Lecture8-Homework

1. Let 2n (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number Cn.

Answer: Label the points 1 , 2, . . . , 2n clockwise around the circle. Let M = Mn denote the set of matchings of the 2n points. For a matching in M let t denote the point matched to point k (k is odd). Note that t is even. We can let point 2k-1 marked by +1,let point 2k marked by -1.

Point i	Marked as Pi
i= 2k-1 (k belongs to [1,2,3,...,n])	Pi=+1
i=2k (k belongs to [1,2,3,...,n])	Pi=-1

These points in pairs(odd point precedes even point) that satisfy the requirement can form a sequence:$P_{t1},P_{t2},P_{t3},...,P_{t,2n}$ (1)

The partial sums of the sequence (1) are always positive:

$P_{t1}+P_{t2}+...+P_{tk}>=0$ (k=1,2,...,2n)

According to the definition of Catalan number Cn,the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number Cn.

7. The general term hn of a sequence is a polynomial in n of degree 3. If the first four entries of the Oth row of its difference table are 1, -1, 3, 10, determine hn and a formula for $\sum_{k=0}^{n} h_k$.

Answer: The difference table is

1		-1		3		10	...
	-2		4		7	...	
		6		3	...		
			-3	...			
				0	...		

From the diagonal sequence 3, 1, 4, 0, . . . we see that:

$$h_n = \binom{n}{0} - 2\binom{n}{1} + 6\binom{n}{2} - 3\binom{n}{3} \quad n=0,1,2,\cdots$$

Therefore:

$$\sum_{k=0}^{n} h_k = \binom{n+1}{1} - 2\binom{n+1}{2} + 6\binom{n+1}{3} - 3\binom{n+1}{4}, \quad n=0,1,2,\cdots$$

25. Let t1, t2, ... , tm be distinct positive integers, and let

$$q_n = q_n(t_1, t_2, \ldots, t_m)$$

equal the number of partitions of n in which all parts are taken from t1, t2,...,tm.Define q_0 = 1.

Show that the generating function for $q_0, q_1, \ldots, q_n, \ldots$ is $$\prod_{k=1}^{m}(1-x^{t_k})^{-1}.$$

Answer:

25. We show : $\prod_{k=1}^{m}(1-x^{t_k})^{-1} = \sum_{n=0}^{\infty} q_n x^n$

Note that for $n \geq 0$, q_n is equal to the number of nonnegative integral solutions $n_1, n_2, \cdots, n_m$ to

$$n_1 t_1 + n_2 t_2 + \cdots + n_m t_m = n.$$

Recall that for $1 \leq k \leq m$:

$$(1-x^{t_k})^{-1} = 1 + x^{t_k} + x^{2t_k} + \cdots$$

Therefore :

$$\prod_{k=1}^{m}(1-x^{t_k})^{-1} = \prod_{k=1}^{m}(1 + x^{t_k} + x^{2t_k} + \cdots)$$

$$= \left(\sum_{n_1=0}^{\infty} x^{n_1 t_1}\right)\left(\sum_{n_2=0}^{\infty} x^{n_2 t_2}\right) \cdots \left(\sum_{n_m=0}^{\infty} x^{n_m t_m}\right)$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} x^{n_1 t_1 + n_2 t_2 + \cdots + n_m t_m}$$

$$= \sum_{n=0}^{\infty} q_n x^n$$