EXAMINATION IN SF2943 TIME SERIES ANALYSIS

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Lecturer: Fredrik Armerin, tel. 070-251 75 55, email: armerin@math.kth.se

Allowed technical aids: Calculator and "Formulas and survey, Time series analysis" by Jan Grandell, without notes

Any notation must be explained and defined. Arguments and computations must be detailed so that they are easy to follow. Write only on one side of the page.

Good luck!

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Problem 1

The sample PACF for the first 10 lags from a data serie of 1 000 observations from a causal AR(p) process is given by

Lag	PACF
1	0.5587
2	-0.5342
3	-0.0427
4	0.0169
5	0.0178
6	0.0027
7	-0.0498
8	-0.0084
9	-0.0422
10	0.0075

(a) Argue that it is reasonable to assume that the order $p = 2$. (3 p)

The 15 first observations in the sample are

-0.2753, 0.1143, 0.1430, 0.4547, -0.0498, -0.8943, -1.3266, -0.3958, 0.2673, 0.3241, 0.7604, 0.5766, 0.1696, 0.5574, 0.0014

(b) Use these 15 observations to find the Yule-Walker estimates of the parameters in a casual AR(2)-model. (7 p)

Problem 2

The causal AR(1) process defined by

$$X_t - \varphi X_{t-1} = Z_t,\ \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

can be written

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

for a sequence $\{\psi_j\}$ satisfying

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

(a) Determine the sequence $\{\psi_j\}$. (5 p)

(b) Show that the ACVF for the causal AR(1) process above is given by

$$\gamma_X(h) = \sigma^2 \frac{\varphi^{|h|}}{1-\varphi^2},\ h = 0, \pm 1, \pm 2, \ldots$$

(5 p)

Problem 3

Let $\{X_t\}$ be the times series defined by

$$X_t + 0.60 X_{t-1} - 0.16 X_{t-2} = Z_t + 0.15 Z_{t-1},\ \{Z_t\} \sim \mathrm{WN}(0, 0.4).$$

(a) Show that $\{X_t\}$ is a well defined ARMA(2, 1) process. (4 p)

(b) Is $\{X_t\}$ a causal process? (3 p)

(c) Is $\{X_t\}$ an invertible process? (3 p)

Problem 4

Consider the MA(1) process

$$X_t = Z_t + 0.2 Z_{t-1},\ \{Z_t\} \sim \mathrm{WN}(0, 0.1).$$

The first 3 observations X_1, X_2 and X_3 from a sample generated by this MA-model are

`0.6301, 0.2652, 0.7592`

Determine the one-step predictor $\hat{X}_4$. (10 p)

Problem 5

Let $\{\mathbf{X}_t\}$ be a 2-variate AR(1) process:

$$\mathbf{X}_t - \Phi_1 \mathbf{X}_{t-1} = \mathbf{Z}_t, \ \{\mathbf{Z}_t\} \sim \mathrm{WN}(\mathbf{0}, \Sigma).$$

Consider this model when

$$\Phi_1 = \begin{bmatrix} \alpha & 0 \\ \beta & \beta \end{bmatrix}$$

with $\alpha, \beta \in \mathbb{R}$ and such that $\alpha, \beta \neq 0$. For which values on α and β is this a causal 2-variate AR(1) process? (10 p)

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Suggested solutions

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Problem 1

(a) The PACF drops down close to zero after the second lag – which is what we expect from an AR(2)-process.

(b) To estimate the parameters in an AR(2)-process using Yule-Walker, we need to solve for $\hat{\varphi} = [\hat{\varphi}_1, \hat{\varphi}_2]^T$ and $\hat{\sigma}^2$ in

$$\hat{\Gamma}_2\hat{\varphi} = \gamma_2 \text{ and } \hat{\sigma}^2 = \gamma(0) - \hat{\varphi}^T\gamma_2,$$

or

$$\begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{bmatrix} \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix} = \begin{bmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{bmatrix} \text{ and } \hat{\sigma^2} = \gamma(0) - \hat{\varphi}^T\gamma_2.$$

To find the $\hat{\gamma}(h)$'s we use

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_t - \bar{x}_n)(x_{t+h} - \bar{x}_n),\ h = 0, 1, 2.$$

We get

$$\hat{\gamma}(0) = 0.2996,\ \hat{\gamma}(1) = 0.1748 \text{ and } \hat{\gamma}(2) = 0.0250.$$

Solving the system of equations yields

$$\begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix} = \begin{bmatrix} 0.8265 \\ -0.4028 \end{bmatrix}.$$

Finally we get

$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\varphi}^T\hat{\gamma}_2 = 0.1622.$$

Problem 2

(a) Since $\{X_t\}$ is a causal process we must have

$$\psi_j = 0,\ j = 0, -1, -2, \ldots.$$

We also know that causality for the given AR(1) process is equivalent to $|\varphi| < 1$. Iterating the time series equation yields

$$\begin{aligned} X_t &= \varphi X_{t-1} + Z_t \\ &= \varphi^2 X_{t-2} + \varphi Z_{t-1} + Z_t \\ &= \cdots \\ &= \varphi^n X_{t-n} + \sum_{j=0}^{n-1}\varphi^j Z_{t-j}. \end{aligned}$$

Since $|\varphi| < 1$ we have $\varphi^n \to 0$ as $n \to \infty$. Hence, a candidate for the ψ_j's are

$$\psi_j = \varphi^j, \ j = 0, 1, \ldots.$$

We now show that

$$X_t = \sum_{j=0}^{\infty} \psi^j Z_{t-j}$$

indeed satisfies the time series equation. We get

$$\begin{aligned} X_t &= \sum_{j=0}^{\infty} \varphi^j Z_{t-j} \\ &= \sum_{j=1}^{\infty} \varphi^j Z_{t-j} + Z_t \\ &= \varphi \sum_{j=1}^{\infty} \varphi^{j-1} Z_{t-j} + Z_t \\ &= \varphi \sum_{j=0}^{\infty} \varphi^j Z_{t-1-j} + Z_t \\ &= \varphi X_{t-1} + Z_t. \end{aligned}$$

To summarize:

$$\psi_j = \begin{cases} \varphi^j & \text{if } j = 0, 1, \ldots \\ 0 & \text{if } j = -1, -2, \ldots. \end{cases}$$

(b)

$\boxed{h = 0}$ We get

$$\begin{aligned} \gamma(0) &= \mathrm{Var}(X_t) \\ &= \mathrm{Cov}(X_t, X_t) \\ &= \mathrm{Cov}(\varphi X_{t-1} + Z_t, \varphi X_{t-1} + Z_t) \\ &= \varphi^2 \mathrm{Var}(X_t) + \sigma^2 \\ &= \varphi^2 \gamma(0) + \sigma^2. \end{aligned}$$

It follows that

$$\gamma(0) = \frac{\sigma^2}{1 - \varphi^2}.$$

$\boxed{h > 0}$ Now

$$\begin{aligned}
\gamma(h) &= \mathrm{Cov}(X_t, X_{t-h}) \\
&= \mathrm{Cov}(\varphi X_{t-1} + Z_t, X_{t-h}) \\
&= \varphi \mathrm{Cov}(X_{t-1}, X_{t-h}) + 0 \\
&= \cdots \\
&= \varphi^h \mathrm{Cov}(X_{t-h}, X_{t-h}) \\
&= \varphi^h \mathrm{Var}(X_{t-h}) \\
&= \varphi^h \mathrm{Var}(X_t) \\
&= \varphi^h \gamma(0) \\
&= \sigma^2 \frac{\varphi^h}{1-\varphi^2}.
\end{aligned}$$

$\boxed{h < 0}$ Since

$$\gamma(-h) = \gamma(h)$$

we get

$$\gamma(h) = \sigma^2 \frac{\varphi^{-h}}{1-\varphi^2}$$

when $h < 0$.

Problem 3

We can write the time series equation as

$$\varphi(B)X_t = \theta(B)Z_t,$$

where B is the lag operator,

$$\varphi(z) = 1 + 0.60z - 0.16z^2 = (1 + 0.80z)(1 - 0.20z)$$

and

$$\theta(z) = 1 + 0.15z$$

respectively.

(a) The zeros of $\varphi(z)$ are

$$z_1 = -\frac{1}{0.8} = -1.25 \text{ and } z_2 = \frac{1}{0.2} = 5,$$

and the zero of $\theta(z)$ is

$$z_3 = -\frac{1}{0.15} = 6.67.$$

Since the two polynomials $\varphi(z)$ and $\theta(z)$ does not have any common zeros and since $\varphi(z) \neq 0$ when $|z| = 1$, the time series equation represents a well defined ARMA$(2,1)$ process.

(b) The zeros of $\varphi(z)$ are outside the unit circle; hence $\{X_t\}$ is a causal process.

(c) The zero of $\theta(z)$ is outside the unit circle; hence $\{X_t\}$ is an invertible process.

Problem 4

To find $\hat{X}_4$ we use the fact that

$$\hat{X}_4 = \sum_{i=1}^{3} \varphi_{3,i} X_{4-i},$$

where

$$\varphi_3 = \begin{bmatrix} \varphi_{3,1} \\ \varphi_{3,2} \\ \varphi_{3,3} \end{bmatrix}$$

solves

$$\Gamma_3 \varphi_3 = \gamma_3$$

with

$$\Gamma_3 = \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{bmatrix}$$

and

$$\gamma_3 = \begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \gamma(3) \end{bmatrix}.$$

Since $\{X_t\}$ is an MA(1) process we have

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 & \text{if } h = 0 \\ \theta\sigma^2 & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases} = \begin{cases} 0.1040 & \text{if } h = 0 \\ 0.0200 & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1. \end{cases}$$

Solving the system of equations yields

$$\varphi_3 = \begin{bmatrix} 0.1040 & 0.0200 & 0 \\ 0.0200 & 0.1040 & 0.0200 \\ 0 & 0.0200 & 0.1040 \end{bmatrix}^{-1} \begin{bmatrix} 0.0200 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2000 \\ -0.0399 \\ 0.0077 \end{bmatrix}.$$

Using the data

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.6301 \\ 0.2652 \\ 0.7592 \end{bmatrix}$$

we get

$$\hat{X}_4 = 0.2000 \cdot 0.7592 - 0.0399 \cdot 0.2652 + 0.0077 \cdot 0.6301 = 0.1461.$$

Problem 5

We write the time series equation as

$$\Phi(B)\mathbf{X}_t = \mathbf{Z}_t,$$

where B is the lag operator and

$$\Phi(z) = I - \Phi_1 z = \begin{bmatrix} 1-\alpha z & 0 \\ \beta & 1-\beta z \end{bmatrix}.$$

The time series $\{\mathbf{X}_t\}$ is a causal 2-variate AR(1) process if and only if

$$\det \Phi(z) \neq 0 \text{ for every } z \text{ such that } |z| \leq 1.$$

Here

$$\det \Phi(z) = (1-\alpha z)(1-\beta z),$$

and the zeros of $\det \Phi(z)$ are

$$z_1 = \frac{1}{\alpha} \text{ and } z_2 = \frac{1}{\beta}.$$

We see that if $|\alpha| < 1$ and $|\beta| < 1$ then $\det \Phi(z) \neq 0$ for every $|z| \leq 1$. Since $\alpha, \beta \neq 0$ we find that $\{\mathbf{X}_t\}$ is a causal 2-variate AR(1) process if and only if

$$\alpha \in (-1,0) \cup (0,1) \text{ and } \beta \in (-1,0) \cup (0,1).$$