Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2945 TIME SERIES ANALYSIS/TIDSSERIEANALYS
TISDAGEN DEN 16 DECEMBER 2008 KL 14.00–19.00.

Examinator: Timo Koski, tel. 7907136

Tillåtna hjälpmedel: Formulas and survey, Time series analysis. Handheld calculator.

Införda beteckningar skall förklaras och definieras. Resonemang och uträkningar skall vara så utförliga och väl motiverade att de är lätta att följa.

Varje korrekt lösning ger 10 poäng. Gränsen för godkänt är 25 poäng. De som erhåller 23 eller 24 poäng på tentamen kommer att erbjudas möjlighet att komplettera till betyget E. Den som är aktuell för komplettering skall till examinator anmäla önskan att få en sådan inom en vecka från publicering av tentamensresultatet.

Lösningarna får givetvis skrivas på svenska.

Resultatet rapporteras till Ladok senast tisdagen den 13 januari 2009 och finns därefter tillgängligt via ”Mina sidor”.
Tentamen kommer att finnas tillgänglig på studentexpeditionen till den 1 mars 2009.

Lösningarna får givetvis skrivas på svenska.

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Quantiles of the normal distribution
(Normalfördelningens kvantiler)

$P(X > \lambda_\alpha) = \alpha$ where $X \sim N(0,1)$

α	λ_α	α	λ_α
0.10	1.2816	0.001	3.0902
0.05	1.6449	0.0005	3.2905
0.025	1.9600	0.0001	3.7190
0.010	2.3263	0.00005	3.8906
0.005	2.5758	0.00001	4.2649

Problem 1

Let $\{X_t, t \in \mathbb{Z}\}$ be an MA(1) process

$$X_t = Z_t + \theta Z_{t-1},$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

(a) Find the value θ^* for θ such that the coefficient of correlation between X_t and X_{t-1} is maximal. (7 p)

(b) Find the spectral density of $\{X_t, t \in \mathbb{Z}\}$ under θ^*. Are high or low frequencies dominating in the spectrum? Why is your conclusion about the frequency contents reasonable? (3 p)

Problem 2

Let $X, Y_1, Y_2, Y_3, \ldots, Y_n$ be any set of random variables with means 0 and known covariances.

Prove that

$$\mathrm{Var}(X) \geq \mathrm{Var}(\widehat{X}),$$

where $\widehat{X}$ is the linear minimal mean square predictor (or estimator) of X in terms of $Y_1, Y_2, Y_3, \ldots, Y_n$. Justify your proof in sufficient detail. (10 p)

Problem 3

Let $\{X_t, t \in \mathbb{Z}\}$ be a Gaussian and stationary AR(1) time series

$$X_t = 0.4 \cdot X_{t-1} + Z_t,$$

where the white noise $\{Z_t\}$ I.I.D $\sim N(0, 1)$.

Determine two constants a and b such that

$$P\left(a \leq X_t \leq b \mid X_{t-1} = 0.3\right) = 0.95.$$

(10 p)

Problem 4

A GARCH(1,1) time series $\{X_t, t = 1, 2, \ldots\}$ satisfies

$$X_t = \sigma_t \cdot Z_t, \quad t = 1, 2, \ldots$$

where $\{Z_t, t = 1, 2, \ldots\} \sim$ I.I.D. $N(0, 1)$, and the conditional variance σ_t^2 satisfies

$$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \alpha_1 X_{t-1}^2.$$

where $\alpha_0 > 0, \beta_1 \geq 0, \alpha_1 \geq 0$.

(a) Show that the time series $\{\varepsilon_t\}$ defined by

$$\varepsilon_t = X_t^2 - \sigma_t^2$$

is a white noise. You do not need to find an explicit expression for the second moment of σ_t^2, which is assumed to exist. *Aid*: $E\left[Z_t^4\right] = 3$. (5 p)

(b) Show that the time series $\{X_t^2\}$ is an ARMA(1,1) process

$$X_t^2 = \alpha_0 + (\alpha_1 + \beta_1)\, X_{t-1}^2 + \varepsilon_t - \beta_1 \varepsilon_{t-1},$$

where $\{\varepsilon_t\}$ is the white noise defined in (a). (4 p)

(c) What is the condition for stationarity and causality of the ARMA(1,1) process in (b)? (1 p)

Problem 5

We have an AR(1) process

$$X_n = \phi X_{n-1} + Z_n, n = 0, 1, 2, \ldots, \tag{1}$$

where $\{Z_n\} \sim \mathrm{WN}(0, \sigma^2)$. We assume that $\mathrm{Var}(X_0) = \sigma_0^2$. The true state X_n is not observed directly, but we observe X_n with added white measurement noise $\{V_n\} \sim \mathrm{WN}(0, \sigma_V^2)$, as Y_n in

$$Y_n = cX_n + V_n, n = 0, 1, 2, \ldots, \tag{2}$$

We have that that $\{Z_n\}$ and $\{V_n\}$ are independent.

The predictor $\widehat{X}_{n+1} = E\left[X_{n+1} \mid Y_0, Y_1, \ldots, Y_n\right]$ is computed recursively by the Kalman recursions,

$$\widehat{X}_{n+1} = \phi \widehat{X}_n + \frac{\theta_n}{\nabla_n} \varepsilon_n, \tag{3}$$

where

$$\varepsilon_n = Y_n - c\widehat{X}_n, \tag{4}$$

and

$$e_{n+1} = X_{n+1} - \widehat{X}_{n+1},$$

and

$$E\left[e_{n+1}^2\right] = \phi^2 E\left[e_n^2\right] + \sigma^2 - \frac{\theta_n^2}{\nabla_n}, \tag{5}$$

where

$$E\left[e_0^2\right] = \sigma_0^2, \tag{6}$$

and

$$\theta_n = \phi c E\left[e_n^2\right], \tag{7}$$

and

$$\nabla_n = \sigma_V^2 + c^2 E\left[e_n^2\right]. \tag{8}$$

We consider now a special case of the preceding. Assume now in (1) that $\phi = 1$ and $\sigma^2 = 0$, i.e.,

$$X_n = X_0, \quad n = 0, 1, \ldots, \tag{9}$$

Assume also that $c = 1$, so that

$$Y_n = X_n + V_n, \quad n = 0, 1, 2, \ldots, \tag{10}$$

In following assignments (a) - (c) you are expected to find the Kalman recursions for predicting (or estimating) X_0 using $Y_0, Y_1, \ldots, Y_n, \ldots$, when the model in (9) and (10) holds.

(a) Show that

$$\frac{1}{E\left[e_{n+1}^2\right]} = \frac{1}{E\left[e_n^2\right]} + \frac{1}{\sigma_V^2}, \tag{11}$$

and then that

$$\frac{1}{E\left[e_n^2\right]} = \frac{1}{\sigma_0^2} + \frac{n}{\sigma_V^2}. \tag{12}$$

(3 p)

(b) Show next that

$$\widehat{X}_{n+1} = \widehat{X}_n + \frac{\sigma_0^2}{(n+1)\sigma_0^2 + \sigma_V^2}\left(Y_n - \widehat{X}_n\right). \tag{13}$$

(3 p)

(c) Show next that

$$\widehat{X}_{n+1} = \frac{1}{(n+1)\sigma_0^2 + \sigma_V^2}\left(\sigma_0^2 \sum_{t=0}^{n} Y_t + \sigma_V^2 \widehat{X}_0\right), \tag{14}$$

Aid: it may be helpful to define

$$U_n = \left(n\sigma_0^2 + \sigma_V^2\right)\widehat{X}_n$$

and then to use the fact (check this) that

$$U_{n+1} = U_n + \sigma_0^2 Y_n.$$

What is the interperation of the result in (14) ? What happens when $n \to \infty$? (4 p)

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KTH Matematik

LÖSNINGAR TILL
TENTAMEN I 5B1545 TIDSSERIEANALYS
ONSDAGEN DEN 18 APRIL 2001 KL 14.00–19.00.

Problem 1

(a) An MA(1) process defined by

$$X_t = Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

has ACVF, see the Collection of Formulas (CF),

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 & \text{if } h = 0, \\ \theta\sigma^2 & \text{if } |h| = 1, \\ 0 & \text{if } |h| > 1. \end{cases}$$

The coefficient of correlation between X_t and X_{t-1} is due to stationarity

$$\rho_{X_t, X_{t-1}} = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta}{(1+\theta^2)}.$$

Let us set $\phi(\theta) = \frac{\theta}{(1+\theta^2)}$. Then we have

$$\frac{d}{d\theta}\phi(\theta) = \frac{(1+\theta^2) - 2\theta^2}{(1+\theta^2)} = \frac{1-\theta^2}{(1+\theta^2)}.$$

Thus

$$\frac{d}{d\theta}\phi(\theta) = 0 \Leftrightarrow \theta = \pm 1.$$

We see that $\phi(\theta) < 0$ for $\theta < 0$ and $\phi(\theta) > 0$ for $\theta > 0$. Hence $\theta^* = +1$ gives the maximum for $\rho_{X_t, X_{t-1}}$.

ANSWER (a): $\underline{\theta^* = +1.}$

(b) By CF MA(1) process defined by

$$X_t = Z_t + \theta^* Z_{t-1},$$

has the spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left(1 + (\theta^*)^2 + 2\theta^* \cos(\lambda)\right), \quad -\pi \leq \lambda \leq \pi.$$

When we insert from part (a) of this problem, we get

$$f(\lambda) = \frac{\sigma^2}{2\pi} 2(1+\cos(\lambda)), \quad -\pi \leq \lambda \leq \pi.$$

This is clearly a spectrum where the low frequencies (= frequencies around $\lambda = 0$) dominate (draw a graph). Since we are maximizing the coeffient of correlation between X_t and X_{t-1}, we should also expect the process to change slowly.

ANSWER (b): $\underline{f(\lambda) = \frac{\sigma^2}{2\pi} 2(1+\cos(\lambda)), \quad -\pi \leq \lambda \leq \pi.}$

Problem 2

The orthogonality principle (theorem) of linear estimation in minimal mean square gives that $\mathrm{Cov}(X - \widehat{X}, Y_k) = 0$ for all $k = 1, 2, \ldots, n$ which implies that $\mathrm{Cov}(X - \widehat{X}, \widehat{X}) = 0$. Thus we get

$$\mathrm{Var}(X) = \mathrm{Var}(X - \widehat{X} + \widehat{X}) = \mathrm{Var}(X - \widehat{X}) + \mathrm{Var}(\widehat{X}) \geq \mathrm{Var}(\widehat{X}),$$

since all variances are non-negative. Thus the desired inequality has been shown as claimed.

Problem 3

Since $\{X_t, t \in \mathbb{Z}\}$ is stationary with

$$X_t = 0.4 \cdot X_{t-1} + Z_t,$$

where the white noise variance is $= 1$, the CF gives its ACVF as

$$\gamma(h) = \frac{0.4^{|h|}}{1 - 0.4^2}.$$

Hence the coefficient of autocorrelation is

$$\rho(h) = \gamma(h)/\gamma(0) = 0.4^{|h|}.$$

Since the time series is Gaussian, the joint distribution of X_t, X_{t-1} is a bivariate Gaussian distribution. We have by CF and, as means are zero and $\rho = \rho(1) = 0.4$, that

$$X_t \text{ conditional on } X_{t-1} = 0.3$$

$$\sim N\left(0.4 \cdot 0.3, \left(1 - 0.4^2\right) \frac{1}{1 - 0.4^2}\right) = N(0.12, 1)\,.$$

Hence

$$X_t - 0.12 \text{ conditional on } X_{t-1} = 0.3 \sim N(0, 1)\,.$$

We are led to consider

$$P\left(a - 0.12 \leq X_t - 0.12 \leq b - 0.12 \mid X_{t-1} = 0.3\right) = 0.95.$$

From the table of quantiles of the standard normal distribution $N(0,1)$ we get $\lambda_{0.025} = 1.96$ so that

$$-1.96 = a - 0.12, 1.96 = b - 0.12$$

which gives $a = -1.84, b = 2.08$.

ANSWER: $\underline{a = -1.84, b = 2.08.}$

Problem 4

(a)We have

$$\varepsilon_t = X_t^2 - \sigma_t^2 = \sigma_t^2 \cdot (Z_t^2 - 1). \tag{15}$$

Then we get

$$E[\varepsilon_t] = E[\sigma_t^2]E[Z_t^2 - 1] = E[\sigma_t^2] \cdot (1-1) = 0$$

We are assuming that $E[\sigma_t^4] < \infty$, and with the "hint" that $E(Z_t^4) = 3$ we obtain

$$\mathrm{Cov}[\varepsilon_s, \varepsilon_t] = \begin{cases} E[\sigma_t^4]E[(Z_t^2-1)^2] = E[\sigma_t^4]E[Z_t^4 - 2Z_t^2 + 1] = 2E[\sigma_t^4] & \text{if } s = t, \\ E[\sigma_s^2\sigma_t^2(Z_s^2-1)]E[Z_t^2 - 1] = E[\ldots] \cdot 0 = 0 & \text{if } s < t. \end{cases}$$

Thus $\{\varepsilon_t\}$ is a white noise.
(b)

$$\begin{aligned} \varepsilon_t &= X_t^2 - \sigma_t^2 = \\ &= X_t^2 - \alpha_0 - \beta_1\sigma_{t-1}^2 - \alpha_1 X_{t-1}^2 \\ &= X_t^2 - \alpha_0 - \beta_1\left(X_{t-1}^2 + \varepsilon_{t-1}\right) - \alpha_1 X_{t-1}^2 \\ &= X_t^2 - \alpha_0 - (\alpha_1 + \beta_1) X_{t-1}^2 + \beta_1\varepsilon_{t-1} \end{aligned}$$

$$\Leftrightarrow$$

$$X_t^2 = \alpha_0 + (\alpha_1 + \beta_1) X_{t-1}^2 + \varepsilon_t - \beta_1\varepsilon_{t-1}$$

as was to be shown. The condition for stationarity and causality is simply that

$$\alpha_1 + \beta_1 < 1.$$

Problem 5

(a) In this case (5) becomes

$$E\left[e_{n+1}^2\right] = E\left[e_n^2\right] - \frac{E\left[e_n^2\right]^2}{\sigma_V^2 + E\left[e_n^2\right]},$$

and this gives by some elementary algebra

$$E\left[e_{n+1}^2\right] = \frac{\sigma_V^2 E\left[e_n^2\right]^2}{\sigma_V^2 + E\left[e_n^2\right]}.$$

From this

$$\frac{1}{E\left[e_{n+1}^2\right]} = \frac{\sigma_V^2 + E\left[e_n^2\right]}{\sigma_V^2 E\left[e_n^2\right]^2} = \frac{1}{E\left[e_n^2\right]} + \frac{1}{\sigma_V^2}.$$

which proves (11).

As we have $E\left[e_0^2\right] = \sigma_0^2$, it follows by iteration that

$$\frac{1}{E\left[e_n^2\right]} = \frac{1}{E\left[e_{n-1}^2\right]} + \frac{1}{\sigma_V^2} = \frac{1}{E\left[e_{n-2}^2\right]} + \frac{1}{\sigma_V^2} + \frac{1}{\sigma_V^2}$$

$$= \ldots = \frac{1}{\sigma_0^2} + \frac{n}{\sigma_V^2}. \tag{16}$$

(b) To check the recursion

$$\widehat{X}_{n+1} = \widehat{X}_n + \frac{\sigma_0^2}{(n+1)\sigma_0^2 + \sigma_V^2}\left(Y_n - \widehat{X}_n\right).$$

we just need in view of (3) to compute the Kalman gain for the case at hand, or

$$\frac{\theta_n}{\nabla_n} = \frac{E\left[e_n^2\right]}{\sigma_V^2 + E\left[e_n^2\right]} = \frac{1}{\frac{\sigma_V^2}{E\left[e_n^2\right]} + 1},$$

where we now use (16) from (b) to get

$$= \frac{1}{\frac{\sigma_V^2}{\sigma_0^2} + n + 1} = \frac{\sigma_0^2}{\sigma_V^2 + \sigma_0^2\,(n+1)},$$

which gives the asserted formula for the recursive estimator of X_0.

(c) We define

$$U_n = \left(n\sigma_0^2 + \sigma_V^2\right)\widehat{X}_n.$$

Then we get from (13) that

$$\left((n+1)\sigma_0^2 + \sigma_V^2\right)\widehat{X}_{n+1} = \left((n+1)\sigma_0^2 + \sigma_V^2\right)\widehat{X}_n + \sigma_0^2\left(Y_n - \widehat{X}_n\right),$$

which thus gives

$$U_{n+1} = U_n + \sigma_0^2 Y_n.$$

Then we get recursively from the last equation that

$$U_{n+1} = U_n + \sigma_0^2 Y_n = U_{n-1} + \sigma_0^2\left(Y_{n-1} + Y_n\right)$$

$$= \ldots = U_0 + \sigma_0^2 \sum_{t=0}^{n} Y_t,$$

where $U_0 = \sigma_V^2 \widehat{X}_0$ and division by $\left((n+1)\sigma_0^2 + \sigma_V^2\right)$ gives

$$\widehat{X}_{n+1} = \frac{1}{(n+1)\sigma_0^2 + \sigma_V^2}\left(\sigma_0^2 \sum_{t=0}^{n} Y_t + \sigma_V^2 \widehat{X}_0\right).$$

here we see the recursive estimate of X_0 as a weighted mean of arithmetic mean of the measurements Y_t and of the a priori estimate $\widehat{X}_0$ (= a guess of X_0 without any measurements).

Now from (10) we get

$$\widehat{X}_{n+1} = \frac{1}{(n+1)\sigma_0^2 + \sigma_V^2} \left(\sigma_0^2 \sum_{t=0}^{n} (X_0 + V_n) + \sigma_V^2 \widehat{X}_0 \right).$$

$$= \frac{1}{\sigma_0^2 + \sigma_V^2/(n+1)} \left(\sigma_0^2 \frac{1}{n+1} \sum_{t=0}^{n} V_n + \sigma_0^2 X_0 + \frac{1}{n+1} \sigma_V^2 \widehat{X}_0 \right).$$

By the weak law of large numbers for white noise (shown in Lecture 2 (handout on mean square convergence Theorem 2.2),
`http://www.math.kth.se/matstat/gru/sf2945/aktuell08tk.html`,
there is the convergence in probability, as $n \to \infty$,

$$\frac{1}{n+1} \sum_{t=0}^{n} V_n \xrightarrow{p} 0.$$

since $E[V_n] = 0$. Since

$$\frac{1}{n+1} \sigma_V^2 \widehat{X}_0 \to 0,$$

$n \to \infty$, and since

$$\frac{1}{\sigma_0^2 + \sigma_V^2/(n+1)} \to \frac{1}{\sigma_0^2},$$

we get

$$\widehat{X}_{n+1} \xrightarrow{p} X_0.$$

Hence, we are computing with Kalman recursions an asymptotically consistent estimator $\widehat{X}_{n+1}$ of a random variable measured in noise. Because $\widehat{X}_0$ is a guess of X_0 without any measurements, it is a good property that its influence vanishes as the number of measurements increases.