

# MEM6810 Engineering Systems Modeling and Simulation



## 工程系统建模与仿真

Theory Analysis

### Lecture 9: Output Analysis II: Comparison

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(Sino-US Global Logistics Institute)



- 1 Introduction
- 2 Comparison of Two Designs
  - ▶ Significant Difference
  - ▶ Independent Sampling
  - ▶ Common Random Numbers
- 3 Comparison of Multiple Designs
  - ▶ Bechhofer's Procedure
  - ▶ Paulson's Procedure
  - ▶ Ranking and Selection Review
  - ▶ Multi-Arm Bandit Problem

## 1 Introduction

## 2 Comparison of Two Designs

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- We now discuss how to compare two or more simulation models, i.e. to estimate their *relative performance*.
- Here, different simulation models may refer to different designs, operation policies, etc., of a simulated system; in this lecture we simply call them *different (system) designs*.
- It is one of the most important uses of simulation.

- **Key Question:** Are the observed differences due to
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  - the **actual differences** on the expected performance of system designs?
  - or the **random errors** in the simulation outputs?
- The comparison can be classified into two types:
  - Two system designs: using confidence interval of the difference.
  - Multiple (more than two) system designs: selection of the best.



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- Let  $\theta_1$  and  $\theta_2$  be the mean performance of the two system designs in simulation.
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System	Replication				Sample Mean	Sample Variance
	1	2	...	$R_i$		
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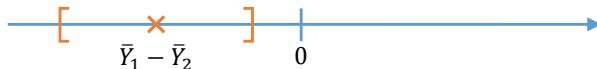
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- Approximate  $1 - \alpha$  CI:  $\bar{Y}_1 - \bar{Y}_2 \pm t_{v, 1-\alpha/2} \times \text{s.e.}(\bar{Y}_1 - \bar{Y}_2)$ .
  - $\text{s.e.}(\bar{Y}_1 - \bar{Y}_2)$  is the estimator of standard error of  $\bar{Y}_1 - \bar{Y}_2$ ; see more details about this quantity and  $v$  later.

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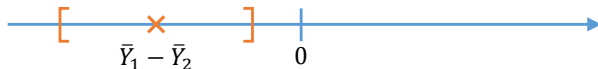
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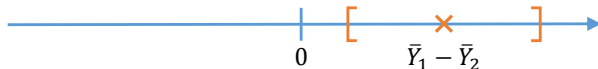


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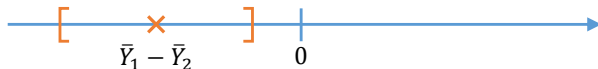


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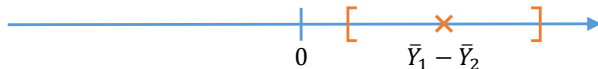


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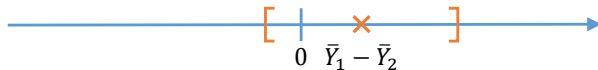
- Case 1 – Strong evidence that  $\theta_1 < \theta_2$ :



- Case 2 – Strong evidence that  $\theta_1 > \theta_2$ :



- Case 3 – No strong evidence that one is larger than the other:



- It does not imply  $\theta_1 = \theta_2$ !





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- The first two cases are conclusive.
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- The first two cases are conclusive.
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- We will shrink the CI until case 1 or 2 is achieved, or the confidence interval is so narrow, which suggests that we do not need to separate them.

- For the comparison of performance of two designs, there is an important distinction between
  - *statistically significant difference* (统计意义上的显著区别);
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- **Statistical** significance answers the following questions:
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  - Is the true difference  $|\theta_1 - \theta_2|$  large enough so it is worthwhile to separate them?

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- Confidence intervals do not answer the question of practical significance directly.
  - Instead, they bound, with probability  $1 - \alpha$ , the true difference  $\theta_1 - \theta_2$  within the range  $\bar{Y}_1 - \bar{Y}_2 \pm t_{v, 1-\alpha/2} \times \text{s.e.}(\bar{Y}_1 - \bar{Y}_2)$ .
  - Whether a difference within these bounds is practically significant depends on the particular problem.



- Independent sampling means that **different** random number streams are used to simulate the two systems.
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- Suppose  $\text{Var}(Y_{r1}) = \sigma_1^2$  and  $\text{Var}(Y_{r2}) = \sigma_2^2$ . Due to the independence,

$$\text{Var}(\bar{Y}_1 - \bar{Y}_2) = \text{Var}(\bar{Y}_1) + \text{Var}(\bar{Y}_2) = \frac{\sigma_1^2}{R_1} + \frac{\sigma_2^2}{R_2}.$$

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- The  $1 - \alpha$  CI is approximated by

$$\bar{Y}_1 - \bar{Y}_2 \pm t_{v, 1-\alpha/2} \times \text{s.e.}(\bar{Y}_1 - \bar{Y}_2). \quad (2)$$

where  $\text{s.e.}(\bar{Y}_1 - \bar{Y}_2)$  is given in (1), and the degree of freedom  $v$  is

$$v = \frac{[S_1^2/R_1 + S_2^2/R_2]^2}{[S_1^2/R_1]^2/(R_1 - 1) + [S_2^2/R_2]^2/(R_2 - 1)}.$$

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- The approximated CI (2) is called the *Welch confidence interval* (Welch 1938).
  - Sometimes, people will round  $v$  to integer for convenience.

- If  $R_1 = R_2 = R$ , or we are willing to discard some observations from the system design on which we actually have more data, we can pair  $Y_{r1}$  with  $Y_{r2}$  to define  $Z_r = Y_{r1} - Y_{r2}$ , for  $r = 1, \dots, R$ .



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- Point estimator of  $\theta_1 - \theta_2$ :  $\bar{Z} = \frac{1}{R} \sum_{r=1}^R Z_r = \bar{Y}_1 - \bar{Y}_2$ .

$$\begin{aligned}\text{Var}(\bar{Z}) &= \frac{\text{Var}(Z_r)}{R} = \frac{\text{Var}(Y_{r1} - Y_{r2})}{R} = \frac{\sigma_1^2 + \sigma_2^2}{R} \\ &= \text{Var}(\bar{Y}_1 - \bar{Y}_2) = \text{Var}(\bar{Y}_1) + \text{Var}(\bar{Y}_2) = \frac{\sigma_1^2 + \sigma_2^2}{R}.\end{aligned}\tag{3}$$

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- To estimate  $\text{Var}(Z_r)$ , instead of estimating  $\sigma_1^2$  and  $\sigma_2^2$  separately, we can directly use

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- Approximate  $1 - \alpha$  CI:

$$\bar{Z} \pm t_{R-1, 1-\alpha/2} \frac{S}{\sqrt{R}}.$$



- Common Random Numbers (CRN, also known as correlated sampling): For each replication, the same random numbers are used to simulate both systems.
  - For each replication  $r$ , the two estimates,  $Y_{r1}$  and  $Y_{r2}$ , are correlated.
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- The purpose of using CRN is to induce a **positive** correlation between  $Y_{r1}$  and  $Y_{r2}$  for each  $r$  and thus to achieve a variance reduction in the point estimator of  $\theta_1 - \theta_2$ ,  $\bar{Z}$ .

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- $\text{Var}(\bar{Z})$  in (6) is smaller than that in (3)  $\implies$  higher precision of point estimator.
- CI is still computed via (4) and (5), but the width will be smaller  $\implies$  higher precision.

- It is never enough to simply use the same seed for the random-number generator(s):
  - The random numbers must be synchronized: each random number used in one model for some purpose should be used for the same purpose in the other model.
  - E.g., if the  $i$ th random number is used to generate a service time at work station 2 for the 5th arrival in model 1, the  $i$ th random number should be used for the very same purpose in model 2.



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- The CRN idea is also used when we validate simulation model via input-output transformation, where we prefer to compare the model and actual system under the same historical input, rather than generate the input from input model.

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  - ② Comparison of each  $\theta_i$  to a control, say,  $\theta_1$  ( $\theta_1$  can represent the mean performance of an existing system).
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- The first three can be achieved by **simultaneous** construction of confidence intervals, whereas the last by some **selection approaches**.

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- Some possible goals:
  - ① Estimation of each parameter  $\theta_i$ .
  - ② Comparison of each  $\theta_i$  to a control, say,  $\theta_1$  ( $\theta_1$  can represent the mean performance of an existing system).
  - ③ All pairwise comparisons.
  - ④ Selection of the best  $\theta_i$  (largest or smallest).
- The first three can be achieved by **simultaneous** construction of confidence intervals, whereas the last by some **selection approaches**.
- From now on, without loss of generality, let's *assume the best  $\theta_i$  is the largest one*.

- Assumption 1: For each design  $i$  with mean performance  $\theta_i$ , the noisy output  $Y_{ri} \sim \mathcal{N}(\theta_i, \sigma_i^2)$ , for  $r = 1, 2, \dots$
- Assumption 2: No CRN is used, i.e.,  $Y_{ri}$  is independent of  $Y_{rj}$  for  $i \neq j$ .
- Assumption 3 (**indifference-zone**): The gap between the largest  $\theta_i$  and the second largest  $\theta_i$  is at least  $\delta$ , *a value known to us*.
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- Assumption 4 (known variance):  $\sigma_i^2$  is known, for  $i = 1, \dots, k$ .
- Bechhofer (1954) first developed a selection procedure, which can ensure the probability of correct selection (PCS):

$$\mathbb{P}\{\text{select the largest } \theta_i\} \geq 1 - \alpha, \quad (7)$$

under Assumptions 1-4, where  $\alpha$  is a user specified value and  $1 - \alpha > 1/k$ .



- Bechhofer's Procedure

- ① Calculate a constant  $h$ , which satisfies

$$\mathbb{P}\{Z_i \leq h, i = 1, 2, \dots, k-1\} = 1 - \alpha, \quad (8)$$

where  $(Z_1, Z_2, \dots, Z_{k-1})^\top$  has a multivariate normal distribution with means 0, variances 1, and common pairwise correlations  $1/2$ .

- ② For  $i = 1, \dots, k$ , let

$$n_i = \left\lceil \frac{2h^2 \sigma_i^2}{\delta^2} \right\rceil. \quad (9)$$

- ③ For  $i = 1, \dots, k$ , run  $n_i$  replications for design  $i$  and calculate

$$\bar{Y}_i = \frac{1}{n_i} \sum_{r=1}^{n_i} Y_{ri}.$$

- ④ Select the design with the largest sample mean  $\bar{Y}_i$  as the best.

Proof.

Without loss of generality, assume  $\theta_k \geq \theta_{k-1} \geq \cdots \geq \theta_1$ . Then Assumption 3 says,  $\theta_k - \theta_{k-1} \geq \delta$ , which implies that

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Proof. (Cont'd)

Now we only need to check that  $\mathbf{Z} = (Z_1, Z_2, \dots, Z_{k-1})^\top$  indeed has a multivariate normal distribution with means 0, variances 1, and common pairwise correlations  $1/2$  (except for some rounding error).

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and  $\mathbf{Y} = (\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_k)^\top$  is a  $k$ -variate normal random vector. So,  $\mathbf{Z}$ , as a linear combination of  $\mathbf{Y}$ , must be a  $(k-1)$ -variate normal random vector.



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Moreover, since  $n_i = \left\lceil \frac{2h^2\sigma_i^2}{\delta^2} \right\rceil$  in (9),  $\frac{\sigma_i^2}{n_i} = \frac{\delta^2}{2h^2}$  approximately,  $i = 1, \dots, k.$



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For  $i \neq j$ ,  $\text{Cov}(Z_i, Z_j) = \text{Cov}\left(\frac{\bar{Y}_i - \bar{Y}_k}{\delta/h}, \frac{\bar{Y}_j - \bar{Y}_k}{\delta/h}\right) = \frac{\text{Cov}(\bar{Y}_i, \bar{Y}_j)}{\delta^2/h^2} = \frac{\sigma_k^2/n_k}{\delta^2/h^2} = \frac{1}{2}.$



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Hence, by (8) and (11),  $\mathbb{P}\{\text{select } k\} \geq 1 - \alpha$ .



- Assumption 3 (indifference-zone) can be **relaxed** by *softening* the selection target to probability of good selection (PGS):

$$\mathbb{P} \left\{ \left| \text{selected } \theta_i - \max_{1 \leq i \leq k} \theta_i \right| < \delta \right\} \geq 1 - \alpha.$$



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- Rinott (1978) proposed a procedure which can still guarantee the PCS in (7) while relaxing Assumption 4 (*known* variance), i.e., allowing *unknown* variances.
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  - It requires an initial stage to estimate  $\sigma_i^2$  by sample variance.
  - The proof is more complicated.
- Procedures like Bechhofer's or Rinott's are simple to implement, but the efficiency may be low.
  - The designed sample size (or, replication number),  $n_i$ , may be larger than necessary (too conservative).

- More sample efficient procedures should be in a sequential manner.
  - Take observations sequentially, i.e., one at a time.
  - Eliminate designs from continued sampling when it is statistically clear that they are inferior.
  - Simulation for a problem with a single dominant alternative may terminate very quickly.

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  - Take observations sequentially, i.e., one at a time.
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  - Simulation for a problem with a single dominant alternative may terminate very quickly.
- [Paulson \(1964\)](#) proposed fully sequential procedures, which can guarantee the PCS in (7), under Assumptions 1-3 and (a) *common known* variance or (b) *common unknown* variance.

- Suppose  $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2$  and  $\sigma^2$  is known (*common known variance*).
- Let  $\bar{Y}_i(r)$  be the sample mean of the first  $r$  observations.

- Suppose  $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_k^2 = \sigma^2$  and  $\sigma^2$  is known (*common known variance*).
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- Paulson's Procedure

- ① Let  $0 < \lambda < \delta$  (a good choice is  $\lambda = \delta/2$ ), and

$$a = \ln\left(\frac{k-1}{\alpha}\right) \frac{\sigma^2}{\delta - \lambda}.$$

Let  $I = \{1, 2, \dots, k\}$  and  $r = 0$ .

- ② Let  $r \leftarrow r + 1$ . Take one observation from each alternative in  $I$  and compute  $\bar{Y}_i(r)$ ,  $\forall i \in I$ .
- ③ Let  $I^{\text{old}} = I$  and

$$I = \left\{ \ell \in I^{\text{old}} : \bar{Y}_\ell(r) \geq \max_{i \in I^{\text{old}}} \bar{Y}_i(r) - \max\{0, a/r - \lambda\} \right\}.$$

If  $|I| > 1$ , then go to Step 2; otherwise, select the alternative left in  $I$  as the best.

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- Commercial simulation software, Simio, implements the  $\mathcal{KN}$  procedure of Kim and Nelson (2001) as an Add-In, to help user to select the best scenario.

- Ranking and Selection (R&S) problem was first introduced in the 1950s by the statistics community:
  - rank all alternatives
  - select a subset of alternatives
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  - rank all alternatives
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- Existing procedures for R&S (selection of the best) problems:
  - frequentist
  - Bayesian

- Frequentist procedures typically aim to deliver the PCS or PGS; see Kim and Nelson (2006) for a review:
  - two-stage procedures: Bechhofer (1954), Rinott (1978)
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- Bayesian procedures often allocate samples to each alternative either to maximize the Bayesian posterior PCS or to minimize the expected opportunity cost; see Chen et al. (2015) for a review:
  - optimal computing budget allocation: Chen et al. (2000), He et al. (2007)
  - value of information: Chick and Inoue (2001), Chick et al. (2010)
  - knowledge gradient: Frazier et al. (2008), Frazier et al. (2009)
  - economics of selection procedures: Chick and Gans (2009), Chick and Frazier (2012)

- Emerging research problems that extend classical R&S from different perspectives; see [Hong et al. \(2021\)](#) for a review:
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- What if the number of candidate designs (feasible solutions) is huge, or countably infinite, or even uncountably infinite?
  - **Simulation Optimization (or called Optimization via Simulation)**

- R&S Problem vs Multi-Arm Bandit (MAB) Problem:

