Lecture7-homework

9. Let *hn* equal the number of different ways in which the squares of a *1-by-n* chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that *hn* satisfies. Then find a formula for *hn* .

Answer:

We have h0=1 and h1=3. If the first squares is white or blue, then the chessboard can be completed in h_{n-1} ways. If the first square is red, then second square should be white or blue. hence, $h_n = 2\, h_{n-1} + 2h_{n-2}$ ($n \geq 2$).

The characteristic equation of the recurrence relation is $x^2 - 2x - 2 = 0$ and its two characteristic roots are $1+\sqrt{3}, 1-\sqrt{3}$. By Th.7.2.1,the general solution is: $h_n = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$ (n=0,1,2...)

The general solution satisfies both the recurrence relation and the initial conditions. Setting n = 0, 1 we can find:

$c_1+c_2=1$

$c_1(1+\sqrt{3})+c_2(1-\sqrt{3})=3$

The we can get: $c1=(3+2\sqrt{3})/6$, $c2=(3-2\sqrt{3})/6$.

Therefore:

$$h_n = \frac{3+2\sqrt{3}}{6}(1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n \quad (n=0,1,2\ldots)$$

16. Formulate a combinatorial problem for which the generating function is

$$(1+x+x^2)(1+x^2+x^4+x^6)(1+x^2+x^4+\cdots)(x+x^2+x^3+\cdots).$$

Answer:

This is the generating function for the sequence $\{h_n\}$ where h_n is the number of n-combinations of the multiset $\{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}$,such that (i) a appears at most 2; (ii) b is even and at most 6; (iii) c is even; (iv) d is nonzero.

***25.* Let *hn* denote the number of ways to color the squares of a I-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence *ha, hl, . ..* , *hn , ...* , and then find a simple formula for *hn* .**

Answer:

For an integer $n \geq 0$, h_n is equal to the number of n-permutations of the multiset $\{\infty \cdot R, \infty \cdot W, \infty \cdot B, \infty \cdot G\}$ such that both (i) R appears an even number of times; (ii) W appears an odd number of times. The exponential generating function is $g^e(x) = G_1(x)G_2(x)G_3(x)G_4(x)$, where

$$G_1(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{e^x + e^{-x}}{2}$$

$$G_2(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{e^x - e^{-x}}{2}$$

$$G_3(x) = G_4(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Using this we obtain :

$$g^e(x) = \frac{e^{4x} - 1}{4} = x + \frac{4x^2}{2!} + \frac{4^2 x^3}{3!} + \cdots$$

Therefore $h_n = 4^{n-1}$ if $n \geq 1$ and $h_0 = 0$.

48. Solve the following recurrence relations by using the method of generating functions as described in Section 7.4:

(b) $h_n = h_{n-1} + h_{n-2}, (n >= 2); h_0 = 1, h_1 = 3$

Answer:

The characteristic equation of the recurrence relation is $x^2 - x - 1 = 0$ and its two characteristic roots are $(1+\sqrt{5})/2$,$(1-\sqrt{5})/2$. By Th.7.2.1,the general solution is:

$$h_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad (n = 0, 1, 2, \cdots)$$

The general solution satisfies both the recurrence relation and the initial conditions. Setting n = 0, 1 we can find:

$$c_1 + c_2 = 1$$

$$\frac{1+\sqrt{5}}{2} c_1 + \frac{1-\sqrt{5}}{2} c_2 = 3$$

The we can get: $c1=(1+\sqrt{5})/2$, $c2=(1-\sqrt{5})/2$.

Therefore:

$$h_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \quad (n = 0, 1, 2, \cdots)$$