Division of Mathematical Statistics

KTH Matematik

EXAM IN SF2943 TIME SERIES ANALYSIS
MONDAY JUNE 3 2019 KL 14:00–19:00.

Examiner: Pierre Nyquist, 08 – 790 7311.

Allowed aids: Pocket calculator, "Formulas and survey, Time series analysis" by Jan Grandell, without notes.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Problem 1

Consider the following time series model:

$$X_t = X_{t-1} - 0.25X_{t-2} + Z_t - 0.25Z_{t-1}, \quad \{Z_t\} \sim WN(0, 1).$$

a) Determine the orders p and q for which $\{X_t\}$ is an ARMA(p, q) process. (3 p)

b) Check whether $\{X_t\}$ is a causal and/or invertible process. If it is causal, find the numerical values of the first three coefficients ψ_0, ψ_1, ψ_2 in the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

If $\{X_t\}$ is invertible, find the numerical values of the first three coefficients π_0, π_1, π_2 in the representation

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}.$$

(7 p)

Problem 2

The six figures A-F (in Figure 1) show a realisation, the sample autocorrelation function (based on a sample size of 200), and the spectral density for two AR(1) processes.

a) Which figures are realisations, sample ACFs and spectral densities, respectively? Motivate your answer. (2 p)

b) Group the figures into two triplets, each containing a realisation, a sample ACF and a spectral density, such that the figures in each triplet correspond to the same time series. What can you say about the AR coefficient ϕ for each triplet? Motivate your answers. (8 p)

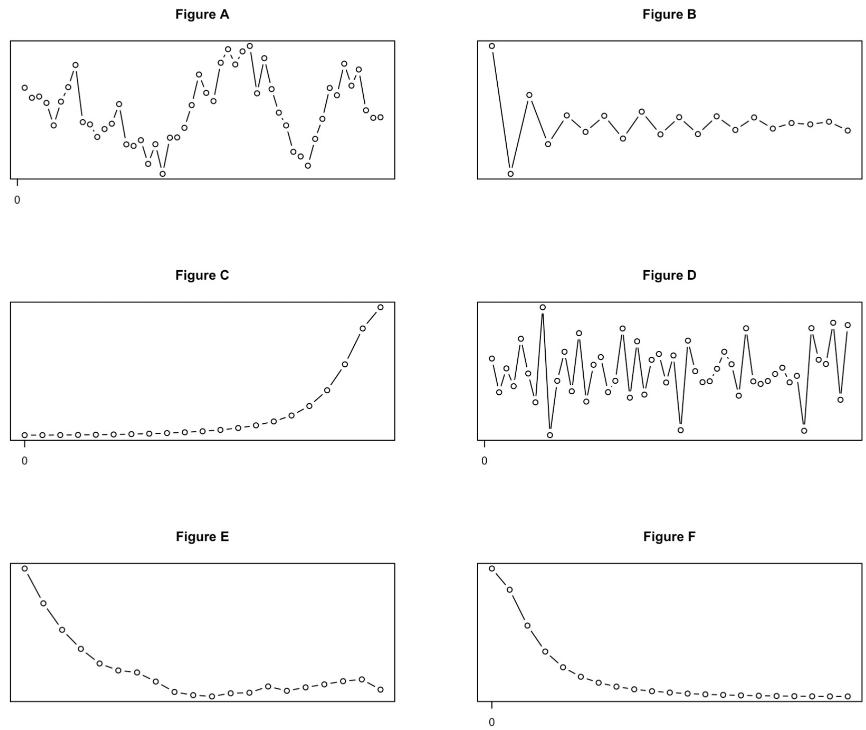

Figure 1: Figures for Problem 2

Problem 3

Consider a stationary process $\{X_t\}$ with mean zero, ACVF γ_X and spectral density f_X.

a) Let $\{\psi_j\}_{j=-\infty}^{\infty}$ be an absolutely summable time-invariant linear filter and define the process $\{Y_t\}$ as the filter ψ applied to $\{X_t\}$: for each $t = 0, \pm 1, \dots$,

$$Y_t = \psi(B)X_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j}.$$

State and prove the form of the spectral density of the process $\{Y_t\}$. (6 p)

b) Consider the process

$$Y_t = (1 - B)X_t.$$

Derive the spectral density for $\{Y_t\}$ and explain the function this filter has in terms of frequencies $\lambda \in [0, \pi)$. For full credit your explanation must be based on mathematical results and quantitative reasoning. (4 p)

Problem 4

a) Your colleague is analysing the time series depicted in Figure 2. To draw conclusions on whether the data is stationary or not the Augmented Dickey-Fuller (ADF) test is used.

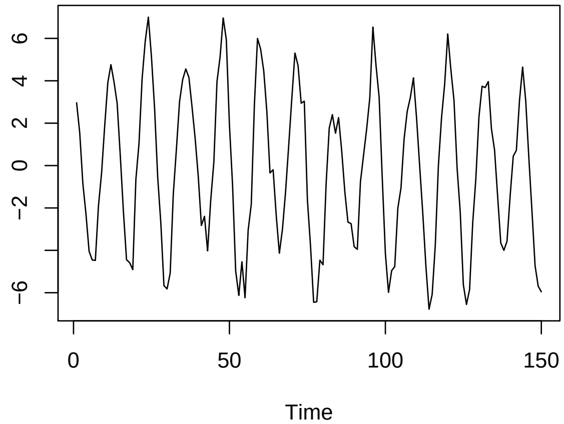

Figure 2: Time series realisation for Problem 4a

The result of the test is a p-value < 0.01 and your colleague draws the conclusion that the null hypothesis of non-stationarity can be rejected, accepts the alternative hypothesis of stationarity and proceeds to fit an ARMA(p, q) model.

Is the interpretation of the test results, and the resulting conclusion, correct? If not, explain the correct interpretation and/or conclusion; your answer should be based on statistical reasoning and the outcome and properties of the ADF test. (5 p)

b) It has been proposed, based on heuristic arguments, that a certain process should be modeled with either an AR(1) or an AR(2) process. Based on a realisation of size $n = 150$, you obtain the Yule-Walker estimates

$$\hat{\phi}_1 = 0.75, \quad \hat{\phi}_2 = -0.14,$$

the estimated variance $\hat{\sigma}^2 = 1.27$ and sample autocovariance function $\hat{\gamma}$ given in Table 1. Given these estimates, decide based on *quantitative statistical reasoning* whether to use an

h	0	1	2	3	4
$\hat{\gamma}(h)$	2.23	1.47	0.79	0.44	0.23

Table 1: Sample autocovariance for Problem 4b

AR(1) or an AR(2) model. You may decide the confidence level (report this) to be used for the decision. (5 p)

Problem 5

Suppose that $\{Z_t\}$ is a stationary GARCH(1, 1) process:

$$\begin{cases} Z_t = \sqrt{h_t} e_t, \quad e_t \sim \text{IID}(0, 1), \\ h_t = \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}, \end{cases}$$

for $\alpha_0 > 0$, $\alpha_1, \beta_1 \geq 0$.

a) Show that the squared process $\{Z_t^2\}$ can be expressed as an ARMA$(1, 1)$ process and use this representation to specify for what parameter values the process is causal. You may assume that $E[Z_t^4] < \infty$. (4 p)

b) Figure 3 shows two realisations of the h_t components in two GARCH$(1, 1)$ models, both using the same Gaussian noise sequence $\{e_t\}$. Explain the differences between the two models and how they give rise to the different behaviours seen in the figure. (3 p)

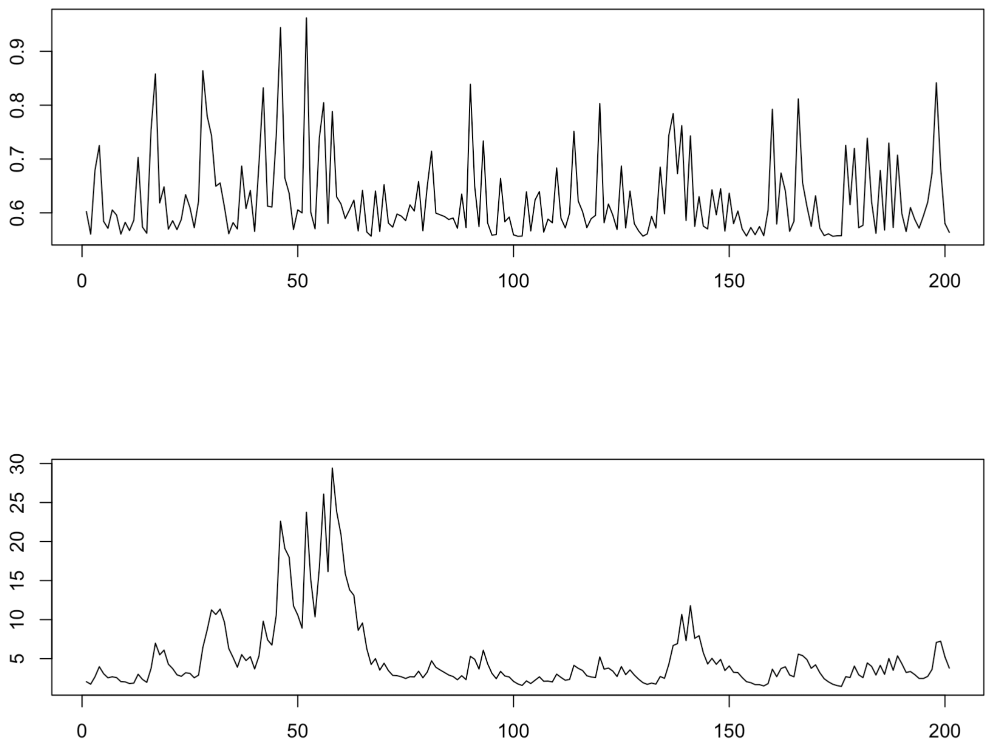

Figure 3: Realisations of h_t in two GARCH(1,1) models for Problem 5b

c) If we set $\beta_1 = 0$ we retrieve the ARCH(1) process. Show that the corresponding $\{Z_t\}$ is a white noise process. (2 p)

d) Explain how, despite being a white noise process (Part c)), the ARCH(1) process can be used to model time series with non-trivial dependence structure (i.e., not i.i.d.). (1 p)

Good luck!

Solutions

Problem 1

a) Start by finding the roots of the AR and MA polynomials: For the ϕ-polynomial we have

$$\phi(z) = 0 \Leftrightarrow 1 - z + 0.25z^2 = 0$$

which corresponds to $z = 2$ (multiplicity 2). Similarly, for the MA polynomial,

$$\theta(z) = 0 \Leftrightarrow 1 - 0.25z = 0,$$

and there is a root at $z = 4$. There are no common factors in the two polynomials and thus this is an ARMA$(2, 1)$ process.

b) By part (a) all zeros lie outside the unit circle (unit interval for the MA part) and the process is therefore both causal and invertible. We start by computing the coefficients in the representation of X in terms of Z: The ψ-polynomial satisfies

$$\left(\psi_0 + \psi_1 z + \psi_2 z^2 + \dots\right)(1 - z + 0.25z^2) = 1 - 0.25z.$$

Because there is a root of multiplicity 2 at $z = 2$, the general form of the ψ_js is

$$\psi_j = (a + bj)2^{-j}, \quad a, b \in \mathbb{R},$$

and we have the initial condition

$$\psi_0 = 1, \quad \psi_1 - \psi_0 = -0.25.$$

Combining these leads to $a = 1$, $a = 0.5$ and

$$\psi_j = (1 + 0.5j)2^{-j}, \quad j = 0, 1, \dots.$$

The first three coefficients are thus

$$\psi_0 = 1, \quad \psi_1 = 0.75, \quad \psi_2 = 0.5.$$

For the MA part, the π-polynomial satisfies

$$\left(\pi_0 + \pi_1 z + \pi_2 z^2 + \dots\right)(1 - 0.25z) = 1 - z + 0.25z^2.$$

The general form of the solution to the homogeneous difference equation is

$$\pi_j = c4^{-j}.$$

and we have the initial conditions

$$\pi_0 = 1, \quad -0.25\pi_0 + \pi_1 = -1, \quad -0.25\pi_1 + \pi_2 = 0.25.$$

Combining the general form with the initial conditions we obtain

$$\pi_0 = 1, \quad \pi_1 = -0.75,$$

and

$$\pi_j = 4^{-j}, , j \geq 2.$$

The three first coefficients are therefore

$$\pi_0 = 1, \quad \pi_1 = -0.75, \quad \pi_2 = 0.0625.$$

Answer: The process is both causal and invertible and the coefficients are $(\psi_0, \psi_1, \psi_2) = (1, 0.75, 0.5)$ and $(\pi_0, \pi_1, \pi_2) = (1, -0.75, 0.0625)$.

Remark: In part (b) you need not find the general form for the coefficients in either polynomial and to receive full credit it is enough to use the equations for the first three coefficients and solve them.

Problem 2

a) The sample ACFs are in Figures B and E - these are the only figures in which the maximum is at 0 on the x-axis and both exhibit a decay along the x-axis which is reminiscent of the time series we are used to. The realisations are in Figures A and D - they exhibit the zig-zag pattern of a function with a stochastic component as opposed to any smooth function, and take on both positive and negative values (hence they can not show spectral densities). Lastly, Figures C and F show spectral densities - the minimum in both figures is ≤ 0 and it is plausible to think that the these are samples of a function with continuous domain (corresponding to frequency or angular frequency).

Answer: B and E are sample ACFs, A and D are realisations and C and F are spectral densities.

b) Having $\phi < 0$ implies that the corresponding AR-process will oscillate rapidly as opposed to reinforcing movements in either direction - "up" or "down" - corresponding to $\phi > 0$). We see the first type of behaviour in Figure D and the second in Figure A. Next, the oscillations in Figure B corresponds precisely to those of an AR(1) process with negative coefficient: positive correlation for even lags and negative for odd lags (recall that the ACF is $\phi^{|h|}$). We conclude that Figure B corresponds to the realisation in Figure D and Figure E to the realisation in Figure A.
For the spectral densities, Figure F shows the spectral density of a process where low frequencies dominate. This is consistent with a slowly varying realisation such as in Figure A. Conversely, Figure C shows the spectral density of a process with high frequencies dominating, consistent with rapid oscillations as in Figure D.

Answer: The triplets are (AEF) and (BCD), with the first triplet corresponding to an AR(1) with coefficient $\phi > 0$ and the second triplet corresponding to $\phi < 0$.

Problem 3

a) See course book or lecture notes.

b) We can view Y_t as the process obtained by applying the linear filter $\psi_0 = 1$, $\psi_1 = -1$ and $\psi_j = 0$, $j \geq 2$, to X_t. The corresponding ACVF is given by

$$\begin{aligned}\gamma_Y(h) &= \mathrm{Cov}(Y_t, Y_{t+h}) \\ &= \mathrm{Cov}(X_t - X_{t-1}, X_{t+h}, X_{t+h-1}) \\ &= 2\gamma_X(h) - \gamma_X(h-1) - \gamma_X(h+1).\end{aligned}$$

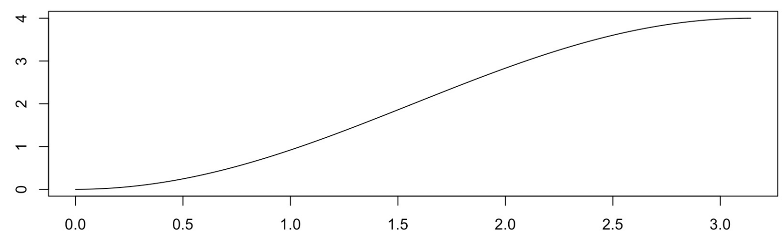

Figure 4: The function $\lambda \mapsto 2(1-\cos(\lambda))$ appearing in Problem 3b.

Next, we compute the spectral density for Y: For $\lambda \in [0, \pi)$,

$$\begin{aligned} f_Y(\lambda) &= \sum_{|h|<\infty} e^{-ih\lambda}\gamma_Y(h) \\ &= \sum_{|h|<\infty} e^{-ih\lambda}(2\gamma_X(h) - \gamma_X(h-1) - \gamma_X(h+1)) \\ &= 2\sum_{|h|<\infty} e^{-ih\lambda}\gamma_X(h) - e^{-i\lambda}\sum_{|h|<\infty} e^{-ih\lambda}\gamma_X(h) - e^{i\lambda}\sum_{|h|<\infty} e^{-ih\lambda}\gamma_X(h) \\ &= 2(1-\cos(\lambda))f_X(\lambda). \end{aligned}$$

The power transfer function of this filter is therefore the function $\lambda \mapsto 2(1-\cos(\lambda)$, that is This function retains frequencies close to π whereas lower frequencies are essentially removed from the spectrum of the process (since $1-\cos(\lambda)$ goes to 0 as λ goes to 0 from above); this is illustrated in Figure 4. The conclusion is that this is a *high-pass filter*, i.e., a filter that lets high frequencies pass through whereas lower frequencies are removed.

Answer: The corresponding filter is a high-pass filter, which preserves high frequencies and diminishes low frequencies.

Problem 4

a) The conclusion that the process is stationary is incorrect - from the realisation it is clear that there is some type of periodic component in the process, a violation of the claim of stationarity. The error lies in the interpretation of the ADF test, specifically the null hypothesis. The null hypothesis is *not* that the process is non-stationary, but rather that there is a unit root in the AR polynomial. That is, if ϕ is the AR polynomial of interest, the null hypothesis is

$$H_0: \quad \phi(1) = 0.$$

Given the outcome of the test - a p-value < 0.01 - we can conclude that there is no unit root and the process is *trend stationary.* However we can not draw conclusions of other types of stationarity based on the ADF test.

Answer: The conclusion is incorrect - the process is not stationary. The ADF test tests the null hypothesis of a unit root in the AR polynomial and this we can reject (e.g., at the level 0.05, or 0.01).

b) We are interested in discerning whether or not to include the second AR component. In the current setting this can be achieved by investigating if there is statistical evidence for the

hypothesis that $\phi_2 = 0$, or if the sample instead supports including a second component. For this we can use the asymptotic normality of Yule-Walker estimates.
The Yule-Walker estimates $(\hat{\phi}_1, \hat{\phi}_2)$ are asympottically normal (in sample size n) with mean vector (ϕ_1, ϕ_2) and covariance matrix

$$\frac{\sigma^2 \Gamma_2^{-1}}{n},$$

where $\Gamma_p = [\gamma(i-j)]_{i,j=0}^{p-1}$ (see Section 11.1 in "Formulas and survey"). To estimate the variance we replace σ^2 and Γ_2 with the sample versions $\hat{\sigma}^2$ and $\hat{\Gamma}_2$. To test whether the second component can be set to zero or not we use the marginal (asymptotic) distribution for that estimate.
Inserting the values for $\hat{\gamma}(h)$ for $h = 0, 1$, we obtain

$$\frac{\sigma^2 \Gamma_2^{-1}}{n} = \begin{pmatrix} 0.0067 & -0.0044 \\ -0.0044 & 0.0067 \end{pmatrix}.$$

Under the null hypothesis that $\phi_2 = 0$, $\hat{\phi}_2 \sim \text{AN}(0, 0.0067)$ and we obtain, for the two-sided test, a p-value

$$\mathbb{P}(|\hat{\phi}_2| > 0.14) = 1 - \mathbb{P}(|\hat{\phi}_2| \leq 0.14) = 0.087.$$

Answer: The p-value for an approximate two-sided test of the null hypothesis that $\phi_2 = 0$ is 0.087 and at the 5% level we do not reject the null hypothesis; at this level we would use an AR(1) model.

Problem 5

a) We define the auxiliary process v_t by

$$v_t = Z_t^2 - h_t.$$

This is a zero mean process:

$$\mathbb{E}[v_t] = \mathbb{E}[Z_t^2 - h_t] = \mathbb{E}\left[\mathbb{E}[Z_t^2 - h_t | Z_s, s < t]\right] = 0.$$

Moreover, under the assumption that $\mathbb{E}[Z_t^4] < \infty$, for any $h \in \mathbb{N}_+$

$$\begin{aligned} \mathbb{E}[v_t v_{t+h}] &= \mathbb{E}[h_t(e_t^2 - 1) h_{t+h}(e_{t+h}^2 - 1)] \\ &= \mathbb{E}[h_t(e_t^2 - 1) h_{t+h}]\mathbb{E}[(e_{t+h}^2 - 1)] = 0. \end{aligned}$$

Under the assumption that $\mathbb{E}[Z_t^4] < \infty$ we then have that $\{v_t\}$ is white noise.
Furthermore, combining the definitions of v_t and h_t yields

$$Z_t^2 - v_t = \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 (Z_{t-1}^2 - v_{t-1})$$

which leads to

$$Z_t^2 - (\alpha_1 + \beta_1) Z_{t-1}^2 = \alpha_0 + v_t - \beta_1 v_{t-1}.$$

That is $\{Z_t^2\}$ is an ARMA(1,1) process with white noise $\{v_t\}$. The process is causal iff $|\alpha_1 + \beta_1| < 1$ (then the zero of the AR polynomial is outside the unit interval).

Answer: The process $\{Z_t^2\}$ is an ARMA(1,1) process with coefficients $\alpha_1 + \beta_1$ and $-\beta_1$ for the AR and MA part, respectively. It is causal iff $\alpha_1 + \beta_1 < 1$.

b) Some notable differences between the figures are the scale of the volatilities (peaks at approximately 1 compared to approximately 30) and the clustering (seen in the bottom but not the top figure). The behaviour in the bottom figure is indicative of a process where past history is allowed to influence future observations to a large extent - $\alpha_1 + \beta_1$ is closer to 1 than to 0, causing the process ot deviate form the unconditional variance $\alpha_0/(1 - \alpha_1 + \beta_1)$.

c) If $\beta_1 = 0$ the equation is reduced to

$$\begin{cases} Z_t = \sqrt{h_t} e_t, \quad e_t \sim \mathrm{IID}(0,1), \\ h_t = \alpha_0 + \alpha_1 Z_{t-1}^2, \end{cases}$$

for $\alpha_0 > 0$, $\alpha_1 \geq 0$. To show that this is a white noise process we can use the following representation:

$$Z_t = e_t \sqrt{\alpha_0 \left(1 + \sum_{j=1}^{\infty} \alpha_1^j \Pi_{k=1}^j e_{t-k}^2 \right)}.$$

It follows immediately from this that $\mathbb{E}[Z_t] = 0$. Moreover, for $h \geq 1$,

$$\mathbb{E}\left[Z_t Z_{t+h}\right] = \mathbb{E}\left[\mathbb{E}\left[Z_t Z_{t+h} | e_s, s < t + h\right]\right] = 0.$$

Thus, $\{Z_t\}$ is a white noise process.

d) The definition fo white noise is in terms of second order properties, specifically the ACVF. However (in)dependence is a much broader concept - although the process is uncorrelated, we have

$$\mathbb{E}[Z_t^2 | Z_{t-1}] = \alpha_0 + \alpha_1 Z_{t-1}^2,$$

and the components of $\{Z_t\}$ are clearly *not* independent. Thus, the process has an interesting, non-trivial dependence structure despite being a white noise process.