

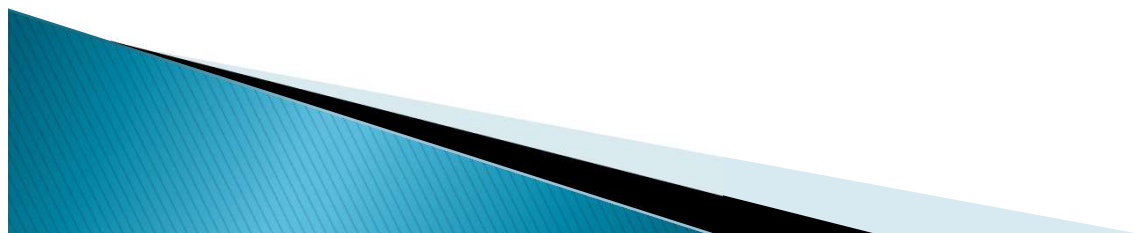
第八章 解线性方程组的迭代法

- ▶ 大稀疏方程组适用迭代法
- ▶ 常用迭代法
- ▶ 迭代法收敛性
- ▶ 误差估计
- ▶ 收敛速率



大稀疏方程组

- ▶ 线性方程组 $Ax = b$
 - 大稀疏矩阵: 非零元素极少
 - 大稀疏方程组: 非零系数极少
 - 微分方程离散化
 - 结构分析
 - 网络排名
- ▶ 大稀疏方程组更适用迭代法求解
 - 直接法: 用有限步计算得到准确解
 - 迭代法: 给出一个近似解序列



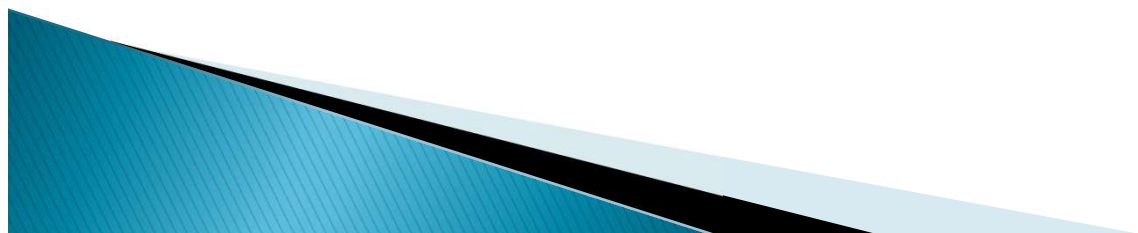
迭代法

▶ 线性方程组

$$Ax = b$$
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

▶ 迭代法

- 给出一个近似解序列: $x^{(k+1)} = Bx^{(k)} + g$, $k = 0, 1, 2, \dots$
(一阶线性定常迭代法)
- 收敛性: $\lim_{k \rightarrow \infty} x^{(k)} = x^*$, 其中 x^* 为方程的解
- 误差估计: $\|x^{(k)} - x^*\|$
- 收敛速率: $\|x^{(k+1)} - x^*\| = r\|x^{(k)} - x^*\|$



Jacobi 迭代法算例

- ▶ 用Jacobi迭代法解线性方程组
- ▶ 例1.方程组

$$\begin{cases} 10x_1 - x_2 - 2x_3 = 7.2 \\ -x_1 + 10x_2 - 2x_3 = 8.3 \\ -x_1 - x_2 + 5x_3 = 4.2 \end{cases}$$

化成

$$x_1 = 0.1x_2 + 0.2x_3 + 0.72$$

$$x_2 = 0.1x_1 + 0.2x_3 + 0.83$$

$$x_3 = 0.2x_1 + 0.2x_2 + 0.84$$

任取初始近似 $x^{(0)}$,对 $k = 0, 1, 2, \dots$ 计算

$$x_1^{(k+1)} = 0.1x_2^{(k)} + 0.2x_3^{(k)} + 0.72$$

$$x_2^{(k+1)} = 0.1x_1^{(k)} + 0.2x_3^{(k)} + 0.83$$

$$x_3^{(k+1)} = 0.2x_1^{(k)} + 0.2x_2^{(k)} + 0.84$$

直至 $\|x^{(k+1)} - x^{(k)}\| \leq \varepsilon$,预定的精度,例如 5×10^{-5}



Jacobi 迭代法算例

► 计算结果

k	x_1^k	x_2^k	x_3^k
0	0	0	0
1	0.720000000000	0.830000000000	0.840000000000
2	0.971000000000	1.070000000000	1.150000000000
3	1.057000000000	1.157100000000	1.248200000000
4	1.085350000000	1.185340000000	1.282820000000
5	1.095098000000	1.195099000000	1.294138000000
6	1.098337500000	1.198337400000	1.298039400000
7	1.099441620000	1.199441630000	1.299334980000
8	1.099811159000	1.199811158000	1.299776650000
9	1.099936445800	1.199936445900	1.299924463400
10	1.099978537270	1.199978537260	1.299974578340
11	1.099992769394	1.199992769395	1.299991414906

Jacobi 迭代法

▶ 线性方程组(1) $Ax = b$

▶ 化成(2) $x = Bx + g$

从第一个方程解出 x_1 ,第二个方程解出 x_2, \dots ,得

$$x_1 = b_{12}x_2 + b_{13}x_3 + \cdots + b_{1n}x_n + g_1$$

$$x_2 = b_{21}x_1 + b_{23}x_3 + \cdots + b_{2n}x_n + g_2$$

... ..

$$x_n = b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{n,n-1}x_{n-1} + g_n$$

▶ 迭代

任取初始近似 $x^{(0)}$,对 $k = 0, 1, 2, \dots$ 计算

$$x_1^{(k+1)} = b_{12}x_2^{(k)} + b_{13}x_3^{(k)} + \cdots + b_{1n}x_n^{(k)} + g_1$$

$$x_2^{(k+1)} = b_{21}x_1^{(k)} + b_{23}x_3^{(k)} + \cdots + b_{2n}x_n^{(k)} + g_2$$

... ..

$$x_n^{(k+1)} = b_{n1}x_1^{(k)} + b_{n2}x_2^{(k)} + \cdots + b_{n,n-1}x_{n-1}^{(k)} + g_n$$

直至 $\|x^{(k+1)} - x^{(k)}\| \leq \varepsilon$, 预定的精度.



Jacobi 迭代法矩阵关系

- 线性方程组

$$(1) Ax = b$$

- 化成

$$(2) x = Bx + g$$

- 迭代

$$x^{k+1} = Bx^{(k)} + g$$

- 迭代矩阵: $B = L + U$

$$L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ b_{21} & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & 0 & \cdots & 0 \\ & & & \ddots & \\ b_{n1} & b_{n2} & b_{n3} & \cdots & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & 0 & b_{23} & \cdots & b_{2n} \\ 0 & 0 & 0 & \cdots & b_{3n} \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$B = - \begin{bmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & \cdots & \frac{a_{2n}}{a_{22}} \\ \frac{a_{31}}{a_{33}} & \frac{a_{32}}{a_{33}} & 0 & \cdots & \frac{a_{3n}}{a_{33}} \\ & & & \ddots & \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \frac{a_{n3}}{a_{nn}} & \cdots & 0 \end{bmatrix}$$



Gauss-Seidel 迭代法算例

- ▶ 用G-S迭代法解线性方程组
- ▶ 例2方程组

$$\begin{aligned}10x_1 - x_2 - 2x_3 &= 7.2 \\ -x_1 + 10x_2 - 2x_3 &= 8.3 \\ -x_1 - x_2 + 5x_3 &= 4.2\end{aligned}$$

化成

$$x_1 = 0.1x_2 + 0.2x_3 + 0.72$$

$$x_2 = 0.1x_1 + 0.2x_3 + 0.83$$

$$x_3 = 0.2x_1 + 0.2x_2 + 0.84$$

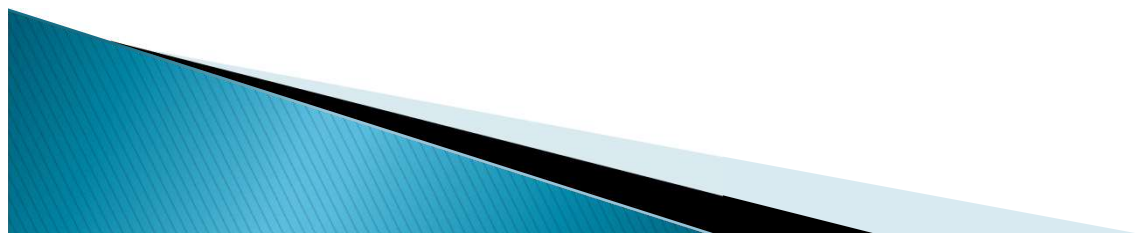
任取初始近似 $x^{(0)}$, 对 $k = 0, 1, 2, \dots$ 计算

$$x_1^{(k+1)} = 0.1x_2^{(k)} + 0.2x_3^{(k)} + 0.72$$

$$x_2^{(k+1)} = 0.1x_1^{(k+1)} + 0.2x_3^{(k)} + 0.83$$

$$x_3^{(k+1)} = 0.2x_1^{(k+1)} + 0.2x_2^{(k+1)} + 0.84$$

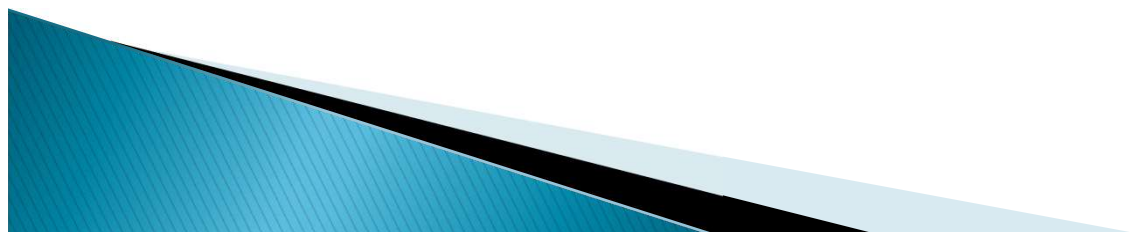
直至 $\|x^{(k+1)} - x^{(k)}\| \leq \varepsilon$, 预定的精度, 例如 5×10^{-5} .



Gauss-Seidel迭代法算例

▶ 计算结果

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	0.720000000000000	0.902000000000000	1.164400000000000
2	1.043080000000000	1.167188000000000	1.282053600000000
3	1.093129520000000	1.195723672000000	1.297770638400000
4	1.099126494880000	1.199466777168000	1.299718654409600
5	1.099890408598720	1.199932771741790	1.299964636068100
6	1.099986204387800	1.199991547652400	1.299995550408040
7	1.099998264846850	1.199998936566290	1.299999440282630



Gauss-Seidel迭代法

▶ 线性方程组(1) $Ax = b$

化成(2) $x = Bx + g$

从第一个方程解出 x_1 , 第二个方程解出 x_2 , ..., 得

$$x_1 = b_{12}x_2 + b_{13}x_3 + \cdots + b_{1n}x_n + g_1$$

$$x_2 = b_{21}x_1 + b_{23}x_3 + \cdots + b_{2n}x_n + g_2$$

... ..

$$x_n = b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{n,n-1}x_{n-1} + g_n$$

▶ 迭代

任取初始近似 $x^{(0)}$, 对 $k = 0, 1, 2, \dots$ 计算

$$x_1^{(k+1)} = b_{12}x_2^{(k)} + b_{13}x_3^{(k)} + \cdots + b_{1n}x_n^{(k)} + g_1$$

$$x_2^{(k+1)} = b_{21}x_1^{(k+1)} + b_{23}x_3^{(k)} + \cdots + b_{2n}x_n^{(k)} + g_2$$

... ..

$$x_n^{(k+1)} = b_{n1}x_1^{(k+1)} + b_{n2}x_2^{(k+1)} + \cdots + b_{n,n-1}x_{n-1}^{(k+1)} + g_n$$

直至 $\|x^{(k+1)} - x^{(k)}\| \leq \varepsilon$, 预定的精度.

G-S迭代法矩阵关系

- ▶ 线性方程组

(1) $Ax = b$

- ▶ 化成

(2) $x = Bx + g, B = L + U$

- ▶ 迭代

$$\begin{aligned}x^{(k+1)} &= Lx^{(k+1)} + Ux^{(k)} + g \\x^{(k+1)} &= (I - L)^{-1}Ux^{(k)} + (I - L)^{-1}g\end{aligned}$$

- ▶ 迭代矩阵: $B_1 = (I - L)^{-1}U$

$$L = \begin{bmatrix} 0 & & & & \\ b_{21} & 0 & & & \\ b_{31} & b_{32} & 0 & & \\ & & & \ddots & \\ b_{n1} & b_{n2} & b_{n3} & \cdots & 0 \end{bmatrix}$$
$$U = \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1n} \\ & 0 & b_{23} & \cdots & b_{2n} \\ & & 0 & \cdots & b_{3n} \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$



SOR迭代法

- ▶ 线性方程组(1) $Ax = b$
- ▶ 化成(2) $x = Bx + g$ (同Jacobi法, G-S法)
- ▶ 迭代(这是G-S法改进与推广. $\omega = 1$ 即G-S法)

任取初始近似 $x^{(0)}$, 对 $k = 0, 1, 2, \dots$ 计算

$$\begin{aligned}x_1^{(k+1)} &= (1 - \omega)x_1^{(k)} + \omega(b_{12}x_2^{(k)} + b_{13}x_3^{(k)} + \dots + b_{1n}x_n^{(k)} + g_1) \\x_2^{(k+1)} &= (1 - \omega)x_2^{(k)} + \omega(b_{21}x_1^{(k+1)} + b_{23}x_3^{(k)} + \dots + b_{2n}x_n^{(k)} + g_2) \\&\dots\dots\dots \\x_n^{(k+1)} &= (1 - \omega)x_n^{(k)} + \omega(b_{n1}x_1^{(k+1)} + b_{n2}x_2^{(k+1)} + \dots + b_{nn}x_n^{(k+1)} + g_n) \\&\text{直至 } \|x^{(k+1)} - x^{(k)}\| \leq \varepsilon, \text{ 预定的精度. } (\omega, \text{ 参数, } 0 < \omega < 2)\end{aligned}$$

矩阵表示 $x^{(k+1)} = (1 - \omega)x^{(k)} + \omega(Lx^{(k+1)} + Ux^{(k)} + g)$

- ▶ 迭代矩阵: $B_\omega = (I - \omega L)^{-1}(1 - \omega)I + \omega U$
- ▶ SOR (successive over-relaxation)



SOR迭代法算例

- ▶ 用G-S迭代法解线性方程组
- ▶ 例3方程组

$$\begin{aligned}10x_1 - x_2 - 2x_3 &= 7.2 \\ -x_1 + 10x_2 - 2x_3 &= 8.3 \\ -x_1 - x_2 + 5x_3 &= 4.2\end{aligned}$$

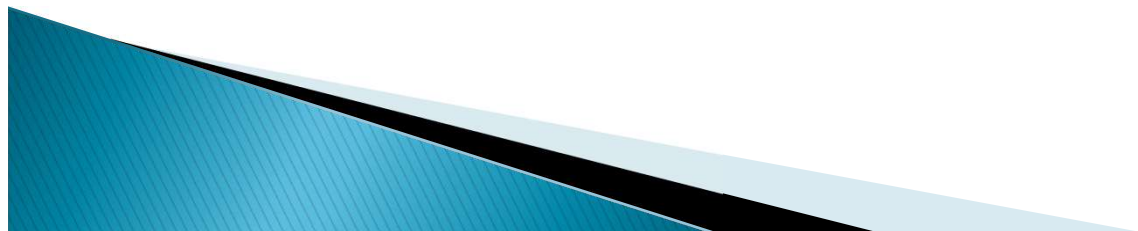
化成

$$\begin{aligned}x_1 &= 0.1x_2 + 0.2x_3 + 0.72 \\ x_2 &= 0.1x_1 + 0.2x_3 + 0.83 \\ x_3 &= 0.2x_1 + 0.2x_2 + 0.84\end{aligned}$$

任取初始近似 $x^{(0)}$, 对 $k = 0, 1, 2, \dots$ 计算

$$\begin{aligned}x_1^{(k+1)} &= (1 - \omega)x_1^{(k)} + \omega(0.1x_2^{(k)} + 0.2x_3^{(k)} + 0.72) \\ x_2^{(k+1)} &= (1 - \omega)x_2^{(k)} + \omega(0.1x_1^{(k)} + 0.2x_3^{(k)} + 0.83) \\ x_3^{(k+1)} &= (1 - \omega)x_3^{(k)} + \omega(0.2x_1^{(k+1)} + 0.2x_2^{(k+1)} + 0.84)\end{aligned}$$

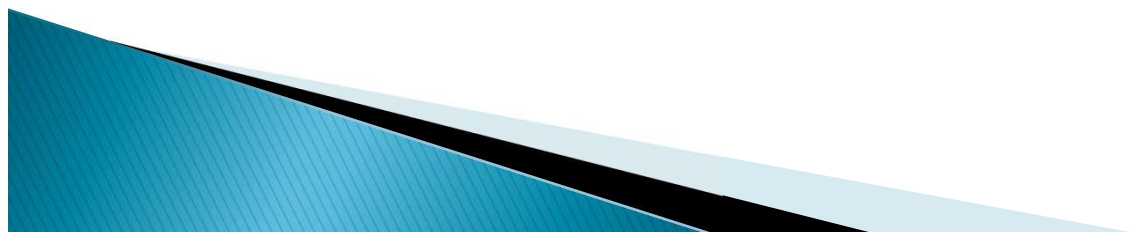
直至 $\|x^{(k+1)} - x^{(k)}\| \leq \varepsilon$, 预定的精度, 例如 5×10^{-5} .



SOR迭代法算例

▶ 计算结果

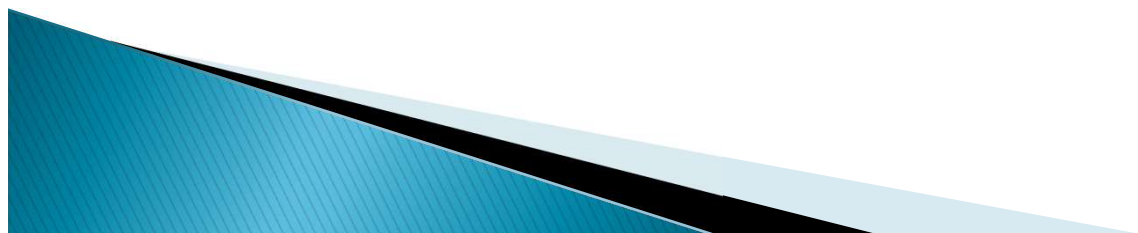
k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	0.7560000000000000	0.9508800000000000	1.2404448000000000
2	1.0785358080000000	1.1976956678400000	1.2979863699264000
3	1.100408392407744	1.199735235495357	1.300130843363331
4	1.099979257212925	1.200038537338889	1.299997194687714
5	1.100004494444357	1.199997955934133	1.300000654845097
6	1.099999698168336	1.200000208028439	1.299999947559068



迭代矩阵

▶ 一阶线性定常迭代

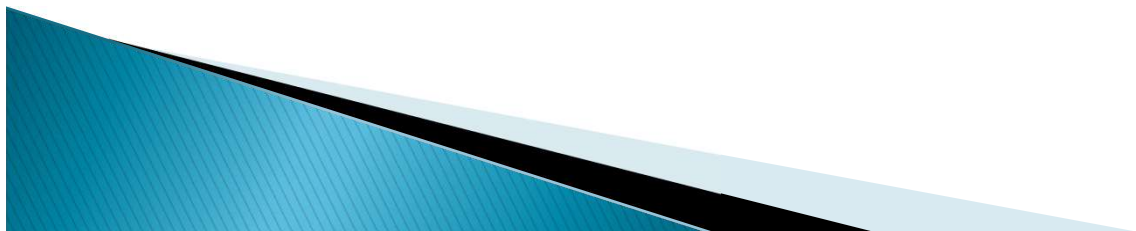
- 两次近似有关系 $x^{(k+1)} = Mx^{(k)} + f$
- 称 M 为迭代矩阵
- Jacobi迭代法
 - $x^{(k+1)} = Bx^{(k)} + g$
 - 迭代矩阵 $B = L + U$
- Gauss-Seidel迭代法
 - $x^{(k+1)} = (I - L)^{-1}Ux^{(k)} + (I - L)^{-1}g$
 - 迭代矩阵 $B_1 = (I - L)^{-1}U$
- SOR迭代法
 - $x^{(k+1)} = (I - \omega L)^{-1} \left(((1 - \omega)I + \omega U)x^{(k)} + \omega g \right)$
 - 迭代矩阵 $B_\omega = (I - \omega L)^{-1}((1 - \omega)I + \omega U)$



迭代收敛性

▶ 收敛充要条件

- 对任何初值 $x^{(0)}$, 迭代格式 $x^{(k+1)} = Mx^{(k)} + f$ 确定的序列 $\{x^{(k)}\}$ 收敛并且极限与初值无关的充分必要条件是迭代矩阵的谱半径小于1, 即 $\rho(M) < 1$.
- 三个算例皆收敛
 - $\rho(B) = 0.337 \dots$
 - $\rho(B_1) = 0.125 \dots$
 - $\rho(B_\omega) = 0.086 \dots$



收敛性判定

▶ 计算迭代矩阵特征值并确定谱半径

◦ 例1.

- -0.1
- 0.33722813232690
- -0.23722813232690

◦ 例2

- 0
- 0.12579720807237
- -0.03179720807237

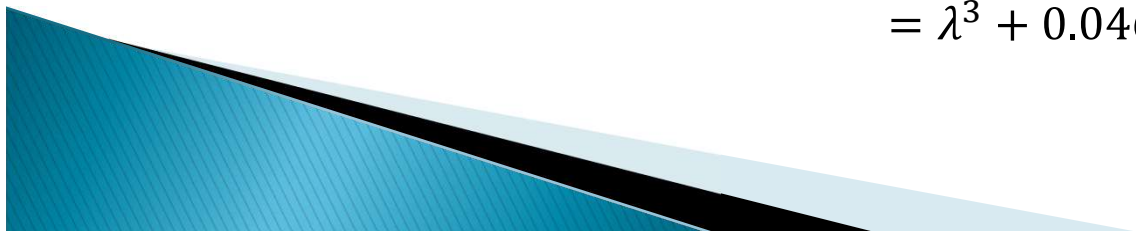
◦ 例3

- -0.086...
- 0.020... ± 0.032...i

$$\begin{aligned}\det(\lambda I - B) &= \begin{vmatrix} \lambda & -0.1 & -0.2 \\ -0.1 & \lambda & -0.2 \\ -0.2 & -0.2 & \lambda \end{vmatrix} \\ &= \lambda^3 - 0.09\lambda - 0.008 \\ &= (\lambda + 0.1)(\lambda^2 - 0.1\lambda - 0.08)\end{aligned}$$

$$\begin{aligned}\det(\lambda I - B_1) &= \det(\lambda I - (I - L)^{-1}U) \\ &= \det(\lambda(I - L) - U) \\ &= \begin{vmatrix} \lambda & -0.1 & -0.2 \\ -0.1\lambda & \lambda & -0.2 \\ -0.2\lambda & -0.2\lambda & \lambda \end{vmatrix} \\ &= \lambda(\lambda^2 - 0.094\lambda - 0.004)\end{aligned}$$

$$\begin{aligned}\det(\lambda I - B_\omega) &= \det((\lambda + 1 - \omega)I - \lambda\omega L + \omega U) \\ &= \lambda^3 + 0.0461445\lambda^2 - 0.00281175\lambda + 0.000125\end{aligned}$$

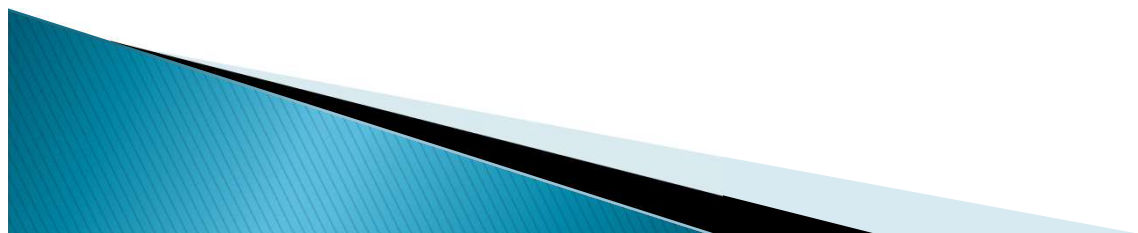


误差估计

- ▶ 定理
- ▶ 如果迭代格式 $x_{k+1} = Mx_k + f$ 的迭代矩阵的范数 $\|M\| = q < 1$, 则迭代收敛, 並且有误差估计:

$$\|x_k - x^*\| \leq \frac{q}{1-q} \|x_k - x_{k-1}\| \leq \frac{q^k}{1-q} \|x_1 - x_0\|$$

- ▶ 常用前一不等式估计当前近似的误差



对角占优阵

▶ 严格对角优势矩阵

- 可逆
- Jacobi 法收敛
- G-S法收敛

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, i = 1, 2, \dots, n$$

▶ 不可约对角优势阵

- 可逆
- Jacobi 法收敛
- SOR法 ($0 < \omega \leq 1$) 收敛

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, i = 1, 2, \dots, n$$

至少有一*i*严格大于.
并且不存在排列阵*P*,

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$$



SOR收敛性

- ▶ 松弛法收敛的必要条件是 $0 < \omega < 2$
- ▶ 若 A 是实对称矩阵或 Hermite 矩阵, 对角元为正, 则当 $0 < \omega < 2$ 时松弛法收敛的充分必要条件是 A 正定

