MEM6804 Modeling and Simulation for Logistics & Supply Chain 物流与供应链建模与仿真

Theory Analysis

Lecture 5: Input Modeling

SHEN Haihui 沈海辉

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

shenhaihui.github.io/teaching/mem6804f
shenhaihui@sjtu.edu.cn

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上海交通大学 SHANGHAI JIAO TONG UNIVERSITY | 董浩云航运与物流研究院 CY TUNG Institute of Maritime and Logistics
中美物流研究院(工程系统管理研究院) Sino-US Global Logistics Institute (Institute of Industrial & System Engineering)

Contents

1. Introduction
2. Data Collection
3. Identifying Distribution
   - Physical Basis of Distributions
   - Histogram
4. Distribution Fitting
   - Method of Moments
   - A Simple Variation of MoM
   - Maximum Likelihood Estimation
5. Goodness of Fit
   - Graphical Methods
   - Statistical Tests
   - Remarks
6. An Illustrative Example

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2 Data Collection

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4 Distribution Fitting
- Method of Moments
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5 Goodness of Fit
- Graphical Methods
- Statistical Tests
- Remarks

6 An Illustrative Example

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- The quality of outputs is no better than the quality of inputs.
  - *"Garbage in, garbage out."*
- *"All models are wrong, but some are useful."* – George Box.
  - There is no "true" model for any stochastic input.
  - The best we can do is to obtain an approximation that yields reasonable and useful results.

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  - can capture the physical properties of the system;
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  - can be easily tuned to the situation at hand;
  - can be efficiently generated with certain random variate generation technique.
- Input modeling is sometimes more of an art than an engineering.
  - It nearly always requires the analysts to use their judgment as well as to apply appropriate statistical tools.
  - Since there is no "true" model, it is sensible to run the simulation with several plausible input models to see if the conclusions are robust or highly sensitive to the choices.

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  5. If the fit is not good, select another candidate and go to Step 3, or use an empirical distribution.

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Data Collection

- People often have the **false** impression that data are readily available, but it is one of the most *challenging* task in solving a real problem.
- **Never** trust data blindly!
  - A common mistake is to simply throw data into a software and ask for a "best" fit model.
  - Always take into account under what context (e.g., time, potential influence of other factors) the data was collected.

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- The collected data can be
  - stale (out of date);
  - "dirty" (containing errors);
  - unexpected;
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- Sometimes the effort or cost to transform data into a usable form, or "clean" data, can be as significant as that required to obtain them.

Data Collection

- Suggestions that may enhance and facilitate data collection.
  - Plan ahead: begin by a practice or pre-observing session, watch for unusual circumstances.

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    – Example: customer may quit the queue due to excessive long waiting. How to find out the patience limit for those who don't experience long waiting and receive service?

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  - Check for autocorrelation.
  - Collect input data, not output data.

    – Example: customer arrival times and service times are input, whereas waiting times are output.

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- Do not ignore the physical characteristics of the process when selecting distributions.
  - Is the process naturally discrete or continuous valued?
  - Is it bounded or is there no natural bound?
- There are literally hundreds of probability distributions that have been created; many were created with some specific physical process in mind.

- Discrete Distributions:
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  - **empirical distribution**: Often used when no theoretical distribution seems appropriate.

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    – *Example 1*: the times between the arrivals from a large population of potential customers who act independently.
    – *Example 2*: the time to failure for a system that is memoryless or has constant failure rate over time.
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  - **Weibull**: Models the time to failure for components.
    – *Note*: the failure rate can be increasing, decreasing, or constant (reduce to exponential distribution).

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  - **Erlang**: Models the time that can be viewed as the sum of several exponentially distributed times.
    – *Example*: a computer network fails when a computer and two backup computers fail, and each has exponentially distributed time to failure.
    – *Note*: Erlang is a special case of gamma.

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    – *Note*: can be shifted away from 0 by adding a constant; can cover a range different from $[0, 1]$ by multiplying by a constant.
  - **triangular**: Models a process for which only the minimum, most likely, and maximum values of the distribution are known.
    – *Example*: only the minimum, most likely, and maximum time required to test a product are known.

- **Histogram** describes frequency (or relative frequency, i.e., ratio) of data in different ranges.
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  - Useful in determining the shape of the distribution from which the data have been sampled.
- For continuous data:
  - Corresponds to the pdf of a theoretical distribution.
  - In terms of the *shape*, not the exact *value*!
- For discrete data:
  - Corresponds to the pmf of a theoretical distribution.
  - In terms of both the *shape* and *value* (if the histogram uses relative frequency).
  - If there are few data points, it could be necessary to combine adjacent cells to eliminate the ragged appearance of the histogram.

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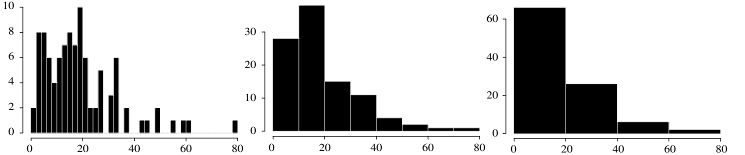

Figure: Ragged, Appropriate and Coarse Histograms (*from Banks et al. (2010)*)

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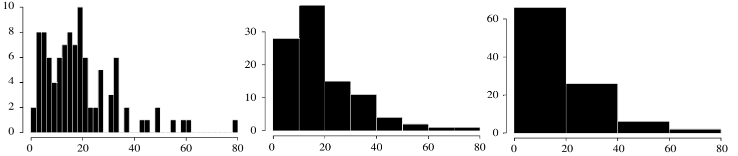

Figure: Ragged, Appropriate and Coarse Histograms (*from* *Banks et al. (2010)*)

- Choosing the number of intervals approximately equal to the square root of the sample size often works well in practice (Hines et al. 2002).

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Distribution Fitting

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- There are many different approaches and we discuss two simple ones:
  - method of moments (MoM)
  - maximum likelihood estimation (MLE)

- For a random variable X, its kth moment is defined as $\mathbb{E}[X^k]$.

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- Let $X_1, \ldots, X_n$ be a random sample of X. The kth sample moment is defined as
$$m_k := \frac{X_1^k + \cdots + X_n^k}{n}.$$

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- Let $X_1, \ldots, X_n$ be a random sample of X. The kth sample moment is defined as
$$m_k := \frac{X_1^k + \cdots + X_n^k}{n}.$$
- Suppose the considered distribution family has s unknown parameters.
  1. Analytically compute $\mathbb{E}[X^1], \ldots, \mathbb{E}[X^s]$, as functions of those parameters.
     – *Note*: the moments of common distributions are well-known.
  2. Compute $m_1, \ldots, m_s$ from the data.
  3. Solve $\mathbb{E}[X^k] = m_k, \ k = 1, \ldots, s$, for s unknown parameters.

- Example 1: Suppose $X_1, \ldots, X_n$ are iid from $\mathrm{Gamma}(\alpha, \lambda)$ (in shape & rate parametrization).
  - Recall: $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $\mathbb{E}[X] = \alpha/\lambda$, $\mathrm{Var}(X) = \alpha/\lambda^2$.

  Estimate α and λ using MoM.

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  Estimate α and λ using MoM.

  Solution. The first two moments are

$$\begin{aligned}\mathbb{E}[X] &= \alpha/\lambda = m_1, \\ \mathbb{E}[X^2] &= \mathrm{Var}(X) + (\mathbb{E}[X])^2 = (\alpha + \alpha^2)/\lambda^2 = m_2.\end{aligned}$$

  Solving two equations yields MoM estimators

$$\widehat{\alpha} = \frac{m_1^2}{m_2 - m_1^2}, \quad \widehat{\lambda} = \frac{m_1}{m_2 - m_1^2}. \qquad \blacksquare$$

- Example 2: Suppose $X_1, \ldots, X_n$ are iid from $\mathrm{Exp}(\lambda)$. Estimate λ using MoM.

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  Solution. The first moment is

  $$\mathbb{E}[X] = 1/\lambda = m_1.$$

  So the MoM estimator of λ is $\widehat{\lambda} = \frac{1}{m_1} = \frac{n}{X_1 + \cdots + X_n}$. ■

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$$\widehat{\mu} = m_1, \quad \widehat{\sigma}^2 = m_2 - m_1^2. \qquad \blacksquare$$

- Remark: $\widehat{\mu} = \frac{\sum_{i=1}^n X_i}{n}$, and

$$\begin{aligned} \widehat{\sigma}^2 = \frac{X_1^2 + \cdots + X_n^2}{n} - \bar{X}^2 &= \frac{\sum_{i=1}^n X_i^2 - n\bar{X}^2}{n} \\ &= \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}. \quad \text{(why?)} \end{aligned}$$

- Many common distributions have no more than 2 parameters: $\mathrm{Ber}(p)$, $\mathrm{B}(n,p)$, $\mathrm{NB}(r,p)$, $\mathrm{Geo}(p)$, $\mathrm{Pois}(\lambda)$, $\mathrm{Unif}[a,b]$, $\mathrm{Exp}(\lambda)$, $\mathrm{Erl}(k,\lambda)$, $\mathrm{Gamma}(\alpha,\lambda)$, $\mathrm{Beta}(\alpha,\beta)$, $\mathrm{Weibull}(\alpha,\beta)$, $\mathcal{N}(\mu,\sigma^2)$, t_p, χ_p^2.

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- Instead of using MoM, another convenient way to estimate the parameters is using sample mean $\bar{X}$ and sample variance S^2:

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n} = m_1,$$
$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1} = \frac{\sum_{i=1}^n X_i^2 - n\bar{X}^2}{n-1} = \frac{n}{n-1}(m_2 - m_1^2),$$

to solve $\mathbb{E}[X] = \bar{X}$, and $\mathrm{Var}(X) = S^2$ (if necessary).

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- Remarks:
  - If $X_1, \dots, X_n$ haven't been observed, $\lambda^* = n/(X_1 + \cdots + X_n)$.
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- For discrete distributions, replace the pdf with pmf.

1 Introduction

2 Data Collection

3 Identifying Distribution
- Physical Basis of Distributions
- Histogram

4 Distribution Fitting
- Method of Moments
- A Simple Variation of MoM
- Maximum Likelihood Estimation

5 Goodness of Fit
- Graphical Methods
- Statistical Tests
- Remarks

6 An Illustrative Example

Goodness of Fit

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- Try more than one plot/test before making conclusion.

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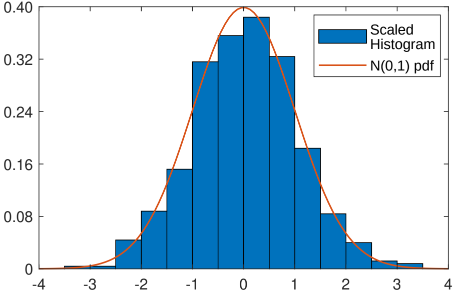

Figure: Example of Scaled Histogram vs. Fitted pdf

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- The q-quantile of X is that value γ such that $\mathbb{P}(X \leq \gamma) = F(\gamma) = q$, for $0 < q < 1$. When $F(x)$ has an inverse, we can write $\gamma = F^{-1}(q)$.
    - Median: 50% quantile.
    - In financial risk management, quantile of the profit-and-loss of a portfolio is also called Value-at-Risk (VaR).

- To make Q-Q plots, given the data $\{x_1, \ldots, x_n\}$ and the fitted distribution with CDF $F(x)$:
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    - If the data is indeed generated from distribution $F(x)$, then

$$y_j \approx F^{-1}\left(\frac{j-0.5}{n}\right),$$

so the plot will be approximately a straight line with slop 1.

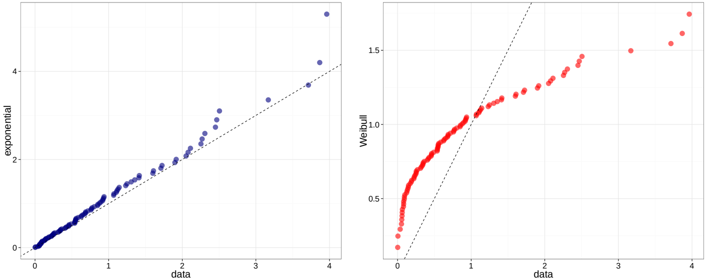
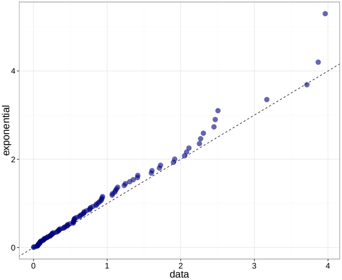
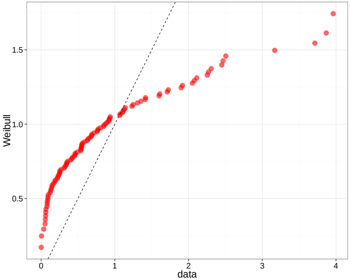


Figure: Examples of Q-Q Plot (*from ZHANG Xiaowei*)





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- The observed values will never fall exactly on a straight line
- The ordered values are not independent because they are ranked. Hence, if one point lies above the line, it is likely that the next one will too.
- The values at the extremes have a much higher variance than those in the middle. So greater discrepancies can be acceptable at the extremes; linearity in the middle is much more important.

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Truth \ Decision	reject H_0	fail to reject H_0
H_0 is true	type I error	correct
H_1 is true	correct	type II error

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  - the computed test statistic falls in certain range (called *rejection region*), which is determined by α.

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  - otherwise, there is no enough evidence to reject H_0 (fail to reject H_0).

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E_i &:= \textbf{expected} \text{ number of points in } [a_{i-1}, a_i) \text{ for fitted dist.} \\
&= n \times \mathbb{P}(a_{i-1} \le X < a_i) \\
&= n \int_{a_{i-1}}^{a_i} f(x)\mathrm{d}x \quad \text{or} \quad n \sum_{a_{i-1} \le x_j < a_i} p(x_j).
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     – Reason: A large value of R indicates a poor fit, whereas a small value indicates a good fit.
     – Question: How large is too large? (i.e., what is the rejection region?)

- View the test statistic R as a random variable.
  - Since we assume the collected data is one observed random sample from some unknown distribution, if we conduct the study multiple times, the values of the statistics will be different because the collected data will be different.
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- If H_0 is true, then R **approximately** follows the chi-square distribution with $k - s - 1$ degrees of freedom (i.e., χ^2_{k-s-1} distribution) when sample size n is large.

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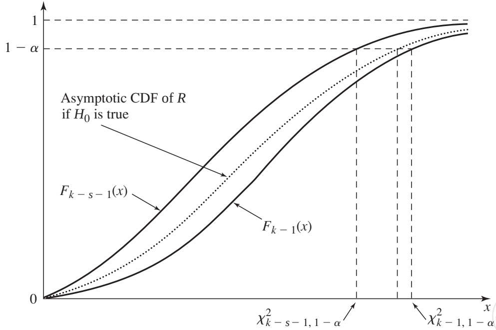

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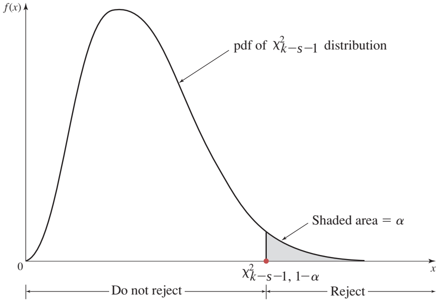

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- In the absence of a definitive guideline for choosing the intervals, it's usually recommended to make E_i equal (or approximately equal) and no less than 5, for all intervals.

- The **Kolmogorov-Smirnov test** (K-S test, 柯尔莫哥洛夫–斯米尔诺夫检验) formally compares the empirical CDF $F_n(x)$ with the CDF of the hypothesized distribution, $F(x)$.

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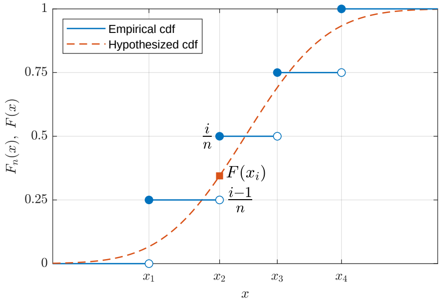

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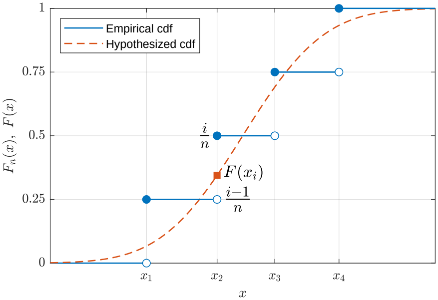

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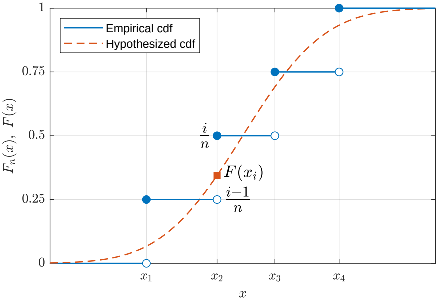

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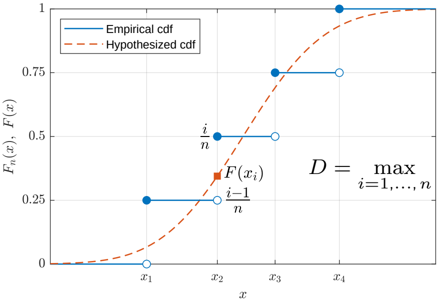

$$D = \max_{i=1,\dots,n} \left\{ \left(\frac{i}{n} - F(x_i) \right) \bigvee \left(F(x_i) - \frac{i-1}{n} \right) \right\}.$$

Note: $x_1, \dots, x_n$ are the **sorted** data points.

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- If no parameter of $F(x)$ is estimated in any way:
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- If $F(x)$ is CDF of distribution such as normal, exponential, or Weibull, and parameters are estimated via MLE (except for normal σ^2, which is estimated by S^2):
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  - Approximations and numerical tables are available.

- Advantage of K-S test:
    - It does not require us to group the data in any way, so no information is lost and no troublesome selection is faced.
    - It is valid (exactly) for any sample size, whereas chi-square test is valid only in an asymptotic sense.
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- K-S test is relatively more convenient to be used in a case where the hypothesized distribution is continuous and no parameter is estimated. For example:
  - Test random number generators.
  - Test a Poisson process (more details later).

- Comments on p-value:
  - p-value can be viewed as a measure of fit: a large p-value tends to indicate a good fit, while a small p-value suggests a poor fit.
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  - Different statistical tests may give different p-values.
  - Whether or not you reject H_0 also depends on the significance level α chosen by yourself.

- Comments on general goodness-of-fit tests:
  - If very little data are available, then a goodness-of-fit test is unlikely to reject any candidate distribution.
    - No enough evidence to reject H_0.
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    – H_0 is virtually never exactly true, and even a tiny departure from the hypothesized distribution will be detected for large n.
  - Do not have blind faith in goodness-of-fit tests!
    – Failing to reject a candidate distribution should be taken as only **one piece of evidence** in favor of that choice.
    – Rejecting a candidate distribution should be taken as only **one piece of evidence** against the choice.

- Graphical Methods vs. Statistical Tests
  - Graphical methods *qualitatively* measure the fitting goodness, while statistical tests *quantitatively* measure the fitting goodness.
  - Statistical tests measure the lack of fit by summary statistics, while graphical methods show where the lack of fit occurs (body, left tail, right tail) and allow users to decide whether it is important.
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- Always combine statistical test results with graphical analysis.
- When no model fits the data satisfactorily, we may end up with the empirical distribution.

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  - It recommends the "best" distribution in its library based on summary measure like the p-value (and perhaps other factors such as discrete or continuous, bounded or unbounded).
- Always keep the following in mind when using such an option:
  - The software might know nothing about the physical basis of the data.
  - Automated best-fit procedures tend to choose the more flexible distributions (gamma over Erlang, Weibull over exponential).
  - But, close conformance to the data does not always lead to the most appropriate input model (overfitting).
  - The limitation of summary measure like p-value.
  - View the automated distribution selection as one suggestion, inspect it using graphical methods, and remember that *the final choice is yours*.

- All the graphical methods and statistical tests can be used to check the uniformity of a random number generator (RNG).
  1. Generate a sequences of numbers (as many as you want) using the RNG.
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- Poisson-Process Test
  - Suppose we observe an arrival process for a time interval $[0, T]$, where T is a constant decided before we start our observation.
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  - Method 2: Test if $\mathrm{Unif}(0, T)$ can fit the data $\{s_1, \ldots, s_n\}$ well. (**Why?**)
    - Given $N(T) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as n independent RVs from $\mathrm{Unif}(0, T)$ that are **sorted**.

1 Introduction

2 Data Collection

3 Identifying Distribution
- Physical Basis of Distributions
- Histogram

4 Distribution Fitting
- Method of Moments
- A Simple Variation of MoM
- Maximum Likelihood Estimation

5 Goodness of Fit
- Graphical Methods
- Statistical Tests
- Remarks

6 An Illustrative Example

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1. Data Collection.
   - Perform life tests on a random sample ($n = 50$) of electronic components and record their lifetime, in days:

79.919	3.081	0.062	1.961	5.845
3.027	6.505	0.021	0.013	0.123
6.769	59.899	1.192	34.760	5.009
18.387	0.141	43.565	24.420	0.433
144.695	2.663	17.967	0.091	9.003
0.941	0.878	3.371	2.157	7.579
0.624	5.380	3.148	7.078	23.960
0.590	1.928	0.300	0.002	0.543
7.004	31.764	1.005	1.147	0.219
3.217	14.382	1.008	2.336	4.562

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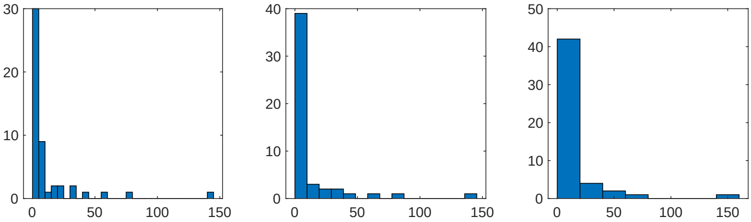

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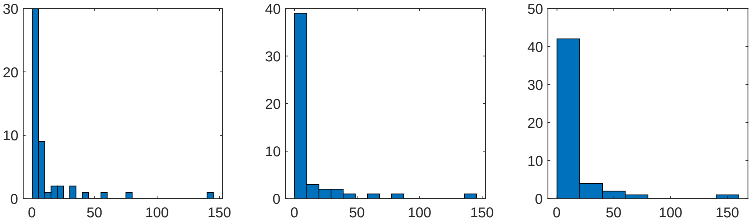

- We decide to first try exponential distribution family $\mathrm{Exp}(\lambda)$.

An Illustrative Example

3. Distribution Fitting.
   - Recall Example 2, MoM (or its variation) and MLE yield the same estimator for λ, which is $\widehat{\lambda} = \frac{n}{X_1+\cdots+X_n}$.
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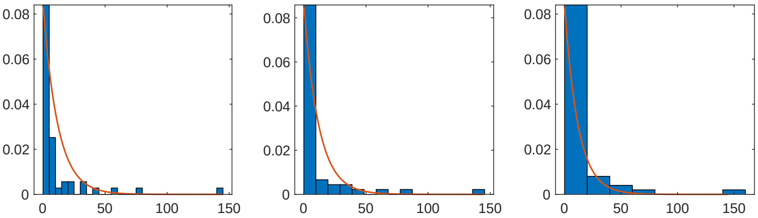
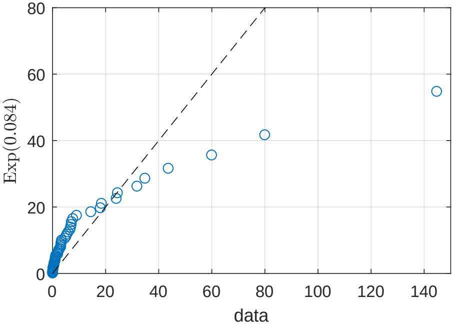
4. Goodness of Fit.
   - Scaled histogram vs. pdf of $\mathrm{Exp}(0.084)$.



An Illustrative Example

4. Goodness of Fit.
    - Q-Q plot.



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  Choose intervals (make E_i equal).

Class Interval	*Observed Frequency* O_i	*Expected Frequency* E_i	$\frac{(O_i - E_i)^2}{E_i}$
[0, 1.590)	19	6.25	26.01
[1.590, 3.425)	10	6.25	2.25
[3.425, 5.595)	3	6.25	0.81
[5.595, 8.252)	6	6.25	0.01
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  So, p-value $= \mathbb{P}(R \geq r) = \mathbb{P}(R \geq 39.6) = 5 \times 10^{-7}$.

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  Hence, at almost any practical level of significance, e.g., $\alpha = 0.05$, $\alpha = 0.01$, we will reject H_0.