Scientific Computing: HW9 Solution

May 30, 2021

Professor Lai

Xu Shengze 3190102721

Problem 1

The augmented matrix of the system of equations is as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -3 & 1 & 3 & 2 \\ 3 & -2 & 1 & 3 \end{array}\right] \tag{1}$$

We perform the first transformation,

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 3 \\ -3 & 1 & 3 & 2 \\ 1 & 2 & -1 & 1 \end{array}\right] \tag{2}$$

Next, we perform the second transformation,

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 3 \\ 0 & -1 & 4 & 5 \\ 0 & \frac{8}{3} & -\frac{4}{3} & 0 \end{array}\right] \tag{3}$$

Then, we perform the third transformation,

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 3 \\ 0 & \frac{8}{3} & -\frac{4}{3} & 0 \\ 0 & -1 & 4 & 5 \end{array}\right] \tag{4}$$

Then, we perform the last transformation,

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 3 \\ 0 & \frac{8}{3} & -\frac{4}{3} & 0 \\ 0 & 0 & \frac{7}{2} & 5 \end{array}\right] \tag{5}$$

Now we can easily get the solution of the system of equations:

$$\begin{cases} x_1 = 1 \\ x_2 = \dfrac{5}{7} \\ x_3 = \dfrac{10}{7} \end{cases} \tag{6}$$

Problem 2

Assume that $A = (a_{ij})_{n\times n}$. According to the condition, A is a matrix with a bandwidth of $2m+1$. $\forall |i-j| > m$, we have $a_{ij} = 0$.
$A = LL^T$, $L = (l_{jk})_{n\times n}$.
We have the relation below:

$$\begin{cases} l_{kk} = (a_{kk} - \displaystyle\sum_{i=1}^{k-1} l_{ki}^2)^{\frac{1}{2}} \\ l_{jk} = \dfrac{a_{jk} - \sum_{i=1}^{k-1} l_{ji} l_{ki}}{l_{kk}}, j = k+1, k+2, \cdots, n \end{cases} \tag{7}$$

(i)When $k = 1$, it's obvious that $l_{j1} = 0$.
(ii)When $k = 2$, $l_{j2} = \frac{a_{j2} - l_{j1} l_{21}}{l_{22}}, j \geq m+3$. $a_{j2} = 0$ and $l_{j1} = 0$, so $l_{j2} = 0$.
(iii)Assume that the conclusion holds for k, consider the situation when $k+1$, at this time $l_{j,k+1} = \frac{a_{j,k+1} - \sum_{i=1}^{k} l_{ji} l_{k+1,i}}{l_{k+1,k+1}}$. $a_{j,k+1} = 0$, $j - i > k+1+m-k = m+1 > m$, $l_{ji} = 0$, so $l_{j,k+1} = 0$.
Therefore, L is a matrix with a bandwidth of $m+1$.

Problem 3

Frobenius norm of matrix A is $||A||_F = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$.
(i)Positivity: $\forall A \in R^{n\times n}, ||A||_F = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}} \geq 0$, $||A||_F = 0$ if and only if $\forall i, j, a_{ij} = 0, i.e. A = 0$.
(ii)Homogeneity: $\forall A \in R^{n\times n}$ and $\alpha \in R$, $||\alpha A||_F = (\sum_{i,j=1}^{n} |\alpha a_{ij}|^2)^{\frac{1}{2}} = |\alpha|(\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}} = |\alpha| \cdot ||A||_F$.
(iii)the Triangle Inequality: $\forall A, B \in R^{n\times n}, ||A+B||_F = (\sum_{i,j=1}^{n} |a_{ij}+b_{ij}|^2)^{\frac{1}{2}} \leq (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}+(\sum_{i,j=1}^{n} |b_{ij}|^2)^{\frac{1}{2}} = ||A||_F + ||B||_F$.
(iv)Compatibility: $\forall A, B \in R^{n\times n}, ||A \cdot B|| = (\sum_{i,j=1}^{n} |a_{ij}b_{ij}|^2)^{\frac{1}{2}} \leq (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}(\sum_{i,j=1}^{n} |b_{ij}|^2)^{\frac{1}{2}} = ||A||_F \cdot ||B||_F$.

Problem 4

(a) $||A||_F = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$, $||A||_2 = max||Ax||_2 = \sqrt{\lambda_1}$, take $x = (x_1, x_2, \cdots, x_n)^T$.
$||Ax||_2 = ||AX||_F \leq ||A||_F||X||_F$, then $||Ax||_2 \leq ||A||_F$, so $||A||_2 \leq ||A||_F$.
$(||A||_F)^2 = tr(A^T A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n \leq n\lambda_1 = n||A||_2^2$, so $||A||_F \leq \sqrt{n}||A||_2$.

(b) $||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$.
Let $x' = Ax$, then $||Ax||_2 \leq \sqrt{n}||Ax||_\infty$, so $||A||_2 \leq \sqrt{n}||A||_\infty$.
$||A||_2 = max\frac{||Ax||_2}{||x||_2} \geq \frac{\sqrt{\sum_{i=1}^{n}(\sum_{j=1}^{n} x_j a_{ij})^2}}{\sum_{j=1}^{n} x_j^2}$, take $x_{ij} = sgn(a_{ij})$, then $||A||_2 \geq \frac{\sqrt{\sum_{i=1}^{n}(\sum_{j=1}^{n} |a_{ij}|)^2}}{\sqrt{n}} \geq \frac{max(\sum_{j=1}^{n} |a_{ij}|^2)}{\sqrt{n}} = \frac{1}{\sqrt{n}}||A||_\infty$.

(c) $||x||_1 \leq \sqrt{n}||x||_2$.
Let $x' = Ax$, then $||Ax||_1 \leq \sqrt{n}||Ax||_2$, so $\frac{1}{\sqrt{n}}||A||_1 \leq ||A||_2$.
Meanwhile, $||x||_\infty \leq ||x||_1$, we can get $||A||_\infty \leq ||A||_1$, so $||A||_2 \leq \sqrt{n}||A||_\infty \leq \sqrt{n}||A||_1$.

Problem 5

First we generate the Hilbert matrix.

```
function H=returnH(n)
        H=zeros(n,n);
        for i=1:n
                for j=1:n
                        H(i,j)=1/(i+j-1);
                end
        end
end
```

Then, we use the column principal element Gaussian elimination method to solve fangchengzu.

```
function [H,x1] = change(H,x1,n)
        for k=1:n-1
                a=H(k:n,k);
                [s,flag]=max(abs(a));
                if s==0
                        break;
```

```
                end
                flag=flag+k-1;
                for j=1:n
                        t=H(flag,j);
                        H(flag,j)=H(k,j);
                        H(k,j)=t;
                end
                t=x1(flag);
                x1(flag)=x1(k);
                x1(k)=t;
                for i=k+1:n
                        m(i,k)=H(i,k)/H(k,k);
                        H(i,k)=m(i,k);
                end
                for i=k+1:n
                        for j=k+1:n
                                H(i,j)=H(i,j)-m(i,k)*H(k,j);
                        end
                        x1(i)=x1(i)-m(i,k)*x1(k);
                end
        end
        for i=n:-1:1
                for j=i+1:n
                        x1(i)=x1(i)-H(i,j)*x1(j);
                end
                x1(i)=x1(i)/H(i,i);
        end
end
```

According to the problem, set the loop condition to solve n.

```
n=1;
m=zeros(n,n);
while 1
        H=returnH(n);H0=H;
        x=ones(n,1);b=H*x;x1=b;
        [H,x1]=change(H,x1,n);
        if max(abs(x-x1))/max(abs(x))>=1
                break;
        end
        n=n+1;
end
```

By running the program, we got the following solution.
n: 13.
$Cond$: 1.5192×10^{16}.
Infinite norm of backward error: 8.8818×10^{-16}.
Infinite norm of forward error: 4.2430.