Student Name:

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答题内容写在边线外视为无效

Major:

哈尔滨工业大学深圳研究生院

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HIT Shenzhen Graduate School Examination Paper

Course Name: 组合数学 **Lecturer:** 黄荷姣

Question	One	Two	Three	Four	Five	Six	Seven	Eight	Nine	Ten	Total
Mark											

The following formula and results may be useful for you in this examination.

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\frac{1}{2}(e^x+e^{-x})$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$$

$$e^{ax}=\sum_{n=0}^{\infty}a^n\frac{x^n}{n!}$$

The number of ways to place n non-attacking, indistinguishable rooks on an n-by-n board with forbidden positions equals $n! - r_1(n-1)! + r_2(n-2)! - \ldots + (-1)^k r_k(n-k)! + \ldots + (-1)^r r_n$.

For a specified grid *i* in chessboard C, let C_i be a chessboard induced from C by deleting the row and the column of grid *i*; let C_e be induced from C by deleting grid *i* from C. Then, $r_k(C)=r_{k-1}(C_i)+r_k(C_e)$, $R(C)=xR(C_i)+R(C_e)$.

Assume that the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ has a difference table whose 0^{th} diagonal equals, $c_0, c_1, c_2, \ldots, c_p \neq 0, 0, 0, \ldots$ Then, $h_n = c_0 C(n, 0) + c_1 C(n, 1) + c_2 C(n, 2) + \ldots + c_p C(n, p)$. and

$$\sum_{k=0}^{n}h_k=c_0\binom{n+1}{1}+c_1\binom{n+1}{2}+\cdots+c_p\binom{n+1}{p+1}.$$

Let $n \geq 2$ be an integer. If there exist n-1 MOLS of order n, then there exists a resolvable BIBD with parameters $b = n^2+n$, $v = n^2$, $k = n$, $r = n+1$, $\lambda = 1$.

$$N(G,C)=\frac{1}{|G|}\sum_{f\in G}|C(f)|.$$

$$P_G(z_1,z_2,\cdots,z_n)=\frac{1}{|G|}\sum_{f\in G}z_1^{e_1}z_2^{e_2}\cdots z_n^{e_n}$$, where type(f) = $(e_1, e_2, \ldots, e_n)$.

Question One: Please give the answers for the following 18 questions **without explanation.**

1. (3') A football team of 11 players is to be selected from a set of players, 5 of whom can play only in the backfield, 8 of them can only play on the line, 2 of them can play either in the backfield, or on the line. Assuming a football team has 7 men on the line and 4 men in the backfield, the number of possible football teams is ( ).

2. (3') Determine an integer m_n such that if m_n points are chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{n}$. m_n = ( )

3. (3') Consider the permutation of {1,2,3,4,5,6,7,8}. The inversion sequence of 83476215 is ( ); The permutation with an inversion sequence 66142100 is ( ).

4. (3') In which position does the combination 1289 occur in the lexicographic order of the 4-combinations of {1, 2, 3, 4, 5, 6, 7, 8, 9}? ( ).

5. (3') Among the combination of $\{x_7, x_6, \ldots, x_1, x_0\}$, what is the combination that immediately follow $\{x_7, x_5, x_4, x_3, x_2, x_1, x_0\}$ by using the base 2 arithmetic generating scheme? ( )

6. (3') In how many ways can six women, eight men and a dog sit around a table such that no two women sit next to each other?

7. (3') How many circular permutations are there of the multiset {3 a, 4 b, 2 c, 1 d}, where for each type of letters, all letters of that type do not appear consecutively.

8. (3') Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, green, and black so that no two squares that are colored red are adjacent. Then, the recurrence relation for h_n is ( ).

9. (3') Determine the generating function for the number h_n of non-negative integral solution of $3e_1+5e_2+e_3+6e_4=n$.

10. (3') Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, blue, white, and green, so that the number of squares colored red is even, and the number of squares colored white is odd. Then h_n = (   ).

11. (3') Let the sequence $h_0, h_1, h_2, \ \ldots h_n \ldots$ be defined by $h_n = 2n^2 - n + 3, (n \geq 0)$.

Then $\sum_{k=0}^{n} h_k$ = (   ).

12. (3') Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$$

Let c=(R, B, B, R, R, R) be a coloring of 1, 2, 3, 4, 5, 6 with the colors R and B. Then $(g \circ f) * c$ = (   ) and $(f \circ g) * c$ = (   )

13. (3') Suppose 20 indistinguishable balls are allocated to 5 persons so that everyone has no less than 3 balls. Then the number of ways is (   ). If the balls are distinguishable, then the number of ways is (   ).

14. (3') In how many ways can 5 red rooks and 3 blue rooks can be placed on 10×10 board so that no two rooks can attack one another, where the first line and the first column should not be empty.

15. (3') There is a party with 4 couples. In how many ways can they choose their partner, where each pair of partners is not couples?

16. (3') Count the permutations $i_1, i_2, i_3, i_4, i_5, i_6$ of {1,2,3,4,5,6} where $i_2 \neq 1,4$; $i_3 \neq 2,3,5$, $i_4 \neq 3,4$, $i_6 \neq 5,6$

17. (3') Apply the deferred acceptance algorithm to obtain a stable complete marriage for the following preferential ranking matrix

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1,3 & 2,3 & 3,2 & 4,3 \\ 1,4 & 4,1 & 3,3 & 2,2 \\ 2,2 & 1,4 & 3,4 & 4,1 \\ 4,1 & 2,2 & 3,1 & 1,4 \end{bmatrix} \end{array}$$

18. (3') Complete the following semi-Latin square or order 5.

$$\begin{pmatrix} \mathbf{1} & & & & \mathbf{2} \\ & \mathbf{2} & \mathbf{1} & & \\ & \mathbf{1} & & \mathbf{2} & \\ \mathbf{2} & & & \mathbf{1} & \\ & & \mathbf{2} & & \mathbf{1} \end{pmatrix}$$

Note: Please give a detailed explanation for the following Five problems.

Question One (6'): Prove that, among any 50 points in a square of size 7cm × 7cm, there are two points whose distance apart is at most $\sqrt{2}$cm.

Question Two (10’): Let $A=\{A_1,A_2,A_3,A_4,A_5,A_6\}$, where $A_1=\{a,b,c\}$, $A_2=\{a,b,c,d,e\}$, $A_3=\{a,b\}$, $A_4=\{b,c\}$, $A_5=\{a\}$, $A_6=\{a,c,e\}$. Does A has SDRs ? Why? What is the largest number of sets of A which can be chosen such that they have an SDR?

Question Three (10’): Let p be a prime number. Determine the number of different necklaces that can be made from p beads of n different colors. Specially, how many different necklaces that are made from 5 beads of 2 red, 1 yellow and 2 white colors?

Question Four (10'): Suppose there are 16 varieties of products which need to be tested by 20 consumers. The test is to have a property that each pair of the 16 varieties is compared by exactly one person. Each consumer is responsible for 4 products and each product is tested 5 times. Please give a proper solution.

Answer:

Let b: the number of consumers.

v: the number of varieties.

k: the number of varieties tested by each consumer.

r: the number of consumers containing each variety.

λ : the number of consumers containing each pair of varieties.

So b=20, v=16, k=4, r=5, λ =1. To construct a prober solution with parameters:

$$b=n^2+n, v=n^2, k=n, r=n+1, \lambda=1$$

n=4.

Let A_1, A_2, A_3 denote three MOLS of order four.

$$A_1=\begin{bmatrix} 0 & 1 & i & 1+i \\ 1 & 0 & 1+i & i \\ i & 1+i & 0 & 1 \\ 1+i & i & 1 & 0 \end{bmatrix} \qquad A_2=\begin{bmatrix} 0 & 1 & i & 1+i \\ i & 1+i & 0 & 1 \\ 1+i & i & 1 & 0 \\ 1 & 0 & 1+i & i \end{bmatrix} \qquad A_3=\begin{bmatrix} 0 & 1 & i & 1+i \\ 1+i & i & 1 & 0 \\ 1 & 0 & 1+i & i \\ i & 1+i & 0 & 1 \end{bmatrix}$$

$$R_4=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ i & i & i & i \\ 1+i & 1+i & 1+i & 1+i \end{bmatrix} \qquad S_4=\begin{bmatrix} 0 & 1 & i & 1+i \\ 0 & 1 & i & 1+i \\ 0 & 1 & i & 1+i \\ 0 & 1 & i & 1+i \end{bmatrix}$$

The varieties are 5 positions of a 5-by-5 array, and the result is pictured as follows:

$$\begin{bmatrix} R_4(0) & R_4(1) & R_4(i) & R_4(1+i) \\ S_4(0) & S_4(1) & S_4(i) & S_4(1+i) \\ A_1(0) & A_1(1) & A_1(i) & A_1(1+i) \\ A_2(0) & A_2(1) & A_2(i) & A_2(1+i) \\ A_3(0) & A_3(1) & A_3(i) & A_3(1+i) \end{bmatrix}$$

Hence the proper solution is:

$B_1=\{0,1,2,3\}, B_2=\{4,5,6,7\}, B_3=\{8,9,10,11\}$

$B_4=\{12,13,14,15\}, B_5=\{0,4,8,12\}, \ldots, B_9=\{0,5,10,15\}, \ldots,$

$B_{13}=\{0,6,11,13\}, \ldots, B_{20}=\{3,4,10,13\}$

Question Five (10'): Determine the number of integral solutions of the following equation

$$x_1+x_2+\ldots+x_5=18$$

which satisfy $0 \le x_1 \le 3$, $3 \le x_2 \le 10$, $2 \le x_3 \le 10$, $1 \le x_4 \le 3$, $5 \le x_5 \le 8$.

Answer:

We introduce new variables

$$y_1=x_1, y_2=x_2-3, y_3=x_3-2, y_4=x_4-1, y_5=x_5-5$$

And our equation becomes

$$y_1+y_2+y_3+y_4+y_5=7$$

The inequalities on the $x_i's$ are satisfied if and only if

$$0 \le y_1 \le 3, 0 \le y_2 \le 7, 0 \le y_3 \le 8, 0 \le y_4 \le 2, 0 \le y_5 \le 3$$

Let S be the set of all nonnegative integral solutions of $y_1 + y_2 + y_3 + y_4 + y_5 = 7$,the size of S is

$$|S| = \binom{5+7-1}{7} = 330$$

Let P_1 be the property that $y_1 \geq 4$, P_2 be the property that $y_2 \geq 8$, P_3 be the property that $y_3 \geq 9$, P_4 be the property that $y_4 \geq 3$, P_5 be the property that $y_5 \geq 4$ .Let A_i denote the subset of S consisting of the solutions satisfying property P_i ,(i=1,2,3,4,5).We wish to evaluate the size of the set $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \bar{A}_5$,and we do so by applying the inclusion-exclusion principle. The set A_1 consist of all those solutions in S for which $y_1 \geq 4$ .Performing a change in variable ($z_1 = y_1 - 4, z_2 = y_{2,} z_3 = y_3, z_4 = y_4$), we see that the number of solutions in A_1 is the same as the number of nonnegative integral solutions of

$$z_1 + z_2 + z_3 + z_4 + z_5 = 3$$

Hence,

$$|A_1| = \binom{3+5-1}{3} = 35$$

In a similar way, we obtain

$$|A_2| = 0 , |A_3| = 0 , |A_4| = \binom{4+5-1}{4} = 70 , |A_5| = \binom{3+5-1}{3} = 35$$

The set $A_1 \cap A_2$ consists of all those solutions in S for which $y_1 \geq 4$ and $y_2 \geq 8$.

In a similar way, we obtain

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_1 \cap A_5| = \ldots = |A_3 \cap A_5| = 0$$
$$|A_1 \cap A_4| = |A_4 \cap A_5| = 1$$

Hence,

$$\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \bar{A}_5\right| = 330 - 35 - 35 - 70 + 2 = 192$$