Scientific Computing: HW14 Solution

June 29, 2021

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Problem 1

Trapezoidal formula is $y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, y_{n+1}))$.
We know that $T_{n+1} = y(x_{n+1}) - [y(x_n) + \frac{h}{2}(f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1})))]$.
According to Taylor expansion, we get the following relationship,

$$\begin{cases} y(x_{n+1}) = y(x_n) + hy'(x_n) + \dfrac{h^2}{2}y''(x_n) + \dfrac{h^3}{6}y'''(\xi_{n_1}) \\ y(x_n) = y(x_{n+1}) - hy'(x_{n+1}) + \dfrac{h^2}{2}y''(x_{n+1}) - \dfrac{h^3}{6}y'''(\xi_{n_2}) \end{cases} \tag{1}$$

We have $T_{n+1} = y(x_{n+1}) - y(x_n) + \frac{h}{2}(y'(x_n) + y'(x_{n+1}))$, substitute the relationship of the above equations and we know that $T_{n+1} = \frac{h^3}{12}[y'''(\xi_{n_1}) + y'''(\xi_{n_2})] + \frac{h^2}{4}[y''(x_n) - y''(x_{n+1})]$.
Meanwhile, we have $y''(x_n) - y''(x_{n+1}) = -hy'''(\xi_{n_3})$, so we have $T_{n+1} = \frac{h^3}{12}[y'''(\xi_{n_1}) + y'''(\xi_{n_2}) - 3y'''(\xi_{n_3})]$, and there must be ξ_n satisfying that $y'''(\xi_n) = -(y'''(\xi_{n_1}) + y'''(\xi_{n_2}) - 3y'''(\xi_{n_3}))$.
Finally, we have $T_{n+1} = -\frac{h^3}{12}y'''(\xi_n)$, where $\xi_n \in [x_n, x_{n+1}]$.

Problem 2

(a) According to the question, we have $\frac{dy}{3-2y} = dx$, integrate both sides and we have $-\frac{1}{2}\ln|3-2y| = x + c_1$. Meanwhile, we have $y(1) = 2$, so $c_1 = -1$, we simplify to get that $y = \frac{3}{2} + \frac{1}{2}e^{-2(x-1)}$.

(b) We have $y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, y_{n+1}))$ and $f(x_n, y_n) = 3 - 2y_n$, so $y_{n+1} = y_n + \frac{h}{2}(3 - 2y_n + 3 - 2y_{n+1})$, then $(1+h)y_{n+1} = (1-h)y_n + 3h$, we simplify to get that $y_{n+1} = \frac{1-h}{1+h}y_n + \frac{3h}{1+h}$.

(c) We have $f(x, y) = 3 - 2y$. The truncation error of step $n+1$ is $|y(x_{n+1}) - y_{n+1}| \leq |y(x_n) - y_n| + \frac{h}{2}|f(x_n, y(x_n)) - f(x_n, y_n)| + \frac{h}{2}|f(x_{n+1}, y(x_{n+1})) - f(x_{n+1}, y_{n+1})| + |T_{n+1}| \leq |y(x_n) - y_n| + \frac{h}{2}\cdot 2|y(x_n) - y_n| + \frac{h}{2}\cdot 2|y(x_{n+1}) - y_{n+1}| + |T_{n+1}|$.

Assume that $|y(x_n) - y_n| = e_n$ and we have $y_{n+1} \leq \frac{1+h}{1-h}y_n + |T_{n+1}|$, so we have the following formula, $e_{n+1} \leq \frac{1+h}{1-h}e_n + \frac{|T_{n+1}|}{1-h} \leq \frac{1+h}{1-h}e_n + \frac{1}{1-h}\frac{h^3}{12}M$, where M is the upper bound of $y'''(\xi_n)$.

Therefore, we know that $e_n \leq \frac{1+h}{1-h}e_{n-1} + \frac{1}{1-h}\frac{h^3}{12}M \leq (\frac{1+h}{1-h})^n e_0 + \frac{1}{1-h}\frac{h^3}{12}M(1 + (\frac{1+h}{1-h})^1 + (\frac{1+h}{1-h})^2 + \cdots + (\frac{1+h}{1-h})^{n-1})$. We can easily know that $e_0 = 0$, so simplify the above formula and we know that $e_n \leq \frac{h^2}{24}M(e^{\frac{2h}{1-h}(n-1)} - 1) \to \frac{h^2}{24}M(e^{\frac{2}{1-h}} - 1)$.

When $h \to 0$, we have $e_n \to 0$, further, $y_n \to y(x)$.

(d) According to the principle, write the following program.

```
format long e;
for m=1:5
        h=10^(-m);
        y=2;
        for i=1:10^m
                y=(1-h)*y/(1+h)+3*h/(1+h);
        end
        real=3/2+1/2*exp(-2);
        disp(m);
        disp(y);
        disp(y-real);
end
```

Run the program, we get the following results.

m : 1, calculated value:1.567215316374655, error:$-4.523252436512415 \times 10^{-4}$.

m : 2, calculated value:1.567663130321896, error:$-4.511296410658616 \times 10^{-6}$.

m : 3, calculated value:1.567667596506614, error:$-4.511169260368320 \times 10^{-8}$.

m : 4, calculated value:1.567667641167568, error:$-4.507381134999378 \times 10^{-10}$.

m : 5, calculated value:1.567667641614879, error:$-3.427702566227708 \times 10^{-12}$.