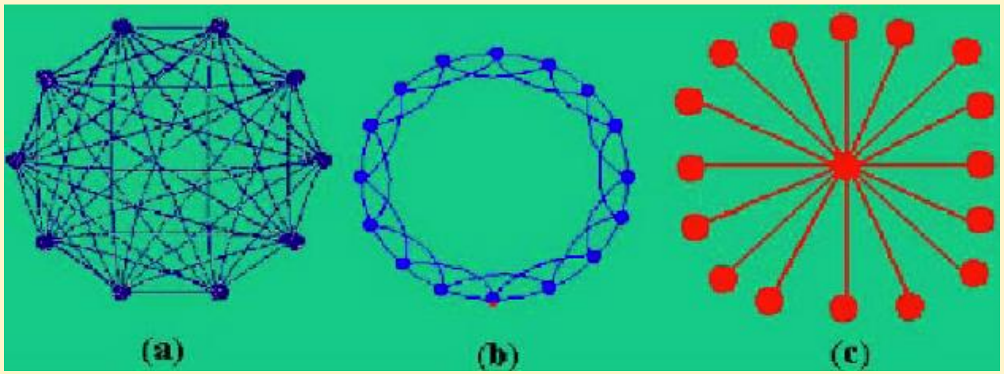
Modeling of Complex Networks

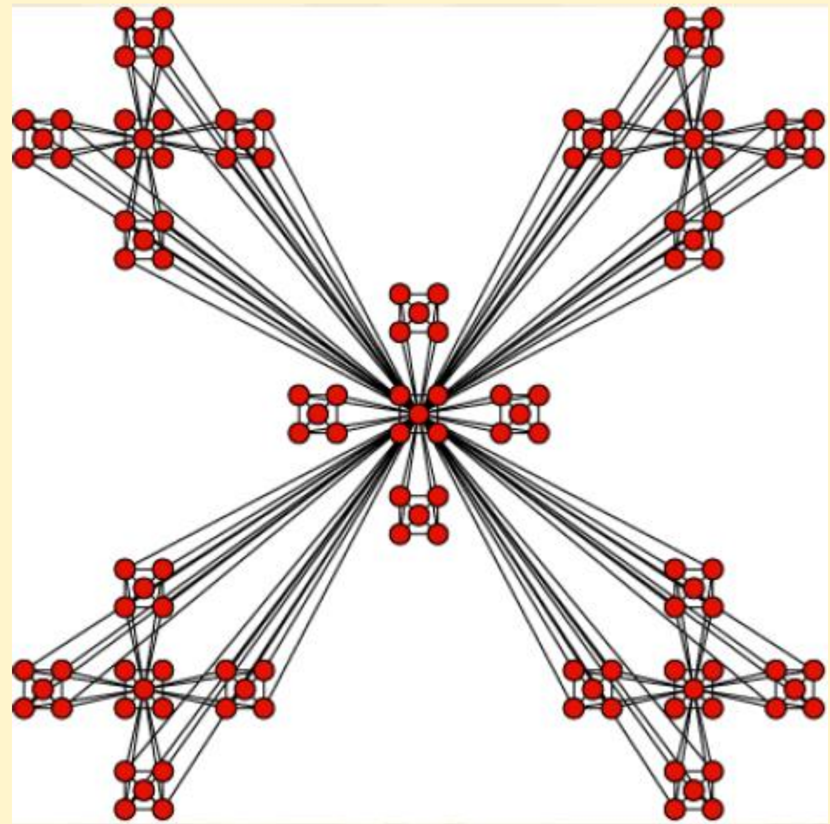
Lecture 3: Network Topologies
--Basic Models and Properties

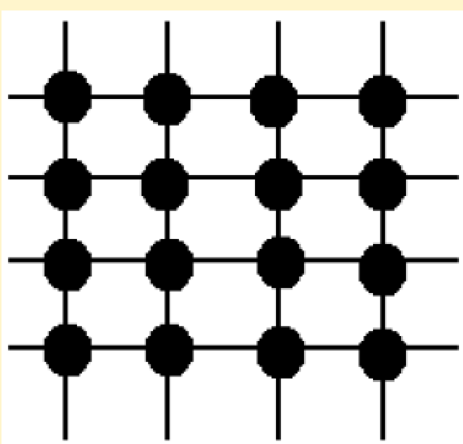
S8101003Q (Sem A, Fall 2020)

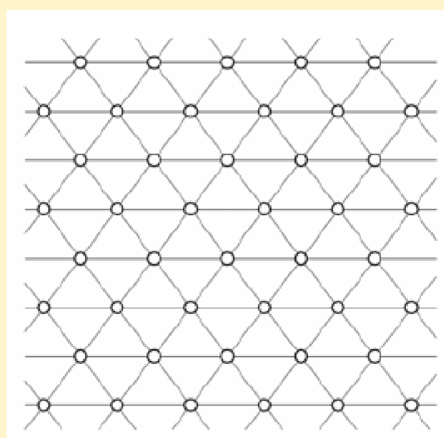
Instructor: Aaron, Haijun Zhang

Regular Networks









Fully-Connected Networks

Theorem 3-1:

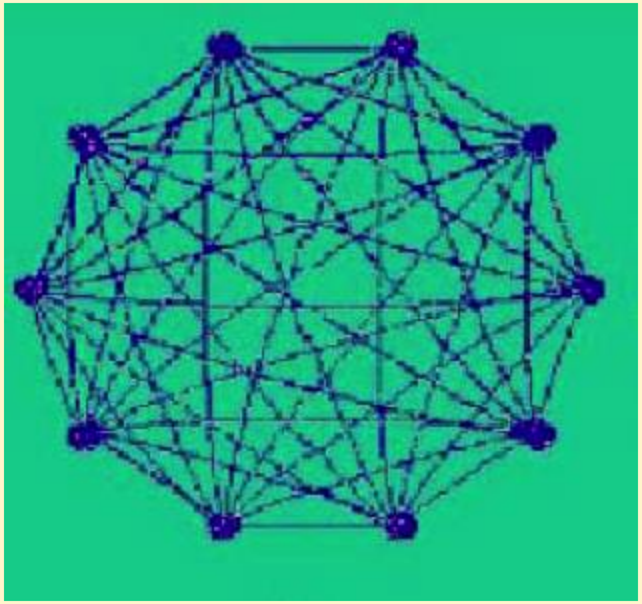
- They have the shortest Average Path Length

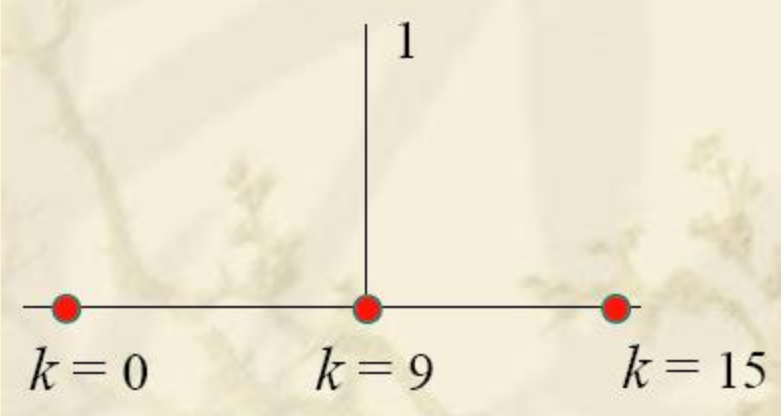
$$L = 1$$

- They have the largest Clustering Coefficient:

$$C = 1$$

- Total Node Degree: $N(N\text{-}1)$
- Total Number of Edges: $N(N\text{-}1)/2$
- Degree Distribution: delta





Ring-Shaped Networks

M – number of edges

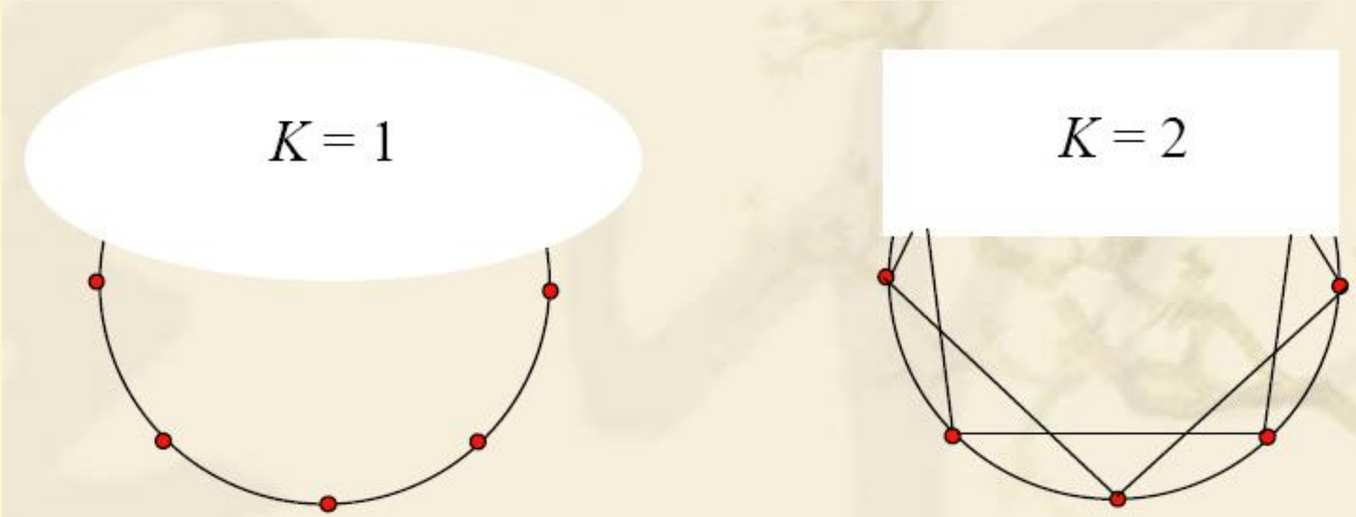
Theorem 3-2: For large enough N,

- Average Path Length:

$$L_{ring} \approx \frac{M(M+1) - 2(K-1)(M-K+1)}{2M} \to \infty \qquad (M \to \infty)$$

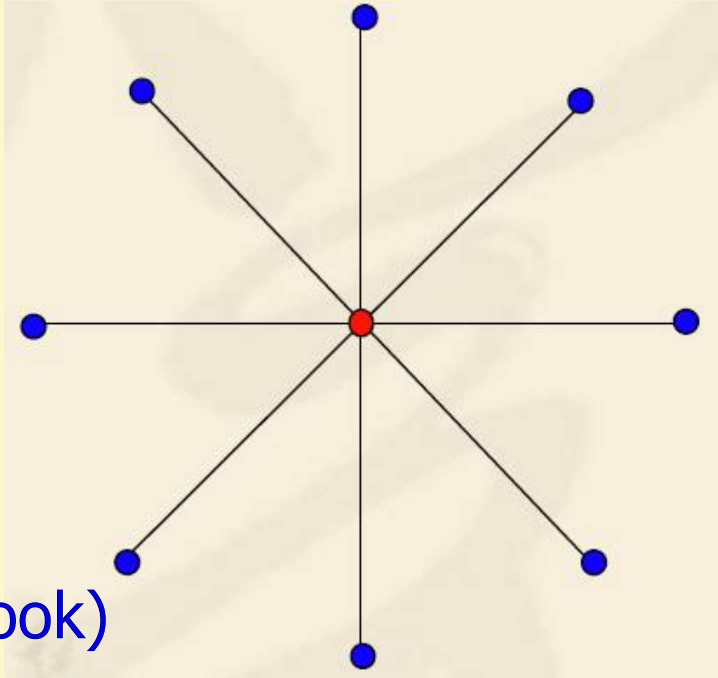
- Clustering Coefficient:

$$C_{ring} \approx \frac{3(K-1)}{2(2K-1)} \to \frac{3}{4} \qquad (K \to \infty)$$



Star-Shaped Networks

Theorem 3-3:



- Average Path Length:

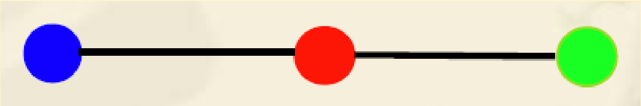
$$L_{star} = 2 - \frac{2}{N} \to 2 \qquad (N \to \infty)$$

- Clustering Coefficient:

$$C_{star} = 0$$

Proof. By simple calculation (see Textbook)

Example: $N = 3$

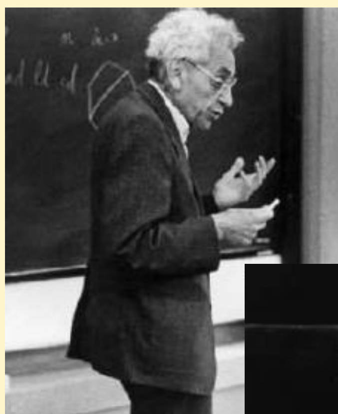


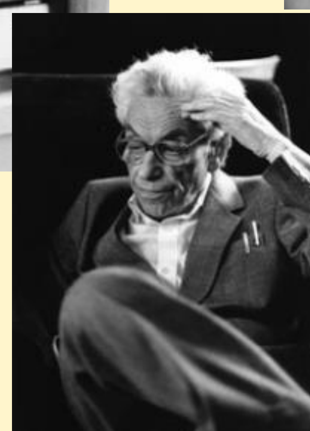
There are 3 different paths with lengths 1, 1, 2

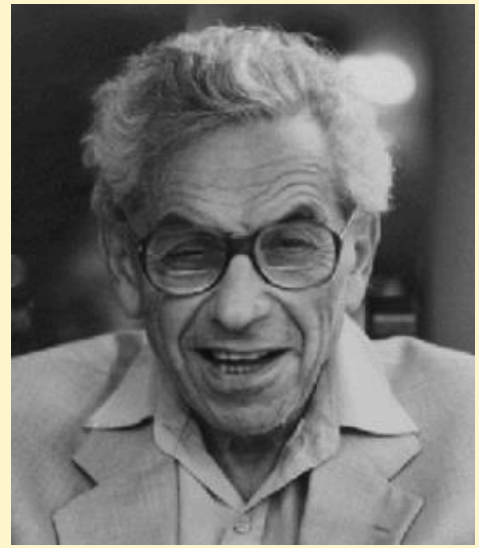
So $L = (1+1+2)/3 = 4/3$

$L = 2\text{-}[2/(N=3)] = 4/3$

Random Graph Theory





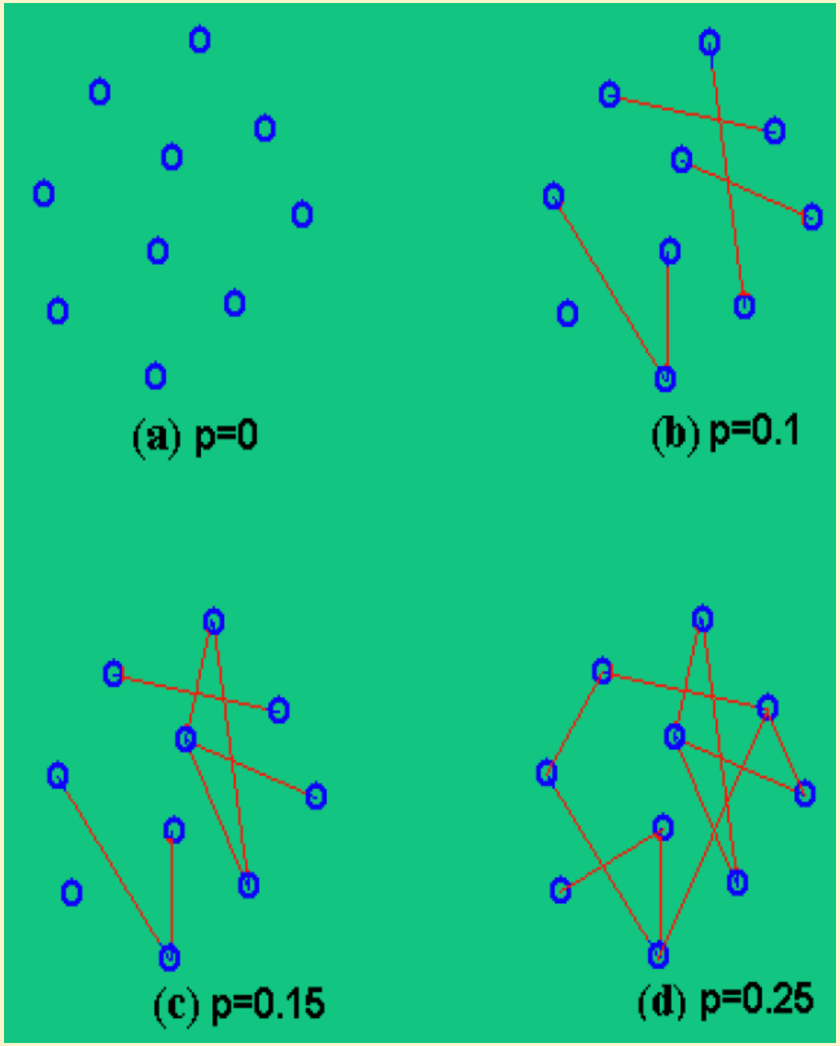


Paul Erdös
(1913-1996)

Alfred Rényi
(1921-1970)

Erdös-Rényi Random-Graph Networks

- Given N isolated nodes
- Add an edge between the two nodes with probability p
- Statistically, $pN(N-1)/2$ edges will have been added



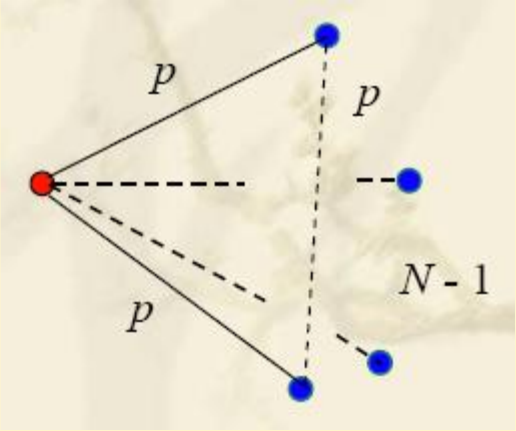
Random-Graph Networks

Theorem 3-4:

- Average degree: $\langle k \rangle = p(N-1) \approx pN$
- Clustering coefficient: $C = p = \langle k \rangle / N$
- Degree distribution: $P(k) = \dfrac{\mu^k}{k!} e^{-\mu}$ (Poisson)
- Average Path Length: $L_{ER} \sim \ln N / \ln \langle k \rangle$

Proof of Clustering coefficient:

$$C = \frac{p.p^2.\binom{N-1}{2}}{p^2.\binom{N-1}{2}} = p = \langle k \rangle / N$$



Random-Graph Networks

Theorem 3-4:

- Average degree: $\langle k \rangle = p(N-1) \approx pN$
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- Average Path Length: $L_{ER} \sim \ln N / \ln\langle k \rangle$

- ***Proof*** . For any node, it can connect to roughly $n_1 \sim \langle k \rangle$ nodes. Its next node can connect to roughly $n_2 \sim \langle k \rangle n_1 = \langle k \rangle^2$ nodes, and so on. Thus, because the average distance between each pair of nodes is L there are roughly $N \sim \langle k \rangle^L$ nodes in the network, yielding $L \sim \ln N / \ln\langle k \rangle$

Proof of Poisson Distribution: $P(k) = \frac{\mu^k}{k!} e^{-\mu}$

- After N pairs of nodes have been randomly picked, probability of obtaining exactly k edges, is $P(k \mid N) = \binom{N}{k} p^k (1-p)^{N-k}$
- Expectation value is the average degree $\mu = Np$
- Viewing the above distribution as a function of μ rather than N and for a fixed probability p, one can rewrite the distribution as

$$P_\mu(k \mid N) = \frac{N!}{k!(N-k)!}\left(\frac{\mu}{N}\right)^k \left(1 - \frac{\mu}{N}\right)^{N-k}$$

- Thus,

$$P(k) = \lim_{N\to\infty} P_\mu(k \mid N)$$

$$= \lim_{N\to\infty} \frac{N(N-1)\ldots(N-k+1)}{N^k} \frac{\mu^k}{k!} \left(1 - \frac{\mu}{N}\right)^N \left(1 - \frac{\mu}{N}\right)^{-k}$$

$$= 1 \cdot \frac{\mu^k}{k!} \cdot e^{-\mu} \cdot 1$$

Random Graph and Poisson Degree Distribution

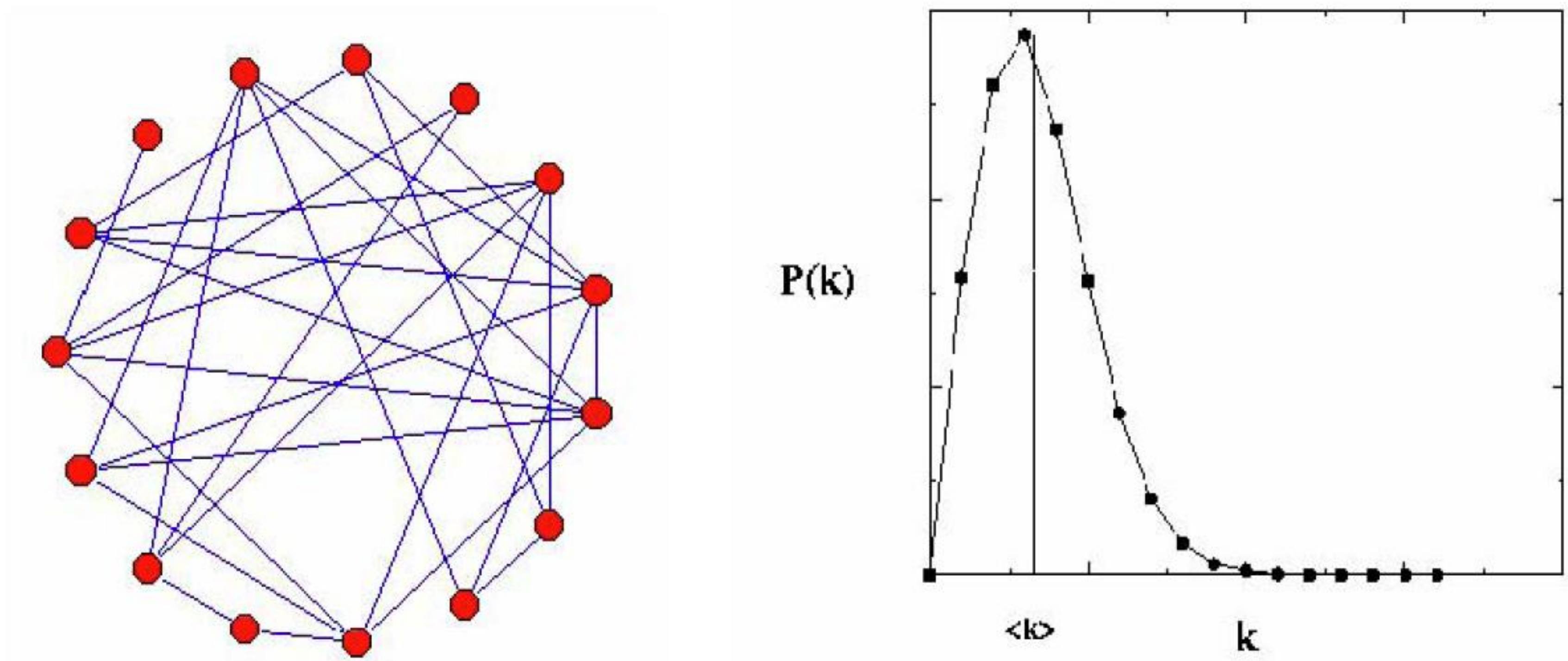


Illustration of Erdös-Rényi randon-graph network model

What is the small world problem

- Often referred to as “Six degree of Separation”
  - “Six degree of separation between us and everyone else on this planet”
  - John Guare, 1990
- An urban myth ? (“Six handshakes to the President”)
- First mentioned in 1920’s by Karinthy
- 30 years later, became a research problem

The small world problem

- In the 1950's, Pool and Kochen asked "what is the probability that two strangers will have a mutual friend ?"
  - i.e. the "small world" of cocktail parties
- Then asked a harder question: "What about when there is no mutual friend – how long would the chain of intermediaries be ?"
- Too hard …

The Small World Experiment

- Stanley Milgram (and student Jeffrey Travers) designed an experiment based on Pool and Kochen’s
  - A single “target” in Boston
  - 300 initial “senders” in Boston and Omaha
  - Each sender asked to forward a packet to a friend who was “closer” to the target
  - The friends got the same instructions

"Six Degrees of Separation"

- Travers and Milgram's protocol generated 300 "letter chains" of which 64 reached the target.
- Found that typical chain length was 6
- Led to the famous phrase (Guare)
- They not much happened for another 30 years
  - Theory was too hard to do with pencil and paper
  - Data were too hard to collect manually

The “New” Science of Networks

- Mid of 90’s, Seteve Strogatz and Duncan Watts working on another problem altogether
- Decided to think about this urban myth
- We had three advantages
  - We didn’t know anything
  - We had MUCH faster computers
  - Our background in physics and mathematics caused us to think about the problem somewhat differently

Small World Networks

- They managed to show that if a network has
  - Some source of "order"
  - The tiniest amount of randomness
- It will be a "small-world" network of the kind that Pool and Kochen were looking for
- They also made the prediction that small world networks should be *everywhere*

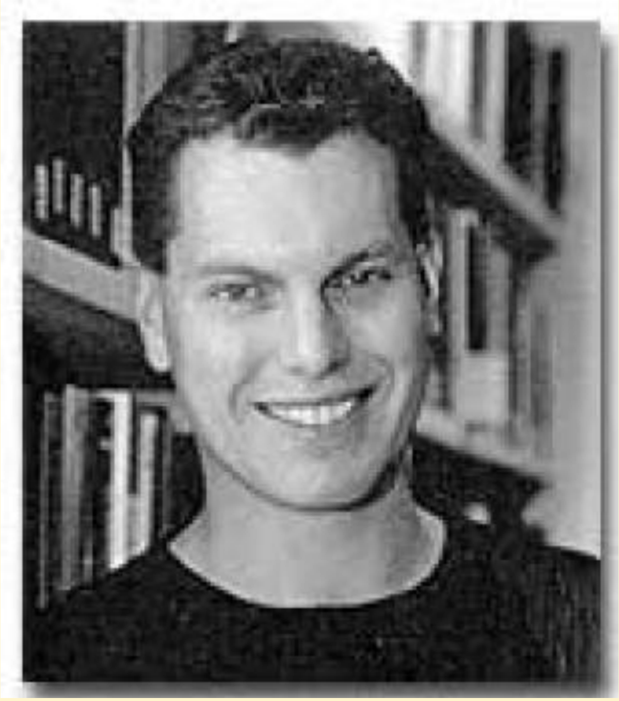
Small World Networks

- Online social networks
- Email networks
- Networks of movie stars, boards of directors, and scientists
- Power transmission grid of the Western US
- Neural networks
- Genetic regulatory networks, protein interaction networks, metabolic reaction networks
- World Wide Web
- Food Webs

Small-World Networks

"Collective dynamics of 'small-world' networks"

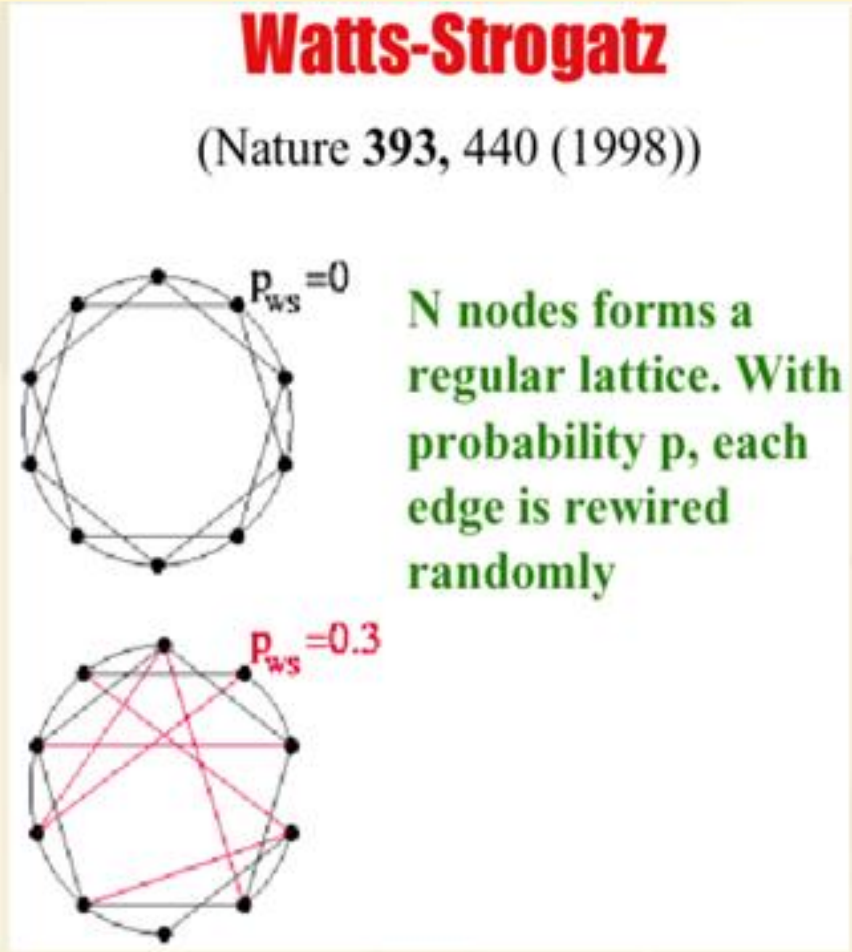
--- Nature, 393: 440-442, 1998



D. J. Watts S. H. Strogatz

(Cornell University)

Small-World Networks

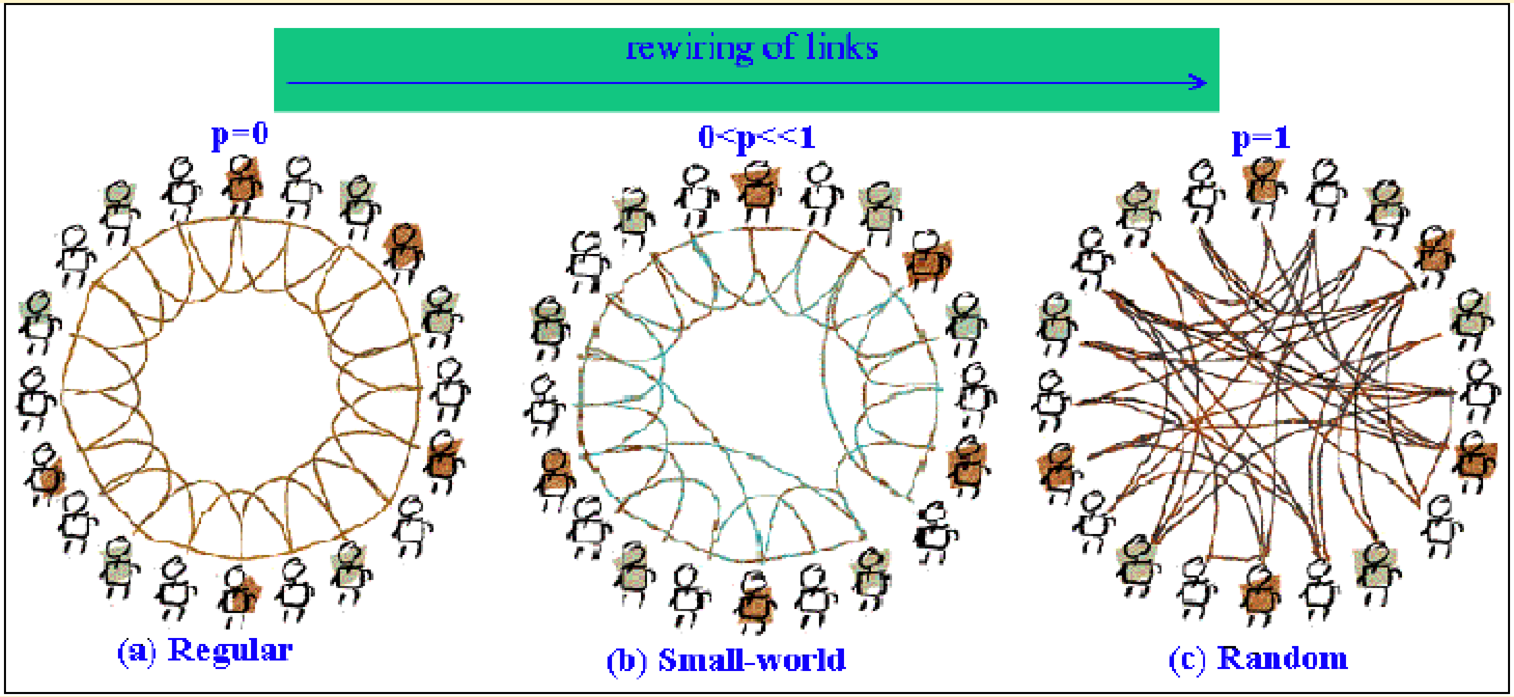


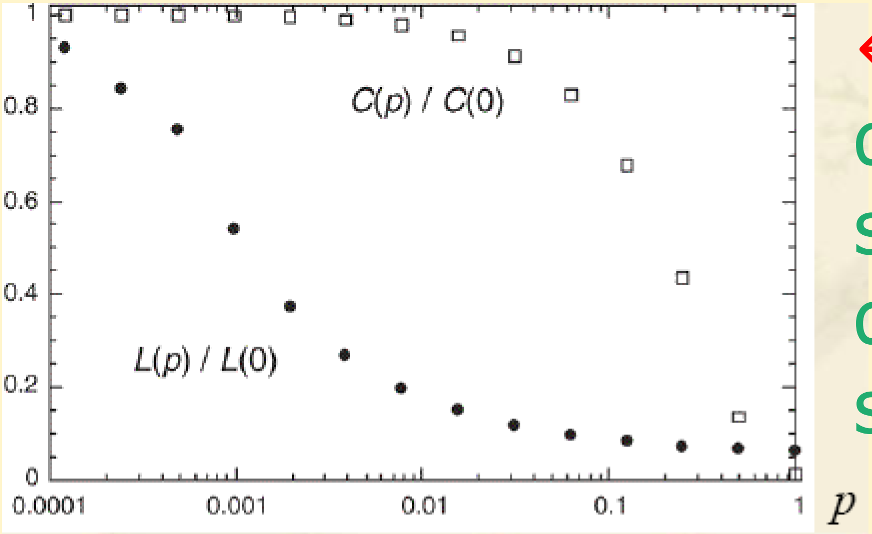
- **Features:**
  - Connectivity Poisson distribution
  - Homogeneous nature:
    - ✓ Each node has roughly the same number of edges
  - Small average path length but large clustering coefficient
  - Not growing

WS Small-World Network Model

Algorithm (rewiring):

- Start from a ring-shaped network with N nodes, in which each node is connected to its $2K$ neighbors, where K is a (usually small) positive integer.
- Go around the ring-shaped network clockwise (or counterclockwise). Pick a node and operate on its connections to the K neighbors one by one: an end connecting this node to its neighbor will be kept unchanged; another end of the edge will be disconnected with probability p and then be re- connected to a randomly-picked node from the network. - ignore self-loops and multiple edges.





← Even a few long-distance edges are sufficient to drastically decrease L without significantly changing C

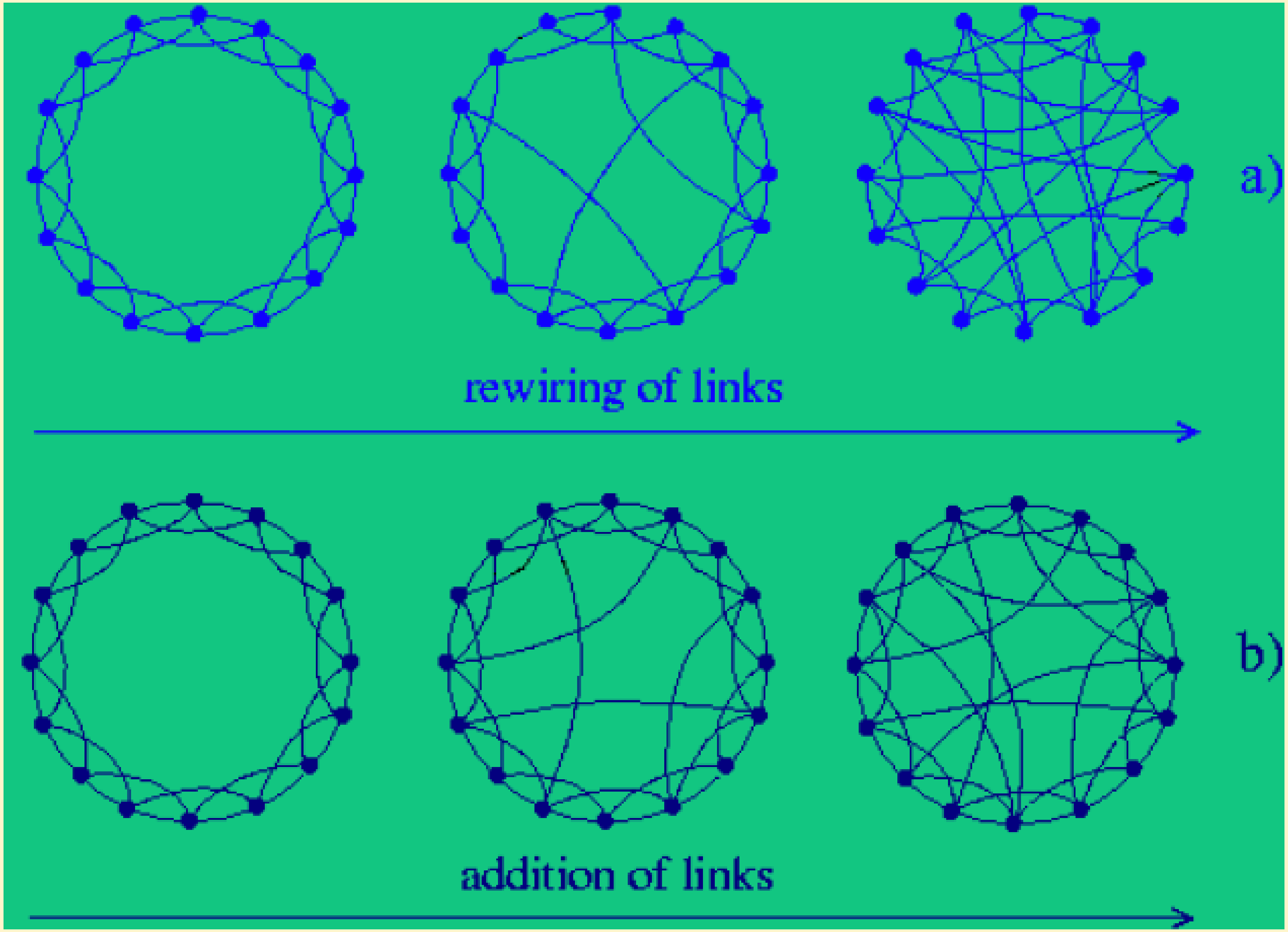
NW Small-World Network Model

Algorithm **(adding edges):**

- Start from a ring-shaped network with N nodes, in which each node is connected to its $2K$ neighbors, where K is a (usually small) positive integer.
- With a (small) probability p, add an edge to each pair of nodes in the network.

Mark E J Newman

Newman-Watts Small-World Network Model



a) **WS small-world network model b) NW small-world network model**

Small-World Networks

Theorem 3-5: For large enough N,

Clustering Coefficient of the WS small-world network model:

$$C(p) = \frac{3(K-2)}{4(K-1)}(1-p)^3$$

Clustering Coefficient of the NW small-world network model:

$$C(p) = \frac{3(K-2)}{4(K-1) + 4Kp(p+2)}$$

Small-World Networks

Theorem 3-6:

Average Path Length of the WS small-world network model:

$$L(p) = \frac{2N}{K} f(NKp/2)$$

$$f(x) = \begin{cases} c & x \ll 1 \\ \ln x / x & x \gg 1 \end{cases} \qquad (c - \text{constant})$$

Average Path Length of the NW small-world network model:

$$L(p) = \frac{2N}{K} f(NKp/2)$$

$$f(x) \approx \frac{1}{2\sqrt{x^2 + 2x}} \tanh^{-1} \sqrt{\frac{x}{x+2}}$$

Small-World Networks

Theorem 3-7:

Degree Distribution of the WS small-world network model:

$$P(k) = \sum_{i=0}^{\min\{k-K/2,K/2\}} \binom{K/2}{i} (1-p)^i \, p^{(K/2)-i} \frac{(pK/2)^{k-i-K/2}}{(k-N-K/2)!} e^{-pK/2} \quad k \geq K/2$$

$$P(k) = 0 \qquad k < K/2$$

Degree Distribution of the NW small-world network model:

$$P(k) = \binom{N}{k-K} \left(\frac{Kp}{N}\right)^{k-K} \left(1-\frac{Kp}{N}\right)^{N-k+K} \qquad k \geq K$$

$$P(k) = 0 \qquad k < K$$

Navigable Small-World Networks

International Congress of Mathematics (ICM)

22-28 August 2006, Madrid, Spain

Jon M Kleinberg (Cornell Univ.) received the Nevanlinna Prize for Applied Mathematics

He gave a 45-minute talk - "Complex Networks and Decentralized Search Algorithms"

J M Kleinberg, "Navigation in a small world," Nature, 2000

J M Kleinberg, The small-world phenomenon: An algorithmic perspective. *Proc. 32nd ACM Symposium on Theory of Computing*, 2000: 163-170

Kleinberg's Navigable Networks

Problem -- To deliver a message from a source to a target on a lattice, in as few steps as possible

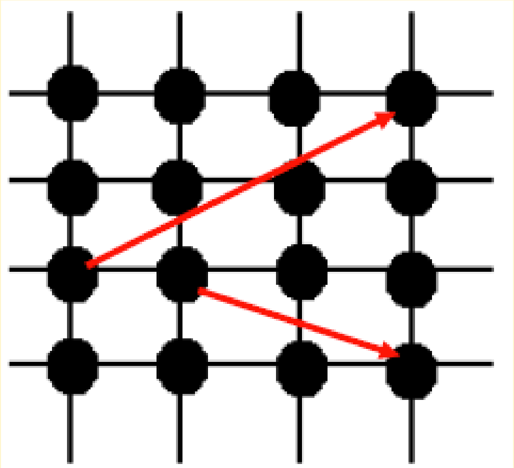
Solution: First generate a small world network, then construct an algorithm capable of finding a short path from the source to the target:

(i) allowed to add m long-range edges anywhere in the lattice

(ii) using only locally available information:

- Location of the target on the lattice
- Locations and long-range neighbors of all nodes that had previously processed the message

Suppose that every node does not know the long-range neighbors of the other nodes which had not previously processed the message. (otherwise, the problem would be easy to solve)



Kleinberg's Navigable Networks

- Recall the NW small-world network model
- One may simply add m long-range edges to the given lattice to make it an NW small-world network
- But Kleinberg showed that a fast algorithm does not exist for NW model
- **Observations:**
- In NW model, each node has an equal probability to receive new edge
- Kleinberg argued that intuitively it would be easier for a new edge to connect to a nearby neighbor than to those far away
- So, he assumed that the connecting probability is <u>reversely proportional</u> to the distance between the two nodes

Kleinberg's Navigable Networks

Assumption: the probability of a new edge connecting a node u to any other node v in the given lattice:

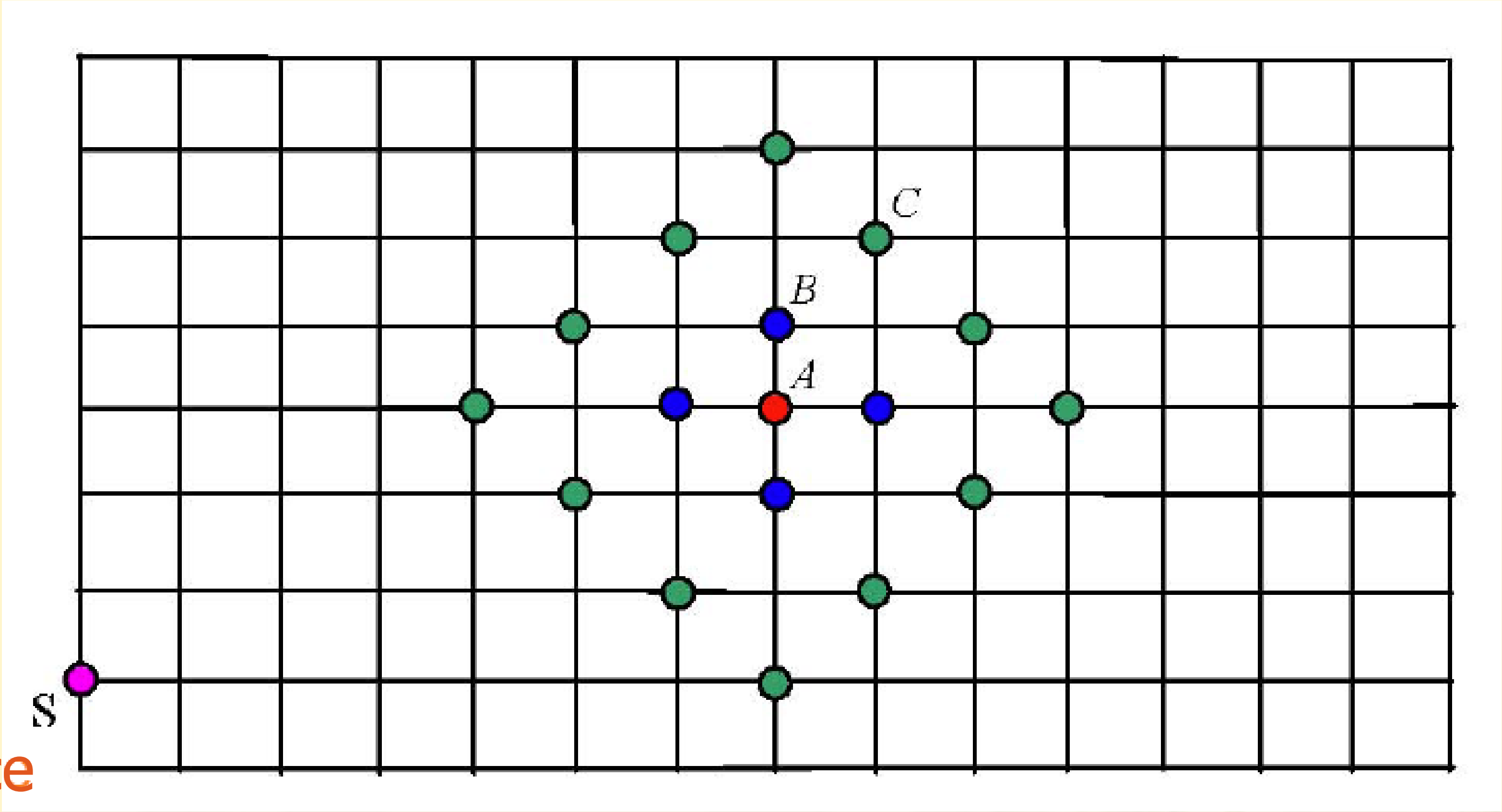
$$P(u,v) = \beta d_{uv}^{-\alpha}$$

where d_{uv} is the distance between u and v with parameters $\alpha \geq 0, \beta > 0$ satisfying $\beta \sum_{(u,v)} d_{u,v}^{-\alpha} = 1$ as a probability measure.

When $\alpha = 0$, the new edge-addition probability is a constant, so every node has the same probability to receive the new edge; therefore, the model reduces to the original NW small- word network model.

Kleinberg's Navigable Networks

Graphical illustration of the new edge-distribution mechanism:



source

$d_{AB}=1$ $d_{AC}=3$ $P(A,B)=\beta \text{x} 1^{-\alpha} > P(A,C)= \beta \text{x} 3^{-\alpha}$

Kleinberg’s Navigable Networks

- Noticing that other algorithms may lead the message to go through some detours in approaching the target
- Kleinberg’s idea was to find a way that guarantees the message to move at least one step closer to the target each time
- He applied a greedy algorithm, where at every step the current message holder will pass the message to a neighbor who is as close to the target as possible

Kleinberg's Navigable Networks

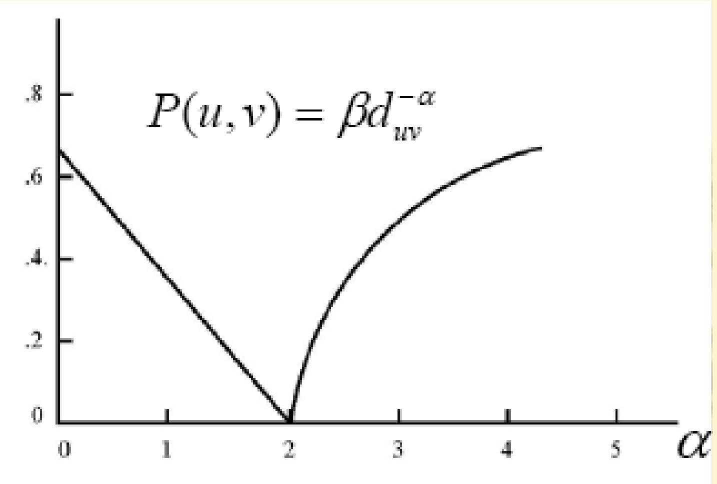
When $\alpha = 0$ the edge-addition probability is a constant, so the model is the original NW small-word network model

As α increases, a decentralized algorithm can take more advantage of the geographic structure with long-range connections

As α further increases, however, the number of long-range edges become very small therefore less useful

Therefore, there is a best value of α in between: $\alpha = 2$

Y-axis represents beta in time-steps O(N^beta)
The figure represents the relation between parameter Alpha and the message delivery time O(N^beta)



Kleinberg's Navigable Networks

A network is *navigable* if it requires at most $poly(logN)$ time-steps to deliver a message to any target from any source on the network.

Kleinberg's Theorem.

- For $\alpha = 2$, the lattice is navigable in $O(logN)^2$ time-steps at a rate depending on m, but for all $\alpha < 0$ it is not navigable.
- For $0 \leq \alpha < 2$, it takes at least $O(N^{(2-\alpha)/3})$ time-steps to deliver a message from any source to any target.
- For $\alpha > 2$, it will take at least $O(N^{(\alpha-2)/(\alpha-1)})$ time-steps.

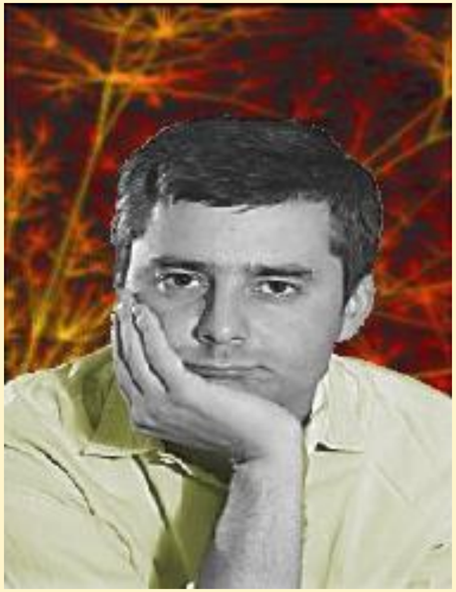
- For the Kleinberg problem formulated in the original NW ring-shaped network setting, see:

M E J Newman: Networks: An Introduction. Oxford, UK, 2010: 713-718

Another Breakthrough: Scale Free

"Emergence of scaling in random networks"

Science 286: 509 (1999)



A.-L. Barabási

R. Albert

(Norte Dame University)

BA Scale-Free Network Model

Start with a fully-connected network having $m_0 \geq 1$ nodes

(i) Add new nodes:

Add 1 new node into the network:

This node is connected to m ($m \leq m_0$) existing nodes simultaneously

(ii) Add new edges:

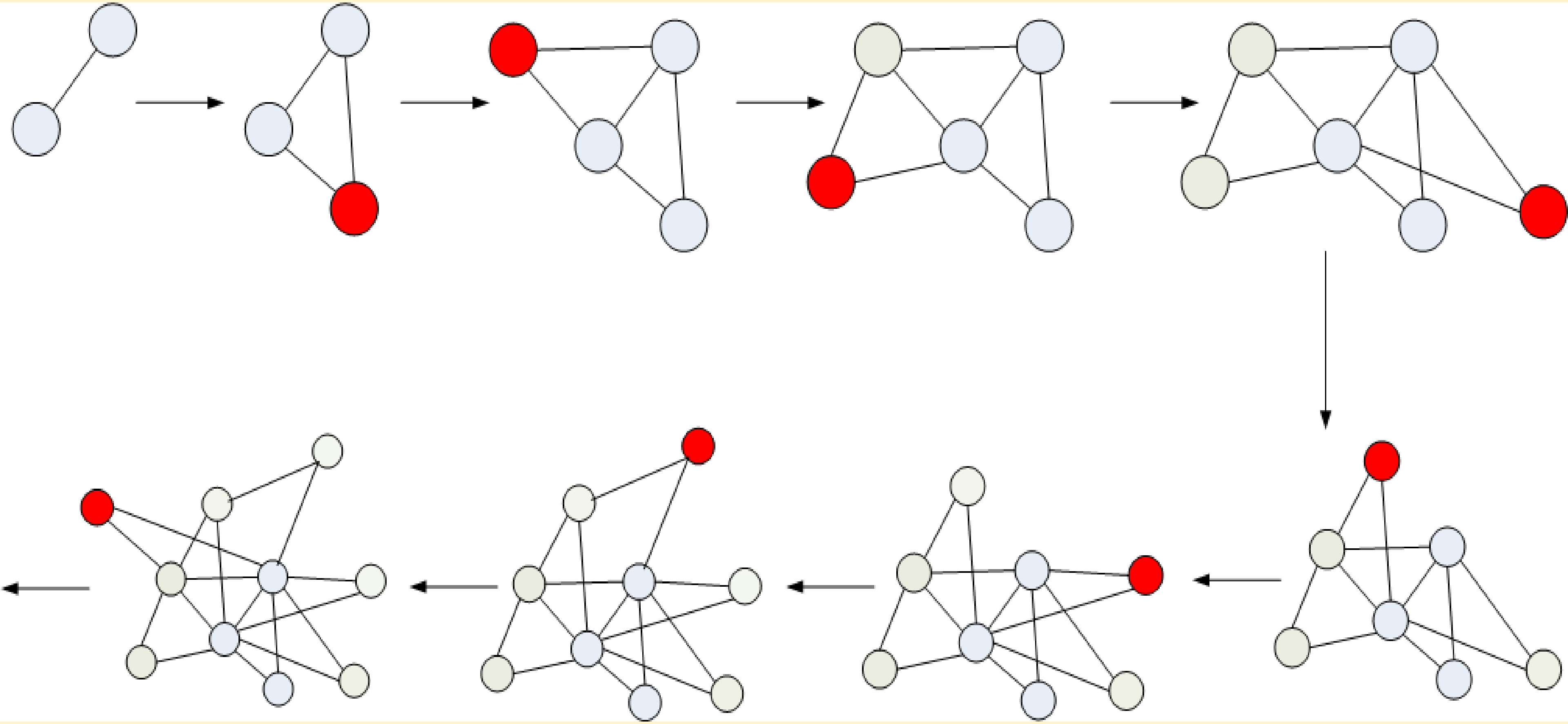
The way to add the m new edges into the network:

Every existing node is to be chosen with probability

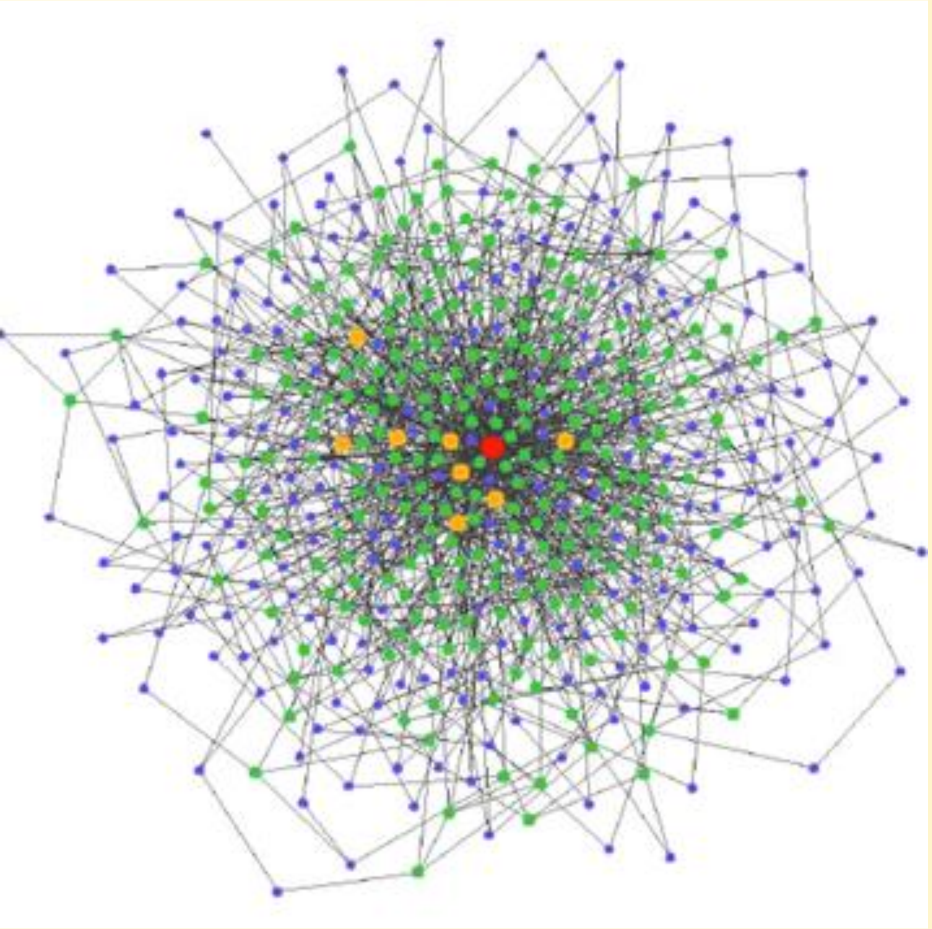
$$\Pi(k_i) = \frac{k_i}{\sum_l k_l}$$

BA Scale-Free Network Model
- An example

- $(m = m_0 = 2)$

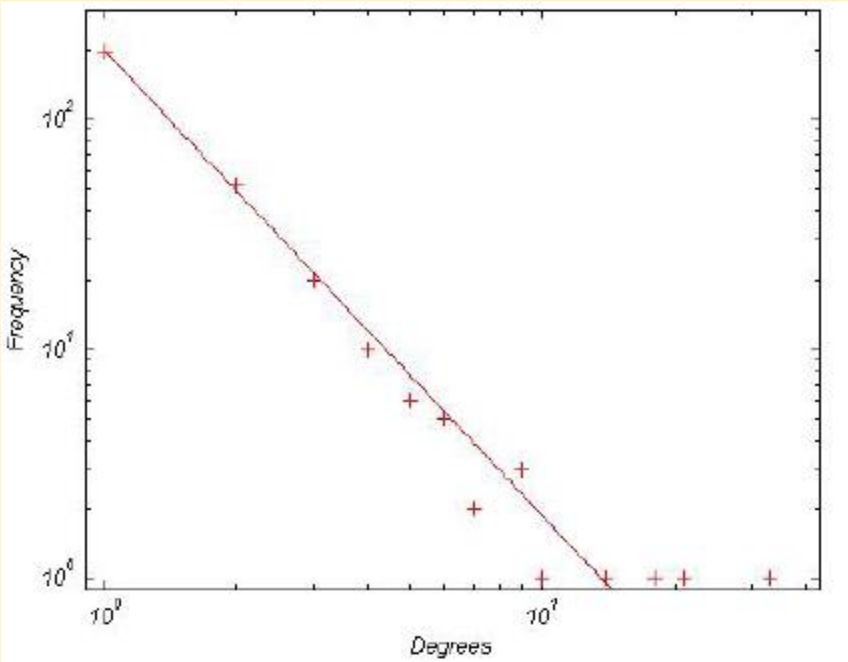


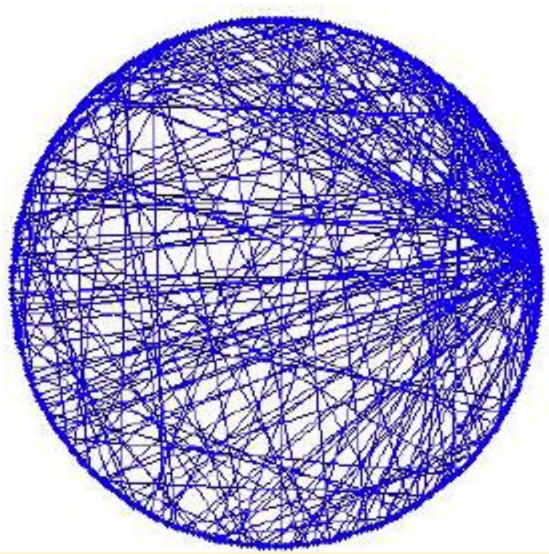
Scale-Free Networks



- **Features:**
  - Connectivity: in Power-law form
  - Non-Homogeneous nature:
    - Very few nodes have many links but most nodes have very few links
  - Growing

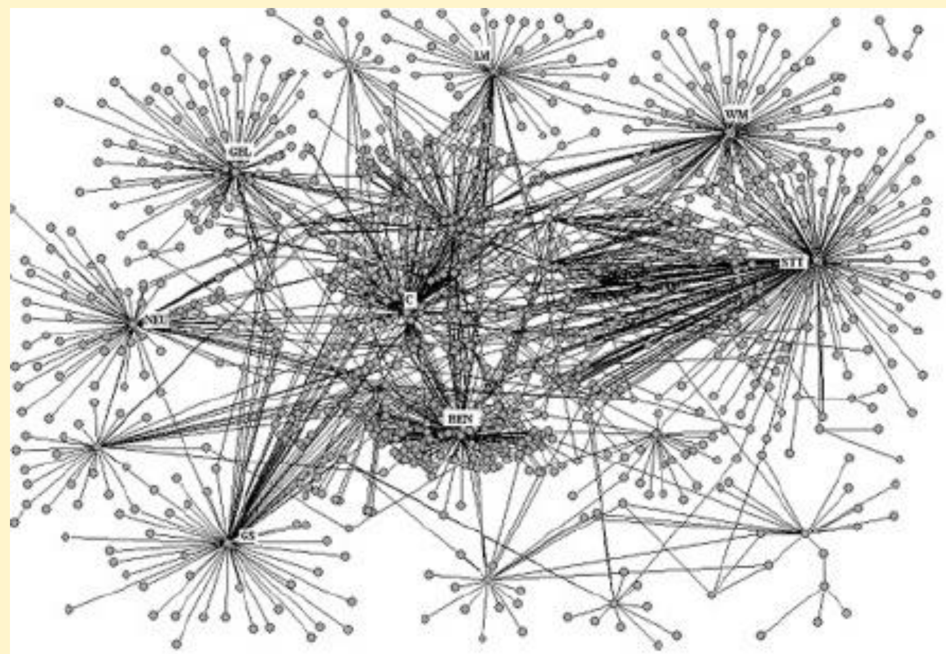
Scale-Free Networks





Power-Law Degree Distribution

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \approx 2m^2 k^{-3}$$



How to derive the power law?-An illustration

Let i be a node, which was added to the network at time t_i. Suppose this node has degree k when it is being picked up at time $t \geq t_i$.

Imaging that k is a continuous variable, so that probability $\Pi(k_i) = k_i / \sum_j k_j$ can be viewed as a continuous rate of change of k_i therefore

$$\frac{\partial k_i}{\partial t} = a\Pi(k_i) = ak_i / \sum_j k_j$$

Since the new node brings m edges in, the change of connectivity at each step is m, one has $a = m$. Also, $\sum_j k_j = 2mt$ so

$$\frac{\partial k_i}{\partial t} = \frac{mk_i}{2mt} = \frac{k_i}{2t}$$

Solving this equation, with the initial condition that node i was added to the network at t_i with connectivity $k_i(t_i) = m$ yields $k_i(t) = m\sqrt{\frac{t}{t_i}}$ namely $t_i = \frac{m^2 t}{k_i^2}$

On the other hand, it follows from $t_i = \frac{m^2 t}{k_i^2}$ that

$$P(k_i(t) < k) = P\left(t_i > \frac{m^2 t}{k^2}\right)$$

Assuming the time intervals $\{t_i\}$ are equally distributed, one has $P(t_i) = \frac{1}{t+m_0}$ so that

$$P\left(t_i > \frac{m^2 t}{k^2}\right) = 1 - P\left(t_i \leq \frac{m^2 t}{k^2}\right) = 1 - \frac{m^2 t}{k^2} \cdot P(t_i) = 1 - \frac{m^2 t}{k^2 (t+m_0)}$$

Consequently

$$P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = 2\frac{m^2 t}{t+m_0} \cdot k^{-3} \approx 2m^2 k^{-3}$$

This is a power law of the form $c \cdot k^{-\gamma}$ with $\gamma = 3$

Power-Law Distribution: Scale-Free

Consider a probability distribution function $f\,(x)$.

If, for any given constant a, there is a constant b such that the following "scale-free" property holds:

$$f\,(a\,x) = b\,f\,(x)$$

then, with the assumption that $f(1)f'(1)\neq 0$, the function $f\,(x)$ is uniquely determined by a power-law:

$$f\,(x) = f\,(1)x^{-\gamma} \qquad \gamma = -f\,'(1)\,/\,f\,(1)$$

BA Model: $P(k) = 2m^2 k^{-3}$

BA Scale-Free Network Model

Theorem 3-8~3-10:

Average Path Length:

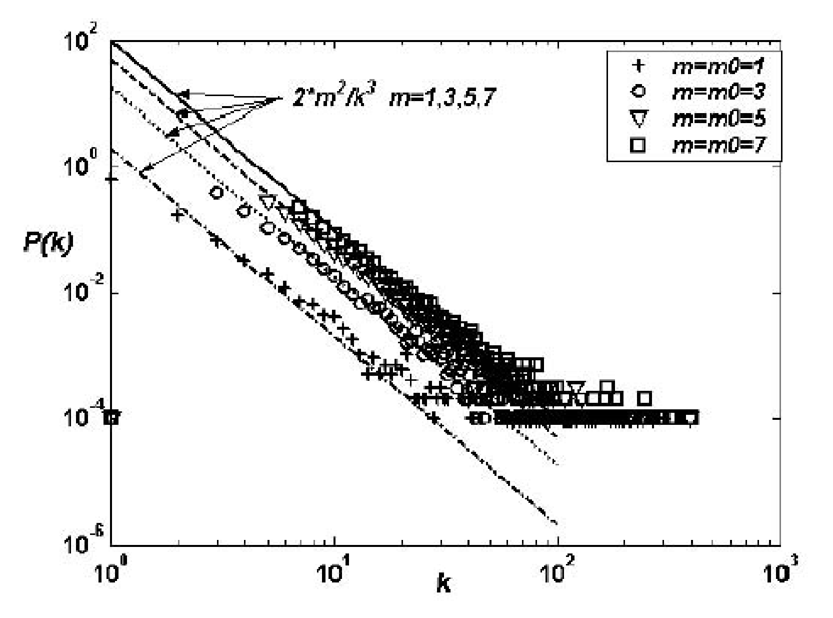
$$L \sim \frac{\ln N}{\ln \ln N}$$

Clustering Coefficient:

$$C = \frac{m^2(m+1)^2}{4(m-1)}\left(\ln\left(\frac{m+1}{m}\right) - \frac{1}{m+1}\right)\frac{(\ln t)^2}{t}$$

Degree Distribution:

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \approx 2m^2k^{-3}$$



Extended BA (EBA) Model

The BA Model can describe scale-free networks having power law with $r=3$, but many real-world networks do <u>not</u> satisfy $r=3$

EBA model (Albert and Barabasi, 2000)

Starting with a fully-connected network with $m_0 \geq 0$ nodes.

(i) Add new nodes (incremental growth):

With probability p, one new node is added into the network

(ii) Re-wiring:

With probability q, m $(m \leq m_0)$ edges are rewired

(ii) Add new edges (preferential attachment):

m $(m \leq m_0)$ new edges are added into the network with probability

$$\Pi(k_i) = \frac{k_i}{\sum_l k_l}$$

EBA Model

In this mode, $0 < p \leq 1$ and $0 \leq q < 1\text{-}p$

If $q < \min(1\text{-}p, (1\text{-}p+m)/(1+2m))$, then the degree distribution of nodes will be in a power-law form:

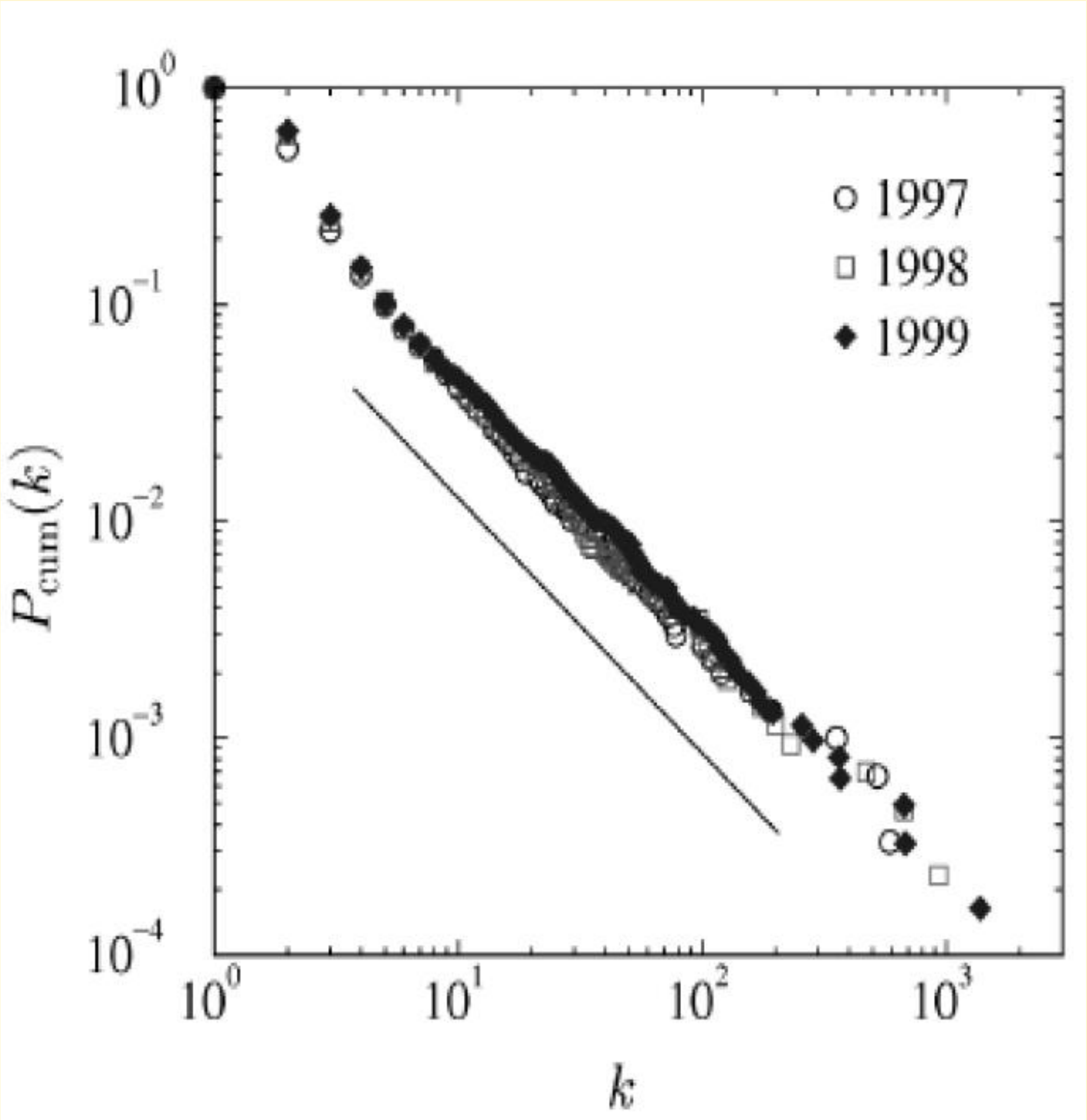
$$P(k) \sim (k + A(p,q,m) + 1)^{-\gamma}$$

where $\gamma = 1+B$ (typically, $2<\gamma<3$), and

$$A(p,q,m) = (p-q)\left(\frac{2m(1-q)}{1-p-q} + 1\right)$$

$$B(p,q,m) = \frac{2m(1-q) + 1 - p - q}{m}$$

Scale-Free Internet



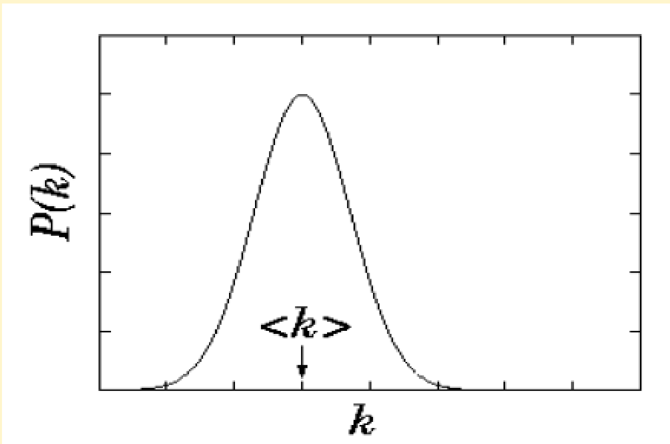
$$P(k) \sim k^{-\gamma}$$

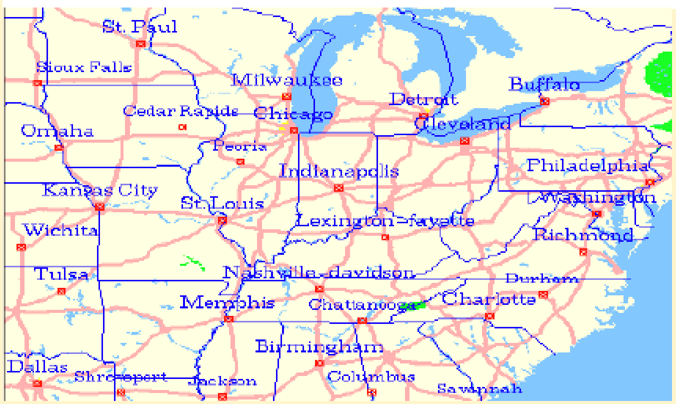
exponent $r = 2.2$

(at the Autonomous Systems level)

Road Map vs Airline Routing Map

Poisson distribution

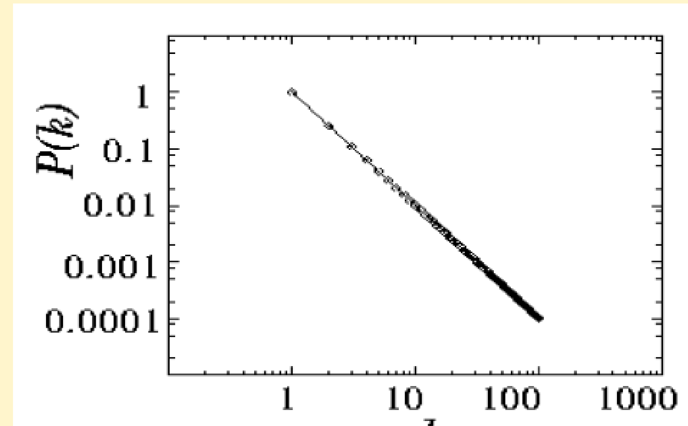


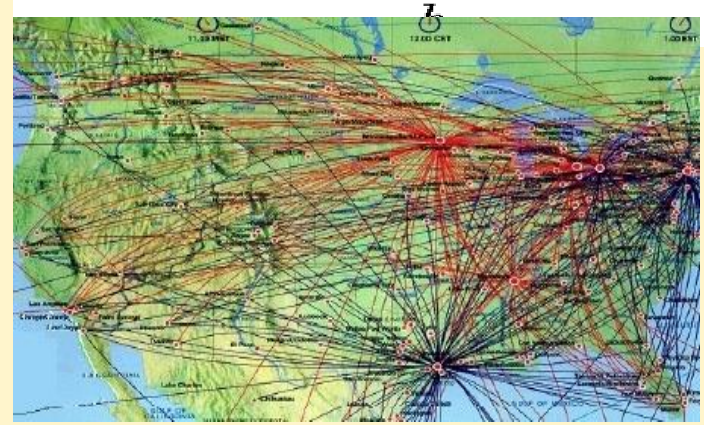


Small-world Network

(nodes: cities edges: highways)

Power-law distribution $P(k) \sim k^{-\gamma}$





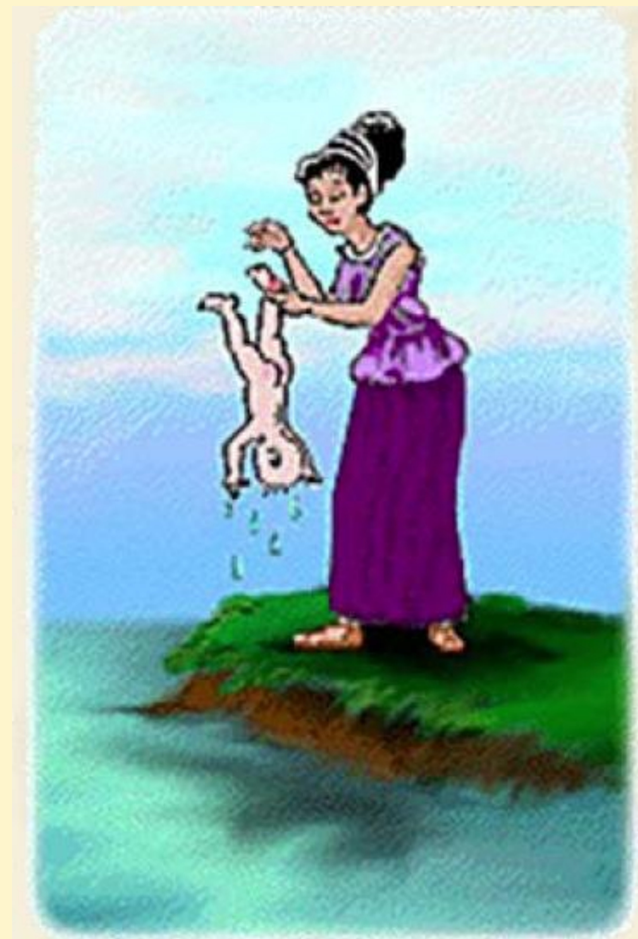
Scale-free Network

(nodes: airports edges: flights)

Robustness and Fragility of Scale-Free Networks

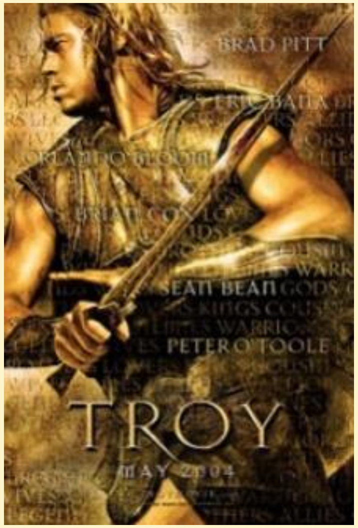
"Achilles' heel"

R. Albert, H. Jeong, A. L. Barabasi, Nature, 406, 387-482 (2000)



“Achilles’ heel”

Corfu, Greece



2004 film:
TROY

Trojan Horse

In the Greek mytholody, Achilles was the son of Thetis and Peleus, the bravest hero in the Trojan war. Stories told that when Achilles was born, his mother Thetis tried to make him immortal by dipping him in the river Styx. As she immersed him, she held him by one heel but forgot to dip him a second time to wet his other heel she held. Thus, as a result, the place where she held him remained untouched by the magic water of the Styx and that part stayed vulnerable. Achilles was killed in a battle by an enemy’s arrow his exactly at his that very heel. Today, any weak point of a strong entity is called an “Achilles’ heel”.

Achilles was invulnerable on all of his body except for his heel. So **“Achilles' heel”** or “Achilles' tendon” has come to mean a person's **principal weakness**.

Robustness versus Fragility

Illustration:

If the network remains being connected after some nodes have been removed, then the network is said to be robust against node-removal. Comparing two networks, when a certain number of nodes are removed, one network stays connected while another lost the connectivity, then the former is said to be more robust than the latter.

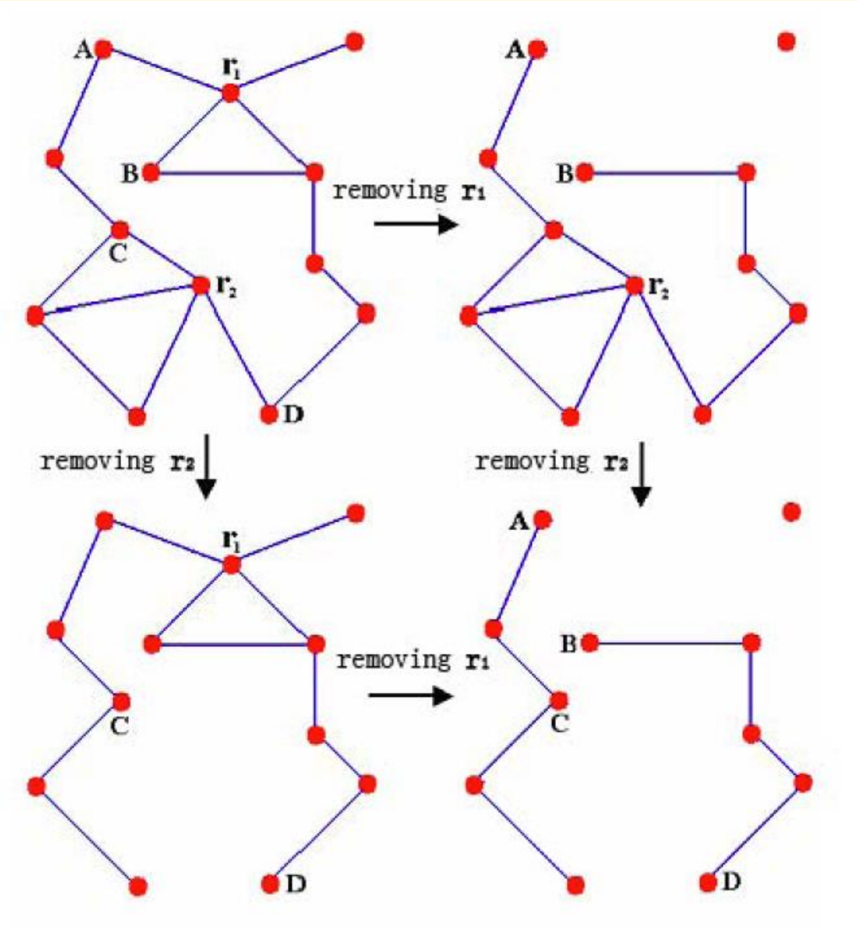
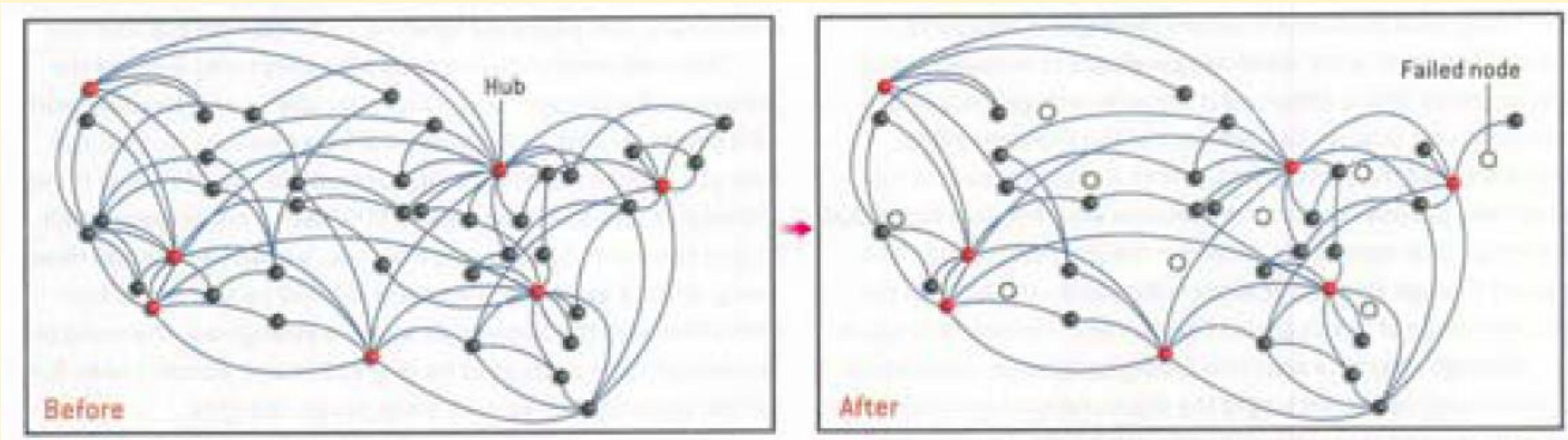
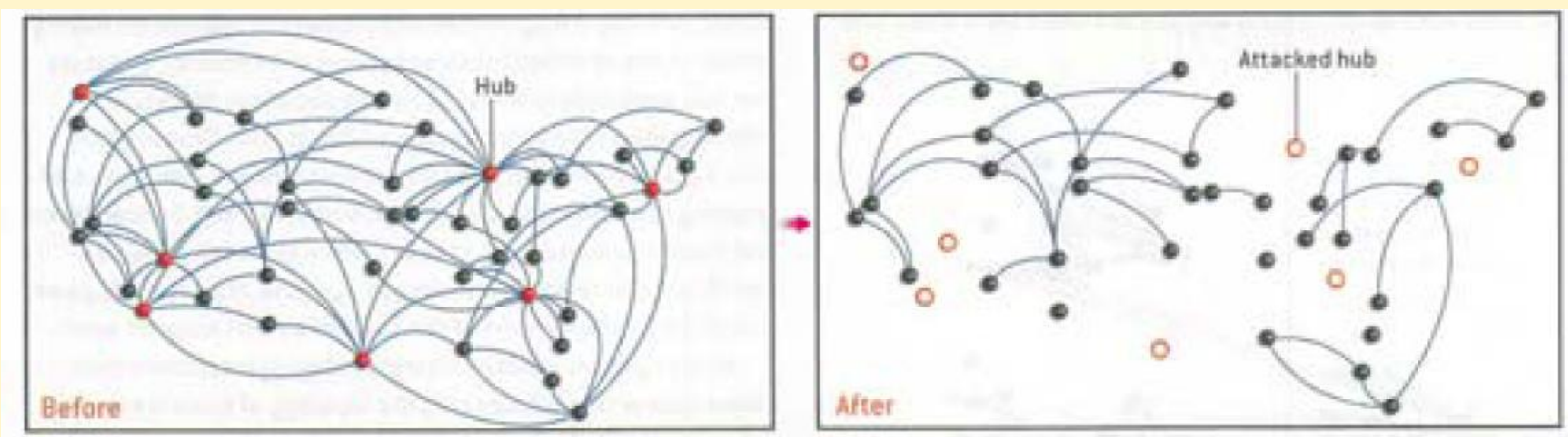


Illustration Example:

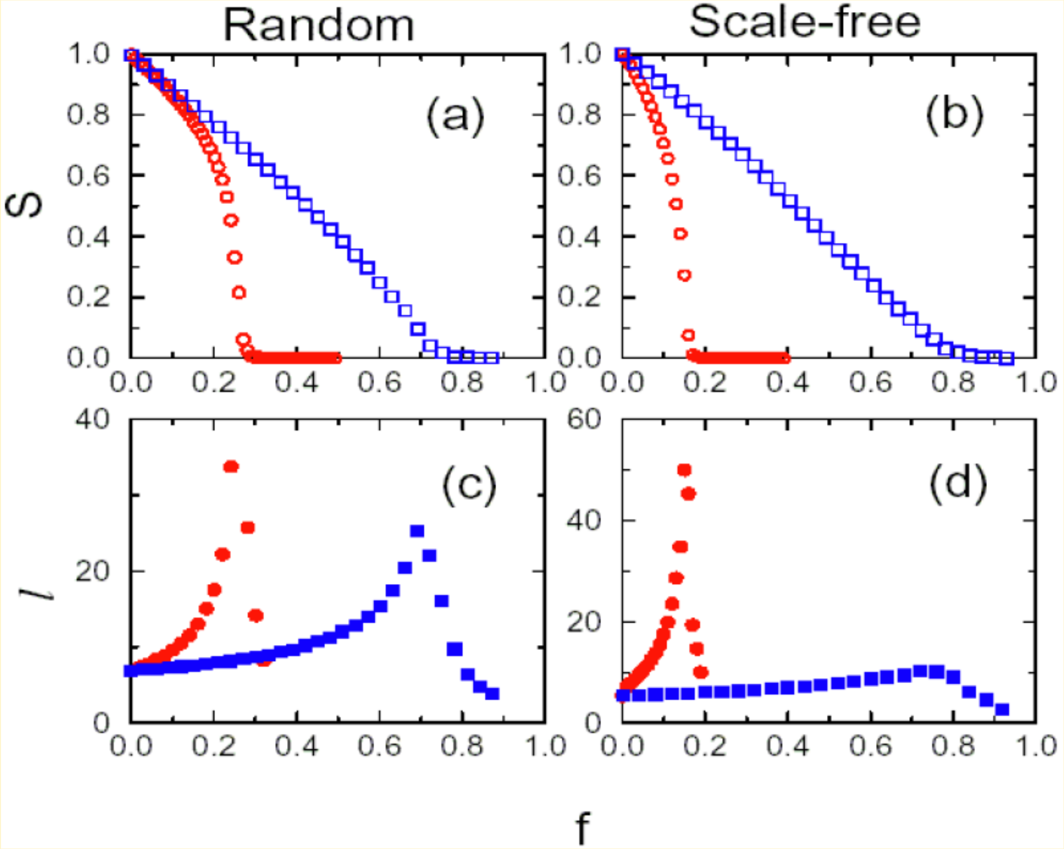


(a) Random failures



(b) Intentional attacks

Simulation Example:



S – size of the largest subgraph

l - average path length

f – fraction of nodes being removed or attacked

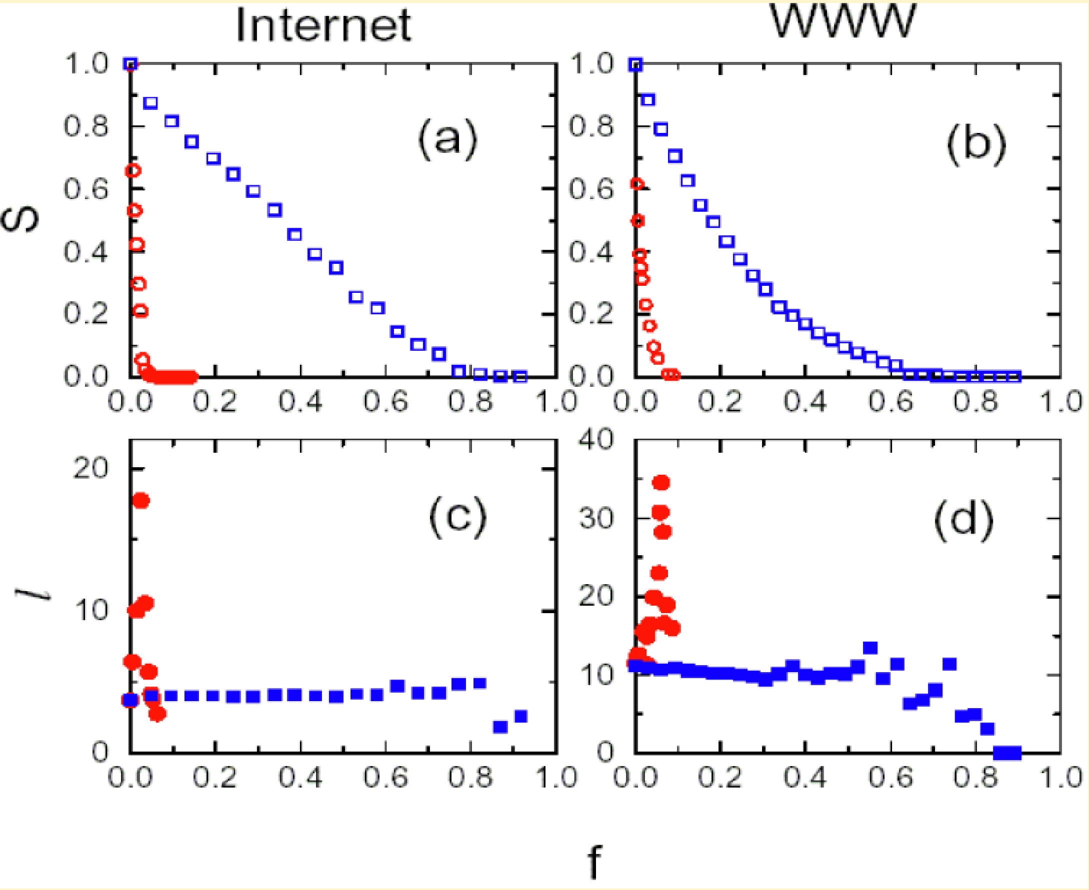
R. Albert, H. Jeong, A. L. Barabasi, Nature, 406, 387-482 (2000)

Robustness and fragility of ER random-graph and BA scale-free networks: (a) and (c): ER random-graph networks; (b) and (d): BA scale-free networks; squares—random removal of nodes; circles—intentional removal of nodes

Observations:

- The BA scale-free networks are very robust against random removal of nodes:
  - ✓ comparing to the ER random-graph network, the size *S* of the largest sub-network in a scale-free network decreases to zero slowly and for a much larger fraction f of the removal nodes;
  - ✓ yet its average path length grows also much slower.
- The above-described robustness of scale-free networks against random removal of nodes is due to the heterogeneous distribution of nodes in the network:
  - ✓ most nodes have very small degrees and only a few nodes have large degrees; thus, randomly removing a fraction of nodes will very likely remove some small nodes, which does not affect the network connectivity by too much.
  - ✓ However, for exactly the same reason, any intentional removal of even a very small fraction of high-degree nodes will significantly affect the topology of the networks, leading to drastic change of the network connectivity.

Internet and WWW: Simulation Results



S – size of the largest subgraph

l - average path length

f – fraction of attacks

R. Albert, H. Jeong, A. L. Barabasi, Nature, 406, 387-482 (2000)

Robustness and fragility of the Internet and WWW against intentional attacks: (a) and (c): Internet; (b) and (d): WWW; squares—random failures; circles—intentional attacks

Observations from Internet and WWW

- The Internet provides a typical example of the robust yet fragile phenomenon.
  - ✓ The Internet has become an indispensable part of human life today. However, the Internet also faces with various failures and attacks everyday. Therefore, it is an extremely important issue to guarantee the robustness of the Internet against both random failures and intentional attacks, leaving a challenging issue for good solutions.
- Broder studied the WWW and found that only when all nodes with degree larger than 5 are removed the WWW can be disconnected, meaning that the WWW is quite robust against intentional attacks.
  - ✓ Since the WWW is so large in size, the total number of degree – 5 nodes is still a relatively small fraction of the entire huge network.