n次插值

- n次插值问题
  - 求$\phi(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
    满足 $\phi(x_i) = y_i, i = 0,1,2,\cdots,n$
  - $n+1$元一次方程组

$$a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n = y_0$$
$$a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n = y_1$$
$$\cdots\cdots$$
$$a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n = y_n$$

n次插值惟一性

- 解的惟一性
  - 根据Cramer法则解存在而且惟一
    由于系数行列式是Vandermonde行列式，非零
  - 由代数基本定理：设$\psi(x)$也是插值函数，则差$h(x)=\phi(x)-\psi(x)$是次数不超过n的多项式，并有$n+1$个零点$x_0, x_1, \cdots, x_n$.由代数基本定理可知$h(x) \equiv 0, \phi(x) \equiv \psi(x)$

n次插值：Newton公式

- Newton公式
  - 易对 $n=2,3,\cdots$ 导出下列各项

$$\phi(x)=f(x_0)+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+\cdots+f[x_0,x_1,\cdots,x_n](x-x_0)\ (x-x_1)\cdots(x-x_{n-1})$$

  - n阶均差（差商）：

$$f[x_0,x_1,\cdots,x_n]=\frac{f[x_1,x_2\cdots,x_n]-f[x_0,x_1,\cdots,x_{n-1}]}{x_n-x_0}$$

$$=\frac{f[x_0,x_1,\cdots,x_{n-2},x_n]-f[x_0,x_1,\cdots,x_{n-2},x_{n-1}]}{x_n-x_{n-1}}$$

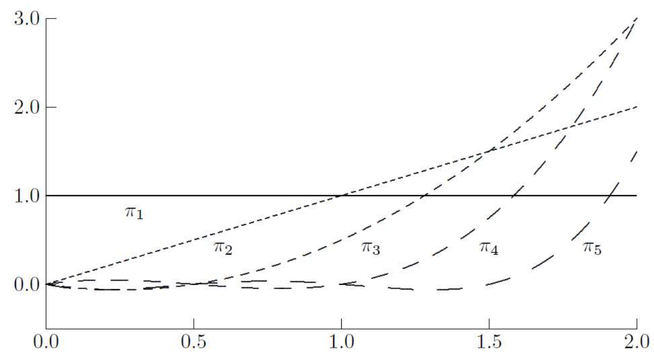
  - 均差对称性：其值是n次插值函数首项系数，与点的次序无关

均差表

- Newton公式计算须均差表.求值应使用秦九韶算法，均差表(n=3)

x_i	y_i	一阶均差	二阶均差	三阶均差
x_0	$\underline{y_0}$			
x_1	y_1	$\underline{f[x_0,x_1]}$		
x_2	y_2	$f[x_1,x_2]$	$\underline{f[x_0,x_1,x_2]}$	
x_3	y_3	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	$\underline{f[x_0,x_1,x_2,x_3]}$

- 基函数： $\omega_i(x)=\prod_{j=0}^{i}(x-x_j)$



n次插值:Lagrange公式

- Lagrange公式

$$\phi(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) + \cdots + y_n l_n(x)$$

  - n次插值基函数

$$l_i(x) = \frac{(x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

$$= \frac{\omega_n(x)}{(x - x_i)\omega_n'(x_i)}, i = 0,1,\cdots,n$$

其中$\omega_n(x) = (x - x_0)\cdots(x - x_{i-1})(x - x_i)\cdots(x - x_n)$

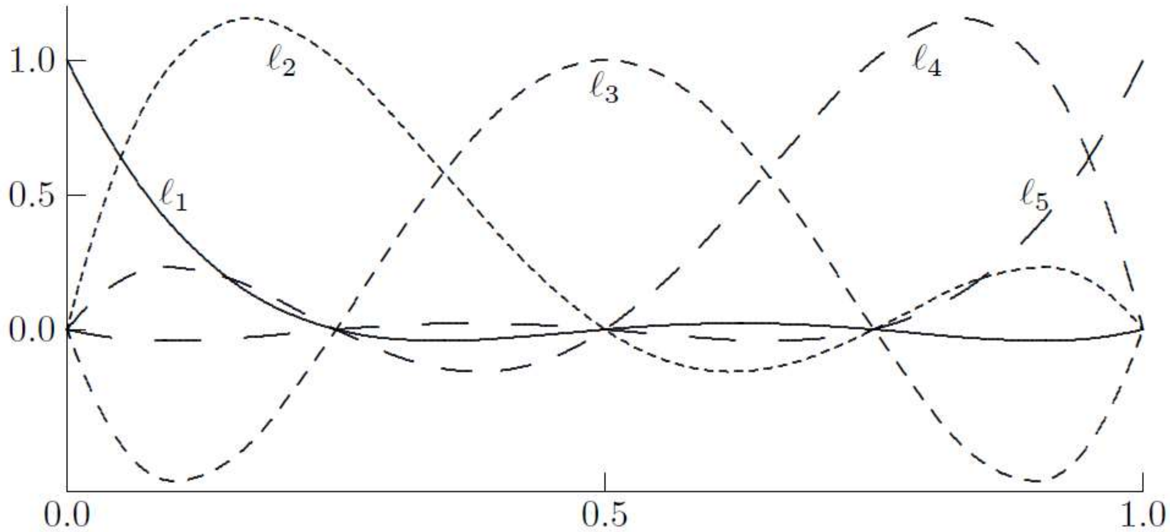
x_i	x_0	x_1	...	x_n
$l_0(x_i)$	1	0	...	0
$l_1(x_i)$	0	1	...	0
...	...	...	...	
$l_n(x_i)$	0	0	...	1

n次插值：Lagrange公式

- Lagrange公式

$$\phi(x) = \sum_{i=0,\cdots,n} \frac{\omega_n(x)}{(x-x_i)\omega_n'(x_i)} y_i = \frac{\sum_{i=0,\cdots,n} \frac{1}{(x-x_i)\omega_n'(x_i)} y_i}{\sum_{i=0,\cdots,n} \frac{1}{(x-x_i)\omega_n'(x_i)}},$$

其中$\omega_n(x) = (x-x_0)\cdots(x-x_{i-1})(x-x_i)\cdots(x-x_n)$



n次插值:Aitken公式

- Aitken公式

$$\phi_{01\cdots n}(x)=\frac{1}{x_{n-1}-x_n}\begin{vmatrix}\phi_{01\cdots(n-2)(n-1)}(x) & x-x_{n-1}\\ \phi_{01\cdots(n-2)n}(x) & x-x_n\end{vmatrix}$$

x_i	y_i	一次插值	二次插值	三次插值	$x-x_i$
x_0	y_0				$x-x_0$
x_1	y_1	$\varphi_{01}(x)$			$x-x_1$
x_2	y_2	$\varphi_{02}(x)$	$\varphi_{012}(x)$		$x-x_2$
x_3	y_3	$\varphi_{03}(x)$	$\varphi_{013}(x)$	$\varphi_{0123}(x)$	$x-x_3$

N次插值:余项

- 余项:微商形式、差商形式
  - 设$\phi(x)$是$f(x)$过$x_0, x_1, \cdots, x_n$的n次插值函数，
    $f(x) \in C^{n+1}[a,b], x_0, x_1, \cdots, x_n \in [a,b]$ 则有$\xi \in (a,b)$，
    使

$$R(x) = f(x) - \phi(x)$$

$$= \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - x_0)(x - x_1) \cdots (x - x_n)$$

$$= f[x_0, x_1, \cdots, x_n, x](x - x_0)(x - x_1) \cdots (x - x_n)$$

- 差商与微商关系
  - $f[x_0, x_1, \cdots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$

等距节点Newton公式

- (向前)差分
  - 设$f_k = f(x_0 + kh), k = 0,1,2,\cdots$则将
    $$\Delta f_k = f_{k+1} - f_k,$$
    称为一阶差分, 一般地
    $$\Delta^m f_k = \Delta^{m-1} f_{k+1} - \Delta^{m-1} f_k, m = 2,3,\cdots$$
    称为m阶差分.
- $\Delta^n f_k = \sum_{j=0}^{n} (-1)^j C_n^j f_{n+k-j}$
- Newton公式
  $$\phi_n(x) = f_0 + t\Delta f_0 + \cdots + \frac{t(t-1)\cdots\big(t-(n-1)\big)}{n!}\Delta^n f_0$$
  $$x = x_0 + th, x_k = x_0 + kh, k = 0,1,2,\cdots, n-1$$

差分表

- Newton公式计算可用差分表

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
x_0	f_0			
		Δf_0		
x_1	f_1		$\Delta^2 f_0$	
		Δf_1		$\Delta^3 f_0$
x_2	f_2		$\Delta^2 f_1$	
		Δf_2		
x_3	f_3			

差分

- 向后差分
  $\nabla f_k = f_k - f_{k-1}, \nabla^m f_k = \nabla^{m-1} f_k - \nabla^{m-1} f_{k-1}, m = 2,3,\cdots$
- 中心差分
  $$\delta f_k = f_{k+1/2} - f_{k-1/2}, \delta^m f_k = \delta^{m-1} f_{k+1/2} - \delta^{m-1} f_{k-1/2}, m = 2,3,\cdots$$
- 差分与均差
  $$f[x_0, x_1, \cdots, x_n] = \frac{1}{\mathrm{n}!\, h^n} \Delta^n f_0 = \frac{1}{n!\, h^n} \nabla^n f_n$$
- 插值节点还可依次取或者另外某些次序安排，从而变化出多种用差分表出的插值公式

三次Hermite插值

- 三次Hermite插值
  - 已知$y = f(x)$及其导函数的表：

x_i	x_0	x_1
y_i	y_0	y_1
$y_i{}'$	$y_0{}'$	$y_1{}'$

  - 求$H(x) = a_0 + a_1x + a_2x^2 + a_3x^3$
    满足　　$H(x_i) = y_i, H'(x_i) = y_i', i = 0,1$
- 三次Hermite插值函数
  - $H(x) = y_0h_0(x) + y_1h_1(x) + y_0'H_0(x) + y_1'H_1(x)$
  - $h_0(x), h_1(x), H_0(x), H_1(x)$三次Hermite插值基函数

三次Hermite插值

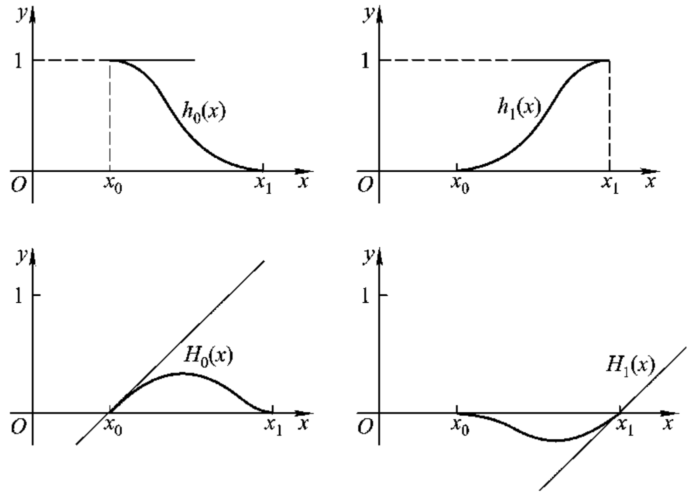
- 三次Hermite插值基函数

$$h_0(x) = (1 + 2\frac{x - x_0}{x_1 - x_0})\left(\frac{x - x_1}{x_0 - x_1}\right)^2$$

$$h_1(x) = (1 + 2\frac{x - x_1}{x_0 - x_1})\left(\frac{x - x_0}{x_1 - x_0}\right)^2$$

$$H_0(x) = (x - x_0)\left(\frac{x - x_1}{x_0 - x_1}\right)^2$$

$$H_1(x) = (x - x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)^2$$



函数	函数值		导数值	
	x_0	x_1	x_0	x_1
$h_0(x)$	1	0	0	0
$h_1(x)$	0	1	0	0
$H_0(x)$	0	0	1	0
$H_1(x)$	0	0	0	1

三次Hermite插值

- 三次Hermite插值函数余项
  - 设$H(x)$是$f(x)$过x_0, x_1 的三次Hermite插值函数，$f(x) \in C^4[a,b], x_0, x_1 \in [a,b]$，则对任一$x \in [a,b]$有$\xi \in (a,b)$，使

$$R(x) = f(x) - H(x)$$
$$= \frac{1}{4!} f^{(4)}(\xi)(x - x_0)^2 (x - x_1)^2$$

特别，若$x_0 \le x \le x_1$，则有

$$|R(x)| \le \frac{1}{384}(x_1 - x_0)^4 \max |f^{(4)}(x)|$$

$n+1$点Hermite插值

- $n+1$点$2n+1$次Hermite插值
  - 已知$y=f(x)$及其导函数的表：

x_{i}	x_0	x_1	$\cdots$	x_n
y_{i}	y_0	y_1	$\cdots$	y_n
$y_i{}'$	$y_0{}'$	$y_1{}'$	$\cdots$	$y_n{}'$

  - 求$H(x)=a_0+a_1x+a_2x^2+\cdots+a_{2n+1}x^{2n+1}$
  - 满足　$H(x_i)=y_i, H'(x_i)=y_i', i=0,1,\cdots,n$

$n+1$点Hermite插值

- 插值函数

$$H(x)=\sum_{i=0}^{n}(y_i h_i(x)+y_i^{(1)}H_i(x))$$

$$h_i(x)=\left(1+2\sum_{\mathrm{j}\neq i}\frac{x-x_i}{x_j-x_i}\right)l_i^2(x)$$

$$H_i(x)=(x-x_i)l_i^2(x)$$

$$l_i(x)=\prod_{j\neq i}\frac{x-x_j}{x_i-x_j}$$

$n+1$点Hermite插值

- 余项
  - 设$H(x)$是$f(x)$过$x_0, x_1, \cdots, x_n$的$2n+1$次Hermite插值函数，$x_0, x_1, \cdots, x_n \in [a,b], f(x) \in C^{2n+2}[a,b]$,
    则对任一$x \in [a,b]$,有$\xi \in (a,b)$,使
    $$R(x) = f(x) - H(x) = \frac{1}{(2n+2)!} f^{(2n+2)}(\xi)(x-x_0)^2(x-x_1)^2 \cdots (x-x_n)^2$$