

MEM6810 Engineering Systems Modeling and Simulation



工程系统建模与仿真

Theory Analysis

Lecture 2: Elements of Probability and Statistics

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上海交通大学
SHANGHAI JIAO TONG UNIVERSITY

董浩云航运与物流研究院

CY TUNG Institute of Maritime and Logistics

中美物流研究院 (工程系统管理研究院)

Sino-US Global Logistics Institute (Institute of Industrial & System Engineering)



- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations
- 4 Common Distributions
- 5 Useful Inequalities
- 6 Convergence
- 7 Properties of a Random Sample

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 - ① $\mathbb{P}(A) \in [0, 1]$ for any $A \in \mathcal{F}$;
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 - ③ Countably additive: If $A_i \in \mathcal{F}$, $i = 1, 2, \dots$, is a **countable** sequence of **disjoint** sets, then $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

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 - $\mathcal{F}_1 = \{\emptyset, [0, a), [a, 1], \Omega\}$, $\mathcal{F}_2 = \{\emptyset, (0, a), \{0\} \cup [a, 1], \Omega\}$...
 - A more practical and interesting \mathcal{F} is the one that contains all intervals (no matter open or closed) on $[0, 1]$.



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- Clearly, mutual independence implies pairwise independence, but not vice versa!



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- Remark: For event A , if $\mathbb{P}(A) = 1$, then we say A happens **almost surely** (a.s.).

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 - A popular convention is to denote the RVs by upper-case letters (e.g., X and Y) and their realizations by lower-case letters (e.g., x and y).

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- The **cumulative distribution function** (CDF) of a RV X , denoted by $F : \mathbb{R} \rightarrow [0, 1]$, is defined by

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$$p(x) := \mathbb{P}(X = x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\}), \quad \forall x \in \mathbb{R},$$

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- The **joint** CDF of RVs X and Y , denoted by $F : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$, is defined by

$$\begin{aligned} F(x, y) &:= \mathbb{P}(X \leq x, Y \leq y) \\ &= \mathbb{P}(\{\omega : X(\omega) \leq x\} \cap \{\omega : Y(\omega) \leq y\}), \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

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- Observe that $\frac{\partial^2 F(x, y)}{\partial x \partial y} = f(x, y)$.

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- For Y , its marginal CDF, and pmf or pdf, can be determined similarly.

Univariate Transformation - Continuous Case

Let X be a continuous RV, and $Y = g(X)$, where g is a **monotone** function. Let

$\mathcal{X} := \{x : f_X(x) > 0\}$ and $\mathcal{Y} := \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}$.

Suppose that $g^{-1}(y)$ has a continuous derivative on \mathcal{Y} . Then,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, & y \in \mathcal{Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Bivariate Transformation - Continuous Case

Let $(X, Y)^T$ be a continuous bivariate random vector, and $U = g_1(X, Y)$ and $V = g_2(X, Y)$. Let

$$\mathcal{A} := \{(x, Y) : f_{X, Y}(x, y) > 0\},$$

$$\mathcal{B} := \{(u, v) : u = g_1(x, y), v = g_2(x, y) \text{ for some } (x, y) \in \mathcal{A}\}.$$

Suppose that $u = g_1(x, y)$ and $v = g_2(x, y)$ define a **one-to-one** transformation of \mathcal{A} **onto** \mathcal{B} , and $x = h_1(u, v)$ and $y = h_2(u, v)$ have continuous partial derivatives on \mathcal{B} . Then,

$$f_{U, V}(u, v) = \begin{cases} f_{X, Y}(h_1(u, v), h_2(u, v)) |J|, & (u, v) \in \mathcal{B}, \\ 0, & \text{otherwise,} \end{cases}$$

given that J is not identically 0 on \mathcal{B} , where J is the Jacobian

Bivariate Transformation - Continuous Case (Cont'd)

of the transformation, i.e.,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v},$$

and

$$\begin{aligned} \frac{\partial x}{\partial u} &= \frac{\partial h_1(u, v)}{\partial u}, & \frac{\partial x}{\partial v} &= \frac{\partial h_1(u, v)}{\partial v}, \\ \frac{\partial y}{\partial u} &= \frac{\partial h_2(u, v)}{\partial u}, & \frac{\partial y}{\partial v} &= \frac{\partial h_2(u, v)}{\partial v}. \end{aligned}$$



- If $(X, Y)^T$ is discrete, for any y such that $\mathbb{P}(Y = y) = p_Y(y) > 0$, the **conditional** pmf of X given that $Y = y$ is defined as

$$p(x|y) := \mathbb{P}(X = x|Y = y) = \frac{p(x, y)}{p_Y(y)}.$$



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② Then, $f(x|y) = \frac{\partial}{\partial x} F(x|Y = y) = \frac{\frac{\partial}{\partial x} \int_{-\infty}^x f(t, y) dt}{f_Y(y)} = \frac{f(x, y)}{f_Y(y)}$.

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- RVs X_1, \dots, X_n are pairwise independent if for any $i \neq j$, $X_i \perp X_j$.

- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations**
- 4 Common Distributions
- 5 Useful Inequalities
- 6 Convergence
- 7 Properties of a Random Sample



- The **expectation**, or **expected value**, or **mean**, of a RV X is defined as

$$\mathbb{E}[X] := \int_{\Omega} X(\omega) d\mathbb{P}(\omega),$$

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 - **Correlation**: $\rho(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}.$
- In general, $X \perp Y \xLeftrightarrow{\neq} \rho(X, Y) = 0 \iff \text{Cov}(X, Y) = 0.$
- If $(X, Y)^\top$ follows a bivariate normal distribution,[†] then

$$X \perp Y \iff \rho(X, Y) = 0.$$

[†] **CAUTION:** It means MORE than that X and Y both follow a normal distribution! More details latter.

- The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X|y] := \begin{cases} \sum_{x \in \mathbb{R}} xp(x|y), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{+\infty} xf(x|y)dx, & \text{if } X \text{ is continuous.} \end{cases}$$



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$$\mathbb{E}[X^n] = \left. \frac{d^n}{dt^n} M_X(t) \right|_{t=0}, \quad n \in \mathbb{N}.$$



- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations
- 4 Common Distributions**
- 5 Useful Inequalities
- 6 Convergence
- 7 Properties of a Random Sample



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 - Given $X_1 + X_2 = n$, $X_1 \sim \text{B}(n, \lambda_1/(\lambda_1 + \lambda_2))$.

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 - $\alpha = p/2$, where p is an integer, and $\lambda = 1/2 \implies$ **chi-square distribution with p degrees of freedom**, denoted as χ_p^2 .



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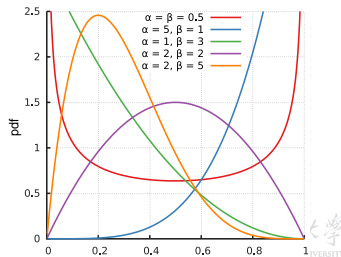
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 - $\alpha > 1, \beta = 1 \implies$ strictly increasing
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 - $\alpha > 1, \beta > 1 \implies$ unimodal



- $X \sim$ Student's t distribution with p degrees of freedom, denoted as t_p , where p is an integer, if its pdf is given by

$$f(x) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1 + x^2/p)^{(p+1)/2}}, \quad x \in \mathbb{R}.$$

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The proof is completed by showing that $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \sqrt{\pi}$, which can be seen if we convert to polar coordinates.



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- To see $\rho = 0 \implies X_1 \perp X_2$, let $\rho = 0$, and note

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left[\left(\frac{x_1-\mu_1}{\sigma_1} \right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2} \right)^2 \right]} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \times \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} = f_{X_1}(x_1)f_{X_2}(x_2). \end{aligned}$$

- If $(X_1, X_2)^\top \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $X_i \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, 2$, then $X_1 + X_2 \perp X_1 - X_2$.



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Since \mathbf{B} has full row rank, $\mathbf{Y} \sim \mathcal{N}(\mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^\top)$, which is non-degenerate. Hence, to prove $X_1 + X_2 \perp X_1 - X_2$, it suffices to show $\text{Cov}(X_1 + X_2, X_1 - X_2) = 0$.



- If $(X_1, X_2)^\top \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $X_i \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, 2$, then $X_1 + X_2 \perp X_1 - X_2$.

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$$\mathbf{Y} := \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} =: \mathbf{B} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Since \mathbf{B} has full row rank, $\mathbf{Y} \sim \mathcal{N}(\mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^\top)$, which is non-degenerate. Hence, to prove $X_1 + X_2 \perp X_1 - X_2$, it suffices to show $\text{Cov}(X_1 + X_2, X_1 - X_2) = 0$. Note that

$$\begin{aligned} \text{Cov}(X_1 + X_2, X_1 - X_2) &= \text{Cov}(X_1, X_1) - \text{Cov}(X_2, X_2) \\ &= \sigma^2 - \sigma^2 = 0. \end{aligned}$$



- <http://www.math.wm.edu/~leemis/chart/UDR/UDR.html>

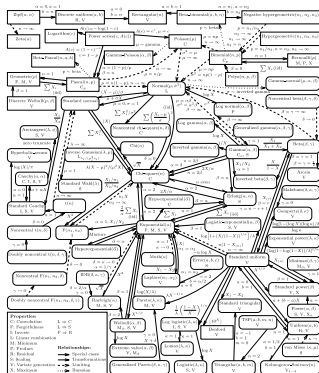


Figure: Relationships Among 76 Distributions (from Leemis & McQueston (2008))

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Markov's Inequality

Let X be a RV. If $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X = 0) < 1$, then, for any $r > 0$,

$$\mathbb{P}(X \geq r) \leq \frac{\mathbb{E}[X]}{r},$$

with equality if and only if

$$X = \begin{cases} r, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$



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- Markov's Inequality has many variations, which are usually called Chebyshev's Inequality.



Chebyshev's Inequality

Let X be a RV and $g(x)$ be a nonnegative function. Then, for any $r > 0$,

$$\mathbb{P}(g(X) \geq r) \leq \frac{\mathbb{E}[g(X)]}{r}.$$



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Let X be a RV and $g(x)$ be a nonnegative function. Then, for any $r > 0$,

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Chebyshev's Inequality

Let X be a RV. Then, for any $r, p > 0$,

$$\mathbb{P}(|X| \geq r) \leq \frac{\mathbb{E}[|X|^p]}{r^p},$$

$$\mathbb{P}(|X - \mu| \geq r) \leq \frac{\sigma^2}{r^2},$$

where $\mu := \mathbb{E}[X]$, and $\sigma^2 := \text{Var}(X)$.

- Chebyshev's Inequality is typically very conservative.

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- If $Z \sim \mathcal{N}(0, 1)$, a tighter bound is available: For any $t > 0$,

$$2\Phi(-t) = \mathbb{P}(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}} \frac{1}{t} e^{-t^2/2},$$

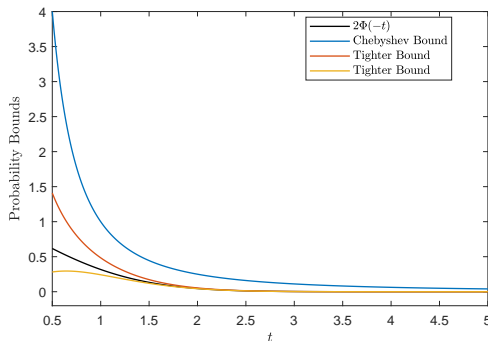
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- A function $g(x)$ is **convex** if

$$g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y),$$

for all x and y , and $\lambda \in (0, 1)$.



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Jensen's Inequality

Let X be a RV. If $g(x)$ is a convex function, then

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X]),$$

with equality if and only if $g(x)$ is a linear function on some set A such that $\mathbb{P}(X \in A) = 1$.



Hölder's Inequality

Let X and Y be any two RVs, and let p and q be any two positive numbers (necessarily greater than 1) satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then,

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq \{\mathbb{E}[|X|^p]\}^{1/p} \{\mathbb{E}[|Y|^q]\}^{1/q}.$$



Cauchy-Schwarz Inequality ($p = q = 2$)

Let X and Y be any two RVs, then

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq \{\mathbb{E}[|X|^2]\}^{1/2} \{\mathbb{E}[|Y|^2]\}^{1/2}.$$



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Liapounov's Inequality ($Y \equiv 1$)

Let X be a RV, then for any $s > r > 1$,

$$\{\mathbb{E}[|X|^r]\}^{1/r} \leq \{\mathbb{E}[|X|^s]\}^{1/s}.$$



Minkowski's Inequality

Let X and Y be any two RVs. Then, for $p \geq 1$,

$$\{\mathbb{E}[|X + Y|^p]\}^{1/p} \leq \{\mathbb{E}[|X|^p]\}^{1/p} + \{\mathbb{E}[|Y|^p]\}^{1/p}.$$



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- **Remark:** The preceding Hölder's Inequality (including its special cases) and Minkowski's Inequality also apply to numerical sums where there is no explicit reference to an expectation.



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where F_n and F are CDF of X_n and X , respectively.



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- **Convergence in L^r Norm** ($r \in [1, \infty)$), $X_n \xrightarrow{L^r} X$:

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_n - X|^r) = 0,$$

given $\mathbb{E}[|X_n|^r] < \infty$ for any $n \geq 1$ and $\mathbb{E}[|X|^r] < \infty$.



- Simple relationships:

$$\begin{array}{ccccc}
 X_n \xrightarrow{a.s.} X & \implies & X_n \xrightarrow{p} X & \implies & X_n \xrightarrow{d} X \\
 & & \uparrow & & \\
 X_n \xrightarrow{L^s} X & \xRightarrow{s>r\geq 1} & X_n \xrightarrow{L^r} X & \implies & \mathbb{E}[|X_n|^r] \rightarrow \mathbb{E}[|X|^r]
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- $X_n \xrightarrow{p} X \iff \text{For every subsequence } X_n(m) \text{ there is a further subsequence } X_n(m_k) \text{ such that } X_n(m_k) \xrightarrow{a.s.} X.$

- Question: If $X_n \xrightarrow{d} X$ or $X_n \xrightarrow{p} X$ or $X_n \xrightarrow{a.s.} X$, does it imply $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$?

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Monotone Convergence Theorem (MCT)

Suppose $X_n \xrightarrow{a.s.} X$, and $0 \leq X_1 \leq X_2 \leq \dots$ a.s.. Then $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$.



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Fatou's Lemma

Suppose $X_n \geq Y$ a.s. for all n where $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[\liminf_{n \rightarrow \infty} X_n] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n]$. In particular, if $X_n \geq 0$ a.s. for all n , then the result holds.

Dominated Convergence Theorem (DCT)

Suppose $X_n \xrightarrow{a.s.} X$, $|X_n| \leq Y$ a.s. for all n , and $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$.



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- The DCT is still true if $\xrightarrow{a.s.}$ is replaced by \xrightarrow{p} .
- An **even more general** result:
Suppose $X_n \xrightarrow{p} X$, $|X_n| \leq Y$ a.s. for all n , and $\mathbb{E}[|Y|^r] < \infty$ with $r \geq 1$. Then, $\mathbb{E}[|X_n|^r] < \infty$, $\mathbb{E}[|X|^r] < \infty$, and $X_n \xrightarrow{L^r} X$.



- $X = Y$ a.s., if *any one* of the following holds:
 - $X_n \xrightarrow{a.s.} X$ and $X_n \xrightarrow{a.s.} Y$;
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- None of the above are true for convergence in distribution.
- If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} \text{constant } c$, then $(X_n, Y_n)^\top \xrightarrow{d} (X, c)^\top$.
 $\implies aX_n + bY_n \xrightarrow{d} aX + bc$; $X_n Y_n \xrightarrow{d} cX$. (Due to CMT; also known as Slutsky's theorem)



Continuous Mapping Theorem (CMT)

Consider a sequence of RVs $\{X_n : n \geq 1\}$ and another RV X . Suppose g is a function that has the set of discontinuity points D such that $\mathbb{P}(X \in D) = 0$. Then,

$$X_n \xrightarrow{a.s.} X \implies g(X_n) \xrightarrow{a.s.} g(X);$$

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- CMT also holds for **random vectors**.
- **Caution:** For convergence in L^r norm, stronger assumption of g than continuity is required to ensure $g(X_n) \xrightarrow{L^r} g(X)$.



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- Let X_1, \dots, X_n be a **random sample** from a distribution with mean μ and variance σ^2 , i.e., X_1, \dots, X_n are iid, and $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$, $i = 1, \dots, n$.

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- Define

$$\bar{X} := \frac{1}{n} \sum_{i=1}^n X_i, \text{ and } S^2 := \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}.$$

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Suppose X_1, \dots, X_n are iid with mean μ and variance $\sigma^2 < \infty$.[†] Then, $\bar{X}_n \xrightarrow{p} \mu$.

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- Note that for **normal** distribution, $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$, regardless of the value of n .
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Central Limit Theorem (CLT)

Suppose X_1, \dots, X_n are iid with mean μ and variance $\sigma^2 \in (0, \infty)$. Then,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1).$$