

# On the Neural Tangent Kernel Analysis of Randomly Pruned Wide Neural Networks

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## Abstract

We study the behavior of ultra-wide neural networks when their weights are randomly pruned at the initialization, through the lens of neural tangent kernels (NTKs). We show that for fully-connected neural networks when the network is pruned randomly at the initialization, as the width of each layer grows to infinity, the empirical NTK of the pruned neural network converges to that of the original (unpruned) network with some extra scaling factor. Further, if we apply some appropriate scaling after pruning at the initialization, the empirical NTK of the pruned network converges to the exact NTK of the original network, and we provide a non-asymptotic bound on the approximation error in terms of pruning probability. Moreover, when we apply our result to an unpruned network (i.e., we set the probability of pruning a given weight to be zero), our analysis is optimal up to a logarithmic factor in width compared with the result in (Arora et al., 2019). We conduct experiments to validate our theoretical results. We further test our theory by evaluating random pruning across different architectures via image classification on MNIST and CIFAR-10 and compare its performance with other pruning strategies.

## 1. Introduction

Neural network pruning has been receiving a lot attention historically since a sparse network can significantly reduce the computational costs when performing inference than its dense counterpart (LeCun et al., 1990; Li et al., 2016). However, such gain seems hard to be directly transferred to the *training* phase until the discovery of the lottery ticket

hypothesis (LTH) (Frankle & Carbin, 2018). LTH states that there exists a sparse subnetwork inside a dense network at the initialization stage such that when trained in isolation, it can achieve almost the same (or sometimes slightly better) accuracy than the original dense network.

A recent line of theoretical work (Malach et al., 2020; Pensia et al., 2020; Sreenivasan et al., 2021) proves that such a subnetwork indeed exists even without further training provided that the network is sufficiently overparameterized but they also pointed out that finding such a subnetwork is computationally hard. Hence, the practical method of finding such a network at the initialization stage is computationally expensive: the current workhorse is still iterative magnitude-based pruning (IMP) with rewinding which requires multiple rounds of pruning and re-training (Frankle & Carbin, 2018; Frankle et al., 2019). Subsequent work has been making effort in finding good sparse subnetwork at the initialization stage with little or no training (Lee et al., 2018; Wang et al., 2020; Tanaka et al., 2020; Frankle et al., 2020). Nonetheless, these methods suffer a degenerate performance than the lottery ticket found by IMP and yields no theoretical guarantees. Interestingly, even random pruning, as the most naive option to obtain sparsity, has been observed to be empirically competitive for sparse training in some settings (Su et al., 2020; Frankle et al., 2020; Liu et al., 2022).

Over the past few years, people have observed that running (stochastic) gradient descent on overparameterized neural network can rapidly drive the training error toward zero while the weights of the network has small change after training, and further, under some conditions, those networks are able to generalize (Du et al., 2018; Allen-Zhu et al., 2018; 2019; Du et al., 2019; Arora et al., 2019; Ji & Telgarsky, 2019; Lee et al., 2019). The extra wideness of the networks plays an essential role in their proof. Since then, people have made tremendous development on understanding ultra-wide neural network based on neural tangent kernel (NTK) (Jacot et al., 2018). It can be shown that once the width of a neural network approaches infinity, training will incur increasingly smaller changes and the neural network

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$f$  starts to behave like a simple kernel defined as:

$$\text{ker}(\mathbf{x}, \mathbf{x}') = \left\langle \frac{\partial f(\boldsymbol{\theta}, \mathbf{x})}{\partial \boldsymbol{\theta}}, \frac{\partial f(\boldsymbol{\theta}, \mathbf{x}')}{\partial \boldsymbol{\theta}} \right\rangle$$

where  $\boldsymbol{\theta}$  is the parameters of the neural network.

Since random pruning is the cheapest avenue towards sparsity, how good a random pruned subnetwork could actually be, compared to the original unpruned network, even in a simplified analysis framework, say the network is very wide? Pursuing this answer has the appeal to establish a “lower bound”-type understanding on the effectiveness of sparse neural network, compared to other sophisticated pruning options; it is also motivated by the recent observation that random sparsity becomes particularly effective for training if the original network is wide and deep (Liu et al., 2022). To take the first step, we ask the following question:

*Can a randomly pruned neural network at the random initialization be put onto the NTK regime if the neural network is wide enough?*

If the answer is yes, then we might hope for formalized results suggesting that the pruned and unpruned neural network can achieve similar fast *convergence* to zero training error and yield similar *generalization* after training since they are both close to the neural tangent kernel (Du et al., 2018; Allen-Zhu et al., 2018; 2019; Du et al., 2019; Arora et al., 2019; Ji & Telgarsky, 2019; Lee et al., 2019). For practitioners, this perhaps surprising result is likely to bring random pruning back to the spotlight of model compression and efficient training, in our era when neural networks are practically scaled a lot wider and deeper, say those gigantic “foundational models” (Bommasani et al., 2021).

This work gives an **affirmative** answer to this question. We shall, however, point out that while the *asymptotic* analysis can be straightforward, it is *misleading* to believe the original *non-asymptotic* proof of (Arora et al., 2019) can be *trivially* generalized to the pruned case considering all we only did is multiplying weights by random masks. We summarize the main results below, and state the **non-trivial technical barriers** as well as solutions in Section 4.

### 1.1. Contributions

We leverage the recent development on NTK and show that if a neural network is wide enough, then there indeed exists a sparse subnetwork at the initialization such that when trained with gradient descent with an appropriate learning rate, the sparse subnetwork possess the same convergence property as the original full dense network, and it can be found by using simple random pruning.

#### Our Contribution 1: Asymptotic Convergence.

**Theorem 1.1** (The NTK of randomly pruned networks). *Consider an  $L$ -hidden-layer fully connected neural network*

*with ReLU activation. Suppose we prune the weights in this neural network except the input layer i.i.d. with probability  $1 - \alpha$  at the initialization. Then, as the width of each layer goes to infinity sequentially,*

$$\lim_{d_1, d_2, \dots, d_L \rightarrow \infty} \tilde{\Theta}(\mathbf{x}, \mathbf{x}') = \alpha^L \Theta_\infty(\mathbf{x}, \mathbf{x}')$$

The theorem suggests that if we perform random pruning at the initialization, as the width of the neural network approaches infinity, the empirical neural tangent kernel of the pruned network indeed approaches the neural tangent kernel up to some multiplicative scaling factor that depends on the pruning probability. We note that despite the scaling, the actual function we learned is the same. We give explanation in Section 3. On the other hand, this scaling factor can be removed simply by rescaling the weights according to the pruning probability which we use in the next theorem.

**Our Contribution 2: Non-Asymptotic Bound.** Given the asymptotic result, once we fix pruning probabilities it would be great to be informed about *how wide our network needs to be* in order to approach the neural tangent kernel regime.

**Theorem 1.2** (Non-asymptotic Bound of Pruned Network’s NTK). *Consider an  $L$ -hidden-layer fully connected neural network with all the weights initialized with i.i.d. standard Gaussian distribution. Suppose all the weights except the input layer are pruned with probability  $1 - \alpha$  at the initialization and after pruning we rescale the weights by  $1/\sqrt{\alpha}$ . Fix  $\epsilon > 0$  and  $\delta \in (0, 1)$ . Suppose we use ReLU activation  $\sigma(z) = \max(0, z)$  and  $d_h \geq \Omega(\frac{1}{\alpha^2} \frac{L^6}{\epsilon^4} \log \frac{L d_{h+1} \sum_{i=1}^h d_i + \sum_{h'=h}^L d_{h'}}{\delta})$ . Then for any inputs  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{d_0}$  such that  $\|\mathbf{x}\|_2 \leq 1$ ,  $\|\mathbf{x}'\|_2 \leq 1$ , with probability at least  $1 - \delta$  we have*

$$\left| \left\langle \frac{\partial f(\boldsymbol{\theta}, \mathbf{x})}{\partial \boldsymbol{\theta}}, \frac{\partial f(\boldsymbol{\theta}, \mathbf{x}')}{\partial \boldsymbol{\theta}} \right\rangle - \Theta_\infty(\mathbf{x}, \mathbf{x}') \right| \leq (L + 1)\epsilon$$

We point out that when we assume each layer has the same width and apply our bound on width to an unpruned network (i.e., we set the probability of pruning a given weight to be zero), our analysis is optimal up to a logarithmic factor in width compared with the result in (Arora et al., 2019).

We validate our theory by comparing the Monte Carlo estimate of NTK value to our theoretically computed NTK value. We further test our theory by evaluating the performance of random pruning on training wide sparse architectures such as fully-connected neural networks and ResNet using MNIST and CIFAR-10 dataset. We also compare random pruning (with and without rescaling) to other pruning techniques. Our result shows that if the architecture is wide enough, random pruning retains much of the performance as the original neural network and can be made as a strong competitor against methods like IMP.

## 1.2. Related Work

### THE LOTTERY TICKET HYPOTHESIS AND BEYOND

The existence of a sparse subnetwork inside a full network was first observed in (Frankle & Carbin, 2018) which is named the lottery ticket hypothesis. However, finding such a subnetwork is expensive: the algorithm the authors proposed is Iterative Magnitude-based Pruning which requires multiple rounds of training and pruning. On the theory side, (Malach et al., 2020) proved that a target network of width  $d$  and depth  $l$  can be indeed approximated by pruning a randomly initialized network that is of a polynomial factor (in  $d, l$ ) wider and twice deeper even without further training. However finding such a network is computationally hard. (Ramanujan et al., 2020) empirically verified this stronger version of LTH. Later, (Pensia et al., 2020) improved the widening factor to a logarithmic bound and (Sreenivasan et al., 2021) proves that with a polylogarithmic widening factor, such a result holds even if the network is binary.

On the practical side, to tame down the computational cost, people have been studying transferring lottery ticket across different dataset (Morcos et al., 2019) and across neural architectures of the same type but different sizes (Chen et al., 2021). As (Chen et al., 2021) showed that on ResNet, it is possible to transfer weights across different layers to make the network deeper while still maintaining comparable performance to IMP. To understand why we can transfer lottery tickets, (Redman et al., 2021) tried to relate IMP to the renormalization group theory in statistical physics.

### SPARSE TRAINING

Since running IMP to find the lottery ticket is expensive, many efforts have been made to develop methods which require little or no retraining. Those methodologies can be divided into two groups: static sparse training, which is trying to locate a fixed sparse mask at the initialization stage, and dynamic sparse training, which adaptively evolves the sparse mask during the training time.

Static sparse training can be based on either *random pruning* and *non-random pruning* at the initialization. As for random pruning, the most naive approach is pruning every layer uniformly with the same pre-defined pruning ratio (Mariet & Sra, 2015; He et al., 2017; Gale et al., 2019). Apart from pre-specified pruning rate, pruning ratio can be varied for different layers such as Erdő-Rényi (Mocanu et al., 2018) and Erdő-Rényi Kernel (Evci et al., 2020a). For non-random pruning at the initialization, those methods usually prune network weights according to some proposed saliency criteria. SNIP (Lee et al., 2018) prunes the network weights according to their sensitivity to loss. Following that, (Wang et al., 2020) proposed GraSP which prunes the weights based on gradient information; (Tanaka et al., 2020)

introduced SynFlow based on synaptic strengths, and (Liu & Zenke, 2020) pruned the weights based on the neural tangent kernel. On the other hand, dynamic sparse training stems from Sparse Evolutionary Training (Mocanu et al., 2018; Liu et al., 2021a) which explores the sparsity pattern in a prune-and-grow scheme according to some criteria (Mocanu et al., 2018; Mostafa & Wang, 2019; Dettmers & Zettlemoyer, 2019; Evci et al., 2020a; Ye et al., 2020; Jayakumar et al., 2021; Liu et al., 2021b).

### NEURAL TANGENT KERNELS

Over the past few years, there is tremendous progress on understanding ultra wide neural networks based on neural tangent kernel (Jacot et al., 2018) either implicitly or explicitly. A series of works (Allen-Zhu et al., 2019; Du et al., 2018; 2019; Zou et al., 2018) have observed that running gradient descent on overparameterized neural network can rapidly reduce the training error toward zero while the weights of the network possess only small changes after training. Extra wideness plays an central role in their proof.

Later, (Jacot et al., 2018; Yang, 2019; Arora et al., 2019) provided both asymptotic and non-asymptotic proofs that as the neural network’s width grows to infinity, the neural network simplifies to a simple tangent kernel regime. Formula for computing such kernels are developed for fully connected neural networks (Lee et al., 2019) and convolutional neural networks (Arora et al., 2019). Fast convergence rate of training loss has been established for such neural networks and, further, under some conditions, these overparameterized networks are able to generalize to unseen data (Du et al., 2018; Allen-Zhu et al., 2018; 2019; Du et al., 2019; Arora et al., 2019; Ji & Telgarsky, 2019; Lee et al., 2019). Overall, the neural tangent kernel regime provides valuable, yet oversimplified, explanation on the neural network’s success (Chizat et al., 2018).

We particularly point out one prior work related to ours (Liu & Zenke, 2020). Their work considers pruning a network at the initialization by ensuring the linearized networks of pruned and unpruned networks are close. However, they did not provide any theoretical guarantee to ensure the two NTKs to be close after training.

## 2. Preliminaries

We use lower case letters to denote scalars and boldface letters and symbols (e.g.  $\mathbf{x}$ ) to denote vectors and matrices. We use  $\odot$  to denote element-wise product and  $\otimes$  to denote the kronecker product.  $\Pi_{\mathbf{x}}$  denote the orthogonal projection onto the vector space generated by  $\mathbf{x}$  and similarly  $\Pi_{\mathbf{A}}$  denote the orthogonal projection onto the column space of  $\mathbf{A}$ . We use  $\text{diag}(\mathbf{x})$  to denote a diagonal matrix where its diagonals are elements from the vector  $\mathbf{x}$ .

## 2.1. Problem Definition

Here we want to study the training dynamics of a sparse sub-network in an ultra-wide neural network. For simplicity, we first apply our analysis on fully-connected neural networks. We denote by  $f(\mathbf{x}) = f(\boldsymbol{\theta}, \mathbf{x})$  the output of the full neural network,  $\tilde{f}(\mathbf{x}) = f(\boldsymbol{\theta} \odot \mathbf{m}, \mathbf{x}) \in \mathbb{R}$  the output of a sparse sub-network obtained by random pruning where  $\boldsymbol{\theta} \in \mathbb{R}^N$  denotes the network parameters,  $\mathbf{m} \in \mathbb{R}^N$  is the sparse mask and  $\mathbf{x} \in \mathbb{R}^d$  is the input. From now on, we distinguish the variables of the original full networks from the sparse sub-networks by adding tilde to the variable symbols. For simplicity, we only assume the network as a single output<sup>1</sup>. We assume that the sparse mask of this sub-network is obtained from sampling each individual weight i.i.d. from a Bernoulli distribution with probability  $\alpha$ . Formally, we adopt the definition of  $L$ -hidden-layer fully connected network from (Arora et al., 2019; Lee et al., 2019): let  $\mathbf{x} \in \mathbb{R}^d$  be the input, and denote  $\tilde{\mathbf{g}}^{(0)}(\mathbf{x}) = \mathbf{x}$  and  $d_0 = d$ . We define an  $L$ -hidden-layer fully connected network recursively:

$$\begin{aligned}\tilde{\mathbf{f}}^{(h)}(\mathbf{x}) &= (\mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}) \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \in \mathbb{R}^{d_h}, \\ \tilde{\mathbf{g}}^{(h)}(\mathbf{x}) &= \sqrt{\frac{c_\sigma}{d_h}} \sigma(\tilde{\mathbf{f}}^{(h)}(\mathbf{x})) \in \mathbb{R}^{d_h}, \quad h = 1, 2, \dots, L\end{aligned}$$

where  $\mathbf{W}^{(h)} \in \mathbb{R}^{d_h \times d_{h-1}}$  is the weight matrix in the  $h$ -th layer,  $\mathbf{m}^{(h)} \in \mathbb{R}^{d_h \times d_{h-1}}$  is the sparse mask for the  $h$ -th layer  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  is a coordinate-wise activation function which we only consider ReLU activation in this work and  $c_\sigma = (\mathbb{E}_{z \sim \mathcal{N}(0,1)} [\sigma(z)^2])^{-1}$ . Under ReLU, simple calculation shows  $c_\sigma = 2$ . The last layer of the network is

$$\tilde{f}(\boldsymbol{\theta}, \mathbf{x}) = \tilde{f}^{(L+1)}(\mathbf{x}) = (\mathbf{W}^{(L+1)} \odot \mathbf{m}^{(L)}) \tilde{\mathbf{g}}^{(L)}(\mathbf{x})$$

where  $\mathbf{W}^{(L+1)} \in \mathbb{R}^{1 \times d_L}$  is the weights in the final layer,  $\mathbf{m} = (\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(L+1)})$  and  $\boldsymbol{\theta} = (\mathbf{W}^{(1)}, \dots, \mathbf{W}^{(L+1)})$  represents all the parameters in the network. All the weights  $\mathbf{W}_{ij}^{(h)}$  are sampled i.i.d. from  $\mathcal{N}(0, 1)$  and the masks  $\mathbf{m}_{ij}^{(h)}$  are sampled i.i.d. from  $\text{Bernoulli}(\alpha)$ .

## 2.2. Review of the Neural Tangent Kernel of the Fully Connected Neural Networks

Recall from (Arora et al., 2019) that for the full neural networks with infinite width, the pre-activations  $\mathbf{f}^{(h)}(\mathbf{x})$  at every hidden layer  $h \in [L]$  has all its coordinates tending to i.i.d. centered Gaussian processes of covariance  $\Sigma^{(h-1)} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  defined recursively as: for  $h \in [L]$ ,

$$\Sigma^{(0)}(\mathbf{x}, \mathbf{x}') = \mathbf{x}^\top \mathbf{x}'$$

$$\boldsymbol{\Lambda}^{(h)}(\mathbf{x}, \mathbf{x}') = \begin{bmatrix} \Sigma^{(h-1)}(\mathbf{x}, \mathbf{x}) & \Sigma^{(h-1)}(\mathbf{x}, \mathbf{x}') \\ \Sigma^{(h-1)}(\mathbf{x}', \mathbf{x}) & \Sigma^{(h-1)}(\mathbf{x}', \mathbf{x}') \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

<sup>1</sup>Without loss of generality, our analysis should be straightforward to extend to the case when the network has multiple outputs.

$$\Sigma^{(h)}(\mathbf{x}, \mathbf{x}') = c_\sigma \mathbb{E}_{(u,v) \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Lambda}^{(h)})} [\sigma(u)\sigma(v)]$$

To give the formula of NTK, we further define a derivative covariance,

$$\dot{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') = c_\sigma \mathbb{E}_{(u,v) \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Lambda}^{(h)})} [\dot{\sigma}(u)\dot{\sigma}(v)]$$

To compute  $\left\langle \frac{\partial f(\boldsymbol{\theta}, \mathbf{x})}{\partial \boldsymbol{\theta}}, \frac{\partial f(\boldsymbol{\theta}, \mathbf{x}')}{\partial \boldsymbol{\theta}} \right\rangle$ , we have

$$\frac{\partial f(\boldsymbol{\theta}, \mathbf{x})}{\partial \mathbf{W}^{(h)}} = \mathbf{b}^{(h)}(\mathbf{x}) \left( \mathbf{g}^{(h-1)}(\mathbf{x}) \right)^\top, \quad h = 1, 2, \dots, L+1$$

where  $\mathbf{b}^{(h)}$  is the gradient back-propagated to layer  $h$ :

$$\mathbf{b}^{(h)}(\mathbf{x}) = \begin{cases} 1 \in \mathbb{R}, & h = L+1 \\ \sqrt{\frac{c_\sigma}{d_h}} \mathbf{D}^{(h)}(\mathbf{x}) (\mathbf{W}^{(h+1)})^\top \mathbf{b}^{(h+1)}(\mathbf{x}) \in \mathbb{R}^{d_h}, & h = 1, \dots, L, \end{cases}$$

and

$$\mathbf{D}^{(h)}(\mathbf{x}) = \text{diag} \left( \dot{\sigma} \left( \mathbf{f}^{(h)}(\mathbf{x}) \right) \right) \in \mathbb{R}^{d_h \times d_h}, \quad h = 1, \dots, L$$

It can be shown that

$$\begin{aligned}\Theta_\infty(\mathbf{x}, \mathbf{x}') &= \lim_{d_1, d_2, \dots, d_L \rightarrow \infty} \Theta(\mathbf{x}, \mathbf{x}') \\ &= \lim_{d_1, d_2, \dots, d_L \rightarrow \infty} \sum_{h=1}^{L+1} \left\langle \frac{\partial f(\boldsymbol{\theta}, \mathbf{x})}{\partial \mathbf{W}^{(h)}}, \frac{\partial f(\boldsymbol{\theta}, \mathbf{x}')}{\partial \mathbf{W}^{(h)}} \right\rangle \\ &= \sum_{h=1}^{L+1} \left( \Sigma^{(h-1)}(\mathbf{x}, \mathbf{x}') \prod_{h'=h}^{L+1} \dot{\Sigma}^{(h')}(\mathbf{x}, \mathbf{x}') \right)\end{aligned}$$

## 3. Asymptotic Result

In this section, we show how to derive the NTK for pruned networks. The proof of our result is simple and straightforward. We give an outline of our analysis in this section and where we did not give out a proof or the detailed computation we defer them to Appendix B.1. Our analysis is similar from the analysis in (Jacot et al., 2018; Arora et al., 2019).

First of all, we compute the gradient of the pruned network. Note that since the weights being pruned are staying at zero always during the training process, the gradient of the pruned network is simply the masked gradient of the unpruned network. Thus, its gradient is given by

$$\begin{aligned}\frac{\partial \tilde{f}(\mathbf{x})}{\partial [\mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}]} &= \frac{\partial \tilde{f}(\mathbf{x})}{\partial \mathbf{W}^{(h)}} \odot \mathbf{m}^{(h)} \\ &= \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \right)^\top \right) \odot \mathbf{m}^{(h)}\end{aligned}$$

where

$$\tilde{\mathbf{b}}^{(L+1)}(\mathbf{x}) = \begin{cases} 1 \in \mathbb{R}, & h = L+1 \\ \sqrt{\frac{c_\sigma}{d_h}} \tilde{\mathbf{D}}^{(h)}(\mathbf{x}) (\mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)})^\top \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}) \in \mathbb{R}^{d_h}, & h = 1, \dots, L \end{cases}$$



and

$$\tilde{\mathbf{D}}^{(h)}(\mathbf{x}) = \text{diag} \left( \dot{\sigma} \left( \tilde{\mathbf{f}}^{(h)}(\mathbf{x}) \right) \right) \in \mathbb{R}^{d_h \times d_h}, \quad h = 1, \dots, L$$

Now, it can be shown that

$$\begin{aligned} & \left\langle \frac{\partial \tilde{\mathbf{f}}(\mathbf{x})}{\partial [\mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}]}, \frac{\partial \tilde{\mathbf{f}}(\mathbf{x}')}{\partial [\mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}]} \right\rangle \\ &= \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \right)^\top \mathbf{G}^{(h-1)} \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \end{aligned} \quad (1)$$

where  $\mathbf{G}^{(h-1)}$  is a diagonal matrix and  $\mathbf{G}_{ii}^{(h-1)} = \left\langle \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \odot \mathbf{m}_i^{(h)}, \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}') \odot \mathbf{m}_i^{(h)} \right\rangle$ . Notice that as  $d_{h-1} \rightarrow \infty$ ,  $\mathbf{G}_{ii}^{(h-1)} \rightarrow \alpha \tilde{\Sigma}^{(h-1)}(\mathbf{x}, \mathbf{x}')$ .

Like the original (unpruned) case, the NTK depends on analyzing both the forward propagation and the backward propagation of the pruned neural network. We show the results in the following two simple lemmas.

**Lemma 3.1.** *Suppose a fully-connected neural network uses ReLU as its activation and  $d_1, d_2, \dots, d_L \rightarrow \infty$  sequentially, then*

$$\tilde{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') = \alpha^{h-1} \Sigma^{(h)}(\mathbf{x}, \mathbf{x}')$$

for  $h = 1, 2, \dots, L$ .

**Lemma 3.2.**

$$\begin{aligned} & \lim_{d_1, \dots, d_L \rightarrow \infty} \left\langle \tilde{\mathbf{b}}^{(h)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \right\rangle \\ &= \alpha^{L+1-h} \lim_{d_1, \dots, d_L \rightarrow \infty} \left\langle \mathbf{b}^{(h)}(\mathbf{x}), \mathbf{b}^{(h)}(\mathbf{x}') \right\rangle \\ &= \alpha^{L+1-h} \prod_{h'=h}^L \dot{\Sigma}^{(h')}(\mathbf{x}, \mathbf{x}') \end{aligned}$$

The proof of Lemma 3.2 assumes that we use an independent Gaussian copy in the backward propagation which we introduce and rigorously justify in the next section.

Combining the two lemmas provided above with Equation 1, we can prove the following theorem.

**Theorem 1.1** (The NTK of randomly pruned networks, Restated). *Consider an  $L$ -hidden-layer fully connected neural network with ReLU activation. Suppose we prune the weights in this neural network except the input layer i.i.d. with probability  $1 - \alpha$  at the initialization. Then, as the width of each layer goes to infinity sequentially,*

$$\lim_{d_1, d_2, \dots, d_L \rightarrow \infty} \tilde{\Theta}(\mathbf{x}, \mathbf{x}') = \alpha^L \Theta_\infty(\mathbf{x}, \mathbf{x}')$$

A special note is that our analysis breaks when we prune the input layer since the input dimension is fixed. The NTK

of the full network depends on  $\Sigma^{(0)}(\mathbf{x}, \mathbf{x}') = \mathbf{x}^\top \mathbf{x}'$ . If we prune the input layer then  $\tilde{\Sigma}^{(0)}(\mathbf{x}, \mathbf{x}') = (\mathbf{m} \odot \mathbf{x})^\top (\mathbf{m} \odot \mathbf{x}')$  which becomes random. In this case, it seems hard to relate  $\tilde{\Sigma}^{(1)}(\mathbf{x}, \mathbf{x}')$  to  $\Sigma^{(1)}(\mathbf{x}, \mathbf{x}')$ .

Another note we make is that despite of the scaling factor in front of the NTK, the actual function we learned is the *same*. This can be seen from the fact that if the training dataset is given by  $(\mathbf{X}, \mathbf{y})$  and  $\mathbf{H}_{ij} = \Theta(\mathbf{x}_i, \mathbf{x}_j)$ , the function induced by NTK is

$$f_{\text{ntk}}(\mathbf{x}) = (\ker(\mathbf{x}, \mathbf{X}))^\top \mathbf{H}^{-1} \mathbf{y}$$

Any scaling factor in front of the NTK is cancelled.

### 3.1. Going from Asymptotic to Non-asymptotic Analysis

Before we give our non-asymptotic analysis, we note that our asymptotic result is obtained from taking the sequential limits of all the hidden layers which is a somewhat limited notion of limits. Sequential limit assumes when we analyze with a given layer, all the previous layers are already at the limit. Non-asymptotic analysis, on the other hand, considers using a large but finite amount of samples to get arbitrarily close to (but not exactly at) the limit. Thus, we need to justify that we are indeed able to approximate the limit using a large but finite amount of samples. In mathematical language, this is the same as justifying taking the limit outside of integration. We justify this by leveraging the tools in measure-theoretic probability theory. Despite its importance, this part will not affect the overall understanding of our result, and hence we refer the readers to Appendix B.2.

## 4. Non-Asymptotic Analysis

Different from previous sections, from this point, since we are only talking about the pruned network, there is no longer ambiguity in distinguishing pruned and unpruned networks. For notation ease, we remove the tilde above all the symbols referring to the quantity in the pruned network. We give an outline of our non-asymptotic analysis and highlight the differences and challenges in this section. From a high level, our proof follows the proof outline of (Arora et al., 2019). We give a complete treatment in Appendix C.

In the non-asymptotic analysis, it is convenient to keep the output at each layer to have unit variance. Hence, for the ease of analysis, a difference between this section and previous section is that now the mask is rescaled with  $\frac{1}{\sqrt{\alpha}}$  to remove the extra scaling in front of the NTK.

### 4.1. Technical Barriers

We list the main technical challenges behind our non-asymptotic analysis on pruned neural networks. First of

all, one may wonder since all we only did is multiplying weights by random masks, the resulting joint distribution of mask and weights still exhibit subgaussian concentration and, thus, the original proof of (Arora et al., 2019) can be trivially generalized. Unfortunately, **this is not the case**. The key argument in the non-asymptotic analysis in (Arora et al., 2019) is its Lemma E.7, which justifies the key trick of using a fresh copy of Gaussian in the backward pass. To prove this lemma, the authors used a key special property of the Gaussian random vectors: given an i.i.d. Gaussian random vector  $\mathbf{w}$  and two fixed vectors  $\mathbf{x}, \mathbf{y}$ , if  $\mathbf{x}^\top \mathbf{y} = 0$ , then  $\mathbf{w}^\top \mathbf{x}$  and  $\mathbf{w}^\top \mathbf{y}$  are independent. This property usually cannot hold for general random variables beyond Gaussian.

Therefore, how can we derive a similar result for the pruned network? If we consider the joint distribution of weights and masks, it is not even spherically symmetric. Therefore, this argument cannot be trivially applied despite that the joint distribution has subgaussian concentration. We make the following key observation: consider the  $i$ -th neuron at layer  $h$ , the input to this neuron is given by  $(\mathbf{w}_i^{(h)} \odot \mathbf{m}_i^{(h)})^\top \mathbf{g}^{(h-1)}$ . We can separate the masks and weights and consider the mask to be part of the output from the previous layer, i.e.,  $(\mathbf{w}_i^{(h)} \odot \mathbf{m}_i^{(h)})^\top \mathbf{g}^{(h-1)} = (\mathbf{w}_i^{(h)})^\top (\mathbf{m}_i^{(h)} \odot \mathbf{g}^{(h-1)})$ . Further, since the mask and weights are independent, conditioning on the mask will not affect the distribution of the weight. Thus, the fresh Gaussian copy trick can be applied when we condition on the realization of masks.

The solution introduces further challenges:

- From Equation (1), we need to analyze the covariance between  $\mathbf{f}_i^{(h)}(\mathbf{x})$  and  $\mathbf{f}_i^{(h)}(\mathbf{x}')$  conditioned on the mask and all previous layers, which is given by  $\langle \mathbf{g}^{(h-1)}(\mathbf{x}) \odot \mathbf{m}_i^{(h)}, \mathbf{g}^{(h-1)}(\mathbf{x}') \odot \mathbf{m}_i^{(h)} \rangle$ . How does the inclusion of mask affect the concentration?
- How does the fresh Gaussian copy trick play out under the presence of mask?

To resolve the first challenge, we derive a new concentration result by exploring the property of the Bernoulli distribution in Lemma 4.1. For the second challenge, we give a detailed outline in Section 4.3.

## 4.2. Analyzing the Forward Propagation

**Lemma 4.1.** *Let  $X$  be a scaled Bernoulli random variable with probability  $\alpha$  being  $\frac{1}{\alpha}$  and zero otherwise, and  $Y$  be an independent subgamma random variable with parameter  $(\sigma^2, c)$  where  $\sigma^2 = O(1), c = O(1)$ , and  $\mathbb{E}[Y] = O(1)$ , then  $XY$  is subgamma with parameter  $(O((\frac{1-\alpha}{\alpha})^2 + 1), O(\frac{1-\alpha}{\alpha} + 1))$ .*

Applying the concentration result, we have if  $d_h \gtrsim$

$\left(\left(\frac{1-\alpha}{\alpha}\right)^2 + 1\right) \frac{\log \frac{8d_{h+1}}{\delta}}{\epsilon^2}$  and  $\epsilon \leq c_2$  then

$$\left| \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) - \mathbb{E} \left[ \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) \right] \right| \leq \epsilon$$

holds for a single  $i \in [d_{h+1}]$ . Taking a union bound over  $i$  can make this holds for all  $i \in [d_{h+1}]$ . Note that this only guarantees that our empirical estimate of the covariance will stay close to its mean, which is not exactly what we want. We want to say the distance between this empirical estimate and  $\Sigma(\mathbf{x}, \mathbf{x}')$  is close. In order to establish such a result, we need to further make sure the distance between the above expectation and  $\Sigma(\mathbf{x}, \mathbf{x}')$  are close. To do this, we open the black box of the proof of Corollary 16 in (Daniely et al., 2016). Finally, by triangle inequality, combining the two upper bounds will give us the following theorem.

**Theorem 4.2.** *Let  $\sigma(z) = \max(0, z)$  and  $[\mathbf{W}^{(h)}]_{ij} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$ ,  $[\mathbf{m}^{(h)}]_{ij} \stackrel{i.i.d.}{\sim} \frac{1}{\sqrt{\alpha}} \text{Bernoulli}(\alpha)$ ,  $\forall h \in [L]$ ,  $i \in [d_{h+1}]$ ,  $j \in [d_h]$ , there exist constants  $c_1, c_2$  such that if  $\left(\left(\frac{1-\alpha}{\alpha}\right)^2 + 1\right) \frac{L^2 \log \frac{8d_{h+1}L}{\delta}}{\epsilon^2} \lesssim d_h$ ,  $\forall h \in \{1, 2, \dots, L\}$  and  $\epsilon \leq \min(c_2, \frac{1}{L})$  then for any fixed  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{d_0}$ ,  $\|\mathbf{x}\|_2, \|\mathbf{x}'\|_2 \leq 1$ , we have with probability  $1 - \delta$ ,  $\forall 0 \leq h \leq L$ ,  $\forall i \in [d_{h+1}]$ ,  $\forall (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \in \{(\mathbf{x}, \mathbf{x}), (\mathbf{x}, \mathbf{x}'), (\mathbf{x}', \mathbf{x}')\}$ ,*

$$\left| \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) - \Sigma^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| \leq \epsilon$$

## 4.3. Analyzing the Backward Propagation

The main goal of this section is to show  $\langle \mathbf{b}^{(h)}(\mathbf{x}), \mathbf{b}^{(h)}(\mathbf{x}') \rangle$  and  $\prod_{h'=h}^L \hat{\Sigma}^{(h')}(\mathbf{x}, \mathbf{x}')$  are close. This is where we justify the fresh Gaussian copy trick. The road map is that we first show the fresh Gaussian copy trick holds when conditioning on a fixed realization of sparse masks and then consider the randomness introduced by the sparse masks.

Let  $\mathbf{G}_i^{(h)} = [(\mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)}) \quad (\mathbf{g}^{(h)}(\mathbf{x}') \odot \mathbf{m}_i^{(h+1)})]$  and  $\mathbf{G}^{(h)} = [\mathbf{G}_1^{(h)} \mathbf{G}_2^{(h)} \dots \mathbf{G}_{d_{h+1}}^{(h)}]$  and  $\mathbf{F}^{(h+1)} = (\mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)}) \mathbf{G}^{(h)}$ . First for each row in  $\mathbf{W}^{(h+1)}$ , we can write  $\mathbf{w}_i^{(h+1)} = \mathbf{w}_i^{(h+1)} (\Pi_{\mathbf{G}_i} + \Pi_{\mathbf{G}_i}^\perp)$ . Notice that conditioned on  $\mathbf{G}_i^{(h)}, \mathbf{F}^{(h+1)}, \mathbf{m}^{(h+1)}$ , we have  $\mathbf{w}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \stackrel{D}{=} \tilde{\mathbf{w}}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp$  where  $\tilde{\mathbf{w}}_i^{(h+1)}$  is an i.i.d. copy of  $\mathbf{w}_i^{(h+1)}$ . We

define

$$\mathbf{b}_\perp^{(h)} = \left( \mathbf{b}^{(h+1)}(\mathbf{x}) \right)^\top \begin{bmatrix} (\tilde{\mathbf{w}}_1^{(h+1)} \odot \mathbf{m}_1^{(h+1)})^\top \Pi_{\mathbf{G}_1}^\perp \\ (\tilde{\mathbf{w}}_2^{(h+1)} \odot \mathbf{m}_2^{(h+1)})^\top \Pi_{\mathbf{G}_2}^\perp \\ \vdots \\ (\tilde{\mathbf{w}}_{d_{h+1}}^{(h+1)} \odot \mathbf{m}_{d_{h+1}}^{(h+1)})^\top \Pi_{\mathbf{G}_{d_{h+1}}}^\perp \end{bmatrix} \mathbf{D}$$

$$\mathbf{b}_\parallel^{(h)} = \left( \mathbf{b}^{(h+1)}(\mathbf{x}) \right)^\top \begin{bmatrix} (\mathbf{w}_1^{(h+1)} \odot \mathbf{m}_1^{(h+1)})^\top \Pi_{\mathbf{G}_1} \\ (\mathbf{w}_2^{(h+1)} \odot \mathbf{m}_2^{(h+1)})^\top \Pi_{\mathbf{G}_2} \\ \vdots \\ (\mathbf{w}_{d_{h+1}}^{(h+1)} \odot \mathbf{m}_{d_{h+1}}^{(h+1)})^\top \Pi_{\mathbf{G}_{d_{h+1}}} \end{bmatrix} \mathbf{D}$$

Notice that  $\mathbf{b}^{(h)} = \mathbf{b}_\perp^{(h)} + \mathbf{b}_\parallel^{(h)}$ . Next, in order to rigorously justify the fresh Gaussian copy trick, we are going to show that the main contribution of  $\langle \mathbf{b}^{(h)}(\mathbf{x}), \mathbf{b}^{(h)}(\mathbf{x}') \rangle$  is from  $\mathbf{b}_\perp^{(h)}$  and the contribution from the dependent part  $\mathbf{b}_\parallel^{(h)}$  is small, which we show in the next two propositions.

**Proposition 4.3** (Informal). *With probability at least  $1 - \delta_2$ , under some appropriate conditions, for any  $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \in \{(\mathbf{x}, \mathbf{x}), (\mathbf{x}, \mathbf{x}'), (\mathbf{x}', \mathbf{x}')\}$ , we have*

$$\left| \left( \mathbf{b}_\perp^{(h)}(\mathbf{x}^{(1)}) \right)^\top \mathbf{b}_\perp^{(h)}(\mathbf{x}^{(2)}) - \frac{2}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \right| \lesssim \sqrt{\frac{\log \frac{1}{\delta_2}}{\alpha d_h}}$$

**Proposition 4.4** (Informal). *With probability  $1 - \delta_2/2$  and under some appropriate conditions,*

$$\|\mathbf{b}_\parallel^{(h)}\|_2 \lesssim \sqrt{\frac{1}{\alpha^2 d_h} \log \frac{1}{\delta_2}}$$

Finally, like in analyzing the forward propagation, we can show  $2 \frac{\text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D}^{(h)}(\mathbf{x}, \mathbf{x}') \mathbf{M}_i^{(h+1)})}{d_h}$  and  $\dot{\Sigma}(\mathbf{x}, \mathbf{x}')$  are close in Lemma C.7. Combining this with Proposition 4.3, 4.4, we are able to prove that  $\langle \mathbf{b}^{(h)}(\mathbf{x}), \mathbf{b}^{(h)}(\mathbf{x}') \rangle$  and  $\prod_{h'=h}^L \dot{\Sigma}^{(h')}(\mathbf{x}, \mathbf{x}')$  are close.

Note that due to the inclusion of the mask, the proof of these two propositions are different from the proof of Claim E.2 and Claim E.3 in (Arora et al., 2019). In particular, we point out the difficulty in proving Proposition 4.4. The proof of Claim E.3 in (Arora et al., 2019) results in upper bounding  $|(\mathbf{g}^{(h)})^\top (\mathbf{W}^{(h+1)})^\top \mathbf{b}^{(h+1)}|$  which nicely simplifies to upper bounding the output of the network  $|f(\theta, \mathbf{x})|$ . However, this is no longer true in our case. It turns out that under the inclusion of masks, we need to upper bound the output of every sparse networks induced by the mask from each layer which we define formally in the Lemma C.15 in the Appendix. A naive way is to expand  $\mathbf{b}^{(h)}$  by its definition and

apply the upper bound on the two norm of weight matrix of each layer which can result in an exponential blow up (in  $L$ ) of the error, and we are trying to avoid that. The observation is that two norm considers the *worst case* blow up which can be replaced by a more careful bounding of the output of each induced networks and then take a union bound, which is where the  $\text{poly}(d_1, d_2, \dots, d_L, L)$  term in the log factor in our final result comes from.

Finally, combining the result on forward and backward propagation gives our main result.

**Theorem 1.2** (Main Theorem, Restated). *Consider an  $L$ -hidden-layer fully connected neural network with all the weights initialized with i.i.d. standard Gaussian distribution. Suppose all the weights except the input layer are pruned with probability  $1 - \alpha$  at the initialization and after pruning we rescale the weights by  $1/\sqrt{\alpha}$ . Fix  $\epsilon > 0$  and  $\delta \in (0, 1)$ . Suppose we use ReLU activation  $\sigma(z) = \max(0, z)$  and  $d_h \geq \Omega(\frac{1}{\alpha^2} \frac{L^6}{\epsilon^4} \log \frac{L d_{h+1} \sum_{i=1}^h d_i + \sum_{h'=h'}^L d_{h'}}{\delta})$ . Then for any inputs  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{d_0}$  such that  $\|\mathbf{x}\|_2 \leq 1$ ,  $\|\mathbf{x}'\|_2 \leq 1$ , with probability at least  $1 - \delta$  we have*

$$\left| \left\langle \frac{\partial f(\theta, \mathbf{x})}{\partial \theta}, \frac{\partial f(\theta, \mathbf{x}')}{\partial \theta} \right\rangle - \Theta^{(L)}(\mathbf{x}, \mathbf{x}') \right| \leq (L + 1)\epsilon$$

## 5. Experiments

In this section we present our empirical results. Our results contain two parts: in the first part we construct a toy example to validate our theory; in the second part, we examine the practical performance of random pruning.

### 5.1. Validating Our Theory

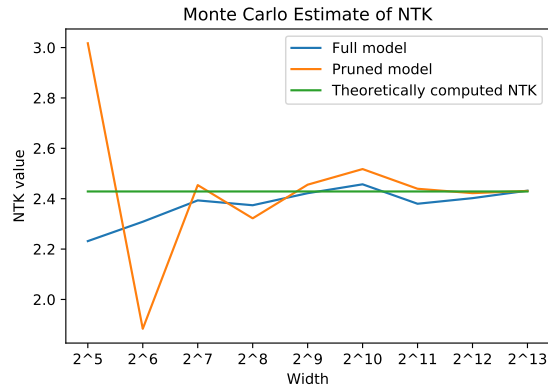


Figure 1. Validation of Theorem 1.1. Experiment results of the empirical NTK value generated by the full model and pruned model with varying width compared with theoretical NTK limit.

**Validation of Theorem 1.1:** We validate this theorem by showing that the empirical NTK value computed from the pruned network converges to the theoretical NTK limit as

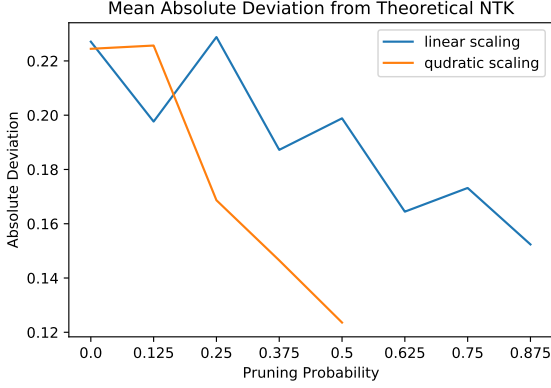


Figure 2. Experiment results of the mean absolute deviation of the empirical NTK value from the theoretical NTK. At each pruning probability, the width of the network is scaled quadratically and linearly with respect to  $1/\alpha$ .

the width increases. We use fully-connected neural networks with 3-hidden layers of the same width as our model. We eliminate the scaling in front of the NTK of pruned network by rescaling the weights by  $1/\sqrt{\alpha}$  after pruning. We first randomly generate two data points  $\mathbf{x}, \mathbf{y}$  and then randomly initialize the networks with Gaussian distribution. We fix the pruning probability to be  $1/2$  and vary the width from 32 to 8192. For each trial, we create 64 samples of the empirical NTK values generated by the unpruned and pruned networks, and plot their mean. Figure 1 shows that, as the width increase, our empirical estimates from both unpruned and pruned model converge to the theoretically computed NTK value.

**Validation of Theorem 1.2:** Our theorem 1.2 suggests that  $d_h$  should scale quadratically with respect to  $1/\alpha$  to maintain the same level of error between the empirical NTK and theoretical limit with probability  $1 - \delta$ . To evaluate our Theorem 1.2, we start with full model of width 1024 and then prune the model with various probability  $p$  while scaling the width by  $1/(1 - p)^2$ . Since quadratically scaling width is expensive, we stop at 0.5 pruning probability. We generate 100 samples for each pruning probability and take their mean absolute deviation from the theoretically computed NTK value. Our experiment suggests that although our result holds, it yields a pessimistic dependence on  $1/\alpha$ . We further investigate how the errors behave if we scale the width linearly with respect to  $1/\alpha$ . The result is shown in Figure 2. Our results suggest that a better dependence of width on  $1/\alpha$  than our result indicated should be possible.

## 5.2. Sparse Training On Real World Data

In this section, we further test our theory by evaluating random pruning on real world dataset. Our theory suggests that

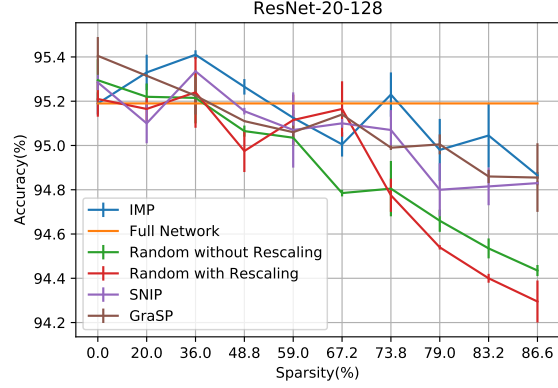


Figure 3. Performance of random pruning with/without rescaling and other popular pruning methods on ResNet-20-128. Sparsity on the x-axis means the fraction of weights remaining in IMP, SNIP and GraSP, and pruning probability in random pruning.

if the network is wide enough, the pruned networks should retain much of the performance of the full networks. Different from our theory, here we prune **all** layers of the neural networks. We adopt the implementation from (Frankle & Carbin, 2018; Chen et al., 2021). As suggested in (Evci et al., 2020b), scaling after pruning preserves the gradient flow of the neural network. Therefore, we include random pruning both with and without scaling in our comparison. We compare random pruning against popular pruning methods such as Iterative Magnitude-based Pruning (IMP), SNIP and GraSP. We train a wide fully-connected neural network on MNIST dataset (Deng, 2012) and ResNet (He et al., 2016) on CIFAR-10 (Krizhevsky et al., 2009). Each data point in the plot is averaged over 2 independent runs. Due to the space limit, we defer the detailed setup to Appendix A.

**Results.** Unsurprisingly, even if the network is wide, IMP, SNIP and GraSP can still outperform random pruning, however, by a small margin. For ResNet-20-128, at sparsity 86.6%, the performance of random pruning and the full model is within 1% on CIFAR-10 as shown in Figure 3. Similar result has been observed on MNIST dataset with wide fully-connected neural networks. We further conduct experiments on ResNet with different width. At the same sparsity level, the one with larger width are seen with less performance drop compared to the full dense network. The experiment results are shown in Appendix A.

## 6. Discussion and Future Direction

In this paper, we apply neural tangent kernel analysis on randomly pruned neural network at the initialization. Both our theoretical result and empirical study show that the randomly pruned neural network can indeed approach the NTK



of the full network. However, one limitation of our result is that we assume the input layer is not pruned. Further, our empirical study suggests that our result’s dependence on  $1/\alpha$  is not optimal. We leave both understanding how pruning the input layer can affect the NTK and improving our result’s dependence on  $1/\alpha$  as future work.

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## A. Additional Experiment Results

### A.1. Experimental Settings

All of our models are trained with SGD and the detailed settings are summarized below.

Table 1. Summary of architectures, dataset and training hyperparameters

MODEL	DATA	EPOCH	BATCH SIZE	LR	MOMENTUM	LR DECAY, EPOCH	WEIGHT DECAY
FULLY-CONNECTED NN	MNIST	40	128	0.1	0	0	0
RESNETS	CIFAR-10	160	128	0.1	0.9	$0.1 \times [80, 120]$	0.0001

### A.2. MNIST

For MNIST dataset, we train a fully-connected neural network with 2-hidden layers of width 2048. The performance is shown in Figure 4.

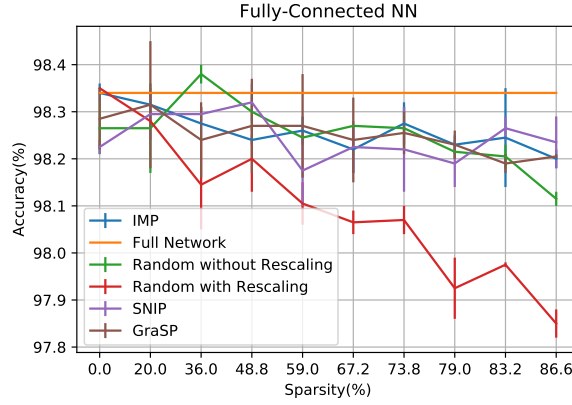


Figure 4. Comparing the performance of random pruning with/without rescaling with IMP, SNIP and GraSP using a fully-connected neural network with 2 hidden layers of width 2048 on MNIST dataset.

### A.3. CIFAR-10

We train ResNet-20 of width 32, 64 and 128 and compare the performance of random pruning with and without rescaling against IMP. The results are shown in Figure 5a, Figure 5b and Figure 5c.



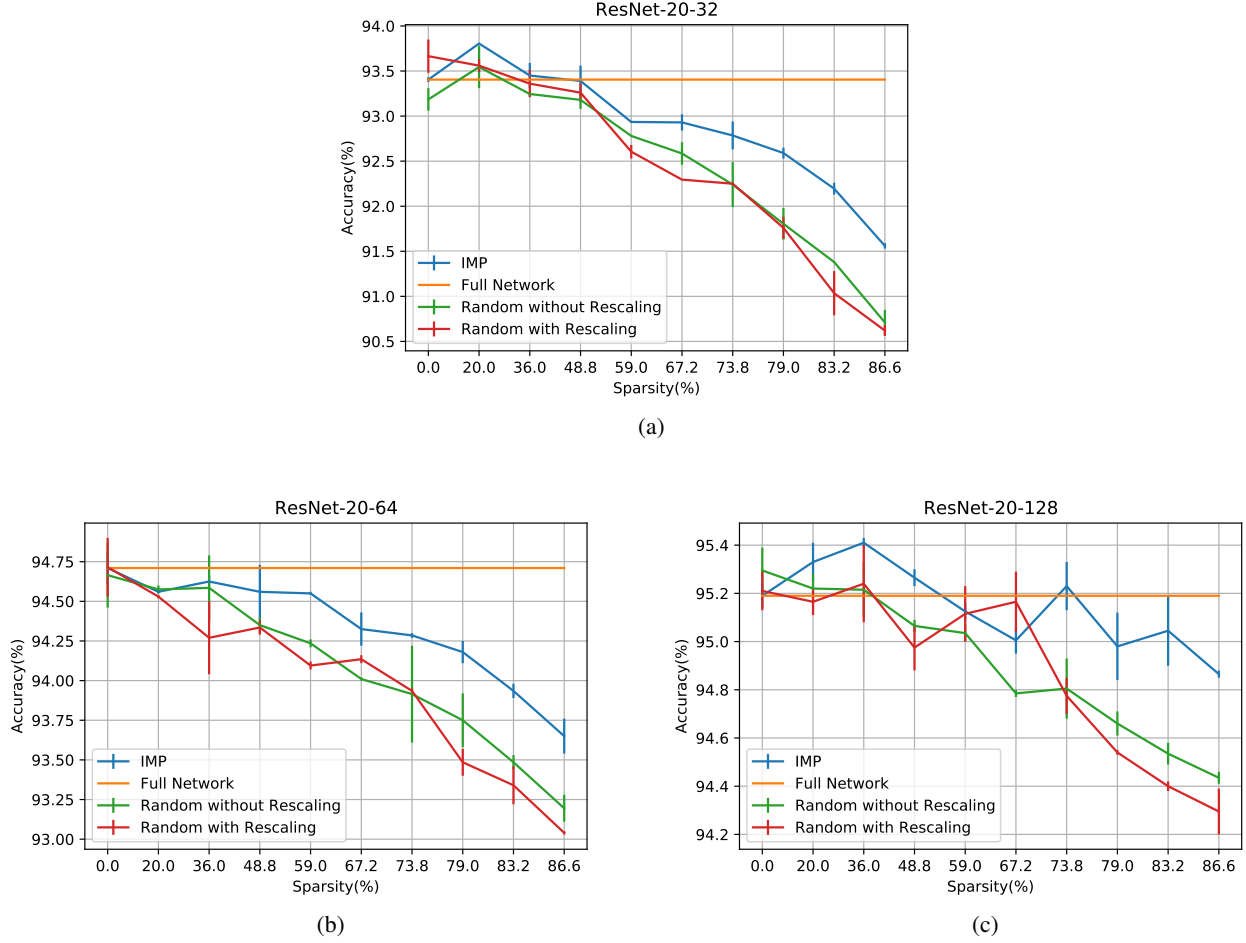


Figure 5. Comparing the performance of random pruning with/without rescaling with IMP using ResNet-20 of different width on CIFAR-10 dataset.

## B. Proofs

### B.1. Asymptotic Analysis

We first give an asymptotic analysis followed from (Arora et al., 2019). We use tilde over a symbol to denote the quantity in the *pruned* network and the corresponding symbol without tilde denotes the quantity in the unpruned network.

We first analyze the forward dynamics of the pruned neural network.

$$\begin{aligned}
 [\tilde{\mathbf{f}}^{(h+1)}(\mathbf{x})]_i &= \sum_{j=1}^{d_h} [\mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)}]_{ij} [\tilde{\mathbf{g}}^{(h)}(\mathbf{x})]_j \\
 &= \sqrt{\frac{c_\sigma}{d_h}} \sum_{j=1}^{d_h} [\mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)}]_{ij} \sigma \left( [\tilde{\mathbf{f}}^{(h)}(\mathbf{x})]_j \right)
 \end{aligned}$$

and note that for finite  $d_h$ , different from unmasked case, this is no longer a centered Gaussian random variable but a centered Gaussian mixture as we apply the mask. However, for  $h \in \{1, \dots, L\}$ , as  $d_h \rightarrow \infty$ , by the central limit theorem,  $[\tilde{\mathbf{f}}^{(h+1)}(\mathbf{x})]_i$  converges to a Gaussian random variable. This is certainly not true for the output in the first layer because the input dimension can't go to infinity. Thus, we make assumption that the pruning only starts from the second layer.

Now by i.i.d assumption of the mask and weights, we have its covariance being

$$\begin{aligned} \mathbb{E}_{\mathbf{W}^{(h+1)}, \mathbf{m}^{(h+1)}} \left[ \left[ \tilde{\mathbf{f}}^{(h+1)}(\mathbf{x}) \right]_i \left[ \tilde{\mathbf{f}}^{(h+1)}(\mathbf{x}') \right]_i \middle| \tilde{\mathbf{f}}^{(h)} \right] &= \alpha \left\langle \tilde{\mathbf{g}}^{(h)}(\mathbf{x}), \tilde{\mathbf{g}}^{(h)}(\mathbf{x}') \right\rangle \\ &= \alpha \frac{c_\sigma}{d_h} \sum_{j=1}^{d_h} \sigma \left( \left[ \tilde{\mathbf{f}}^{(h)}(\mathbf{x}) \right]_j \right) \sigma \left( \left[ \tilde{\mathbf{f}}^{(h)}(\mathbf{x}') \right]_j \right) \end{aligned} \quad (2)$$

which converges as  $d_h \rightarrow \infty$ .

We define

$$\tilde{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') := \lim_{d_h \rightarrow \infty} \left\langle \tilde{\mathbf{g}}^{(h)}(\mathbf{x}), \tilde{\mathbf{g}}^{(h)}(\mathbf{x}') \right\rangle = \lim_{d_h \rightarrow \infty} \frac{c_\sigma}{d_h} \sum_{j=1}^{d_h} \sigma \left( \left[ \tilde{\mathbf{f}}^{(h)}(\mathbf{x}) \right]_j \right) \sigma \left( \left[ \tilde{\mathbf{f}}^{(h)}(\mathbf{x}') \right]_j \right)$$

Therefore, we can characterize the forward dynamics for the masked neural networks similar to the full network.

$$\begin{aligned} \tilde{\Sigma}^{(0)}(\mathbf{x}, \mathbf{x}') &= \mathbf{x}^\top \mathbf{x}' \\ \tilde{\mathbf{\Lambda}}^{(1)} &= \begin{bmatrix} \tilde{\Sigma}^{(h-1)}(\mathbf{x}, \mathbf{x}) & \tilde{\Sigma}^{(h-1)}(\mathbf{x}, \mathbf{x}') \\ \tilde{\Sigma}^{(h-1)}(\mathbf{x}', \mathbf{x}) & \tilde{\Sigma}^{(h-1)}(\mathbf{x}', \mathbf{x}') \end{bmatrix} \\ \tilde{\mathbf{\Lambda}}^{(h)} &= \alpha \begin{bmatrix} \tilde{\Sigma}^{(h-1)}(\mathbf{x}, \mathbf{x}) & \tilde{\Sigma}^{(h-1)}(\mathbf{x}, \mathbf{x}') \\ \tilde{\Sigma}^{(h-1)}(\mathbf{x}', \mathbf{x}) & \tilde{\Sigma}^{(h-1)}(\mathbf{x}', \mathbf{x}') \end{bmatrix} \\ \tilde{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') &= c_\sigma \mathbb{E}_{(u,v) \sim \mathcal{N}(\mathbf{0}, \tilde{\mathbf{\Lambda}}^{(h)})} [\sigma(u)\sigma(v)] \end{aligned}$$

**Lemma 3.1.** Suppose the neural network uses ReLU as its activation and  $d_1, d_2, \dots, d_L \rightarrow \infty$  sequentially, then

$$\tilde{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') = \alpha^{h-1} \Sigma^{(h)}(\mathbf{x}, \mathbf{x}')$$

for  $h = 1, 2, \dots, L$ .

*Proof.* We prove by induction. First, notice that  $\tilde{\Sigma}^{(0)}(\mathbf{x}, \mathbf{x}') = \Sigma^{(0)}(\mathbf{x}, \mathbf{x}')$ . When  $h = 1$ , there is nothing to prove. Now, assume the induction hypothesis holds for all  $h$  such that  $h \leq t$  where  $t \geq 1$  and we want to show that  $\tilde{\Sigma}^{(t+1)}(\mathbf{x}, \mathbf{x}') = \alpha^t \Sigma^{(t+1)}(\mathbf{x}, \mathbf{x}')$ . Notice that Equation 2 is true for  $h \in \{1, \dots, L\}$ . Therefore, as  $d_t \rightarrow \infty$

$$\tilde{\Sigma}^{(t+1)}(\mathbf{x}, \mathbf{x}') = c_\sigma \mathbb{E}_{\left[ \tilde{\mathbf{f}}^{(t+1)}(\mathbf{x}) \right]_1, \left[ \tilde{\mathbf{f}}^{(t+1)}(\mathbf{x}') \right]_1} \left[ \sigma \left( \left[ \tilde{\mathbf{f}}^{(t+1)}(\mathbf{x}) \right]_1 \right) \sigma \left( \left[ \tilde{\mathbf{f}}^{(t+1)}(\mathbf{x}') \right]_1 \right) \right]$$

Assume all the previous layers are already at the limit,

$$\left( \left[ \tilde{\mathbf{f}}^{(t+1)}(\mathbf{x}) \right]_1, \left[ \tilde{\mathbf{f}}^{(t+1)}(\mathbf{x}') \right]_1 \right) \sim \mathcal{N}(\mathbf{0}, \tilde{\mathbf{\Lambda}}^{(t+1)})$$

By induction hypothesis on  $\tilde{\Sigma}^{(t)}(\mathbf{x}, \mathbf{x}')$ , we have  $\tilde{\mathbf{\Lambda}}^{(t+1)} = \alpha^t \mathbf{\Lambda}^{(t+1)}$ . Hence

$$\begin{aligned} \tilde{\Sigma}^{(t+1)}(\mathbf{x}, \mathbf{x}') &= c_\sigma \mathbb{E}_{(u,v) \sim \mathcal{N}(\mathbf{0}, \alpha^t \mathbf{\Lambda}^{(t+1)})} [\sigma(u)\sigma(v)] \\ &= c_\sigma \mathbb{E}_{(u',v') \sim \mathcal{N}(\mathbf{0}, \mathbf{\Lambda}^{(t+1)})} [\sigma(\alpha^{\frac{t}{2}} u') \sigma(\alpha^{\frac{t}{2}} v')] \\ &= \alpha^t c_\sigma \mathbb{E}_{(u',v') \sim \mathcal{N}(\mathbf{0}, \mathbf{\Lambda}^{(t+1)})} [\sigma(u') \sigma(v')] \\ &= \alpha^t \Sigma^{(t+1)}(\mathbf{x}, \mathbf{x}') \end{aligned}$$

where the second last inequality is from our assumption that the activation is ReLU.  $\square$

This lemma implies that

$$\lim_{d_1, \dots, d_h \rightarrow \infty} \langle \tilde{\mathbf{g}}^{(h)}(\mathbf{x}), \tilde{\mathbf{g}}^{(h)}(\mathbf{x}') \rangle = \alpha^{h-1} \lim_{d_1, \dots, d_h \rightarrow \infty} \langle \mathbf{g}^{(h)}(\mathbf{x}), \mathbf{g}^{(h)}(\mathbf{x}') \rangle \quad (3)$$

For the pruned neural networks, its gradient is given by

$$\begin{aligned} \frac{\partial \tilde{f}(\mathbf{x})}{\partial [\mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}]} &= \frac{\partial \tilde{f}(\mathbf{x})}{\partial \mathbf{W}^{(h)}} \odot \mathbf{m}^{(h)} \\ &= \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \right)^\top \right) \odot \mathbf{m}^{(h)}, \quad h = 1, 2, \dots, L+1 \end{aligned}$$

where

$$\tilde{\mathbf{b}}^{(h)}(\mathbf{x}) = \begin{cases} 1 \in \mathbb{R}, & h = L+1 \\ \sqrt{\frac{c_\sigma}{d_h}} \tilde{\mathbf{D}}^{(h)}(\mathbf{x}) (\mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)})^\top \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}) \in \mathbb{R}^{d_h}, & h = 1, \dots, L, \end{cases} \quad (4)$$

and

$$\tilde{\mathbf{D}}^{(h)}(\mathbf{x}) = \text{diag} \left( \dot{\sigma} \left( \tilde{\mathbf{f}}^{(h)}(\mathbf{x}) \right) \right) \in \mathbb{R}^{d_h \times d_h}, \quad h = 1, \dots, L \quad (5)$$

Note that since the weights being pruned are staying at zero always during the training process, the gradient of the pruned network is simply the masked gradient of the unpruned network.

Now, we have

$$\begin{aligned} \left\langle \frac{\partial \tilde{f}(\mathbf{x})}{\partial [\mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}]}, \frac{\partial \tilde{f}(\mathbf{x}')}{\partial [\mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}]} \right\rangle &= \left\langle \frac{\partial \tilde{f}(\mathbf{x})}{\partial \mathbf{W}^{(h)}} \odot \mathbf{m}^{(h)}, \frac{\partial \tilde{f}(\mathbf{x}')}{\partial \mathbf{W}^{(h)}} \odot \mathbf{m}^{(h)} \right\rangle \\ &= \left\langle \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \right)^\top \right) \odot \mathbf{m}^{(h)}, \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}') \right)^\top \right) \odot \mathbf{m}^{(h)} \right\rangle \end{aligned}$$

Now we write

$$\left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \right)^\top \right) \odot \mathbf{m}^{(h)} = \begin{bmatrix} \tilde{\mathbf{b}}_1^{(h)}(\mathbf{x}) \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \odot \mathbf{m}_1^{(h)} \\ \tilde{\mathbf{b}}_2^{(h)}(\mathbf{x}) \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \odot \mathbf{m}_2^{(h)} \\ \vdots \\ \tilde{\mathbf{b}}_{d_h}^{(h)}(\mathbf{x}) \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \odot \mathbf{m}_{d_h}^{(h)} \end{bmatrix}$$

Thus,

$$\begin{aligned} \left\langle \frac{\partial \tilde{f}(\mathbf{x})}{\partial [\mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}]}, \frac{\partial \tilde{f}(\mathbf{x}')}{\partial [\mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}]} \right\rangle &= \left\langle \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \right)^\top \right) \odot \mathbf{m}^{(h)}, \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}') \right)^\top \right) \odot \mathbf{m}^{(h)} \right\rangle \\ &= \sum_{i=1}^{d_h} \tilde{\mathbf{b}}_i^{(h)}(\mathbf{x}) \tilde{\mathbf{b}}_i^{(h)}(\mathbf{x}') \left\langle \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \odot \mathbf{m}_i^{(h)}, \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}') \odot \mathbf{m}_i^{(h)} \right\rangle \\ &= \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \right)^\top \mathbf{G}^{(h-1)} \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \end{aligned} \quad (6)$$

where  $\mathbf{G}^{(h-1)}$  is a diagonal matrix and  $\mathbf{G}_{ii}^{(h-1)} = \left\langle \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \odot \mathbf{m}_i^{(h)}, \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}') \odot \mathbf{m}_i^{(h)} \right\rangle$ . Observe that

$$\begin{aligned} \lim_{d_{h-1} \rightarrow \infty} \left\langle \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \odot \mathbf{m}_i^{(h)}, \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}') \odot \mathbf{m}_i^{(h)} \right\rangle &= \lim_{d_{h-1} \rightarrow \infty} \frac{c_\sigma}{d_{h-1}} \sum_{j=1}^{d_{h-1}} \sigma \left( \tilde{\mathbf{f}}_j^{(h-1)}(\mathbf{x}) \right) \sigma \left( \tilde{\mathbf{f}}_j^{(h-1)}(\mathbf{x}') \right) \left( \mathbf{m}_{ij}^{(h)} \right)^2 \\ &= \mathbb{E} \left[ c_\sigma \sigma \left( \tilde{\mathbf{f}}_j^{(h-1)}(\mathbf{x}) \right) \sigma \left( \tilde{\mathbf{f}}_j^{(h-1)}(\mathbf{x}') \right) \left( \mathbf{m}_{ij}^{(h)} \right)^2 \right] \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E} \left[ c_\sigma \sigma \left( \tilde{\mathbf{f}}_j^{(h-1)}(\mathbf{x}) \right) \sigma \left( \tilde{\mathbf{f}}_j^{(h-1)}(\mathbf{x}') \right) \right] \mathbb{E} \left[ \left( \mathbf{m}_{ij}^{(h)} \right)^2 \right] \\
 &= \alpha \mathbb{E} \left[ c_\sigma \sigma \left( \tilde{\mathbf{f}}_j^{(h-1)}(\mathbf{x}) \right) \sigma \left( \tilde{\mathbf{f}}_j^{(h-1)}(\mathbf{x}') \right) \right]
 \end{aligned}$$

Notice that  $\mathbf{G}_{ii}^{(h-1)} \rightarrow \alpha \tilde{\Sigma}^{(h-1)}(\mathbf{x}, \mathbf{x}')$  as  $d_{h-1} \rightarrow \infty$ . Thus, we have

$$\begin{aligned}
 &\left\langle \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \right)^\top \right) \odot \mathbf{m}^{(h)}, \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}') \right)^\top \right) \odot \mathbf{m}^{(h)} \right\rangle \\
 &= \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \right)^\top \mathbf{G}^{(h-1)} \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \\
 &\xrightarrow{d_{h-1} \rightarrow \infty} \alpha \tilde{\Sigma}^{(h-1)}(\mathbf{x}, \mathbf{x}') \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \right)^\top \tilde{\mathbf{b}}^{(h)}(\mathbf{x}')
 \end{aligned} \tag{7}$$

**Lemma 3.2.**

$$\lim_{d_1, \dots, d_L \rightarrow \infty} \left\langle \tilde{\mathbf{b}}^{(h)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \right\rangle = \alpha^{L+1-h} \lim_{d_1, \dots, d_L \rightarrow \infty} \left\langle \tilde{\mathbf{b}}^{(h)}(\mathbf{x}), \mathbf{b}^{(h)}(\mathbf{x}') \right\rangle = \alpha^{L+1-h} \prod_{h'=h}^L \dot{\Sigma}^{(h')}(\mathbf{x}, \mathbf{x}') \tag{8}$$

*Proof.* For the factor  $\left\langle \tilde{\mathbf{b}}^{(h)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \right\rangle$ , we expand using the definition of  $\tilde{\mathbf{b}}^{(h)}(\mathbf{x})$

$$\begin{aligned}
 &\left\langle \tilde{\mathbf{b}}^{(h)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \right\rangle \\
 &= \left\langle \sqrt{\frac{c_\sigma}{d_h}} \tilde{\mathbf{D}}^{(h)}(\mathbf{x}) \left( \mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)} \right)^\top \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}), \sqrt{\frac{c_\sigma}{d_h}} \tilde{\mathbf{D}}^{(h)}(\mathbf{x}') \left( \mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)} \right)^\top \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}') \right\rangle
 \end{aligned}$$

First we analyze  $\tilde{\mathbf{D}}^{(h)}(\mathbf{x})$ . Since we use ReLU as the activation function, by the definition in Equation (5),  $\dot{\sigma}(x) = \mathbb{I}(x > 0)$  and in particular,  $\dot{\sigma}(cx) = \mathbb{I}(cx > 0) = \mathbb{I}(x > 0) = \dot{\sigma}(x)$  for any positive constant  $c$ . By Lemma 3.1, we show that under sequential limit,  $\tilde{\mathbf{f}}^{(h)}(\mathbf{x})$  has the same distribution as  $\alpha^{h-1} \mathbf{f}^{(h)}(\mathbf{x})$  which implies  $\tilde{\mathbf{D}}^{(h)}(\mathbf{x})$  has the same distribution as  $\mathbf{D}^{(h)}(\mathbf{x})$ .

Observe that  $\mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)}$  and  $\tilde{\mathbf{b}}^{(h+1)}(\mathbf{x})$  are dependent. Now we apply the independent copy trick which is rigorously justified for ReLU network with Gaussian weights by replace  $\mathbf{W}^{(h+1)}$  with a fresh new sample  $\widetilde{\mathbf{W}}^{(h+1)}$  without changing its limit.

$$\begin{aligned}
 \left\langle \tilde{\mathbf{b}}^{(h)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \right\rangle &= \left\langle \sqrt{\frac{c_\sigma}{d_h}} \tilde{\mathbf{D}}^{(h)}(\mathbf{x}) \left( \mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)} \right)^\top \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}), \sqrt{\frac{c_\sigma}{d_h}} \tilde{\mathbf{D}}^{(h)}(\mathbf{x}') \left( \mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)} \right)^\top \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}') \right\rangle \\
 &\approx \left\langle \sqrt{\frac{c_\sigma}{d_h}} \tilde{\mathbf{D}}^{(h)}(\mathbf{x}) \left( \widetilde{\mathbf{W}}^{(h+1)} \odot \mathbf{m}^{(h+1)} \right)^\top \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}), \sqrt{\frac{c_\sigma}{d_h}} \tilde{\mathbf{D}}^{(h)}(\mathbf{x}') \left( \widetilde{\mathbf{W}}^{(h+1)} \odot \mathbf{m}^{(h+1)} \right)^\top \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}') \right\rangle \\
 &\rightarrow \alpha \frac{c_\sigma}{d_h} \text{Tr} \left( \tilde{\mathbf{D}}^{(h)}(\mathbf{x}) \tilde{\mathbf{D}}^{(h)}(\mathbf{x}') \right) \left\langle \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}') \right\rangle \\
 &\rightarrow \alpha \dot{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') \left\langle \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}') \right\rangle
 \end{aligned} \tag{9}$$

where we justify the limit as the following: first let  $\mathbf{D}$  short for  $\tilde{\mathbf{D}}^{(h)}(\mathbf{x}) \tilde{\mathbf{D}}^{(h)}(\mathbf{x}')$

$$\left( \frac{c_\sigma}{d_h} (\widetilde{\mathbf{W}}^{(h+1)} \odot \mathbf{m}^{(h+1)})^\top \mathbf{D} (\widetilde{\mathbf{W}}^{(h+1)} \odot \mathbf{m}^{(h+1)}) \right)_{ij} = \frac{c_\sigma}{d_h} \sum_k \mathbf{D}_{kk} \widetilde{\mathbf{W}}_{ik}^{(h+1)} \mathbf{m}_{ik}^{(h+1)} \widetilde{\mathbf{W}}_{jk}^{(h+1)} \mathbf{m}_{jk}^{(h+1)}$$

which converges to a diagonal matrix as  $d_h \rightarrow \infty$ . Thus, the inner product is given by

$$\begin{aligned}
 &\frac{c_\sigma}{d_h} \sum_{i,j} \tilde{\mathbf{b}}_i^{(h+1)}(\mathbf{x}) \tilde{\mathbf{b}}_j^{(h+1)}(\mathbf{x}') \sum_k \mathbf{D}_{kk} \widetilde{\mathbf{W}}_{ik}^{(h+1)} \mathbf{m}_{ik}^{(h+1)} \widetilde{\mathbf{W}}_{jk}^{(h+1)} \mathbf{m}_{jk}^{(h+1)} \\
 &= \frac{c_\sigma}{d_h} \sum_{i,j} \tilde{\mathbf{b}}_i^{(h+1)}(\mathbf{x}) \tilde{\mathbf{b}}_j^{(h+1)}(\mathbf{x}') (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \mathbf{D} (\tilde{\mathbf{w}}_j^{(h+1)} \odot \mathbf{m}_j^{(h+1)})
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{c_\sigma}{d_h} \sum_{i,j} \tilde{\mathbf{b}}_i^{(h+1)}(\mathbf{x}) \tilde{\mathbf{b}}_j^{(h+1)}(\mathbf{x}') \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_j^{(h+1)} \tilde{\mathbf{w}}_j^{(h+1)} \\
 &\rightarrow \frac{c_\sigma}{d_h} \sum_i \tilde{\mathbf{b}}_i^{(h+1)}(\mathbf{x}) \tilde{\mathbf{b}}_i^{(h+1)}(\mathbf{x}') \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \\
 &\rightarrow \alpha \dot{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') \left\langle \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}') \right\rangle
 \end{aligned}$$

where  $\mathbf{M}_i = \text{diag}(\mathbf{m}_i)$  and  $\tilde{\mathbf{w}}_i$  is the  $i$ -th row of  $\tilde{\mathbf{W}}$  and  $\lim_{d_h \rightarrow \infty} \frac{c_\sigma}{d_h} \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) = \alpha \dot{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}')$ . Now, we can unroll the formula of  $\left\langle \tilde{\mathbf{b}}^{(h)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \right\rangle$  in Equation (9), we have

$$\lim_{d_1, \dots, d_L \rightarrow \infty} \left\langle \tilde{\mathbf{b}}^{(h)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \right\rangle = \alpha^{L+1-h} \lim_{d_1, \dots, d_L \rightarrow \infty} \left\langle \mathbf{b}^{(h)}(\mathbf{x}), \mathbf{b}^{(h)}(\mathbf{x}') \right\rangle = \alpha^{L+1-h} \prod_{h'=h}^L \dot{\Sigma}^{(h')}(\mathbf{x}, \mathbf{x}')$$

□

Thus, combining the result in Equation (7) and Equation (9), we have

$$\begin{aligned}
 &\left\langle \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}) \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}) \right)^\top \right) \odot \mathbf{m}^{(h)}, \left( \tilde{\mathbf{b}}^{(h)}(\mathbf{x}') \left( \tilde{\mathbf{g}}^{(h-1)}(\mathbf{x}') \right)^\top \right) \odot \mathbf{m}^{(h)} \right\rangle \\
 &\xrightarrow{d_{h-1}, d_h \rightarrow \infty} \alpha^2 \tilde{\Sigma}^{(h-1)}(\mathbf{x}, \mathbf{x}') \dot{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') \left\langle \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}), \tilde{\mathbf{b}}^{(h+1)}(\mathbf{x}') \right\rangle
 \end{aligned}$$

Finally combining Equation (3), Equation (6) and Equation (8), we have

$$\left\langle \frac{\partial \tilde{f}(\mathbf{x})}{\partial \mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}}, \frac{\partial \tilde{f}(\mathbf{x}')}{\partial \mathbf{W}^{(h)} \odot \mathbf{m}^{(h)}} \right\rangle \rightarrow \alpha^L \left\langle \frac{\partial f(\mathbf{x})}{\partial \mathbf{W}^{(h)}}, \frac{\partial f(\mathbf{x}')}{\partial \mathbf{W}^{(h)}} \right\rangle$$

and we conclude

$$\tilde{\Theta}_\infty(\mathbf{x}, \mathbf{x}') := \lim_{d_1, d_2, \dots, d_L \rightarrow \infty} \tilde{\Theta}(\mathbf{x}, \mathbf{x}') = \alpha^L \Theta_\infty(\mathbf{x}, \mathbf{x}') \quad (10)$$

## B.2. Going from Asymptotic Regime to Non-Asymptotic Regime

Before we give proof for our non-asymptotic result, we note that our asymptotic result is obtained from taking sequential limits of all the hidden layers which is a somewhat a limited notion of limits since we assume all the layer before is already at the limit when we deal with a given layer. Non-asymptotic analysis, on the other hand, consider using a large but finite amount of samples to get close to (but not exactly at) the limit. Thus, we need to justify that we are indeed able to approximate the limit using a large but finite amount of samples. In mathematical language, this is the same as justifying taking the limit outside of integration.

We invoke two useful theorems from probability theory: the Skorokhod's representation theorem (Billingsley, 1999) and the continuous mapping theorem (Mann & Wald, 1943).

**Theorem B.1** (Skorokhod's Representation Theorem, (Billingsley, 1999)). *Let  $\{\mu_n\}$  be a sequence of probability measure defined on a metric space  $S$  such that  $\mu_n$  converges weakly to some probability measure  $\mu_\infty$  on  $S$  as  $n \rightarrow \infty$ . Suppose that the support of  $\mu_\infty$  is separable. Then there exists  $S$ -valued random variables  $X_n$  defined on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that the law of  $X_n$  is  $\mu_n$  for all  $n$  (including  $n = \infty$ ) and such that  $(X_n)_{n \in \mathbb{N}}$  converges to  $X_\infty$ ,  $\mathbb{P}$ -almost surely.*

**Theorem B.2** (Continuous Mapping Theorem (Mann & Wald, 1943)). *Let  $\{X_n\}, X$  be random variables defined on a metric space  $S$ . Suppose a function  $g : S \rightarrow S'$  (where  $S'$  is another metric space) has the set of discontinuities of measure zero. Then*

$$X_n \xrightarrow{a.s.} X \quad \Rightarrow \quad g(X_n) \xrightarrow{a.s.} g(X)$$

**Lemma B.3.** *Let*

$$X_n = \left[ \sqrt{\frac{c\sigma}{n}} \sum_{j=1}^n \mathbf{W}_{ij}^{(h+1)} \mathbf{m}_{ij}^{(h+1)} \sigma(\tilde{\mathbf{f}}_j^{(h)}(\mathbf{x})) \right] \in \mathbb{R}^2$$

and define  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  to be  $g(x, y) = \sigma(x)\sigma(y)$ . Then,

$$\lim_{n \rightarrow \infty} \mathbb{E}[g(X_n)] = \mathbb{E}[g(\lim_{n \rightarrow \infty} X_n)]$$

*Proof.* By the Central Limit Theorem,  $X_n \xrightarrow{D} \mathcal{N}(\mathbf{0}, \tilde{\mathbf{A}}^{(h+1)})$ . By the Continuous Mapping Theorem,  $g(X_n) \xrightarrow{D} g(\mathcal{N}(\mathbf{0}, \tilde{\mathbf{A}}^{(h+1)}))$ . Then by the Skorokhod's Representation Theorem, there exists  $g(X'_n)$  such that  $g(X_n), g(X'_n)$  have the same distribution and  $g(X'_n) \xrightarrow{a.s.} g(X_\infty)$  with  $g(X_\infty) \sim g(\mathcal{N}(\mathbf{0}, \tilde{\mathbf{A}}^{(h+1)}))$ . Note that we do not assume the invertibility of  $g(\cdot)$  and  $g(X'_n)$  is simply a notation to denote the random variable. Now we use the fact that  $g(X'_n)$  is **uniformly integrable**<sup>2</sup> which can be proved by Lemma 6, Chapter 7.10 (Grimmett & Stirzaker, 2020) by using  $\sup_n \mathbb{E} |g(X'_n)| < \infty$ . This implies convergence in  $L^1$

$$\lim_{n \rightarrow \infty} \mathbb{E}[g(X'_n)] = \mathbb{E}[g(X_\infty)]$$

Since

$$\begin{aligned} \mathbb{E}[g(X_n)] &= \mathbb{E}[g(X'_n)] \\ \mathbb{E}[g(X_\infty)] &= \mathbb{E}[g(\mathcal{N}(\mathbf{0}, \tilde{\mathbf{A}}^{(h+1)}))] \end{aligned}$$

we have

$$\lim_{n \rightarrow \infty} \mathbb{E}[g(X_n)] = \mathbb{E}[g(\mathcal{N}(\mathbf{0}, \tilde{\mathbf{A}}^{(h+1)}))]$$

□

## C. Nonasymptotic Analysis

### C.1. Probabilities

**Definition C.1** (Subgamma Random Variables). A random variables  $X$  is subgamma with parameters  $(\sigma^2, c)$  if

$$\mathbb{E} \left[ e^{\lambda(X - \mathbb{E}[X])} \right] \leq e^{\frac{\lambda^2 \sigma^2}{2}}, \quad \forall |\lambda| < \frac{1}{c}$$

**Lemma C.2.** *Let  $X_1$  and  $X_2$  be two centered subgamma random variables with parameter  $(\sigma_1^2, c_1)$  and  $(\sigma_2^2, c_2)$  respectively that are not necessarily independent. Then  $X_1 + X_2$  is subgamma with parameter  $(2(\sigma_1^2 + \sigma_2^2), \max(c_1, c_2))$ .*

*Proof.* Applying the definition of subgamma random variables and Cauchy-Schwarz inequality, we have for  $\lambda < \frac{1}{\max(c_1, c_2)}$ ,

$$\begin{aligned} \mathbb{E} \left[ e^{\lambda(X_1 + X_2)} \right] &\leq \sqrt{\mathbb{E} [e^{2\lambda X_1}] \mathbb{E} [e^{2\lambda X_2}]} \\ &\leq \sqrt{e^{4\lambda^2 \sigma_1^2 / 2} e^{4\lambda^2 \sigma_2^2 / 2}} \\ &= e^{\lambda^2 2(\sigma_1^2 + \sigma_2^2) / 2} \end{aligned}$$

Thus,  $X_1 + X_2$  is subgamma with parameter  $(2(\sigma_1^2 + \sigma_2^2), \max(c_1, c_2))$ . □

**Lemma 4.1.** *Let  $X$  be a scaled Bernoulli random variable with probability  $\alpha$  being  $\frac{1}{\alpha}$  and zero otherwise, and  $Y$  be an independent subgamma random variable with parameter  $(\sigma^2, c)$  where  $\sigma^2 = O(1), c = O(1)$ , and  $\mathbb{E}[Y] = O(1)$ , then  $XY$  is subgamma with parameter  $(O((\frac{1-\alpha}{\alpha})^2 + 1), O(\frac{1-\alpha}{\alpha} + 1))$ .*

<sup>2</sup>A class  $\mathcal{C}$  of random variables is called uniformly integrable if given  $\epsilon > 0$ , there exists  $K \in \mathbb{R}_+$  such that  $\mathbb{E}[|X| \mathbb{I}_{|X| \geq K}] \leq \epsilon$  for all  $X \in \mathcal{C}$ .

*Proof.* We need to show that the centered random variable  $XY - \mathbb{E}[X] \mathbb{E}[Y]$  concentrates like a subgamma. We write

$$\begin{aligned} XY - \mathbb{E}[X] \mathbb{E}[Y] &= (X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) + (X \mathbb{E}[Y] - \mathbb{E}[X] \mathbb{E}[Y]) + (\mathbb{E}[X]Y - \mathbb{E}[X] \mathbb{E}[Y]) \\ &= (X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) + (X - \mathbb{E}[X]) \mathbb{E}[Y] + (Y - \mathbb{E}[Y]) \mathbb{E}[X] \end{aligned}$$

We show the first two terms is subgamma and thus the sum is subgamma. We apply Theorem 2.3 in (Boucheron et al., 2013). Recall that the  $k$ -th centered moment of a standard Bernoulli random variable with parameter  $p$  is given by  $\mu_k = (1-p)(-p)^k + p(1-p)^k$  and thus

$$\mathbb{E}[(X - \mathbb{E}[X])^{2k}] = (1-\alpha) + \alpha \left( \frac{1-\alpha}{\alpha} \right)^{2k}, \quad \forall k \geq 1$$

which implies  $X$  is subgamma with parameter  $(4 + 4 \left( \frac{1-\alpha}{\alpha} \right)^2, 2 \frac{1-\alpha}{\alpha})$  and thus  $(X - \mathbb{E}[X]) \mathbb{E}[Y]$  is subgamma with parameter  $(O(1 + \left( \frac{1-\alpha}{\alpha} \right)^2), O(\frac{1-\alpha}{\alpha}))$ . Since  $Y$  is subgamma, we get

$$\mathbb{E}[(Y - \mathbb{E}[Y])^{2k}] \leq k!(8\sigma^2)^k + (2k)!(4c)^{2k}, \quad \forall k \geq 1$$

and thus  $\forall k \geq 1$ ,

$$\begin{aligned} \mathbb{E}[(X - \mathbb{E}[X])^{2k} (Y - \mathbb{E}[Y])^{2k}] &= \mathbb{E}[(X - \mathbb{E}[X])^{2k}] \mathbb{E}[(Y - \mathbb{E}[Y])^{2k}] \\ &\leq \left( (1-\alpha) + \alpha \left( \frac{1-\alpha}{\alpha} \right)^{2k} \right) (k!(8\sigma^2)^k + (2k)!(4c)^{2k}) \\ &\leq k!(8\sigma^2)^k + (2k)!(4c)^{2k} + k! \left( 8\sigma^2 \left( \frac{1-\alpha}{\alpha} \right)^2 \right)^k + (2k)! \left( 4c \frac{1-\alpha}{\alpha} \right)^{2k} \\ &\leq k! \left( 16\sigma^2 \left( \frac{1-\alpha}{\alpha} \right)^2 \right)^k + (2k)! \left( 8c \frac{1-\alpha}{\alpha} \right)^{2k} \end{aligned}$$

Therefore,  $(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])$  is subgamma with parameter  $(4(16\sigma^2 + 64c^2)(\frac{1-\alpha}{\alpha})^2, 16c\frac{1-\alpha}{\alpha})$ .

Hence, by Lemma C.2,  $XY$  is subgamma with parameter  $(O((\sigma^2 + c^2 + 1)(\frac{1-\alpha}{\alpha})^2 + \sigma^2 + 1), O(\max(c\frac{1-\alpha}{\alpha}, \frac{1-\alpha}{\alpha}, c)))$ .  $\square$

**Lemma C.3** (Gaussian Chaos of Order 2 (Boucheron et al., 2013)). *Let  $\xi \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$  be an  $n$ -dimensional unit Gaussian random vector;  $\mathbf{A} \in \mathbb{R}^{n \times n}$  be a symmetric matrix, then for any  $t > 0$ ,*

$$\mathbb{P} \left[ |\xi^\top \mathbf{A} \xi - \mathbb{E}[\xi^\top \mathbf{A} \xi]| > 2 \|\mathbf{A}\|_F \sqrt{t} + 2 \|\mathbf{A}\|_2 t \right] \leq 2 \exp(-t)$$

Equivalently,

$$\mathbb{P} \left[ |\xi^\top \mathbf{A} \xi - \mathbb{E}[\xi^\top \mathbf{A} \xi]| > t \right] \leq 2 \exp \left( - \frac{t^2}{4 \|\mathbf{A}\|_F^2 + \|\mathbf{A}\|_2 t} \right)$$

## C.2. Other Auxiliary Results

**Lemma C.4** (Lemma E.2 in (Arora et al., 2019)). *For events  $\mathcal{A}, \mathcal{B}$ , define the event  $\mathcal{A} \Rightarrow \mathcal{B}$  as  $\neg \mathcal{A} \vee \mathcal{B}$ . Then  $\mathbb{P}[\mathcal{A} \Rightarrow \mathcal{B}] \geq \mathbb{P}[\mathcal{B} | \mathcal{A}]$ .*

**Lemma C.5** (Lemma E.3 in (Arora et al., 2019)). *Let  $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d)$ ,  $\mathbf{G} \in \mathbb{R}^{d \times k}$  be some fixed matrix, and random vector  $\mathbf{F} = \mathbf{w}^\top \mathbf{G}$ , then conditioned on the value of  $\mathbf{F}$ ,  $\mathbf{w}$  remains Gaussian in the null space of the column space of  $\mathbf{G}$ , i.e.,*

$$\Pi_{\mathbf{G}}^\perp \mathbf{w} \stackrel{D}{=}_{\mathbf{F}=\mathbf{w}^\top \mathbf{G}} \Pi_{\mathbf{G}}^\perp \tilde{\mathbf{w}}$$

where  $\tilde{\mathbf{w}} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d)$  is a fresh i.i.d. copy of  $\mathbf{w}$ .

### C.3. Proof of the Main Result

Now we prove our main result. Notice that we modify the setting by letting the mask  $\mathbf{m}_{ij}^{(h)} \sim \sqrt{\frac{1}{\alpha}} \text{Bernoulli}(\alpha)$  so that  $\mathbb{E}(\mathbf{m}_{ij}^{(h)})^2 = 1$ . From a high level, our proof follows the proof outline of (Arora et al., 2019).

**Theorem 1.2** (Main Theorem). *Consider an  $L$ -hidden-layer fully connected neural network with all the weights initialized with i.i.d. standard Gaussian distribution. Suppose all the weights except the input layer are pruned with probability  $1 - \alpha$  at the initialization and after pruning we rescale the weights by  $1/\sqrt{\alpha}$ . Fix  $\epsilon > 0$  and  $\delta \in (0, 1)$ . Suppose we use ReLU activation  $\sigma(z) = \max(0, z)$  and  $d_h \geq \Omega(\frac{1}{\alpha^2} \frac{L^6}{\epsilon^4} \log \frac{Ld_{h+1} \sum_{i=1}^h d_i + \sum_{h'=h'}^L d_{h'}}{\delta})$ . Then for any inputs  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{d_0}$  such that  $\|\mathbf{x}\|_2 \leq 1$ ,  $\|\mathbf{x}'\|_2 \leq 1$ , with probability at least  $1 - \delta$  we have*

$$\left| \left\langle \frac{\partial f(\boldsymbol{\theta}, \mathbf{x})}{\partial \boldsymbol{\theta}}, \frac{\partial f(\boldsymbol{\theta}, \mathbf{x}')}{\partial \boldsymbol{\theta}} \right\rangle - \boldsymbol{\Theta}^{(L)}(\mathbf{x}, \mathbf{x}') \right| \leq (L+1)\epsilon$$

Let  $\mathbf{m}_i^{(h)}$  denote the  $i$ -th row of the mask in  $h$ -th layer. We first define the following events:

- $\mathcal{A}_i^h(\mathbf{x}, \mathbf{x}', \epsilon_1) := \left\{ \left| \left( \mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}') \odot \mathbf{m}_i^{(h+1)} \right) - \Sigma^{(h)}(\mathbf{x}, \mathbf{x}') \right| \leq \epsilon_1 \right\}$ .
- $\mathcal{A}^h(\mathbf{x}, \mathbf{x}', \epsilon_1) = \bigcap_{i=1}^{d_{h+1}} \mathcal{A}_i^h(\mathbf{x}, \mathbf{x}', \epsilon_1) \cap \left\{ \left| \left( \mathbf{g}^{(h)}(\mathbf{x}) \right)^\top \mathbf{g}^{(h)}(\mathbf{x}') - \Sigma^{(h)}(\mathbf{x}, \mathbf{x}') \right| \leq \epsilon_1 \right\}$ .
- $\overline{\mathcal{A}}^h(\mathbf{x}, \mathbf{x}', \epsilon_1) = \mathcal{A}^h(\mathbf{x}, \mathbf{x}, \epsilon_1) \cap \mathcal{A}^h(\mathbf{x}, \mathbf{x}', \epsilon_1) \cap \mathcal{A}^h(\mathbf{x}', \mathbf{x}', \epsilon_1)$ .
- $\overline{\mathcal{A}}(\mathbf{x}, \mathbf{x}', \epsilon_1) = \bigcap_{h=0}^L \overline{\mathcal{A}}^h(\mathbf{x}, \mathbf{x}', \epsilon_1)$ .
- $\mathcal{B}^h(\mathbf{x}, \mathbf{x}', \epsilon_2) = \left\{ \left| \langle \mathbf{b}^{(h)}(\mathbf{x}), \mathbf{b}^{(h)}(\mathbf{x}') \rangle - \prod_{h=h}^L \dot{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') \right| < \epsilon_2 \right\}$ .
- $\overline{\mathcal{B}}^h(\mathbf{x}, \mathbf{x}', \epsilon_2) = \mathcal{B}^h(\mathbf{x}, \mathbf{x}, \epsilon_2) \cap \mathcal{B}^h(\mathbf{x}, \mathbf{x}', \epsilon_2) \cap \mathcal{B}^h(\mathbf{x}', \mathbf{x}', \epsilon_2)$ .
- $\overline{\mathcal{B}} = \bigcap_{h=1}^{L+1} \overline{\mathcal{B}}^h(\mathbf{x}, \mathbf{x}', \epsilon_2)$ .
- $\overline{\mathcal{C}}(\epsilon_3)$ : a special event defined in Lemma C.15.
- $\mathcal{D}_i^h(\mathbf{x}, \mathbf{x}', \epsilon_4) = \left\{ \left| 2 \frac{\text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D}^{(h)}(\mathbf{x}, \mathbf{x}') \mathbf{M}_i^{(h+1)})}{d_h} - \dot{\Sigma}^{(h)}(\mathbf{x}, \mathbf{x}') \right| < \epsilon_4 \right\}$  where  $\mathbf{M}_i^{(h+1)} = \text{diag}(\mathbf{m}_i^{(h+1)})$ .
- $\mathcal{D}^h(\mathbf{x}, \mathbf{x}', \epsilon_4) = \bigcap_{i=1}^{d_{h+1}} \mathcal{D}_i^h(\mathbf{x}, \mathbf{x}', \epsilon_4)$ .
- $\overline{\mathcal{D}}(\mathbf{x}, \mathbf{x}', \epsilon_4) = \mathcal{D}^h(\mathbf{x}, \mathbf{x}, \epsilon_4) \cap \mathcal{D}^h(\mathbf{x}, \mathbf{x}', \epsilon_4) \cap \mathcal{D}^h(\mathbf{x}', \mathbf{x}', \epsilon_4)$ .
- $\overline{\mathcal{D}}(\mathbf{x}, \mathbf{x}', \epsilon_4) = \bigcap_{h=1}^{L+1} \overline{\mathcal{D}}^h(\epsilon_4)$ .

The proof of our main theorem is based on the following theorem.

**Theorem C.6.** *Let  $\sigma(z) = \max(0, z)$ ,  $z \in \mathbb{R}$ , if  $[\mathbf{W}^{(h)}]_{ij} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$ ,  $[\mathbf{m}^{(h)}]_{ij} \stackrel{i.i.d.}{\sim} \frac{1}{\sqrt{\alpha}} \text{Bernoulli}(\alpha)$ ,  $\forall h \in [L+1]$ ,  $i \in [d_{h+1}]$ ,  $j \in [d_h]$ , then if  $d_h \gtrsim \frac{1}{\alpha^2} \frac{L^6}{\epsilon^4} \log \frac{Ld_{h+1} \sum_{i=1}^h d_i + \sum_{h'=h'}^L d_{h'}}{\delta}$ ,  $h \in [L+1]$  and  $\epsilon \leq \frac{c}{L}$  for some constant  $c$ , then for any fixed  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{d_0}$ ,  $\|\mathbf{x}\|_2, \|\mathbf{x}'\|_2 \leq 1$ , we have with probability  $1 - \delta$ ,  $\forall 0 \leq h \leq L$ ,  $\forall (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \in \{(\mathbf{x}, \mathbf{x}), (\mathbf{x}, \mathbf{x}'), (\mathbf{x}', \mathbf{x}')\}$ ,*

$$\left| \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) - \Sigma^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| \leq \epsilon^2/2, \quad \forall i \in [d_{h+1}]$$

and

$$\left| \langle \mathbf{b}^{(h)}(\mathbf{x}^{(1)}), \mathbf{b}^{(h)}(\mathbf{x}^{(2)}) \rangle - \prod_{h'=h}^L \dot{\Sigma}^{(h')}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| < 3L\epsilon$$



In other words, if  $d_h \gtrsim \frac{1}{\alpha^2} \frac{L^6}{\epsilon^4} \log \frac{L d_{h+1} \sum_{i=1}^h d_i + \sum_{h=h'}^L d_h}{\delta}$ ,  $h \in [L+1]$  and  $\epsilon \leq \frac{c}{L}$  for some constant  $c$ , then for fixed  $\mathbf{x}, \mathbf{x}'$ ,

$$\mathbb{P} \left[ \bar{\mathcal{A}} \left( \frac{\epsilon_1^2}{2} \cap \bar{\mathcal{B}}(3L\epsilon) \right) \right] \geq 1 - \delta$$

**Lemma C.7.**

$$\mathbb{P} \left[ \bar{\mathcal{A}}^{h+1}(\epsilon_1^2/2) \Rightarrow \bar{\mathcal{D}}^h \left( \epsilon_1 + C \sqrt{\left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{\log \frac{2d_{h+1}}{\delta}}{d_h}} \right) \right] \geq 1 - \delta_4$$

and by taking a union bound,

$$\mathbb{P} \left[ \bar{\mathcal{A}}(\epsilon_1^2/2) \Rightarrow \bar{\mathcal{D}} \left( \epsilon_1 + \max_h C \sqrt{\left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{\log \frac{2d_{h+1}}{\delta}}{d_h}} \right) \right] \geq 1 - \delta_4$$

for some constant  $C$ .

**Lemma C.8.** If  $d_{h'} \geq c_1 \left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{L^2 \log \frac{8 \sum_{h \leq h'} d_h d_{h'+1} L}{\delta_3}}{\epsilon^2}$ , with probability  $1 - \delta_3$ , the event  $\bar{\mathcal{C}}(\sqrt{\log \frac{\sum d_h}{\delta_3}})$  holds. Further, there exists constant  $C, C'$  such that for any  $\epsilon_2, \epsilon_4 \in [0, 1]$ , we have

$$\mathbb{P} \left[ \bar{\mathcal{A}}^L(\epsilon_1^2/2) \cap \bar{\mathcal{B}}^{h+1}(\epsilon_2) \cap \bar{\mathcal{C}}(\epsilon_3) \cap \bar{\mathcal{D}}^h(\epsilon_4) \Rightarrow \bar{\mathcal{B}}^h \left( \epsilon_2 + \frac{C\epsilon_3}{\sqrt{\alpha^2 d_h}} + 2\epsilon_4 + C' \sqrt{\frac{\log \frac{1}{\delta_2}}{\alpha d_h}} \right) \right] \geq 1 - \delta_2/2$$

*Proof of Theorem C.6.* The proof of the theorem largely follows the proof of Theorem E.2. in (Arora et al., 2019) with some slight modification. Nonetheless, we give out the proof for completeness.

We prove by induction on Lemma C.8. In the statement of Theorem C.9, we set  $\delta_1 = \delta/4$ ,  $\epsilon_1 = \frac{\epsilon^2}{8}$ , we have

$$\mathbb{P}[\bar{\mathcal{A}}(\epsilon^2/8)] \geq 1 - \delta/4$$

In the statement of Lemma C.7, we set  $\delta_4 = \delta/4$  and  $\epsilon_1 = \epsilon/2$ . We can pick  $c_1$  large enough such that  $\max_h C \sqrt{\left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{\log \frac{2d_{h+1}}{\delta}}{d_h}} \leq \frac{\epsilon}{2}$  and we have

$$\mathbb{P}[\bar{\mathcal{A}}(\epsilon^2/8) \Rightarrow \bar{\mathcal{D}}(\epsilon)] \geq \mathbb{P} \left[ \bar{\mathcal{A}}(\epsilon_1^2/2) \Rightarrow \bar{\mathcal{D}} \left( \epsilon_1 + \max_h C \sqrt{\left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{\log \frac{2d_{h+1}}{\delta}}{d_h}} \right) \right] \geq 1 - \delta/4$$

In the statement of Lemma C.8, if we set  $\delta_3 = \delta/4$ , then

$$\mathbb{P} \left[ \bar{\mathcal{C}} \left( \sqrt{\log \frac{4 \sum_h d_h}{\delta}} \right) \right] \geq 1 - \delta/4$$

Take a union bound we have

$$\mathbb{P} \left[ \bar{\mathcal{A}}(\epsilon^2/2) \cap \bar{\mathcal{C}} \left( \sqrt{\log \frac{4 \sum_h d_h}{\delta}} \right) \cap \bar{\mathcal{D}}(\epsilon) \right] \geq 1 - \frac{3\delta}{4}$$

Now we begin the induction. First of all,  $\mathbb{P}[\bar{\mathcal{B}}^{L+1}(0)] = 1$  by definition. For  $1 \leq h \leq L$ , in the statement of Lemma C.8,

set  $\epsilon_2 = 3(L+1-h)\epsilon$ ,  $\epsilon_3 = \sqrt{\log \frac{\sum d_h}{\delta}}$ ,  $\delta_2 = \frac{\delta}{4L}$ . For  $c_1$  large enough, we have  $C \sqrt{\frac{\log \frac{4 \sum d_h}{\delta}}{\alpha^2 d_h}} + C' \sqrt{\frac{\log \frac{L}{\delta}}{\alpha d_h}} < \epsilon$ . Thus we have

$$\mathbb{P} \left[ \bar{\mathcal{B}}^{(h+1)}((3L-3h)\epsilon) \cap \bar{\mathcal{C}} \left( 3 \sqrt{\log \frac{\sum d_h}{\delta}} \right) \cap \bar{\mathcal{D}}(\epsilon) \Rightarrow \bar{\mathcal{B}}^h \left( (3L+2-3h)\epsilon + C \sqrt{\frac{\log \frac{4 \sum d_h}{\delta}}{\alpha^2 d_h}} + C' \sqrt{\frac{\log \frac{L}{\delta}}{\alpha d_h}} \right) \right]$$

$$\begin{aligned}
 &\geq \mathbb{P} \left[ \bar{\mathcal{B}}^{(h+1)}((3L-3h)\epsilon) \cap \bar{\mathcal{C}} \left( 3\sqrt{\log \frac{\sum d_h}{\delta}} \right) \cap \bar{\mathcal{D}}(\epsilon) \Rightarrow \bar{\mathcal{B}}^h((3L+3-3h)\epsilon) \right] \\
 &\geq 1 - \frac{\delta}{4L}
 \end{aligned}$$

Applying union bound for every  $h \in [L]$ , we have

$$\begin{aligned}
 &\mathbb{P} \left[ \bar{\mathcal{A}}^L(\epsilon^2/8) \cap \bar{\mathcal{B}}(3L\epsilon) \cap \bar{\mathcal{C}}(\epsilon) \cap \bar{\mathcal{D}}(\epsilon) \right] \\
 &\geq \mathbb{P} \left[ \bar{\mathcal{A}}^L(\epsilon^2/8) \bigcap_{h=1}^L \bar{\mathcal{B}}^h(3(L+1-h)\epsilon) \cap \bar{\mathcal{C}}(\epsilon) \cap \bar{\mathcal{D}}(\epsilon) \right] \\
 &\geq 1 - \mathbb{P} \left[ \neg \left( \bar{\mathcal{A}}(\epsilon^2/8) \cap \bar{\mathcal{C}}(\epsilon) \cap \bar{\mathcal{D}}^h(\epsilon) \right) \right] \\
 &\quad - \sum_{h=1}^L \mathbb{P} \left[ \neg \left( \bar{\mathcal{B}}^{(h+1)}((3L-3h)\epsilon) \cap \bar{\mathcal{C}} \left( 3\sqrt{\log \frac{\sum d_h}{\delta}} \right) \cap \bar{\mathcal{D}}(\epsilon) \Rightarrow \bar{\mathcal{B}}^h((3L+3-3h)\epsilon) \right) \right] \\
 &\geq 1 - \delta
 \end{aligned}$$

□

#### C.4. Forward Propagation

First we want to obtain bounds on  $\left| (\mathbf{m}^{(h+1)} \odot \mathbf{g}^{(h)}(\mathbf{x}))^\top (\mathbf{m}^{(h+1)} \odot \mathbf{g}^{(h)}(\mathbf{x}') - \Sigma^{(h)}(\mathbf{x}, \mathbf{x}') \right|$ . We first prove a theorem corresponding to Corollary 16 in (Daniely et al., 2016).

**Theorem C.9.** *Let  $\sigma(z) = \max(0, z)$ ,  $z \in \mathbb{R}$  and  $[\mathbf{W}^{(h)}]_{ij} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$ ,  $[\mathbf{m}^{(h)}]_{ij} \stackrel{i.i.d.}{\sim} \frac{1}{\sqrt{\alpha}} \text{Bernoulli}(\alpha)$ ,  $\forall h \in [L]$ ,  $i \in [d_{h+1}]$ ,  $j \in [d_h]$ , there exist constants  $c_1, c_2$  such that if  $c_1 \left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{L^2 \log \frac{8d_{h+1}L}{\delta}}{\epsilon^2} \leq d_h$ ,  $\forall h \in \{1, 2, \dots, L\}$  and  $\epsilon \leq \min(c_2, \frac{1}{L})$  then for any fixed  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{d_0}$ ,  $\|\mathbf{x}\|_2, \|\mathbf{x}'\|_2 \leq 1$ , we have with probability  $1 - \delta$ ,  $\forall 0 \leq h \leq L$ ,  $\forall i \in [d_{h+1}]$ ,  $\forall (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \in \{(\mathbf{x}, \mathbf{x}), (\mathbf{x}, \mathbf{x}'), (\mathbf{x}', \mathbf{x}')\}$ ,*

$$\left| \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) - \Sigma^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| \leq \epsilon$$

In other words, if  $d_h \geq c_1 \left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{L^2 \log \frac{8d_{h+1}L}{\delta}}{\epsilon^2}$ ,  $\forall h \in \{1, 2, \dots, L\}$  and  $\epsilon \leq \min(c_2, \frac{1}{L})$  then for fixed  $\mathbf{x}, \mathbf{x}'$ ,

$$\mathbb{P} [\bar{\mathcal{A}}(\epsilon_1)] \geq 1 - \delta_1$$

We remark that when taking  $\alpha = 1$  (i.e., no pruning is done), our analysis is optimal compared to the original analysis in Corollary 16 of (Daniely et al., 2016) up to a logarithmic factor in  $d_{h+1}$  which arises in taking a union bound over the rows of mask  $\mathbf{m}^{(h+1)}$ . The extra variance factor  $\left( \frac{1-\alpha}{\alpha} \right)^2$  is precisely due to the presence of random mask.

*Proof.* Our analysis follows from the proof of Theorem 14 in (Daniely et al., 2016). First of all, we introduce some notations in the referenced paper:

$$\bar{\sigma}(\Sigma) = c_\sigma \mathbb{E}_{(X,Y) \sim \mathcal{N}(\mathbf{0}, \Sigma)} \sigma(X) \sigma(Y)$$

and the set

$$\mathcal{M}_+^\gamma := \left\{ \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix} \in \mathcal{M}_+ \mid 1 - \gamma \leq \Sigma_{11}, \Sigma_{22} \leq 1 + \gamma \right\}$$

where  $\mathcal{M}_+$  denote the set of positive semi-definite matrices. From Lemma 13 in (Daniely et al., 2016), we have that  $\bar{\sigma}$  is  $(1 + o(\epsilon))$ -Lipschitz on  $\mathcal{M}_+^\epsilon$  with respect to  $\infty$ -norm. Define the quantity  $B_d = \sum_{i=1}^d (1 + o(\epsilon))^i$ .

We begin our proof by saying the  $h$ -th layer of a neural network is well-initialized if  $\forall i \in [d_{h+1}]$ , we have

$$\left| \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) - \Sigma^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| \leq \epsilon \frac{B_h}{B_L}$$

We prove the result by induction. Since we don't prune the input layer, the result trivially holds for  $h = 0$ . Assume all the layers first  $h - 1$  layers are well-initialized.

Now, conditioned on  $\mathbf{g}^{(h-1)}(\mathbf{x}^{(1)}), \mathbf{g}^{(h-1)}(\mathbf{x}^{(2)}), \mathbf{m}^{(h)}$ , we have

$$\begin{aligned} & \mathbb{E}_{\mathbf{W}^{(h)}, \mathbf{m}^{(h+1)}} \left[ \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) \right] \\ &= \mathbb{E}_{\mathbf{W}^{(h)}} \left[ \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \right)^\top \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \right] \\ &= \frac{c_\sigma}{d_h} \sum_{i=1}^{d_h} \mathbb{E}_{\mathbf{W}^{(h)}} \left[ \sigma \left( \langle \mathbf{W}_i^{(h)}, \mathbf{m}^{(h)} \odot \mathbf{g}^{(h-1)}(\mathbf{x}^{(1)}) \rangle \right) \sigma \left( \langle \mathbf{W}_i^{(h)}, \mathbf{m}^{(h)} \odot \mathbf{g}^{(h-1)}(\mathbf{x}^{(2)}) \rangle \right) \right] \end{aligned}$$

Define

$$\begin{aligned} \widehat{\Sigma}_i^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) &= \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) \\ \widehat{\Lambda}_i^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) &= \begin{bmatrix} \widehat{\Sigma}_i^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(1)}) & \widehat{\Sigma}_i^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \\ \widehat{\Sigma}_i^{(h)}(\mathbf{x}^{(2)}, \mathbf{x}^{(1)}) & \widehat{\Sigma}_i^{(h)}(\mathbf{x}^{(2)}, \mathbf{x}^{(2)}) \end{bmatrix} \end{aligned}$$

Now apply Lemma 4.1, we know for a given  $i, j$ ,  $\mathbf{g}_j^{(h)}(\mathbf{x}^{(1)}) \mathbf{g}_j^{(h)}(\mathbf{x}^{(2)}) \left( \mathbf{m}_{ij}^{(h+1)} \right)^2$  is subgamma with parameters  $(\frac{1}{d_h^2} O((\frac{1-\alpha}{\alpha})^2 + 1), \frac{1}{d_h} O(\frac{1-\alpha}{\alpha} + 1))$  and thus  $\widehat{\Sigma}_i^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$  is subgamma with parameters  $(\frac{1}{d_h} O((\frac{1-\alpha}{\alpha})^2 + 1), \frac{1}{d_h} O(\frac{1-\alpha}{\alpha} + 1))$ . By the property of subgamma random variables, we have

$$\mathbb{P} \left[ \left| \widehat{\Sigma}_i^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) - \mathbb{E}_{\mathbf{W}^{(h)}, \mathbf{m}^{(h+1)}} \widehat{\Sigma}_i^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| > \epsilon \right] \leq 2 \exp \left\{ \frac{d_h \epsilon^2}{2 O((\frac{1-\alpha}{\alpha})^2 + 1)} \right\}$$

for some constant  $c_2$  such that  $\epsilon < c_2$ .

Taking a union bound over  $i \in [d_{h+1}]$ , we have if  $d_h \geq c_1 \left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{B_L^2 \log \frac{8d_{h+1}L}{\delta}}{\epsilon^2}$ , then with probability  $1 - \frac{\delta}{L}$  for all  $i \in [d_{h+1}]$ ,

$$\left| \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) - \frac{c_\sigma}{d_h} \sum_{j=1}^{d_h} \mathbb{E}_{(u,v) \sim \mathcal{N}(\mathbf{0}, \widehat{\Lambda}_j^{(h-1)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}))} [\sigma(u)\sigma(v)] \right| \leq \epsilon/B_L$$

Now apply triangle inequality

$$\begin{aligned} & \left| \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) - \Sigma^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| \\ & \leq \left| \left( \mathbf{g}^{(h)}(\mathbf{x}^{(1)}) \odot \mathbf{m}_i^{(h+1)} \right)^\top \left( \mathbf{g}^{(h)}(\mathbf{x}^{(2)}) \odot \mathbf{m}_i^{(h+1)} \right) - \frac{c_\sigma}{d_h} \sum_{j=1}^{d_h} \mathbb{E}_{(u,v) \sim \mathcal{N}(\mathbf{0}, \widehat{\Lambda}_j^{(h-1)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}))} [\sigma(u)\sigma(v)] \right| \\ & \quad + \left| \frac{c_\sigma}{d_h} \sum_{j=1}^{d_h} \mathbb{E}_{(u,v) \sim \mathcal{N}(\mathbf{0}, \widehat{\Lambda}_j^{(h-1)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}))} [\sigma(u)\sigma(v)] - \Sigma^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| \\ & \leq \epsilon/B_L + \frac{1}{d_h} \sum_{i=1}^{d_h} \left| c_\sigma \mathbb{E}_{(u,v) \sim \mathcal{N}(\mathbf{0}, \widehat{\Lambda}_j^{(h-1)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}))} [\sigma(u)\sigma(v)] - \Sigma^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| \end{aligned}$$

$$\leq \epsilon/B_L + \frac{1}{d_h} \sum_{i=1}^{d_h} (1 + o(\epsilon)) \epsilon \frac{B_{h-1}}{B_L} = \epsilon \frac{B_h}{B_L}$$

where the last inequality applies by the fact that  $\bar{\sigma}$  is  $(1 + o(\epsilon))$ -Lipschitz on  $\mathcal{M}_+^\gamma$  with respect to the  $\infty$ -norm and the induction hypothesis that the first  $h - 1$  layers are well-initialized.

Finally we expand  $B_d = \sum_{i=1}^d (1 + o(\epsilon))^i$  and take  $\epsilon = \min(c_2, \frac{1}{L})$ , we have

$$B_L = \sum_{i=1}^L (1 + o(\epsilon))^i \leq \sum_{i=1}^L e^{o(\epsilon)L} = O(L)$$

□

**Lemma C.10.** *Given a fixed vector  $\mathbf{x}$  and a vector  $\mathbf{w}$  whose elements are i.i.d. Gaussian random variables. With probability  $1 - \delta$ , we have*

$$|\mathbf{w}^\top \mathbf{x}| \leq \sqrt{2 \log \frac{2}{\delta}} \|\mathbf{x}\|_2$$

*Proof.* Since  $\mathbf{w}^\top \mathbf{x} \sim \mathcal{N}(0, \|\mathbf{x}\|_2^2)$ . By subgaussian concentration, with probability at least  $1 - \delta$ , we have  $|\mathbf{w}^\top \mathbf{x}| \leq \sqrt{2 \log \frac{2}{\delta}} \|\mathbf{x}\|_2$ . □

## C.5. Backward Propagation

**Lemma C.7.**

$$\mathbb{P} \left[ \bar{\mathcal{A}}^{h+1}(\epsilon_1^2/2) \Rightarrow \bar{\mathcal{D}}^h \left( \epsilon_1 + C \sqrt{\left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1\right) \frac{\log \frac{2d_{h+1}}{\delta}}{d_h}} \right) \right] \geq 1 - \delta_4$$

for some constant  $C$ .

In order to prove this lemma, we use a known result.

**Lemma C.11** (Lemma E.8. (Arora et al., 2019)). *Let*

$$t_{\dot{\sigma}}(\Sigma) = c_{\sigma} \mathbb{E}_{(u,v) \sim \mathcal{N}(\mathbf{0}, \Sigma')} [\dot{\sigma}(u) \dot{\sigma}(v)] \quad \text{with} \quad \Sigma' = \begin{bmatrix} 1 & \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \\ \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} & 1 \end{bmatrix}$$

then

$$\left\| \mathbf{G}^{(h)}(\mathbf{x}, \mathbf{x}') - \mathbf{\Lambda}^{(h)}(\mathbf{x}, \mathbf{x}') \right\|_{\infty} \leq \frac{\epsilon^2}{2} \Rightarrow \left| t_{\dot{\sigma}} \left( \mathbf{G}^{(h)}(\mathbf{x}, \mathbf{x}') \right) - t_{\dot{\sigma}} \left( \mathbf{\Lambda}^{(h)}(\mathbf{x}, \mathbf{x}') \right) \right| \leq \epsilon$$

*Proof of Lemma C.7.* Conditioned on  $\hat{\mathbf{\Lambda}}_i^{(h)}$ ,  $\forall i \in [d_h]$  and consider the randomness of  $\mathbf{W}^{(h)}$ ,  $\mathbf{m}^{(h+1)}$ , we have

$$\begin{aligned} & \mathbb{E}_{\mathbf{W}^{(h)}, \mathbf{m}^{(h+1)}} \left[ 2 \frac{\text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D}^{(h)}(\mathbf{x}, \mathbf{x}') \mathbf{M}_i^{(h+1)})}{d_h} \right] \\ &= \mathbb{E}_{\mathbf{W}^{(h)}} \left[ 2 \frac{\text{Tr}(\mathbf{D}^{(h)}(\mathbf{x}, \mathbf{x}'))}{d_h} \right] \\ &= \frac{1}{d_h} \sum_{i=1}^{d_h} \mathbb{E}_{\mathbf{W}^{(h)}} \left[ \dot{\sigma} \left( \left\langle \mathbf{W}_i^{(h)}, \mathbf{m}_i^{(h)} \odot \mathbf{g}^{(h-1)}(\mathbf{x}) \right\rangle \right) \dot{\sigma} \left( \left\langle \mathbf{W}_i^{(h)}, \mathbf{m}_i^{(h)} \odot \mathbf{g}^{(h-1)}(\mathbf{x}') \right\rangle \right) \right] \\ &= \frac{1}{d_h} \sum_{i=1}^{d_h} t_{\dot{\sigma}} \left( \hat{\mathbf{\Lambda}}_i^{(h)} \right) \end{aligned}$$



Now, by triangle inequality and our assumption on  $\widehat{\mathbf{\Lambda}}_i$ ,  $\forall i \in [d_h]$ , apply Lemma C.11

$$\left| t_{\dot{\sigma}} \left( \mathbf{\Lambda}^{(h)}(\mathbf{x}, \mathbf{x}') \right) - \frac{1}{d_h} \sum_{i=1}^{d_h} t_{\dot{\sigma}} \left( \widehat{\mathbf{\Lambda}}_i^{(h)} \right) \right| \leq \frac{1}{d_h} \sum_{i=1}^{d_h} \left| t_{\dot{\sigma}} \left( \mathbf{\Lambda}^{(h)}(\mathbf{x}, \mathbf{x}') \right) - t_{\dot{\sigma}} \left( \widehat{\mathbf{\Lambda}}_i^{(h)} \right) \right| \leq \epsilon$$

Finally, since  $\dot{\sigma}(\mathbf{f}_j^{(h)}(\mathbf{x}))\dot{\sigma}(\mathbf{f}_j^{(h)}(\mathbf{x}'))$  is a 0-1 random variable, it is subgaussian with variance proxy  $\frac{1}{4}$  which is the same as subgamma with parameters  $(\frac{1}{4}, 0)$ . Apply Lemma 4.1, we have  $\left( \mathbf{m}_{ij}^{(h+1)} \right)^2 \dot{\sigma}(\mathbf{f}_j^{(h)}(\mathbf{x}))\dot{\sigma}(\mathbf{f}_j^{(h)}(\mathbf{x}'))$  is subgamma with parameters  $(O((\frac{1-\alpha}{\alpha})^2 + 1), O(\frac{1-\alpha}{\alpha} + 1))$ . Thus,  $\frac{2}{d_h} \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D}^{(h)}(\mathbf{x}, \mathbf{x}') \mathbf{M}_i^{(h+1)})$  is subgamma with parameters  $(\frac{1}{d_h} O((\frac{1-\alpha}{\alpha})^2 + 1), \frac{1}{d_h} O(\frac{1-\alpha}{\alpha} + 1))$ . By the concentration of subgamma random variables, for a given  $i$  and small enough  $t$ ,

$$\mathbb{P} \left[ \left| 2 \frac{\text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D}^{(h)}(\mathbf{x}, \mathbf{x}') \mathbf{M}_i^{(h+1)})}{d_h} - \frac{1}{d_h} \sum_{i=1}^{d_h} t_{\dot{\sigma}} \left( \widehat{\mathbf{\Lambda}}_i^{(h)} \right) \right| > t \right] \leq 2 \exp \left\{ -\frac{d_h t^2}{2O((\frac{1-\alpha}{\alpha})^2 + 1)} \right\}$$

Finally, by taking a union bound over  $i \in [d_{h+1}]$ , with probability  $1 - \delta$  over the randomness of  $\mathbf{W}^{(h)}, \mathbf{m}^{(h+1)}$ , we have  $\forall i \in [d_{h+1}]$ ,

$$\left| 2 \frac{\text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D}^{(h)}(\mathbf{x}, \mathbf{x}') \mathbf{M}_i^{(h+1)})}{d_h} - \frac{1}{d_h} \sum_{i=1}^{d_h} t_{\dot{\sigma}} \left( \widehat{\mathbf{\Lambda}}_i^{(h)} \right) \right| < \sqrt{2O \left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{\log \frac{2d_{h+1}}{\delta}}{d_h}}$$

□

## C.6. The Fresh Gaussian Copy Trick

**Lemma C.8.** If  $d_{h'} \geq c_1 \left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{L^2 \log \frac{8 \sum_{h \leq h'} d_h d_{h'+1} L}{\delta_3}}{\epsilon^2}$ , with probability  $1 - \delta_3$ , the event  $\bar{\mathcal{C}}(\sqrt{\log \frac{\sum d_h}{\delta_3}})$  holds. Further, there exists constant  $C, C'$  such that for any  $\epsilon_2, \epsilon_4 \in [0, 1]$ , we have

$$\mathbb{P} \left[ \bar{\mathcal{A}}^L(\epsilon_1^2/2) \cap \bar{\mathcal{B}}^{h+1}(\epsilon_2) \cap \bar{\mathcal{C}}(\epsilon_3) \cap \bar{\mathcal{D}}^h(\epsilon_4) \Rightarrow \bar{\mathcal{B}}^h \left( \epsilon_2 + \frac{C\epsilon_3}{\sqrt{\alpha^2 d_h}} + 2\epsilon_4 + C' \sqrt{\frac{\log \frac{1}{\delta_2}}{\alpha d_h}} \right) \right] \geq 1 - \delta_2/2$$

*Proof.* The proof is based on Proposition C.12, Proposition 4.3 and Proposition 4.4. □

**Proposition C.12.** If  $\bar{\mathcal{A}}^L(\epsilon_1^2/2) \cap \bar{\mathcal{B}}^{h+1}(\epsilon_2) \cap \bar{\mathcal{C}}(\epsilon_3) \cap \bar{\mathcal{D}}^h(\epsilon_4)$ , then we have

$$\left| \frac{2}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) - \prod_{h'=h}^L \dot{\Sigma}^{(h')}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| \leq \epsilon_2 + 2\epsilon_4$$

*Proof.*

$$\begin{aligned} & \left| \frac{2}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) - \prod_{h'=h}^L \dot{\Sigma}^{(h')}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right| \\ & \leq \left| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \left( \frac{2}{d_h} \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) - \dot{\Sigma}^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right) \right| \\ & \quad + \left| \dot{\Sigma}^{(h)}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \left( \left\langle \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}) \right\rangle - \prod_{h'=h+1}^L \dot{\Sigma}^{(h')}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \right) \right| \\ & \leq \left\| \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}) \right\|_2 \left\| \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}) \right\|_2 \epsilon_4 + \epsilon_2 \\ & = 2\epsilon_4 + \epsilon_2 \end{aligned}$$

□

Let  $\mathbf{G}_i^{(h)} = [(\mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)}) \quad (\mathbf{g}^{(h)}(\mathbf{x}') \odot \mathbf{m}_i^{(h+1)})]$  and  $\mathbf{G}^{(h)} = [\mathbf{G}_1^{(h)} \mathbf{G}_2^{(h)} \dots \mathbf{G}_{d_{h+1}}^{(h)}]$  and  $\mathbf{F}^{(h+1)} = (\mathbf{W}^{(h+1)} \odot \mathbf{m}^{(h+1)}) \mathbf{G}^{(h)}$ . Notice that conditioned on  $\mathbf{F}^{(h+1)}, \mathbf{m}^{(h+1)}, \mathbf{G}^{(h)}$ , we have

$$\left( \mathbf{b}^{(h+1)}(\mathbf{x}) \right)^\top \begin{bmatrix} (\mathbf{w}_1^{(h+1)} \odot \mathbf{m}_1^{(h+1)})^\top \Pi_{\mathbf{G}_1}^\perp \\ (\mathbf{w}_2^{(h+1)} \odot \mathbf{m}_2^{(h+1)})^\top \Pi_{\mathbf{G}_2}^\perp \\ \vdots \\ (\mathbf{w}_{d_{h+1}}^{(h+1)} \odot \mathbf{m}_{d_{h+1}}^{(h+1)})^\top \Pi_{\mathbf{G}_{d_{h+1}}}^\perp \end{bmatrix} \in \mathbb{R}^{d_h}$$

has multivariate Gaussian distribution and by Lemma C.5 it has the same distribution as

$$\left( \mathbf{b}^{(h+1)}(\mathbf{x}) \right)^\top \begin{bmatrix} (\tilde{\mathbf{w}}_1^{(h+1)} \odot \mathbf{m}_1^{(h+1)})^\top \Pi_{\mathbf{G}_1}^\perp \\ (\tilde{\mathbf{w}}_2^{(h+1)} \odot \mathbf{m}_2^{(h+1)})^\top \Pi_{\mathbf{G}_2}^\perp \\ \vdots \\ (\tilde{\mathbf{w}}_{d_{h+1}}^{(h+1)} \odot \mathbf{m}_{d_{h+1}}^{(h+1)})^\top \Pi_{\mathbf{G}_{d_{h+1}}}^\perp \end{bmatrix} = \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}) (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \Pi_{\mathbf{G}_i}^\perp$$

where  $\tilde{\mathbf{w}}_i^{(h+1)}$  is a fresh copy of i.i.d. Gaussian. First of all, let  $\mathbf{M}_i^{(h+1)} = \text{diag}(\mathbf{m}_i^{(h+1)})$ , and we have

$$\begin{aligned} & \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} (\Pi_{\mathbf{G}_i} + \Pi_{\mathbf{G}_i}^\perp) \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) (\Pi_{\mathbf{G}_j} + \Pi_{\mathbf{G}_j}^\perp) \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \\ &= \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) \Pi_{\mathbf{G}_j}^\perp \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \\ &+ \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) \Pi_{\mathbf{G}_j} \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \\ &+ \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i} \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) \Pi_{\mathbf{G}_j}^\perp \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \\ &+ \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i} \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) \Pi_{\mathbf{G}_j} \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \end{aligned}$$

**Proposition C.13.**  $\mathbf{M}_i^{(h+1)}$  commutes with  $\Pi_{\mathbf{G}_i}$  (and thus  $\Pi_{\mathbf{G}_i}^\perp$ ).

*Proof.* We can decompose  $\Pi_{\mathbf{G}_i} = \Pi_{\mathbf{M}_i \mathbf{g}_1} + \Pi_{\mathbf{G}_i / \mathbf{M}_i \mathbf{g}_1}$ . Observe that  $\Pi_{\mathbf{G}_i / \mathbf{M}_i \mathbf{g}_1}$  is projecting a vector into the space spanned by  $\mathbf{M}_i \mathbf{g}_2 - \langle \mathbf{M}_i \mathbf{g}_1, \mathbf{M}_i \mathbf{g}_2 \rangle \mathbf{M}_i \mathbf{g}_1 = \mathbf{M}_i (\mathbf{g}_2 - \langle \mathbf{M}_i \mathbf{g}_1, \mathbf{M}_i \mathbf{g}_2 \rangle \mathbf{g}_1)$ . Thus, we can first prove  $\mathbf{M}_i$  commutes with  $\Pi_{\mathbf{M}_i \mathbf{g}_1}$  and the same result follows for  $\Pi_{\mathbf{G}_i / \mathbf{M}_i \mathbf{g}_1}$ . Notice that  $\mathbf{M}_i \Pi_{\mathbf{M}_i \mathbf{g}_1} = \mathbf{M}_i \frac{\mathbf{M}_i \mathbf{g}_1 (\mathbf{M}_i \mathbf{g}_1)^\top}{\|\mathbf{M}_i \mathbf{g}_1\|_2^2} = \frac{1}{\sqrt{\alpha}} \frac{\mathbf{M}_i \mathbf{g}_1 (\mathbf{M}_i \mathbf{g}_1)^\top}{\|\mathbf{M}_i \mathbf{g}_1\|_2^2} = \frac{\mathbf{M}_i \mathbf{g}_1 (\mathbf{M}_i \mathbf{g}_1)^\top}{\|\mathbf{M}_i \mathbf{g}_1\|_2^2} \mathbf{M}_i = \Pi_{\mathbf{M}_i \mathbf{g}_1} \mathbf{M}_i$ .  $\square$

We want to prove a claim corresponding to Claim E.2 of (Arora et al., 2019).

**Proposition 4.3.** With probability at least  $1 - \delta_2$ , if  $\overline{\mathcal{A}}^L(\epsilon_1^2/2) \cap \overline{\mathcal{B}}^{h+1}(\epsilon_2) \cap \overline{\mathcal{C}}(\epsilon_3) \cap \overline{\mathcal{D}}^h(\epsilon_4)$ , then for any  $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \in \{(\mathbf{x}, \mathbf{x}), (\mathbf{x}, \mathbf{x}'), (\mathbf{x}', \mathbf{x}')\}$ , we have

$$\begin{aligned} & \left| \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D} \Pi_{\mathbf{G}_j}^\perp \mathbf{M}_j^{(h+1)} \tilde{\mathbf{w}}_j^{(h+1)} \right. \\ & \quad \left. - \frac{2}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \right| \leq 3 \sqrt{\frac{8 \log \frac{6}{\delta_2}}{\alpha d_h}} \end{aligned}$$

which implies for any  $\mathbf{x}^{(1)} \in \{\mathbf{x}, \mathbf{x}'\}$ ,

$$\left\| \sqrt{\frac{c_\sigma}{d_h}} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D} \right\|_2 \leq \sqrt{4 + 3 \sqrt{\frac{8 \log \frac{6}{\delta_2}}{\alpha d_h}}}$$

*Proof.* Next, we compute the difference between the projected version of the inner product and normal inner product in expectation: First we have

$$\begin{aligned} & \mathbb{E}_{\tilde{\mathbf{w}}^{(h+1)}} \left( \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_j^{(h+1)} \tilde{\mathbf{w}}_j^{(h+1)} \right) \\ &= \frac{c_\sigma}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \end{aligned}$$

Then,

$$\begin{aligned} & \mathbb{E}_{\tilde{\mathbf{w}}^{(h+1)}} \left( \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D} \Pi_{\mathbf{G}_j}^\perp \mathbf{M}_j^{(h+1)} \tilde{\mathbf{w}}_j^{(h+1)} \right. \\ & \quad \left. - \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_j^{(h+1)} \tilde{\mathbf{w}}_j^{(h+1)} \right) \\ &= \frac{c_\sigma}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D} \Pi_{\mathbf{G}_i}^\perp \mathbf{M}_i^{(h+1)} - \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \\ &= \frac{c_\sigma}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\Pi_{\mathbf{G}_i}^\perp \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp - \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \\ &= \frac{c_\sigma}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}((\Pi_{\mathbf{G}_i}^\perp - I) \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \\ &= \frac{c_\sigma}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\Pi_{\mathbf{G}_i} \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \end{aligned}$$

where the third last equality is true because we can interchange between  $\mathbf{M}_i^{(h+1)}$  and  $\Pi_{\mathbf{G}_i}^\perp$ . And the second last equality is because  $\text{Tr}(\Pi_{\mathbf{G}_i}^\perp \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp - \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) = \text{Tr}(\Pi_{\mathbf{G}_i}^\perp \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp) - \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) = \text{Tr}(\Pi_{\mathbf{G}_i}^\perp \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) - \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) = \text{Tr}((\Pi_{\mathbf{G}_i}^\perp - I) \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) = \text{Tr}(\Pi_{\mathbf{G}_i} \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)})$ . Since  $\text{rank}(\Pi_{\mathbf{G}_i}) \leq 2$  and  $\left\| \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)} \right\|_2 \leq \frac{1}{\alpha}$ , we have

$$0 \leq \text{Tr}(\Pi_{\mathbf{G}_i} \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \leq \frac{2}{\alpha}$$

Now notice that

$$\sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\Pi_{\mathbf{G}_i} \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) = \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)})^\top \mathbf{T} \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)})$$

where

$$\mathbf{T} = \begin{bmatrix} \text{Tr}(\Pi_{\mathbf{G}_1} \mathbf{M}_1^{(h+1)} \mathbf{D} \mathbf{M}_1^{(h+1)}) & 0 & \dots & 0 \\ 0 & \text{Tr}(\Pi_{\mathbf{G}_2} \mathbf{M}_2^{(h+1)} \mathbf{D} \mathbf{M}_2^{(h+1)}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \text{Tr}(\Pi_{\mathbf{G}_{d_{h+1}}} \mathbf{M}_{d_{h+1}}^{(h+1)} \mathbf{D} \mathbf{M}_{d_{h+1}}^{(h+1)}) \end{bmatrix}$$

Notice that  $\|\mathbf{T}\|_2 \leq \frac{2}{\alpha}$  and thus,  $|\mathbf{b}^{(h+1)}(\mathbf{x}^{(1)})^\top \mathbf{T} \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)})| \leq \frac{2}{\alpha} \|\mathbf{b}^{(h+1)}(\mathbf{x}^{(1)})\|_2 \|\mathbf{b}^{(h+1)}(\mathbf{x}^{(2)})\|_2$ . Therefore, we have

$$\begin{aligned} & \mathbb{E}_{\tilde{\mathbf{w}}^{(h+1)}} \left( \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D} \Pi_{\mathbf{G}_j}^\perp \mathbf{M}_j^{(h+1)} \tilde{\mathbf{w}}_j^{(h+1)} \right. \\ & \quad \left. - \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_j^{(h+1)} \tilde{\mathbf{w}}_j^{(h+1)} \right) \end{aligned}$$

$$\leq \frac{c_\sigma}{d_h} \frac{2}{\alpha} \left\| \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}) \right\|_2 \left\| \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}) \right\|_2 \leq \frac{c_\sigma}{d_h} \frac{8}{\alpha} \quad (11)$$

Next since the following new random vector has multivariate Gaussian distribution, we can write

$$\left[ \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \Pi_{\mathbf{G}_i}^\perp \quad \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \Pi_{\mathbf{G}_i}^\perp \right]^\top \stackrel{D}{=} \mathbf{M} \boldsymbol{\xi}$$

where  $\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_{2d_h})$ , and  $\mathbf{M} \in \mathbb{R}^{2d_h \times 2d_h}$  and its covariance matrix is given by a blocked symmetric matrix

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}(\mathbf{x}^{(1)}, \mathbf{x}^{(1)}) & \mathbf{C}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \\ \mathbf{C}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) & \mathbf{C}(\mathbf{x}^{(2)}, \mathbf{x}^{(2)}) \end{bmatrix} = \mathbf{M} \mathbf{M}^\top$$

where each block is given by

$$\begin{aligned} \mathbf{C}(\mathbf{x}^{(p)}, \mathbf{x}^{(q)}) &= \mathbb{E}_{\tilde{\mathbf{w}}^{(h+1)}} \left( \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(p)}) (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \Pi_{\mathbf{G}_i}^\perp \right)^\top \left( \sum_{j=1}^{d_{h+1}} \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(q)}) (\tilde{\mathbf{w}}_j^{(h+1)} \odot \mathbf{m}_j^{(h+1)})^\top \Pi_{\mathbf{G}_j}^\perp \right) \\ &= \mathbb{E}_{\tilde{\mathbf{w}}^{(h+1)}} \left( \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(p)}) \Pi_{\mathbf{G}_i}^\perp (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)}) \right) \left( \sum_{j=1}^{d_{h+1}} \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(q)}) (\tilde{\mathbf{w}}_j^{(h+1)} \odot \mathbf{m}_j^{(h+1)})^\top \Pi_{\mathbf{G}_j}^\perp \right) \\ &= \mathbb{E}_{\tilde{\mathbf{w}}^{(h+1)}} \left( \sum_{i=1}^{d_{h+1}} \sum_{j=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(p)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(q)}) \Pi_{\mathbf{G}_i}^\perp (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)}) (\tilde{\mathbf{w}}_j^{(h+1)} \odot \mathbf{m}_j^{(h+1)})^\top \Pi_{\mathbf{G}_j}^\perp \right) \\ &= \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(p)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(q)}) \Pi_{\mathbf{G}_i}^\perp \left( \mathbb{E}_{\tilde{\mathbf{w}}^{(h+1)}} (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)}) (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \right) \Pi_{\mathbf{G}_i}^\perp \\ &= \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(p)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(q)}) \Pi_{\mathbf{G}_i}^\perp \text{diag} \left( \left( \mathbf{m}_i^{(h+1)} \right)^2 \right) \Pi_{\mathbf{G}_i}^\perp \end{aligned}$$

where the square on a vector is applied element-wise. Therefore, we can write

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} \mathbf{C}(\mathbf{x}^{(1)}, \mathbf{x}^{(1)}) & \mathbf{C}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \\ \mathbf{C}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) & \mathbf{C}(\mathbf{x}^{(2)}, \mathbf{x}^{(2)}) \end{bmatrix} \\ &= \sum_{i=1}^{d_{h+1}} \begin{bmatrix} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \\ \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \end{bmatrix} \otimes \Pi_{\mathbf{G}_i}^\perp \text{diag} \left( \left( \mathbf{m}_i^{(h+1)} \right)^2 \right) \Pi_{\mathbf{G}_i}^\perp \end{aligned}$$

#### BOUNDING THE OPERATOR NORM OF $\mathbf{A}$

Next, we want to show that

$$\sum_{i=1}^{d_{h+1}} \begin{bmatrix} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \\ \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \end{bmatrix} \otimes \left( \frac{1}{\alpha} \mathbf{I} - \Pi_{\mathbf{G}_i}^\perp \text{diag} \left( \left( \mathbf{m}_i^{(h+1)} \right)^2 \right) \Pi_{\mathbf{G}_i}^\perp \right) \succeq \mathbf{0} \quad (12)$$

Given this, since Kronecker product preserves two norm we have that

$$\begin{aligned} \|\mathbf{C}\|_2 &\leq \frac{1}{\alpha} \left\| \sum_{i=1}^{d_{h+1}} \begin{bmatrix} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \\ \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \end{bmatrix} \right\|_2 \\ &= \frac{1}{\alpha} \left\| \left[ \begin{array}{cc} \langle \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}), \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}) \rangle & \langle \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}), \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}) \rangle \\ \langle \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}), \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}) \rangle & \langle \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}), \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}) \rangle \end{array} \right] \right\|_2 \\ &\leq \frac{1}{\alpha} \sqrt{2} \left( \langle \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}), \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}) \rangle + \langle \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}), \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}) \rangle \right) \end{aligned}$$

where the last inequality is by applying  $\|\mathbf{A}\|_2 \leq \sqrt{m} \|\mathbf{A}\|_\infty$ .

We prove the matrix in Equation (12) is positive semi-definite by constructing a multivariate Gaussian distribution such that its covariance matrix is exactly the matrix and exploring the fact that the covariance matrix of two independent Gaussian distribution is the sum of the two covariance matrix. First, notice that

$$\begin{aligned}
 \frac{1}{\alpha} \mathbf{I} - \Pi_{\mathbf{G}_i}^\perp \text{diag} \left( \left( \mathbf{m}_i^{(h+1)} \right)^2 \right) \Pi_{\mathbf{G}_i}^\perp &= \frac{1}{\alpha} (\Pi_{\mathbf{G}_i}^\perp + \Pi_{\mathbf{G}_i}) - \Pi_{\mathbf{G}_i}^\perp \text{diag} \left( \left( \mathbf{m}_i^{(h+1)} \right)^2 \right) \Pi_{\mathbf{G}_i}^\perp \\
 &= \Pi_{\mathbf{G}_i}^\perp \left( \frac{1}{\alpha} \mathbf{I} - \text{diag} \left( \left( \mathbf{m}_i^{(h+1)} \right)^2 \right) \Pi_{\mathbf{G}_i}^\perp \right) + \frac{1}{\alpha} \Pi_{\mathbf{G}_i} \\
 &= \Pi_{\mathbf{G}_i}^\perp \left( \frac{1}{\alpha} (\Pi_{\mathbf{G}_i}^\perp + \Pi_{\mathbf{G}_i}) - \text{diag} \left( \left( \mathbf{m}_i^{(h+1)} \right)^2 \right) \Pi_{\mathbf{G}_i}^\perp \right) + \frac{1}{\alpha} \Pi_{\mathbf{G}_i} \\
 &= \Pi_{\mathbf{G}_i}^\perp \left( \frac{1}{\alpha} \Pi_{\mathbf{G}_i} + \left( \frac{1}{\alpha} \mathbf{I} - \text{diag} \left( \left( \mathbf{m}_i^{(h+1)} \right)^2 \right) \right) \Pi_{\mathbf{G}_i}^\perp \right) + \frac{1}{\alpha} \Pi_{\mathbf{G}_i} \\
 &= \Pi_{\mathbf{G}_i}^\perp \left( \frac{1}{\alpha} \mathbf{I} - \text{diag} \left( \left( \mathbf{m}_i^{(h+1)} \right)^2 \right) \right) \Pi_{\mathbf{G}_i}^\perp + \frac{1}{\alpha} \Pi_{\mathbf{G}_i}
 \end{aligned}$$

The final Gaussian is constructed by the sum of the following two groups of Gaussian: let  $\mathbf{W}_1, \mathbf{W}_2$  be two independent standard Gaussian matrices,

$$\begin{aligned}
 &\left[ \sum_{i=1}^{d_{h+1}} \mathbf{b}_i(\mathbf{x}^{(1)}) (\mathbf{w}_{1,i}^{(h+1)})^\top \left( \frac{1}{\sqrt{\alpha}} \mathbf{I} - \text{diag} \left( \mathbf{m}_i^{(h+1)} \right) \right) \Pi_{\mathbf{G}_i}^\perp \quad \sum_{i=1}^{d_{h+1}} \mathbf{b}_i(\mathbf{x}^{(2)}) (\mathbf{w}_{1,i}^{(h+1)})^\top \left( \frac{1}{\sqrt{\alpha}} \mathbf{I} - \text{diag} \left( \mathbf{m}_i^{(h+1)} \right) \right) \Pi_{\mathbf{G}_i}^\perp \right] \\
 &\left[ \sum_{i=1}^{d_{h+1}} \mathbf{b}_i(\mathbf{x}^{(1)}) (\mathbf{w}_{2,i}^{(h+1)})^\top \frac{1}{\sqrt{\alpha}} \Pi_{\mathbf{G}_i} \quad \sum_{i=1}^{d_{h+1}} \mathbf{b}_i(\mathbf{x}^{(2)}) (\mathbf{w}_{2,i}^{(h+1)})^\top \frac{1}{\sqrt{\alpha}} \Pi_{\mathbf{G}_i} \right]
 \end{aligned}$$

where  $\mathbf{w}_{i,j}$  denote the  $j$ -th row of  $\mathbf{W}_i$ .

Now conditioned on  $\{\mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}), \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}), \mathbf{g}^{(h)}(\mathbf{x}^{(1)}), \mathbf{g}^{(h)}(\mathbf{x}^{(2)})\}$ , we have

$$\begin{aligned}
 &\left( \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) (\mathbf{w}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \Pi_{\mathbf{G}_i}^\perp \right) \mathbf{D} \left( \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) (\mathbf{w}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \Pi_{\mathbf{G}_i}^\perp \right) \\
 &\stackrel{D}{=} \left( \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \Pi_{\mathbf{G}_i}^\perp \right) \mathbf{D} \left( \sum_{i=1}^{d_{h+1}} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) (\tilde{\mathbf{w}}_i^{(h+1)} \odot \mathbf{m}_i^{(h+1)})^\top \Pi_{\mathbf{G}_i}^\perp \right) \\
 &\stackrel{D}{=} ([\mathbf{I}_{d_h} \quad \mathbf{0}] \mathbf{M} \boldsymbol{\xi})^\top \mathbf{D} ([\mathbf{0} \quad \mathbf{I}_{d_h}] \mathbf{M} \boldsymbol{\xi}) \\
 &\stackrel{D}{=} \frac{1}{2} \boldsymbol{\xi}^\top \mathbf{M}^\top \begin{bmatrix} \mathbf{0} & \mathbf{D} \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \mathbf{M} \boldsymbol{\xi}
 \end{aligned}$$

Now, let

$$\mathbf{A} = \frac{1}{2} \mathbf{M}^\top \begin{bmatrix} \mathbf{0} & \mathbf{D} \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \mathbf{M}$$

and we have

$$\begin{aligned}
 \|\mathbf{A}\|_2 &\leq \frac{1}{2} \|\mathbf{M}\|_2^2 \|\mathbf{D}\|_2 \\
 &= \frac{1}{2} \|\mathbf{M} \mathbf{M}^\top\|_2 \|\mathbf{D}\|_2 \\
 &\leq \frac{1}{2\alpha} \sqrt{2} \left( \left\langle \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}), \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}) \right\rangle + \left\langle \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}), \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}) \right\rangle \right) \\
 &\leq \frac{2\sqrt{2}}{\alpha}
 \end{aligned}$$

BOUNDING THE FROBENIUS NORM OF  $\mathbf{A}$ 

Naively apply 2-norm-Frobenius-norm bound for matrices will give us

$$\|\mathbf{A}\|_F \leq \sqrt{2d_h} \|\mathbf{A}\|_2 \leq \frac{4\sqrt{d_h}}{\alpha}$$

We prove a better bound. Observe that

$$\frac{1}{d_h} \|\mathbf{A}\|_F = \frac{1}{d_h} \left\| \frac{1}{2} \mathbf{M}^\top \begin{bmatrix} \mathbf{0} & \mathbf{D} \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \mathbf{M} \right\|_F \leq \frac{1}{2d_h} \|\mathbf{M}\|_2 \|\mathbf{M}\|_F \|\mathbf{D}\|_2 = \frac{1}{2d_h\sqrt{\alpha}} \|\mathbf{M}\|_F = \frac{1}{2d_h\sqrt{\alpha}} \sqrt{\text{Tr}(\mathbf{M}\mathbf{M}^\top)}$$

Using the similar idea from bounding the 2-norm of  $\mathbf{M}\mathbf{M}^\top$ , we want to show that

$$\begin{aligned} & \sum_{i=1}^{d_{h+1}} \begin{bmatrix} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \\ \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \end{bmatrix} \otimes \left( \left( \mathbf{M}_i^{(h+1)} \right)^2 - \Pi_{\mathbf{G}_i}^\perp \left( \mathbf{M}_i^{(h+1)} \right)^2 \Pi_{\mathbf{G}_i}^\perp \right) \\ &= \sum_{i=1}^{d_{h+1}} \begin{bmatrix} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \\ \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \end{bmatrix} \otimes \Pi_{\mathbf{G}_i} \left( \mathbf{M}_i^{(h+1)} \right)^2 \Pi_{\mathbf{G}_i} \succeq \mathbf{0} \end{aligned} \quad (13)$$

If this equation is true, then we have

$$\begin{aligned} \frac{1}{d_h} \text{Tr}(\mathbf{M}\mathbf{M}^\top) &\leq \frac{1}{d_h} \text{Tr} \left( \sum_{i=1}^{d_{h+1}} \begin{bmatrix} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \\ \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) & \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \end{bmatrix} \otimes \mathbf{M}_i^2 \right) \\ &= \frac{1}{d_h} \sum_{i=1}^{d_{h+1}} \left[ \left( \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \right)^2 + \left( \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \right)^2 \right] \text{Tr} \left( \left( \mathbf{M}_i^{(h+1)} \right)^2 \right) \\ &\leq \frac{1}{d_h} \sum_{i=1}^{d_{h+1}} \left[ \left( \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \right)^2 + \left( \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \right)^2 \right] \max_i \text{Tr} \left( \left( \mathbf{M}_i^{(h+1)} \right)^2 \right) \\ &= \max_i \frac{1}{d_h} \text{Tr} \left( \left( \mathbf{M}_i^{(h+1)} \right)^2 \right) \left( \left\| \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}) \right\|_2^2 + \left\| \mathbf{b}^{(h+1)}(\mathbf{x}^{(2)}) \right\|_2^2 \right) \\ &\leq 4 \max_i \frac{1}{d_h} \text{Tr} \left( \left( \mathbf{M}_i^{(h+1)} \right)^2 \right) \end{aligned}$$

Since each  $\mathbf{m}_{ij}^2$  is subgamma with parameters  $(4 + 4 \left( \frac{1-\alpha}{\alpha} \right)^2, 2 \frac{1-\alpha}{\alpha})$ , if  $d_h \gtrsim \left( \frac{1-\alpha}{\alpha} \right)^2 \log \frac{2d_{h+1}}{\delta}$ , then with probability  $\geq 1 - \delta$ ,  $\max_i \frac{1}{d_h} \text{Tr} \left( \left( \mathbf{M}_i^{(h+1)} \right)^2 \right) \leq 1 + 1 = 2$ . Thus, we have

$$\frac{1}{d_h} \|\mathbf{A}\|_F \leq \sqrt{\frac{2}{d_h \alpha}}$$

To prove Equation (13), since  $\mathbf{M}_i$  commutes with  $\Pi_{\mathbf{G}_i}^\perp$ , we have

$$\mathbf{M}_i^2 - \Pi_{\mathbf{G}_i}^\perp \mathbf{M}_i^2 \Pi_{\mathbf{G}_i}^\perp = \mathbf{M}_i^2 - \mathbf{M}_i^2 \Pi_{\mathbf{G}_i}^\perp = \mathbf{M}_i^2 \Pi_{\mathbf{G}_i} = \Pi_{\mathbf{G}_i} \mathbf{M}_i^2 \Pi_{\mathbf{G}_i}$$

The Gaussian vector given by

$$\left[ \sum_{i=1}^{d_{h+1}} \mathbf{b}_i(\mathbf{x}^{(1)}) (\mathbf{w}_{2,i}^{(h+1)})^\top \mathbf{M}_i \Pi_{\mathbf{G}_i} \quad \sum_{i=1}^{d_{h+1}} \mathbf{b}_i(\mathbf{x}^{(2)}) (\mathbf{w}_{2,i}^{(h+1)})^\top \mathbf{M}_i \Pi_{\mathbf{G}_i} \right]$$

has the covariance matrix.

Now apply Gaussian chaos concentration bound (Lemma C.3), we have with probability  $1 - \frac{\delta_2}{6}$ ,

$$\frac{1}{d_h} |\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi} - \mathbb{E}[\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi}]| \leq \frac{1}{d_h} \left( 2 \|\mathbf{A}\|_F \sqrt{\log \frac{6}{\delta_2}} + 2 \|\mathbf{A}\|_2 \log \frac{6}{\delta_2} \right)$$



$$\leq \sqrt{\frac{8 \log \frac{6}{\delta_2}}{\alpha d_h}} + 4\sqrt{2} \frac{\log \frac{6}{\delta_2}}{\alpha d_h} \quad (14)$$

Finally, combining Equation 11 and Equation 14, we have

$$\begin{aligned} & \left| \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D} \Pi_{\mathbf{G}_j}^\perp \mathbf{M}_j^{(h+1)} \tilde{\mathbf{w}}_j^{(h+1)} \right. \\ & \quad \left. - \frac{2}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \right| \\ & \leq \frac{2}{d_h} \left| \boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi} - \mathbb{E}[\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi}] \right| + \left| \frac{2}{d_h} \mathbb{E}[\boldsymbol{\xi}^\top \mathbf{A} \boldsymbol{\xi}] - \frac{2}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)}) \right| \\ & \leq \frac{c_\sigma}{d_h} \frac{8}{\alpha} + \sqrt{\frac{8 \log \frac{6}{\delta_2}}{\alpha d_h}} + 4\sqrt{2} \frac{\log \frac{6}{\delta_2}}{\alpha d_h} \leq 3\sqrt{\frac{8 \log \frac{6}{\delta_2}}{\alpha d_h}} \end{aligned}$$

where we choose  $d_h \geq \frac{8}{\alpha} \log \frac{6}{\delta_2}$ . Then take a union bound over  $(\mathbf{x}, \mathbf{x}), (\mathbf{x}, \mathbf{x}'), (\mathbf{x}', \mathbf{x}')$ . Finally, taking  $\mathbf{x}^{(1)} = \mathbf{x}^{(2)}$ , we have

$$\begin{aligned} & \left\| \sqrt{\frac{c_\sigma}{d_h}} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D} \right\|_2 \\ & \leq \sqrt{\left| \frac{c_\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \tilde{\mathbf{w}}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D} \Pi_{\mathbf{G}_j}^\perp \mathbf{M}_j^{(h+1)} \tilde{\mathbf{w}}_j^{(h+1)} \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(1)}) \right|} \\ & \leq \sqrt{\frac{2}{d_h} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(2)}) \text{Tr}(\mathbf{M}_i^{(h+1)} \mathbf{D} \mathbf{M}_i^{(h+1)})} + 3\sqrt{\frac{8 \log \frac{6}{\delta_2}}{\alpha d_h}} \\ & \leq \sqrt{4 + 3\sqrt{\frac{8 \log \frac{6}{\delta_2}}{\alpha d_h}}} \leq 6 \end{aligned}$$

□

Next, we want to upper bound

**Proposition 4.4.** *If  $d_{h'} \geq c_1 \left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{L^2 \log \frac{8 \sum_{h \leq h'} d_h d_{h'+1} L}{\delta}}{\epsilon^2}$ , with probability  $1 - \delta_3/2$ , the event  $\bar{\mathcal{C}}(\sqrt{\log \frac{\sum d_h}{\delta_3}})$  (which we define in the proof) holds and at layer  $h'$ , for all  $j \in [d_h]$ ,*

$$\left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i} \right\|_2 \leq 2\sqrt{\frac{1}{\alpha} \log \frac{8}{\delta_2}} + \frac{4}{\alpha} \sqrt{\log \frac{\sum d_h}{\delta_3}}$$

*Proof.* By triangle inequality, combining the result from Lemma C.14 and Lemma C.15, we have

$$\begin{aligned} \left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i} \right\|_2 & \leq \left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{(\mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)})} \right\|_2 \\ & \quad + \left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i / (\mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)})} \right\|_2 \\ & \leq 2\sqrt{\frac{1}{\alpha} \log \frac{8}{\delta_2}} + \frac{4}{\alpha} \sqrt{\log \frac{\sum d_h}{\delta_3}} \end{aligned}$$

□

Thus, we need to upper bound the above two terms.

**Lemma C.14.** *With probability  $1 - \delta_2$ ,*

$$\left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i / (\mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)})} \right\|_2 \lesssim \sqrt{\frac{1}{\alpha} \log \frac{8}{\delta_2}}$$

*Proof.* Notice that  $\mathbf{G}_i / (\mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)})$  is spanned by the vector

$$\mathbf{g}^{(h)}(\mathbf{x}') \odot \mathbf{m}_i^{(h+1)} - \left\langle \mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)}, \mathbf{g}^{(h)}(\mathbf{x}') \odot \mathbf{m}_i^{(h+1)} \right\rangle \mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)}$$

Naively apply the argument in (Arora et al., 2019) we can get

$$\left\| \sum_i \mathbf{b}_i w_i \mathbf{u}^{(i)} \right\|_2 \leq \sum_i \left\| \mathbf{b}_i w_i \mathbf{u}^{(i)} \right\|_2 = \sum_i |\mathbf{b}_i w_i| \leq \sqrt{d_h} \|\mathbf{b}\|_2$$

which is going to increase the dependence on  $\log \frac{1}{\delta}$  by a polylogarithmic factor.

Now conditioned on  $\mathbf{g}^{(h)}(\mathbf{x}), \mathbf{f}^{(h+1)}, \{\mathbf{W}^{(h')}\}_{h+2}^{L+1}$ . Observe that  $\sum_i \mathbf{b}_i (\tilde{\mathbf{w}}_i^\top \mathbf{u}^{(i)}) \mathbf{u}^{(i)} = \sum_i \mathbf{b}_i w_i \mathbf{u}^{(i)}$  where  $w_i \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$  is a Gaussian vector. Its covariance matrix is given by

$$\mathbb{E}_{\tilde{\mathbf{w}}} \left( \sum_i \mathbf{b}_i w_i \mathbf{u}^{(i)} \right) \left( \sum_j \mathbf{b}_j w_j \mathbf{u}^{(j)} \right)^\top = \mathbb{E}_{\tilde{\mathbf{w}}} \sum_{i,j} \mathbf{b}_i \mathbf{b}_j w_i w_j \mathbf{u}^{(i)} \left( \mathbf{u}^{(j)} \right)^\top = \sum_i \mathbf{b}_i^2 \mathbf{u}^{(i)} \left( \mathbf{u}^{(i)} \right)^\top$$

Let the eigenvalue decomposition of this matrix be  $\mathbf{U} \mathbf{D} \mathbf{U}^\top$ , then the vector  $\sum_i \mathbf{b}_i w_i \mathbf{u}^{(i)}$  has the same distribution as  $\mathbf{U} \mathbf{D}^{1/2} \tilde{\mathbf{w}}$  where  $\tilde{\mathbf{w}} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ . Thus,

$$\mathbb{E}_{\tilde{\mathbf{w}}} \left[ \left\| \sum_i \mathbf{b}_i w_i \mathbf{u}^{(i)} \right\|_2^2 \right] = \mathbb{E}_{\tilde{\mathbf{w}}} \left[ \tilde{\mathbf{w}}^\top \mathbf{D}^{1/2} \mathbf{U}^\top \mathbf{U} \mathbf{D}^{1/2} \tilde{\mathbf{w}} \right] = \text{Tr}(\mathbf{D})$$

Now, we use the fact that the sum of the eigenvalues is the trace of the original matrix and we have

$$\text{Tr}(\mathbf{D}) = \text{Tr} \left( \sum_i \mathbf{b}_i^2 \mathbf{u}^{(i)} \left( \mathbf{u}^{(i)} \right)^\top \right) = \sum_j \sum_i \mathbf{b}_i^2 \left( \mathbf{u}_j^{(i)} \right)^2 = \sum_i \mathbf{b}_i^2 = \|\mathbf{b}\|_2^2$$

By Jensen's inequality, we have

$$\mathbb{E}_{\tilde{\mathbf{w}}} \left[ \left\| \sum_i \mathbf{b}_i w_i \mathbf{u}^{(i)} \right\|_2 \right] \leq \sqrt{\mathbb{E}_{\tilde{\mathbf{w}}} \left[ \left\| \sum_i \mathbf{b}_i w_i \mathbf{u}^{(i)} \right\|_2^2 \right]} = \|\mathbf{b}\|_2$$

Further, use the definition of two norm we can write

$$\left\| \sum_i \mathbf{b}_i w_i \mathbf{u}^{(i)} \right\|_2 = \sup_{\|\mathbf{x}\|_2=1} \left\langle \mathbf{x}, \sum_i \mathbf{b}_i w_i \mathbf{u}^{(i)} \right\rangle \stackrel{D}{=} \sup_{\|\mathbf{x}\|_2=1} \left\langle \mathbf{x}, \mathbf{U} \mathbf{D}^{1/2} \tilde{\mathbf{w}} \right\rangle = \sup_{\|\mathbf{x}\|_2=1} \left\langle \mathbf{x} \mathbf{D}^{1/2}, \tilde{\mathbf{w}} \right\rangle$$

The last quantity is in form of a Gaussian complexity and is known to have subgaussian concentration with variance proxy  $\sigma^2 = \max_i \mathbf{D}_{ii} \leq \text{Tr}(\mathbf{D})$ , see Example 2.30 (Wainwright, 2019). Thus, with probability  $1 - \delta_2/4$ ,

$$\left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i / (\mathbf{g}^{(h)}(\mathbf{x}) \odot \mathbf{m}_i^{(h+1)})} \right\|_2 \leq \sqrt{\frac{2}{\alpha} \log \frac{8}{\delta_2}} \|\mathbf{b}^{(h+1)}\|_2 \leq 2 \sqrt{\frac{1}{\alpha} \log \frac{8}{\delta_2}}$$

□

Now we bound the first term.

**Lemma C.15.** *With probability  $1 - \delta$ ,*

$$\left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})} \right\|_2 \leq \frac{4}{\alpha} \sqrt{\log \frac{\sum_h d_h}{\delta}}$$

*Proof.* Since  $\Pi_{(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})} = \frac{(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})^\top}{\|(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})\|_2^2}$  we have

$$\begin{aligned} & \left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})} \right\|_2 \\ &= \left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \frac{(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})^\top}{\|(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})\|_2^2} \right\|_2 \end{aligned}$$

Now let's look at the  $j$ -th coordinate of this vector:

$$\begin{aligned} & \left( \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \frac{(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})^\top}{\|(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})\|_2^2} \right)_j \\ &= \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \frac{(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)}) \mathbf{m}_{ij}^{(h+1)} \mathbf{g}_j^{(h)}}{\|(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})\|_2^2} \\ &= \frac{1}{\alpha} \mathbf{g}_j^{(h)} \left( \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}) \right)^\top \text{diag} \left( \frac{\sqrt{\alpha} \mathbf{m}_{ij}^{(h+1)}}{\|(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})\|_2^2} \right) \begin{bmatrix} \left( \mathbf{w}_1^{(h+1)} \odot \mathbf{m}_1^{(h+1)} \right)^\top (\mathbf{g}^{(h)} \odot \mathbf{m}_1^{(h+1)}) \sqrt{\alpha} \\ \left( \mathbf{w}_2^{(h+1)} \odot \mathbf{m}_2^{(h+1)} \right)^\top (\mathbf{g}^{(h)} \odot \mathbf{m}_2^{(h+1)}) \sqrt{\alpha} \\ \vdots \\ \left( \mathbf{w}_{d_{h+1}}^{(h+1)} \odot \mathbf{m}_{d_{h+1}}^{(h+1)} \right)^\top (\mathbf{g}^{(h)} \odot \mathbf{m}_{d_{h+1}}^{(h+1)}) \sqrt{\alpha} \end{bmatrix} \\ &= \frac{1}{\alpha} \mathbf{g}_j^{(h)} \left( \mathbf{b}^{(h+1)}(\mathbf{x}^{(1)}) \right)^\top \text{diag} \left( \frac{\sqrt{\alpha} \mathbf{m}_{ij}^{(h+1)}}{\|(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})\|_2^2} \right) \mathbf{f}^{(h+1)}(\mathbf{x}^{(1)}) \\ &= \frac{1}{\alpha} \mathbf{g}_j^{(h)} \left( \mathbf{w}^{(L+1)} \odot \mathbf{m}^{(L+1)} \right)^\top \sqrt{\frac{c_\sigma}{d_L}} \mathbf{D}^{(L)}(\mathbf{x}^{(1)}) \left( \mathbf{W}^{(L)} \odot \mathbf{m}^{(L)} \right) \\ &\quad \cdots \sqrt{\frac{c_\sigma}{d_{h+1}}} \mathbf{D}^{(h+1)}(\mathbf{x}^{(1)}) \text{diag} \left( \frac{\sqrt{\alpha} \mathbf{m}_{ij}^{(h+1)}}{\|(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})\|_2^2} \right) \mathbf{f}^{(h+1)}(\mathbf{x}^{(1)}) \\ &= \frac{1}{\alpha} \mathbf{g}_j^{(h)} f^{(h+1,j,L+1)}(\mathbf{x}^{(1)}) \end{aligned}$$

where we define the network induced by the sparse masks  $\mathbf{m}_j^{(h+1)}$  for  $h' \in \{h+2, h+3, \dots, L+1\}$  to be

$$\begin{aligned} \mathbf{g}^{(h+1,j,h+1)}(\mathbf{x}) &= \sqrt{\frac{c_\sigma}{d_{h+1}}} \mathbf{D}^{(h+1)}(\mathbf{x}) \text{diag} \left( \frac{\mathbb{I}(\mathbf{m}_{ij} \neq \mathbf{0})}{\|\mathbf{g} \odot \mathbf{m}_i\|_2^2} \right) \mathbf{f}^{(h+1)}(\mathbf{x}) \\ \mathbf{f}^{(h+1,j,h') }(\mathbf{x}) &= \left( \mathbf{W}^{(h')} \odot \mathbf{m}^{(h')} \right) \mathbf{g}^{(h+1,j,h'-1)}(\mathbf{x}) \\ \mathbf{g}^{(h+1,j,h') }(\mathbf{x}) &= \sqrt{\frac{c_\sigma}{d_{h'}}} \mathbf{D}^{(h')}(\mathbf{x}) \mathbf{f}^{(h+1,j,h') }(\mathbf{x}) \end{aligned}$$

We are going to show that for all  $j \in [d_h]$ ,  $|f^{(h+1,j,L+1)}(\mathbf{x})|$  is small, which implies the entire vector has small norm. We define the event

$$\mathcal{F} = \left\{ \left| \left\| \mathbf{g}^{(h,j,h')} \right\|_2^2 - \mathbb{E} \left\| \mathbf{g}^{(h,j,h')} \right\|_2^2 \right| < \sqrt{\log \frac{L d_{h'+1} \sum_{h \leq h'} d_h}{\delta}}, \quad |f^{(h,j,L+1)}| < \sqrt{\log \frac{\sum_{h=1}^L d_h}{\delta}}, \quad \forall h, j, h' \right\}$$

We are going to show that the event  $\mathcal{F}$  holds with probability  $1 - \delta$ .

We would like to bound  $|f^{(h+1,j,L+1)}(\mathbf{x})|$ . Observe that without the diagonal matrix in the middle we have

$$\mathbf{g}^{(h+1)}(\mathbf{x}) = \sqrt{\frac{c_\sigma}{d_{h+1}}} \mathbf{D}^{(h+1)}(\mathbf{x}) \mathbf{f}^{(h+1)}(\mathbf{x})$$

and since by our assumption  $\left\| \mathbf{g}^{(h)} \odot \mathbf{m}^{(h+1)} \right\|_2^2 \geq 1 - \epsilon_1^2 \geq 1/2$  we have  $\left\| \mathbf{g}^{(h+1,j,h+1)}(\mathbf{x}) \right\|_2^2 \leq 2 \left\| \mathbf{g}^{(h+1)}(\mathbf{x}) \right\|_2^2$ . Conditioned on  $\mathbf{g}^{(h+1,j,L)}(\mathbf{x})$ ,  $f^{(h+1,j,L+1)}(\mathbf{x})$  has distribution  $\mathcal{N}(0, \left\| \mathbf{g}^{(h+1,j,L)}(\mathbf{x}) \right\|_2^2)$ . Therefore, the magnitude of  $|f^{(h+1,j,L+1)}(\mathbf{x})|$  would depend on  $\left\| \mathbf{g}^{(h+1,j,L)}(\mathbf{x}) \right\|_2$ .

We are going to show that

$$\mathbb{E}_{\mathbf{w}^{(h+2)}, \dots, \mathbf{w}^{(L)}} \left[ \left\| \mathbf{g}^{(h+1,j,L)}(\mathbf{x}) \right\|_2^2 \left\| \mathbf{g}^{(h+1,j,h+1)} \right\|_2^2 \right] \leq 2 \mathbb{E}_{\mathbf{w}^{(h+2)}, \dots, \mathbf{w}^{(L)}} \left[ \left\| \mathbf{g}^{(L)}(\mathbf{x}) \right\|_2^2 \left\| \mathbf{g}^{(h+1)}(\mathbf{x}) \right\|_2^2 \right]$$

*Proof.* Consider a single coordinate given by  $(\mathbf{w} \odot \mathbf{m})^\top \tilde{\mathbf{g}} \sqrt{\frac{c_\sigma}{d_h}} \dot{\sigma}((\mathbf{w} \odot \mathbf{m})^\top \mathbf{g})$ . We are going to use the fact that

$$\mathbb{E}_{\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} \left[ (\mathbf{w}^\top \mathbf{x})^2 \mathbb{I}(\mathbf{w}^\top \mathbf{y} > 0) \right] = \mathbb{E}_{\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} \left[ (\mathbf{w}^\top \mathbf{x})^2 \mathbb{I}(\mathbf{w}^\top \mathbf{x} > 0) \right]$$

The equation is a simple consequence of the proposition below where we prove a stronger result: the distribution of the two random variables are the same. This equation tells us that the direction of  $\mathbf{y}$  doesn't matter which implies

$$\mathbb{E}_{\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} \left[ (\mathbf{w}^\top \mathbf{x})^2 \frac{c_\sigma}{d_{h+1}} \dot{\sigma}(\mathbf{w}^\top \mathbf{y}) \right] = \mathbb{E}_{\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} \left[ (\mathbf{w}^\top \mathbf{x})^2 \frac{c_\sigma}{d_{h+1}} \dot{\sigma}(\mathbf{w}^\top \mathbf{x}) \right] = \frac{c_\sigma \left\| \mathbf{x} \right\|_2^2}{d_{h+1}}$$

**Proposition C.16.** *For any given nonzero vectors  $\mathbf{x}, \mathbf{y}$ , the distribution of  $(\mathbf{w}^\top \mathbf{x})^2 \mathbb{I}(\mathbf{w}^\top \mathbf{y} > 0)$  is the same as  $(\mathbf{w}^\top \mathbf{x})^2 \mathbb{I}(\mathbf{w}^\top \mathbf{x} > 0)$  where  $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ .*

*Proof.* Define random variables  $z_1 = (\mathbf{w}^\top \mathbf{x})^2 \mathbb{I}(\mathbf{w}^\top \mathbf{y} > 0)$  and  $z_2 = (\mathbf{w}^\top \mathbf{x})^2 \mathbb{I}(\mathbf{w}^\top \mathbf{x} > 0)$ . Let  $F_1, F_2$  be the cumulative distribution function of  $z_1, z_2$ . It is easy to see that both  $z_1$  and  $z_2$  has probability  $1/2$  of being zero and thus we consider the probability that  $z_1$  and  $z_2$  are not identically zero. Then for  $z > 0$ ,

$$\begin{aligned} \mathbb{P}[0 < z_1 \leq z] &= \int_{\{\mathbf{w}: \mathbf{w}^\top \mathbf{y} > 0, |\mathbf{w}^\top \mathbf{x}| \leq \sqrt{z}\}} \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2} \|\mathbf{w}\|_2^2} d\mathbf{w} \\ &= \int_{\{\mathbf{w}: \mathbf{w}^\top \mathbf{x} > 0, \mathbf{w}^\top \mathbf{y} > 0, |\mathbf{w}^\top \mathbf{x}| \leq \sqrt{z}\} \cup \{\mathbf{w}: \mathbf{w}^\top \mathbf{x} \leq 0, \mathbf{w}^\top \mathbf{y} > 0, |\mathbf{w}^\top \mathbf{x}| \leq \sqrt{z}\}} \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2} \|\mathbf{w}\|_2^2} d\mathbf{w} \\ &= \int_{\{\mathbf{w}: \mathbf{w}^\top \mathbf{x} > 0, \mathbf{w}^\top \mathbf{y} > 0, |\mathbf{w}^\top \mathbf{x}| \leq \sqrt{z}\}} \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2} \|\mathbf{w}\|_2^2} d\mathbf{w} \\ &\quad + \int_{\{\mathbf{w}: \mathbf{w}^\top \mathbf{x} \leq 0, \mathbf{w}^\top \mathbf{y} > 0, |\mathbf{w}^\top \mathbf{x}| \leq \sqrt{z}\}} \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2} \|\mathbf{w}\|_2^2} d\mathbf{w} \\ &= \int_{\{\mathbf{w}: \mathbf{w}^\top \mathbf{x} > 0, \mathbf{w}^\top \mathbf{y} > 0, |\mathbf{w}^\top \mathbf{x}| \leq \sqrt{z}\}} \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2} \|\mathbf{w}\|_2^2} d\mathbf{w} \\ &\quad + \int_{\{\mathbf{w}: \mathbf{w}^\top \mathbf{x} > 0, \mathbf{w}^\top \mathbf{y} \leq 0, |\mathbf{w}^\top \mathbf{x}| \leq \sqrt{z}\}} \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2} \|\mathbf{w}\|_2^2} d\mathbf{w} \end{aligned}$$

$$\begin{aligned}
 &= \int_{\{\mathbf{w}: \mathbf{w}^\top \mathbf{x} > 0, |\mathbf{w}^\top \mathbf{x}| \leq \sqrt{z}\}} \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2}\|\mathbf{w}\|_2^2} d\mathbf{w} \\
 &= \mathbb{P}[0 < z_2 \leq z]
 \end{aligned}$$

where the third last equality is by spherical symmetry of Gaussian and take  $\mathbf{w} := -\mathbf{w}$  over the region.  $\square$

Now, this implies that conditioned on  $\mathbf{m}$ ,

$$\begin{aligned}
 \mathbb{E}_{\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} \left[ \left( (\mathbf{w} \odot \mathbf{m})^\top \mathbf{x} \right)^2 \frac{c_\sigma}{d_{h+1}} \dot{\sigma} \left( (\mathbf{w} \odot \mathbf{m})^\top \mathbf{y} \right) \right] &= \mathbb{E}_{\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} \left[ \left( (\mathbf{w} \odot \mathbf{m})^\top \mathbf{x} \right)^2 \frac{c_\sigma}{d_{h+1}} \dot{\sigma} \left( (\mathbf{w} \odot \mathbf{m})^\top \mathbf{x} \right) \right] \\
 &= \frac{c_\sigma \|\mathbf{x} \odot \mathbf{m}\|_2^2}{2d_{h+1}}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 &\mathbb{E}_{\mathbf{m}^{(h+2)}, \mathbf{W}^{(h+2)}} \left[ \left\| \mathbf{g}^{(h+1, j, h+2)} \right\|_2^2 \right] \\
 &= \mathbb{E}_{\mathbf{m}^{(h+2)}} \left[ \sum_{i=1}^{d_{h+2}} \mathbb{E}_{\mathbf{w}_i^{(h+2)}} \left[ \left( \left( \mathbf{w}_i^{(h+2)} \odot \mathbf{m}_i^{(h+2)} \right)^\top \mathbf{g}^{(h+1, j, h+1)} \right)^2 \frac{c_\sigma}{d_{h+2}} \dot{\sigma} \left( \left( \mathbf{w}_i^{(h+2)} \odot \mathbf{m}_i^{(h+2)} \right)^\top \mathbf{g}^{(h+1)} \right) \middle| \mathbf{m}^{(h+2)} \right] \right] \\
 &= \left\| \mathbf{g}^{(h+1, j, h+1)} \right\|_2^2
 \end{aligned}$$

Hence, by induction and iterated expectation, we have

$$\begin{aligned}
 \mathbb{E}_{\mathbf{W}^{(h+2)}, \dots, \mathbf{W}^{(L)}} \left[ \left\| \mathbf{g}^{(h+1, j, L)}(\mathbf{x}) \right\|_2^2 \middle| \mathbf{g}^{(h+1, j, h+1)} \right] &= \left\| \mathbf{g}^{(h+1, j, h+1)} \right\|_2^2 \\
 \mathbb{E}_{\mathbf{W}^{(h+2)}, \dots, \mathbf{W}^{(L)}} \left[ \left\| \mathbf{g}^{(L)}(\mathbf{x}) \right\|_2^2 \middle| \mathbf{g}^{(h+1)}(\mathbf{x}) \right] &= \left\| \mathbf{g}^{(h+1)}(\mathbf{x}) \right\|_2^2
 \end{aligned}$$

$\square$

Now, since we proved that conditioned on  $\mathbf{g}, \tilde{\mathbf{g}}, \mathbf{m}$ , the random variable  $((\mathbf{w} \odot \mathbf{m})^\top \tilde{\mathbf{g}} \sqrt{\frac{c_\sigma}{d_h}})^2 \dot{\sigma}((\mathbf{w} \odot \mathbf{m})^\top \mathbf{g})$  has the same distribution as  $((\mathbf{w} \odot \mathbf{m})^\top \tilde{\mathbf{g}} \sqrt{\frac{c_\sigma}{d_h}})^2 \dot{\sigma}((\mathbf{w} \odot \mathbf{m})^\top \tilde{\mathbf{g}})$ , and, thus, their concentration properties are the same. Therefore, by

Theorem C.9, if  $d_{h'} \geq c_1 \left( \left( \frac{1-\alpha}{\alpha} \right)^2 + 1 \right) \frac{L^2 \log \frac{8 \sum_{h=1}^{h'} d_h d_{h'+1}^L}{\delta_3}}{\epsilon^2}$ , with probability  $1 - \delta_3$ , at layer  $h'$ , for all  $h \in [h']$  and  $j \in [d_h]$ ,

$$\left\| \mathbf{g}^{(h+1, j, h')}(\mathbf{x}) \right\|_2^2 \leq 2 \left\| \mathbf{g}^{(h+1)}(\mathbf{x}) \right\|_2^2 + \epsilon \leq 3$$

By Lemma C.10, this implies with probability  $1 - \delta_3$ , for all  $j \in [d_h]$ ,

$$|f^{(h+1, j, L+1)}(\mathbf{x})| \leq 2 \sqrt{\log \frac{\sum_h d_h}{\delta_3}}$$

Finally, this implies

$$\left\| \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{(\mathbf{g}^{(h)} \odot \mathbf{m}_i^{(h+1)})} \right\|_2 \leq \frac{4}{\alpha} \sqrt{\log \frac{\sum_h d_h}{\delta_3}}$$

$\square$

Wrapping things up, we have

$$\begin{aligned}
 & \left| \frac{c\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \right. \\
 & \quad \left. - \frac{c\sigma}{d_h} \sum_{i,j} \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) \Pi_{\mathbf{G}_j}^\perp \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \right| \\
 & \leq \left\| \sqrt{\frac{c\sigma}{d_h}} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i}^\perp \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \right\| \left\| \sqrt{\frac{c\sigma}{d_h}} \sum_j \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) \Pi_{\mathbf{G}_j} \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \right\| \\
 & \quad + \left\| \sqrt{\frac{c\sigma}{d_h}} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i} \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \right\| \left\| \sqrt{\frac{c\sigma}{d_h}} \sum_j \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) \Pi_{\mathbf{G}_j}^\perp \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \right\| \\
 & \quad + \left\| \sqrt{\frac{c\sigma}{d_h}} \sum_i \mathbf{b}_i^{(h+1)}(\mathbf{x}^{(1)}) \left( \mathbf{w}_i^{(h+1)} \right)^\top \mathbf{M}_i^{(h+1)} \Pi_{\mathbf{G}_i} \mathbf{D}^{(h)}(\mathbf{x}^{(1)}) \right\| \left\| \sqrt{\frac{c\sigma}{d_h}} \sum_j \mathbf{b}_j^{(h+1)}(\mathbf{x}^{(2)}) \mathbf{D}^{(h)}(\mathbf{x}^{(2)}) \Pi_{\mathbf{G}_j} \mathbf{M}_j^{(h+1)} \mathbf{w}_j^{(h+1)} \right\| \\
 & \leq \left( 12 \sqrt{\frac{1}{\alpha} \frac{\log \frac{8}{\delta_2}}{d_h}} + \frac{24}{\alpha} \frac{\sqrt{\log \frac{\sum d_h}{\delta_3}}}{\sqrt{d_h}} \right) + \left( 12 \sqrt{\frac{1}{\alpha} \frac{\log \frac{8}{\delta_2}}{d_h}} + \frac{24}{\alpha} \frac{\sqrt{\log \frac{\sum d_h}{\delta_3}}}{\sqrt{d_h}} \right) + \left( 12 \sqrt{\frac{1}{\alpha} \frac{\log \frac{8}{\delta_2}}{d_h}} + \frac{24}{\alpha} \frac{\sqrt{\log \frac{\sum d_h}{\delta_3}}}{\sqrt{d_h}} \right) \\
 & \leq 36 \sqrt{\frac{1}{\alpha} \frac{\log \frac{8}{\delta_2}}{d_h}} + \frac{72}{\alpha} \frac{\sqrt{\log \frac{\sum d_h}{\delta_3}}}{\sqrt{d_h}}
 \end{aligned}$$