SF2943 Time Series Analysis: Lecture 1

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1 Introduction

A time series is a temporal date $\{x_1, x_2, \cdots, x_n\}$. In other words, a sequence of observations at different times consists of a time series. A well-known time series is the price of the stock market. However, the scope of application for time series is much broader than just in finance. Examples include bus GPS data, yearly average temperatures in Sweden, every year baby birth rates in Stockholm, etc. One of the goals of time series analysis is to predict (forecast) the future based on the temporal data we observe. In order to make a prediction, we have to build a model for the temporal data. In this course, our focus is to create a time series model. The exact definition of a time series model is given below.

Definition 1.1 *A time series model for the observed data $\{x_t\}$ is a specification of the joint distributions of a sequence of random variables $\{X_t\}$ of which $\{x_t\}$ is a realization.*

Remark 1.2 *In probability theory, a capital letter is often used to denote a random object; a lower letter is used to denote a deterministic object.*

Remark 1.3 *Given a specification of the joint distributions of a sequence of random variables $\{X_t\}$. The joint distributions of $\{X_t\}$ are determined by*

$$F_n(x_1, \cdots, X_n) = P(X_1 \le x_1 \cdots, X_n \le x_n),$$

for all $n \in \mathbb{N}$ and $x_1, \cdots, x_n \in \mathbb{R}$.

Since it is too complicated to specify such a large class of joint distributions, it is impractical to model our time series in this general form. On the other hand, recall that for any two random variables X, Y, a useful way to measure their relation is through covariance of X and Y. Therefore, for a time series $\{X_t\}$, the covariance of X at any two time points t, s is an important quantify to study.

Definition 1.4 *Let $\{X_t\}$ be a time series with $EX_t^2 < \infty$*[1]*. The mean function of $\{X_t\}$ is*

$$\mu_X(t) = EX_t \text{ for all } t.$$

The covariance function of $\{X_t\}$ is

$$\gamma_X(s,t) = Cov(X_s, X_t) \text{ for all } s, t.$$

[1] Otherwise, the covariance might be ∞.

In this course, most of the time, we will focus on building a time series model of the following form:

$$X_t = m_t + s_t + Y_t,$$

where m_t is called the trend component, s_t is the seasonal component, and Y_t is a "stationary" noise. In order to make an "educational guess" when picking our models, we have to understand various types of models and understand their properties so that we can justify our choice.

2 Zero-Mean Models

Let's start by introducing a few models with no trend and no seasonal component.

Example 2.1 (iid noise) *Let $\{X_t\}_t$ be a sequence of independent and identically distributed (iid) random variables with means 0 and variance σ^2. We will use the notation $\{X_t\}_t \sim IID(0, \sigma^2)$ to denote such process. The kind of time series has the following two properties:*

- $P(X_{n+h} \leq x | X_1 = x_1, \cdots, X_n = x_n) = P(X_{n+h} \leq x)$.
- $\min_f E[(X_{n+h} - f(X_1, \cdots, X_n)^2)]$ *is achieved at* $f = 0$.

The first property says that the past observations $x_1, \cdots, x_n$ does affect the future X_{n+h}. The intuition for the second property is that since the past information $X_1, \cdots, X_n$ can not assist our prediction/estimation of the future X_{n+h}, and the only information we know about X_{n+h} is that $EX_{n+h} = 0$, our best guess must be $f = 0$.

We will use the notation $\{X_t\}_t \sim \text{IID}(0, \sigma^2)$ to denote a sequence of independent and identically distributed (iid) random variables with means 0 and variance σ^2.

The following is a special example of iid noise.

Example 2.2 (Binary Process) *Let $\{X_t\}_t$ be a sequence of independent and identically distributed (iid) random variables satisfying*

$$P(X_t = 1) = P(X_t = -1) = \frac{1}{2}.$$

Such a time series is called a binary process. Moreover, $EX_t = 0$. One can view $\{X_t\}_t$ as the results of a sequence of coin-flipping with a fair coin, where 1 refers to a head and -1 refers to a tail.

Example 2.3 (Random Walk) *Let $\{X_t\}_t$ be a binary process. An important time series $\{S_t\}$ associated with $\{X_t\}_t$, known as a simple symmetric random walk, is defined as $S_0 = 0$ and*

$$S_t = X_1 + \cdots + X_t.$$

This process is no longer iid.

Another important zero-mean model is white noise. White noise is similar to iid noise, but "weaker".

Example 2.4 (White noise) *A time series $\{X_t\}_t$ is a white noise (with mean 0 and variance σ^2) if*

- $EX_t = 0$,
- X_s, X_t *are uncorrelated for any s, t such that $s \neq t$, i.e., $Cov(X_s, X_t) = 0$ for $s \neq t$.*
- $EX_t^2 = \sigma^2$ *for all t.*

We use the notation $\{X_t\}_t \sim WN(0, \sigma^2)$.

Remark 2.1 *Recall that for any two random variables X, Y, if they are independent, it implies they are uncorrelated, but the reverse does not always hold. Therefore, $\{X_t\}_t \sim IID(0, \sigma^2)$ implies $\{X_t\}_t \sim WN(0, \sigma^2)$, but not vice versa.*

3 Models with Trend and Seasonality

Example 3.1 (Models only with trend component) *An example of model with trend is $X_t = m_t + Y_t$. Often we still need to further make a model for the trend component. For example, by assuming it is linear $m_t = a_0 + a_1 t$ with unknown constant a_0, a_1, which can be estimated by tools like linear regression with our observed data.*

Example 3.2 (Models only with seasonal component) *An example of model with trend is $X_t = s_t + Y_t$. In this case, we still need to further make a model for the seasonal component. A convenient choice is using a sum of trigonometric functions*

$$s_t = a_0 + \sum_{j=1}^{k} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)),$$

where a_0, a_j, b_j are unknown constant and λ_j are known periodicity. For example, if our data is monthly average daylight time for the past 10 years, then a reasonable choice of λ is 12.

4 Stationary Models

Recall that $\mu_X(t) = EX_t$ and $\gamma_X(s, t) = \mathrm{Cov}(X_s, X_t)$.

Definition 4.1 *A time series $\{X_t\}$ is (weakly) stationary if*

- $\mu_X(t)$ *is invariant with time.*
- *For any given $h \geq 0$, $\gamma_X(t+h, t)$ is also invariant with time.* [2]

Definition 4.2 *A time series $\{X_t\}$ is strictly stationary if any finite dimensional distribution of $\{X_t\}$ is invariant with time. Namely, for any $n \in \mathbb{N}$, for any $t_1, \ldots, t_n$, and for any k,*

$$(X_{t_1}, X_{t_2}, \cdots, X_{t_n}) \stackrel{d}{=} (X_{t_1+k}, X_{t_2+k}, \cdots, X_{t_n+k}).$$

[2]Here we implicitly assume that $EX_t^2 < \infty$ for all t.

Remark 4.3 *A strictly stationary time series $\{X_t\}$ satisfying $EX_0^2 < \infty$ is (weakly) stationary.*

Since for a stationary time series $\{X_t\}$, the covariance function $\gamma_X(t+h,t)$ is invariant with time. Abuse of notation, we denote $\gamma_X(t+h,t)$ by $\gamma_X(h)$ in this case, and call $\gamma_X(h)$ the autocovariance function of $\{X_t\}$ at lag h.

Definition 4.4 *Let $\{X_t\}$ be a stationary time series. The autocovariance function (acvf) of $\{X_t\}$ at lag h is defined as*

$$\gamma_X(h) = Cov(X_{t+h}, X_t).$$

In particular, $\gamma_X(0) = Cov(X_t, X_t) = Var(X_t) = EX_t^2$. Moreover, the autocorrelation function (acf) of $\{X_t\}$ at lag h is defined as

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}.$$

Remark 4.5 *Let $\{X_t\}$ be a stationary time series. If we define the autocorrelation function (acf) of $\{X_t\}$ at lag h as correlation of X_{t+h} and X_t, then we have*

$$Cor(X_{t+h}, X_t) = \frac{Cov(X_{t+h}, X_t)}{\sqrt{Var(X_{t+h})\,Var(X_t)}} = \frac{\gamma_X(h)}{\gamma_X(0)}.$$

4.1 Examples of stationary time series

Example 4.1 (iid noise and white noise) *Let $\{X_t\} \sim IID(0, \sigma^2)$. Then since $\mu_X(t) = EX_t = 0$ and*

$$\gamma_X(t+h,t) = Cov(X_{t+h}, X_t) = \begin{cases} 0, & \text{if } h \neq 0 \\ \sigma^2, & \text{if } h = 0. \end{cases}$$

Thus both the mean function and the covariance function are invariant with time, which implies that $\{X_t\}$ is stationary. Notice that the acf of $\{X_t\}$ is

$$\gamma_\rho(t+h,t) = \frac{\gamma_X(h)}{\gamma_X(0)} = Cov(X_{t+h}, X_t) = \begin{cases} 0, & \text{if } h \neq 0 \\ 1, & \text{if } h = 0. \end{cases}$$

Let $\{X_t\} \sim WN(0, \sigma^2)$. For the same reason, we will have the same mean function and the same covariance function as iid noise case. Therefore, $\{X_t\}$ is stationary.

Example 4.2 (Counterexample: random walk) *Let $\{S_t\}$ be the simple symmetric random walk. Then $ES_t = EX_1 + \cdots + EX_t = 0$, and for any $h \neq 0$*

$$\gamma_S(t+h,t) = Cov(S_{t+h}, S_t) = Cov(S_t + X_{t+1} + \cdots + X_{t+h}, S_t) = Cov(S_t, S_t) = t\sigma^2.$$

Therefore, the covariance function is NOT invariant with time. This means that $\{S_t\}$ is NOT stationary (despite the mean function being invariant with time).