Lecture14-homework

13. 设 f 与 g 为练习题 1 中的置换，$\mathbf{c}=(R, B, B, R, R, R)$ 是用颜色 R 与 B 对 1，2，3，4，5，6 进行的一种着色。求以下对 $\mathbf{c}$ 的作用：

(a) $f * \mathbf{c}$

$\boxed{\text{(b) } f^{-1} * \mathbf{c}}$

(c) $g * \mathbf{c}$

$\boxed{\text{(d) } (g \circ f) * \mathbf{c} \text{ 与 } (f \circ g) * \mathbf{c}}$

(e) $(g^2 \circ f) * \mathbf{c}$

(R, B, B, R, R, R)

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$$

Answer:

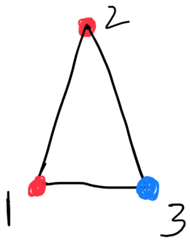
$$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 2 & 5 & 1 \end{pmatrix}, \quad f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}$$

$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 3 & 4 & 6 \end{pmatrix}$$

(b) $f^{-1} * c = (R, R, B, R, R, B)$

(d) $(g \circ f) * c = (R, B, R, R, B, R)$

20. 用红色与蓝色对等腰但非等边的三角形的顶点进行着色，利用定理 14.2.3 求非等价的着色数。用 p 种颜色（参考练习题 4）重做此练习题。



The permutation group has two permutation:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

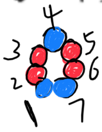
(1) With the two colors (red and blue), the inequivalent colorings is $N(G, C) = \frac{1}{2}(2^3 + 2^2) = 6$

(2) With the p colors, the inequivalent colorings is

$$N(G, C) = \frac{1}{2}(p^3 + p^2)$$

26. 用 4 个红色珠子与 3 个蓝色珠子镶成项链，问有多少种不同的项链？

Answer:



ρ: rotation $\quad$ r: axis-symmetry flip

D_7	Cycle factorization	type	monomial
ρ_7^0	[1]∘[2]∘[3]∘[4]∘[5]∘[6]∘[7]	(7, 0, 0, 0, 0, 0, 0)	z_1^7
ρ_7^1	[1, 2, 3, 4, 5, 6, 7]	(0, 0, 0, 0, 0, 0, 1)	z_7
ρ_7^2	[1, 3, 5, 7, 2, 4, 6]	(0, 0, 0, 0, 0, 0, 1)	z_7
ρ_7^3	[1, 4, 7, 3, 6, 2, 5]	(0, 0, 0, 0, 0, 0, 1)	z_7
ρ_7^4	[1, 5, 2, 6, 3, 7, 4]	(0, 0, 0, 0, 0, 0, 1)	z_7
ρ_7^5	[1, 6, 4, 2, 7, 5, 3]	(0, 0, 0, 0, 0, 0, 1)	z_7
ρ_7^6	[1, 7, 6, 5, 4, 3, 2]	(0, 0, 0, 0, 0, 0, 1)	z_7
r_1	[1]∘[2, 7]∘[3, 6]∘[4, 5]	(1, 3, 0, 0, 0, 0, 0)	$z_1 z_2^3$
r_2	[1, 3]∘[2], [4, 7]∘[5, 6]	(1, 3, 0, 0, 0, 0, 0)	$z_1 z_2^3$
r_3	[1, 5]∘[2, 4]∘[3]∘[6, 7]	(1, 3, 0, 0, 0, 0, 0)	$z_1 z_2^3$
r_4	[1, 7]∘[2, 6]∘[3, 5]∘[4]	(1, 3, 0, 0, 0, 0, 0)	$z_1 z_2^3$
r_5	[1, 2]∘[3, 7]∘[4, 6]∘[5]	(1, 3, 0, 0, 0, 0, 0)	$z_1 z_2^3$
r_6	[1, 4]∘[2, 3]∘[5, 7]∘[6]	(1, 3, 0, 0, 0, 0, 0)	$z_1 z_2^3$
r_7	[1, 6]∘[2, 5]∘[3, 4]∘[7]	(1, 3, 0, 0, 0, 0, 0)	$z_1 z_2^3$

$$\therefore G = \frac{1}{14}\left(z_1^7 + 6z_7 + 7z_1z_2^3\right)$$

$$P_{D_7}(r+b, r^2+b^2, r^3+b^3, r^4+b^4, r^5+b^5, r^6+b^6, r^7+b^7)$$

$$= \frac{1}{14}\left((r+b)^7 + 6(r^7+b^7) + 7(r+b)(r^2+b^2)^3\right)$$

$\because (a+b)^n = C_n^0 a^0 b^n + C_n^1 a b^{n-1} + \cdots + C_n^n a^n b^0$

$\therefore$ We can get the coefficient of $r^4 b^3$ is:

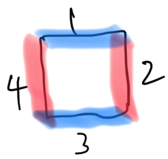
$$\frac{1}{14}(C_7^4 + 7 \times C_3^2) = 4$$

$\therefore$ There are 4 different necklaces using four red and three blue beads.

44. 用红色和蓝色对正方形的边进行着色，求非等价着色的生成函数。使用 k 种颜色时有多少种非等价的着色？（参考练习题 43。）

Answer:

(1)



D_4	Cycle factorization	type	monomial
ρ_4^0	$[1]\circ[2]\circ[3]\circ[4]$	$[4,0,0,0]$	z_1^4
ρ_4^1	$[1,2,3,4]$	$[0,0,0,1]$	z_4
ρ_4^2	$[1,3]\circ[2,4]$	$[0,2,0,0]$	z_2^2
ρ_4^3	$[1,4,3,2]$	$[0,0,0,1]$	z_4
r_1	$[1]\circ[3]\circ[2,4]$	$[2,1,0,0]$	$z_1^2 z_2$
r_2	$[2]\circ[4]\circ[1,3]$	$[2,1,0,0]$	$z_1^2 z_2$
r_3	$[1,4]\circ[2,3]$	$[0,2,0,0]$	z_2^2
r_4	$[1,2]\circ[3,4]$	$[0,2,0,0]$	z_2^2

$$\therefore P_{D_4}(z_1, z_2, z_3, z_4) = \frac{1}{8}(z_1^4 + 2z_4 + 3z_2^2 + 2z_1^2 z_2)$$

$\therefore P_{D_4}(r+b,\ r^2+b^2,\ r^3+b^3,\ r^4+b^4)$

$$= \frac{1}{8}\left((r+b)^4 + 2(r^4+b^4) + 3(r^2+b^2)^2 + 2(r+b)^2(r^2+b^2)\right)$$

$$= \frac{1}{8}(8r^4 + 8r^3b + 16r^2b^2 + 8rb^3 + 8b^4)$$

(2) If we use k colors,

$$P_{D_4}\left(\sum_{i=1}^{k} c_i,\ \sum_{i=1}^{k} c_i^2,\ \sum_{i=1}^{k} c_i^3,\ \sum_{i=1}^{k} c_i^4\right)$$

$$= \frac{1}{8}\left[\left(\sum_{i=1}^{k} c_i\right)^4 + 2\left(\sum_{i=1}^{k} c_i^4\right) + 3\left(\sum_{i=1}^{k} c_i^2\right)^2 + 2\left(\sum_{i=1}^{k} c_i\right)^2\left(\sum_{i=1}^{k} c_i^2\right)\right]$$

$\therefore$ the number of inequivalent colorings is:

$$\frac{1}{8}(k^4 + 2k + 3k^2 + 2k^3)$$