SF2943 Time Series Analysis: Lecture 11

May 10, 2022

This lecture will introduce two discrete-time time series models for financial data: ARCH and GARCH.

If P_t denotes the price of a stock or other financial asset at time t, then the series of log returns, $\{Z_t\}$ with $Z_t = \log P_t - \log P_{t-1}$, is typically modeled as a stationary time series. The quantity Z_t is related to the continuously compounded rate since we can rewrite $Z_t = \log P_t - \log P_{t-1}$ in terms of

$$P_t = e^{Z_t} P_{t-1} = \lim_{n\to\infty} \left(1 + \frac{Z_t}{n}\right)^n P_{t-1}.$$

Features of $\{Z_t\}$:

1. Heavy-tailed, i.e., Z_t is more likely to take larger values compared to "light-tailed" distribution such as Gaussian.
2. Asymmetry, i.e., Z_t and $-Z_t$ have different distributions.
3. Volatility clustering: large (small) fluctuations in the data tend to followed by fluctuations of comparable magnitude.
4. The conditional variance of Z_t given $\{Z_s, s < t\}$ varies with t.

It turns out that any ARMA model does not satisfy the last feature. Actually, one can show that for any ARMA(p, q) process $\{X_t\}$, the conditional variance

$$\mathrm{Var}(X_t | X_s, s < t)$$

is invariant with t. Therefore, ARMA models are not suitable for describing financial data.

1 Autoregressive Conditional Heteroscedasticity

In this section, we will introduce a model which has non-constant condtional variance.

Definition 1.1 *A process $\{Z_t\}$ is an autoregressive conditional heteroscedasticity process of order p, denoted as ARCH(p), for $p \in \mathbb{N}$ if it is a stationary process and satisfying*

$$\begin{cases} Z_t = \sqrt{h_t} e_t, & \text{with } \{e_t\} \sim \text{IID } N(0,1) \\ h_t = \alpha_0 + \sum_{i=1}^p \alpha_i Z_{t-i}^2 \end{cases}, \tag{1.1}$$

where $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i = 1, \ldots, p$. Here h_t is the conditional variance, i.e., $h_t = Var(Z_t | Z_s, s < t)$, and it is also known as the volatility.

Remark 1.2 *By the definition, h_t is always positive, so $\sqrt{h_t}$ is always a real number.*

Remark 1.3 *The name "autoregressive" comes from rewriting* (1.1) *as the following:*

$$Z_t^2 = h_t e_t^2 = \left(\alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2\right) e_t^2.$$

Notice that Z_t^2 is defined recursively interms of Z_s^2 with $s = t-1, \ldots, t-p$.

Just like an AR(p) process X_t can be written as a function of underlying white noise, we can also use (1.1) to find the explicit solution Z_t in terms of $\{e_t\}$. For simplicity, we will focus on the case of $p = 1$, namely, consider $\{Z_t\}$ an ARCH(1) process. In this case, (1.1) becomes

$$\begin{cases} Z_t = \sqrt{h_t} e_t, \quad \text{with } \{e_t\} \sim \text{IID } N(0,1) \\ h_t = \alpha_0 + \alpha_1 Z_{t-1}^2 \end{cases} \tag{1.2}$$

which implies that for any t,

$$Z_t^2 = h_t e_t^2 = \left(\alpha_0 + \alpha_1 Z_{t-1}^2\right) e_t^2. \tag{1.3}$$

Using (1.3) iteratively we find that

$$\begin{aligned} Z_t^2 = &= \left(\alpha_0 + \alpha_1 Z_{t-1}^2\right) e_t^2 = \left[\alpha_0 + \alpha_1 \left(\alpha_0 + \alpha_1 Z_{t-2}^2\right) e_{t-1}^2\right] e_t^2 \\ &= \alpha_0 e_t^2 + \alpha_1 \alpha_0 e_t^2 e_{t-1}^2 + \alpha_1^2 Z_{t-2}^2 e_t^2 e_{t-1}^2 \\ &= \cdots \\ &= \alpha_0 \sum_{j=0}^{n} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2 + \alpha_1^{n+1} Z_{t-n-1}^2 e_t^2 e_{t-1}^2 \cdots e_{t-n}^2. \end{aligned}$$

Suppose $\alpha_1 \in (0,1)$ and $\{Z_t\}$ is stationary and causal with respect to $\{e_t\}$ (i.e., Z_t depends on e_s with $s \leq t$), then because $\{e_t\} \sim$ IID $N(0,1)$

$$E(\alpha_1^{n+1} Z_{t-n-1}^2 e_t^2 e_{t-1}^2 \cdots e_{t-n}^2) = \alpha_1^{n+1} E Z_{t-n-1}^2 E e_t^2 \cdots E e_{t-n}^2 = \alpha_1^{n+1} E Z_t^2 \to 0$$

as $n \to \infty$. This implies that

$$Z_t^2 = \alpha_0 \sum_{j=0}^{\infty} \alpha_1^j e_t^2 e_{t-1}^2 \cdots e_{t-j}^2.$$

Equivalently,

$$Z_t = e_t \sqrt{\alpha_0 \sum_{j=0}^{\infty} \alpha_1^j e_{t-1}^2 \cdots e_{t-j}^2} = e_t \sqrt{\alpha_0 \left(1 + \sum_{j=1}^{\infty} \alpha_1^j e_{t-1}^2 \cdots e_{t-j}^2\right)}. \tag{1.4}$$

We can apply this explicit expression of Z_t or Z_t^2 to compute many quantities such as the expectation and the variance of Z_t. Indeed, due to $\{e_t\} \sim$ IID $N(0,1)$

$$EZ_t = E\left[e_t\sqrt{\alpha_0\left(1+\sum_{j=1}^{\infty}\alpha_1^j e_{t-1}^2\cdots e_{t-j}^2\right)}\right] = Ee_tE\left[\sqrt{\alpha_0\left(1+\sum_{j=1}^{\infty}\alpha_1^j e_{t-1}^2\cdots e_{t-j}^2\right)}\right] = 0$$

and

$$\begin{aligned}\mathrm{Var}(Z_t) = EZ_t^2 &= E\left[\alpha_0\sum_{j=0}^{\infty}\alpha_1^j e_t^2 e_{t-1}^2\cdots e_{t-j}^2\right] = \alpha_0\sum_{j=0}^{\infty}\alpha_1^j E\left[e_t^2 e_{t-1}^2\cdots e_{t-j}^2\right] \\ &= \alpha_0\sum_{j=0}^{\infty}\alpha_1^j = \frac{\alpha_0}{1-\alpha_1}.\end{aligned}$$

Remark 1.4 *As we mentioned in our previous lectures, one way to create a strictly stationary process is by considering a function of an iid sequence. From* (1.4) *we know that* $\{Z_t\}$ *is a function of the iid sequence of* $\{e_t\}$*, thus it is strictly stationary. Since we also know that* $EZ_t^2 < \infty$*, this indicates that* $\{Z_t\}$ *is stationary.*

Moreover, the covariance function of $\{Z_t\}$ is

$$\begin{aligned}\gamma(t+h,t) = E[Z_{t+h}Z_t] &= E[E[Z_{t+h}Z_t|e_s, s<t+h]] = E[Z_tE[Z_{t+h}|e_s, s<t+h]] \\ &= E\left[Z_tE\left[e_{t+h}\sqrt{\alpha_0\left(1+\sum_{j=1}^{\infty}\alpha_1^j e_{t+h-1}^2\cdots e_{t+h-j}^2\right)}\middle| e_s, s<t+h\right]\right] \\ &= E\left[Z_t\sqrt{\alpha_0\left(1+\sum_{j=1}^{\infty}\alpha_1^j e_{t+h-1}^2\cdots e_{t+h-j}^2\right)}E\left[e_{t+h}\mid e_s, s<t+h\right]\right] = 0\end{aligned}$$

for $h>0$.

To summarize, we have the following result.

Solution of the ARCH(1) Equations:

If $\alpha_1 < 1$, the unique causal stationary solution of the ARCH(1) equations (1.1) is given by

$$Z_t = e_t\sqrt{\alpha_0\left(1+\sum_{j=1}^{\infty}\alpha_1^j e_{t-1}^2\cdots e_{t-j}^2\right)}.$$

In addition, $EZ_t = 0$,

$$\mathrm{Var}(Z_t) = \frac{\alpha_0}{1-\alpha_1} \text{ and } E[Z_{t+h}Z_t] = 0 \text{ for } h \neq 0.$$

Remark 1.5 *From the above result, we know that the acvf for an ARCH(1) process is*

$$\gamma(h) = \begin{cases} 0, & \text{for } h \neq 0 \\ \frac{\alpha_0}{1-\alpha_1}, & \text{for } h = 0 \end{cases}.$$

This indicates that $\{Z_t\}$ is white noise. However, it is not an iid sequence cause

$$E[Z_t^2|Z_{t-1}] = (\alpha_0 + \alpha_1 Z_{t-1}^2)E[e_t^2|Z_{t-1}] = (\alpha_0 + \alpha_1 Z_{t-1}^2)E[e_t^2] = \alpha_0 + \alpha_1 Z_{t-1}^2.$$

If $\{Z_t\}$ were iid, we would instead have

$$E[Z_t^2|Z_{t-1}] = EZ_t^2 = \frac{\alpha_0}{1-\alpha_1}.$$

Furthermore, this also shows that Z_t is not Gaussian since for Gaussian distributions, independence is the same as uncorrelated.

Remark 1.6 *Z_t is symmetric, i.e., Z_t and $-Z_t$ have the same distribution. Moreover, one can show that for any $\alpha_1 \in (0,1)$, $EZ^{2k} = \infty$ for some $k \in \mathbb{N}$. This says that Z_t is heavy-tailed.*

Remark 1.7 *In general, the ARCH(p) process is conditionally Gaussian, namely, for given values of $\{Z_s, s = t-1, \ldots, t-p\}$, Z_t is Gaussian. Hence, it is easy to write down the likelihood of $Z_{p+1}, \ldots, Z_n$ conditional on $\{Z_1, \ldots, Z_p\}$. As a result, we can use the maximum likelihood estimator to estimate the parameters of the model. For example, the conditional likelihood of observations $\{z_2, \ldots, z_n\}$ of the ARCH(1) process given $Z_1 = z_1$ is*

$$L = \prod_{t=2}^{n} \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 z_{t-1}^2)}} \exp\left\{-\frac{z_t^2}{2(\alpha_0 + \alpha_1 z_{t-1}^2)}\right\}.$$

1.1 Connection between ARCH(1) and AR(1)

There is a connection between ARCH(1) and AR(1). Consider $U_t = Z_t^2 - h_t$, i.e., $h_t = Z_t^2 - U_t$. One can verify that $\{U_t\}$ is white noise (if $EZ_t^4 < \infty$). Moreover, by (1.1) we have $h_t = \alpha_0 + \alpha_1 Z_{t-1}^2$, so

$$Z_t^2 - U_t = h_t = \alpha_0 + \alpha_1 Z_{t-1}^2.$$

Subtracting both sides by $\alpha_0/(1-\alpha_1)$ (which equals EZ_t^2) gives

$$Z_t^2 - \frac{\alpha_0}{1-\alpha_1} = \alpha_1 \left(Z_{t-1}^2 - \frac{\alpha_0}{1-\alpha_1} \right) + U_t,$$

which shows that $\{Z_t^2 - \alpha_0/(1-\alpha_1)\}$ is an AR(1) process.

Remark 1.8 *This connection holds between ARCH(p) and AR(p) for any $p \in \mathbb{N}$.*

2 Generalized ARCH Model

We can consider a generalization of the ARCH model, which connects to the ARMA(p, q) process.

Definition 2.1 *A process $\{Z_t\}$ is a generalized autoregressive conditional heteroscedasticity process of order (p, q), denoted as GARCH(p, q), for $p, q \in \mathbb{N}$ if it is a stationary process and satisfying*

$$\begin{cases} Z_t = \sqrt{h_t} e_t, & \text{with } \{e_t\} \sim IID\ (0, 1) \\ h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} \end{cases}, \tag{2.1}$$

where $\alpha_0 > 0$ and $\alpha_i, \beta_j \geq 0$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$.

Remark 2.2 *The differences between (1.1) and (2.1) are the assumptions on $\{e_t\}$ and the equation for the volatility h_t.*

Remark 2.3 *In the analysis of empirical financial data, it is usually found that using $\{e_t\}$ with heavier-tailed distribution such as Student's t-distribution can fit the data better. It is commonly assumed that*

$$\sqrt{\frac{\nu}{\nu - 2}} e_t \sim t_\nu, \quad \nu > 2,$$

where t_ν denotes Student's t-distribution with ν degrees of freedom.

The GARCH model with order $(1, 1)$ has been widely used in financial time series. Suppose $0 < \alpha_1 + \beta_1 < 1$. We can find the explicit expression for GARCH$(1, 1)$ in terms of $\{e_t\}$ by using a similar iterative method. The expression is

$$Z_t = e_t \sqrt{\alpha_0 \left(1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} (\alpha_1 e_{t-i}^2 + \beta_1) \right)}.$$

Moreover, one can prove that the process Z_t is stationary and causal with

$$EZ_t = 0 \quad \text{and} \quad EZ_t^2 = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}.$$

2.1 Connection between GARCH$(1, 1)$ and ARMA$(1, 1)$

There is a connection between GARCH$(1, 1)$ and ARMA$(1, 1)$. Consider once again $U_t = Z_t^2 - h_t$, i.e., $h_t = Z_t^2 - U_t$. One can verify that $\{U_t\}$ is white noise (if $EZ_t^4 < \infty$). Moreover, by (2.1) we have $h_t = \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}$, so

$$Z_t^2 - U_t = h_t = \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}.$$

Subtracting both sides by $\alpha_0/(1 - \alpha_1 - \beta_1)$ (which equals EZ_t^2) gives

$$Z_t^2 - \frac{\alpha_0}{1 - \alpha_1 - \beta_1} = (\alpha_1 + \beta_1) \left(Z_{t-1}^2 - \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \right) + U_t - \beta U_{t-1},$$

which shows that $\{Z_t^2 - \alpha_0/(1 - \alpha_1 - \beta_1)\}$ is an ARMA$(1, 1)$ process.

Remark 2.4 *If we define* $\sigma^2 \doteq EZ_t^2 = \alpha_0/(1-\alpha_1-\beta_1)$*, then*

$$\begin{aligned} h_t &= \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1} \\ &= (1-\alpha_1-\beta_1)\sigma^2 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}, \end{aligned}$$

which shows that the conditional variance at time t *(i.e.,* h_t*) is a weighted average of*

- *the (unconditional) variance* σ^2*,*
- *the square of the observation at* $t-1$*,* Z_{t-1}^2*, and*
- *the conditional variance at time* $t-1$*.*

Therefore, GARCH models can reflect the "persistence of volatility".

Remark 2.5 *In general, the ARMA models are used to describe conditional mean, and the GARCH models are for conditional variance. These two types of models take care of different perspectives of data, so they can be used in combination as an ARMA*(p,q)*-GARCH*(r,s) *model, where the GARCH part takes care of the underlying white noise appearing in an ARMA model.*