MEM6810 Engineering Systems Modeling and Simulation

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

Spring 2025 (full-time)

Assignment 1

Due Date: April 11 (in class)

Instruction

(a) You can answer in English or Chinese or both.
(b) Show **enough** intermediate steps.
(c) Write by hand.

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Question 1 ($10 + 5 = 15$ points)

(1) Prove the result in Buffon's Needle: $\mathbb{P}(\text{needle crosses a line}) = \dfrac{2l}{\pi d}$. (Lec 1 page 25/33)
(2) If the straight needle is bent to V shape, and let X denote the number of intersection points between a needle and the lines, prove that $\mathbb{E}[X] = \dfrac{2l}{\pi d}$.

Question 2 (10 points)

Prove that $\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y])$.

Question 3 (5 points)

If $X_1, X_2, \dots, X_n$ are n independent random variables, and $X_i \sim \mathrm{Exp}(\lambda_i)$, $i = 1, \dots, n$, prove that

$$\min\{X_1, \dots, X_n\} \sim \mathrm{Exp}(\lambda_1 + \cdots + \lambda_n).$$

Question 4 (10 points)

Use Hölder's Inequality to prove that, for real numbers a_i, b_i, $i = 1, 2, \dots, n$, and positive real numbers p and q such that $1/p + 1/q = 1$,

$$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}.$$

Question 5 (10 points)

Use Minkowski's Inequality to prove that, for real numbers a_i, b_i, $i = 1, 2, \ldots, n$, and real number $p \geq 1$,

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}.$$

Question 6 (10 points)

For *positive* real numbers a_i, $i = 1, 2, \ldots, n$, define

$$\begin{aligned} \text{arithmetic mean: } a_A &= \frac{1}{n}(a_1 + \cdots + a_n), \\ \text{geometric mean: } a_G &= (a_1 \times \cdots \times a_n)^{1/n}, \\ \text{harmonic mean: } a_H &= \frac{1}{\frac{1}{n}(\frac{1}{a_1} + \cdots + \frac{1}{a_n})}. \end{aligned}$$

Use Jensen's Inequality to prove that $a_H \leq a_G \leq a_A$. (Hint: Use the log() function.)

Question 7 (10 points)

Prove the Weak Law of Large Numbers with iid assumption and $\sigma^2 < \infty$. (Hint: Use Chebyshev's Inequality.)

Question 8 (10 points)

Recall the Numerical Integration example in Lec 1 page 28/33. Suppose that $f(x)$ is continuous on $[a, b]$. Let

$$Y_n := \frac{b-a}{n}[f(X_1) + \cdots + f(X_n)].$$

Prove that $Y_n \xrightarrow{a.s.} \int_a^b f(x)\mathrm{d}x$ as $n \to \infty$.

Question 9 ($5 + 5 + 10 = 20$ points)

Background: To prove that a sequence of random variables $\{X_n : n \geq 1\}$ converge to a random variable X a.s. is often not easy. One nice strategy is as follows.

(a) First try to prove for any $\epsilon > 0$, $\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \epsilon) < \infty$.

(b) Then by Borel-Cantelli Lemma, we can conclude that for any $\epsilon > 0$, $\mathbb{P}(|X_n - X| > \epsilon \text{ i.o.}) = 0$.

(c) Finally we realize that if $\mathbb{P}(|X_n - X| > \epsilon \text{ i.o.}) = 0$ for any $\epsilon > 0$, then $X_n \xrightarrow{a.s.} X$.

So, to prove $X_n \xrightarrow{a.s.} X$, it suffices to prove $\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \epsilon) < \infty$ for any $\epsilon > 0$, which is often more mathematically traceable.

Now let us prove the fact (c). This task can be further decomposed into three subtasks. Recall the definition of a.s. convergence:

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

(1) Prove the above definition of a.s.convergence is equivalent to the following definition.

$$\mathbb{P}\left\{\omega \in \Omega : \forall \epsilon > 0,\ \exists N_{\epsilon,\omega} \text{ s.t. } \forall n \geq N_{\epsilon,\omega},\ |X_n(\omega) - X(\omega)| \leq \epsilon\right\} = 1,$$

where $\forall$ means "for any", $\exists$ means "there exists", and s.t. means "such that".

(2) Prove that, for any fixed ω, the following two things are equivalent:

$$\forall \epsilon > 0,\ \exists N_{\epsilon,\omega} \text{ s.t. } \forall n \geq N_{\epsilon,\omega},\ |X_n(\omega) - X(\omega)| \leq \epsilon,$$

$$\forall m \in \mathbb{N}^+,\ \exists N_{m,\omega} \text{ s.t. } \forall n \geq N_{m,\omega},\ |X_n(\omega) - X(\omega)| \leq 1/m,$$

where $\mathbb{N}^+$ denotes the set of natural numbers.

(3) Finish the proof of fact (c).

[Here are some hints. De Morgan's laws: Consider sets $A_i,\ i \in I$, where I can be either a countable set or an uncountable set. Let $\overline{A}$ denote the complement of a set A. Then $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$, and $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$. Boole's inequality: When I is a countable set, $\mathbb{P}(\bigcup_{i\in I} A_i) \leq \sum_{i\in I} \mathbb{P}(A_i)$.]