Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2945 TIME SERIES ANALYSIS/TIDSSERIEANALYS
ONSDAGEN DEN 16 DECEMBER 2009 KL 08.00–13.00.

Examinator: Timo Koski, tel. 7907134

Tillåtna hjälpmedel: Formulas and survey, Time series analysis. Handheld calculator.

Införda beteckningar skall förklaras och definieras. Resonemang och uträkningar skall vara så utförliga och väl motiverade att de är lätta att följa.

Varje korrekt lösning ger 10 poäng. Gränsen för godkänt är 25 poäng. De som erhåller 23 eller 24 poäng på tentamen kommer att erbjudas möjlighet att komplettera till betyget E. Den som är aktuell för komplettering skall till examinator anmäla önskan att få en sådan inom en vecka från publicering av tentamensresultatet.

Lösningarna får givetvis skrivas på svenska.

Resultatet skall vara klart senast torsdag den 7 januari 2010 och blir tillgängligt via ”Mina sidor”.

Lösningarna får givetvis skrivas på svenska.

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Quantiles of the normal distribution
(Normalfördelningens kvantiler)

$P(X > \lambda_\alpha) = \alpha$ where $X \sim N(0, 1)$

α	λ_α	α	λ_α
0.10	1.2816	0.001	3.0902
0.05	1.6449	0.0005	3.2905
0.025	1.9600	0.0001	3.7190
0.010	2.3263	0.00005	3.8906
0.005	2.5758	0.00001	4.2649

Problem 1

The process $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ satisfies the equation

$$X_t - \frac{5}{6}X_{t-1} + \frac{1}{6}X_{t-2} = Z_t - \frac{9}{20}Z_{t-1} + \frac{1}{20}Z_{t-2},$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$.

(a) Show that $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is an ARMA(2,2) -process. (3 p)

(b) Is $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ a causal process ? (3 p)

(c) Show that $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is an invertible process. What do we mean by the statement that $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is an invertible process ? (4 p)

Problem 2

The Gaussian stationary process $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ has mean function zero and the autocovariance function (ACVF) $\gamma(h)$ equal to

$$\gamma(h) = \frac{5}{3}\left(\frac{1}{2}\right)^h - \frac{2}{3}\left(\frac{1}{4}\right)^h, \quad h \geq 0.$$

(a) Find the probability distribution of the column vector $(X_{t+2}, X_{t+1}, X_t)^T$. Show your calculations. (4 p)

(b) Find the probability distribution of the column vector $(X_{t+22}, X_{t+21}, X_{t+20})^T$. Show your reasoning. (1 p)

(c) Find a constant a such that

$$P\left(X_{t+1} > a \mid X_t = 2.0\right) = 0.95.$$

Problem 3

The process $\{Y_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is an MA(2) process with

$$Y_t = Z_t - 2Z_{t-1} + Z_{t-2},$$

where $Z_t \sim \mathrm{WN}(0, 2)$. The process $\{Y_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is the input to an LTI -filter, where the output is $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ given by

$$X_t = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j Y_{t-j}.$$

Show that the spectral density $f_X(\lambda)$ of $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is

$$f_X(\lambda) = \frac{16(1 - \cos(\lambda))^2}{\pi(5 - 4\cos(\lambda))}.$$

Aid: You may at some stage need the formula

$$\cos(2\lambda) = 2\cos^2(\lambda) - 1.$$

(10 p)

Problem 4

Let $|\phi| < 1$ and

$$W_t = \phi W_{t-1} + Z_t, t = 0, \pm 1, \pm 2 \ldots,$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$. Let for $h \geq 1$

$$\widehat{W_t}(h) = E\left[W_{t+h} \mid \mathbf{W}_t\right],$$

where $\mathbf{W}_t = (W_t, W_{t-1}, \ldots)$.

(a) Set

$$\widehat{W_t}(\infty) = \lim_{n \to \infty} \sum_{h=1}^{n} \widehat{W_t}(h),$$

and show that

$$\widehat{W_t}(\infty) = \frac{\phi}{1-\phi} W_t. \tag{1}$$

(3 p)

(b) Let $\tilde{\psi}_j = \sum_{l=j+1}^{\infty} \phi^l$, $j = 0, 1, 2, \ldots$. Show that

$$\tilde{\psi}_j = \frac{1}{1-\phi} \phi^{j+1} \tag{2}$$

Set

$$\tilde{\psi}(B) = \sum_{j=0}^{\infty} \tilde{\psi}_j B^j, \tag{3}$$

and show that

$$\widehat{W_t}(\infty) = \tilde{\psi}(B) Z_t, \tag{4}$$

You are allowed to use (2) even if you have not shown this. (2 p)

Hint: Express W_t as a linear process with

$$\psi(B) = \sum_{j=0}^{\infty} \phi^j B^j. \tag{5}$$

(c) Set $\psi(1) = \frac{1}{1-\phi}$, and let $\tilde{\psi}(B)$ and $\psi(B)$ be as in (3) and (5), respectively. Verify that

$$\psi(B) = \psi(1) - (I - B)\tilde{\psi}(B). \tag{6}$$

Hint: Expand $(I - B)\tilde{\psi}(B)$. (4 p)

(d) In view of (6), what do we know about W_t ? (1 p)

Problem 5

An ARIMA(1,1,0) process $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ with a differential trend μ satisfies

$$\phi(B)(1-B)X_t = (1-\phi)\mu + Z_t,$$

where $\phi(B) = 1 - \phi B$, $\mid \phi \mid < 1$ and $Z_t \sim \mathrm{WN}(0, \sigma^2)$.

(a) Set $Y_t = (1-B)X_t$ and check that

$$Y_t - \mu = \phi\left(Y_{t-1} - \mu\right) + Z_t. \tag{7}$$

Recall that the backward shift for a constant sequence satisfies $B\mu = \mu$. (1 p)

(b) Set $W_t = Y_t - \mu$ in (7). Check now that

$$X_{t+n} = X_t + n\mu + \sum_{l=1}^{n} W_{t+l}.$$

(1 p)

(c) Use (b) to show that

$$\lim_{n\to\infty} E\left[X_{t+n} - n\mu \mid \mathbf{W}_t\right] = X_t + \frac{\phi}{1-\phi}\left(Y_t - \mu\right), \tag{8}$$

where $\mathbf{W}_t$ is $\mathbf{W}_t = (W_t, W_{t-1}, \ldots)$. (You are allowed to use (1) even if you have not shown this). (2 p)

(d) We introduce in (8)

$$BN_t = X_t + \frac{\phi}{1-\phi}\left(Y_t - \mu\right).$$

Show that

$$BN_t = BN_{t-1} + \mu + \psi(1)Z_t. \tag{9}$$

(5 p)

Hint: The identity (6) on W_t may turn out to be useful.

(d) Explain what we have found about X_t in an ARIMA(1,1,0) process. Think of

$$X_t = BN_t - \frac{\phi}{1-\phi}\left(Y_t - \mu\right). \tag{10}$$

(1 p)

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LÖSNINGAR TILL TENTAMEN
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Problem 1

(a) We note that the linear stochastic difference equation with constant coefficients

$$X_t - \frac{5}{6}X_{t-1} + \frac{1}{6}X_{t-2} = Z_t - \frac{9}{20}Z_{t-1} + \frac{1}{20}Z_{t-2},$$

can be written as

$$\phi(B)X_t = \theta(B)Z_t,$$

where

$$\phi(B) = 1 - \frac{5}{6}B + \frac{1}{6}B^2, \quad \theta(B) = 1 - \frac{9}{20}B + \frac{1}{20}B^2.$$

In order to answer the question about ARMA(2,2) we consider zeros of the polynomials $\phi(z)$ and $\theta(z)$ in $z \in \mathbf{C}$,

$$\phi(z) = 1 - \frac{5}{6}z + \frac{1}{6}z^2, \quad \theta(z) = 1 - \frac{9}{20}z + \frac{1}{20}z^2$$

We get that

$$\phi(\xi_i) = 0 \Leftrightarrow \xi_1 = 2, \xi_2 = 3 \Leftrightarrow \phi(z) = \left(1 - \frac{z}{2}\right)\left(1 - \frac{z}{3}\right).$$

Thus we see that all roots of $\phi(z) = 0$ are satisfy $\mid z \mid \neq 1$. Hence $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is **stationary**. Next,

$$\theta(\eta_i) = 0 \Leftrightarrow \eta_1 = 4, \eta_2 = 5 \Leftrightarrow \theta(z) = \left(1 - \frac{z}{4}\right)\left(1 - \frac{z}{5}\right).$$

and we see that $\phi(z)$ and $\theta(z)$ have **no common factors/ common zeros** (c.f., definition 3.1.1. in Brockwell and Davis). Hence $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is an ARMA(2,2)-process by definition 3.1.1. in Brockwell and Davis).

(b) The ARMA(2,2)-process $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is a causal process, since all roots of $\phi(z) = 0$ are satisfy $\mid z \mid > 1$.

(c) We have seen above the roots of $\theta(z) = 0$ satisfy $\mid z \mid > 1$. By defintion this means that $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is an invertible process. By (b) we have shown that

$$\phi(B)X_t = \theta(B)Z_t \Leftrightarrow X_t = \frac{\theta(B)}{\phi(B)}Z_t$$

Invertibility means that

$$Z_t = \frac{\phi(B)}{\theta(B)} X_t = \pi(B) X_t,$$

or, in other words, we can write Z_t as the output of a causal linear filter with X_t as input.

Note about grading: The definition of ARMA(p,q) in Brockwell and Davis requires the condition of no common factors in $\phi(z)$ and $\theta(z)$. First the definition of causality and invertibility in the **Collection of Formulas** pp- 8−9 requires the condition of no common zeros in $\phi(z)$ and $\theta(z)$. A solution of the problem that recognizes the condition of no common zeros at some point gives credit points.

Problem 2

(a) Since the stationary process $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is Gaussian, the probability distribution of the column vector $(X_{t+2}, X_{t+1}, X_t)^T$ is multivariate normal. Because the mean function is zero, the mean vector of $(X_{t+2}, X_{t+1}, X_t)^T$ is the 3×1 zero vector. Thus we have

$$\begin{pmatrix} X_{t+2} \\ X_{t+1} \\ X_t \end{pmatrix} \sim N_3 \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{pmatrix} \right)$$

where the entries of the covariance matrix are obtained from the ACVF $\gamma(h)$. From what is given in the **Problem 2**,

$$\gamma(0) = \frac{5}{3}\left(\frac{1}{2}\right)^0 - \frac{2}{3}\left(\frac{1}{4}\right)^0 = \frac{5}{3} - \frac{2}{3} = 1,$$

$$\gamma(1) = \frac{5}{3}\frac{1}{2} - \frac{2}{3}\frac{1}{4} = \frac{2}{3},$$

and

$$\gamma(2) = \frac{5}{3}\left(\frac{1}{2}\right)^2 - \frac{2}{3}\left(\frac{1}{4}\right)^2 =$$

$$= \frac{5}{3}\frac{1}{4} - \frac{2}{3}\frac{1}{16} = \frac{20-2}{3 \cdot 16} = \frac{18}{3 \cdot 16} = \frac{3}{8}.$$

ANSWER (a) : $\underline{\begin{pmatrix} X_{t+2} \\ X_{t+1} \\ X_t \end{pmatrix} \sim N_3 \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{8} \\ \frac{2}{3} & 1 & \frac{2}{3} \\ \frac{3}{8} & \frac{2}{3} & 1 \end{pmatrix} \right)}$

(b) Because the stationary process $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is Gaussian, it is also strictly stationary. This means amongst other things that the distribution of $(X_{t+2}, X_{t+1}, X_t)^T$ is not changed by a common shift of the time indices. As $(X_{t+22}, X_{t+21}, X_{t+20})^T$ contains these random variables shifted by twenty time units, we get the answer.

ANSWER (b): $\underline{\begin{pmatrix} X_{t+22} \\ X_{t+21} \\ X_{t+20} \end{pmatrix} \sim \begin{pmatrix} X_{t+2} \\ X_{t+1} \\ X_t \end{pmatrix}}.$

(c) In order to find a constant a such that

$$P\left(X_{t+1} > a \mid X_t = 2.0\right) = 0.95,$$

we need to find probability distribution of $X_{t+1} \mid X_t = 2.0$. As the process $\{X_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is Gaussian, the joint distribution of $(X_{t+1}, X_t)^T$ is a bivariate normal distribution. We have by **Collection of Formulas**,

$$X_{t+1} \mid X_t = 2.0 \sim N\left(\rho \frac{\sigma_{X_{t+1}}}{\sigma_{X_t}} \cdot 0.2, \sigma^2_{X_{t+1}}\left(1 - \rho^2\right)\right)$$

where, as means are zero, and from part (a) $\sigma_{X_{t+1}} = \sigma_{X_t} = \sqrt{\gamma(0)} = 1$ and the coefficient of correlation is by part (a)

$$\rho = \frac{\gamma(1)}{\gamma(0)} = \frac{2}{3}.$$

Thereby

$$X_{t+1} \mid X_t = 2 =$$
$$\sim N\left(\frac{2}{3} \cdot 2, \left(1 - \left(\frac{2}{3}\right)^2\right)\right) = N\left(\frac{4}{3}, \frac{5}{9}\right).$$

Then

$$P\left(X_{t+1} > a \mid X_t = 2.0\right) = P\left(\frac{X_{t+1} - \frac{4}{3}}{\sqrt{\frac{5}{9}}} > \frac{a - \frac{4}{3}}{\sqrt{\frac{5}{9}}} \mid X_t = 2.0\right)$$

and since now $\frac{X_{t+1} - \frac{4}{3}}{\sqrt{\frac{5}{9}}} \mid X_t = 2.0 \sim N(0,1)$, we obtain

$$= P\left(\xi > \frac{a - \frac{4}{3}}{\sqrt{\frac{5}{9}}}\right),$$

where $\xi \sim N(0,1)$. Hence we want to find a such that

$$P\left(\xi > \frac{a - \frac{4}{3}}{\sqrt{\frac{5}{9}}}\right) = 0.95.$$

By symmetry of $N(0,1)$ we have that $\frac{a - \frac{4}{3}}{\sqrt{\frac{5}{9}}} = -\lambda_{0.05}$, where

$$P\left(\xi > \lambda_{0.05}\right) = 0.05,$$

and $\lambda_{0.05} = 1.6449$ is given in the Quantiles of the normal distribution recapitulated in the ingress.

In other words,

$$a = -\lambda_{0.05}\sqrt{\frac{5}{9}} + \frac{4}{3} = -1.6449 \cdot 0.894 + \frac{4}{3} = 0.1073.$$

ANSWER: $\underline{a = 0.1073.}$

Problem 3

We have X_t as a causal operation on Y_t

$$X_t = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j B^j Y_t = \psi(B) Y_t.$$

Then with $\phi(B) = 1 - \frac{1}{2}B$ we have

$$\psi(B) = \frac{1}{\phi(B)}.$$

As the process $\{Y_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is an MA(2) process

$$Y_t = Z_t - 2Z_{t-1} + Z_{t-2},$$

where $Z_t \sim \mathrm{WN}(0, 2)$, we set $\theta(B) = 1 - 2B + B^2$, and thus we have

$$X_t = \frac{\theta(B)}{\phi(B)} Z_t.$$

Then

$$f_X(\lambda) = \frac{1}{\pi} \frac{\mid \theta\left(e^{-i\lambda}\right) \mid^2}{\mid \phi\left(e^{-i\lambda}\right) \mid^2} \tag{11}$$

by 4.4 in Brockwell and Davis (or Theorem 6.2 in **Collection of Formulas**), where, since $Z_t \sim \mathrm{WN}(0, 2)$, we have that

$$f_Z(\lambda) = \frac{2}{2\pi} = \frac{1}{\pi}$$

First we compute the transfer function

$$\phi\left(e^{-i\lambda}\right) = 1 - \frac{1}{2} e^{-i\lambda}$$

Hence

$$\mid \phi\left(e^{-i\lambda}\right) \mid^2 = \phi\left(e^{-i\lambda}\right) \phi\left(e^{i\lambda}\right) = \left(1 - \frac{1}{2}e^{-i\lambda}\right)\left(1 - \frac{1}{2}e^{i\lambda}\right).$$

Some simple algebra yields

$$\left(1 - \frac{1}{2}e^{-i\lambda}\right) \cdot \left(1 - \frac{1}{2}e^{i\lambda}\right)$$

$$= 1 - \frac{1}{2}e^{i\lambda} - \frac{1}{2}e^{-i\lambda} + \frac{1}{4},$$

where now the Euler formula $2\cos(\lambda) = e^{i\lambda} + e^{-i\lambda}$ gives

$$= \frac{5}{4} - \cos(\lambda).$$

We summarize this by

$$\frac{1}{\mid \phi\left(e^{-i\lambda}\right) \mid^2} = \frac{4}{5 - 4\cos(\lambda)}. \tag{12}$$

Next

$$\theta\left(e^{-i\lambda}\right) = 1 - 2e^{-i\lambda} + e^{-i2\lambda}.$$

Then some diligent algebra entails

$$\mid \theta\left(e^{-i\lambda}\right) \mid^2 = \left(1 - 2e^{-i\lambda} + e^{-i2\lambda}\right)\left(1 - 2e^{i\lambda} + e^{i2\lambda}\right)$$

$$= 1 - 2e^{i\lambda} + e^{i2\lambda} - 2e^{-i\lambda} + 4 - 2e^{i\lambda} + e^{-i2\lambda} - 2e^{-i\lambda} + 1$$

$$= 6 - 4\left(e^{i\lambda} + e^{-i\lambda}\right) + \left(e^{2i\lambda} + e^{-2i\lambda}\right)$$

and by the Euler formula $2\cos(\lambda) = e^{i\lambda} + e^{-i\lambda}$ again

$$= 6 - 4 \cdot 2\cos(\lambda) + 2\cos(2\lambda).$$

Now we invoke the hint

$$\cos(2\lambda) = 2\cos^2(\lambda) - 1.$$

to get

$$\mid \theta\left(e^{-i\lambda}\right) \mid^2 = 6 - 8\cos(\lambda) + 4\cos^2(\lambda) - 2$$

$$= 4 - 8\cos(\lambda) + 4\cos^2(\lambda) = 4\left(1 - 2\cos(\lambda) + \cos^2(\lambda)\right)$$

$$= 4\left(1 - \cos(\lambda)\right)^2,$$

or summarized as

$$\mid \theta\left(e^{-i\lambda}\right) \mid^2 = 4\left(1 - \cos(\lambda)\right)^2. \tag{13}$$

When we insert (12) and (13) in (11), we obtain

$$f_X(\lambda) = \frac{16(1 - \cos(\lambda))^2}{\pi(5 - 4\cos(\lambda))},$$

as was desired.

Alternatively one can evoke the *super formula*[1], or Proposition 4.3.1. in Brockwell and Davis, which says that

$$f_X(\lambda) = \mid \Psi\left(e^{-i\lambda}\right) \mid^2 f_Y(\lambda) \tag{14}$$

where $\Psi\left(e^{-i\lambda}\right)$ is the transfer function of the LTI-filter $X_t = \psi(B)Y_t$ and $f_Y(\lambda)$ is the spectral density of the process $\{Y_t \mid t = 0, \pm 1, \pm 2, \ldots\}$. By another application of the super formula we get:

$$f_Y(\lambda) = \mid \theta\left(e^{-i\lambda}\right) \mid^2 f_Z(\lambda) \tag{15}$$

where $\theta(z) = 1 - 2z + z^2$. From here on the computations are the same as above.

Problem 4

(a) For $h \geq 1$

$$W_{t+h} = \phi W_{t+h-1} + Z_{t+h},$$

we get by iteration

$$= \phi\left(\phi W_{t+h-2} + Z_{t+h-1}\right) + Z_{t+h}$$

[1] this name has been used at the KTH department of teletransmission.

$$\vdots$$

$$= \phi^h W_t + \sum_{j=1}^{h} \phi^j Z_{t+j}.$$

Then

$$\widehat{W}_t(h) = E\left[W_{t+h} \mid \mathbf{W}_t\right] = E\left[\phi^h W_t + \sum_{j=1}^{h} \phi^j Z_{t+j} \mid \mathbf{W}_t\right]$$

$$= E\left[\phi^h W_t \mid \mathbf{W}_t\right] + E\left[\sum_{j=1}^{h} \phi^j Z_{t+j} \mid \mathbf{W}_t\right]$$

$$= \phi^h W_t,$$

by properties of conditional expectation, and since W is a causal AR-process ($\mid \phi \mid < 1$) and $Z_t \sim \mathrm{WN}(0, \sigma^2)$. Then

$$\widehat{W}_t(\infty) = \lim_{n\to\infty} \sum_{h=1}^{n} \phi^h W_t,$$

$$= W_t \lim_{n\to\infty} \sum_{h=1}^{n} \phi^h = W_t \left(\sum_{h=0}^{\infty} \phi^h - 1 \right)$$

$$= W_t \left(\frac{1}{1-\phi} - 1 \right) = \frac{\phi}{1-\phi} W_t,$$

which is (1), as asserted.

(b) We have

$$\tilde{\psi}_j = \sum_{l=j+1}^{\infty} \phi^l = \sum_{l-j=1}^{\infty} \phi^l \overbrace{=}^{k=l-j}$$

$$= \sum_{k=1}^{\infty} \phi^{k+j} = \phi^j \sum_{k=1}^{\infty} \phi^k = \phi^j \left(\sum_{k=0}^{\infty} \phi^k - 1 \right) = \phi^j \left(\frac{1}{1-\phi} - 1 \right) =$$

$$= \phi^j \left(\frac{1 - 1 + \phi}{1-\phi} \right) = \frac{1}{1-\phi} \phi^{j+1},$$

which is (2), as claimed. Next, let us observe that with $\psi(B) = \sum_{j=0}^{\infty} \phi^j B^j$ we have

$$W(t) = \psi(B) Z_t.$$

Then from (a) and (1) we have

$$\widehat{W}_t(\infty) = \frac{\phi}{1-\phi} W_t = \frac{\phi}{1-\phi} \psi(B) Z_t$$

$$= \frac{\phi}{1-\phi} \sum_{j=0}^{\infty} \phi^j B^j Z_t = \sum_{j=0}^{\infty} \frac{\phi}{1-\phi} \phi^j B^j Z_t$$

and in view of (2),

$$= \sum_{j=0}^{\infty} \frac{1}{1-\phi}\phi^{j+1} B^j Z_t = \sum_{j=0}^{\infty} \tilde{\psi}_j B^j Z_t$$
$$= \tilde{\psi}(B) Z_t,$$

which is (4), as was to be shown.

(c) As hinted, we expand $(I-B)\tilde{\psi}(B)$,

$$(I-B)\tilde{\psi}(B) = \sum_{j=0}^{\infty} \tilde{\psi}_j B^j - \sum_{j=0}^{\infty} \tilde{\psi}_j B^{j+1}$$

$$\underbrace{=}_{k=j+1} \sum_{j=0}^{\infty} \tilde{\psi}_j B^j - \sum_{k=1}^{\infty} \tilde{\psi}_{k-1} B^k$$

$$= \tilde{\psi}_0 + \sum_{j=1}^{\infty} \tilde{\psi}_j B^j - \sum_{k=1}^{\infty} \tilde{\psi}_{k-1} B^k$$

$$\underbrace{=}_{k=j} \tilde{\psi}_0 + \sum_{j=1}^{\infty} \left[\tilde{\psi}_j - \tilde{\psi}_{j-1}\right] B^j$$

Here we have

$$\tilde{\psi}_j - \tilde{\psi}_{j-1} = \sum_{l=j+1}^{\infty} \phi^l - \sum_{l=j}^{\infty} \phi^l = -\phi^j.$$

Hence

$$\tilde{\psi}_0 + \sum_{j=1}^{\infty} \tilde{\psi}_j B^j - \sum_{j=1}^{\infty} \tilde{\psi}_{j-1} B^j = \tilde{\psi}_0 - \sum_{j=1}^{\infty} \phi^j B^j = \tilde{\psi}_0 - \sum_{j=0}^{\infty} \phi^j B^j + \phi^0.$$

Now, by our definitions

$$\tilde{\psi}_0 + \phi^0 = \sum_{l=1}^{\infty} \phi^l + \phi^0 = \sum_{l=0}^{\infty} \phi^l = \frac{1}{1-\phi} = \psi(1).$$

Hence we have found that

$$(I-B)\tilde{\psi}(B) = \psi(1) - \psi(B).$$

In other words

$$\psi(1) - (I-B)\tilde{\psi}(B) = \psi(1) - \psi(1) + \psi(B) = \psi(B),$$

which is (6) as was claimed.

(d) We have by (6) that

$$W(t) = \psi(B)Z_t = \psi(1)Z_t - (I-B)\tilde{\psi}(B)Z_t = \psi(1)Z_t + B\tilde{\psi}(B)Z_t - \tilde{\psi}(B)Z_t,$$

and by (4)

$$\widehat{W}_t(\infty) = \tilde{\psi}(B)Z_t,$$

which we insert to get

$$\begin{aligned} &= \psi(1)Z_t + B\widehat{W}_t(\infty) - \widehat{W}_t(\infty) \\ &= \psi(1)Z_t + \widehat{W}_{t-1}(\infty) - \widehat{W}_t(\infty) \\ &= \psi(1)Z_t - (\widehat{W}_t(\infty) - \widehat{W}_{t-1}(\infty)). \end{aligned}$$

In summary,

$$W(t) = \psi(1)Z_t - (\widehat{W}_t(\infty) - \widehat{W}_{t-1}(\infty)).$$

Here we see the AR(1) process as a sum of the white noise term $\psi(1)Z_t$, and a term which corresponds to a change in the accumulated long-term predictions between t and $t-1$.

Problem 5

(a) With $Y_t = (1-B)X_t$ we get from ARIMA(1,1,0)

$$\phi(B)Y_t = (1-\phi)\mu + Z_t,$$

i.e.,

$$Y(t) - \phi Y_{t-1} = (1-\phi)\mu + Z_t$$

$$\Leftrightarrow$$

$$Y(t) - \mu = \phi\,(Y_{t-1} - \mu) + Z_t.$$

(b) If we take $W_t = Y_t - \mu$, we have

$$Y_t = (1-B)X_t = X_t - X_{t-1}$$

or

$$X_t = X_{t-1} + \mu + W_t$$

When we iterate this we get

$$X_{t+n} = X_{t+n-1} + \mu + W_{t+n} = (X_{t+n-2} + \mu + W_{t+n-1}) + \mu + W_{t+n}$$

$$\vdots$$

$$= X_t + n\mu + \sum_{l=1}^{n} W_{t+l}.$$

as was to be shown.

(c) From (b) we get

$$E\left[X_{t+n} - n\mu \mid \mathbf{W}_t\right] = E\left[X_t + \sum_{l=1}^{n} W_{t+l} \mid \mathbf{W}_t\right] = X_t + E\left[\sum_{l=1}^{n} W_{t+l} \mid \mathbf{W}_t\right]$$

as X_t is a function of the random variables in $\mathbf{W}_t$. Hence

$$\lim_{n\to\infty} E\left[X_{t+n} - n\mu \mid \mathbf{W}_t\right] = X_t + \lim_{n\to\infty} E\left[\sum_{l=1}^{n} W_{t+l} \mid \mathbf{W}_t\right].$$

Now, since $W_t = Y_t - \mu$ and $Y(t) - \mu = \phi\left(Y_{t-1} - \mu\right) + Z_t$ we get

$$W_t = \phi W_{t-1} + Z_t.$$

and then by (1) we get

$$\lim_{n\to\infty} E\left[X_{t+n} - n\mu \mid \mathbf{W}_t\right] = X_t + \frac{\phi}{1-\phi} W_t = X_t + \frac{\phi}{1-\phi}\left(Y_t - \mu\right),$$

which is (8), as claimed.

(d) We start with

$$X_t = X_{t-1} + \mu + W_t$$

and note that again $W(t) = \psi(B)Z_t$, or $W(t) = \sum_{j=0}^{\infty} \phi^j B^j Z_t$ as in Problem 4 above. Thus by (6)

$$\begin{aligned} X_t = X_{t-1} + \mu + \psi(B)Z_t &= X_{t-1} + \mu + \psi(1)Z_t - (I - B)\tilde{\psi}(B)Z_t \\ &= X_{t-1} + \mu + \psi(1)Z_t + B\tilde{\psi}(B)Z_t - \tilde{\psi}(B)Z_t \end{aligned} \tag{16}$$

where $B\tilde{\psi}(B)Z_t = \tilde{\psi}(B)Z_{t-1}$. But recalling (4) we have that

$$\widehat{W}_t(\infty) = \tilde{\psi}(B)Z_t, \widehat{W}_{t-1}(\infty) = \tilde{\psi}(B)Z_{t-1}$$

and that by (1)

$$\widehat{W}_t(\infty) = \frac{\phi}{1-\phi} W_t, \widehat{W}_{t-1}(\infty) = \frac{\phi}{1-\phi} W_{t-1}$$

Thus we get in (16) that

$$X_t = X_{t-1} + \mu + \psi(1)Z_t + \frac{\phi}{1-\phi} W_{t-1} - \frac{\phi}{1-\phi} W_t$$

which can be rearranged as

$$X_t + \frac{\phi}{1-\phi} W_t = X_{t-1} + \frac{\phi}{1-\phi} W_{t-1} + \mu + \psi(1)Z_t.$$

By our definition of BN_t and as $W_t = Y_t - \mu$, we have thus that

$$BN_t = BN_{t-1} + \mu + \psi(1)Z_t,$$

as was claimed in (9).

(d) We have established that if X_t in an ARIMA(1,1,0) - process, then

$$X_t = BN_t - \frac{\phi}{1-\phi}\left(Y_t - \mu\right), \tag{17}$$

where BN_t is a random walk

$$BN_t = BN_{t-1} + \mu + \psi(1)Z_t.$$

In other words, an ARIMA(1,1,0) - process is decomposed as a sum of a stochastic trend BN_t and a transitory term $\frac{\phi}{1-\phi}\left(Y_t - \mu\right)$. The trend and the disturbance terms are correlated. This is known as the *Beveridge-Nelson decomposition*[2] of ARIMA(1,1,0). The decomposition holds more generally for ARIMA(p,r,q) - processes with $r = 1, 2$ (the decomposition is (?) so far not known or disproved for $r > 2$).

[2]S. Beveridge and C. Nelson: A new approach to decomposition of economic times series into permanent and transitory components with particular attention to the measurement of the business cycle. *Journal of Monetary Economics*, 1981, vol. 7, pp. 151– 174.
There is is a fairly extensive literature on the topic.