Scientific Computing: HW8 Solution

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Problem 1

The augmented matrix of the system of equations is as follows:

$$\left[\begin{array}{cccc|c} 6 & 2 & 1 & -1 & 6 \\ 2 & 4 & 1 & 0 & -1 \\ 1 & 1 & 4 & -1 & 5 \\ -1 & 0 & -1 & 3 & -5 \end{array}\right] \tag{1}$$

We perform the first transformation,

$$\left[\begin{array}{cccc|c} 6 & 2 & 1 & -1 & 6 \\ 0 & \frac{10}{3} & \frac{2}{3} & \frac{1}{3} & -3 \\ 0 & \frac{2}{3} & \frac{23}{6} & -\frac{5}{6} & 4 \\ 0 & \frac{1}{3} & -\frac{5}{6} & \frac{17}{6} & -4 \end{array}\right] \tag{2}$$

Next, we perform the second transformation,

$$\left[\begin{array}{cccc|c} 6 & 2 & 1 & -1 & 6 \\ 0 & \frac{10}{3} & \frac{2}{3} & \frac{1}{3} & -3 \\ 0 & 0 & \frac{37}{10} & -\frac{9}{10} & \frac{23}{5} \\ 0 & 0 & -\frac{9}{10} & \frac{14}{5} & -\frac{37}{10} \end{array}\right] \tag{3}$$

Then, we perform the third transformation,

$$\left[\begin{array}{cccc|c} 6 & 2 & 1 & -1 & 6 \\ 0 & \frac{10}{3} & \frac{2}{3} & \frac{1}{3} & -3 \\ 0 & 0 & \frac{37}{10} & -\frac{9}{10} & \frac{23}{5} \\ 0 & 0 & 0 & \frac{191}{74} & -\frac{191}{74} \end{array}\right] \tag{4}$$

Now we can easily get the solution of the system of equations:

$$\begin{cases} x_1 = 1 \\ x_2 = -1 \\ x_3 = 1 \\ x_4 = -1 \end{cases} \tag{5}$$

Problem 2

We write A as follow:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \tag{6}$$

Assume that B is the inverse of A and we define B like A above, then we have:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & \cdots & b_{2n} \\ & & \ddots & \vdots \\ & & & b_{nn} \end{bmatrix} = I \tag{7}$$

We get the relation:

$$\sum_{k=i}^{n} a_{ik}b_{kj} = c_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases} \tag{8}$$

Firstly, by observing the last line we know $b_{nn} = \frac{1}{a_{nn}}$, further we could easily get $b_{ii} = \frac{1}{a_{ii}}$.
Next we consider the penultimate line and get the relation $a_{n-1,n-1}b_{n-1,n} + a_{n-1,n}b_{n,n} = 0$, then we know $b_{n-1,n} = -\frac{a_{n-1,n}}{a_{n-1,n-1}a_{n,n}}$.
For the l^{th} line from the bottom, we have $\sum_{k=l}^{n} a_{lk}b_{kj} = c_{lj}$. When $j \geq l+1$, we have $\sum_{k=l}^{n} a_{lk}b_{kj} = 0$, then $a_{ll}b_{lj} + \sum_{k=l+1}^{n} a_{lk}b_{kj} = 0$, so $b_{lj} = -\frac{\sum_{k=l+1}^{n} a_{lk}b_{kj}}{a_{ll}}$ for the other elements are known quantities.
Replace subscript, we get the recurrence expression, $b_{ii} = \frac{1}{a_{ii}}$ and $b_{ij} = -\frac{\sum_{k=i+1}^{n} a_{ik}b_{kj}}{a_{ii}}(j > i)$.
What's interesting is that, unlike usual, this recursive expression is pushed from back to front.

Problem 3

Assume that the LU decomposition of A is not unique, then we can write A as $A = L_1U_1 = L_2U_2$.
Because $L_i, U_i(i = 1, 2)$ are upper triangular matrices, they have inverse matrix, and $L_2^{-1}L_1 = U_1^{-1}U_2$.
Meanwhile, the inverse of the upper triangular matrix is still the upper triangular matrix, and the inverse of the lower triangular matrix is still the lower triangular matrix.
Therefore, $L_2^{-1}L_1 = U_1^{-1}U_2 = E$, then we have $L_1 = L_2, U_1 = U_2$, the proposition is proved.

Problem 4

Assume that $A = LL^T$ and write the code according to the formula.

```
A=zeros(10,10);
A(1,1)=9;A(1,2)=-4;A(1,3)=1;
A(2,1)=-4;A(2,2)=6;A(2,3)=-4;A(2,4)=1;
for i=3:10-2
        A(i,i-2)=1;A(i,i-1)=-4;A(i,i)=6;A(i,i+1)=-4;A(i,i+2)=1;
end
A(9,7)=1;A(9,8)=-4;A(9,9)=5;A(9,10)=-2;
A(10,8)=1;A(10,9)=-2 ;A(10,10)=1;
disp(A);

L=zeros(10,10);
for i=10:-1:1
        L(i,i)=A(i,i);
        for k=i+1:1:10
                L(i,i)=L(i,i)-L(i,k)*L(i,k);
        end
        L(i,i)=sqrt(L(i,i));
        for j=1:1:i-1
                L(j,i)=A(i,j);
                for k=i+1:1:10
                        L(j,i)=L(j,i)-L(i,k)*L(j,k);
                end
        L(j,i)=L(j,i)*1.0/L(i,i);
        end
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```
end
disp(L);
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By running the program, we get the matrix L:

$$L = \begin{bmatrix} 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$