Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2945 TIME SERIES ANALYSIS/TIDSSERIEANALYS
FREDAGEN DEN 5 JUNI 2008 KL 08.00–13.00.

Examinator: Timo Koski, tel. 7907134

Tillåtna hjälpmedel: Formulas and survey, Time series analysis. Handheld calculator.

Införda beteckningar skall förklaras och definieras. Resonemang och uträkningar skall vara så utförliga och väl motiverade att de är lätta att följa.

Varje korrekt lösning ger 10 poäng. Gränsen för godkänt är 25 poäng. De som erhåller 23 eller 24 poäng på tentamen kommer att erbjudas möjlighet att komplettera till betyget E. Den som är aktuell för komplettering skall till examinator anmäla önskan att få en sådan inom en vecka från publicering av tentamensresultatet.

Lösningarna får givetvis skrivas på svenska.

Resultatet skall vara klart senast ONSDAG DEN 17 juni 2009 och blir tillgängligt via ”Mina sidor”.

Lösningarna får givetvis skrivas på svenska.

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Quantiles of the normal distribution
(Normalfördelningens kvantiler)

$P(X > \lambda_\alpha) = \alpha$ where $X \sim N(0,1)$

α	λ_α	α	λ_α
0.10	1.2816	0.001	3.0902
0.05	1.6449	0.0005	3.2905
0.025	1.9600	0.0001	3.7190
0.010	2.3263	0.00005	3.8906
0.005	2.5758	0.00001	4.2649

Problem 1

Let $\{X_t, t \in \mathbb{Z}\}$ be an MA(1) process

$$X_t = Z_t + \frac{1}{2} Z_{t-1},$$

where $\{Z_t\} \sim \mathrm{WN}(0, 0.81)$. Find the linear minimal mean square predictor (or estimator) of X_t in terms of X_{t-1}, X_{t-2} and give the optimal prediction error. (10 p)

Problem 2

Let $\{X_t, t \in \mathbb{Z}\}$ be a Gaussian and stationary AR(1) time series

$$X_t = 0.9 \cdot X_{t-1} + Z_t,$$

where the white noise $\{Z_t\}$ I.I.D. $\sim N(0, 2)$.

Determine two constants a and b such that

$$P\left(a \le X_t \le b\right) = 0.95.$$

(10 p)

Problem 3

Let $\{X_t, t \in \mathbb{Z}\}$ be a Gaussian and stationary ($| \phi |< 1$) AR(1) time series with the unknown expectation μ, i.e.,

$$X_t - \mu = \phi \cdot (X_{t-1} - \mu) + Z_t,$$

where the white noise $\{Z_t\}$ I.I.D $\sim N(0, 1)$. To estimate μ by with an observed finite sample path $\{X_t, 0 \le t \le n\}$ we use the arithmetic mean

$$\bar{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t.$$

(a) What is the distribution of $\bar{X}_n$? It is allowed to give asympotic expressions for the parameters of the sought distribution. (5 p)

(b) Show that $\bar{X}_n$ is a consistent estimate of μ. *Hint:* Recall section 2.2 in *Formulas and Survey.* (5 p)

Problem 4

Let $\{X_t : t \in \mathbb{Z}\}$ be an ARCH(1) process. Show that $\{X_t\}$ is a white noise. The parameters are assumed to have been chosen so that necessary moments are finite. (10 p)

Problem 5

In an econometric model it is assumed that the price X_t of a certain product year t depends on the supply that year, while the supply Y_t year t depends on the price year $t-1$, where X_t and Y_t are counted from fixed mean levels. More formally it is assumed that $\{X_t,\ t \in \mathbb{Z}\}$ and $\{Y_t,\ t \in \mathbb{Z}\}$ are stationary time series, given by

$$X_t = -aY_t + U_t, \quad \{U_t\} \sim \mathrm{WN}(0, \sigma^2)$$

$$Y_t = bX_{t-1} + V_t, \quad \{V_t\} \sim \mathrm{WN}(0, \tau^2),$$

where a and b are positive constants, and $\{U_t\}$ and $\{V_t\}$ are independent. Show that $\{X_t\}$ and $\{Y_t\}$ are AR processes. Determine the parameters and give conditions on a and b so that $\{X_t\}$ and $\{Y_t\}$ are causal. (10 p)

Avd. Matematisk statistik

KTH Matematik

LÖSNINGAR TILL TENTAMEN
SF2945 TIME SERIES ANALYSIS/TIDSSERIEANALYS FREDAGEN DEN FEMTE JUNI 2001 KL 08.00–13.00.

Problem 1

An MA(1) $\{X_t, t \in \mathbb{Z}\}$

$$X_t = Z_t + \frac{1}{2} Z_{t-1},$$

with $\{Z_t\} \sim \mathrm{WN}(0, 0.81)$, has ACVF $\gamma(h)$, see the Collection of Formulas (CF),

$$\gamma(h) = \begin{cases} (1 + 1/4)0.81 & \text{if } h = 0, \\ 0.81/2 & \text{if } |h| = 1, \\ 0 & \text{if } |h| > 1. \end{cases}$$

The linear minimal mean square predictor (or estimator) of X_t in terms of X_{t-1}, X_{t-2} is given by

$$\widehat{X}_t = a_1 X_{t-1} + a_2 X_{t-2}$$

where

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

satisfies $\boldsymbol{\gamma} = \Gamma \boldsymbol{a}$, where

$$\Gamma = \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix} = \begin{pmatrix} \mathrm{Cov}(X_{t-1}, X_{t-1}) & \mathrm{Cov}(X_{t-1}, X_{t-2}) \\ \mathrm{Cov}(X_{t-1}, X_{t-2}) & \mathrm{Cov}(X_{t-1}, X_{t-1}) \end{pmatrix}$$

and

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma(1) \\ \gamma(2) \end{pmatrix} = \begin{pmatrix} \mathrm{Cov}(X_{t-1}, X_t) \\ \mathrm{Cov}(X_{t-2}, X_t) \end{pmatrix}.$$

Here we use the fact that the mean function is zero. From the ACVF we get

$$\Gamma = \begin{pmatrix} 1.0125 & 0.4050 \\ 0.4050 & 1.0125 \end{pmatrix}$$

and

$$\boldsymbol{\gamma} = \begin{pmatrix} 0.405 \\ 0 \end{pmatrix}.$$

Then

$$\boldsymbol{a} = \Gamma^{-1} \boldsymbol{\gamma}$$

or

$$a_1 = 0.4762, a_2 = -0.1905.$$

Then the variance of the optimal error is (Collection of Formulas and Survey)

$$\sigma^2_{opt} = \gamma(0) - \boldsymbol{\gamma}^T \Gamma^{-1} \boldsymbol{\gamma} = \gamma(0) - \boldsymbol{\gamma}^T \boldsymbol{a}$$

$$= 0.8196.$$

ANSWER **a)**: $\underline{a_1 = 0.48, a_2 = -0.19, \sigma^2_{opt} = 0.82.}$

Problem 2

Since $\{X_t, t \in \mathbb{Z}\}$ be a Gaussian and stationary AR(1) time series

$$X_t = 0.9 \cdot X_{t-1} + Z_t,$$

where the white noise $\{Z_t\}$ I.I.D. $\sim N(0,2)$, the Collection of Formulas gives its ACVF as

$$\gamma(h) = 2\frac{0.9^{|h|}}{1 - 0.9^2}.$$

Since the time series is Gaussian, the distribution of X_t is a Gaussian distribution with mean zero and variance $\gamma(0)$. Hence $\frac{X_t}{\sqrt{\gamma(0)}} \in N(0,1)$.
We are led to compute

$$P(a \le X_t \le b) = 0.95.$$

$$\Leftrightarrow$$

$$P\left(\frac{a}{\sqrt{\gamma(0)}} \le \frac{X_t}{\sqrt{\gamma(0)}} \le \frac{b}{\sqrt{\gamma(0)}}\right) = 0.95.$$

From the table of quantiles of the standard normal distribution $N(0,1)$ we get $\lambda_{0.025} = 1.96$ so that

$$-1.96 = \frac{a}{\sqrt{\gamma(0)}}, 1.96 = \frac{b}{\sqrt{\gamma(0)}}$$

which gives $a = -1.96 \cdot \sqrt{\gamma(0)}, b = 1.96 \cdot \sqrt{\gamma(0)}$. Since $\gamma(0) = 2\frac{1}{1-0.9^2} = 10.5$ we get

$$a = -6.3591, b = 6.3591.$$

ANSWER: $\underline{a = -6.34, b = 6.34.}$

Problem 3

(a) The distribution of $\bar{X}_n$ is clearly Gaussian, as a linear combination of Gaussian random variables is again Gaussian. We need to find the mean and the variance.

$$E\left[\bar{X}_n\right] = \frac{1}{n}\sum_{t=1}^{n} E\left[X_t\right] = \mu$$

We use the asymptotic approximation of the variance as found in section 10.1 of CF. Then we have

$$\bar{X}_n \sim AN\left(\mu, \frac{v}{n}\right)$$

where

$$v = \sum_{h=-\infty}^{\infty} \gamma(h).$$

Here

$$\gamma(h) = \frac{\phi^{|h|}}{1-\phi^2}.$$

Therefore,as $\gamma(h) = \gamma(-h)$, we can write

$$\sum_{h=-\infty}^{\infty} \gamma(h) = \frac{1}{1-\phi^2}\left(1 + 2\sum_{h=1}^{\infty}\phi^{|h|}\right)$$

$$= \frac{1}{1-\phi^2}\left(1 + 2\left(\frac{1}{1-\phi} - 1\right)\right)$$

$$= \frac{1}{1-\phi^2}\left(\frac{2}{1-\phi} - 1\right)$$

$$= \frac{1}{(1-\phi)^2}.$$

$$\text{ANSWER: } \underline{\bar{X}_n \sim AN\left(\mu, \tfrac{1}{n(1-\phi)^2}\right).}$$

(b) $\bar{X}_n$ is a consistent estimate of μ, if it holds for any $\varepsilon > 0$ that

$$P\left(\mid \bar{X}_n - \mu \mid > \varepsilon\right) \to 0,$$

as $n \to \infty$. By Chebysjev's inequality we get

$$P\left(\mid \bar{X}_n - \mu \mid > \varepsilon\right) \leq \frac{1}{\varepsilon^2}\mathrm{Var}\left[\bar{X}_n\right].$$

But then we see from part (a) that we can take approximately

$$\mathrm{Var}\left[\bar{X}_n\right] \approx \frac{1}{n\left(1-\phi\right)^2}$$

Hence

$$\mathrm{Var}\left[\bar{X}_n\right] \to 0.$$

as $n \to \infty$. Hence we have (more or less) shown or argued for the desired convergence.

Problem 4

We have $E[X_t] = E[\sigma_t]E[Z_t] = 0$, provided that $E[\sigma_t] < \infty$, and

$$\mathrm{Cov}[X_s, X_t] = \begin{cases} \mathrm{Var}[X_t] = E[X_t^2] = E[\sigma_t^2]E[Z_t^2] = E[\sigma_t^2], & \text{if } s = t, \\ E[X_sX_t] = E[\sigma_sZ_s\sigma_tZ_t] = E[\sigma_sZ_s\sigma_t]E[Z_t] = 0, & \text{if } s < t. \end{cases}$$

Thus $\{X_t\}$ is WN provided that all moments above are finite.

Problem 5

We have

$$X_t = -a(bX_{t-1} + V_t) + U_t = -abX_{t-1} - aV_t + U_t$$

$$Y_t = b(-aY_{t-1} + U_{t-1}) + V_t = -abY_{t-1} + bU_{t-1} + V_t.$$

Since $\{U_t\}$ and $\{V_t\}$ are independent is follows that

$$\{-aV_t + U_t\} \sim \mathrm{WN}(0, \sigma^2 + a^2\tau^2) \quad \text{and} \quad \{bU_{t-1} + V_t\} \sim \mathrm{WN}(0, b^2\sigma^2 + \tau^2).$$

Thus both $\{X_t\}$ and $\{Y_t\}$ are AR(1) processes with "AR-parameter"$= -ab$. They are both causal if $\underline{ab < 1}$.