SF2943 Time Series Analysis: Lecture 10

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In this lecture we will continue the introduction of spectral analysis.

1 Spectral Density for a Linear Filter

Proposition 1.1 *Let $\{X_t\}$ be a stationary time series with mean zero and spectral density $f_X(\lambda)$. Suppose $\{\psi_j\}$ is absolutely summable, and thus forms a linear filter. Then the time series*

$$Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j}$$

is stationary with mean zero and spectral density

$$f_Y(\lambda) = |\psi(e^{-i\lambda})|^2 f_X(\lambda)$$

where

$$\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j.$$

(The function $\psi(e^{-i\lambda})$ is called the transfer function of the filter, and $|\psi(e^{-i\lambda})|^2$ is called the power transfer function of the filter.)

Proof. We have shown before that if we apply a linear filter to a stationary time series, then we end up with another stationary time series. Thus, it remains to show that the spectral density of $\{Y_t\}$ is $f_Y(\lambda) = |\psi(e^{-i\lambda})|^2 f_X(\lambda)$, which is true if we can show that for any λ

$$\gamma_Y(\lambda) = \int_{-\pi}^{\pi} e^{ih\lambda} f_Y(\lambda) d\lambda.$$

To prove this, notice that $\{X_t\}$ has the spectral density f_X, namely, for any λ

$$\gamma_X(\lambda) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda) d\lambda. \tag{1.1}$$

On the other hand, using one of our previous results on the acvf of a linear filtered process, we know that

$$\gamma_Y(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h+k-j) \tag{1.2}$$

Plugging (1.1) into (1.2) gives

$$\begin{aligned}
\gamma_Y(h) &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\psi_j\psi_k\left(\int_{-\pi}^{\pi}e^{i(h+k-j)\lambda}f_X(\lambda)d\lambda\right)\\
&= \int_{-\pi}^{\pi}e^{ih\lambda}\left(\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\psi_j\psi_k e^{i(k-j)\lambda}\right)f_X(\lambda)d\lambda\\
&= \int_{-\pi}^{\pi}e^{ih\lambda}\left(\sum_{j=-\infty}^{\infty}\psi_j e^{-ij\lambda}\right)\overline{\left(\sum_{k=-\infty}^{\infty}\psi_k e^{-ik\lambda}\right)}f_X(\lambda)d\lambda\\
&= \int_{-\pi}^{\pi}e^{ih\lambda}\left|\sum_{j=-\infty}^{\infty}\psi_j e^{-ij\lambda}\right|^2 f_X(\lambda)d\lambda\\
&= \int_{-\pi}^{\pi}e^{ih\lambda}\left|\psi(e^{-ih\lambda})\right|^2 f_X(\lambda)d\lambda = \int_{-\pi}^{\pi}e^{ih\lambda}f_Y(\lambda)d\lambda.
\end{aligned}$$

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Remark 1.2 *There exists a similar result for a stationary time series $\{X_t\}$ with spectral distribution function F_X.*

We can use Proposition 1.1 to find the spectral density for an ARMA(p,q) process. But before we state the result for an ARMA(p,q) process, we need to find the spectral density for a white noise.

Example 1.1 (White noise) *Consider $\{Z_t\} \sim WN(0,\sigma^2)$. The acvf is*

$$\gamma_Z(h) = \begin{cases} \sigma^2, & \text{for } h = 0 \\ 0, & \text{for } h \neq 0 \end{cases}.$$

Obviously

$$\sum_{h=-\infty}^{\infty}|\gamma_Z(h)| = \gamma_Z(0) = \sigma^2 < \infty.$$

Therefore, the spectral density according to the definition is

$$f_Z(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}e^{-ih\lambda}\gamma_Z(h) = e^{-i\cdot 0\cdot\lambda}\gamma_Z(0) = \frac{\sigma^2}{2\pi}.$$

*Notice that the white noise contains all frequencies with equal amplitude. This is why the process is called **white** noise, just like **white** light.*

Proposition 1.3 *Let $\{X_t\}$ be an ARMA(p,q) process, i.e.,*

$$\phi(B)X_t = \theta(B)Z_t$$

with $\{Z_t\} \sim WN(0, \sigma^2)$*. The spectral density of* $\{X_t\}$ *is*

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}.$$

(This spectral density is also known as a rational spectral density.)

Proof. As we mentioned in our previous lecture, an ARMA(p, q) process $\{X_t\}$ is well-defined when $\phi(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$. In addition, when it is well-defined, it can be written as a linear process

$$X_t = \psi(B) Z_t$$

with a linear filter

$$\psi(z) = \frac{\theta(z)}{\phi(z)}.$$

Hence, we can apply Proposition 1.1 and recall the spectral density of a white noise to find that the spectral density of $\{X_t\}$ is

$$f_X(\lambda) = |\psi(e^{-i\lambda})|^2 f_Z(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}.$$

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Let's consider a concrete example of applying Proposition 1.3 to find the spectral density for a certain ARMA(p, q) process.

Example 1.2 (AR(1) process) *Consider an AR(1) process, i.e.,*

$$X_t = \phi X_{t-1} + Z_t$$

with $\{Z_t\} \sim WN(0, \sigma^2)$ *and* $|\phi| < 1$*.*

Then in this case, $\phi(z) = 1 - \phi z$ and $\theta(z) \equiv 1$. Moreover,

$$\begin{aligned} |\phi(e^{-i\lambda})|^2 &= \phi(e^{-i\lambda})\overline{\phi(e^{-i\lambda})} = (1 - \phi e^{-i\lambda})\overline{(1 - \phi e^{-i\lambda})} = (1 - \phi e^{-i\lambda})(1 - \phi e^{i\lambda}) \\ &= 1 - \phi(e^{-i\lambda} + e^{i\lambda}) + \phi^2 |e^{-i\lambda}|^2 = 1 - 2\phi \cos\lambda + \phi^2. \end{aligned}$$

Thus, the spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{1 - 2\phi\cos\lambda + \phi^2}.$$

Remark 1.4 *For* $\phi \in (0, 1)$*, the spectral density* $f(\lambda)$ *is decreasing with the maximizer at frequency* $\lambda = 0$*. This is because* X_t *is sort of similiar to* X_{t-1}*, we expect to see a smooth trajectory in the time domain, which leads to relatively few high-frequency components. For* $\phi \in (-1, 0)$*, the spectral density* $f(\lambda)$ *is increasing with the maximizer at frequency* $\lambda = \pi$*. This is because* X_t *switches signs all the time and fluctuates rapidly about its mean value, we expect to see a rough trajectory in the time domain, which corresponds to a large contribution from high-frequency components.*

Example 1.3 (MA(1) process) *Consider a MA(1) process, i.e.,*

$$X_t = Z_t + \theta Z_{t-1}$$

with $\{Z_t\} \sim WN(0, \sigma^2)$.

Then in this case, $\phi(z) \equiv 1$ and $\theta(z) = 1 + \theta z$. Moreover,

$$\begin{aligned}|\theta(e^{-i\lambda})|^2 &= \theta(e^{-i\lambda})\overline{\theta(e^{-i\lambda})} = (1+\theta e^{-i\lambda})\overline{(1+\theta e^{-i\lambda})} = (1+\theta e^{-i\lambda})(1+\theta e^{i\lambda}) \\ &= 1 + \theta(e^{-i\lambda} + e^{i\lambda}) + \theta^2|e^{-i\lambda}|^2 = 1 + 2\theta\cos\lambda + \theta^2.\end{aligned}$$

Thus, the spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi}(1 + 2\theta\cos\lambda + \theta^2).$$

2 Periodogram

In this section, we introduce an estimator for the spectral density $f(\lambda)$ given observations $\{x_1, \ldots, x_n\}$. Before proceeding, we need to define some notations.

Definition 2.1 *The Fourier frequencies associated with sample size n are given by*

$$\omega_k = \frac{2\pi k}{n}, \quad k = -\left[\frac{n-1}{2}\right], \ldots, \left[\frac{n}{2}\right],$$

where $[x]$ denotes the largest integer less than or equal to x. Moreover, we define

$$F_n = \left\{\omega_k \quad k = -\left[\frac{n-1}{2}\right], \ldots, \left[\frac{n}{2}\right]\right\}.$$

Remark 2.2 *Notice that for any $\omega_k \in F_n$, we have $\omega_k \in (-\pi, \pi]$.*

For any $k = -\left[\frac{n-1}{2}\right], \ldots, \left[\frac{n}{2}\right]$, we define a vector

$$\mathbf{e}_k = \frac{1}{\sqrt{n}}(e^{i\omega_k}, e^{2i\omega_k}, \ldots, e^{ni\omega_k})'.$$

One can show that $\{\mathbf{e}_k : k = -\left[\frac{n-1}{2}\right], \ldots, \left[\frac{n}{2}\right]\}$ forms an orthonormal basis in $\mathbb{C}^n$, i.e.,

$$\langle \mathbf{e}_k, \mathbf{e}_j \rangle = \begin{cases} 1, & \text{for } k = j \\ 0, & \text{for } k \neq j \end{cases}.$$

This implies that for any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$,

$$\mathbf{x} = \sum_{k=-[(n-1)/2]}^{[n/2]} a_k \mathbf{e}_k \tag{2.1}$$

with

$$a_k = \langle \mathbf{x}, \mathbf{e}_k \rangle = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{it\omega_k}. \tag{2.2}$$

The sequence $\{a_k\}$ is called the *discrete Fourier transform* of the sequence $\{x_1, \ldots, x_n\}$.

Definition 2.3 *The periodogram of* $\{x_1, \ldots, x_n\}$ *is defined as*

$$I_n(\lambda) = \frac{1}{n}\left|\sum_{t=1}^{n} x_t e^{it\lambda}\right|^2$$

for all $\lambda \in [-\pi, \pi]$.

Remark 2.4 *If* λ *is one of the Fourier frequencies* ω_k*, then by* (2.2)

$$I_n(\omega_k) = \frac{1}{n}\left|\sum_{t=1}^{n} x_t e^{it\omega_k}\right|^2 = |a_k|^2.$$

This implies that the squared length of $\boldsymbol{x}$ *is*

$$|\boldsymbol{x}|^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \sum_{k=-[(n-1)/2]}^{[n/2]} |a_k|^2 = \sum_{k=-[(n-1)/2]}^{[n/2]} I_n(\omega_k) = \sum_{\omega \in F_n} I_n(\omega),$$

where the second equality comes from (2.1) *and since* $\{\boldsymbol{e}_k\}$ *is orthonormal.*

The next proposition connects the relation between the periodogram $I_n(\lambda)$ and the spectral density $f(\lambda)$.

Proposition 2.5 *If* $x_1, \ldots, x_n \in \mathbb{R}$ *and* $\omega_k \in F_n \setminus \{0\}$*, then the periodogram* $I_n(\omega_k)$ *can be written as*

$$I_n(\omega_k) = \sum_{h=-n}^{n} \hat{\gamma}(h) e^{-ih\omega_k}, \tag{2.3}$$

where $\hat{\gamma}(h)$ *is the sample acvf of* $\{x_1, \ldots, x_n\}$.

Remark 2.6 *Recall that if* $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$*, then the spectral density*

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\lambda}.$$

This indicates that for any ω_k

$$2\pi f(\omega_k) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-i\omega_k\lambda}.$$

Comparing the above equation with (2.3)*, we find that* $I_n(\lambda)$ *can be regarded as a sample analogue of* $2\pi f(\lambda)$.

Remark 2.7 *The periodogram* $I_n(\omega_k)$ *is not a consistent estimator of* $2\pi f(\lambda)$*. It is not surprising since* $I_n(\omega_k)$ *always uses the values of the sample acvf* $\hat{\gamma}$ *at large lag, for example,* $h = n$*, which as we pointed out before, are not reliable estimation of the true acvf* γ.