Division of Mathematical Statistics

KTH Matematik

EXAM IN SF2943 TIME SERIES ANALYSIS
FRIDAY AUGUST 20 2021, 08:00–13:00

Examiner: Joakim Andén, 08-790 80 52.

Allowed aids: Pocket calculator, "Formulas and survey, Time series analysis" by Jan Grandell, without notes.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Problem 1

The sample acf is computed for a time series consisting of $n = 198$ samples. The following values are obtained

$$\hat{\rho}(1) = 0.580, \quad \hat{\rho}(2) = 0.205, \quad \hat{\rho}(3) = -0.076.$$

(a) Assuming the data arose from a white noise model, provide 95% confidence intervals for $\hat{\rho}(1)$, $\hat{\rho}(2)$, and $\hat{\rho}(3)$. Can the null hypothesis of a white noise model be rejected? (2 p)

(b) Provide a 95% confidence interval for $\hat{\rho}(3)$ under an MA(2) model. Can the null hypothesis of an MA(2) model be rejected? State any approximations or assumptions made. (8 p)

Problem 2

Let $\kappa(h)$ be given by

$$\kappa(h) = \begin{cases} 1, & h = 0, \\ -0.3, & |h| = 1, \\ 0.2, & |h| = 2, \\ 0, & |h| > 2. \end{cases}$$

(a) Show that there exists a stationary process whose acvf equals $\kappa(h)$. *Aid: Remember that* $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. (8 p)

(b) If $\kappa(h)$ is the acvf of a stationary process, what would be an appropriate model for that process? (2 p)

Problem 3

Suppose that $\{X_t\}$ follows a causal AR(2) model. That is, there exists ϕ_1 and ϕ_2 such that

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and Z_s is uncorrelated with X_t for $s > t$. Prove that the pacf $\alpha(h)$ has the form

$$\alpha(h) = \begin{cases} 1, & h = 0, \\ \phi_1/(1-\phi_2), & h = 1, \\ \phi_2, & h = 2, \\ 0, & h > 2. \end{cases}$$

You do not need to prove results present in Formulas and survey. (10 p)

Good luck!

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SOLUTIONS TO EXAM IN SF2943 TIME SERIES ANALYSIS
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Problem 1

(a) Under a white noise model, we have $\{X_t\} \sim \mathrm{WN}(0, \sigma^2)$ for some $\sigma > 0$. This gives the acf

$$\rho(h) = \begin{cases} 1, & h = 0, \\ 0, & |h| > 0. \end{cases}$$

According to Theorem 10.4 in Formulas and survey, we have that for $h > 0$, $\hat{\rho}(h)$ is approximately distributed as $\mathrm{N}(\rho(h), n^{-1}w_{hh})$ for large n, where (setting $i = j = h$)

$$w_{hh} = \sum_{k=-\infty}^{+\infty} (\rho(k+h)^2 + \rho(k-h)\rho(k+h) + 2\rho(h)^2\rho(k)^2 - 4\rho(h)\rho(k)\rho(k+h)).$$

This simplifies to $w_{hh} = 1$ for all $h > 0$ with $\rho(h)$ given above.
With $n = 198$ we can assume that n is large enough. We also assume that the the fourth moment of our white noise process is finite so that Theorem 10.4 holds. This gives the approximate distribution $\mathrm{N}(0, n^{-1})$ for $\hat{\rho}(1)$, $\hat{\rho}(2)$, and $\hat{\rho}(3)$, which in turn yields the 95% confidence interval

$$I = \left[-\frac{\lambda_{0.025}}{\sqrt{n}}, \frac{\lambda_{0.025}}{\sqrt{n}}\right] \approx \left[-1.96/\sqrt{198}, 1.96/\sqrt{198}\right] \approx [-0.139, 0.139].$$

Since $\hat{\rho}(1)$ and $\hat{\rho}(2)$ are outside this interval, we reject the null hypothesis.

(b) Under and MA(2) model, we have a different acf $\rho(h)$, where $\rho(\pm 1)$ and $\rho(\pm 2)$ are non-zero, but unknown, while $\rho(h) = 0$ for $|h| > 2$. Plugging this into the formula above for w_{hh} (and using symmetry of $\rho(h)$) then gives

$$w_{hh} = 1 + 2\rho(1)^2 + 2\rho(2)^2,$$

for $h > 2$. Since n is large, we assume that $\hat{\rho}(1)$ and $\hat{\rho}(2)$ are reasonable approximations of $\rho(1)$ and $\rho(2)$. Plugging this in, we obtain

$$w_{hh} \approx 1 + 2 \times 0.580^2 + 2 \times 0.205^2 = 1.76,$$

which gives the 95% confidence interval

$$I = \left[-\frac{\lambda_{0.025}\sqrt{w_{hh}}}{\sqrt{n}}, \frac{\lambda_{0.025}\sqrt{w_{hh}}}{\sqrt{n}}\right] \approx [-0.185, 185],$$

for $\hat{\rho}(3)$. Since the given value of $\hat{\rho}(3)$ is in this interval, we do not reject the null hypothesis.

Problem 2

(a) To prove the desired result, it suffices to show that $\kappa(h)$ is non-negative definite (or positive semidefinite), which is equivalent to showing that the corresponding spectral density

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \kappa(h) e^{-ih\lambda}$$

is non-negative for all $\lambda \in [-\pi, \pi]$.
For the given $\kappa(h)$, we have

$$f(\lambda) = \frac{1}{2\pi}\left(1 - 0.3(e^{-i\lambda} + e^{i\lambda}) + 0.2(e^{-i\lambda} + e^{i\lambda})\right) = \frac{1}{2\pi}(1 - 0.6\cos(\lambda) + 0.4\cos(2\lambda)),$$

so it suffices to show that $f(\lambda) \geq 0$ for all $\lambda \in [-\pi, \pi]$. We know that the $|\cos(\lambda)| \leq 1$ and $|\cos(2\lambda)| \leq 1$, so the lowest value attained by $f(\lambda)$ is

$$\frac{1}{2\pi}(1 - 0.6 \times 1 + 0.4 \times -1) = 0.$$

As a result, we have that $f(\lambda) \geq 0$ for all $\lambda \in [-\pi, \pi]$ and $\kappa(h)$ is therefore non-negative definite and the acvf of a stationary process.

(b) Since the acvf cuts off at $h = 2$, the process is 2-correlated, which means that it can be written as an MA(2) model.

Problem 3

We know from the definition of the pacf (Formulas and survey, Definition 8.2) that $\alpha(0) = 1$ and $\alpha(1) = \rho(1)$. For an AR(2) model, we multiply the recurrence relation with X_{t-1} to obtain

$$X_t X_{t-1} - \phi_1 X_{t-1} X_{t-1} - \phi_2 X_{t-2} X_{t-1} = Z_t X_{t-1}.$$

Taking expectation then gives

$$\gamma(1) - \phi_1\gamma(0) - \phi_2\gamma(1) = 0,$$

since $\gamma(-1) = \gamma(1)$ and Z_t is uncorrelated with X_{t-1}. As a result

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\phi_1}{1 - \phi_2},$$

and we have

$$\alpha(1) = \frac{\phi_1}{1 - \phi_2}$$

Since $\{X_t\}$ is causal, we know that X_t can be written as an infinte sum of Z_s terms

$$X_t = \sum_{j=0}^{+\infty} \psi_j Z_{t-j}$$

for some sequence $\{\psi_j\}$. Since $\{X_t\}$ follows an AR(2) model, it is a stationary process and therefore has $\mathrm{Var}[X_t] < +\infty$. Now since

$$\mathrm{Var}[X_t] = \sigma^2 \sum_{j=0}^{+\infty} |\psi_j|^2,$$

this implies that $\{\psi_j\}$ is square summable, and therefore also absolutely summable. Consequently, there exists M such that $\psi_j \leq M$ for all j and we thus have, for $h > 0$,

$$\begin{aligned}\gamma(h) = \mathbb{E}[X_t X_{t-h}] &= \sigma^2 \sum_{j=0}^{+\infty} \psi_j \psi_{j+h} \\ &\leq \sigma^2 M \sum_{j=0}^{+\infty} \psi_{j+h} = \sigma^2 M \sum_{j=h}^{+\infty} \psi_j.\end{aligned}$$

Since $\{\psi_j\}$ is absolutely summable, the sum tends to zero as $h \to +\infty$. Consequently $\gamma(h) \to 0$ as $h \to +\infty$, which lets us apply Theorems 7.1 and 8.2 from Formulas and Survey.
For $\alpha(2)$, we use the fact that $\alpha(h) = \phi_{hh}$ (Theorem 8.2), where ϕ_{hh} is given by the expansion (Theorem 7.1)

$$\hat{X}_{n+1} = \phi_{n1} X_n + \phi_{n2} X_{n-1} + \ldots + \phi_{nn} X_1.$$

For an AR(2) process, we have that

$$\hat{X}_{n+1} = \phi_1 X_n + \phi_2 X_{n-1},$$

for $n \geq 2$ (Formulas and survey, Section 7.2). Setting these equations equal to one another for $n = 2$, we have that

$$\phi_{21} X_2 + \phi_{22} X_1 = \phi_1 X_2 + \phi_2 X_1,$$

which gives $\phi_{22} = \phi_2$. Consequently

$$\alpha(2) = \phi_2.$$

Finally, we use the same equality, but for $n = h > 2$, which gives

$$\phi_{h1} X_h + \phi_{h2} X_{h-1} + \ldots + \phi_{hh} X_1 = \phi_1 X_h + \phi_2 X_{h-1}.$$

As a result, we have $\alpha(h) = \phi_{hh} = 0$ for $h > 2$.