第四、五次作业反馈

第四次作业

9. Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.

We can color the first square white or blue then we have $2 * h_{n-1}$. And if we color the first square red then we could only color the second square blue or white. So we have $2 * h_{n-2}$. Therefore, the recursive relation is

$$h_n = 2 * h_{n-1} + 2 * h_{n-2}$$

We have the corresponding function:

$$x^2 - 2x - 2 = 0$$

and we get $x_1 = 1 - \sqrt{3}$, $x_2 = 1 + \sqrt{3}$. Now we have

$$h_n = c_1(1 - \sqrt{3})^n + c_2(1 + \sqrt{3})^n$$

,and $h_0 = 1$, $h_1 = 3$. So we have:

$$\begin{cases} c_1 + c_2 = 1 \\ c_1(1 - \sqrt{3}) + c_2(1 + \sqrt{3}) = 3 \end{cases}$$

Solve the equations above we get

$$\begin{cases} c_1 = \dfrac{\sqrt{3} - 2}{2\sqrt{3}} \\ c_2 = \dfrac{\sqrt{3} + 2}{2\sqrt{3}} \end{cases}$$

At the end we get

$$h_n = \frac{\sqrt{3} - 2}{2\sqrt{3}}(1 - \sqrt{3})^n + \frac{\sqrt{3} + 2}{2\sqrt{3}}(1 + \sqrt{3})^n$$

16. Formulate a combinatorial problem for which the generating function is

$$(1 + x + x^2)(1 + x^2 + x^4 + x^6)(1 + x^2 + x^4 + \cdots)(x + x^2 + x^3 + \cdots).$$

h_n is equal to the number of n-permutations of the multiset $S\ \{\infty \cdot x_1, \infty \cdot x_2, \infty \cdot x_3, \infty \cdot x_4\}$, and x_1 appears at most twice, x_2 is even and at most 6, x_3 is even, x_4 is nonzero.

25. Let h_n denote the number of ways to color the squares of a 1-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence $h_0, h_1, \ldots, h_n, \ldots$, and then find a simple formula for h_n.

Answer: Define $h_0 = 1$. Then h_n equals the number of n-permutations of a multiset of four colors, each with an infinite repetition number, in which red occurs an even number of times and white occurs an odd number of times. Thus the exponential generating function for $h_0, h_1, h_2, \ldots, h_n, \ldots$ is the product of red, white, blue, and green factors:

$$g^{(e)} = \left(1 + \frac{x^2}{2!} + \frac{x^4}{2!} + \cdots\right)\left(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$
$$= \frac{1}{4}\left(e^x + e^{-x}\right)\left(e^x - e^{-x}\right)e^x e^x = \frac{1}{4}\left(e^{4x} - 1\right)$$
$$= \frac{1}{4}\sum_{n=1}^{\infty} 4^n \frac{x^n}{n!}$$

Hence, $h_n = 4^{n-1}, \quad n \geq 1, h_0 = 1$.

48. Solve the following recurrence relations by using the method of generating functions as described in Section 7.4:

(a) $h_n = 4h_{n-2}$, $(n \geq 2)$; $h_0 = 0, h_1 = 1$

(b) $h_n = h_{n-1} + h_{n-2}$, $(n \geq 2)$; $h_0 = 1, h_1 = 3$

Solution: Let $g(x) = h_0 + h_1 x + h_2 x^2 + \cdots$

$$(1-x-x^2)g(x) = h_0 + (h_1 - h_0)x + (h_2 - h_1 - h_0)x^2 + \cdots + (h_n - h_{n-1} - h_{n-2})x^n + \cdots$$

$$= h_0 + (h_1 - h_0)x$$

$$= 1 + 2x$$

$$g(x) = \frac{1+2x}{1-x-x^2}$$

Assume there are two numbers a, b satisfying

$$ab = -1, \quad a + b = 1$$

Then $g(x) = \dfrac{1+2x}{(1-ax)(1-bx)} = \dfrac{A}{1-ax} + \dfrac{B}{1-bx}$

To solve the simultaneous equations,

$$\begin{cases} a = \frac{1+\sqrt{5}}{2} \\ b = \frac{1-\sqrt{5}}{2} \end{cases}, \quad \text{and thus} \quad \begin{cases} A + B = 1 \\ -aB - Ab = 2 \end{cases} \Rightarrow \begin{cases} A = \frac{1+\sqrt{5}}{2} \\ B = \frac{1-\sqrt{5}}{2} \end{cases}$$

Thus, $g(x) = \sum_{n=0}^{\infty} (Aa^n + Bb^n)x^n$

$$h_n = Aa^n + Bb^n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

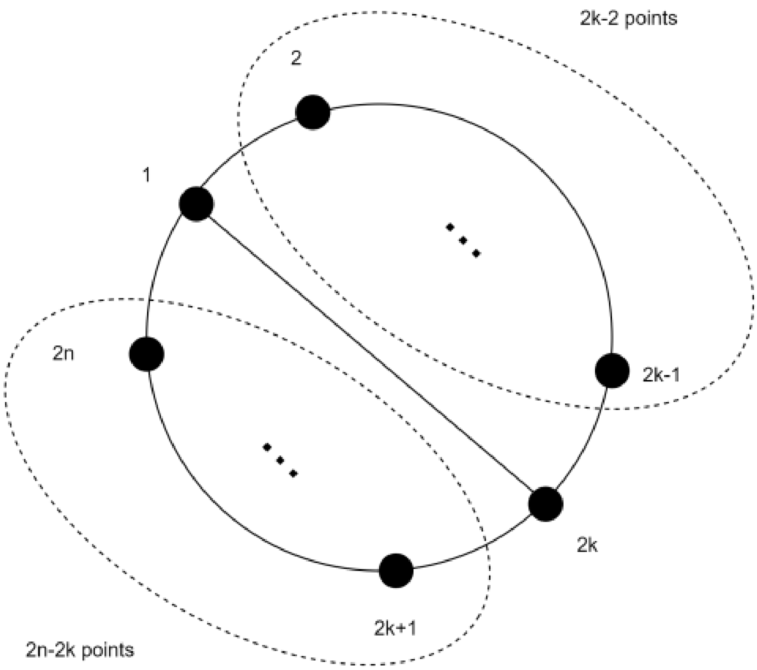
1. Let $2n$ (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number C_n.

Solution:

Using n lines to connect 2n points indicates that the degree of each point is only 1. We choose any point from 2n points as the 1st point, named starting point. To ensure all connected lines are not interacted, there must be an even number of points between 1st point and its connected point (marked as 2k-th point, $k = 1, 2, \ldots, n$). Becasue the two parts separated by the connected line between 1st and 2k-th point are independent, the problem can be transformed into calculating the product of the number of connecting schemes on both smaller enclosed point sets, and we can recursively do it like this until the point size of every part is 0.

h_{2n} is set as the number of ways to join these 2n points in pairs without intersecting, then $h_{2n} = h_0 h_{2n-2} + h_2 h_{2n-4} + \cdots + h_{2n-2} h_0$, and $h_0 = 1$, $h_2 = 1$.

We set $g_n = h_{2n}$, so $g_n = g_0 g_{n-1} + g_1 g_{n-2} + \cdots + g_{n-1} g_0$, $g_0 = 1$, $g_1 = 1$. According to Equation 8.7, g_n is equal to n-th Catalan Number.



7. The general term h_n of a sequence is a polynomial in n of degree 3. If the first four entries of the 0th row of its difference table are $1, -1, 3, 10$, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.

Solution:

The first four numbers in row 0 of the finite differences table are 1, -1, 3, 10. Knowing that h_n is a cubic polynomial in n, the entries in row five and beyond in the differences table are all zero.

A partial representation of the differences table can be illustrated as follows:

$$\begin{array}{ccccccccc}
1 & & -1 & & 3 & & 10 & & \cdots\cdots \\
& -2 & & 4 & & 7 & & \cdots\cdots & \\
& & 6 & & 3 & & \cdots\cdots & & \\
& & & -3 & & \cdots\cdots & & & \\
& & & & 0 & & & &
\end{array}$$

"According to the properties of the differences table, we can derive from the coefficients on the left diagonal: 1, -2, 6, -3, 0, 0, 0..."

$$h_n = \binom{n}{0} - 2\binom{n}{1} + 6\binom{n}{2} - 3\binom{n}{3} + 0 \ldots \quad n = 0, 1, 2 \ldots$$

According to the formula

$$\sum_{k=0}^{n} h_k = c_0\binom{n+1}{1} + c_1\binom{n+1}{2} + \cdots + c_p\binom{n+1}{p+1}.$$

We can obtain:

$$\sum_{k=0}^{n} h_k = \binom{n+1}{1} - 2\binom{n+1}{2} + 6\binom{n+1}{3} - 3\binom{n+1}{4} \quad n = 0, 1, 2 \ldots$$

25. Let $t_1, t_2, \ldots, t_m$ be distinct positive integers, and let

$$q_n = q_n(t_1, t_2, \ldots, t_m)$$

equal the number of partitions of n in which all parts are taken from $t_1, t_2, \ldots, t_m$. Define $q_0 = 1$. Show that the generating function for $q_0, q_1, \ldots, q_n, \ldots$ is

$$\prod_{k=1}^{m} (1 - x^{t_k})^{-1}.$$

Solution:

By expanding the generating function of the proof to be established

$$\prod_{k=1}^{m}(1-x^{t_k})^{-1}=\sum_{n=0}^{\infty}q_n x^n$$

We note that for n>=0, q_n is equal to the number of nonnegative integral solutions $n_1, n_2, n_{3,} \ldots n_m$ to

$$n_1t_1+n_2t_2+n_3t_3+\cdots+n_mt_m=n$$

As for $1 \le K \le m$。

$$(1-x^{t_k})^{-1}=1+x^{t_k}+x^{2t_k}+\cdots$$

So

$$\prod_{k=1}^{m}(1-x^{t_k})^{-1}=\prod_{k=1}^{m}(1+x^{t_k}+x^{2t_k}+\cdots)$$

$$=\left(\sum_{n_1=0}^{\infty}x^{n_1t_1}\right)\left(\sum_{n_2=0}^{\infty}x^{n_2t_2}\right)\ldots\left(\sum_{n_m=0}^{\infty}x^{n_mt_m}\right)$$

$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\sum_{n_m=0}^{\infty}x^{n_1t_1+n_2t_2+n_3t_3+\cdots+n_mt_m}$$

$$=\sum_{n=0}^{\infty}q_n x^n$$

第五次作业

7. Let $\mathcal{A}=(A_1,A_2,A_3,A_4,A_5,A_6)$, where

$$A_1=\{a,b,c\}, A_2=\{a,b,c,d,e\}, A_3=\{a,b\},$$
$$A_4=\{b,c\}, A_5=\{a\}, A_6=\{a,c,e\}.$$

Does the family $\mathcal{A}$ have an SDR? If not, what is the largest number of sets in the family with an SDR?

Solution:

Answer

The family A does NOT have SDR.

According to Theorem 9.3.3, with $n=6$

$k=1$, $\min_{i=1,2,\cdots,6} |A_i| + 6-1 = 1+6-1 = 6$

$k=2$, $\min_{i_1,i_2=1,2,\cdots,6} |A_{i_1} \cup A_{i_2}| + 6-2 = 2+6-2 = 6$

$k=3$, $\min_{i_1,i_2,i_3=1,2,3,\cdots,6} |A_{i_1} \cup A_{i_2} \cup A_{i_3}| + 6-3 = 3+6-3 = 6$

$k=4$, $\min_{i_1,i_2,i_3,i_4=1,2,3,\cdots,6} |A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4}| + 6-4 = 3+6-4 = 5$

$k=5$, $\min_{i_1,i_2,i_3,i_4,i_5=1,2,3,\ldots,6} |A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4} \cup A_{i_5}| + 6-5 = 4+6-5 = 5$

$k=6$, $\min_{i_1,i_2,\cdots,i_6=1,2,3,\cdots,6} |A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4} \cup A_{i_5} \cup A_{i_6}| + 6-6 = 5+6-6 = 5$

Hence, 5 is the largest number of sets with an SDR

family A doesn't have an SDR

11. Let $n > 1$, and let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ be the family of subsets of $\{1, 2, \ldots, n\}$, where

$$A_i = \{1, 2, \ldots, n\} - \{i\}, (i = 1, 2, \ldots, n).$$

Prove that $\mathcal{A}$ has an SDR and that the number of SDRs is the nth derangement number D_n.

Solution:

11. $A_i = \{1,2,\cdots,n\} - \{i\}$. $(i=1,\cdots n)$

So. $|A_i| = n-1$ and. ~~A_{i_k}~~ $\left|\bigcup_{k=2}^{n} A_{i_k}\right| = n$.

So for each $k=1,2,\cdots n$. and each choice of k distinct indices $i_1,\cdots i_k$ from $\{1,\cdots n\}$. $|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| \geq k$. The marriage condition is established. So ~~the~~ A has an SDR.

According to the definition of SDR. we need to choose an item a_i from A_i. and ~~for~~ each a_i $(i=1,\cdots n)$. should be ~~dist~~ distinct.

Cause $A_i = \{1,2,\cdots n\} - \{i\}$. so $a_i \neq i$.

So ~~ai fr~~ choose a_i from A_i in distinct. is equal with. set $\{1,2,\cdots n\}$ to a permutation where the number ~~of~~ i is not set in the i^{th} place.

So. the number of SDRs of A. is D_n

19. Use the deferred acceptance algorithm to obtain both the women-optimal and men-optimal stable complete marriages for the preferential ranking matrix

	a	b	c	d
A	1, 3	2, 3	3, 2	4, 3
B	1, 4	4, 1	3, 3	2, 2
C	2, 2	1, 4	3, 4	4, 1
D	4, 1	2, 2	3, 1	1, 4

Conclude that, for the given preferential ranking matrix, there is only one stable complete marriage.

Solution:

19. For women-optimal.

	a	b	c	d
A	1,3	2,3	3,2	4,3
B	1,4	4,1	3,3	2,2
C	2,2	1,4	3,4	4,1
D	4,1	2,2	3,1	1,4

① $A \to a$ $B \to a$. $C \to b$ $D \to d$.
B is ~~tog~~ rejected.
② $B \to d$ D is rejected.
③ $D \to b$ C is rejected.
④ $C \to a$. A is rejected.
⑤ $A \to b$. A is rejected.
⑥ $A \to c$. no rejection.

Hence (A, c) (B, d). (C, a) (D, b). is the women-optimal stable complete marriages.

For men-optimal.

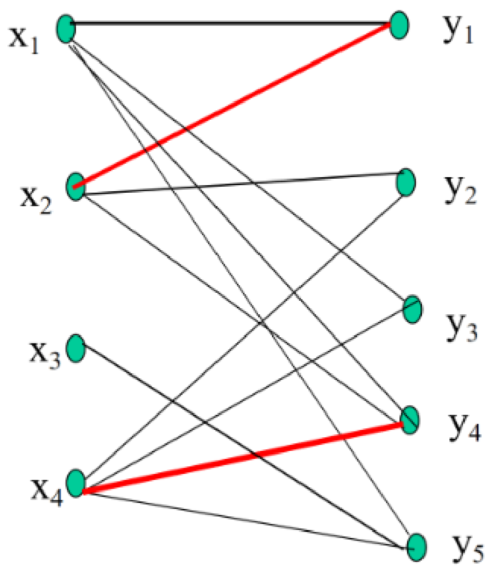
① $a \to D$, $b \to B$, $c \to D$, $d \to C$.: a is rejected.
② $a \to C$. d is rejected.
③ $d \to B$, b is rejected.
④ $b \to D$, c is rejected.
⑤ $c \to A$, no rejection.

Hence (A, c) (B, d) (C, a) (D, b). is the men-optimal stable complete marriages.

Because the result of women-optimal is equal to the result of men-optimal. for each woman, she pairs with the man she ranks highest in women-optimal case and pairs with the man she ranks lowest in men-optimal case. And ranks of two cases is the same. It means that all the partners' rank that are possible for her in a stable complete marriage is the same. Same thing with each man. So all the stable complete marriage' rank matrices are the same, which means that only one stable complete marriage exists.

(a) Determine the max-matching and the min-cover of the right graph by applying the matching algorithm. We choose the red edges and obtain a matching M^1.

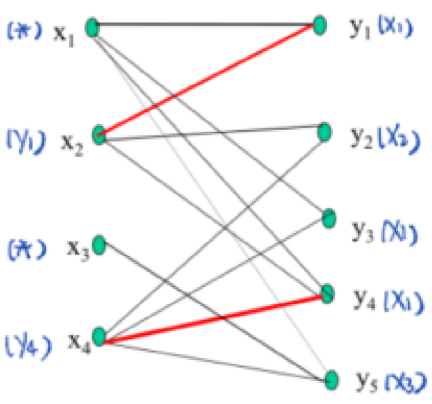
(b) Find a minimum edge cover for the right graph.



Solution:

Answer

Use matching algorithm.



matching $M^1 = \{(x_2, y_1), (x_4, y_4)\}$

(0) set $U = \{x_1, x_3\}$. so label x_1, x_3 with (*)

(1) scan x_1, label $y_1\ y_3\ y_4\ y_5$ with (x_1)

(2) scan x_3, label y_5 with (x_3)

(3) scan y_1, label x_2 with (y_1)

(4) scan y_3, no label

(5) scan y_4, label x_4 with (y_4)

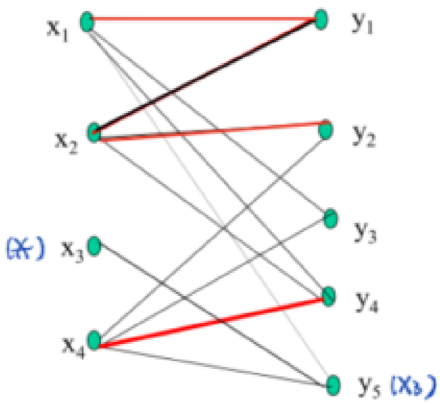
(6) scan y_5, no label

(7) scan x_2, label y_2 with (x_2)

(8) scan x_4, no new label

(9) scan y_2, no new label

Stop. M^1-augmenting path is $r = y_2 x_2 y_1 x_1$ using the labels as a guide. Then $M^2 = [(x_1, y_1), (x_2, y_2), (x_4, y_4)]$ is a matching of 3 edges. Continue to apply the algorithm to M^2



(0) set $U = [x_3]$. so label x_3 with (*)

(1) scan x_3, label y_5 with (x_3)

(2) scan y_5, no new label

Stop. M^2-augmenting path is $r = y_5 x_3$ using the labels as a guide.

Then $M^3 = [(x_1, y_1), (x_2, y_2), (x_3, y_5), (x_4, y_4)]$

All the vertices in X are saturated.

Hence, max-matching is $[(x_1, y_1), (x_2, y_2), (x_3, y_5), (x_4, y_4)]$

min-cover is $[x_1, x_2, x_4, y_5]$

minimum edge cover $[(x_1, y_1), (x_1, y_3), (x_2, y_2), (x_3, y_5), (x_4, y_4)]$