

## Lecture7-homework

9. Let  $h_n$  equal the number of different ways in which the squares of a  $1$ -by- $n$  chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that  $h_n$  satisfies. Then find a formula for  $h_n$ .

**Answer:**

We have  $h_0=1$  and  $h_1=3$ . If the first square is white or blue, then the chessboard can be completed in  $h_{n-1}$  ways. If the first square is red, then second square should be white or blue. hence,  
 $h_n = 2 h_{n-1} + 2h_{n-2}$  ( $n \geq 2$ ).

The characteristic equation of the recurrence relation is  $x^2 - 2x - 2 = 0$  and its two characteristic roots are  $1+\sqrt{3}, 1-\sqrt{3}$ . By Th.7.2.1, the general solution is:  $h_n = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$  ( $n=0,1,2,\dots$ )

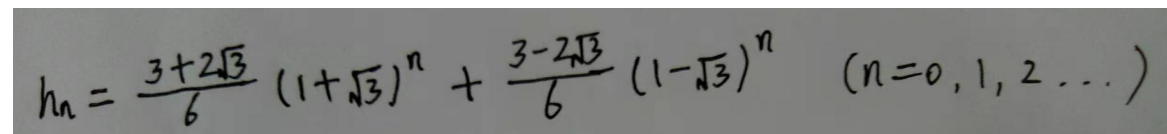
The general solution satisfies both the recurrence relation and the initial conditions. Setting  $n = 0, 1$  we can find:

$$c_1 + c_2 = 1$$

$$c_1(1+\sqrt{3}) + c_2(1-\sqrt{3}) = 3$$

The we can get:  $c_1 = (3+2\sqrt{3})/6$ ,  $c_2 = (3-2\sqrt{3})/6$ .

Therefore:



$$h_n = \frac{3+2\sqrt{3}}{6} (1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6} (1-\sqrt{3})^n \quad (n=0, 1, 2, \dots)$$

16. Formulate a combinatorial problem for which the generating function is

$$(1+x+x^2)(1+x^2+x^4+x^6)(1+x^2+x^4+\dots)(x+x^2+x^3+\dots).$$

**Answer:**

This is the generating function for the sequence  $\{h_n\}$  where  $h_n$  is the number of  $n$ -combinations of the multiset  $\{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}$ , such that (i)  $a$  appears at most 2; (ii)  $b$  is even and at most 6; (iii)  $c$  is even; (iv)  $d$  is nonzero.

25. Let  $h_n$  denote the number of ways to color the squares of a  $1$ -by- $n$  board with the colors red, white, blue, and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence  $h_0, h_1, \dots, h_n, \dots$ , and then find a simple formula for  $h_n$ .

**Answer:**

For an integer  $n \geq 0$ ,  $h_n$  is equal to the number of  $n$ -permutations of the multiset  $\{\infty \cdot R, \infty \cdot W, \infty \cdot B, \infty \cdot G\}$  such that both (i)  $R$  appears an even number of times; (ii)  $W$  appears an odd number of times. The exponential generating function is  $g^e(x) = G_1(x)G_2(x)G_3(x)G_4(x)$ , where

$$\begin{aligned}
 G_1(x) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \frac{e^x + e^{-x}}{2} \\
 G_2(x) &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \frac{e^x - e^{-x}}{2} \\
 G_3(x) &= G_4(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x
 \end{aligned}$$

Using this we obtain :

$$g^e(x) = \frac{e^{4x} - 1}{4} = x + \frac{4x^2}{2!} + \frac{4^2 x^3}{3!} + \dots$$

Therefore  $h_n = 4^{n-1}$  if  $n \geq 1$  and  $h_0 = 0$ .

48. Solve the following recurrence relations by using the method of generating functions as described in Section 7.4:

(b)  $h_n = h_{n-1} + h_{n-2}$ , ( $n \geq 2$ );  $h_0 = 1$ ,  $h_1 = 3$

**Answer:**

The characteristic equation of the recurrence relation is  $x^2 - x - 1 = 0$  and its two characteristic roots are  $(1+\sqrt{5})/2$ ,  $(1-\sqrt{5})/2$ . By Th.7.2.1, the general solution is:

$$h_n = c_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + c_2 \left( \frac{1-\sqrt{5}}{2} \right)^n \quad (n=0, 1, 2, \dots)$$

The general solution satisfies both the recurrence relation and the initial conditions. Setting  $n = 0, 1$  we can find:

$$\begin{aligned}
 c_1 + c_2 &= 1 \\
 \frac{1+\sqrt{5}}{2} c_1 + \frac{1-\sqrt{5}}{2} c_2 &= 3
 \end{aligned}$$

The we can get:  $c_1 = (1+\sqrt{5})/2$ ,  $c_2 = (1-\sqrt{5})/2$ .

Therefore:

$$h_n = \left( \frac{1+\sqrt{5}}{2} \right)^{n+1} + \left( \frac{1-\sqrt{5}}{2} \right)^{n+1} \quad (n=0, 1, 2, \dots)$$