The Fourth Assignment

3190102721 Xu Shengze

Problem 1. Suppose that we have the following (mixed) quantum state:

$$\frac{1}{3}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

a) Write density matrix expression for the mentioned quantum system.

b) Write expression with trace and projection and estimate the probabilities for state 0 and state 1.

Answer 1.

a)

$$\rho = \frac{1}{3}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

b) We have $\Pi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\Pi_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

The probability for state 0 is $P(|0\rangle) = \mathrm{Tr}(\rho\Pi_0) = \mathrm{Tr}(\begin{pmatrix} \frac{1}{6} & 0 \\ -\frac{1}{6} & 0 \end{pmatrix}) = \frac{1}{6}$

The probability for state 1 is $P(|1\rangle) = \mathrm{Tr}(\rho\Pi_1) = \mathrm{Tr}(\begin{pmatrix} 0 & -\frac{1}{6} \\ 0 & \frac{5}{6} \end{pmatrix}) = \frac{5}{6}$

Problem 2. Consider the decoherence operator D that we discussed in lecture 19. Apply D to the following quantum state represented by density matrix:

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Please write each step of D affecting the quantum system.

Answer 2.

First, we append a null qubit: $\rho \to \rho \otimes |0\rangle\langle 0|$, then for this quantum system, the density matrix is:

$$\rho_0 = \begin{pmatrix} \frac{3}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we "copy" the original qubit into the ancilla. This can be achieved by applying the operator $\Lambda(\sigma^x) : |a, b\rangle \to |a, a \otimes b\rangle$.

We get the following formula:

$$\rho \otimes |0\rangle\langle 0| \xrightarrow{\Lambda(\sigma^x)} \sum_{j,k} \rho_{j,k} |j, j\rangle\langle k, k|$$

So for this quantum system, the density matrix becomes:

$$\rho_0^* = \begin{pmatrix} I & 0 \\ 0 & \sigma^x \end{pmatrix} \rho_0 \begin{pmatrix} I & 0 \\ 0 & \sigma^x \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Finally, we take the partial trace over the ancilla, which yields the diagonal matrix $\sum_k \rho_{kk} |k\rangle\langle k|$.

So for this quantum system, the final diagonal matrix result is:

$$\rho^* = (I \otimes \langle 0|)\rho_0^*(I \otimes |0\rangle) + (I \otimes \langle 1|)\rho_0^*(I \otimes |1\rangle) = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

Problem 3. Consider the topic of the lecture 21. Apply measuring operator

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

to the state $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Remember that we start by joining the system with $|0\rangle\langle 0|$. What is the outcome?

Answer 3. We denote $\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. First we add subsystem: joint state is $\rho \otimes |0\rangle\langle 0|$.

Then we apply the measuring operator $W = \sum_j \Pi_{L_j} \otimes U_j$, we will get:

$$
\begin{aligned}
W(\rho \otimes |0\rangle\langle 0|)W^\dagger &= \sum_j (\Pi_{L_j}\rho\Pi_{L_j}) \otimes (U_j|0\rangle\langle 0|U_j^\dagger) \\
&= (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \otimes (\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} |0\rangle\langle 0| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}) \\
&+ (\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \otimes (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |0\rangle\langle 0| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \\
&= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Problem 4. Write down step-by-step application of Shor's algorithm to the number 15. When you describe steps, use $\alpha = 7$ when random number is picked in step 3.

Answer 4.

1. 15 is not even.

2. $15 \neq m^k$, where m is an integer and $k = 2, 3$, that's because $\log_2 15 < 4$.

3. Choose $a = 7$, $\gcd(7, 15) = 1$.

4. $7^4 \equiv 1(\text{mod} 15)$, $r = \text{per}_{15} 7 = 4$, even.

5. $a^{r/2} - 1 = 7^2 - 1 = 48$, $d = \gcd(48, 15) = 3$, output the nontrivial divisor 3.

After the above process, the Shor's algorithm finally outputs the nontrivial divisor 3 for the input number 15, which means 15 is not prime.