gamma}(h)$. Rewriting the equation for $\gamma(0)$ and inserting the estimates provides an estimate:

$$\hat{\sigma}^2 = \gamma(0)(1 - \phi_1^2) \approx 18.54(1 - \phi_1^2).$$

Combined with the above this produces an estimate of σ^2:

$$18.54(1 - 0.68^2) \approx 10.02.$$

Answer: For the AR(1) model, suggested in (a), the obtained estimates are $\hat{\phi}_1 = 0.68$ and $\sigma^2 = 10.02$

Remark: The data was obtained from a simulation of an AR(1) process with AR coefficient $\phi_1 = 0.70$ and $\sigma^2 = 10$.

Problem 3

a) We recognize that Figures C and F both have a continuous index on the x-axis, whereas the other four figures have a discrete index. This suggests that Figures C and F show spectral densities, since the spectral density is a function of a continuous variable (angular frequency or frequency). Next, recall that the ACF of a time series is maximal and equal to one at lag zero. This is consistent with Figures B and E, whereas Figures A and D are *not* maximal at index $x = 0$. Thus, Figures B and E show sample ACFs and Figures A and D realisations of the time series.

Answer: C and F show spectral densities, B and E ACFs, A and D realisations.

b) Studying the ACF plots, Figure B shows a (decreasing) correlation of alternating sign for even lags - no correlation for odd lags - whereas Figure E suggests a process with positive correlation for even lags and little to no correlation for odd lags. Studying the realisations in Figures A and D, Figure D shows a time series for which there seems to be a (strong) negative correlation at lag 2 and positive correlation at lag 4, consistent with Figure B. Figure A shows a time series with a positive correlation for lags 2, 4 etc., consistent with Figure E.

Lastly, a negative correlation for lag 2 and positive correlation for lag 4 (seen in Figure B) suggests a cyclic behavior with period length $2\pi/\lambda \approx 4$. This corresponds to a frequency $\lambda \approx \pi/2$, which should be in the middle of the frequency interval over which the spectral density is plotted (recall that we can restrict ourselves to $\lambda \in [0, \pi]$). This is consistent with Figure C. Similarly, Figure F suggests that the corresponding time series should show a cyclical behavior with period $2\pi/\pi = 2$, the spectral density being large for large values of the argument. This is consistent with a positive correlation for lag 2, and thus with Figures

A and E.

Regarding the coefficients in the AR polynomial, note that both time series seem to have no correlation for lag 1 (and really any odd lag). Similarly, there does not seem to be any strong evidence for a particular type of correlation for one lag in the realisations A and D. This suggests that the first coefficient ϕ_1 is zero, or at least small, for both time series. Moreover, the alternating sign of the ACF in Figure B suggests a negative value for ϕ_2, consistent with the negative two-step correlation seen in Figure D. Similarly, the fact that all values of the ACF in Figure F are positive indicates a positive value for ϕ_2 (also consistent with the behavior seen in Figure A).

Answer: The triplets are C-B-D and F-E-A, and the AR parameters are such that ϕ_1 is zero (or small) for both triplets, and the second coefficient ϕ_2 is negative and positive, respectively, for the two triplets. [*The two series have* $\phi_1 = 0$ *and* $\phi_2 = \pm 0.7$.]

Remark: One way to draw any conclusions regarding the coefficient ϕ_2 for either time series is to use the following set of equations for the autocovariance function γ for an AR(2) process:

$$\begin{aligned}\gamma(k) &= \phi_1\gamma(k-1) + \phi_2\gamma(k-2),\ k \geq 1,\\ \gamma(0) &= \phi_1^2\gamma(0) + \phi_2^2\gamma(0) + 2\phi_1\phi_2\gamma(0) + \sigma^2.\end{aligned}$$

If we set ϕ_1 to zero, motivated by the previous argument(s), then this suggests that

$$\gamma(0) = \frac{\sigma^2}{1-\phi_2^2},$$

and from $\gamma(k) = \phi_2\gamma(k-2)$ we obtain

$$\gamma(k) = \begin{cases}\phi_2^{|k/2|}\sigma^2(1-\phi_2^2)^{-1}, & k \text{ even},\\ 0, & k \text{ odd}.\end{cases}$$

That is, not surprisingly, the autocovariance function resembles that of an AR(1) process with coefficient ϕ_2 for even lags and is zero for odd lags. This is consistent with alternating sign of the ACF (for even lags) for $\phi_2 < 0$, and strictly positive for $\phi_2 > 0$. Note also that, under the assumption that $\phi_1 = 0$, the equations for the autocovariance function γ state that, e.g., $\gamma(1) = \phi_2\gamma(1)$ (the same equation holds for all odd lags), which suggests that either $\gamma(1) = 0$ - the same for all other odd lags - or $\phi_2 = 1$. However, the latter does not correspond to a stationary time series and thus we discard that possibility.

Problem 4

a) At a first glance there is a clear time dependence in the expectation

$$E[Y_t] = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + g(t) + \mu,$$

where $\mu = E[X_t]$. It remains to show that $f(t) := \alpha_1 t + \alpha_2 t^2 + g(t)$ does not amount to a constant function. However, $\alpha_1 t + \alpha_2 t^2$ is not constant and not a periodic function and since g has period 12, f cannot be a constant function. Thus, $E[Y_t]$ depends on t and therefore $\{Y_t\}$ is not a stationary process.

b) We decompose the problem into two parts: Removing the polynomial terms and removing the periodic term. For the latter, since g has period 12 an appropriate choice of transform to apply is $1 - B^{12}$:

$$\begin{aligned}(1-B^{12})Y_t &= \alpha_0 + \alpha_1 t + \alpha_2 t^2 + g(t) + X_t - \alpha_0 - \alpha_1(t-12) - \alpha_2(t-12)^2 - g(t-12) - X_{t-12} \\ &= 12\alpha_1 + \alpha_2(24t - 144) + X_t - X_{t-12},\end{aligned}$$

where we have used that $g(t) = g(t-12)$ for all t. Next, to remove the remaining polynomial term we can apply $1 - B$:

$$(1-B)(1-B^{12})Y_t = 24\alpha_2 + X_t - X_{t-1} - X_{t-12} + X_{t-13}.$$

Alternatively, an equally appropriate choice of (second) transform is $(1-B)^2$, motivated by the fact that Y_t has a quadratic term (one can suggest this without first applying $1 - B^{12}$, which causes the degree of the polynomial to decrease by one). This choice results in the process

$$(1-B)^2(1-B^{12})Y_t = X_t + X_{t-2} - X_{t-12} + 2X_{t-13} - X_{t-14}.$$

Answer: The two most natural choices are $\psi(B) = (1-B)(1-B^{12})$ and $\psi(B) = (1-B)^2(1-B^{12})$.
Remark: Note that these two are by no means the only possible choices. Any choice that renders a stationary process Y_t will of course receive full credit.

c) In accordance with (b), the transform $\psi(B)$ is taken as either

$$\psi(B) = (1-B)(1-B^{12}),$$

or

$$\psi(B) = (1-B)^2(1-B^{12}).$$

Note that the first choice renders a process with mean $E[Y_t] = 24\alpha_2$. Henceforth we consider the mean-adjusted process $Y_t - 24\alpha_2$ and the discussion of ARMA structure and causality and invertibility refers to this “new” process; for the second choice of ψ the resulting process has mean zero

Let ϕ and θ denote the AR and MA polynomial, respectively, of the causal and invertible ARMA(p,q) process $\{X_t\}$. By causality we can express X_t as

$$X_t = \frac{\theta(B)}{\phi(B)} Z_t,$$

where $\{Z_t\}$ is the white noise process associated with X_t. Applying a linear transform $\psi(B)$ thus results in a process

$$\psi(B)\frac{\theta(B)}{\phi(B)} Z_t = \frac{\tilde{\theta}(B)}{\tilde{\phi}(B)} Z_t,$$

where $\tilde{\phi}(z) = \phi(z)$ and

$$\tilde{\theta}(z) = \psi(z)(1 + \theta_1 z + \cdots + \theta_q z^q).$$

For the choice $\psi(z) = (1-z)(1-z^{12})$, this amounts to

$$\begin{aligned}\tilde{\theta}(z) &= (1-z)(1-z^{12})(1 + \theta_1 z + \cdots + \theta_q z^q) \\ &= (1 - z - z^{12} + z^{13})(1 + \theta_1 z + \cdots + \theta_q z^q)\end{aligned}$$

and it is easily seen that this is a polynomial of degree $q + 13$. Similarly, for the choice $\psi(z) = (1-z)^2(1-z^{12})$,

$$\begin{aligned}\tilde{\theta}(z) &= (1-z)^2(1-z^{12})(1 + \theta_1 z + \cdots + \theta_q z^q) \\ &= (1 - 2z + z^2 - z^{12} + 2z^{13} - z^{14})(1 + \theta_1 z + \cdots + \theta_q z^q),\end{aligned}$$

which is a polynomial of degree $q + 14$. In either case the AR polynomial remains the same and the resulting time series is still causal. However, both choices of ψ amounts to introducing a root at $z = 1$ (multiplicity two in the second case) in the MA polynomial, and the resulting time series is therefore *not* invertible.

Answer: For the two choices of ψ from (b), the process is a causal, non-invertible ARMA$(p, \tilde{q})$ process, with $\tilde{q}$ equal to $q + 13$ and $q + 14$, respectively.

Problem 5

a) Start with $E[Z_t]$. By causality,

$$\begin{aligned}E[Z_t] &= E\left[\sqrt{\alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}}\, e_t\right] \\ &= E\left[\sqrt{\alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}}\right] E[e_t] \\ &= 0,\end{aligned}$$

because $E[e_t] = 0$ and h_t is stationary (and thus has a mean $< \infty$). Note that for this to be completely rigorous we must work a bit harder, or for example appeal to part (b) of the problem. However this is not required for full credit.

Next, we consider $E[Z_t^2]$. By definition,

$$\begin{aligned}E[Z_t^2] &= E[h_t e_t^2] \\ &= E\left[(\alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}) e_t^2\right] \\ &= E\left[(\alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1})\right] E\left[e_t^2\right],\end{aligned}$$

where in the last step we have used causality. Again, for a fully rigorous treatment this needs to be better motivated (part (b) is enough here as well) but we ignore that for now. Next, by stationarity $E[Z_{t-1}^2] = E[Z_t^2]$, and because $E[e_t^1] = 1$ we have

$$\begin{aligned}E[Z_t^2] &= E\left[(\alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1})\right] \\ &= \alpha_0 + \alpha_1 E[Z_t^2] + \beta_1 E[h_{t-1}].\end{aligned}$$

By stationarity for h_t and Z_t,

$$E[h_t] = \alpha_0 + \alpha_1 E[Z_t^2] + \beta_1 E[h_t],$$

which is equivalent to

$$E[h_t] = \frac{\alpha_0 + \alpha_1 E[Z_t^2]}{1 - \beta_1}.$$

Inserting this into the expression for $E[Z_t^2]$,

$$E[Z_t^2] = \alpha_0 + \alpha_1 E[Z_t^2] + \beta_1 \frac{\alpha_0 + \alpha_1 E[Z_t^2]}{1 - \beta_1}],$$

which gives

$$E[Z_t^2] \left(1 - \alpha_1 - \frac{\alpha_1 \beta_1}{1 - \beta_1}\right) = \alpha_0 + \frac{\alpha_0 \beta_1}{1 - \beta_1}.$$

Some algebra then shows that this is equivalent to

$$E[Z_t^2] = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}.$$

Lastly, consider $E[Z_t^2 | Z_s, s < t]$. Using the definition of h_t and causality once more (again, not fully rigorous but sufficient for now),

$$\begin{aligned} E[Z_t^2 | Z_s, s < t] &= E\left[(\alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}) e_t^2 | Z_s, s < t\right] \\ &= \left(\alpha_0 + \alpha_1 E\left[Z_{t-1}^2 | Z_s, s < t\right] + \beta_1 E\left[h_{t-1} | Z_s, s < t\right]\right) E[e_t^2] \\ &= \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 E\left[h_{t-1} | Z_s, s < t\right]. \end{aligned}$$

Moreover, based on the the definition of h_{t-1} we can make the claim that $E[h_{t-1} | Z_s, s < t] = h_{t-1}$; for a fully rigorous derivation one can use, for example, part (b). The expression for $E[Z_t^2 | Z_s, s < t]$ then becomes

$$E[Z_t^2 | Z_s, s < t] = \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1} = h_t,$$

which is precisely what we expect. That is, h_t is the conditional variance of Z_t given previous values $Z_{t-1}, Z_{t-2}, \ldots$.
Answer: $E[Z_t] = 0$, $E[Z_t^2] = \alpha_0/(1 - \alpha_1 - \beta_1)$ and $E[Z_t^2 | Z_s, s < t] = h_t$.

b) To show that it is an ARCH(∞) process we must show that h_t can be expressed as an infinite sum with summands containing $Z_{t-1}, Z_{t-2}, \ldots$ (see the definition of an ARCH(p)). Start by using the recursive definition of h_t:

$$\begin{aligned} h_t &= \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1} \\ &= \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1(\alpha_0 + \alpha_1 Z_{t-2}^2 + \beta_1 h_{t-2}) \\ &= \alpha_0(1 + \beta_1) + \alpha_1 Z_{t-1}^2 + \alpha_1 \beta_1 Z_{t-2}^2 + \beta_1^2 h_{t-2} \\ &= \alpha_0(1 + \beta_1 + \beta_1^2) + \alpha_1 Z_{t-1}^2 + \alpha_1 \beta_1 Z_{t-2}^2 + \alpha_1 \beta_1^2 Z_{t-3}^2 + \beta_1^3 h_{t-3}, \end{aligned}$$

where we have applied the definition twice (for h_{t-1} and h_{t-2}). Proceeding like this we obtain after n "steps"

$$h_t = \alpha_0 \sum_{i=0}^{n} \beta_1^i + \alpha_1 \sum_{i=1}^{n+1} \beta_1^{i-1} Z_{t-i}^2 + \beta_1^{n+1} h_{t-n-1}.$$

Similar to the ARCH(1) case studied in class (and the additional lecture notes), we now want to take the limit as n goes to infinity. The conjecture is that the first two sums converge and the third term converges to 0. This can indeed be shown using an argument similar to that used for linear processes:

$$\begin{aligned} E\left[\left|h_t - \alpha_0 \sum_{i=0}^{n} \beta_1^i - \alpha_1 \sum_{i=1}^{n+1} \beta_1^{i-1} Z_{t-i}^2\right|^2\right] & \\ = E\left[\left|\beta_1^{n+1} h_{t-n-1}\right|^2\right] & \\ = \beta_1^{2(n+1)} E[h_{t-n-1}^2]. & \end{aligned}$$

Since h_t is stationary (and has finite second moment), $E[h_{t-n-1}^2] = E[h_t^2]$ for all t and n. Combined with $\beta_1 < 1$ this gives

$$E\left[\left|h_t - \alpha_0 \sum_{i=0}^{n} \beta_1^i - \alpha_1 \sum_{i=1}^{n+1} \beta_1^{i-1} Z_{t-i}^2\right|^2\right] = \beta_1^{2(n+1)} E[h_t^2] \to 0, \ n \to \infty.$$

That is, h_t is equal to the mean-square limit

$$\alpha_0 \sum_{i=0}^{\infty} \beta_1^i + \alpha_1 \sum_{i=1}^{\infty} \beta_1^{i-1} Z_{t-i}^2,$$

which can be expressed as

$$h_t = \frac{\alpha_0}{1-\beta_1} + \sum_{j=0}^{\infty} \alpha_1 \beta_1^j Z_{t-j-1}^2.$$

Answer: A GARCH$(1,1)$ process is, under the given conditions, an ARCH(∞) process with parameters as given in the last display.

c) From part (b) it is clear that h_{t+1} is measurable w.r.t. the σ-algebra generated by $\{Z_s, s \leq t\}$. That is, given $Z_t, Z_{t-1}, \dots$, h_{t+1} is completely known. We will make repeated use of this fact.

Start by considering $\hat{h}_{t+1}$. The best linear predictor, in the sense considered here, is

$$\hat{h}_{t+1} = E\left[h_{t+1} | Z_s, s \leq t\right] = h_{t+1}.$$

Next,

$$\begin{aligned} \hat{h}_{t+2} &= E[h_{t+2} | Z_s, s \leq t] \\ &= E\left[\alpha_0 + \alpha_1 Z_{t+1}^2 + \beta_1 h_{t+1} | Z_s, \leq t\right] \\ &= \alpha_0 + \alpha_1 E[Z_{t+1}^2 | Z_s, s \leq t] + \beta_1 E[h_{t+1} | Z_s, \leq t] \\ &= \alpha_0 + \alpha_1 h_{t+1} + \beta_1 h_{t+1} \\ &= \alpha_0 + (\alpha_1 + \beta_1) h_{t+1}. \end{aligned}$$

Set $\sigma^2 = \alpha_0/(1-\alpha_1-\beta_1)$ (the unconditional variance of the process). We can rewrite the last display as

$$\hat{h}_{t+2} = \sigma^2 + (\alpha_1+\beta_1)\left(h_{t+1}-\sigma^2\right).$$

Computing the best linear prediction of h_{t+3}, given $\{Z_s, s \le t\}$, we obtain

$$\begin{aligned}\hat{h}_{t+3} &= \alpha_0 + \alpha_1 E[Z_{t+2}^2|Z_s, s\le t] + \beta_1 E[h_{t+2}|Z_s.s \le t]\\ &= \alpha_0 + \alpha_1 E[h_{t+2}e_{t+2}^2|Z_s, s\le t] + E[h_{t+2}|Z_s, s\le t]\\ &= \alpha_0 + (\alpha_1+\beta_1)\hat{h}_{t+2},\end{aligned}$$

where in the last step we have used part (b), causality for the Z_t process and that $\hat{h}_{t+2} = E[h_{t+2}|Z_s, s\le t]$. Completely analogous to before we can rewrite this as

$$\hat{h}_{t+3} = \sigma^2 + (\alpha_1+\beta_1)(\hat{h}_{t+2}-\sigma^2).$$

Inserting the expression for $\hat{h}_{t+2}$,

$$\hat{h}_{t+3} = \sigma^2 + (\alpha_1+\beta_1)^2(h_{t+1}-\sigma^2).$$

Proceeding like this yields, for $l \ge 2$,

$$\hat{h}_{t+l} = E[h_{t+l}|Z_s, s\le t] = \sigma^2 + (\alpha_1+\beta_1)^{l-1}(h_{t+1}-\sigma^2),$$

which converges to σ^2 as l grows to infinity.

Answer: The best predictor, in mean-square sense, is

$$\hat{h}_{t+l} = \sigma^2 + (\alpha_1+\beta_1)^{l-1}(h_{t+1}-\sigma^2),$$

which converges to the unconditional variance σ^2 as the number of steps l grows.