SF2943 Time Series Analysis: Lecture 5

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In this lecture, we will introduce ARMA(p, q) processes for general $p, q \in \mathbb{N}$. This class of time series models is vital since it can "almost" approximate any stationary time series in the following sense: For any acvf $\gamma : \mathbb{Z} \to \mathbb{R}$ (i.e., even and nonnegative definition function γ) and for any $k \in \mathbb{N}$, there exists an ARMA process $\{X_t\}$ for some p, q such that its acvf, γ_X, satisfies $\gamma_X(h) = \gamma(h)$ for all $h = 0, 1, \cdots, k$.

1 ARMA(p, q) Process

Definition 1.1 *A time series $\{X_t\}$ is an ARMA(p, q) process if $\{X_t\}$ is stationary and*

$$\underbrace{X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}}_{p\text{-th order autoregressive part}} = \underbrace{Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}}_{q\text{-th order moving average part}} \text{ for all } t, \tag{1.1}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and the polynomials

$$\text{AR polynomial: } \phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$$

and

$$\text{MA polynomial: } \theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$$

have no common factors.

Remark 1.2 *Recall that when we define ARMA$(1, 1)$ process, we require $\phi + \theta \neq 0$. This is equivalent to asking the AR polynomial $\phi(z) = 1 - \phi z$ and the MA polynomial $\theta(z) = 1 + \theta z$ have no common factors.*

Remark 1.3 *When the AR polynomial $\phi(z) \equiv 1$, we have a moving process of order q. When the MA polynomial $\theta(z) \equiv 1$, we have an autoregressive process of order q.*

Remark 1.4 *If we use the backward shift operator notation B, then (1.1) is equivalent to*

$$\phi(B)X_t = \theta(B)Z_t \qquad \text{for all } t \tag{1.2}$$

with $\phi(z)$ is the AR polynomial and $\theta(z)$ is the MA polynomial.

Just like the case of ARMA$(1,1)$ processes, we would like to understand what kinds of $\phi(z)$ and/or $\theta(z)$ lead to the existence, uniqueness, causality, as well as invertibility. Before the discussion, let us recall the results we have for an ARMA$(1,1)$ process

$$\phi(B)X_t = \theta(B)Z_t \qquad \text{for all } t$$

with $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$.

ARMA$(1,1)$ process:

1. (Existence & Uniqueness) If $|\phi| \neq 1$, then there exists an unique stationary process satisfying (1.2).
2. If $|\phi| = 1$, then there does not exist any stationary process satisfying (1.2).
3. (Causality) The ARMA$(1,1)$ process $\{X_t\}$ is causal if and only if $|\phi| < 1$.
4. (Invertibility) The ARMA$(1,1)$ process $\{X_t\}$ is invertible if and only if $|\theta| < 1$.

Observe that $\phi(z) = 1 - \phi z = 0$ if and only if $z = 1/\phi$. Therefore, the condition $|\phi| \neq 1$ is equivalent to saying that $\phi(z) = 1 - \phi z$ has no roots on the (complex) unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Namely, $\phi(z) \neq 0$ for any $z \in \mathbb{C}$ with $|z| = 1$. Similarly, the condition $|\phi| < 1$ is equivalent to saying that the AR polynomial $\phi(z) = 1 - \phi z$ has no roots inside the unit circle. The condition $|\theta| < 1$ is equivalent to saying that the MA polynomial $\theta(z) = 1 + \theta z$ has no roots inside the unit circle. It turns out that that we have analogous results for ARMA(p,q) processes.

Theorem 1.5 (Existence & Uniqueness) *There exists an ARMA(p,q) process with polynomials $\phi(z)$ and $\theta(z)$ if $\phi(z)$ has no roots on the unit circle, i.e., $\phi(z) \neq 0$ for any $z \in \mathbb{C}$ with $|z| = 1$.*

Proof. We will only provide a sketch of the proof. The main tool comes from a result in complex analysis, which ensures that under the assumption $\phi(z) \neq 0$ for any $z \in \mathbb{C}$ with $|z| = 1$, there exists a $\delta > 0$ such that the inverse function of ϕ

$$\chi(z) = \frac{1}{\phi(z)}$$

is well-defined on $\{z \in \mathbb{C} : 1 - \delta < |z| < 1 + \delta\}$. Moreover, $\chi(z)$ is analytic on $\{z \in \mathbb{C} : 1 - \delta < |z| < 1 + \delta\}$ and can be written as

$$\chi(z) = \sum_{-\infty}^{\infty} \chi_j z^j$$

with

$$\sum_{-\infty}^{\infty} |\chi_j| < \infty$$

for any $1 - \delta < |z| < 1 + \delta$. This suggests that $\chi(B)$ is a well-defined linear filter so that we can apply it to (1.2) to find

$$X_t = \chi(B)\theta(B)Z_t.$$

■

Theorem 1.6 (Causality) *$\{X_t\}$ is causal if and only if $\phi(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| \leq 1$.*

Proof. We will only provide a sketch of the proof for the "if" direction. The main tool (once again) comes from a result in complex analysis, which ensures that under the assumption $\phi(z) \neq 0$ for any $z \in \mathbb{C}$ with $|z| \leq 1$, there exists a $\delta > 0$ such that the inverse function of ϕ

$$\chi(z) = \frac{1}{\phi(z)}$$

is well-defined on $\{z \in \mathbb{C} : |z| < 1 + \delta\}$. Moreover, $\chi(z)$ can be written as a power series

$$\chi(z) = \sum_{j=0}^{\infty} \chi_j z^j$$

with

$$\sum_{j=0}^{\infty} |\chi_j| < \infty$$

for any $|z| < 1 + \delta$. This suggests that $\chi(B)$ is a well-defined linear filter so that we can apply it to (1.2) to find

$$X_t = \chi(B)\theta(B)Z_t = \left(\sum_{j=0}^{\infty} \chi_j B^j\right)(I + \theta B)Z_t.$$

Since the orders of the B terms are all nonnegative, this means that the time points of Z's are included in $\{s, s \leq t\}$. Thus, $\{X_t\}$ is causal. ■

Theorem 1.7 (Invertibility) *$\{X_t\}$ is invertible if and only if $\theta(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| \leq 1$.*

We summarize the results for ARMA(p, q) processes in the following:

ARMA(p, q) process:

1. (Existence & Uniqueness) If $\phi(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = 1$, then there exists an unique stationary process satisfying (1.2).

2. If $\phi(z) = 0$ for some $z \in \mathbb{C}$ with $|z| = 1$, then there does not exist any stationary process satisfying (1.2).

3. (Causality) The ARMA(p, q) process $\{X_t\}$ is causal if and only if $\phi(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| \leq 1$.

4. (Invertibility) The ARMA$(1, 1)$ process $\{X_t\}$ is invertible if and only if $\theta(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| \leq 1$.

Below we will show how to apply these theorems to analyze various time series models.

Example 1.1 *Consider an ARMA$(2,1)$ process $\{X_t\}$ defined by*

$$X_t - 0.75X_{t-1} + 0.5625X_{t-2} = Z_t + 1.25Z_{t-1},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$.

The process exists and is unique since its AR polynomial $\phi(z) = 1 - 0.75z + 0.5625z^2 \neq 0$ for any $z \in \mathbb{C}$ with $|z| = 1$. Indeed, by simple calculation one can find that $\phi(z) = 0$ if and only if $9z^2 - 12z + 16 = 0$, which has two roots

$$z_1 = \frac{2(1 + i\sqrt{3})}{3} \text{ and } z_2 = \frac{2(1 - i\sqrt{3})}{3}.$$

And $|z_1| = |z_2| = 4/3 \neq 1$. Moreover, because these two roots are greater than 1, this implies that $\{X_t\}$ is causal.

As for the invertibility, we observe there is only one root for the MA polynomial $\theta(z) = 1 + 1.25z$ which is $z = -0.8$. Since $|-0.8| < 1$, we conclude that $\{X_t\}$ is noninvertible.

Example 1.2 *Consider an AR(2) process $\{X_t\}$ defined by*

$$X_t = 0.7X_{t-1} - 0.1X_{t-2} + Z_t,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$.

The process exists and is unique since its AR polynomial $\phi(z) = 1 - 0.7z + 0.1z^2 \neq 0$ for any $z \in \mathbb{C}$ with $|z| = 1$. Indeed, by simple calculation one can find that $\phi(z) = 0$ if and only if $z^2 - 7z + 10 = (z-5)(z-2) = 0$, which has two roots 2 and 5. Both are not equal to 1. Moreover, because these two roots are greater than 1, this implies that $\{X_t\}$ is causal.

As for the invertibility, we observe that there is no root for the MA polynomial $\theta(z) \equiv 1$ within the unit circle. Thus, we conclude that $\{X_t\}$ is invertible. It is also easy to see the invertibility from rewriting the equation and find $Z_t = X_t - 0.7X_{t-1} + 0.1X_{t-2}$.

2 Acvf for causal ARMA(p,q) process

For a causal ARMA(p,q) process $\{X_t\}$, since it is ARMA(p,q) process, it satisfies

$$\phi(B)X_t = \theta(B)Z_t.$$

Due to the causality, it can also be represented as

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j},$$

where

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}.$$

In particular, we know that $\{X_t\}$ is a linear process (which comes from applying a linear filter $\psi(B)$ to white noise $\{Z_t\}$). Therefore, to find its acvf we can apply a formula we mentioned in lecture 3

$$\gamma_X(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

as long as we figure out the values of $\{\psi_j\}$.

Example 2.1 *Consider a MA(q) process $\{X_t\}$.*

In this case, $\phi(z) \equiv 1$ and $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$. This implies that

$$\psi(z) = \frac{\theta(z)}{\phi(z)} = 1 + \theta_1 z + \cdots + \theta_q z^q$$

and

$$\psi_j = \begin{cases} 1, & \text{for } j = 0 \\ \theta_j, & \text{for } j = 1, \ldots, q \\ 0, & \text{for} j > q \end{cases}.$$

Thus, using the formula, we find

$$\gamma_X(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}, & \text{for } |h| \le q \\ 0, & \text{for } |h| > q \end{cases}.$$

Remark 2.1 *$\gamma_X(h) = 0$ for $|h| > q$ is consistent with the fact that a MA(q) process is q-correlated.*

Example 2.2 *Consider an ARMA$(1,1)$ process $\{X_t\}$ with $|\phi| < 1$.*

In this case, $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$. This implies that

$$\begin{aligned}\psi(z) = \frac{\theta(z)}{\phi(z)} = \frac{1+\theta z}{1-\phi z} &= (1+\theta z)\sum_{j=0}^{\infty} \phi^j z^j = \sum_{j=0}^{\infty} \phi^j z^j + \theta \sum_{j=0}^{\infty} \phi^j z^{j+1} \\ &= \sum_{j=0}^{\infty} \phi^j z^j + \theta \sum_{j=1}^{\infty} \phi^{j-1} z^j = 1 + \sum_{j=1}^{\infty} \phi^{j-1}(\theta+\phi) z^j.\end{aligned}$$

and

$$\psi_j = \begin{cases} 1, & \text{for } j = 0 \\ \phi^{j-1}(\theta+\phi), & \text{for } j > 0 \end{cases}.$$

Observe that for any $j \ge 2$, $\psi_j = \phi\psi_{j-1}$. Thus, using the formula, we find

$$\gamma_X(0) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma^2 \left(1 + \sum_{j=1}^{\infty} (\phi^{j-1}(\theta+\phi))^2\right) = \sigma^2 \left(1 + \frac{(\theta+\phi)^2}{1-\phi^2}\right)$$

and

$$
\begin{aligned}
\gamma_X(1) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+1} = \sigma^2 \left(\psi_0 \psi_1 + \sum_{j=1}^{\infty} \psi_j \psi_{j+1} \right) \\
&= \sigma^2 \left((\theta + \phi) + \sum_{j=1}^{\infty} [\phi^{j-1}(\theta + \phi)\phi^j(\theta + \phi)] \right) \\
&= \sigma^2 \left((\theta + \phi) + (\theta + \phi)^2 \sum_{j=1}^{\infty} \phi^{2j-1} \right) = \sigma^2 \left(\theta + \phi + \frac{(\theta + \phi)^2 \phi}{1 - \phi^2} \right)
\end{aligned}
$$

and for any $h \geq 2$

$$
\begin{aligned}
\gamma_X(h) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} = \sigma^2 \sum_{j=0}^{\infty} \psi_j [\phi \psi_{j+h-1}] = \phi \left(\sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h-1} \right) \\
&= \phi \gamma_X(h-1) = \cdots = \phi^h \gamma_X(1).
\end{aligned}
$$