第三次作业反馈

第三次作业参考

> 3. Find the number of integers between 1 and 10,000 that are neither perfect squares nor perfect cubes.

Using the principle of inclusion-exclusion, this can be written as:

$$|\overline{A_1} \cap \overline{A_2}| = |U| - |A_1 \cup A_2|$$

where:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

The universal set U is the set of all integers from 1 to 10,000. ｜U｜=10000

Perfect squares are of the form n^2, where $n^2 \leq 10,000$.

$$n \leq \sqrt{10,000} = 100$$ ｜A1｜=100

Perfect cubes are of the form n^3, where $n^3 \leq 10,000$.

$$n \leq \sqrt[3]{10,000} \approx 21.5$$ ｜A2｜=21

$\sqrt[6]{10,000} \approx 4.64$, so ｜A1∩A2｜=4

$|\overline{A_1} \cap \overline{A_2}| =$ ｜U｜ - (｜A1｜+｜A2｜) +｜A1∩A2｜

= 10000 – (100 + 21) + 4 = 9883

8. Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 14$ in positive integers x_1, x_2, x_3, x_4 and x_5 not exceeding 5.

Let y_i = x_i -1, i =1,2,3,4,5.

The equation is changed to $y_1 + y_2 + y_3 + y_4 + y_5 = 14 - 5 = 9, 0 \leq y_i \leq 4$.

Let S be the set of all non-negative interger solutions of the above equation, $0 \leq y_i$.

Define A_i as the cases that $y_i \geq 5$.

$|S| = C^9_{9+5-1} = C^9_{13}$

For $|A_i|$, let $z_i = y_i - 5, z_j = y_j (j = 1, 2, 3, 4, 5 \wedge j \neq i)$.

$\sum z_i = 9 - 5 = 4$. Similarly, $|A_i| = C^4_{4+5-1} = C^4_8$

The intersection between more than two (inclusive) A_i is the empty set, because the sum of the corresponding y_i is already greater than 9 at this time, and the equation has no non-negative integer solution.

According to the inclusion-exclusion principle,

$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}| = |S| - \sum |A_i| + ... = C^9_{13} - C^1_5 * C^4_8 = 365$

11. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which no even integer is in its natural position.

Define Aias the set of permutations where the even number 2iappears in its natural position.
For example, A1represents the set where 2 is in position 2.

We need to find:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$$

By the principle of inclusion-exclusion:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |U| - |A_1 \cup A_2 \cup A_3 \cup A_4|$$

For $|A_1 \cup A_2 \cup A_3 \cup A_4|$, we expand using inclusion-exclusion:

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{i=1}^{4} |A_i| - \sum_{1 \leq i < j \leq 4} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 4} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

If the even number 2is fixed in its natural position, the other 7 elements can be permuted freely: ｜Ai｜=7!

$$\sum_{i=1}^{4} |A_i| = 4 \times 7! = 4 \times 5040 = 20160$$

If both 2iand 2j are fixed in their natural positions, the remaining 6 elements can be permuted:
｜Ai∩Aj｜=6!

Since there are $\binom{4}{2} = 6$ pairs of even numbers:

$$\sum_{1\le i<j\le 4} |A_i \cap A_j| = 6 \times 6! = 6 \times 720 = 4320$$

If 2i, 2j and 2k are fixed in their natural positions, the remaining 5 elements can be permuted: | Ai∩Aj∩Ak | =5!

Since there are $\binom{4}{3} = 4$ triples of even numbers:

$$\sum_{1\le i<j<k\le 4} |A_i \cap A_j \cap A_k| = 4 \times 5! = 4 \times 120 = 480$$

If all 4 even numbers are fixed in their natural positions, the remaining 4 elements can be permuted: | A1∩A2∩A3∩A4 | =4!=24

Using the inclusion-exclusion formula:

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{i=1}^{4} |A_i| - \sum_{1\le i<j\le 4} |A_i \cap A_j| + \sum_{1\le i<j<k\le 4} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Substitute the values:

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 20160 - 4320 + 480 - 24 = 16296$$

Finally:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |U| - |A_1 \cup A_2 \cup A_3 \cup A_4|$$

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 40320 - 16296 = 24024$$

25.

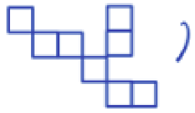
Count the permutations $i_1i_2i_3i_4i_5i_6$ of $\{1,2,3,4,5,6\}$, where $i_1 \neq 1,5$; $i_3 \neq 2,3,5$; $i_4 \neq 4$; and $i_6 \neq 5,6$.

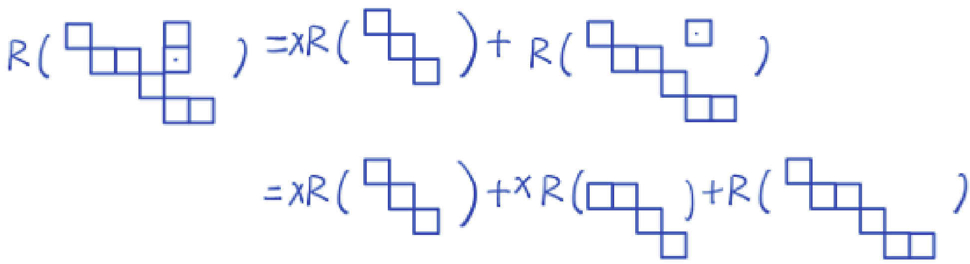
resolution：

Answer

Use chessboard Polynomial. Draw a 6-by-6 board with forbidden positions as shown.

	1	2	3	4	5	6
1	X				X	
2		X	X		X	
3						
4				X		
5						
6					X	X

then calculate R()



$= x(1+x)^3 + x(1+2x)(1+x)^2 + (1+x)^2(1+2x)^2$

$= 1+8x+20x^2+20x^3+7x^4$

$R_6(C) = 6! - 8\times5! + 20\times4! - 20\times3! + 7\times2!$

$= 134$

∴ The result is 134

31.

How many circular permutations are there of the multiset

$$\{2\cdot a, 3\cdot b, 4\cdot c, 5\cdot d\},$$

where, for each type of letter, all letters of that type do not appear consecutively?

resolution：

Answer.

Use inclusion and exclusion principle.

Let S be all circular permutations of the set.

$$|S| = \frac{(2+3+4+5-1)!}{2!\,3!\,4!\,5!}$$

A_i $(i=1,2,3,4)$ is subset of S, and in A_1 all the 'a's appear consecutively, in A_2 all the 'b's appear consecutively, in A_3 all the 'c's appear consecutively, in A_4 all the 'd's appear consecutively.

$$|A_1| = \frac{(1+3+4+5-1)!}{1!\,3!\,4!\,5!} = \frac{12!}{3!\,4!\,5!}$$

$$|A_2| = \frac{(2+1+4+5-1)!}{2!\,1!\,4!\,5!} = \frac{11!}{2!\,4!\,5!}$$

$$|A_3| = \frac{(2+3+1+5-1)!}{2!\,3!\,1!\,5!} = \frac{10!}{2!\,3!\,5!}$$

$$|A_4| = \frac{(2+3+4+1-1)!}{2!\,3!\,4!\,1!} = \frac{9!}{2!\,3!\,4!}$$

$$|A_1 \cap A_2| = \frac{(1+1+4+5-1)!}{1!\,1!\,4!\,5!} = \frac{10!}{4!\,5!}$$

$$|A_1 \cap A_3| = \frac{(1+3+1+5-1)!}{1!\,3!\,1!\,5!} = \frac{9!}{3!\,5!}$$

$$|A_1 \cap A_4| = \frac{(1+3+4+1-1)!}{1!\,3!\,4!\,1!} = \frac{8!}{3!\,4!}$$

$$|A_2 \cap A_3| = \frac{(2+1+1+5-1)!}{2!\,1!\,1!\,5!} = \frac{8!}{2!\,5!}$$

$$|A_2 \cap A_4| = \frac{(2+1+4+1-1)!}{2!\,1!\,4!\,1!} = \frac{7!}{2!\,4!}$$

$$|A_3 \cap A_4| = \frac{(2+3+1+1-1)!}{2!\,3!\,1!\,1!} = \frac{6!}{2!\,3!}$$

$$|A_1 \cap A_2 \cap A_3| = \frac{(1+1+1+5-1)!}{1!\,1!\,1!\,5!} = \frac{7!}{5!}$$

$$|A_1 \cap A_2 \cap A_4| = \frac{(1+1+4+1-1)!}{1!\,1!\,4!\,1!} = \frac{6!}{4!}$$

$$|A_1 \cap A_3 \cap A_4| = \frac{(1+3+1+1-1)!}{1!\,3!\,1!\,1!} = \frac{5!}{3!}$$

$$|A_2 \cap A_3 \cap A_4| = \frac{(2+1+1+1-1)!}{2!\,1!\,1!\,1!} = \frac{4!}{2!}$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = \frac{4!}{4} = 3!$$

According to IEP

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |S| - \sum |A_i| + \sum_{1 \le i < j \le 4} |A_i \cap A_j| - \sum_{1 \le i < j < k \le 4} |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= 144029$$

$\therefore$ The result is 144029