

# MEM6804 Modeling and Simulation for Logistics and Supply Chain: Theory & Analysis

Sino-US Global Logistics Institute  
Shanghai Jiao Tong University

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## Assignment 2

*Due Date: April 7 (in class)*

### Instruction

- (a) You can answer in English or Chinese or both.
  - (b) Show enough intermediate steps.
  - (c) Write your answers independently.
- .....

### Question 1 (10 points)

Use Hölder's Inequality to prove that, for real numbers  $a_i, b_i, i = 1, 2, \dots, n$ , and positive real numbers  $p$  and  $q$  such that  $1/p + 1/q = 1$ ,

$$\sum_{i=1}^n |a_i b_i| \leq \left( \sum_{i=1}^n |a_i|^p \right)^{1/p} \left( \sum_{i=1}^n |b_i|^q \right)^{1/q}.$$

### Question 2 (10 points)

Use Minkowski's Inequality to prove that, for real numbers  $a_i, b_i, i = 1, 2, \dots, n$ , and real number  $p \geq 1$ ,

$$\left( \sum_{i=1}^n |a_i + b_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^n |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^n |b_i|^p \right)^{1/p}.$$

### Question 3 (10 points)

For real numbers  $a_i, b_i, i = 1, 2, \dots, n$ , define

$$\begin{aligned} \text{arithmetic mean: } a_A &= \frac{1}{n}(a_1 + \dots + a_n), \\ \text{geometric mean: } a_G &= (a_1 \times \dots \times a_n)^{1/n}, \\ \text{harmonic mean: } a_H &= \frac{1}{\frac{1}{n}(\frac{1}{a_1} + \dots + \frac{1}{a_n})}. \end{aligned}$$

Use Jensen's Inequality to prove that  $a_H \leq a_G \leq a_A$ . (Hint: Use the  $\log()$  function.)

**Question 4** (10 points)

Prove that, if  $X_n \xrightarrow{L^s} X$  and  $s > r \geq 1$ , then  $X_n \xrightarrow{L^r} X$ . (Hint: Use Hölder's Inequality.)

**Question 5** (10 points)

Prove that, if  $X_n \xrightarrow{L^r} X$  for  $r \geq 1$ , then  $\mathbb{E}[|X_n|^r] \rightarrow \mathbb{E}[|X|^r]$ . (Hint: Use Minkowski's Inequality. Also note that for a sequence of numbers  $a_1, a_2, \dots$ , and a continuous function  $f()$ , as  $n \rightarrow \infty$ ,  $a_n \rightarrow a$  implies that  $f(a_n) \rightarrow f(a)$ .)

**Question 6** (10 points)

Prove that  $S^2$  is an unbiased estimator of  $\sigma^2$ , i.e.,  $\mathbb{E}[S^2] = \sigma^2$ , but  $S$  is a biased estimator of  $\sigma$ , i.e.,  $\mathbb{E}[S] \neq \sigma$ . (Hint: Use Jensen's Inequality.)

**Question 7** (10 points)

Prove that  $\bar{X}$  is independent of  $S^2$  in the normal distribution case. To make it simple, only consider the case where sample size  $n = 2$ . (Note: For general case, the analysis is more tedious, but the intuition behind is similar.) (Hint: Try to show that  $\bar{X}$  is a function only of  $X_1 + X_2$  and  $S^2$  is a function only of  $X_2 - X_1$ .)

**Question 8** (10 points)

Prove that  $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$  in the normal distribution case. [Hint: Let  $S_n^2$  denote the sample variance when the sample size is  $n$ . First show that  $\frac{S_2^2}{\sigma^2} \sim \chi_1^2$ . Then reach the conclusion for  $\frac{(n-1)S_n^2}{\sigma^2}$  by induction.]

**Question 9** (10 points)

Prove the Weak Law of Large Numbers with iid assumption and  $\sigma^2 < \infty$ . (Hint: Use Chebyshev's Inequality.)

**Question 10** (10 points)

Recall the Numerical Integration example in Lec 1 page 27/32. Let  $Y_N = \frac{b-a}{N}[f(X_1) + \dots + f(X_N)]$ . Prove that: (1)  $\mathbb{E}[Y_N] = \int_a^b f(x)dx$ ; and (2)  $Y_N \xrightarrow{a.s.} \int_a^b f(x)dx$  as  $N \rightarrow \infty$ .