第七次作业参考

13. Let f and g be the permutations in Exercise 1. Consider the coloring $\mathbf{c} = (R, B, B, R, R, R)$ of $1, 2, 3, 4, 5, 6$ with the colors R and B. Determine the following actions on $\mathbf{c}$:

    (a) $f * \mathbf{c}$

    (b) $f^{-1} * \mathbf{c}$

    (c) $g * \mathbf{c}$

    (d) $(g \circ f) * \mathbf{c}$ and $(f \circ g) * \mathbf{c}$

    (e) $(g^2 \circ f) * \mathbf{c}$

Solution:

Answer

$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix}$ $\quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$

(b) $f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 2 & 5 & 1 \end{pmatrix}$

$\therefore f^{-1} * c = (R, R, B, R, R, B)$

(d) $g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 3 & 4 & 6 \end{pmatrix}$

$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}$

$\therefore (g \circ f) * c = (R, B, R, R, B, R)$

$(f \circ g) * c = (R, R, B, R, B, R)$

20. Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of a triangle that is isoceles, but not equilateral, with the colors red and blue. Do the same with p colors (cf. Exercise 4).

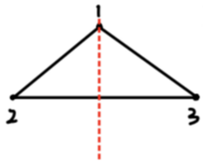
Solution:

Answer:

The group of symmetries of an isoceles triangle is the dihedral group $D=\{\iota, r\}$.

The identity ι fix all colors, thus $|C(\iota)| = 2^3$

The reflection r is like below. as the side 12 = side 13



In order that a coloring be fixed by r. corners 2 and 3 must have the same color

Hence, the colorings fixed by r are picking a color for 1 and picking one color for 2 and 3. $|C(r)| = 2^2$

Therefore, the number of inequivalent colorings is:

$N(D, C) = (2^3+2^2)/2 = 6$ When there are P colors, $N(D, C) = \frac{P^3+P^2}{2}$

26. How many different necklaces are there that contain four red and three blue beads?

Solution:

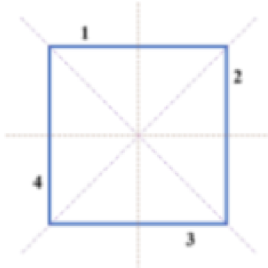
The coefficient of r^4b^3 is 4

So there are 4 different necklaces which contains four red and three blue beads

44. Determine the generating function for nonequivalent colorings of the edges of a square with the colors red and blue. How many nonequivalent colorings are there with k colors (cf. Exercise 43)?

Solution:

The edge permutation group of a square (As shown in the figure on the right. Without loss of generality, name the four edges as $1, 2, 3$ and 4) is the dihedral group:



$$D_4 = \{\rho_4^0 = \iota, \rho_4, \rho_4^2, \rho_4^3, \tau_1, \tau_2, \tau_3, \tau_4\}$$

Among them, τ_1 and τ_2 are reflections about the lines connecting the midpoints of two opposite edges respectively; τ_3 and τ_4 are reflections about the two diagonals, respectively.

We provide the types of each permutation and their corresponding monomials in the table below:

D_4	Cycle factorization	Type	Monomials
$\rho_4^0 = \iota$	$[1] \circ [2] \circ [3] \circ [4]$	$(4, 0, 0, 0)$	z_1^4
ρ_4	$[1\ 2\ 3\ 4]$	$(0, 0, 0, 1)$	z_4^1
ρ_4^2	$[1\ 3] \circ [2\ 4]$	$(0, 2, 0, 0)$	z_2^2
ρ_4^3	$[1\ 4\ 3\ 2]$	$(0, 0, 0, 1)$	z_4^1
τ_1	$[1] \circ [3] \circ [2\ 4]$	$(2, 1, 0, 0)$	$z_1^2 z_2^1$
τ_2	$[2] \circ [4] \circ [1\ 3]$	$(2, 1, 0, 0)$	$z_1^2 z_2^1$
τ_3	$[1\ 4] \circ [2\ 3]$	$(0, 2, 0, 0)$	z_2^2
τ_4	$[1\ 2] \circ [3\ 4]$	$(0, 2, 0, 0)$	z_2^2

The cycle index of D_4 is

$$P_{D_4}(z_1, z_2, z_3, z_4) = \frac{1}{8}(z_1^4 + 2z_4 + 3z_2^2 + 2z_1^2 z_2)$$

If two colors are denoted as r and b, then the generating function is:

$$P_{D_4}(r + b, r^2 + b^2, r^3 + b^3, r^4 + b^4) = \frac{1}{8}(8r^4 + 8r^3 b + 16r^2 b^2 + 8rb^3 + 8b^4)$$

Hence, we get

$$\boldsymbol{P_{D_4}(r + b, r^2 + b^2, r^3 + b^3, r^4 + b^4) = r^4 + r^3 b + 2r^2 b^2 + rb^3 + b^4}$$

If we use k different colors, according to **Theo.14.3.2**, the number of non-equivalent colorings is

$$\boldsymbol{P_{D_4}(k, k, k, k) = \frac{k^4 + 2k + 3k^2 + 2k^2 k}{8} = \frac{k^4 + 2k^3 + 3k^2 + 2k}{8}}$$