Lecture6-homework

3. Find the number of integers between 1 and 10,000 that are neither perfect squares nor perfect cubes.

Define a set S = {1, 2, . . . , 10000}. Let A (resp. B) denote the set of integers in S that are perfect squares (resp. perfect cubes). We seek $|\overline{A} \cap \overline{B}|$. We have:

set	size	justification
S	10000	
A	100	$100^2=10000$
B	21	$21^3=9261$ and $22^3=10648$
$A \cap B$	4	$4^6=4096$ and $5^6=15625$

By inclusion/exclusion:
$|\overline{A} \cap \overline{B}| = 10000-(100+21-4) = 9883.$

8. Determine the number of solutions of the equation $X_1 + X_2 + X_3 + X_4 + X_5 = 14$ in positive integers X_1, X_2, X_3, X_4 and X_5 not exceeding 5.

Let S denote the set of positive integral solutions for $x_1 + x_2 + x_3 + x_4 + x_5 = 14$. For $1 \leqslant i \leqslant 5$ let A_i denote the set of elements in S with $x_i \geqslant 6$. We seek $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}|$. We have:

set	size	justification
S	C(13, 4)	13=14-5+(5-1)
A_i	C(8, 4)	8=9-5+(5-1)
$A_i \cap A_j$	0	

By inclusion/exclusion :
$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}|=C(13,4)-5*C(8,4)=365$

11. Determine the number of permutations of {I, 2, ... ,8} in which no even integer is in its natural position.

Let the set S consist of the permutations of {1, 2, . . . , 8}. For $i \in \{2, 4, 6, 8\}$ let A_i denote the set of permutations in S for which i is in its natural position. We seek $|\overline{A2} \cap \overline{A4} \cap \overline{A6} \cap \overline{A8}|$. We have :

set	size
S	8!
A_i	7!
$A_i \cap A_j$	6!
$A_i \cap A_j \cap A_k$	5!
$A_i \cap A_j \cap A_k \cap A_l$	4!

By inclusion/exclusion :
$|\bar{A}2 \cap \bar{A}4 \cap \bar{A}6 \cap \bar{A}8|$
=8!-[C(4,1)*7!-C(4,2)*6!+C(4,3)*5!-C(4,4)*4!]
=8!-4*7!+6*6! -4*5! +4!

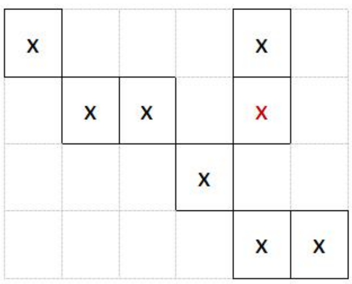
25. Count the permutations $i_1i_2i_3i_4i_5i_6$ of {l, 2, 3, 4, 5, 6}, where $i_1 \neq 1,5$; $i_3 \neq 2,3,5$; $i_4 \neq 4$; and $i_6 \neq 5,6$.

x				x	
	x	x		x	
			x		
				x	x

We interpret this problem in terms of placing six nonattacking rooks on a 6×6 chessboard. The answer is $\sum_{k=0}^{6} r_k(-1)^k\ (6-k)!$ where:

k	0	1	2	3	4	5	6
r_k	1	8	20	20	7	0	0

Chessboard polynomial:



$R(C) = xR(\ \ ) + R(\ \ )$

$= x(1+x)^3 + xR(\ \ ) + R(\ \ ) * R(\ \ )$

$= x(1+x)^3 + xR(\ \ ) * R(\ \ ) + (1+x)^2(1+2x)^2$

$= x(1+x)^3 + x(1+2x)(1+x)^2 + (1+x)^2(1+2x)^2$

$= 1+8x+20x^2+20x^3+7x^4$

31. How many circular permutations are there of the multiset {2 . a, 3· b, 4· c,5 . d}, where, for each type of letter, all letters of that type do not appear consecutively?

Let S denote the set of circular permutations of {2 · a, 3 · b, 4 · c, 5 · d}.

Let A denote the set of elements in S such that all occurrences

of a are consecutive.B,C,D is similar to A. We seek | $\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}$|.

We have :

Set X	X contains CP of	| X |
S	{2 · a, 3 · b, 4 · c, 5 · d}	13!/(2!3!4!5!)
A	{1 · aa, 3 · b, 4 · c, 5 · d}	12!/(1!3!4!5!)
B	{2 · a, 1 · bbb, 4 · c, 5 · d}	11!/(2!1!4!5!)
C	{2 · a, 3 · b, 1 · cccc, 5 · d}	10!/(2!3!1!5!)
D	{2 · a, 3 · b, 4 · c, 1 · ddddd}	9!/(2!3!4!1!)
A∩B	{1 · aa, 1 · bbb, 4 · c, 5 · d}	10!/(1!1!4!5!)
A∩C	{1 · aa, 3 · b, 1· cccc, 5 · d}	9!/(1!3!1!5!)
A∩D	{1 · aa, 3 · b, 4 · c, 1· ddddd}	8!/(1!3!4!1!)
B∩C	{2 · a, 1 · bbb, 1 · cccc, 5 · d}	8!/(2!1!1!5!)
B∩D	{2 · a, 1 · bbb, 4 · c, 1 · ddddd}	7!/(2!1!4!1!)
C∩D	{2 · a, 3 · b, 1 · cccc, 1 · ddddd}	6!/(2!3!1!1!)
A∩B∩C	{1 · aa, 1 · bbb, 1 · cccc, 5 · d}	7!/(1!1!1!5!)
A∩B∩D	{1 · aa, 1 · bbb, 4 · c, 1· ddddd}	6!/(1!1!4!1!)
A∩C∩D	{1 · aa, 3 · b, 1 · cccc, 1· ddddd}	5!/(1!3!1!1!)
B∩C∩D	{2 · a, 1 · bbb, 1 · cccc, 1 · ddddd}	4!/(2!1!1!1!)
A∩B∩C∩D	{1 · aa, 1 · bbb, 1 · cccc, 1 · ddddd}	3!/(1!1!1!1!)

By inclusion/exclusion :

$$|\ \overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}|$$

$=13!/(2!3!4!5!) -12!/(1!3!4!5!) -11!/(2!1!4!5!) -10!/(2!3!1!5!)$

$-9!/(2!3!4!1!) +10!/(1!1!4!5!)$

$+9!/(1!3!1!5!)+8!/(1!3!4!1!)+8!/(2!1!1!5!)+7!/(2!1!4!1!)+6!/(2!3!1!1!)-$

$7!/(1!1!1!5!)-6!/(1!1!4!1!)-5!/(1!3!1!1!)-4!/(2!1!1!1!)+3!/(1!1!1!1!)$.