

MEM6810 Engineering Systems Modeling and Simulation



工程系统建模与仿真

Theory Analysis

Lecture 2: Elements of Probability and Statistics

SHEN Haihui 沈海辉

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

 shenhaihui.github.io/teaching/mem6810f
 shenhaihui@sjtu.edu.cn

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上海交通大学
SHANGHAI JIAO TONG UNIVERSITY

董浩云智能制造与服务管理研究院
CY TUNG Institute of Intelligent Manufacturing and Service Management
(中美物流研究院)
(Sino-US Global Logistics Institute)



- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations
- 4 Common Distributions
- 5 Useful Inequalities
- 6 Convergence
- 7 Properties of a Random Sample

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A **probability space** is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$:

- Ω , sample space: A set of *all* possible outcomes.
 - A set of *some* outcomes, as a subset of Ω , is called an **event**.
- \mathcal{F} , σ -algebra (or σ -field): A set of events, i.e., a set of some subsets of Ω , such that:
 - ① $\Omega \in \mathcal{F}$;
 - ② Closed under complementation: If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;
 - ③ Closed under countable unions:[†] If $A_i \in \mathcal{F}$, $i = 1, 2, \dots$, is a **countable** sequence of sets, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.
- $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$, probability function (or probability measure): A function that assigns probabilities to events, such that:
 - ① $\mathbb{P}(A) \in [0, 1]$ for any $A \in \mathcal{F}$;
 - ② $\mathbb{P}(\Omega) = 1$;
 - ③ Countably additive: If $A_i \in \mathcal{F}$, $i = 1, 2, \dots$, is a **countable** sequence of **disjoint** sets, then $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

[†] It implies that \mathcal{F} is also closed under countable intersections.



- Example 1: Flip a fair coin.
 - $\Omega = \{H \text{ (head)}, T \text{ (tail)}\}$;
 - $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \Omega\}$;
 - $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\{H\}) = 1/2$, $\mathbb{P}(\{T\}) = 1/2$, and $\mathbb{P}(\Omega) = 1$.
- Example 2: Draw a ball out of 3 balls (red, green, blue).
 - $\Omega = \{R \text{ (red)}, G \text{ (green)}, B \text{ (blue)}\}$;
 - $\mathcal{F} = \{\emptyset, \{R\}, \{G\}, \{B\}, \{R,G\}, \{R,B\}, \{G,B\}, \Omega\}$;
 - $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\{R\}) = \mathbb{P}(\{G\}) = \mathbb{P}(\{B\}) = 1/3$,
 $\mathbb{P}(\{R,G\}) = \mathbb{P}(\{R,B\}) = \mathbb{P}(\{G,B\}) = 2/3$, and $\mathbb{P}(\Omega) = 1$;
 - $\mathcal{F}_1 = \{\emptyset, \{R\}, \{G,B\}, \Omega\}$, $\mathcal{F}_2 = \{\emptyset, \{G\}, \{R,B\}, \Omega\}$...
- Example 3: Randomly “draw” a number in $[0, 1]$.
 - $\Omega = [0, 1]$;
 - $\mathcal{F}_1 = \{\emptyset, [0, a), [a, 1], \Omega\}$, $\mathcal{F}_2 = \{\emptyset, (0, a), \{0\} \cup [a, 1], \Omega\}$...
 - A more practical and interesting \mathcal{F} is the one that contains all intervals (no matter open or closed) on $[0, 1]$.



- **Independence** of Events: Two events A and B in \mathcal{F} are called statistically independent events when

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B).$$

- **Conditional Probability**: If A and B are events in \mathcal{F} and $\mathbb{P}(B) > 0$, then the conditional probability of A given B , denoted as $\mathbb{P}(A|B)$, is

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

- Bayes' Rule:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \mathbb{P}(A)}{\mathbb{P}(B)}.$$

- Events A and B are independent $\iff \mathbb{P}(A|B) = \mathbb{P}(A)$.



- For more than two events:
 - **Mutual independence** (or collective independence) intuitively means that each event is independent of any combination of other events;
 - **Pairwise independence** means any two events in the collection are independent of each other.
- Sets A_1, \dots, A_n are (mutually) independent if for any $I \subset \{1, \dots, n\}$ we have $\mathbb{P}(\cap_{i \in I} A_i) = \prod_{i \in I} \mathbb{P}(A_i)$.
- **Warning:** Only having $\mathbb{P}(\cap_{i=1}^n A_i) = \prod_{i=1}^n \mathbb{P}(A_i)$ is not sufficient!
- Sets A_1, \dots, A_n are pairwise independent if for any $i \neq j$ we have $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \mathbb{P}(A_j)$.
- Clearly, mutual independence implies pairwise independence, but not vice versa!



Consider a sequence of sets $\{A_n : n \geq 1\}$.

(The First) Borel-Cantelli Lemma

If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A_n \text{ i.o.}) = 0$, where “i.o.” denotes “infinitely often”.

The Secon Borel-Cantelli Lemma

If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and $\{A_n\}$ are independent,[†] then $\mathbb{P}(A_n \text{ i.o.}) = 1$.

- Remark: For event A , if $\mathbb{P}(A) = 1$, then we say A happens **almost surely** (a.s.).

[†]The assumption of independence can be weakened to pairwise independence, with more difficult proof.

- 1 Probability Space
- 2 Random Variables & Distributions**
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- A **random variable** (RV) is a function from a sample space Ω into the set of real numbers \mathbb{R} .
- Formally, given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a RV X is a function $X : \Omega \rightarrow \mathbb{R}$, such that for any $a \in \mathbb{R}$,

$$\{\omega \in \Omega : X(\omega) \leq a\} \in \mathcal{F}.$$

- For a particular element $\omega \in \Omega$, $X(\omega)$ is called a *realization* of X .
 - Usually, we will simply denote $X(\omega)$ as x when ω is not explicitly shown.
 - A popular convention is to denote the RVs by upper-case letters (e.g., X and Y) and their realizations by lower-case letters (e.g., x and y).

- Example 1': Let $X(H) = 0$, $X(T) = 1$.
- Example 2':
 - Under $(\Omega, \mathcal{F}, \mathbb{P})$, let $X(R) = 0$, $X(G) = 1$, and $X(B) = 2$.
 - Under $(\Omega, \mathcal{F}_1, \mathbb{P})$, let $X(R) = 0$, $X(G) = 1$, and $X(B) = 1$.
- Example 3':
 - Under $(\Omega, \mathcal{F}_1, \mathbb{P})$, let $X(\omega) := \begin{cases} 0, & \text{if } \omega \in [0, a), \\ 1, & \text{if } \omega \in [a, 1]. \end{cases}$
 - Under $(\Omega, \mathcal{F}, \mathbb{P})$, let $X(\omega) = \omega$ for $\omega \in [0, 1]$.



- The **cumulative distribution function** (CDF) of a RV X , denoted by $F : \mathbb{R} \rightarrow [0, 1]$, is defined by

$$F(x) := \mathbb{P}(X \leq x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}), \quad \forall x \in \mathbb{R},$$

and the following is satisfied:

- $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow +\infty} F(x) = 1$;
- $F(x)$ is nondecreasing in x ;
- $F(x)$ is right-continuous, that is, for any $x_0 \in \mathbb{R}$,

$$\lim_{x \downarrow x_0} F(x) = F(x_0).$$



- A RV X is said to be **discrete** if the set of its possible values is countable.
- The **probability mass function** (pmf) of a discrete RV X is given by

$$p(x) := \mathbb{P}(X = x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\}), \quad \forall x \in \mathbb{R},$$

and the following is satisfied:

- $p(x) \geq 0$ for all $x \in \mathbb{R}$;
 - $\sum_{x \in \mathbb{R}} p(x) = 1$.
- It is easy to see that $F(x) = \sum_{y \in (-\infty, x]} p(y)$.



- A RV X is said to be **continuous** if there exists a **probability density function** (pdf) $f(x)$ such that

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x f(t)dt, \quad \forall x \in \mathbb{R},$$

and the following is satisfied:

- $f(x) \geq 0$ for all $x \in \mathbb{R}$;
 - $\int_{-\infty}^{+\infty} f(t)dt = 1$.
- Observe that $\frac{d}{dx}F(x) = f(x)$.



- The **joint** CDF of RVs X and Y , denoted by $F : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$, is defined by

$$\begin{aligned} F(x, y) &:= \mathbb{P}(X \leq x, Y \leq y) \\ &= \mathbb{P}(\{\omega : X(\omega) \leq x\} \cap \{\omega : Y(\omega) \leq y\}), \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

- For discrete RVs X and Y , the **joint** pmf is given by

$$\begin{aligned} p(x, y) &:= \mathbb{P}(X = x, Y = y) \\ &= \mathbb{P}(\{\omega : X(\omega) = x\} \cap \{\omega : Y(\omega) = y\}), \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

- For continuous RVs X and Y , the **joint** pdf is $f(x, y)$ such that

$$F(x, y) = \int_{-\infty}^y \int_{-\infty}^x f(t, u) dt du, \quad \forall x, y \in \mathbb{R}.$$

- Observe that $\frac{\partial^2 F(x, y)}{\partial x \partial y} = f(x, y)$.



- Given the random vector $(X, Y)^T$, the distribution of X or Y is called the **marginal distribution**.

- The marginal CDF of X is $F_X(x) = F(x, +\infty)$.

- If $(X, Y)^T$ is discrete, the marginal pmf of X is

$$p_X(x) = \sum_{y \in \mathbb{R}} p(x, y).$$

- If $(X, Y)^T$ is continuous, the marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y) dy.$$

- For Y , its marginal CDF, and pmf or pdf, can be determined similarly.

Univariate Transformation - Continuous Case

Let X be a continuous RV, and $Y = g(X)$, where g is a **monotone** function. Let

$\mathcal{X} := \{x : f_X(x) > 0\}$ and $\mathcal{Y} := \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}$.

Suppose that $g^{-1}(y)$ has a continuous derivative on \mathcal{Y} . Then,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, & y \in \mathcal{Y}, \\ 0, & \text{otherwise.} \end{cases}$$



Bivariate Transformation - Continuous Case

Let $(X, Y)^T$ be a continuous bivariate random vector, and $U = g_1(X, Y)$ and $V = g_2(X, Y)$. Let

$$\mathcal{A} := \{(x, y) : f_{X, Y}(x, y) > 0\},$$

$$\mathcal{B} := \{(u, v) : u = g_1(x, y), v = g_2(x, y) \text{ for some } (x, y) \in \mathcal{A}\}.$$

Suppose that $u = g_1(x, y)$ and $v = g_2(x, y)$ define a **one-to-one** transformation of \mathcal{A} **onto** \mathcal{B} , and $x = h_1(u, v)$ and $y = h_2(u, v)$ have continuous partial derivatives on \mathcal{B} . Then,

$$f_{U, V}(u, v) = \begin{cases} f_{X, Y}(h_1(u, v), h_2(u, v)) |J|, & (u, v) \in \mathcal{B}, \\ 0, & \text{otherwise,} \end{cases}$$

given that J is not identically 0 on \mathcal{B} , where J is the Jacobian

Bivariate Transformation - Continuous Case (Cont'd)

of the transformation, i.e.,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v},$$

and

$$\begin{aligned} \frac{\partial x}{\partial u} &= \frac{\partial h_1(u, v)}{\partial u}, & \frac{\partial x}{\partial v} &= \frac{\partial h_1(u, v)}{\partial v}, \\ \frac{\partial y}{\partial u} &= \frac{\partial h_2(u, v)}{\partial u}, & \frac{\partial y}{\partial v} &= \frac{\partial h_2(u, v)}{\partial v}. \end{aligned}$$



- If $(X, Y)^T$ is discrete, for any y such that $\mathbb{P}(Y = y) = p_Y(y) > 0$, the **conditional** pmf of X given that $Y = y$ is defined as

$$p(x|y) := \mathbb{P}(X = x|Y = y) = \frac{p(x, y)}{p_Y(y)}.$$

- If $(X, Y)^T$ is continuous, for any y such that $f_Y(y) > 0$, the **conditional** pdf of X given that $Y = y$ is defined as

$$f(x|y) := \frac{f(x, y)}{f_Y(y)}.$$



Intuitively, $f(x|y)$ can be understood as follows (although it is not the most rigorous approach):

① Note that

$$\begin{aligned} F(x|Y=y) &= \lim_{\Delta \rightarrow 0} F(x|Y \text{ between } y \text{ and } y+\Delta) \\ &= \lim_{\Delta \rightarrow 0} \frac{\mathbb{P}(X \leq x, Y \text{ between } y \text{ and } y+\Delta)}{\mathbb{P}(Y \text{ between } y \text{ and } y+\Delta)} \\ &= \frac{\lim_{\Delta \rightarrow 0} [F(x, y+\Delta) - F(x, y)]/\Delta}{\lim_{\Delta \rightarrow 0} [F_Y(y+\Delta) - F_Y(y)]/\Delta} \\ &= \frac{\frac{\partial}{\partial y} F(x, y)}{\frac{d}{dy} F_Y(y)} = \frac{\frac{\partial}{\partial y} \int_{-\infty}^y \int_{-\infty}^x f(t, u) dt du}{f_Y(y)} \\ &= \frac{\int_{-\infty}^x f(t, y) dt}{f_Y(y)}. \end{aligned}$$

② Then, $f(x|y) = \frac{\partial}{\partial x} F(x|Y=y) = \frac{\frac{\partial}{\partial x} \int_{-\infty}^x f(t, y) dt}{f_Y(y)} = \frac{f(x, y)}{f_Y(y)}$.



- Two RVs X and Y are said to be statistically **independent**, which can be denoted as $X \perp Y$, when, for any $x, y \in \mathbb{R}$,

$$F(x, y) = F_X(x)F_Y(y), \text{ or,}$$

$$p(x, y) = p_X(x)p_Y(y), \text{ or,}$$

$$f(x, y) = f_X(x)f_Y(y).$$

- X and Y are independent \iff
 - $p(x|y) \equiv p_X(x)$ or $f(x|y) \equiv f_X(x)$ regardless of the value y ;
 - $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$ for any $A, B \subset \mathbb{R}$.



- For more than two RVs X_1, \dots, X_n , the joint CDF, joint pmf or pdf, and the marginal pmf or pdf, are defined analogically.
- RVs X_1, \dots, X_n are (mutually) independent if

$$F(x_1, \dots, x_n) \equiv F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n), \text{ or,}$$

$$p(x_1, \dots, x_n) \equiv p_{X_1}(x_1) \times \cdots \times p_{X_n}(x_n), \text{ or,}$$

$$f(x_1, \dots, x_n) \equiv f_{X_1}(x_1) \times \cdots \times f_{X_n}(x_n).$$

- RVs X_1, \dots, X_n are pairwise independent if for any $i \neq j$, $X_i \perp X_j$.

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- The **expectation**, or **expected value**, or **mean**, of a RV X is defined as

$$\mathbb{E}[X] := \int_{\Omega} X(\omega) d\mathbb{P}(\omega),$$

provided that $\int_{\Omega} |X(\omega)| d\mathbb{P}(\omega) < \infty$ or $X \geq 0$ a.s., where the integral is the Lebesgue integral, rather than the Riemann integral.

- For function $h : \mathbb{R} \rightarrow \mathbb{R}$, $\mathbb{E}[h(X)] = \int_{\Omega} h(X(\omega)) d\mathbb{P}(\omega)$.
- If X is a discrete RV:
 - $\mathbb{E}[X] = \sum_{x \in \mathbb{R}} xp(x)$;
 - $\mathbb{E}[h(X)] = \sum_{x \in \mathbb{R}} h(x)p(x)$.
- If X is a continuous RV:
 - $\mathbb{E}[X] = \int_{-\infty}^{+\infty} xf(x)dx$;
 - $\mathbb{E}[h(X)] = \int_{-\infty}^{+\infty} h(x)f(x)dx$.



- For integer n , $\mathbb{E}[X^n]$ is called the n th **moment** of X , and $\mathbb{E}[(X - \mathbb{E}[X])^n]$ is called the n th **central moment** of X .
- Some special moments:
 - Mean (1st moment): $\mu := \mathbb{E}[X]$.
 - **Variance** (2nd central moment):

$$\sigma^2 := \text{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$
- **Linear** association:
 - **Covariance**:

$$\text{Cov}(X, Y) := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y].$$
 - **Correlation**: $\rho(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}.$
- In general, $X \perp Y \xLeftrightarrow{\neq} \rho(X, Y) = 0 \iff \text{Cov}(X, Y) = 0.$
- If $(X, Y)^\top$ follows a bivariate normal distribution,[†] then

$$X \perp Y \iff \rho(X, Y) = 0.$$

[†] **CAUTION:** It means MORE than that X and Y both follow a normal distribution! More details latter.

- The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X|y] := \begin{cases} \sum_{x \in \mathbb{R}} xp(x|y), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{+\infty} xf(x|y)dx, & \text{if } X \text{ is continuous.} \end{cases}$$

- The conditional variance of X given $Y = y$ is

$$\text{Var}(X|y) := \mathbb{E}[(X - \mathbb{E}[X])^2|y] = \mathbb{E}[X^2|y] - (\mathbb{E}[X|y])^2.$$

- If $X \not\perp Y$, then $\mathbb{E}[X|y]$ and $\text{Var}(X|y)$ are functions of y .
- If $X \not\perp Y$, then $\mathbb{E}[X|Y]$ and $\text{Var}(X|Y)$ are also RVs, whose value depends on the value of Y .
- If $X \perp Y$, then $\mathbb{E}[X|y] = \mathbb{E}[X|Y] = \mathbb{E}[X]$, and $\text{Var}(X|y) = \text{Var}(X|Y) = \text{Var}(X)$.

- $\mathbb{E}[aX + bY] = a \mathbb{E}[X] + b \mathbb{E}[Y].$
- $\text{Var}(aX + bY) = a^2 \text{Var}(X) + 2ab \text{Cov}(X, Y) + b^2 \text{Var}(Y).$
- $\text{Cov}(aX + bY, cW + dV) = ac \text{Cov}(X, W) + ad \text{Cov}(X, V) + bc \text{Cov}(Y, W) + bd \text{Cov}(Y, V).$
- $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X].$
- $\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y]).$
- If $X \perp Y$, then $\mathbb{E}[XY] = \mathbb{E}[X] \mathbb{E}[Y].$

- For a RV X , the **moment generating function** (mgf), denoted by $M_X(t)$, is

$$M_X(t) = \mathbb{E} [e^{tX}], \quad t \in \mathbb{R}.$$

- If $M_X(t)$ is finite for t in some neighborhood of 0 (i.e., there is an $h > 0$ such that for all $t \in (-h, h)$, $M_X(t) < \infty$), then,

$$\mathbb{E}[X^n] = \left. \frac{d^n}{dt^n} M_X(t) \right|_{t=0}, \quad n \in \mathbb{N}.$$



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- $X \sim \text{Bernoulli}(p)$ or $\text{Ber}(p)$, if

$$X = \begin{cases} 1, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p, \end{cases} \quad p \in [0, 1].$$

- $\mathbb{E}[X] = p$, $\text{Var}(X) = p(1 - p)$.
- The value $X = 1$ is often termed a “success” and p is referred to as the success probability.
- $Y \sim \text{binomial}(n, p)$ or $\text{B}(n, p)$: The number of successes among n (mutually) **independent and identically distributed** (iid) $\text{Ber}(p)$ trials.
 - $Y = \sum_{i=1}^n X_i$, where $X_i \sim \text{Ber}(p)$ are iid.
 - $p(y) = \mathbb{P}(Y = y) = \binom{n}{y} p^y (1 - p)^{n-y}$, $y = 0, 1, \dots, n$.
 - $\mathbb{E}[Y] = np$, $\text{Var}(Y) = np(1 - p)$.
- If $Y_1 \sim \text{B}(n_1, p)$ and $Y_2 \sim \text{B}(n_2, p)$ are independent, then $Y_1 + Y_2 \sim \text{B}(n_1 + n_2, p)$.



- $Y \sim \text{negative binomial}(r, p)$ or $\text{NB}(r, p)$: The number of iid $\text{Ber}(p)$ trials to obtain r successes.
 - $p(y) = \mathbb{P}(Y = y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}, \quad y = r, r+1, \dots$
 - $\mathbb{E}[Y] = r + r(1-p)/p, \text{Var}(Y) = r(1-p)/p^2$.
 - When $r = 1$, it becomes the geometric distribution.
- $Y \sim \text{geometric}(p)$ or $\text{Geo}(p)$: The number of iid $\text{Ber}(p)$ trials to obtain the first success.
 - $p(y) = \mathbb{P}(Y = y) = p(1-p)^{y-1}, \quad y = 1, 2, \dots$
 - $\mathbb{E}[Y] = 1/p, \text{Var}(Y) = (1-p)/p^2$.
 - **Memoryless Property**: For integers $s > t$,

$$\begin{aligned}\mathbb{P}(Y > s | Y > t) &= \frac{\mathbb{P}(Y > s, Y > t)}{\mathbb{P}(Y > t)} = \frac{\mathbb{P}(Y > s)}{\mathbb{P}(Y > t)} = \frac{(1-p)^s}{(1-p)^t} = (1-p)^{s-t} \\ &= \mathbb{P}(Y > s-t).\end{aligned}$$

- If $Y_1 \sim \text{NB}(r_1, p)$ and $Y_2 \sim \text{NB}(r_2, p)$ are independent, then $Y_1 + Y_2 \sim \text{NB}(r_1 + r_2, p)$.

- Poisson distribution is often used to model the number of occurrence in a given time interval.
- One of the basic assumptions is that, *for very small time intervals, the probability of an occurrence is proportional to the length of the time interval.*[†]
- $X \sim \text{Poisson}(\lambda)$ or $\text{Pois}(\lambda)$, with $\lambda > 0$, if

$$p(x) = \mathbb{P}(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, \dots$$

- It can be verified that $\sum_{x=0}^{\infty} p(x) = 1$.
- $\mathbb{E}[X] = \lambda$, $\text{Var}(X) = \lambda$.
- If $X_1 \sim \text{Pois}(\lambda_1)$ and $X_2 \sim \text{Pois}(\lambda_2)$ are independent,
 - $X_1 + X_2 \sim \text{Pois}(\lambda_1 + \lambda_2)$;
 - Given $X_1 + X_2 = n$, $X_1 \sim \text{B}(n, \lambda_1/(\lambda_1 + \lambda_2))$.

[†] See more detailed discussion in Lec 3.



- $X \sim \text{uniform}(a, b)$ or $\text{Unif}(a, b)$ with $a < b$, if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a, b], \\ 0, & \text{otherwise.} \end{cases}$$

- $\mathbb{E}[X] = \frac{b+a}{2}$, $\text{Var}(X) = \frac{(b-a)^2}{12}$.
- $X \sim \text{exponential}(\lambda)$ or $\text{Exp}(\lambda)$, with $\lambda > 0$, if its pdf is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \in [0, \infty).$$

- λ is called the rate parameter.
- $F(x) = 1 - e^{-\lambda x}$, $\mathbb{P}(X > x) = 1 - F(x) = e^{-\lambda x}$.
- $\mathbb{E}[X] = 1/\lambda$, $\text{Var}(X) = 1/\lambda^2$.
- **Memoryless Property:** For $s > t \geq 0$,

$$\begin{aligned} \mathbb{P}(X > s | X > t) &= \frac{\mathbb{P}(X > s, X > t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X > s)}{\mathbb{P}(X > t)} = \frac{e^{-\lambda s}}{e^{-\lambda t}} = e^{-\lambda(s-t)} \\ &= \mathbb{P}(X > s - t). \end{aligned}$$

- If $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ are independent, then $\min\{X_1, X_2\} \sim \text{Exp}(\lambda_1 + \lambda_2)$.
- If $X \sim \text{Exp}(\lambda)$, then for $\alpha > 0$, $Y := X^{1/\alpha} \sim \text{Weibull}(\alpha, \beta)$ in shape & scale parametrization with $\beta = (1/\lambda)^{1/\alpha}$, whose pdf is

$$f(y) = \alpha \beta^{-\alpha} y^{\alpha-1} e^{-(y/\beta)^\alpha}, \quad y \in (0, \infty).$$

- Erlang(k, λ) or $\text{Erl}(k, \lambda)$, with k being a positive integer, is a generalized version of $\text{Exp}(\lambda)$, whose pdf is

$$f(x) = \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}, \quad x \in [0, \infty).$$

- $\mathbb{E}[X] = k/\lambda$, $\text{Var}(X) = k/\lambda^2$.
 - $k = 1 \implies \text{Exp}(\lambda)$.
- If $X_1 \sim \text{Erl}(k_1, \lambda)$ and $X_2 \sim \text{Erl}(k_2, \lambda)$ are independent, then $X_1 + X_2 \sim \text{Erl}(k_1 + k_2, \lambda)$.
- If $X \sim \text{Erl}(k, \lambda)$, then $cX \sim \text{Erl}(k, \lambda/c)$ for $c > 0$.



- $X \sim \text{Gamma}(\alpha, \lambda)$ in shape & rate parametrization with $\alpha, \lambda > 0$, if its pdf is given by

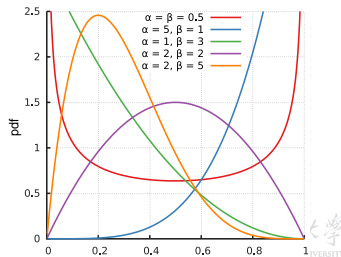
$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x \in (0, \infty).$$

- $\mathbb{E}[X] = \alpha/\lambda$, $\text{Var}(X) = \alpha/\lambda^2$.
- $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt$ is known as the gamma function.
 - $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$; $\Gamma(n) = (n-1)!$, for integer $n > 0$.
- If $X_1 \sim \text{Gamma}(\alpha_1, \lambda)$ and $X_2 \sim \text{Gamma}(\alpha_2, \lambda)$ are independent, then $X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \lambda)$.
- If $X \sim \text{Gamma}(\alpha, \lambda)$, then $cX \sim \text{Gamma}(\alpha, \lambda/c)$ for $c > 0$.
- Important special cases of $\text{Gamma}(\alpha, \lambda)$:
 - α is an integer $\implies \text{Erl}(\alpha, \lambda)$; $\alpha = 1 \implies \text{Exp}(\lambda)$;
 - $\alpha = p/2$, where p is an integer, and $\lambda = 1/2 \implies$ **chi-square distribution with p degrees of freedom**, denoted as χ_p^2 .

- Beta distribution is a very flexible distribution that in a finite interval.
- $X \sim \text{Beta}(\alpha, \beta)$ with $\alpha, \beta > 0$, if its pdf is given by

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad x \in (0, 1).$$

- $\mathbb{E}[X] = \alpha/(\alpha + \beta)$, $\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.
- $B(\alpha, \beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$ is known as the beta function.
 - $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.
- The Beta(α, β) pdf is quite flexible
 - $\alpha = 1, \beta = 1 \implies \text{Unif}(0, 1)$
 - $\alpha > 1, \beta = 1 \implies$ strictly increasing
 - $\alpha = 1, \beta > 1 \implies$ strictly decreasing
 - $\alpha < 1, \beta < 1 \implies$ U-shaped
 - $\alpha > 1, \beta > 1 \implies$ unimodal



- $X \sim$ Student's t distribution with p degrees of freedom, denoted as t_p , where p is an integer, if its pdf is given by

$$f(x) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1 + x^2/p)^{(p+1)/2}}, \quad x \in \mathbb{R}.$$

- $\mathbb{E}[X] = 0$ if $p > 1$;
- $\text{Var}(X) = p/(p-2)$ if $p > 2$.
- t_1 is also known as the standard Cauchy distribution, or Cauchy(0, 1), whose pdf is simply

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R}.$$



- The **normal distribution** (sometimes called the Gaussian distribution) plays a **central role** in a large body of statistics.
- $X \sim$ normal distribution with mean μ and variance σ^2 , denoted as $\mathcal{N}(\mu, \sigma^2)$, with $\sigma > 0$, if its pdf is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

- $\mathbb{E}[X] = \mu$, $\text{Var}(X) = \sigma^2$.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z := (X - \mu)/\sigma \sim \mathcal{N}(0, 1)$.
 - Z is also known as the **standard normal** RV.
 - We often use $\Phi(z)$ and $\phi(z)$ to denote the CDF and pdf of Z .
 - $\mathbb{P}(X \leq x) = \Phi((x - \mu)/\sigma)$.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $a + bX \sim \mathcal{N}(a + b\mu, b^2\sigma^2)$ for $b > 0$.
- If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent, then $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

- If $Z \sim \mathcal{N}(0, 1)$, then $Z^2 \sim \chi_1^2$.

Proof. Let $Y := Z^2$. For $y \in [0, \infty)$,

$$\mathbb{P}(Y \leq y) = \mathbb{P}(Z^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq Z \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \phi(t) dt =: F(y).$$

Then,

$$\begin{aligned} f(y) &= \frac{d}{dy} F(y) = \phi(\sqrt{y}) \frac{d}{dy} \sqrt{y} - \phi(-\sqrt{y}) \frac{d}{dy} (-\sqrt{y}) \\ &= 2\phi(\sqrt{y}) \frac{d}{dy} \sqrt{y} = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} y^{-\frac{1}{2}}. \end{aligned}$$

If $Y \sim \chi_1^2$, i.e., $Y \sim \text{Gamma}(1/2, 1/2)$, it means its pdf is

$$f(y) = \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})} y^{-\frac{1}{2}} e^{-\frac{y}{2}}.$$

The proof is completed by showing that $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \sqrt{\pi}$, which can be seen if we convert to polar coordinates.

- If $Z \sim \mathcal{N}(0, 1)$ and $V \sim \chi_p^2$ are independent, then $\frac{Z}{\sqrt{V/p}} \sim t_p$.

Proof. Since $V \sim \chi_p^2$, by definition, its pdf is

$$f_V(v) = \frac{\left(\frac{1}{2}\right)^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} v^{\frac{p}{2}-1} e^{-\frac{1}{2}v}, \quad v \in (0, \infty).$$

Let $Y := \sqrt{V/p}$. For $y \in (0, \infty)$,

$$f_Y(y) = \frac{d}{dy} \mathbb{P}(Y \leq y) = \frac{d}{dy} \mathbb{P}(V \leq py^2) = \frac{d}{dy} \int_0^{py^2} f_V(v) dv = 2py f_V(py^2).$$

Let $T := \frac{Z}{\sqrt{V/p}} = \frac{Z}{Y}$. For $t \in \mathbb{R}$,

$$\mathbb{P}(T \leq t) = \mathbb{P}\left(\frac{Z}{Y} \leq t\right) = \mathbb{P}(Z \leq tY) = \int_0^\infty \mathbb{P}(Z \leq ty) f_Y(y) dy. \quad (\text{Why?})$$

Then,

$$f_T(t) = \frac{d}{dt} \mathbb{P}(T \leq t) = \int_0^\infty \frac{d}{dt} \mathbb{P}(Z \leq ty) f_Y(y) dy.$$



Proof. (Cont'd) Note that $\frac{d}{dt} \mathbb{P}(Z \leq ty) = \frac{d}{dt} \int_{-\infty}^{ty} \phi(z) dz = y\phi(ty)$. So,

$$\begin{aligned} f_T(t) &= \int_0^\infty y\phi(ty) f_Y(y) dy = \int_0^\infty y\phi(ty) 2py f_V(py^2) dy \\ &= \int_0^\infty 2py^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2 y^2}{2}} \cdot \frac{(\frac{1}{2})^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} (py^2)^{\frac{p}{2}-1} e^{-\frac{1}{2}py^2} dy \\ &= \frac{1}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} 2^{\frac{1-p}{2}} p^{\frac{p+1}{2}} \int_0^\infty y^p e^{-\frac{1}{2}(t^2+p)y^2} dy. \end{aligned}$$

Let $x := y^2$. Then, integration by substitution shows that

$$\int_0^\infty y^p e^{-\frac{1}{2}(t^2+p)y^2} dy = \frac{1}{2} \int_0^\infty x^{\frac{p-1}{2}} e^{-\frac{1}{2}(t^2+p)x} dx =: \frac{1}{2} \int_0^\infty x^{\alpha-1} e^{-\lambda x} dx,$$

where $\alpha := \frac{p+1}{2}$ and $\lambda := \frac{1}{2}(t^2+p)$. Recalling the pdf of $\Gamma(\alpha, \lambda)$, it is easy to see that $\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \Gamma(\alpha)/\lambda^\alpha$. Finally,

$$\begin{aligned} f_T(t) &= \frac{1}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} 2^{\frac{1-p}{2}} p^{\frac{p+1}{2}} \cdot \frac{1}{2} \frac{\Gamma(\frac{p+1}{2})}{(1/2)^{(p+1)/2} (t^2+p)^{(p+1)/2}} \\ &= \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1+t^2/p)^{(p+1)/2}}. \end{aligned}$$



- $\mathbf{X} := (X_1, \dots, X_k)^\top$ is said to follow a k -variate normal distribution, if **every** linear combination of X_1, \dots, X_k follows a (univariate) normal distribution.
 - \mathbf{X} is also called a (k dimensional) normal random vector.
 - If $k = 2$, $\mathbf{X} = (X_1, X_2)^\top$ is also said to follow a *bivariate* normal distribution.
- $\mathbf{X} \sim$ a k -variate normal distribution, denoted as $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, if its joint pdf is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{k/2} |\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}, \mathbf{x} \in \mathbb{R}^k,$$

where $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$.

- $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)^\top = \mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \dots, \mathbb{E}[X_k])^\top \in \mathbb{R}^k$.
- $\boldsymbol{\Sigma} = (\Sigma_{ij}) = \text{Cov}(\mathbf{X}, \mathbf{X}) = (\text{Cov}(Z_i, Z_j)) \in \mathbb{R}^{k \times k}$.
- $\boldsymbol{\Sigma}$ is a symmetric and positive definite matrix.
- $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $i = 1, \dots, k$.



- If $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is k dimensional, then
 - $\mathbf{Z} := \mathbf{A}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, where \mathbf{A} satisfies $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^\top$ (Cholesky decomposition), $\mathbf{0} \in \mathbb{R}^k$, and $\mathbf{I} \in \mathbb{R}^{k \times k}$ denotes the identity matrix.
 - $\mathbf{Z} = (Z_1, \dots, Z_k)^\top$, where $Z_i \sim \mathcal{N}(0, 1)$, $i = 1, \dots, k$, **iid**.
 - $\mathbf{a} + \mathbf{B}\mathbf{X} \sim \mathcal{N}(\mathbf{a} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^\top)$.[†]
- Suppose \mathbf{X} is a k dimensional random vector. Then, $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \iff$
 There exist $\boldsymbol{\mu} \in \mathbb{R}^k$ and $\mathbf{A} \in \mathbb{R}^{k \times \ell}$ such that $\mathbf{X} = \boldsymbol{\mu} + \mathbf{A}\mathbf{Z}$,
 where $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ with $\mathbf{0} \in \mathbb{R}^\ell$ and $\mathbf{I} \in \mathbb{R}^{\ell \times \ell}$.
 - Such \mathbf{A} must satisfy $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^\top$.

[†]The multivariate normal distribution will be degenerate if \mathbf{B} does not have full row rank (\mathbf{B} 不行满秩).

- Bivariate normal distribution: $(X_1, X_2)^T \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$, and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \text{Cov}(X_1, X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_2, X_1) & \text{Cov}(X_2, X_2) \end{bmatrix} =: \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

and the joint pdf is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times e^{-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x_1-\mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{x_1-\mu_1}{\sigma_1} \right) \left(\frac{x_2-\mu_2}{\sigma_2} \right) + \left(\frac{x_2-\mu_2}{\sigma_2} \right)^2 \right]}.$$

- To see $\rho = 0 \implies X_1 \perp X_2$, let $\rho = 0$, and note

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left[\left(\frac{x_1-\mu_1}{\sigma_1} \right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2} \right)^2 \right]} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \times \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} = f_{X_1}(x_1)f_{X_2}(x_2). \end{aligned}$$

- If $(X_1, X_2)^\top \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $X_i \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, 2$, then $X_1 + X_2 \perp X_1 - X_2$.

Proof. Note that

$$\mathbf{Y} := \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} =: \mathbf{B} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Since \mathbf{B} has full row rank, $\mathbf{Y} \sim \mathcal{N}(\mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^\top)$, which is non-degenerate. Hence, to prove $X_1 + X_2 \perp X_1 - X_2$, it suffices to show $\text{Cov}(X_1 + X_2, X_1 - X_2) = 0$. Note that

$$\begin{aligned} \text{Cov}(X_1 + X_2, X_1 - X_2) &= \text{Cov}(X_1, X_1) - \text{Cov}(X_2, X_2) \\ &= \sigma^2 - \sigma^2 = 0. \end{aligned}$$



- There are many other relationships among various probability distributions.
 - See, for example, [Song \(2005\)](#);
 - Or, [Leemis & McQueston \(2008\)](#) and their online interactive graph <http://www.math.wm.edu/~leemis/chart/UDR/UDR.html>

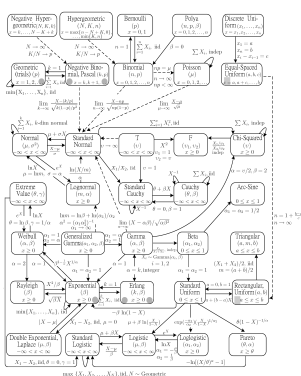


Figure: Relationships Among 35 Distributions (from [Song \(2005\)](#))

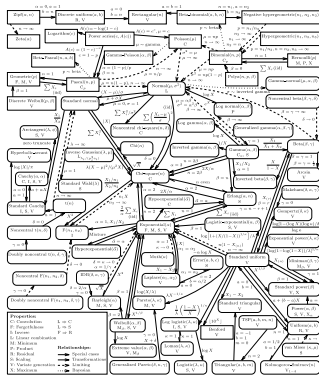


Figure: Relationships Among 76 Distributions (from [Leemis & McQueston \(2008\)](#))

- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations
- 4 Common Distributions
- 5 Useful Inequalities**
- 6 Convergence
- 7 Properties of a Random Sample



Markov's Inequality

Let X be a RV. If $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X = 0) < 1$, then, for any $r > 0$,

$$\mathbb{P}(X \geq r) \leq \frac{\mathbb{E}[X]}{r},$$

with equality if and only if

$$X = \begin{cases} r, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$

- Markov's Inequality has many variations, which are usually called Chebyshev's Inequality.



Chebyshev's Inequality

Let X be a RV and $g(x)$ be a nonnegative function. Then, for any $r > 0$,

$$\mathbb{P}(g(X) \geq r) \leq \frac{\mathbb{E}[g(X)]}{r}.$$

Chebyshev's Inequality

Let X be a RV. Then, for any $r, p > 0$,

$$\mathbb{P}(|X| \geq r) \leq \frac{\mathbb{E}[|X|^p]}{r^p},$$

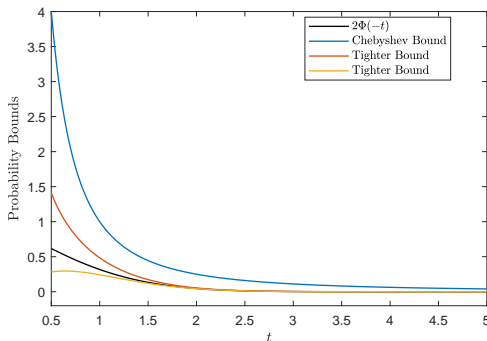
$$\mathbb{P}(|X - \mu| \geq r) \leq \frac{\sigma^2}{r^2},$$

where $\mu := \mathbb{E}[X]$, and $\sigma^2 := \text{Var}(X)$.

- Chebyshev's Inequality is typically very conservative.
- If $Z \sim \mathcal{N}(0, 1)$, a tighter bound is available: For any $t > 0$,

$$2\Phi(-t) = \mathbb{P}(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}} \frac{1}{t} e^{-t^2/2},$$

$$2\Phi(-t) = \mathbb{P}(|Z| \geq t) \geq \sqrt{\frac{2}{\pi}} \frac{t}{1+t^2} e^{-t^2/2}.$$



- A function $g(x)$ is **convex** if

$$g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y),$$

for all x and y , and $\lambda \in (0, 1)$.

- A function $g(x)$ is concave if $-g(x)$ is convex.

Jensen's Inequality

Let X be a RV. If $g(x)$ is a convex function, then

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X]),$$

with equality if and only if $g(x)$ is a linear function on some set A such that $\mathbb{P}(X \in A) = 1$.



Hölder's Inequality

Let X and Y be any two RVs, and let p and q be any two positive numbers (necessarily greater than 1) satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then,

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq \{\mathbb{E}[|X|^p]\}^{1/p} \{\mathbb{E}[|Y|^q]\}^{1/q}.$$



Cauchy-Schwarz Inequality ($p = q = 2$)

Let X and Y be any two RVs, then

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq \{\mathbb{E}[|X|^2]\}^{1/2} \{\mathbb{E}[|Y|^2]\}^{1/2}.$$

Liapounov's Inequality ($Y \equiv 1$)

Let X be a RV, then for any $s > r > 1$,

$$\{\mathbb{E}[|X|^r]\}^{1/r} \leq \{\mathbb{E}[|X|^s]\}^{1/s}.$$



Minkowski's Inequality

Let X and Y be any two RVs. Then, for $p \geq 1$,

$$\{\mathbb{E}[|X + Y|^p]\}^{1/p} \leq \{\mathbb{E}[|X|^p]\}^{1/p} + \{\mathbb{E}[|Y|^p]\}^{1/p}.$$

- **Remark:** The preceding Hölder's Inequality (including its special cases) and Minkowski's Inequality also apply to numerical sums where there is no explicit reference to an expectation.



- 1 Probability Space
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Consider a sequence of RVs $\{X_n : n \geq 1\}$ and another RV X .

- **Convergence Almost Surely** (a.s.), $X_n \xrightarrow{a.s.} X$:

$$\mathbb{P} \left(\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \right\} \right) = 1.$$

- **Convergence in Probability**, $X_n \xrightarrow{p} X$:

$$\lim_{n \rightarrow \infty} \mathbb{P}(\{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}) = 0, \text{ for any } \epsilon > 0.$$

- **Convergence in Distribution**, $X_n \xrightarrow{d} X$, $X_n \Rightarrow X$, or $X_n \xrightarrow{d}$ distribution of X :

$$\lim_{n \rightarrow \infty} F_n(x) = F(x), \text{ for any continuous point } x \text{ of } F(x),$$

where F_n and F are CDF of X_n and X , respectively.

- **Convergence in L^r Norm** ($r \in [1, \infty)$), $X_n \xrightarrow{L^r} X$:

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_n - X|^r) = 0,$$

given $\mathbb{E}[|X_n|^r] < \infty$ for any $n \geq 1$ and $\mathbb{E}[|X|^r] < \infty$.



- Simple relationships:

$$\begin{array}{ccccc}
 X_n \xrightarrow{a.s.} X & \implies & X_n \xrightarrow{p} X & \implies & X_n \xrightarrow{d} X \\
 & & \uparrow & & \\
 X_n \xrightarrow{L^s} X & \xRightarrow{s > r \geq 1} & X_n \xrightarrow{L^r} X & \implies & \mathbb{E}[|X_n|^r] \rightarrow \mathbb{E}[|X|^r]
 \end{array}$$

- $X_n \xrightarrow{d} \text{a constant } c \implies X_n \xrightarrow{p} c.$
- $X_n \xrightarrow{L^1} X \implies \mathbb{E}[X_n] \rightarrow \mathbb{E}[X].$
- $X_n \xrightarrow{a.s.} X \iff \sup_{j \geq n} |X_j - X| \xrightarrow{p} 0.$
- $X_n \xrightarrow{p} X \iff \text{For every subsequence } X_n(m) \text{ there is a further subsequence } X_n(m_k) \text{ such that } X_n(m_k) \xrightarrow{a.s.} X.$

- Question: If $X_n \xrightarrow{d} X$ or $X_n \xrightarrow{p} X$ or $X_n \xrightarrow{a.s.} X$, does it imply $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$?

Monotone Convergence Theorem (MCT)

Suppose $X_n \xrightarrow{a.s.} X$, and $0 \leq X_1 \leq X_2 \leq \dots$ a.s.. Then $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$.

Fatou's Lemma

Suppose $X_n \geq Y$ a.s. for all n where $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[\liminf_{n \rightarrow \infty} X_n] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n]$. In particular, if $X_n \geq 0$ a.s. for all n , then the result holds.

Dominated Convergence Theorem (DCT)

Suppose $X_n \xrightarrow{a.s.} X$, $|X_n| \leq Y$ a.s. for all n , and $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$.

- The DCT is still true if $\xrightarrow{a.s.}$ is replaced by \xrightarrow{p} .
- An **even more general** result:
Suppose $X_n \xrightarrow{p} X$, $|X_n| \leq Y$ a.s. for all n , and $\mathbb{E}[|Y|^r] < \infty$ with $r \geq 1$. Then, $\mathbb{E}[|X_n|^r] < \infty$, $\mathbb{E}[|X|^r] < \infty$, and $X_n \xrightarrow{L^r} X$.



- $X = Y$ a.s., if *any one* of the following holds:
 - $X_n \xrightarrow{a.s.} X$ and $X_n \xrightarrow{a.s.} Y$;
 - $X_n \xrightarrow{p} X$ and $X_n \xrightarrow{p} Y$;
 - $X_n \xrightarrow{L^r} X$ and $X_n \xrightarrow{L^r} Y$.
- If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then $(X_n, Y_n)^\top \xrightarrow{a.s.} (X, Y)^\top$.
 $\implies aX_n + bY_n \xrightarrow{a.s.} aX + bY$; $X_n Y_n \xrightarrow{a.s.} XY$. (Due to CMT)
- If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, then $(X_n, Y_n)^\top \xrightarrow{p} (X, Y)^\top$.
 $\implies aX_n + bY_n \xrightarrow{p} aX + bY$; $X_n Y_n \xrightarrow{p} XY$. (Due to CMT)
- If $X_n \xrightarrow{L^r} X$ and $Y_n \xrightarrow{L^r} Y$, then $(X_n, Y_n)^\top \xrightarrow{L^r} (X, Y)^\top$.
 $\implies aX_n + bY_n \xrightarrow{L^r} aX + bY$.
- **None of the above are true for convergence in distribution.**
- If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} \text{constant } c$, then $(X_n, Y_n)^\top \xrightarrow{d} (X, c)^\top$.
 $\implies aX_n + bY_n \xrightarrow{d} aX + bc$; $X_n Y_n \xrightarrow{d} cX$. (Due to CMT; also known as Slutsky's theorem)



Continuous Mapping Theorem (CMT)

Consider a sequence of RVs $\{X_n : n \geq 1\}$ and another RV X . Suppose g is a function that has the set of discontinuity points D such that $\mathbb{P}(X \in D) = 0$. Then,

$$X_n \xrightarrow{a.s.} X \implies g(X_n) \xrightarrow{a.s.} g(X);$$

$$X_n \xrightarrow{p} X \implies g(X_n) \xrightarrow{p} g(X);$$

$$X_n \xrightarrow{d} X \implies g(X_n) \xrightarrow{d} g(X).$$

- CMT also holds for **random vectors**.
- **Caution:** For convergence in L^r norm, stronger assumption of g than continuity is required to ensure $g(X_n) \xrightarrow{L^r} g(X)$.



- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations
- 4 Common Distributions
- 5 Useful Inequalities
- 6 Convergence
- 7 Properties of a Random Sample**

Properties of a Random Sample

- Let X_1, \dots, X_n be a **random sample** from a distribution with mean μ and variance σ^2 , i.e., X_1, \dots, X_n are iid, and $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$, $i = 1, \dots, n$.

- Define

$$\bar{X} := \frac{1}{n} \sum_{i=1}^n X_i, \text{ and } S^2 := \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}.$$

- For a **general** distribution, the following is true:

- ① \bar{X} is an **unbiased** estimator of μ , i.e., $\mathbb{E}[\bar{X}] = \mu$;
- ② S^2 is an **unbiased** estimator of σ^2 , i.e., $\mathbb{E}[S^2] = \sigma^2$;
- ③ $\text{Var}(\bar{X}) = \sigma^2/n$.

- If the distribution is $\mathcal{N}(\mu, \sigma^2)$, we further have:

- ④ $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$, i.e., $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$;
- ⑤ $\bar{X} \perp S^2$;
- ⑥ $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2$;
- ⑦ $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$.



- For a **general** distribution, *what can we say about the distribution of \bar{X} ?*
- $\text{Var}(\bar{X}) = \sigma^2/n$ intuitively means that the randomness of \bar{X} vanishes and \bar{X} concentrates around μ when n gets large.
- Denote \bar{X} as \bar{X}_n , to explicitly indicate the effect of **sample size** n .

Weak Law of Large Numbers (WLLN)

Suppose X_1, \dots, X_n are iid with mean μ and variance $\sigma^2 < \infty$.[†] Then, $\bar{X}_n \xrightarrow{p} \mu$, as $n \rightarrow \infty$.

Strong Law of Large Numbers (SLLN)

Suppose X_1, \dots, X_n are iid with mean μ and variance $\sigma^2 < \infty$.[†] Then, $\bar{X}_n \xrightarrow{a.s.} \mu$, as $n \rightarrow \infty$.

[†] Mutual independence can be weakened to pairwise independence; $\sigma^2 < \infty$ can be weakened to $\mathbb{E}[|X_i|] \leq \infty$.

- Note that for **normal** distribution, $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$, regardless of the value of n .
- For a **general** distribution, *what can we say about the distribution of $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$?*
- Note that $\mathbb{E} \left[\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \right] = 0$ and $\text{Var} \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \right) = 1$, regardless of the distribution and the value of n .

Central Limit Theorem (CLT)

Suppose X_1, \dots, X_n are iid with mean μ and variance $\sigma^2 \in (0, \infty)$. Then, as $n \rightarrow \infty$,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1).$$