Usage of RC Circuits in Denoising Filters

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***Abstract*—This case study mainly focuses on dynamical systems through the rigorous analysis of RC circuits. By using computational tools and mathematical analysis, our task is to explore the behavior of RC circuits under various configurations and input conditions. The main goal is to model the response of RC circuits under conditions involving both DC and AC inputs. The study also involves manipulating the time constant $\tau = RC$ to observe its effect on the output current of the circuit and the voltage across the capacitor, thus providing insights into the theoretical and practical effects of the circuit design parameters. After that, we further extended to practical applications of these circuits in audio signal denoising, specifically aimed at reducing noise in recorded music. These applications highlight the relevance of theoretical electrical engineering concepts in solving practical problems.**

I. RC Circuits

Suppose there is an RC circuit where $R = 1000\ \Omega$, $C = 10^{-6}$ F, and a constant 5V DC power supply. From a microscopic point of view, in an electric circuit, electrons flow through the wires and do work. By analyzing and modeling the trend of how V_{out} will behave in a specific time range, we can describe it in terms of DE with current.

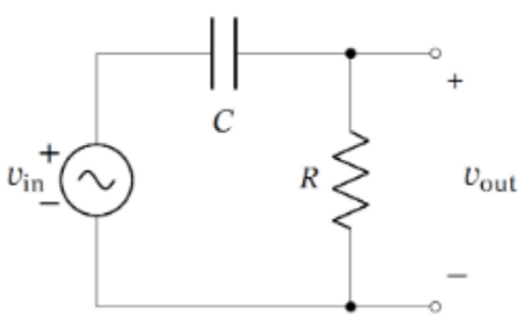

Fig. 1. Basic RC Circuit. [1]

As shown above, this is a conventional RC circuit schematic. Kirchoff's rule states that: The sum of potential differences in each electrical appliance are equal to zero. To comply with Kirschoff's rule, we list the following equations:

$$V_{\text{in}}' = \frac{Q_c'}{C} + RI' \tag{1}$$

Where V_{in} is the input voltage source; the current through the capacitor is equal to the derivative of the amount of charge through the capacitor. We know the V_{in} is a fixed value, V_{in}' will be zero. Also, we know when the time is zero, current will flow through the circuit. Based on the information given above, we can list an IVP as shown below:

$$\begin{cases} 0 = \frac{Q_c'}{C} + RI' \\ I(0) = \frac{V_{\text{in}}}{C} = 0.005 \end{cases} \tag{2}$$

By solving the IVP, we will get:

$$I(t) = 0.005 \cdot e^{-\frac{t}{C \cdot R}} \tag{3}$$

and thus by Ohm's law,

$$V_{\text{in}} = IR = R(0.005 \cdot e^{-\frac{t}{C \cdot R}}) \tag{4}$$

Looking at equations 3 and 4, we can observe that current, voltage, and time are in an exponential decay relationship, that is, as time passes (time equals infinity) the current passing through the circuit decreases dramatically and eventually approaches zero, and the same is true for the voltage. From physics's perspective also suggests that: ideally, no current will pass through the resistor when the capacitor is fully charged, so the output voltage of the resistor will also become zero. We use Matlab to draw these two sets of equations versus time as shown below:

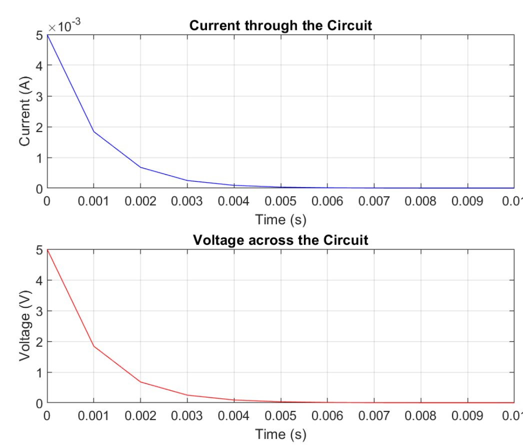

Fig. 2. Voltage and Current across the RC circuit

As we see in Fig 2, it is consistent with our hypothesis: the voltage, current, and time are in an exponential decay, with the voltage and current decreasing over time and tending to zero. In an RC circuit, $\tau = RC$, where τ is a time constant. When we modify the value of τ, the basic trend for volume and current vs. time in an RC circuit will not change. However, the larger τ, the longer time it takes to fully charge the capacitor. This is because when R is higher, the current flow through the circuit is lower, resulting in slower charging speed; when C is higher, the capacitor can hold more charge, so naturally it takes a longer time to charge up the capacitor.

Now, consider we have another RC circuit shown below:

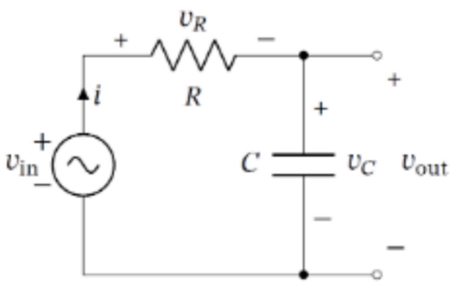

Fig. 3. Switched Positions of The Capacitor and Resistor in An RC Circuit [1]

Compared to Fig 1, the capacitor and resistor have swapped places, and V_{out} shows the capacitor's output voltage in the circuit. $V_{in} = 5\sin(2\pi\omega t)$, $R = 1000\ \Omega$, $C = 10^{-6}$ F, and ω is alternating from 50 to 1000 Hz. Same as we described above, from equation 1, we derive an alternating form of dI/dt:

$$\frac{dI}{dt} = \frac{V_{\text{in}}'}{R} - \frac{I}{RC} \tag{5}$$

If we plug in all the variables we know, we get:

$$\frac{dI}{dt} = \left(\frac{10\pi\omega}{R}\right)\cos(2\pi\omega t) - \frac{1}{\tau}I \tag{6}$$

In Matlab, we use the ODE45 function to calculate the amount of time corresponding to the current through the capacitor. We integrate the calculated current and divide it by the capacitance value and we get V_{out}.

Finally, we performed a Fourier transform on the output voltage, in which we transitioned from a time domain analysis to a frequency domain analysis. This step resulted in a plot of frequency versus amplitude as shown below:

Note that while both signals do seem to pass through the circuit, as shown in the respective Fourier transforms above, the high frequency signal seems to be much more impeded compared to the low frequency signal. The fft output of the high frequency signal is only 200 compared to the 300 for the low frequency one. This suggests that the RC circuit as shown in Fig. 3 can act as a filter to decrease the amplitude of high frequency signals.

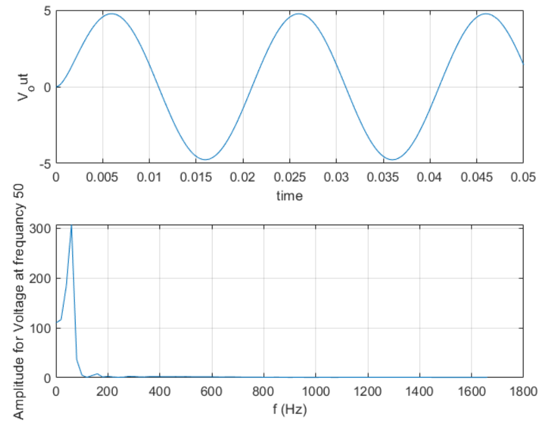

Fig. 4. Amplitude vs frequency and time vs Vout for 50Hz

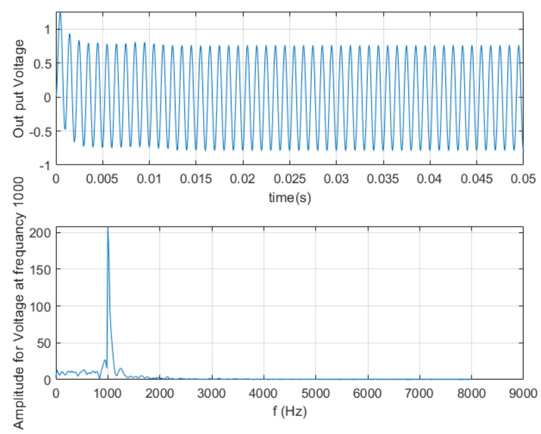

Fig. 5. Amplitude vs frequency and time vs Vout for 1000Hz

Now consider there are two RC circuits, shown in Figure 6 and 7:

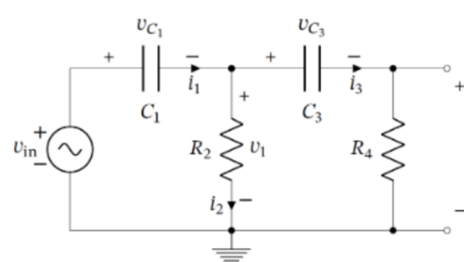

Fig. 6. Combined RC Circuit 1 [1]

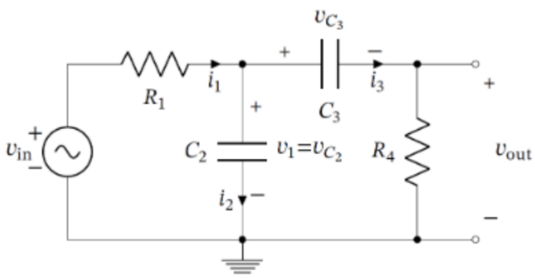

Fig. 7. Combined RC Circuit 2 [1]

Fig 7 is an alternating version of the RC circuit from Fig 6. Assume the resistance, and capacitance are unchanged, for $V_{\text{in}} = a_1 \sin(2\pi \cdot 50t) + a_2 \sin(2\pi \cdot 10^5 t)$. Our goal is to get the graph for output voltage (V_{out}) vs. time and amplitude for V_{out} vs. frequency. For the RC circuit in Figure 6, we see that there are two stages, C_1, R_2 and V_{in} are the first stage, and R_2, C_3, R_4 are the second stage. Notice that, V_1 (voltage across R_2) will be the output voltage from the first stage and is also the input for the second stage. We can simulate the behavior of the full circuit by first looking at the output current of the first stage. Notice that since all components in the first stage are in series, the current across both components will be equal. Thus, we can calculate the output current over R_2 through an expression for the current over C_1 derived from the definition of capacitance:

$$I_1 = CV' = C_1(V'_{in} - V'_1) \tag{7}$$

Since we are measuring voltage over a resistor, we can get the output voltage of the first stage using Ohm's Law:

$$V'_1 = \frac{R_2 C_1 V'_{in} - V_1}{R_2 C_1} \tag{8}$$

Notice that since the second stage is identical to the first, we can get the output voltage for the second stage (and thus, the circuit as a whole) via the same process:

$$V'_{out} = \frac{R_4 C_3 V'_1 - V_{out}}{R_4 C_3} \tag{9}$$

Solving for the differential equation for V_out we then get the following graphs:

In the Fourier transform of the output voltage graph we can clearly observe a much higher peak at 10^5 Hz than at 50 Hz (the two component frequencies of the input), suggesting that this circuit acts as a filter to remove excess lower frequencies and allows high frequencies to pass through. Though the low frequency component is impeded here, it is still clearly present in the Fourier transform. However, this could just be a result of the circuit's cutoff frequency being far closer to 50 Hz than the much higher frequency at 10^5 Hz.

Following the same logic, we split the two stages for the circuit in Fig 7. As we previously described, both stages contain the same circuit components. However, the output for the first stage is now the voltage across the capacitor instead of the resistor. We can again derive V_{out} by finding the current over the capacitor using equation (7). We can then derive the output voltage over the capacitor using Kirchoff's Voltage Law:

$$V_1 = V_{in} - V_{R1} \tag{10}$$

Note that we can get an expression for V_{R1} in the same way that we did for V_{R2} in the previous circuit by using Ohm's Law. Rearranging the expression in terms of V'_1, we have the following:

$$V'_1 = \frac{V_{in} - V_1}{R_1 C_2} \tag{11}$$

The second stage of this circuit is the same as the second stage of the previous circuit, so we can derive an expression for V_{out} using equation (9).

Again, solving for the differential equation for V_{out} we then get the following graphs:

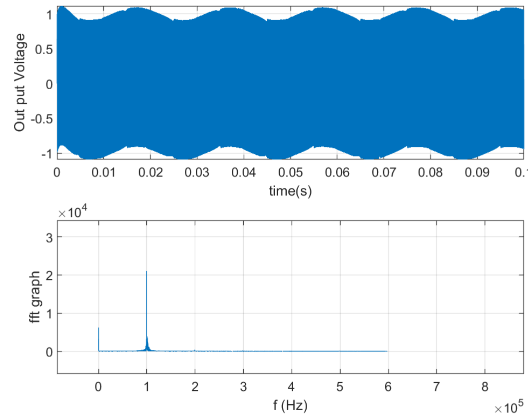

Fig. 8. frequency vs amplitude plot for the circuit in Fig 6

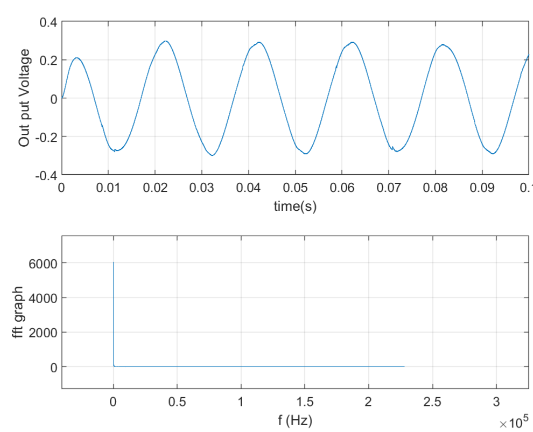

Fig. 9. frequency vs amplitude plot for the circuit in Fig 7

We noticed the circuit filtered the higher frequency, which is reasonable because the first stage of the circuit is identical to the circuit described in Fig 3 which, as we have seen, is effective in filtering out frequencies above a specific cutoff. As such, even though we expect the second stage of the filter (which should behave similarly to the circuit described in Fig. 6) to filter out low-frequency signals, because the first stage had already filtered out the high-frequency signal it makes sense that there are no component signals there. Furthermore, as established in the discussion for the Fig. 6 circuit, it is expected to have some of the lower-frequency signals pass through this stage as well if the cutoff frequency is far closer to the lower-frequency signals than the high-frequency ones.

Referencing the prior Fourier transforms of the various signals in Fig. 4, 5, 8, and 9, it seems that the circuit described by Fig. 3 is most effective at isolating lower frequencies (there is a sharper drop in amplitude at higher frequencies for the Fourier transform of the output of that circuit). This also makes sense as the circuit described in that figure resembles a basic low-pass filter. To attenuate signals at 10 KHz and allow lower frequencies to pass we would need to set the circuit components' parameters so as to achieve a cutoff frequency of 10 KHz for the filter. Recall that the cutoff frequency for passive filters such as these is given by the following:

$$f_c = \frac{1}{2\pi RC} \tag{12}$$

Solving the equation with the previously defined values of R = 1000 Ω and C = $1.6 * 10^{-6}$ F gives a cutoff frequency of approximately 100 Hz, so decreasing either resistance or capacitance by a factor of 10 should give an appropriate filter for attenuating the given input signal at 10 KHz.

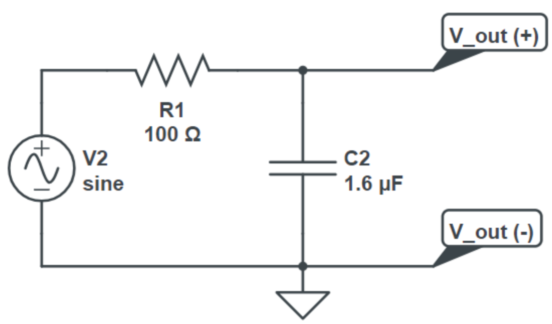

Fig. 10. Circuit diagram of the proposed filter showing changes to the resistance and capacitance

II. Applying RC Circuits to Denoising

In the second part of this case study, we explore the application of RC circuits to the denoising of a 'noisy' audio file. These audio interference typically manifest themselves as a low-frequency hum corresponding to the frequency of the electrical grid as well as a white noise hiss across many frequencies.

The basic setup for experimentation is for the noisy audio file to be fed as the input signal into either of the cascaded RC circuits as described in Fig. 6 and 7. The output voltage is the measured voltage across R_4, which in this case has a fixed resistance of 16 Ω.

Note that the circuits described in Fig. 6 and 7 resemble a second-order high-pass and a first-order band-pass filter respectively. In the case of the first circuit, this means that frequencies below a certain point (the cutoff frequency, f_c) will be filtered by the circuit or have lower contribution to the output, as will be evident in Fourier analysis later (frequencies higher than the cutoff will pass through the circuit, hence the name 'high-pass'). The second circuit resembling a first-order band-pass filter is, here, comprised of cascading a first-order low-pass filter and a first-order high-pass filter with the output being read from the output of the high-pass filter. This circuit has two cutoff frequencies, one high and one low, resulting from the cascaded low/high-pass filter setup. Similar to the prior example, frequencies below the low cutoff frequency will be filtered and frequencies above the high cutoff frequency will be filtered (from the low-pass portion of the circuit).

Given the fixed nature of the resistance of the output resistor, it was decided that it would be easiest to manipulate the capacitors during testing. The values of capacitance to be tested were given by the following expression derived from the cutoff frequency formula for RC circuit-based filters expressed in equation 11.

Where f_c is the associated target cutoff frequency for the circuit. In the case of the second-order high-pass filter, the impedance of the first stage of the circuit was lowered by a factor of 10 (ie. from Fig. 6, $R_2 = 0.1R_4, C_1 = 10C_3$). This was done to avoid having the two cascaded circuits be identical so that the second stage of the circuit can output something that is meaningfully different from the output of the first stage alone.

For the second-order high-pass filter, we get a reference capacitance from equation 11 by plugging in $R = 16\Omega$ (as specified in the constraints) and $f_c = 100$ Hz. A higher frequency was chosen for the cutoff as though the frequency of the hum matches the frequency of the 60 Hz electrical grid, filters such as these in practice do not cleanly cut sound at exactly the cutoff frequency and have some 'response time' built in. As such, the 100 Hz value was chosen to leave enough room for the filter to effectively filter out the 60 Hz low frequency hum. The equation gives a reference capacitance of $C_1 \approx 9.947 * 10^{-5}$ F.

The capacitance for the high-pass stage of the first-order band-pass filter was determined in the same way as above. For setting a reference capacitance for the capacitor on the low-pass stage, we first need to find an appropriate centre frequency for setting an appropriate upper cutoff frequency f_{cH} for the band-pass filter. For this we first do a Fourier analysis on the original noisy input to find the component frequencies of the input:

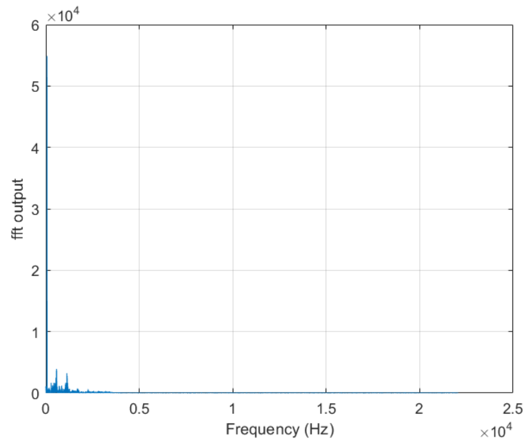

Fig. 11. Fourier transform of the noisy sound input

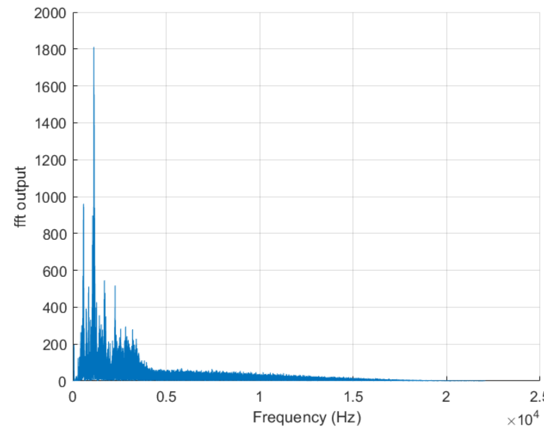

Fig. 12. Fourier transform of the second-order high-pass filter output

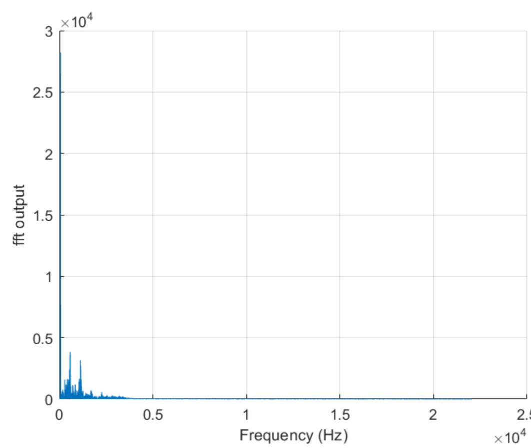

Fig. 13. Fourier transform of the first-order band-pass filter output

Ignoring the large peaks from the 60 Hz hum, we find that the dominant component frequency lies around 567 Hz, which is what we use for the centre frequency. This is confirmed by sorting the data points in the Fourier transform in descending order of amplitude and mapping the points to their original corresponding frequencies. We can now find an appropriate upper cutoff frequency by finding a frequency that, when geometrically averaged with the lower cutoff of 100 Hz, gives the target midpoint of 567 Hz. Note that geometric average is used here instead of mean average as, for the purposes of pitch, we are concerned with the relative difference of frequencies rather than absolute difference. Thus, from the above, we arrive at the following expression for choosing an appropriate upper cutoff:

$$f_{mid} = \sqrt{f_{cL} f_{cH}} \tag{13}$$

Where f_{mid}, f_{cL}, f_{cH} are the midpoint/centre frequency, lower cutoff frequency and upper cutoff frequency respectively. From this, we arrive at an upper cutoff frequency of $f_{cH} \approx 5358.15$ Hz and, from equation 11, a capacitance of $1.856 * 10^{-5}$ F for the capacitor in the low-pass stage of the circuit.

Using the same system of DEs to model the circuit behaviour in the previous circuits described in Fig. 6 and 7, we can then model the performance of the two filters:

From the above Fourier transforms, we see very clearly that while the high-pass filter does a very good job at filtering out the 60 Hz low frequency hum, the band-pass filter seems to struggle a bit more (though still greatly reducing the relative peak compared to the original noisy signal). This is likely because the second-order nature of the high-pass filter makes it far more effective in reducing signal strength pass its cutoff frequency (ie. the drop in signal strength is steeper) compared to the first-order high-pass stage in the band-pass filter.

This is confirmed when sampling and playing the output signals, with the 60 Hz hum barely noticeable on the high-pass filter but still very much present on the band-pass filter. However, unlike the band-pass filter, the high-pass filter was unable to filter out much of the variable frequency hissing (being particularly noticeable at frequencies - which makes sense given the nature of a high-pass filter). The band-pass filter was able to more effectively filter out a wider scope of unwanted frequencies. Because of this, while the band-pass filter cannot completely eliminate the humming and hissing in the original noisy signal, it does a fair bit to reduce the strength both the hissing and humming rather than just the humming, as is the case with the high-pass filter. Furthermore, the band-pass filter is also able to better preserve clarity of the music recording because it having both a low and high cutoff frequency creates a far more focused passband.

Finally, we also experimented with running the 2nd order high-pass circuit's output through the first-order band-pass filter. Though the plots for this output show some unexpected behaviour in comparison to previous plots gathered during this case study, with the signal plot resembling a sine wave centred on a decreasing linear line instead of the time axis:

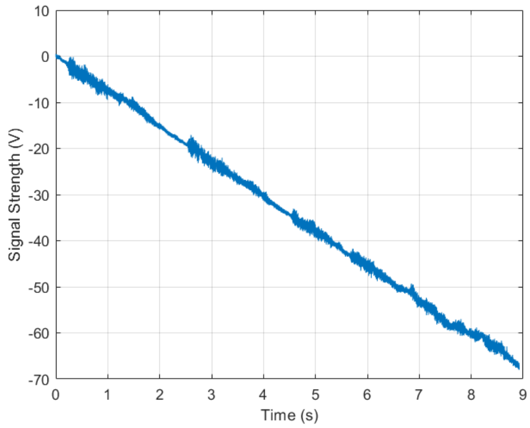

Fig. 14. Output signal generated by passing the output of the second-order high-pass filter through the first-order band-pass filter

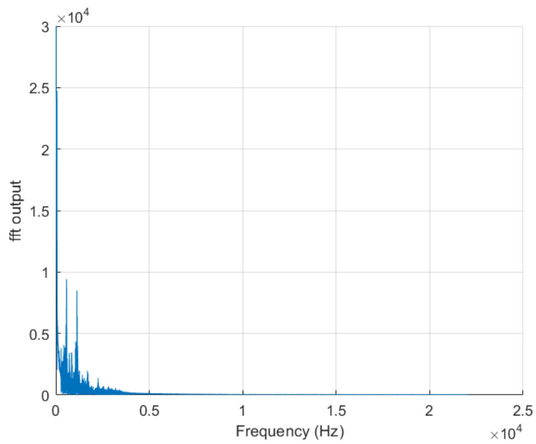

Fig. 15. Fourier transform of the above plot

The lack of a 60 Hz peak on the Fourier transform of this output seems to suggest that the ‘combined’ circuit was able to filter out most of the 60 Hz humming like with the second-order high-pass filter. The sharp drop in signal strengths past 2000 Hz or so also seems to suggest that this circuit preserves the tighter passband of the band-pass filter and is potentially an indication that, like in the oririnal band-pass filter, high frequency hissing has been greatly reduced.

When sampling and playing back the output, however, we notice that though both low frequency humming and high frequency have been reduced as expected, the output signal in general is far weaker and the audio generated is excessively quiet as a result. Because of this, the relative strength of the music and noisiness is far closer and so the humming and hissing are actually more noticeable compared to the previous outputs. Furthermore, this signal also introduced a significant ‘crackling’ noise in the background that was not present on the prior outputs, making this output signal far less usable than was initially expected.

References

[1] B. Y. Wormleighton - Case Study #2. (2024, Spring). ESE 217. Saint Louis, MO, USA: Washington University in Saint Louis.