Division of Mathematical Statistics

KTH Matematik

EXAM IN SF2943 TIME SERIES ANALYSIS
MONDAY AUGUST 16 2019 KL 08:00–13:00.

Examiner: Pierre Nyquist, 08 – 790 7311.

Allowed aids: Pocket calculator, “Formulas and survey, Time series analysis” by Jan Grandell, without notes.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Problem 1

A record of average monthly temperatures at a certain location in Sweden yields, after removing trend and seasonal components, sample ACF and PACF shown in Figure 1 and sample ACVF and PACF given (lags 0 to 5) in Table 1: sample size was $n = 500$.

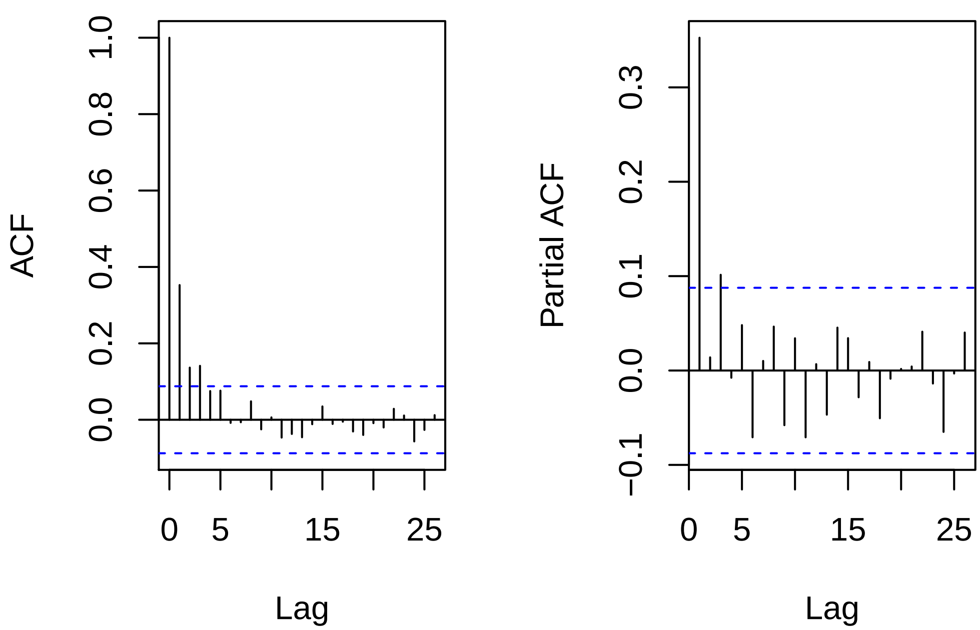

Figure 1: Sample ACF (left) and PACF (right) for Problem 1

h	0	1	2	3	4	5
$\hat{\gamma}(h)$	1.09	0.383	0.148	0.153	0.0815	0.0827
$\hat{\alpha}(h)$	-	0.353	0.0139	0.101	-0.00766	0.0481

Table 1: Sample ACVF and PACF for Problem 1

a) Decide on a suitable time series model for the (adjusted) temperature data given the above information. Choose between the model classes AR(p) and MA(q), and suggest a suitable model order p or q. Motivate your choices properly. (4 p)

b) Estimate the parameters $\phi_1, \ldots, \phi_p$ or $\theta_1, \ldots, \theta_q$ and σ^2 for the model chosen in Part (a). (5 p)

c) Is the model from Parts (a) and (b) a plausible model for the (adjusted) temperature data? Motivate your answer. (1 p).

Problem 2

Let $\{X_t\}$ be an ARMA(p, q) process

$$\phi(B)X_t - \theta(B)Z_t, \quad \{Z_t\} \sim WN(0, \sigma_z^2),$$

for $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$, $\theta(z) = 1 + \theta_1 z + \ldots \theta_q z^q$. Define the process $\{Y_t\}$ by

$$Y_t = X_t + W_t, \quad \{W_t\} \sim WN(0, \sigma_w^2),$$

where W_s and Z_t are uncorrelated for all s, t.

a) Is $\{Y_t\}$ a stationary process? If so, compute its autocovariance function in terms of σ_w^2 and the autocovariance function γ_X of $\{X_t\}$. (5 p)

b) Define the process U_t by applying the AR-polynomial of X_t to Y_t:

$$U_t = \phi(B)Y_t.$$

Determine what type of process - WN, AR, MA or ARMA - $\{U_t\}$ is. From this, what conclusion can you draw about what type of process $\{Y_t\}$ is? Motivate your answers. (5 p)

Problem 3

a) Show that there exists a unique stationary solution to the AR(1) equation (1):

$$X_t - \phi X_{t-1} = Z_t, \tag{1}$$

where $\{Z_t\}$ is a white noise sequence and $|\phi| < 1$. (7 p)

b) Show that for $|\phi| = 1$ there is no stationary solution to (1). (3 p)

Problem 4

a) Explain the limitations of ARMA processes that can be overcome by using an ARCH or GARCH model. You are not required to provide a full mathematical derivation to support your answer for this part. (2 p)

b) In applications, e.g., in financial settings, it is common to consider a combination of ARMA and ARCH/GARH processes. As an example, consider the process

$$X_t - \phi X_{t-1} = Z_t,$$

where $|\phi| < 1$ and $\{Z_t\}$ is a stationary ARCH(1) process:

$$Z_t = h_t^{1/2} e_t, \quad h_t = \alpha_0 + \alpha_1 Z_{t_1}^2,$$

where $\{e_t\}$ are IID $N(0,1)$, $\alpha_0 > 0$, $\alpha_1 > 0$ and e_t and $Z_{t-1}, Z_{t-2}, \ldots$ are independent for all $t \in \mathbb{Z}$. You may assume that α_0, α_1 are such that all necessary moments are finite.

Derive the autocovariance function of $\{X_t\}$ in terms of α_0, α_1 and ϕ. (4 p)

c) For the process $\{X_t\}$ defined in Part (b), compute the conditional second moment

$$E\left[X_t^2 | X_{t-1}, Z_{t-1}, \ldots\right].$$

How, if at all, does this relate to your answer to Part (a)? (4 p)

Problem 5

Consider the ARMA$(1,1)$ process

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim WN(0, \sigma^2),$$

where $|\phi| > 1$, $|\theta| > 1$, $\phi \neq \theta$. Construct new AR and MA polynomials $\tilde{\phi}$ and $\tilde{\theta}$ according to

$$\tilde{\phi}(z) = 1 - \frac{1}{\phi} z,$$
$$\tilde{\theta}(z) = 1 + \frac{1}{\theta} z.$$

Define a new process $\{W_t\}$ by

$$W_t = \tilde{\theta}^{-1}(B) \tilde{\phi}(B) X_t.$$

a) Is $\{X_t\}$ a causal and/or invertible process? (1 p)

b) Determine what type of process - e.g. WN, AR, MA, ARMA etc. - $\{W_t\}$ is and express the parameters defining the process (such as, for example, AR or MA parameters and/or the associated variance) explicitly in terms of ϕ, θ and σ^2. (7 p)

c) Consider the equation

$$\tilde{\phi}(B) Y_t = \tilde{\theta}(B) W_t.$$

Is $\{Y_t\}$ an ARMA-process? If so, is it causal and/or invertible? (2 p)

Hint: It may be helpful to consider the spectral density of $\{W_t\}$.

Good luck!

Solutions

Problem 1

a) A suitable model is an AR(1) model: The ACF exhibits an exponential decay, whereas the PACF is non-zero for lag 1 and zero (up to random fluctuations) for all other lags shown. Recall the property that an AR(p) model has PACF $\alpha(p) = \phi_p$ and $\alpha(h) = 0$ for $h > p$. Thus, the combination of the ACF and PACF plots and Table 1 supports the choice of an AR(1) model.

Answer: An AR(1) model is suitable based on the given information.

b) Using the property of the PACF described in (a), an estimate for $\phi = \phi_1$ is $\hat{\phi} = 0.353$. With this estimate, we can find an estimate for the variance σ^2 from the covariance function for an AR(1):

$$\gamma(0) = \frac{\sigma^2}{1 - \phi^2}.$$

Using the sample ACVF at lag 0 we obtain the estimate $\hat{\sigma}^2 = 0.954$.

The above estimates are obtained in a seemingly ad hoc way, useful mainly because we are dealing with an AR(1) process. We could also use any of the standard, more principled methods, such as Yule-Walker estimation. Because $p = 1$ the equations simplify to

$$\hat{\gamma}(0)\hat{\phi} = \hat{\gamma}(1) \Leftrightarrow \hat{\phi} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)},$$
$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}\hat{\gamma}(1).$$

This leads to $\hat{\phi} = 0.351$ and 0.956.

Answer: Yule-Walker estimates are $\hat{\phi} = 0.351$ and $\hat{\sigma}^2 = 0.956$.

c) Without any additional information, it is plausible to have an AR(1) model with positive coefficient ϕ for the weather data: The autoregressive structure reinforces current temperature - that is, if it is hot this month, this will carry over to next month as well - at a low level, whereas a seasonal component moves the temperature from "hot" to "cold" regimes over the course of a year. This interpretation relies on $\phi > 0$, as otherwise we would see the opposite effect, corresponding to an oscillating temperature between months.

Problem 2

Problem 3

a) See textbook or lecture notes.

b) Assume there exists a stationary solution $\{X_t\}$ to the AR(1) equation when $|\phi| = 1$. Repeating the recursion in the definition n times,

$$X_t = \phi X_{t-1} + Z_t = \phi^2 X_{t-2} + \phi Z_{t-1} + Z_t = \cdots = \phi^{n+1} X_{t-(n+1)} + \sum_{i=0}^{n} \phi^i Z_{t-i},$$

or equivalently

$$X_t - \phi^{n+1} X_{t-(n+1)} = \sum_{i=0}^{n} \phi^i Z_{t-i}.$$

Computing the variance of both sides, for the right-hand side we have

$$\mathrm{Var}\left(\sum_{i=0}^{n} \phi^i Z_{t-i}\right) = \sum_{i=0}^{n} \phi^{2i} \mathrm{Var}(Z_{t-i}) = (n+1)\sigma^2.$$

For the left-hand-side, using the assumption of stationarity and $|\phi| = 1$,

$$\begin{aligned}
\mathrm{Var}\left(X_t - \phi^{n+1} X_{t-(n+1)}\right) &= 2\gamma(0) - 2\gamma(n+1) \\
&\leq 2\gamma(0) + 2\gamma(n+1) \\
&\leq 4\gamma(0).
\end{aligned}$$

Combining the two, we have

$$(n+1)\sigma^2 \leq 4\gamma(0), \forall n.$$

Letting $n \to \infty$ then implies that $\gamma(0) = \infty$. This is a contradiction and thus there exists no stationary solution when $|\phi| = 1$.