

MEM6810 Engineering Systems Modeling and Simulation

工程系统建模与仿真

Theory Analysis

Lecture 1: Introduction to Simulation

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上海交通大学
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董浩云航运与物流研究院

CY TUNG Institute of Maritime and Logistics

中美物流研究院 (工程系统管理研究院)

Sino-US Global Logistics Institute (Institute of Industrial & System Engineering)



- ① What is Simulation?
- ② Why Simulation?
- ③ How to Do Simulation?
- ④ Models
 - ▶ Definition
 - ▶ Types of Simulation Models
- ⑤ Examples
 - ▶ Estimate π : Buffon's Needle
 - ▶ Estimate π : Random Points
 - ▶ Numerical Integration
 - ▶ System Time to Failure
- ⑥ Course Outline

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- Simulation is EVERYWHERE!

What is Simulation?

Figure: Physical Simulation of Solid-Fluid Interaction (*from Ruan et al. (2021)*)



What is Simulation?



Figure: Pilot Training in Boeing 787 Flat Panel Trainer (*from [Boeing](#)*)

What is Simulation?

Figure: Airport Simulation (*by Vancouver Airport Services*)
[Video: <https://www.youtube.com/watch?v=JuXwEbAvk2Q>]



What is Simulation?

Figure: Typhoon Simulation ([image](#) by [Atmoz](#) / [CC BY 3.0](#))



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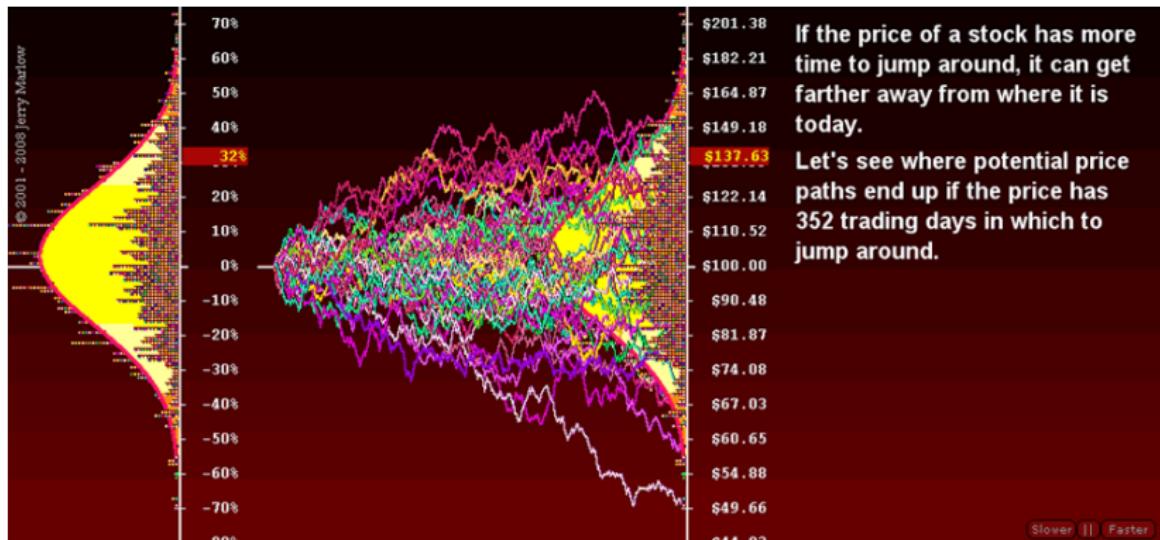


Figure: Financial Analysis

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- With simulation technique, we can easily make change and observe the effect, while keeping high fidelity.

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- Simulation is also an important type of numerical methods.



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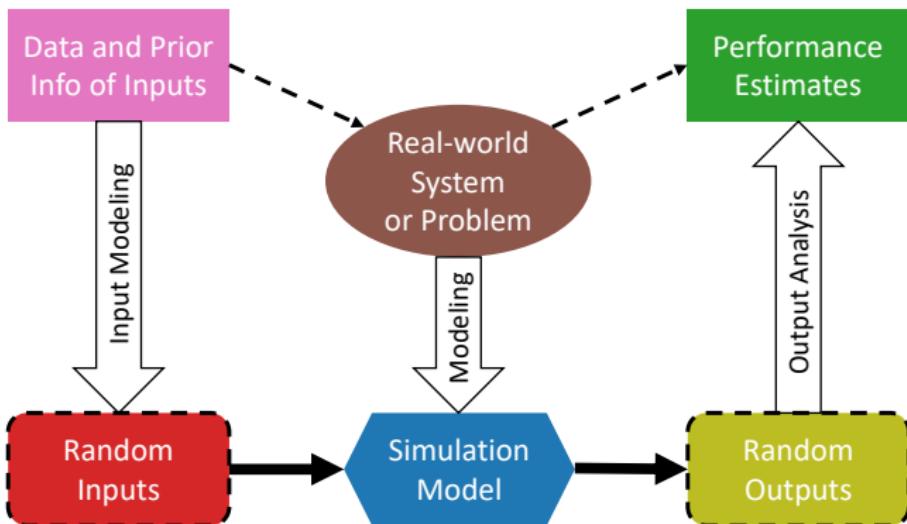


Figure: Basic Paradigm of A Simulation Study

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- A **simulation model** is a particular type of **mathematical model**.





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George E. P. Box (1919.10 – 2013.03) was a British statistician, who worked in the areas of quality control, time-series analysis, design of experiments, and Bayesian inference. He has been called “one of the great statistical minds of the 20th century”.

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- Essentially, running simulation is still one type of numerical methods.
 - Real-world simulation models can be large, and such runs are usually conducted with the aid of a computer.

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Figure: Monte Carlo Casino ([photo](#) by [Cristian Lorini](#) / [CC BY-SA 3.0](#))



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- Example 1 – Logistics Management: evaluate the efficiency of a terminal.
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 - Often used to simulate the logistics/transportation/service systems, whose status naturally changes over time.

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 - Used much more often (uncertainty is more or less involved in a real-world system).

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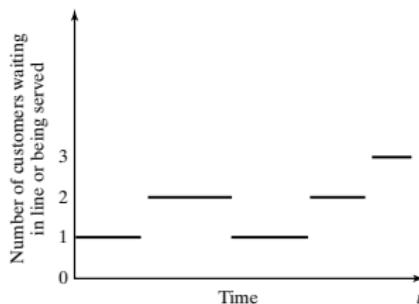


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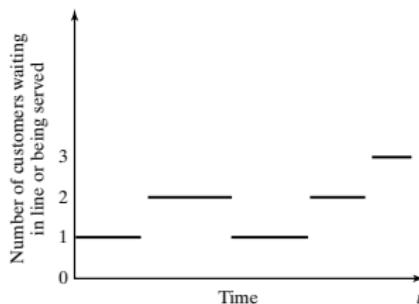


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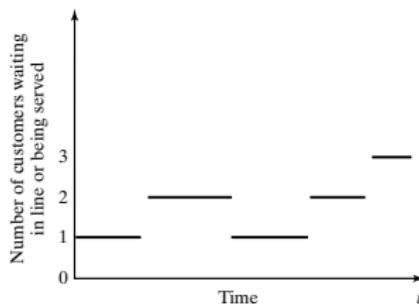


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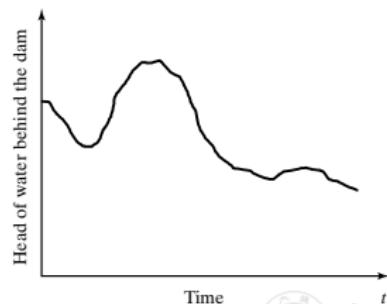


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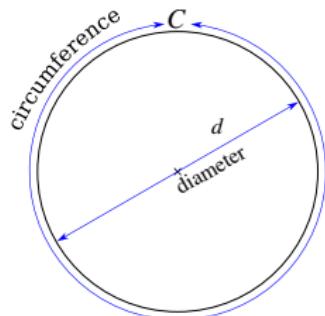
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- For most operational decision-making problems, the suitable simulation models are *dynamic*, *stochastic* and *discrete*.
 - The simulation is called Discrete-Event System Simulation (离散事件系统仿真).
 - It is the main **focus** of this course.

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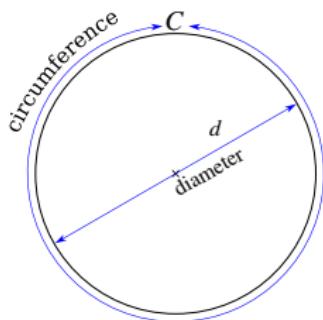
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- It was considered as a quite difficult problem in the history of mankind to find the value of π .

- The earliest written approximations of π :
 - Babylon: A clay tablet (1900–1600 BC), $\pi \approx \frac{25}{8} = 3.1\textcolor{red}{25}\dots$;
 - Egypt: The Rhind Papyrus (莱因德纸草书, 1650 BC, 1850 BC), $\pi \approx (\frac{16}{9})^2 = 3.1\textcolor{red}{60}\dots$

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Figure: Archimedes of Syracuse
(287–212 BC) ([Source/Photographer](#))

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Figure: Zu Chongzhi
(祖冲之, 南北朝时期, 429-500 AD)
(statue image by 三猫 / CC BY 4.0)

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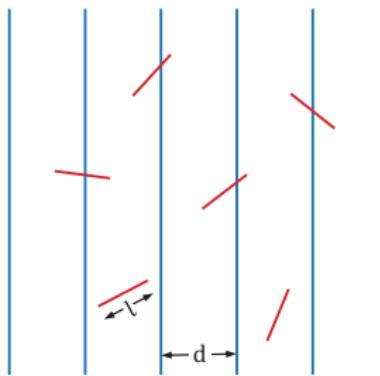
$$\pi \approx 3.1416$$

$$\pi \approx \frac{355}{113} = 3.14159\ 292\dots$$

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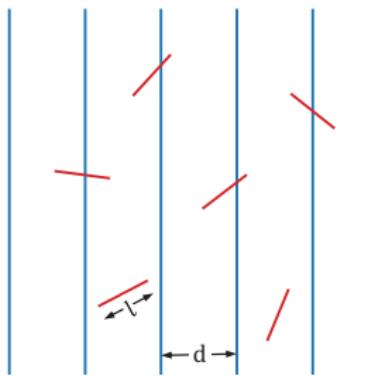
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- Drop a needle of length l onto the floor with parallel lines d apart, where $l < d$.
- Suppose the needle is *equally likely* to fall anywhere.



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- Buffon, a French mathematician, in 1733 (1777) did a static simulation (by hand), which can be used to estimate π .
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- Suppose the needle is *equally likely* to fall anywhere.



- $\mathbb{P}(\text{needle crosses a line}) = \frac{2l}{\pi d}$.

- If Buffon repeats the experiment for n times (i.e., drops n needles), and let h denote the number of needles crossing a line, then,

$$\mathbb{P}(\text{needle crosses a line}) = \frac{2l}{\pi d} \approx \frac{h}{n}.$$

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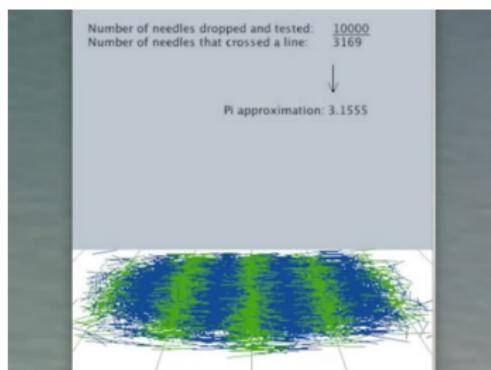
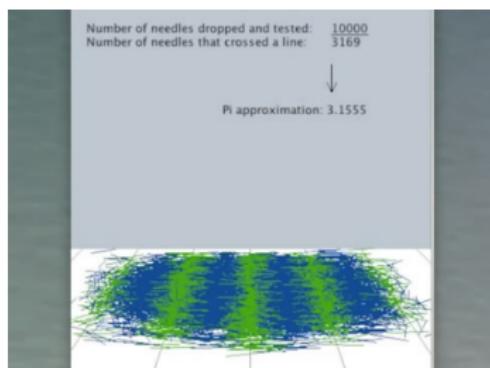


Figure: A Computer Simulation (by Jeffrey Ventrella)
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- Try it out!

<https://mste.illinois.edu/activity/buffon>

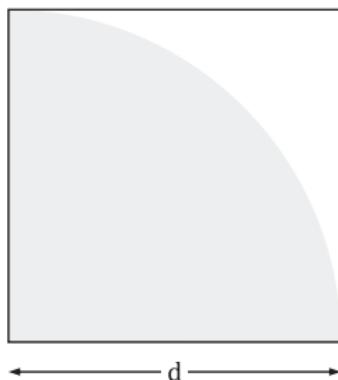
<http://datagenetics.com/blog/may42015/index.html>

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Figure: Animation ([image](#) by [nicoguardo](#)) / [CC BY 3.0](#)

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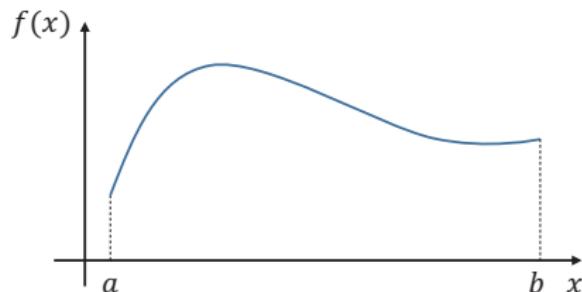


Figure: Animation ([image](#)) by [nicoguardo](#) / [CC BY 3.0](#)

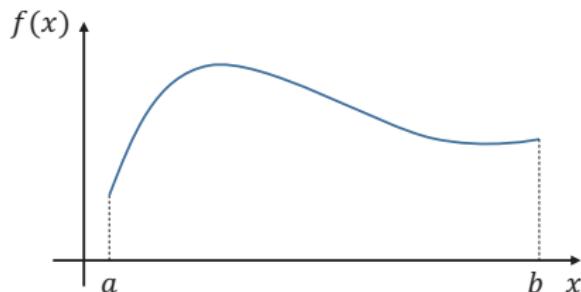
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- Visit <https://xiaoweiz.shinyapps.io/calPi> for interaction.

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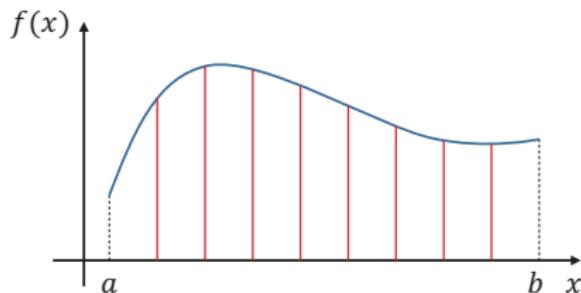


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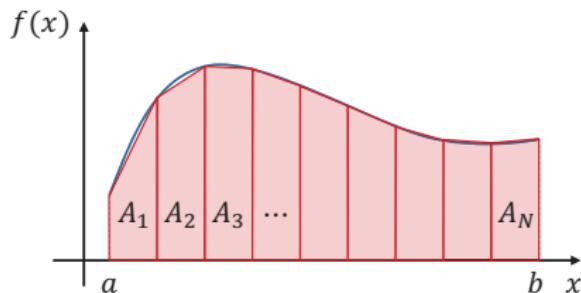
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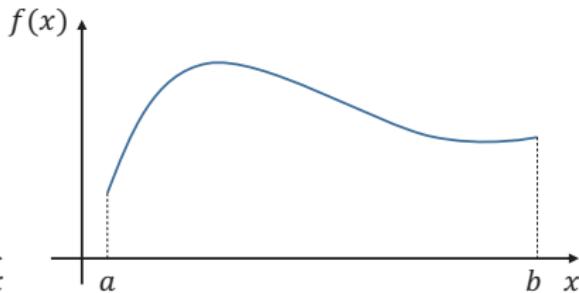
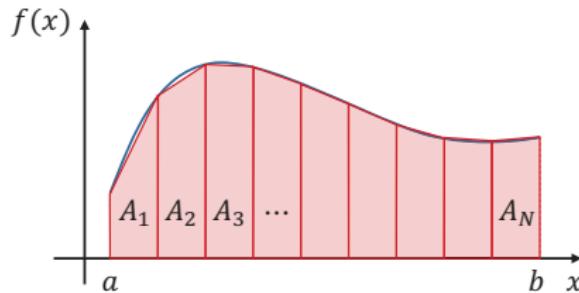
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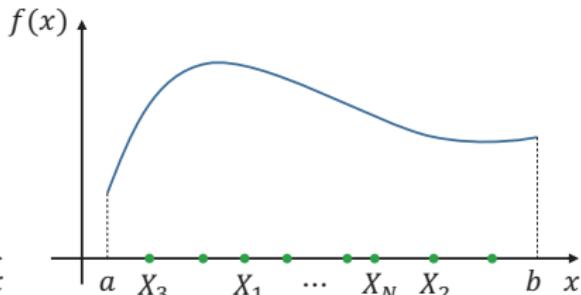
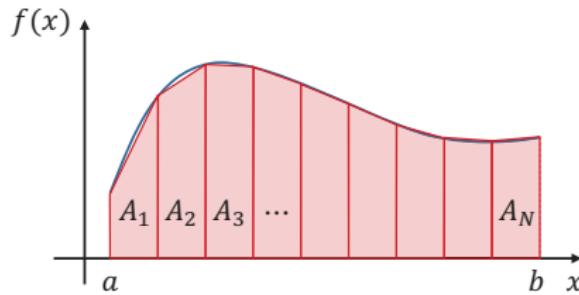
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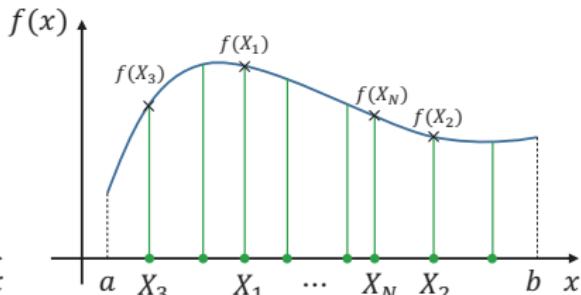
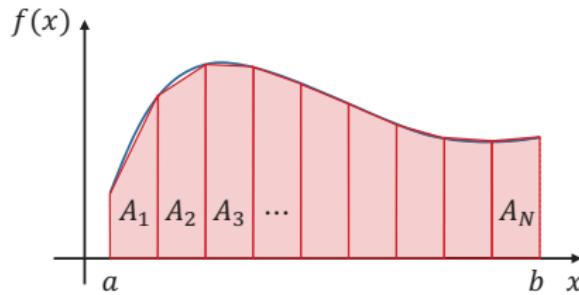
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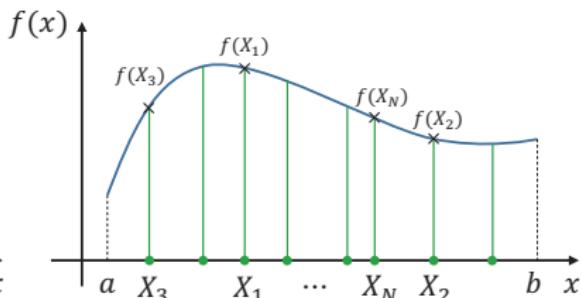
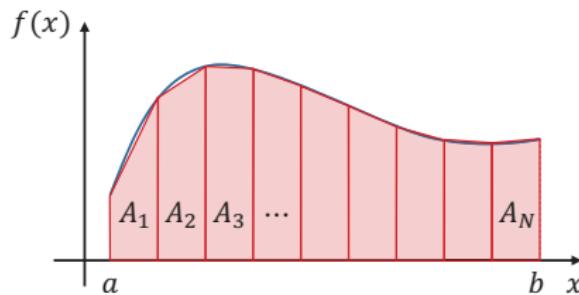
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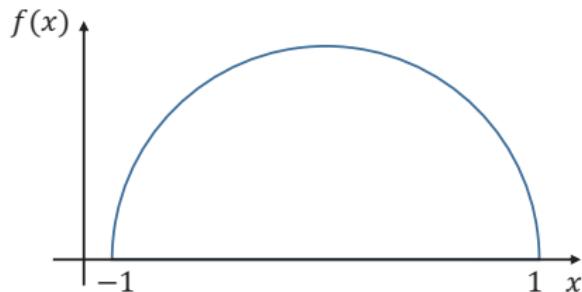
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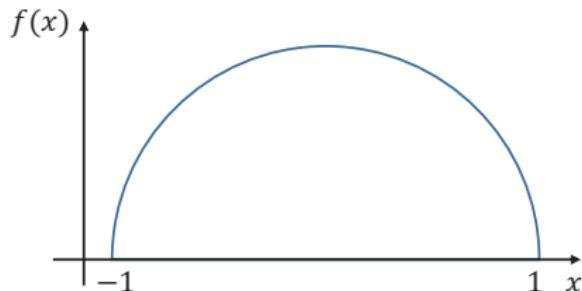
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- Monte Carlo method will be much more **efficient** when the dimension is high! (E.g., $\int_{[a, b]^d} f(x)dx$ for large d .)

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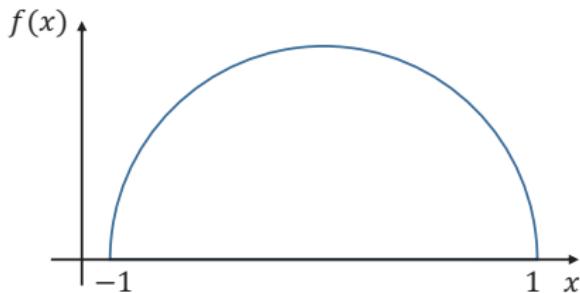


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- Then, $\int_{-1}^1 \sqrt{1 - x^2} dx = \pi/2$.
- So we have another way to estimate π using Monte Carlo simulation (provided we know how to compute square root).

- There is a system:
 - Two components work as active and spare, so the system fails if both components are failed.
 - Suppose the time to next component failure is random (when there is at least one functional components), which follows a known distribution, and we know how to generate it.
 - To make it simple, suppose the time to next failure is equally likely 1, 2, 3, 4, 5 or 6 days (no memory).
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- What can we say about the time to failure for this system?
- Let's run a simulation by hand!
 - Let the system **state** denote the number of functional components.
 - The **events** are the failure of a component and the completion of repair.

Event Calendar			
Clock	System State	Next Failure	Next Repair
0	2		



Event Calendar			
Clock	System State	Next Failure	Next Repair
0	2	$0 + \textcolor{red}{5} = 5$	



Event Calendar			
Clock	System State	Next Failure	Next Repair
0	2	$0 + \textcolor{red}{5} = 5$	∞

Event Calendar			
Clock	System State	Next Failure	Next Repair
0	2	$0 + \textcolor{red}{5} = 5$	∞
5	1		



Event Calendar			
Clock	System State	Next Failure	Next Repair
0	2	$0 + \textcolor{red}{5} = 5$	∞
5	1		$5 + 2.5 = 7.5$

Event Calendar			
Clock	System State	Next Failure	Next Repair
0	2	$0 + \textcolor{red}{5} = 5$	∞
5	1	$5 + \textcolor{red}{3} = 8$	$5 + 2.5 = 7.5$

Event Calendar				
Clock	System State	Next Failure	Next Repair	
0	2	$0 + \textcolor{red}{5} = 5$	∞	
5	1	$5 + \textcolor{red}{3} = 8$	$5 + 2.5 = 7.5$	
7.5	2			

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7.5	2		∞	

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8	1			



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8	1	$8 + \textcolor{red}{6} = 14$	$8 + 2.5 = 10.5$

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10.5	2	14	∞
14	1		$14 + 2.5 = 16.5$

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14	1	$14 + \textcolor{red}{1} = 15$	$14 + 2.5 = 16.5$
15	0		

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- We can observe:

- Time to failure = 15
- Average number of functional components =

$$\begin{aligned} \frac{1}{15-0} [2(5-0) + 1(7.5-5) + 2(8-7.5) + 1(10.5-8) + 2(14-10.5) + 1(15-14)] \\ = \frac{24}{15} \end{aligned}$$

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- Some questions:
 - How to deal with the randomness?
 - How to generate the time interval of component failure?



- ① What is Simulation?
- ② Why Simulation?
- ③ How to Do Simulation?
- ④ Models
 - ▶ Definition
 - ▶ Types of Simulation Models
- ⑤ Examples
 - ▶ Estimate π : Buffon's Needle
 - ▶ Estimate π : Random Points
 - ▶ Numerical Integration
 - ▶ System Time to Failure
- ⑥ Course Outline

Course Outline

- Introduction to Simulation
- Elements of Probability and Statistics
- Queueing Models
- Random Variate Generation
- Input Modeling
- Verification and Validation of Simulation Models
- Output Analysis I: Single Model
- Simulation in Excel and FlexSim
- Output Analysis II: Comparison
- Output Analysis III: Optimization