LINEAR PROGRAMMING

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OUTLINE

- General Linear Programs
- An Overview of Linear Programming
- Standard Form
- Slack Form

A POLITICAL PROBLEM

Policy	urban	suburban	rural
Build roads	-2	5	3
Gun control	8	2	-5
Farm subsidies	0	0	10
Gasoline tax	10	0	-2

- The effect of policies on voters
  - Each **entry** describes the number of thousands of voters who could be won over by spending $1,000 on advertising support of a policy on a particular issue.
  - Negative entries denote votes that would be lost.

We Format This Problem as

Minimize:

$$x_1+x_2+x_3+x_4$$

Subject to

$$-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$$

$$5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$$

$$3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Policy	urban	suburban	rural
Build roads	-2	5	3
Gun control	8	2	-5
Farm subsidies	0	0	10
Gasoline tax	10	0	-2

General Linear Programs

We can define:

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n = \sum_{i=1}^{n} a_i x_i$$

Linear equality

$$f(x_1, x_2, \ldots, x_n) = b$$

Linear inequality

$$f(x_1, x_2, \ldots, x_n) \le b$$

or

$$f(x_1, x_2, \ldots, x_n) \ge b$$

General Linear Programs

- Linear Constraints: We use the general term **linear constraints** to denote either linear equalities or linear inequalities.
- Formally, a **linear-programming problem** is the problem of either minimizing or maximizing a linear function subject to a finite set of linear constraints.
- If we are to minimize, then we call the linear program a minimization linear program, and if we are to maximize, then we call the linear program a maximization linear program.

AN OVERVIEW OF LINEAR PROGRAMMING

In order to describe properties of the algorithms for linear programs, we shall use two forms, **standard** and **slack**.

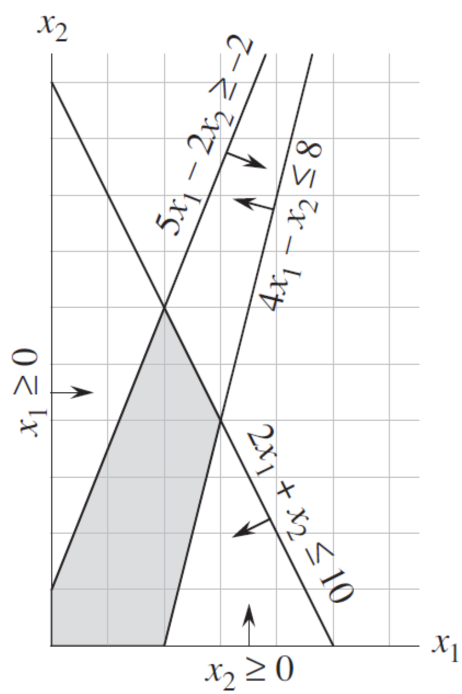
Considering **one** linear program with **two** variables:

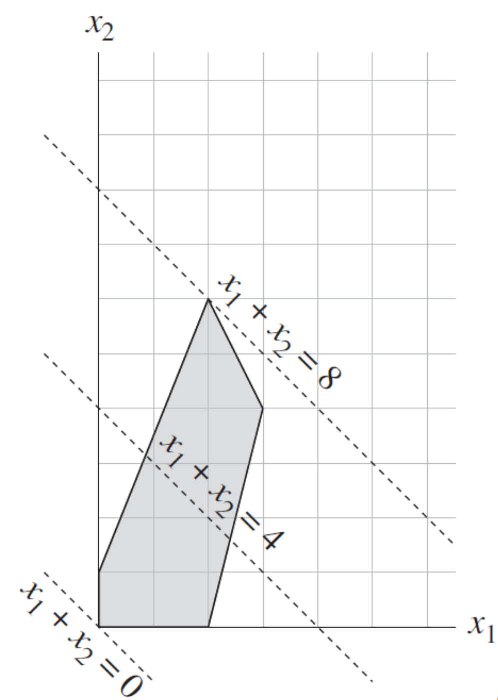
Maximize:

$$f(x_1, x_2) = x_1 + x_2$$

Subject to :

$$
\begin{aligned}
4x_1 - x_2 &\le 8 \\
2x_1 + x_2 &\le 10 \\
5x_1 - 2x_2 &\ge -2 \\
x_1, x_2 &\ge 0
\end{aligned}
$$





INTUITIONS

- Although we cannot easily ***graph*** linear programs with more than two variables, the same intuition holds.
  - If we have three variables, then each constraint is described by a **half-space** in three dimensional space. The **intersection** of these half-space forms **the feasible region**.
  - If we have *n* variables, then each constraint is described by a **half-space** in *n* dimensional space. We call the **feasible region** formed by the intersection of these half-spaces **a simplex**.
  - The objective function is now a hyperplane and, because of **convexity**, an optimal solution still occurs at a vertex of the **simplex**.

Standard Form

- In standard form, we have
  - Given ***n*** real numbers $c_1, c_2, \ldots, c_n$;
  - ***m*** real numbers $b_1, b_2, \ldots, b_m$;
  - ***mn*** real number ***a_{ij}*** for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;
- Finding ***n*** real numbers $x_1, x_2, \ldots, x_n$ that

Maximize:

$$\sum_{j=1}^{n} c_j x_j$$

Subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_j \quad for \quad i = 1, 2, \ldots, m$$

$$x_j \ge 0 \qquad for \quad j = 1, 2, \ldots, n$$

Standard Form

The above linear program can be rewritten as:

$$\textit{Maximize} \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \quad \textit{s.t.} \quad \boldsymbol{Ax} \leq \boldsymbol{b}, \ \ \boldsymbol{x} \geq \boldsymbol{0}$$

where $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{b} = (b_i)$, $\boldsymbol{c} = (c_j)$, $\boldsymbol{x} = (x_j)$.

A tuple ($\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$) can represent a linear program in **standard form**.

Converting linear programs into standard form

For example, if we have the linear program:

Minimize: $-2x_1 + 3x_2$

Subject to: $x_1 + x_2 = 7$; $x_1 - 2x_2 \leq 4$; $x_1 \geq 0$

By negating the **coefficients** of the objective function, we have

Maximize: $2x_1 - 3x_2$

Subject to: $x_1 + x_2 = 7$; $x_1 - 2x_2 \leq 4$; $x_1 \geq 0$

Standard Form

The above linear program can be rewritten as:

$$\textit{Maximize}\ \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}\quad \textit{s.t.}\quad \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},\ \ \boldsymbol{x} \ge \boldsymbol{0}$$

where $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{b} = (b_i)$, $\boldsymbol{c} = (c_j)$, $\boldsymbol{x} = (x_j)$.

A tuple ($\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$) can represent a linear program in **standard form**.

Converting linear programs into standard form

To ensure that each variable has a non-negativity constraint, we have

$(x_1 + x_2 = 7\ ;\ x_1 - 2x_2 \le 4;\ x_1 \ge 0)$ $\qquad x_2 = x'_2 - x''_2$

Maximize: $\quad 2x_1 - 3x'_2 + 3x''_2$

Subject to:

$$x_1 + x'_2 - x''_2 = 7$$

$$x_1 - 2x'_2 + 2x''_2 \le 4$$

$$x_1, x'_2, x''_2 \ge 0$$

$\Rightarrow$

$$x_1 + x'_2 - x''_2 \le 7$$

$$x_1 + x'_2 - x''_2 \ge 7$$

STANDARD FORM

Maximize: $2x_1 - 3x'_2 + 3x''_2$

Subject to:

$$x_1 + x'_2 - x''_2 = 7$$

$$x_1 - 2x'_2 + 2x''_2 \leq 4$$

$$x_1, x'_2, x''_2 \geq 0$$

$$x_1 + x'_2 - x''_2 \leq 7$$

$$x_1 + x'_2 - x''_2 \geq 7$$

$$x'_2 = \mathbf{x_2}$$

$$x''_2 = \mathbf{x_3}$$

For consistency in variable names, we have

Maximize: $2x_1 - 3x_2 + 3x_3$

Subject to:

$$x_1 + x_2 - x_3 \leq 7$$

$$-x_1 - x_2 + x_3 \leq -7$$

$$x_1 - 2x_2 + 2x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0$$

Standard Form

A linear program might not be in standard form for any of **four possible reasons**:

1. The objective function might be a **minimization** rather than a **maximization** (*target*).

2. There might be variables **without nonnegativity** constraints

($x_k = x'_k - x''_k$ & $x'_k , x''_k \geq 0$).

3. There might be **equality constraints**, which have an equal sign rather than a less-than-or-equal-to sign ($\geq$ & $\leq$).

4. There might be **inequality constraints**, but instead of having a less-than-or-equal-to sign, they have a greater-than-or-equal-to sign (*negation, alternate sign direction*).

SLACK FORM

Converting linear programs into slack form.

- We shall convert it into a form in which the nonnegativity constraints are the only inequality constraints, and the remaining constraints are equalities. An inequality constraint is

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

- We introduce a new variable ***s*** and rewrite inequality as the two constraints

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$s \geq 0$$

- We call *s* a slack variable because it measures the **slack**, or **difference**, between the left-hand and right-hand sides of equation.

Slack Form

We introduce slack variables x_4, x_5, x_6, the linear programs we just discussed can be written as:

Maximize: $2x_1 - 3x_2 + 3x_3$

Subject to:

$$x_1 + x_2 - x_3 \leq 7$$

$$-x_1 - x_2 + x_3 \leq -7$$

$$x_1 - 2x_2 + 2x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0$$

Maximize: $Z = 2x_1 - 3x_2 + 3x_3$

Subject to:

$$x_4 = 7 - x_1 - x_2 + x_3$$

$$x_5 = -7 + x_1 + x_2 - x_3$$

$$x_6 = 4 - x_1 + 2x_2 - 2x_3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

For linear programs that satisfy these conditions, we shall sometimes **omit the words** "maximize" and "subject to," as well as the explicit nonnegativity constraints.

Slack Form

As in standard form, we use b_i, c_j and a_{ij} to denote constant terms and coefficients. Thus, we can concisely define a **slack form** by a tuple (N, B, A, b, c, v) to denote the slack form

$$Z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad for \quad i \in B$$

Here, the **equations** is indexed by ***B*** and the **variables** on the right-hand is indexed by ***N***.

Slack Form

For example, in the **slack form**

$$Z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}$$

$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}$$

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$

$$x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2}$$

We have $\boldsymbol{B} = \{1, 2, 4\}$ and $\boldsymbol{N} = \{3, 5, 6\}$

$$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix} \quad \boldsymbol{b} = \begin{Bmatrix} b_1 \\ b_2 \\ b_4 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 4 \\ 18 \end{Bmatrix}$$

$$\boldsymbol{c} = \{c_3, c_5, c_6\}^T = \left\{-\frac{1}{6}, -\frac{1}{6}, -\frac{2}{3}\right\}^T \qquad \boldsymbol{v} = 28$$

Simplex Review

- Although we cannot easily ***graph*** linear programs with more than two variables, the same intuition holds.
  - If we have three variables, then each constraint is described by a **half-space** in three dimensional space. The **intersection** of these half-space forms **the feasible region**.
  - If we have *n* variables, then each constraint is described by a **half-space** in *n* dimensional space. We call the **feasible region** formed by the intersection of these half-spaces **a simplex**.
  - The objective function is now a hyperplane and, because of **convexity**, an optimal solution still occurs at a vertex of the **simplex**.

Simplex Algorithm

- The simplex algorithm takes as input a linear program and returns an optimal solution. Its running time is **not polynomial in the worst case and** often remarkably fast.
- It **starts at** some vertex of the simplex and performs a sequence of iterations. **In each iteration**, it moves <u>along an edge</u> of the simplex from a current vertex to a neighboring vertex whose objective value is **no smaller than** that of the current vertex (and usually is larger.)
- The simplex algorithm **terminates** when it reaches a local maximum, which is a vertex from which **all neighboring vertices** have a smaller objective value.
- Because <u>the feasible region is convex</u> and the objective function is <u>linear</u>, this local optimum is actually a global optimum.

SIMPLEX ALGORITHM

- We take an algebraic view
  - We first write the given **linear program in slack form**, which is a set of linear equalities.
  - These linear equalities express some of the variables, called "**basic variables**," in terms of other variables, called "**nonbasic variables**."
  - We move from one vertex to another by making a basic variable become **nonbasic** and **making a nonbasic variable become basic**. We call this operation a "pivot" and, viewed algebraically, it is nothing more than rewriting the linear program in an equivalent slack form.

SIMPLEX ALGORITHM

- **Idea**: For an iteration of the simplex algorithm, we have
  - Associated with each iteration will be a "basic solution" that easily obtains from **the slack form of the linear program**.
    - Set each **nonbasic variable to 0** and compute the values of **the basic variables** from the equality constraints.
  - An iteration **converts** one slack form into an equivalent slack form.
  - We **choose** a nonbasic variable and **increase** the variable's value from 0, until some basic variable becomes 0.
  - We then **rewrite** the slack form, **exchanging** the roles of that basic variable and the chosen nonbasic variable.

Simplex Algorithm

Consider the following linear program in standard form:

Maximize: $3x_1 + x_2 + 2x_3$

Subject to: $x_1 + x_2 + 3x_3 \leq 30$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

In order to use the **simplex algorithm**, we must convert the linear program into **a slack form**:

Maximize: $Z = 3x_1 + x_2 + 2x_3$

Subject to: $x_4 = 30 - x_1 - x_2 - 3x_3$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

SIMPLEX ALGORITHM

The slack form:

Maximize: $Z = 3x_1 + x_2 + 2x_3$

Subject to: $x_4 = 30 - x_1 - x_2 - 3x_3$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

This system has **3 equations and 6 variables**, and therefore has an infinite number of solutions:

Basic solution: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6) = (0, 0, 0, 30, 24, 36)$

If **a basic solution** is also feasible, we call it a basic feasible solution.

BASIC FEASIBLE SOLUTION

Since the third constraint is the **tightest** constraint, we **switch** $\boldsymbol{x_1}$ (nonbasic) and x_6 (basic).

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

The linear program is rewritten as

$$Z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$

$$x_5 = 6 - \frac{3x_2}{4} - 4x_3 + \frac{x_6}{2}$$

The operation is call pivot.

Maximize: $Z = 3x_1 + x_2 + 2x_3$

Subject to:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

New basic solution: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6) = (9, 0, 0, 21, 6, 0)$

New objective function: $Z = 27$

Basic Feasible Solution

Since the third constraint is the tightest constraint, we **switch** $\boldsymbol{x_3}$ and x_5.

$$Z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$

$$x_5 = 6 - \frac{3x_2}{4} - 4x_3 + \frac{x_6}{2}$$

The linear program is rewritten as

$$Z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5}{8}x_5 - \frac{x_6}{16}$$

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

New basic solution: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6) = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$

New objective function: $Z = \frac{111}{4}$

BASIC FEASIBLE SOLUTION

Now the only way to increase the objective value is to increase ***x_2***.

The linear program is rewritten as

$$Z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}$$

$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}$$

$$x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2}$$

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$

$$Z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5}{8}x_5 - \frac{x_6}{16}$$

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

All coefficients in the objective function are **negative**, as means the basic solution is the optimal solution.

Basic solution(also optimal solution): $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6) = (8, 4, 0, 18, 0, 0)$

Objective function(also final solution) : $Z = 28$

EXERCISE 1

- A company plans to manufacture ***two products***. It is known that the time of **equipment *A*** and equipment ***B*** respectively occupied by manufacturing **one ton**, the time of the **debugging** process and the time available for these two products **each day**, and the **profit** when ***selling one ton*** each are shown in the following table.
- ***Ask*** the company ***how many tons*** each of the two products should be *manufactured* to **maximize the profit**. Solve with **simplex algorithm**.

Project	Product 1	Product 2	Available Time
Equipment A /h	0	5	15
Equipment B /h	6	2	24
Debugging Process/h	1	1	5
Profit /10000 yuan	2	1	

Exercise 1

- Use **variables** x_1 and x_2 to represent the number of products *1* and *2* manufactured by the company.
- At this time, the company can **obtain a profit** of ($2x_1$+x_2) ten thousand yuan, and the maximum profit required is maximize ($2x_1$+x_2).
- The value of x_1 , x_2 is **restricted** by equipment ***A***, ***B*** and the ability of debugging process.
- So for this problem, we have linear programming:

maximize

$$2x_1 + x_2$$

subject to

$$\begin{aligned} 5x_2 &\leq 15 \\ 6x_2 + 2x_2 &\leq 24 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Project	Product 1	Product 2	Available Time
Equipment A /h	0	5	15
Equipment B /h	6	2	24
Debugging Process/h	1	1	5
Profit /10000 yuan	2	1	

EXERCISE 1

Slack form:

maximize

$$Z = 2x_1 + x_2$$

subject to

$$x_3 = 15 - 5x_2$$
$$x_4 = 24 - 6x_1 - 2x_2$$
$$x_5 = 5 - x_1 - x_2$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}) = (0, 0, 15, 24, 5)$

Exercise 1

Switch x_1 and x_4:

$$x_1 = 4 - \frac{1}{3}x_2 - \frac{1}{6}x_4$$

The linear program :

$$Z = 8 + \frac{1}{3}x_2 - \frac{1}{3}x_4$$

$$x_1 = 4 - \frac{1}{3}x_2 - \frac{1}{6}x_4$$

$$x_3 = 15 - 5x_2$$

$$x_5 = 1 - \frac{2}{3}x_2 + \frac{1}{6}x_4$$

maximize

$$\boldsymbol{Z = 2x_1 + x_2}$$

subject to

$$\boldsymbol{x_3 = 15 - 5x_2}$$
$$\boldsymbol{x_4 = 24 - 6x_1 - 2x_2}$$
$$\boldsymbol{x_5 = 5 - x_1 - x_2}$$

New basic solution: $(\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}) = (4, 0, 15, 0, 1)$

Objective function(also final solution) : $\boldsymbol{Z} = 8$

Exercise 1

$$Z = 8 + \frac{1}{3}x_2 - \frac{1}{3}x_4$$
$$x_1 = 4 - \frac{1}{3}x_2 - \frac{1}{6}x_4$$
$$x_3 = 15 - 5x_2$$
$$x_5 = 1 - \frac{2}{3}x_2 + \frac{1}{6}x_4$$

Switch $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$:

$$x_2 = 3 - \frac{x_3}{5}$$

The linear program :

$$Z = 8 + \frac{1}{3}(3 - \frac{x_3}{5}) - \frac{1}{3}x_4 = 9 - \frac{x_3}{15} - \frac{1}{3}x_4$$

subject to

$$x_1 = 4 - \frac{1}{3}(3 - \frac{x_3}{5}) - \frac{1}{6}x_4 = 3 + \frac{x_3}{15} - \frac{1}{6}x_4$$

$$x_3 = 15 - 5x_2 \Rightarrow x_2 = 3 - \frac{x_3}{5}$$

$$x_5 = 1 - \frac{2}{3}(3 - \frac{x_3}{5}) + \frac{1}{6}x_4 = -1 + \frac{2x_3}{15} + \frac{1}{6}x_4$$

New basic solution: $(\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5})$ =(3, 3, 0, 0, -1)

Objective function **(also final solution ????)** : ***Z* = 9**

Exercise 1

Switch $\boldsymbol{x_2}$ and $\boldsymbol{x_5}$:

$$x_2 = \frac{3}{2} + \frac{1}{4}x_4 - \frac{3}{2}x_5$$

The linear program :

$$Z = 8.5 - \frac{1}{4}x_4 - \frac{1}{2}x_5$$

subject to

$$x_1 = \frac{7}{2} - \frac{1}{3}x_4 + \frac{1}{2}x_5$$

$$x_2 = \frac{3}{2} + \frac{1}{4}x_4 - \frac{3}{2}x_5$$

$$x_3 = \frac{15}{2} - \frac{5}{4}x_4 + \frac{15}{2}x_5$$

$$Z = 8 + \frac{1}{3}x_2 - \frac{1}{3}x_4$$

$$x_1 = 4 - \frac{1}{3}x_2 - \frac{1}{6}x_4$$

$$x_3 = 15 - 5x_2$$

$$x_5 = 1 - \frac{2}{3}x_2 + \frac{1}{6}x_4$$

New basic solution: $(\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}) = (\frac{7}{2}, \frac{3}{2}, \frac{15}{2}, 0, 0)$

Objective function(also final solution) : ***Z* = 8.5**

Exercise 2

- According to the contract, a company provides products to the sales company at the end of each quarter. The relevant information is as follows.
- If there are too many products in the current season and there is a surplus at the end of the season, a storage fee of 2,000 yuan will be paid for each ton of products in a quarter.
- Now the factory considers the **best production plan**, so that the factory has the **lowest annual production cost** when the contract is completed (and there is no surplus at the end of the year).
- Try to define three different forms of decision variables for this problem, so as to construct linear programming in different ways.

Quarter j	Production capacity (ton)	Production cost (ten thousand yuan/ton)	Demand (ton)
1	30	15.0	20
2	40	14.0	20
3	20	15.3	30
4	10	14.8	10

Exercise 2

(1) Set up a factory to produce x_j tons of products in the **j** quarter.

- **First**, consider the constraints: The factory must **deliver 20 tons** at the end of the first quarter. Therefore, there should be $x_1 \geq 20$;
- The surplus **after delivery** at the end **of the first quarter** $(x_1 - 20)\ tons$;
- The factory needs to deliver 20 tons at the end of **the second quarter**, so there should be

$$x_1 - 20 + x_2 \geq 20;$$

- Similarly, there should be $x_1 + x_2 - 40 + x_3 \geq 30$;
- After the end of the fourth quarter, the factory cannot overstock products, so there should be

$$x_1 + x_2 + x_3 - 70 + x_4 = 10;$$

- Also considering the factory's production capacity in each quarter, it should have

$$0 \leq x_j \leq a_j.$$

Quarter j	Production capacity (ton)	Production cost (ten thousand yuan/ton)	Demand (ton)
1	30	15.0	20
2	40	14.0	20
3	20	15.3	30
4	10	14.8	10

EXERCISE 2

(1) Set up a factory to produce x_j tons of products in the **j** quarter.

Second, consider the objective function:

- The production cost of the factory in the **first quarter** is 15.0 x_1,
- The cost of the factory production in the second quarter includes the production cost 14 x_2 and the storage cost of overstocked products 0.2(x_1-20);
- Similarly, the cost in the third quarter is $15.3x_3 + 0.2(x_1 + x_2 - 40)$;
- The fourth quarter cost is $14.8x_4 + 0.2(x_1 + x_2 + x_3 - 70)$.
- The annual cost of the factory is the sum of these four quarters. After finishing, the following linear programming model is obtained:

$$
\begin{aligned}
\textbf{min} \quad & 15.6x_1 + 14.4x_2 + 15.5x_3 + 14.8x_4 - 26 \\
\text{s.t.} \quad & x_1 + x_2 \geq 40 \\
& x_1 + x_2 + x_3 \geq 70 \\
& x_1 + x_2 + x_3 + x_4 = 80 \\
& 20 \leq x_1 \leq 30,\ 0 \leq x_2 \leq 40,\ 0 \leq x_3 \leq 20,\ 0 \leq x_4 \leq 10
\end{aligned}
$$

EXERCISE 2

(2) The product produced by the factory in the j quarter is x_j tons, and the product stored at the beginning of the j quarter is y_j tons (obviously, $y_1 = 0$).

- Because the storage volume at the beginning of each quarter is the difference between the storage volume and production volume of the previous quarter and the demand volume of the previous quarter, and considering that the storage volume at the end of the fourth quarter is 0, there are:

$$x_1 - 20 = y_2, \qquad y_2 + x_2 - 20 = y_3$$
$$y_3 + x_3 - 30 = y_4, \qquad y_4 + x_4 = 10$$

- At the same time, the production volume per quarter cannot exceed the production capacity: $x_j \leq a_j$; and the total cost of the four quarters of the factory consists of quarterly production costs and storage costs, so the linear planning:

$$\min\ 15.0x_1 + 0.2y_2 + 14x_2 + 0.2y_3 + 15.3x_3 + 0.2y_4 + 14.8x_4$$

$$\begin{aligned} \text{s.t}\quad & x_1 - y_2 = 20 \\ & y_2 + x_2 - y_3 = 20 \\ & y_3 + x_3 - y_4 = 30 \\ & y_4 + x_4 = 10 \\ & 0 \leq x_1 \leq 30, \quad 0 \leq x_2 \leq 40, \quad 0 \leq x_3 \leq 20, \quad 0 \leq x_4 \leq 10 \\ & y_j \geq 0, \quad j = 1,2,3,4 \end{aligned}$$

EXERCISE 2

(3) Suppose the number of products produced in the i quarter and used for delivery at the end of the j quarter is x_{ij} tons.

- According to contract requirements, there must be:

$$x_{11} = 20, \qquad x_{12} + x_{22} = 20,$$

$$x_{13} + x_{23} + x_{33} = 30 \qquad x_{14} + x_{24} + x_{34} + x_{44} = 10$$

- In addition, the number of products produced every quarter and used for delivery in the current season and subsequent seasons cannot exceed the production capacity of the factory in that season, so there should be:

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 30$$

$$x_{22} + x_{23} + x_{24} \leq 40$$

$$x_{33} + x_{34} \leq 20, \qquad x_{44} \leq 10$$

EXERCISE 2

- The cost per ton of products produced in the i quarter for delivery in the j quarter $c_{ij} = d_i + 0.2(j-i)$,so there is a linear programming:

$$
\begin{aligned}
\min \quad & 15.0x_{11} + 15.2x_{12} + 15.4x_{13} + 15.6x_{14} + 14x_{22} + 14.2x_{23} + 14.4x_{24} \\
& +15.3x_{33} + 15.5x_{34} + 14.8x_{44} \\
\text{s.t.} \quad & x_{11} = 20, \qquad x_{12} + x_{22} = 20, \\
& x_{13} + x_{23} + x_{33} = 30 \qquad x_{14} + x_{24} + x_{34} + x_{44} = 10 \\
& x_{11} + x_{12} + x_{13} + x_{14} \le 30 \\
& x_{22} + x_{23} + x_{24} \le 40 \\
& x_{33} + x_{34} \le 20, \qquad x_{44} \le 10 \\
& x_{ij} \ge 0 \qquad i=1,\ldots,4;\ j=1,\ldots,4,\ \ j \ge i
\end{aligned}
$$