SF2943 Time Series Analysis: Lecture 4

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In this lecture, we will revisit the AR(1) process and introduce the ARMA$(1,1)$, which combines the features of the AR(1) process and the MA(1) process. In particular, we will discuss the existence and uniqueness of these two processes.

1 Linear Process and Linear Filter

Before proceeding, we introduce a convenient notation.

Definition 1.1 *The backward shift operator B is given by*

$$BX_t = X_{t-1}.$$

Also, define B^0 as the identity operator I which satisfies $IX_t = X_t$, and for any $q \in \mathbb{N}$, denote

$$B^q X_t = B(B^{q-1}X_t).$$

With this notation, we know that $B^q X_t = X_{t-q}$ for any q. Furthermore, for any linear process $\{X_t\}$, it can be written as

$$X_t = \psi(B) Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and

$$\psi(B) \doteq \sum_{j=-\infty}^{\infty} \psi_j B^j$$

with

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

This $\psi(B)$ is called a linear filter. Using this terminology, we can restate the last proposition and corollary mentioned in the previous lecture as the following:

1. If one applies a linear filter $\psi(B)$ to $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, then one gets a linear process $\{X_t\}$ with acvf

$$\gamma_X(h) = \sum_{\ell=-\infty}^{\infty} \psi_\ell \psi_{\ell+h} \sigma^2. \tag{1.1}$$

2. If one applies a linear filter $\psi(B)$ to a stationary process $\{Y_t\}$ with acvf γ_Y, then the resulting process $\{X_t\}$ is a stationary process with acvf
$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j).$$

Another feature of linear filters is that the resulting process after applying two linear filters is independent of the order. To be more precise, let $\alpha(B) = \sum_{j=\infty}^{\infty} \alpha_j B^j$ and $\beta(B) = \sum_{j=\infty}^{\infty} \beta_j B^j$ be two linear filters, and $\{Y_t\}$ be a stationary process. Then $\alpha(B)[\beta(B)Y_t] = \beta(B)[\alpha(B)Y_t]$.

2 AR(1) process revisit

In the second lecture, we defined AR(1) process as a stationary process $\{X_t\}$ such that
$$X_t = \phi X_{t-1} + Z_t \text{ for all } t \tag{2.1}$$
with $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, $|\phi| < 1$ and we assume that Z_t and X_s are uncorrelated for any $s < t$. However, we did not discuss if such a process exists or not. In this section, we will show the existence of such process. In fact, we will find a solution as a linear process and show this solution is only solution.

2.1 Existence

To find a possible solution, we make the following observation by using (2.1) repeatedly:
$$\begin{aligned} X_t &= \phi X_{t-1} + Z_t \\ &= \phi(\phi X_{t-2} + Z_{t-1}) + Z_t = \phi^2 X_{t-2} + (\phi Z_{t-1} + Z_t) \\ &= \phi^2(\phi X_{t-3} + Z_{t-2}) + \phi Z_{t-1} + Z_t = \phi^3 X_{t-3} + (\phi^2 Z_{t-2} + \phi Z_{t-1} + Z_t) \\ &= \cdots = \phi^n X_{t-n} + (\phi^{n-1} Z_{t-(n-1)} + \cdots + \phi Z_{t-1} + Z_t). \end{aligned}$$
This suggests that
$$X_t = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \cdots = \sum_{j=0}^{\infty} \phi^j Z_{t-j} \tag{2.2}$$
could be a solution to (2.1). Since $|\phi| < 1$, we have
$$\sum_{j=0}^{\infty} |\phi|^j = \frac{1}{1-|\phi|} < \infty.$$
Thus, $\{X_t\}$ defined as (2.2) is a stationary linear process. Moreover, this $\{X_t\}$ satisfies (2.1) because for any t,
$$X_t - \phi X_{t-1} = \sum_{j=0}^{\infty} \phi^j Z_{t-j} - \sum_{j=0}^{\infty} \phi^j Z_{t-1-j} = \sum_{j=0}^{\infty} \phi^j Z_{t-j} - \sum_{k=1}^{\infty} \phi^k Z_{t-k} = Z_t.$$
Therefore, we show the existence of AR(1) process.

2.2 Uniqueness

For the uniqueness, we need to define a proper metric space.

Definition 2.1 $\sum_{k=1}^{n} X_k$ *converges in mean square if there exists a random variable S such that*

$$E\left[\left(\sum_{k=1}^{n} X_k - S\right)^2\right] \to 0 \text{ as } n \to \infty.$$

We say that this random variable S is the limit of $\{\sum_{k=1}^{n} X_k\}$ and use the notation $S \doteq \sum_{k=1}^{\infty} X_k$.

Remark 2.2 *For those who know real analysis, this limit is unique in L^2 space.*

For uniquesness of AR(1) process, suppose $\{Y_t\}$ is another AR(1) process, it means that $\{Y_t\}$ is stationary and $Y_t = \phi Y_{t-1} + Z_t$ for all t. By using this recursion repeatedly, we find

$$Y_t = \phi Y_{t-1} + Z_t = \cdots = \sum_{j=0}^{k} \phi^j Z_{t-j} + \phi^{k+1} Y_{t-(k+1)}.$$

This implies that

$$E\left[\left(Y_t - \sum_{j=0}^{k} \phi^j Z_{t-j}\right)^2\right] = E\left[\left(\phi^{k+1} Y_{t-(k+1)}\right)^2\right] = \phi^{2k+2} E\left[\left(Y_{t-(k+1)}\right)^2\right] = \phi^{2k+2} \gamma_Y(0) \to 0,$$

as $k \to \infty$. Therefore,

$$Y_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j} = X_t.$$

2.3 Autocovariance function

In the second lecture, we showed how to derive the acvf for an AR(1) process by deriving a recursive equation for $\gamma_X(h)$. In this subsection, we will use another method that simply utilizes the formula for acvf of a linear process (1.1). Recall that

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}.$$

In the notation of a linear process

$$X_t = \sum_{j=-\infty}^{\infty} \psi^j Z_{t-j}$$

with

$$\psi_j = \begin{cases} \phi^j, & \text{for } j \geq 0 \\ 0, & \text{for} j < 0. \end{cases}$$

Hence, for any $h \geq 0$,

$$\gamma_X(h) = \sum_{\ell=-\infty}^{\infty} \psi_\ell \psi_{\ell+h} \sigma^2 = \sum_{\ell=0}^{\infty} \phi^\ell \phi^{\ell+h} \sigma^2 = \sum_{\ell=0}^{\infty} \phi^{2\ell+h} \sigma^2 = \frac{\sigma^2 \phi^h}{1-\phi^2}.$$

Since acvf is even, for $h < 0$, we have

$$\gamma_X(h) = \gamma_X(-h) = \frac{\sigma^2 \phi^{-h}}{1-\phi^2}.$$

2.4 Formal derivation of the solution

In this subsection we introduce a formal way to derive the solution. We will use this technique many times in our next few lectures. Recall that B is the backward shift operator. We can rewrite the recursion (2.1) as

$$\phi(B) X_t = Z_t, \tag{2.3}$$

where $\phi(B) = I - \phi B$. Since $|\phi| < 1$, we know that for any $|x| \leq 1$

$$\chi(x) = \frac{1}{1 - \phi x} = \sum_{j=0}^{\infty} \phi^j x^j.$$

Consider a linear filter

$$\chi(B) = \sum_{j=0}^{\infty} \phi^j B^j.$$

We know $\chi(B)\phi(B) = I$ and apply the linear filter $\chi(B)$ to both sides of (2.3) to find the solution

$$X_t = \chi(B) Z_t = \left(\sum_{j=0}^{\infty} \phi^j B^j \right) Z_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}.$$

3 Causality and Invertibility

Let $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Recall that an AR(1) process $\{X_t\}$ can be represented as

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}.$$

And a MA(1) process $\{X_t\}$ can be represented as

$$X_t = Z_t + \theta Z_{t-1}.$$

From here we see that both processes at time t can be represented as a function of the white noise at "the past and present", i.e., Z_s with $s \leq t$. We call processes with this property causal.

Definition 3.1 *We say that a time series $\{X_t\}$ is a causal (or future-independent) function of $\{Z_t\}$ if X_t has a representation in terms of $\{Z_s, s \leq t\}$. If $\{X_t\}$ is not causal, we call it noncausal.*

Causality means that X_t can be expressed in terms of $\{Z_s, s \leq t\}$. A dual concept, called invertibility, refers to that Z_t can be expressed in terms of $\{X_s, s \leq t\}$.

Definition 3.2 *We say that a time series $\{X_t\}$ is a invertible function of $\{Z_t\}$ if Z_t has a representation in terms of $\{X_s, s \leq t\}$. If $\{X_t\}$ is not invertible, we call it noninvertible.*

4 ARMA$(1, 1)$ Process

We can combine the features of both an AR(1) process and a MA(1) process to create ARMA$(1, 1)$ process.

Definition 4.1 *A time series $\{X_t\}$ is an autoregressive moving average process of order $(1, 1)$ (or an ARMA$(1, 1)$ process) if $\{X_t\}$ is stationary and satisfies*

$$\underbrace{X_t - \phi X_{t-1}}_{\textit{autoregressive part}} = \underbrace{Z_t + \theta Z_{t-1}}_{\textit{moving average part}} \quad \textit{for all } t, \tag{4.1}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\phi + \theta \neq 0$.

Remark 4.2 *If we use the backward shift operator notation B, then (4.1) is equivalent to*

$$\phi(B)X_t = \theta(B)Z_t \qquad \textit{for all } t \tag{4.2}$$

with $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$. Notice that if $\phi + \theta = 0$, then $\phi(z) = 1 - \phi z = 1 + \theta z = \theta(z)$, which indicates that $X_t = Z_t$. Therefore, we require $\phi + \theta \neq 0$ to ensure that $\{X_t\}$ is not simply white noise.

4.1 Existence, uniqueness, and causality

A natural question to ask is for what values of ϕ and θ, there exists such a stationary process satisfying (4.1). We will answer this question by analyzing different ϕ.

Case 1: $|\phi| < 1$
In this case, we can consider the inverse function of $\phi(z)$,

$$\chi(z) = \frac{1}{\phi(z)} = \frac{1}{1 - \phi z} = \sum_{j=0}^{\infty} \phi^j z^j.$$

This function is well-defined for all $|z| \leq 1$ and $\chi(z)\phi(z) = 1$. As a result, we can apply the linear filter $\chi(B)$ to both sides of (4.2) can get

$$X_t = \chi(B)\phi(B)X_t = \chi(B)\theta(B)Z_t.$$

If we define $\psi(B) = \chi(B)\theta(B)$, then

$$\begin{aligned}\psi(z) = \chi(z)\theta(z) &= \left(\sum_{j=0}^{\infty} \phi^j z^j\right)(1+\theta z) = \sum_{j=0}^{\infty} \phi^j z^j + \theta \sum_{j=0}^{\infty} \phi^j z^{j+1} \\ &= \sum_{j=0}^{\infty} \phi^j z^j + \theta \sum_{\ell=1}^{\infty} \phi^{\ell-1} z^{\ell} = 1 + \sum_{j=1}^{\infty} \phi^j z^j + \theta \sum_{j=1}^{\infty} \phi^{j-1} z^j \\ &= 1 + \sum_{j=1}^{\infty} (\theta+\phi)\phi^{j-1} z^j.\end{aligned}$$

Notice that $\psi(B)$ is a well-defined linear filter since

$$\sum_{j=1}^{\infty} (\theta+\phi)\phi^{j-1} = (\theta+\phi)\sum_{j=1}^{\infty} \phi^{j-1} = (\theta+\phi)\frac{1}{1-\phi} < \infty.$$

Thus,

$$X_t = \chi(B)\theta(B)Z_t = \psi(B)Z_t = \left(I + \sum_{j=1}^{\infty}(\theta+\phi)\phi^{j-1}B^j\right) Z_t = Z_t + \sum_{j=1}^{\infty}(\theta+\phi)\phi^{j-1}Z_{t-j}.$$

This indicates that $\{X_t\}$ is a linear process; thus, it is stationary. Moreover, $\{X_t\}$ is causal. The uniqueness follows from the same argument as we did for the AR(1) process.

Remark 4.3 *$\{X_t\}$ is also a MA(∞) process with $\theta_j = (\theta+\phi)\phi^{j-1}$ for all $j \in \mathbb{N}$.*

Case 2: $|\phi| > 1$
In this case, we can consider the inverse function of $\phi(z)$,

$$\begin{aligned}\chi(z) = \frac{1}{\phi(z)} = \frac{1}{1-\phi z} = \frac{-(\phi z)^{-1}}{-(\phi z)^{-1}+1} &= -(\phi z)^{-1}\frac{1}{1-(\phi z)^{-1}} \\ = -(\phi z)^{-1}\sum_{j=0}^{\infty}\phi^{-j}z^{-j} = -\sum_{j=1}^{\infty}\phi^{-j}z^{-j}.\end{aligned}$$

This function is well-defined for all $|z| \geq 1$ and $\chi(z)\phi(z) = 1$. As a result, we can apply the linear filter $\chi(B)$ to both sides of (4.2) can get

$$X_t = \chi(B)\phi(B)X_t = \chi(B)\theta(B)Z_t.$$

If we define $\psi(B) = \chi(B)\theta(B)$, then

$$\begin{aligned}\psi(z) = \chi(z)\theta(z) &= \left(-\sum_{j=1}^{\infty} \phi^{-j} z^{-j}\right)(1+\theta z) = -\sum_{j=1}^{\infty} \phi^{-j} z^{-j} - \theta \sum_{j=1}^{\infty} \phi^{-j} z^{-j+1} \\ &= -\sum_{j=1}^{\infty} \phi^{-j} z^{-j} - \theta \sum_{\ell=0}^{\infty} \phi^{-\ell-1} z^{-\ell} = -\sum_{j=1}^{\infty} \phi^{-j} z^{-j} - \theta\phi^{-1} - \theta \sum_{j=1}^{\infty} \phi^{-j-1} z^{-j} \\ &= -\theta\phi^{-1} - \sum_{j=1}^{\infty} (\theta+\phi)\phi^{-j-1} z^{-j}.\end{aligned}$$

Notice that $\psi(B)$ is a well-defined linear filter since

$$\sum_{j=1}^{\infty} (\theta+\phi)\phi^{-j-1} = (\theta+\phi)\sum_{j=1}^{\infty} \phi^{-j-1} = (\theta+\phi)\frac{\phi^{-2}}{1-\phi^{-1}} < \infty.$$

Thus,

$$\begin{aligned}X_t = \chi(B)\theta(B)Z_t = \psi(B)Z_t &= \left(-\theta\phi^{-1} I - \sum_{j=1}^{\infty} (\theta+\phi)\phi^{-j-1} B^{-j}\right) Z_t \\ &= -\theta\phi^{-1} Z_t - \sum_{j=1}^{\infty} (\theta+\phi)\phi^{-j-1} Z_{t+j}.\end{aligned}$$

This indicates that $\{X_t\}$ is a linear process, thus, it is stationary. Moreover, $\{X_t\}$ is noncausal since X_t is not a function of $\{Z_s, s \le t\}$. The uniqueness follows by the same argument as we did for AR(1) process.
Case 3: $|\phi| = 1$
There is no stationary solution of (4.1). Consequently, there is no well-defined ARMA$(1,1)$ process in this case.

Remark 4.4 *To sum up, the existence, uniqueness and causality of an ARMA$(1,1)$ process depends only on the value ϕ (independent of the value of θ as long as $\phi + \theta \neq 1$). When $|\phi| \neq 1$, we have the existence and uniqueness. Moreover, when $|\phi| < 1$, the ARMA$(1,1)$ process is causal. When $|\phi| > 1$, the ARMA$(1,1)$ process is noncausal.*

4.2 Invertibility

It turns out that the invertibility depends only on the value of θ.
Case 1: $|\theta| < 1$
In this case, we can consider the inverse function of $\theta(z)$,

$$\xi(z) = \frac{1}{\theta(z)} = \frac{1}{1+\theta z} = \sum_{j=0}^{\infty} (-\theta)^j z^j.$$

This function is well-defined for all $|z| \leq 1$ and $\xi(z)\theta(z) = 1$. As a result, we can apply the linear filter $\xi(B)$ to both sides of (4.2) can get

$$Z_t = \xi(B)\theta(B)Z_t = \xi(B)\phi(B)X_t.$$

If we define $\pi(B) = \xi(B)\phi(B)$, then

$$\begin{aligned}
\pi(z) = \xi(z)\phi(z) &= \left(\sum_{j=0}^{\infty}(-\theta)^j z^j\right)(1-\phi z) = \sum_{j=0}^{\infty}(-\theta)^j z^j - \phi\sum_{j=0}^{\infty}(-\theta)^j z^{j+1} \\
&= \sum_{j=0}^{\infty}(-\theta)^j z^j - \phi\sum_{\ell=1}^{\infty}(-\theta)^{\ell-1} z^{\ell} = 1 + \sum_{j=1}^{\infty}(-\theta)^j z^j - \phi\sum_{j=1}^{\infty}(-\theta)^{j-1} z^j \\
&= 1 - \sum_{j=1}^{\infty}(\theta+\phi)(-\theta)^{j-1} z^j.
\end{aligned}$$

Notice that $\pi(B)$ is a well-defined linear filter since

$$\sum_{j=1}^{\infty}(\theta+\phi)(-\theta)^{j-1} = (\theta+\phi)\sum_{j=1}^{\infty}(-\theta)^{j-1} = (\theta+\phi)\frac{1}{1+\theta} < \infty.$$

Thus,

$$Z_t = \xi(B)\phi(B)X_t = \pi(B)X_t = \left(I - \sum_{j=1}^{\infty}(\theta+\phi)(-\theta)^{j-1}B^j\right)X_t = X_t - \sum_{j=1}^{\infty}(\theta+\phi)(-\theta)^{j-1}X_{t-j}.$$

This indicates that $\{X_t\}$ is invertible.
Case 2: $|\theta| > 1$
In this case, using a similar argument we can find that

$$Z_t = -\phi\theta^{-1}X_t + \sum_{j=1}^{\infty}(\theta+\phi)(-\theta)^{-j-1}X_{t+j}.$$

Hence, $\{X_t\}$ is noninvertible.

Remark 4.5 *To sum up, the invertibility of an ARMA$(1,1)$ process depends only on the value θ (independent of the value of ϕ as long as $\phi + \theta \neq 1$). The ARMA$(1,1)$ process is invertible if and only if $|\theta| < 1$.*