第六次作业参考

16. Apply the algorithm for the GCD in Section 10.1 to 15 and 46, and then use the results to determine the multiplicative inverse of 15 in Z_{46}.

Step	Equation	q	r
1	46 = 15 × 3 + 1	3	1
2	15 = 1 × 15 + 0	15	0
inverse	1 = (-3) × 15 + 1 × 46		

Thus the multiplicative inverse of 15 in Z_{46} is 43.

21. Determine the complementary design of the BIBD with parameters $b = v = 7, k = r = 3, \lambda = 1$ in Section 10.2.

Considering the original design, $Z_7 = \{0,1,2,3,4,5,6\}$

Let $B = \{0,1,3\}$ be the start block.

$B+0 = \{0,1,3\}$ $B+1 = \{1,2,4\}$ $B+2 = \{2,3,5\}$

$B+3 = \{3,4,6\}$ $B+4 = \{4,5,0\}$ $B+5 = \{5,6,1\}$

$B+6 = \{6,0,2\}$. So we get a design of BIBD.

	0	1	2	3	4	5	6
B_0	1	1	0	1	0	0	0
B_1	0	1	1	0	1	0	0
B_2	0	0	1	1	0	1	0
B_3	0	0	0	1	1	0	1
B_4	1	0	0	0	1	1	0
B_5	0	1	0	0	0	1	1
B_6	1	0	1	0	0	0	1

So the complementary design of this BIBD is:

$\{2,4,5,6\}, \{0,3,5,6\}, \{4,6,0,1\}$

$\{5,0,1,2\}, \{6,1,2,3\}, \{0,2,3,4\}$

$\{1,3,4,5\}$

28. Show that $B = \{0, 1, 3, 9\}$ is a difference set in Z_{13}, and use this difference set as a starter block to construct an SBIBD. Identify the parameters of the block design.

$B = \{0, 1, 3, 9\}$

$-$	0	1	3	9
0	0	12	10	4
1	1	0	11	5
3	3	2	0	7
9	9	8	6	0

Because each of the non-zero integers in Z_{13} occurs exactly once in the off-diagonal positions and hence exactly once as a difference. Hence, B is a different set in Z_{13}.

Use B as a starter block to construct an SBIBD.

$B+0=\{0,1,3,9\}$ $B+1=\{1,2,4,10\}$ $B+2=\{2,3,5,11\}$ $B+3=\{3,4,6,12\}$

$B+4=\{4,5,7,0\}$ $B+5=\{5,6,8,1\}$ $B+6=\{6,7,9,2\}$ $B+7=\{7,8,10,3\}$

$B+8=\{8,9,11,4\}$ $B+9=\{9,10,12,5\}$ $B+10=\{10,11,0,6\}$ $B+11=\{11,12,1,7\}$ $B+12=\{12,0,2,8\}$

The parameters of the block design:

$b=v=13$, $k=r=4$, $\lambda=\frac{k(k-1)}{v-1}=1$

32. Use Theorem 10.3.2 to construct a Steiner triple system of index 1 having 21 varieties.

Let $X=\{a_1, a_2, a_3\}$ and $Y=\{b_1, b_2, b_3, b_4, b_5, b_6\}$ be two sets of varieties. Let $B_1=\{a_1, a_2, a_3\}$ and $B_2=\{\{b_0, b_1, b_3\}, \{b_1, b_2, b_4\}, \{b_2, b_3, b_5\}, \{b_3, b_4, b_6\}, \{b_4, b_5, b_0\}, \{b_5, b_6, b_1\}, \{b_6, b_0, b_2\}\}$ be the steiner triple systems of X and Y respectively. So we can get a 3-by-7 array as follows.

	a_1	a_2	a_3
b_0	0	1	2

b_1	3	4	5
b_2	6	7	8
b_3	9	10	11
b_4	12	13	14
b_5	15	16	17
b_6	18	19	20

(1) The entries in each of the same rows:

$\mathrm{B}_1 = \{\{0,1,2\}, \{3,4,5\}, \{6,7,8\}, \{9,10,11\}, \{12,13,14\}, \{15,16,17\}, \{18,19,20\}\}$

(2) The entries in each of three columns:

$B_2 = \{\{0,3,9\}, \{1,4,10\}, \{2,5,11\}, \{3,6,12\}, \{4,7,13\}, \{5,8,14\}, \{6,9,15\}, \{7,10,16\}, \{8,11,17\}, \{9,12,18\}, \{10,12,19\}, \{11,14,20\}, \{0,12,15\}, \{1,13,16\}, \{2,14,17\}, \{3,15,18\}, \{4,16,19\}, \{5,17,20\}, \{0,6,18\}, \{1,7,19\}, \{2,8,20\}\}$

(3) Three entries, each two of which are not from the same row or column:

B_3 ={{0,4,11}, {0,5,10}, {1,3,11}, {1,5,3}, {2,3,10}, {2,4,9}, {3,7,14}, {3,8,13}, {4,6,14}, {4,8,12}, {5,6,13}, {5,7 12}, {6,10,17}, {6,11,16}, {7,9,17}, {7,11,15}, {8,10,15}, {8,9,16}, {9,13,20}, {9,14,19}, {10,14,18}, {10,12,20}, {11,12,19}, {11,13,18}, {12,16,2}, {12,17,1}, {13,17,0}, {13,15,2}, {14,16,0}, {14,15,1}, {15,19,5}, {15,20,4}, {16,20,3}, {16,18,5}, {17,19,3}, {17,18,4}, {0,7,20}, {0,8,19}, {1,6,20}, {1,8,18}, {2,6,19}, {2,7,18}

Hence, $B_1 \cup B_2 \cup B_3 = B$ is a steiner triple system of index 1 having 21 varieties.

52. Construct a completion of the 3-by-6 Latin rectangle

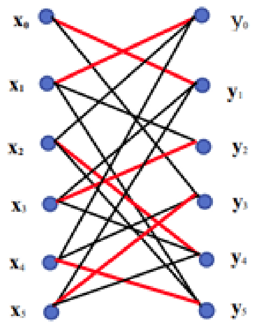
$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 & 0 \\ 5 & 4 & 3 & 0 & 1 & 2 \end{bmatrix}.$$

Solution:

Answer: Define the bigraph G =(X, Δ, Y), X = {x_0, x_1, x_2, x_3, x_4, x_5} corresponds to columns 0, 1, 2, 3, 4, 5 of the rectangle L, Y = {0, 1, 2, 3, 4, 5} is the elements on which L is based. Δ = {(x_i, j): j does not occur in column i of L}. G has a perfect matching. Suppose the edges of a perfect matching are {(x_0, i_0), (x_1, i_1), (x_2, i_2), (x_3, i_3), (x_4, i_4), (x_5, i_5)}. Then 4-by-6 array obtained by adjoining i_0, i_1, …,i_5 as a new row is a Latin rectangle.

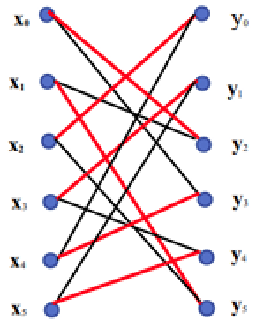
$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 & 0 \\ 5 & 4 & 3 & 0 & 1 & 2 \\ 1 & 0 & 4 & 2 & 5 & 3 \end{bmatrix}$$

Δ = {(x_i, j): j does not occur in column i of L}. G has a perfect matching. Suppose the edges of a perfect matching are {(x_0, i_0), (x_1, i_1) (x_2, i_2), (x_3, i_3), (x_4, i_4), (x_5, i_5)}. Then 5-by-6 array obtained by adjoining i_0, i_1, …,i_5 as a new row is a Latin rectangle.

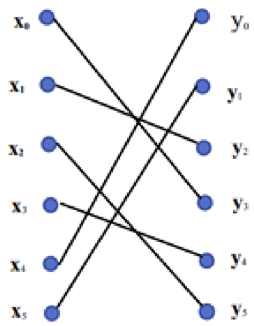


$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 & 0 \\ 5 & 4 & 3 & 0 & 1 & 2 \\ 1 & 0 & 4 & 2 & 5 & 3 \\ 2 & 5 & 0 & 1 & 3 & 4 \end{bmatrix}$$

Δ = {(x_i, j): j does not occur in column i of L}. G has a perfect matching. Suppose the edges of a perfect matching are {(x_0, i_0), (x_1, i_1), (x_2, i_2), (x_3, i_3), (x_4, i_4), (x_5, i_5)}. Then 6-by-6 array obtained by adjoining i_0, i_1, …,i_5 as a new row is a Latin rectangle.



$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 & 0 \\ 5 & 4 & 3 & 0 & 1 & 2 \\ 1 & 0 & 4 & 2 & 5 & 3 \\ 2 & 5 & 0 & 1 & 3 & 4 \\ 3 & 2 & 5 & 4 & 0 & 1 \end{bmatrix}$$



56. Construct a completion of the semi-Latin square

$$\begin{bmatrix} 0 & 2 & 1 & & & & 3 \\ 2 & 0 & & 1 & & 3 & \\ 3 & & 0 & 2 & 1 & & \\ & 3 & 2 & 0 & & 1 & \\ & & 3 & & 0 & 2 & 1 \\ 1 & & & & 3 & 0 & 2 \\ & 1 & & 3 & 2 & & 0 \end{bmatrix}.$$

Answer

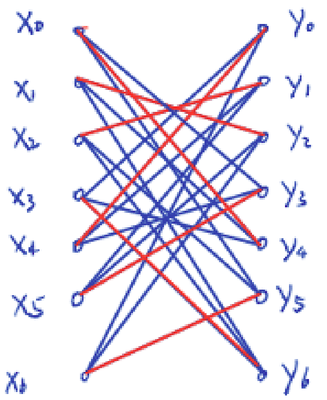
Use the construction method

Let L be the semi-Latin square of order 7

Define a bigraph $G=(X,\Delta,Y)$, $X=\{x_0,x_1,\cdots,x_6\}$, corresponds to the rows $0,1,\cdots,6$ of the rectangle L, $Y=\{y_0,y_1,\cdots,y_6\}$ correspond to the columns $0,1,\cdots,6$ of L.

$\Delta=\{(x_i,y_j)$: the position at row i column j is unoccupied.

$n=7$, $n-m=3 \Rightarrow m=4$

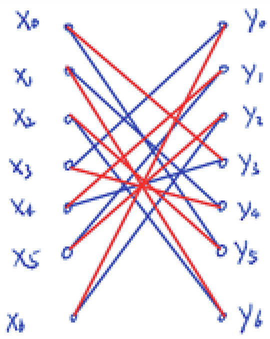


Then find a perfect match

$\{(x_0,y_4),(x_1,y_2),(x_2,y_1),(x_3,y_6),(x_4,y_0),(x_5,y_3),(x_6,y_5)\}$

This matching identifies the desired position of number $m=4$.

$$\begin{bmatrix} 0 & 2 & 1 & & 4 & & 3 \\ 2 & 0 & 4 & 1 & & 3 & \\ 3 & 4 & 0 & 2 & 1 & & \\ & 3 & 2 & 0 & & 1 & 4 \\ 4 & & 3 & & 0 & 2 & 1 \\ 1 & & & 4 & 3 & 0 & 2 \\ & 1 & & 3 & 2 & 4 & 0 \end{bmatrix}$$

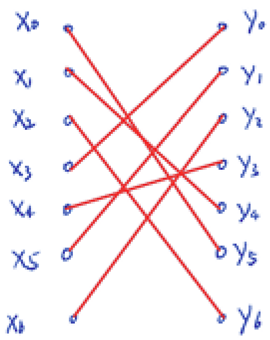


Then find another perfect match

$[(x_0,y_3),(x_1,y_6),(x_2,y_5),(x_3,y_4),(x_4,y_1),(x_5,y_2),(x_6,y_0)]$

This matching identifies the desired position of number $m+1=5$

$$\begin{bmatrix} 0 & 2 & 1 & 5 & 4 & & 3 \\ 2 & 0 & 4 & 1 & & 3 & 5 \\ 3 & 4 & 0 & 2 & 1 & 5 & \\ & 3 & 2 & 0 & 5 & 1 & 4 \\ 4 & 5 & 3 & & 0 & 2 & 1 \\ 1 & & 5 & 4 & 3 & 0 & 2 \\ 5 & 1 & & 3 & 2 & 4 & 0 \end{bmatrix}$$



Then find another perfect match

$[(x_0,y_5),(x_1,y_4),(x_2,y_6),(x_3,y_0),(x_4,y_3),(x_5,y_1),(x_6,y_2)]$

This matching identifies the desired position of number $m+2=6$

$$\begin{bmatrix} 0 & 2 & 1 & 5 & 4 & 6 & 3 \\ 2 & 0 & 4 & 1 & 6 & 3 & 5 \\ 3 & 4 & 0 & 2 & 1 & 5 & 6 \\ 6 & 3 & 2 & 0 & 5 & 1 & 4 \\ 4 & 5 & 3 & 6 & 0 & 2 & 1 \\ 1 & 6 & 5 & 4 & 3 & 0 & 2 \\ 5 & 1 & 6 & 3 & 2 & 4 & 0 \end{bmatrix}$$

$\therefore$ The result is

$$\begin{bmatrix} 0 & 2 & 1 & 5 & 4 & 6 & 3 \\ 2 & 0 & 4 & 1 & 6 & 3 & 5 \\ 3 & 4 & 0 & 2 & 1 & 5 & 6 \\ 6 & 3 & 2 & 0 & 5 & 1 & 4 \\ 4 & 5 & 3 & 6 & 0 & 2 & 1 \\ 1 & 6 & 5 & 4 & 3 & 0 & 2 \\ 5 & 1 & 6 & 3 & 2 & 4 & 0 \end{bmatrix}$$