Scientific Computing: HW6 Solution

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Problem 1

For $T_n(f)$, divide $[a,b]$ into n equal parts, we have the equation $T_n(f) = \frac{h}{2}(f(x_0)+2f(x_1)+...+2f(x_{n-1})+f(x_n)) = \frac{h}{2}\sum_{k=0}^{n-1}(f(x_k)+f(x_{k+1}))$, where $h = \frac{b-a}{n}$.
$f(x)$ is continuous on $[a,b]$, so there is $\xi_k \in (x_k, x_{k+1})$, satisfying that $f(\xi_k) = \frac{f(x_k)+f(x_{k+1})}{2}$, then $T_n(f) = \sum_{k=0}^{n-1} h \cdot f(\xi_k)$.
According to the definition, we could easily know that $\lim\limits_{n\to\infty} T_n(f) = \lim\limits_{n\to\infty}\sum_{k=0}^{n-1} h \cdot f(\xi_k) = \int_a^b f(x)dx$.
For $S_n(f)$, divide $[a,b]$ into $2n$ equal parts, we also have $S_n(f) = \frac{h}{6}\sum_{k=1}^{n}(f(x_{2k-2})+4f(x_{2k-1})+f(x_{2k}))$.
$f(x)$ is continuous on $[a,b]$, so there is $\eta_k \in (x_{2k-2}, x_{2k})$, satisfying that $f(\eta_k) = \frac{f(x_{2k-2})+4f(x_{2k-1})+f(x_{2k})}{6}$, then $S_n(f) = \sum_{k=1}^{n} h \cdot f(\eta_k)$.
$\lim\limits_{n\to\infty} S_n(f) = \lim\limits_{n\to\infty}\sum_{k=1}^{n} h \cdot f(\eta_k) = \int_a^b f(x)dx$.

Problem 2

Divide $[a,b]$ into $2n$ equal parts, assume that h is $\frac{b-a}{n}$.
According to $Problem 1$, we already know that $S_n(f) = \frac{h}{6}\sum_{k=1}^{n}(f(x_{2k-2})+4f(x_{2k-1})+f(x_{2k}))$.
The same, $T_{2n}(f) = \frac{h}{4}\sum_{k=1}^{n}(f(x_{2k-2})+2f(x_{2k-1})+f(x_{2k})))$ and $T_n(f) = \frac{h}{2}\sum_{k=1}^{n}(f(x_{2k-2})+f(x_{2k}))$.
Therefore, according to the expression of $S_n(f)$, $T_n(f)$ and $T_{2n}(f)$, we have $\frac{4}{3}T_{2n}(f) - \frac{1}{3}T_n(f) = \frac{h}{6}\sum_{k=1}^{n}(2f(x_{2k-2})+4f(x_{2k-1})+2f(x_{2k})-f(x_{2k-2})-f(x_{2k})) = \frac{h}{6}\sum_{k=1}^{n}(f(x_{2k-2})+4f(x_{2k-1})+f(x_{2k})) = S_n(f)$.

Problem 3

$T_n(x) = \cos(n \arccos x)$.

(a) $T_{m+n}(x)+T_{m-n}(x) = \cos((m+n)\arccos x)+\cos((m-n)\arccos x) = 2\cos(m\arccos x)\cos(n\arccos x) = 2T_m(x)T_n(x)$.

(b) $T_m(T_n(x)) = \cos(m\arccos(\cos(n\arccos x))) = \cos(mn\arccos x) = T_{mn}(x) = \cos(n\arccos(\cos(m\arccos x))) = T_n(T_m(x))$

(c) $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, $T_0(x) = 1$, $T_1(x) = x$.

Next, we use mathematical induction to prove the proposition.

Apparently, when $n = 1$, $n = 2$, the proposition holds.

Assume that for $n \le k$ the propositon holds.Then for $n = k+1$, $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$, the degree of $T_{k+1}(x)$ is $k+1$, whose coefficient is $2^{k-1}\cdot 2 = 2^k$. Therefore, the proposition holds for $n = k+1$, inductive hypothesis holds.

In summary, the original proposition is established.

Problem 4

Define $f(y) = MT_n(y) - P_n(y)$. For $M = max|P(x)|$ on $[-1,1]$, we have $-M \le P_n(y) \le M$.
According to the properties of Chebyshev polynomials, we know that $-M \le MT_n(y) \le M$ and there is $n+1$ values let $MT_n(y)$ take $-M$ or M.
Therefore, $MT_n(y)$ and $P_n(y)$ have n intersections, then $f(y)$ has n zero points on $[-1,1]$. Meanwhile, the degree of $f(y)$ is no more than n, so there are no more than n zero points.
Obviously, there is no zero point when $y > 1$ or $y < -1$. $T_n(1) = 1$, so $f(1) = M - P_n(y) \ge 0$, then when $y \ge 1$, $f(1) \ge 0$. Therefore, we have $P_n(y) \le MT_n(y)$.

The same, if we define $g(y) = MT_n(y) + P_n(y)$, we can also get the conclusion that $g(1) \geq 0$, then $P_n(y) \geq -MT_n(y)$.
In summary, we have the conclusion that $|P_n(y)| \leq M|T_n(y)|$ for every $y > 1$.

Problem 5

First, we define the function $comtrapezium(f, n, a, b)$.

```
function ans=comtrapezium(f,n,a,b)
        h=(b-a)/n;
        sum=f(a);
        for i=a+h:h:b-h
                sum=sum+2*f(i);
        end
        sum=sum+f(b);
        ans=sum*h/2;
end
```

Then, we should determine the value of n and get the answer.

```
format long e
syms x1;
f1=exp(x1)*sin(x1);
ans=1.0/(1*10^(-6))/12*double(max(abs(subs(diff(f1,x1,2),x1,[1:0.001:2]))));
n=ceil(sqrt(ans));
f=inline('exp(x)*sin(x)','x');
ans1=comtrapezium(f,n,1,2);
true=int(f1,x1,1,2);
disp(n);
disp(ans1);
disp(vpa(true));
```

According to the error formula, we get the value of n is 716.
By running the program, the calculated result we get is 4.487560317, the true value is 4.487560335.

Problem 6

First, we define the function $comsimpson(f, n, a, b)$.

```
function ans=comsimpson(f,n,a,b)
        format long;
        h=(b-a)/n;
        sum=f(a);
        for i=a+h:h:b-h
                sum=sum+2*f(i);
        end
        for i=a+h/2:h:b-h/2
                sum=sum+4*f(i);
        end
```

```
        sum=sum+f(b);
        ans=sum*h/6;
end
```

Next, we get the value of n according to the formula and get the answer by calculation.

```
format long e
f=inline('exp(x)*sin(x)','x');
n=1;
ans1=comsimpson(f,n,1,3);
ans2=comsimpson(f,2*n,1,3);
while abs(ans1-ans2)>10^(-8)
        n=n+1;
        ans1=comsimpson(f,n,1,3);
        ans2=comsimpson(f,2*n,1,3);
end
disp(ans1);
disp(ans2);
```

By running the program, the calculated result we get is 10.95017031.