

MEM6804 Modeling and Simulation for Logistics & Supply Chain

物流与供应链建模与仿真

Theory Analysis

Lecture 2: Elements of Probability and Statistics

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Spring 2021 (full-time)



上海交通大学
SHANGHAI JIAO TONG UNIVERSITY

董浩云航运与物流研究院
CY TUNG Institute of Maritime and Logistics
中美物流研究院 (工程系统管理研究院)
Sino-US Global Logistics Institute (Institute of Industrial & System Engineering)



- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations
- 4 Common Distributions
- 5 Useful Inequalities
- 6 Convergence
- 7 Properties of a Random Sample

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 - ③ Countably additive: If $A_i \in \mathcal{F}$, $i = 1, 2, \dots$, is a **countable** sequence of **disjoint** sets, then $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

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- Example 1: Flip a fair coin.
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 - $\mathcal{F} = \{\emptyset, \{\text{H}\}, \{\text{T}\}, \Omega\};$
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 - $\mathcal{F}_1 = \{\emptyset, [0, a), [a, 1], \Omega\}$, $\mathcal{F}_2 = \{\emptyset, (0, a), \{0\} \cup [a, 1], \Omega\}$...
 - A more practical and interesting \mathcal{F} is the one that contains all intervals (no matter open or closed) on $[0, 1]$.



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- Clearly, mutual independence implies pairwise independence, but not vice versa!



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- Remark: For event A , if $\mathbb{P}(A) = 1$, then we say A happens **almost surely** (a.s.).

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 - A popular convention is to denote the RVs by upper-case letters (e.g., X and Y) and their realizations by lower-case letters (e.g., x and y).

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- The **cumulative distribution function** (CDF) of a RV X , denoted by $F : \mathbb{R} \rightarrow [0, 1]$, is defined by

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$$p(x) := \mathbb{P}(X = x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\}), \quad \forall x \in \mathbb{R},$$

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$$\begin{aligned} F(x, y) &:= \mathbb{P}(X \leq x, Y \leq y) \\ &= \mathbb{P}(\{\omega : X(\omega) \leq x\} \cap \{\omega : Y(\omega) \leq y\}), \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

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② Then, $f(x|y) = \frac{\partial}{\partial x} F(x|Y=y) = \frac{\frac{\partial}{\partial x} \int_{-\infty}^x f(t, y) dt}{f_Y(y)} = \frac{f(x, y)}{f_Y(y)}$.



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- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations**
- 4 Common Distributions
- 5 Useful Inequalities
- 6 Convergence
- 7 Properties of a Random Sample

- The **expectation**, or **expected value**, or **mean**, of a RV X is defined as

$$\mathbb{E}[X] := \int_{\Omega} X(\omega) d\mathbb{P}(\omega),$$

provided that $\int_{\Omega} |X(\omega)| d\mathbb{P}(\omega) < \infty$ or $X \geq 0$ a.s., where the integral is the Lebesgue integral, rather than the Riemann integral.

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 - **Covariance**:
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 - **Correlation**: $\rho(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}$.
- In general, $X \perp Y \xLeftrightarrow{\neq} \rho(X, Y) = 0 \iff \text{Cov}(X, Y) = 0$.
- If $(X, Y)^\top$ follows a bivariate normal distribution,[†] then $X \perp Y \iff \rho(X, Y) = 0$.

[†] **CAUTION:** It means MORE than that X and Y both follow a normal distribution! More details latter.

- The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X|y] := \begin{cases} \sum_{x \in \mathbb{R}} xp(x|y), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{+\infty} xf(x|y), & \text{if } X \text{ is continuous.} \end{cases}$$



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- If $X \not\perp Y$, then $\mathbb{E}[X|Y]$ and $\text{Var}(X|Y)$ are also RVs, whose value depends on the value of Y .

- The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X|y] := \begin{cases} \sum_{x \in \mathbb{R}} xp(x|y), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{+\infty} xf(x|y), & \text{if } X \text{ is continuous.} \end{cases}$$

- The conditional variance of X given $Y = y$ is

$$\text{Var}(X|y) := \mathbb{E}[(X - \mathbb{E}[X])^2|y] = \mathbb{E}[X^2|y] - (\mathbb{E}[X|y])^2.$$

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- If $X \perp Y$, then $\mathbb{E}[X|y] = \mathbb{E}[X|Y] = \mathbb{E}[X]$, and $\text{Var}(X|y) = \text{Var}(X|Y) = \text{Var}(X)$.



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- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations
- 4 Common Distributions**
- 5 Useful Inequalities
- 6 Convergence
- 7 Properties of a Random Sample

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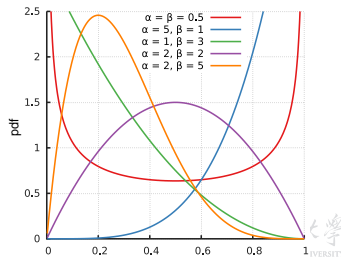
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 - $\alpha = 1, \beta = 1 \implies \text{Unif}(0, 1)$
 - $\alpha > 1, \beta = 1 \implies$ strictly increasing
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 - $\alpha > 1, \beta > 1 \implies$ unimodal



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$$f(x) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1 + x^2/p)^{(p+1)/2}}, \quad x \in \mathbb{R}.$$

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- t_1 is also known as the standard Cauchy distribution, or Cauchy(0, 1), whose pdf is simply

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R}.$$



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The proof is completed by showing that $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \sqrt{\pi}$, which can be seen if we convert to polar coordinates.

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Let $T := \frac{Z}{\sqrt{V/p}} = \frac{Z}{Y}$. For $t \in \mathbb{R}$,

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- If $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is k dimensional, then
 - $\mathbf{Z} := \mathbf{A}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, where \mathbf{A} satisfies $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^\top$ (Cholesky decomposition), $\mathbf{0} \in \mathbb{R}^k$, and $\mathbf{I} \in \mathbb{R}^{k \times k}$ denotes the identity matrix.



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SHANGHAI JIAO TONG UNIVERSITY

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 There exist $\boldsymbol{\mu} \in \mathbb{R}^k$ and $\mathbf{A} \in \mathbb{R}^{k \times \ell}$ such that $\mathbf{X} = \boldsymbol{\mu} + \mathbf{A}\mathbf{Z}$,
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- Bivariate normal distribution: $(X_1, X_2)^T \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$, and

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and the joint pdf is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times e^{-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x_1-\mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{x_1-\mu_1}{\sigma_1} \right) \left(\frac{x_2-\mu_2}{\sigma_2} \right) + \left(\frac{x_2-\mu_2}{\sigma_2} \right)^2 \right]}.$$



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- To see $\rho = 0 \implies X_1 \perp X_2$, let $\rho = 0$, and note

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left[\left(\frac{x_1-\mu_1}{\sigma_1} \right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2} \right)^2 \right]} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \times \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} = f_{X_1}(x_1)f_{X_2}(x_2). \end{aligned}$$

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Proof. Note that

$$\mathbf{Y} := \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} =: \mathbf{B} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$



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$$\begin{aligned} \text{Cov}(X_1 + X_2, X_1 - X_2) &= \text{Cov}(X_1, X_1) - \text{Cov}(X_2, X_2) \\ &= \sigma^2 - \sigma^2 = 0. \end{aligned}$$



- There are many other relationships among various probability distributions.
 - See, for example, [Song \(2005\)](#);
 - Or, [Leemis & McQueston \(2008\)](#) and their online interactive graph <http://www.math.wm.edu/~leemis/chart/UDR/UDR.html>

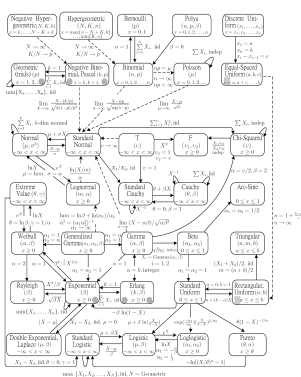


Figure: Relationships Among 35 Distributions (from [Song \(2005\)](#))

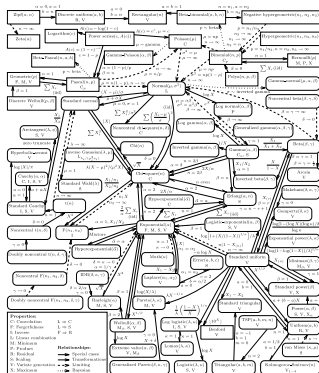


Figure: Relationships Among 76 Distributions (from [Leemis & McQueston \(2008\)](#))

- 1 Probability Space
- 2 Random Variables & Distributions
- 3 Expectations
- 4 Common Distributions
- 5 Useful Inequalities**
- 6 Convergence
- 7 Properties of a Random Sample



Markov's Inequality

Let X be a RV. If $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X = 0) < 1$, then, for any $r > 0$,

$$\mathbb{P}(X \geq r) \leq \frac{\mathbb{E}[X]}{r},$$

with equality if and only if

$$X = \begin{cases} r, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$



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- Markov's Inequality has many variations, which are usually called Chebyshev's Inequality.

Chebyshev's Inequality

Let X be a RV and $g(x)$ be a nonnegative function. Then, for any $r > 0$,

$$\mathbb{P}(g(X) \geq r) \leq \frac{\mathbb{E}[g(X)]}{r}.$$



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Chebyshev's Inequality

Let X be a RV. Then, for any $r, p > 0$,

$$\mathbb{P}(|X| \geq r) \leq \frac{\mathbb{E}[|X|^p]}{r^p},$$

$$\mathbb{P}(|X - \mu| \geq r) \leq \frac{\sigma^2}{r^2},$$

where $\mu := \mathbb{E}[X]$, and $\sigma^2 := \text{Var}(X)$.

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- If $Z \sim \mathcal{N}(0, 1)$, a tighter bound is available: For any $t > 0$,

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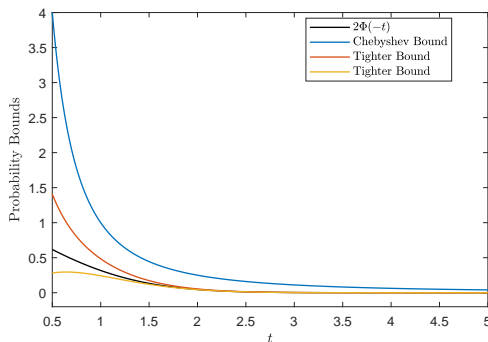
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- A function $g(x)$ is **convex** if

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Jensen's Inequality

Let X be a RV. If $g(x)$ is a convex function, then

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X]),$$

with equality if and only if $g(x)$ is a linear function on some set A such that $\mathbb{P}(X \in A) = 1$.



Hölder's Inequality

Let X and Y be any two RVs, and let p and q be any two positive numbers (necessarily greater than 1) satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then,

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq \{\mathbb{E}[|X|^p]\}^{1/p} \{\mathbb{E}[|Y|^q]\}^{1/q}.$$



Cauchy-Schwarz Inequality ($p = q = 2$)

Let X and Y be any two RVs, then

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Liapounov's Inequality ($Y \equiv 1$)

Let X be a RV, then for any $s > r > 1$,

$$\{\mathbb{E}[|X|^r]\}^{1/r} \leq \{\mathbb{E}[|X|^s]\}^{1/s}.$$



Minkowski's Inequality

Let X and Y be any two RVs. Then, for $p \geq 1$,

$$\{\mathbb{E}[|X + Y|^p]\}^{1/p} \leq \{\mathbb{E}[|X|^p]\}^{1/p} + \{\mathbb{E}[|Y|^p]\}^{1/p}.$$



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- **Remark:** The preceding Hölder's Inequality (including its special cases) and Minkowski's Inequality also apply to numerical sums where there is no explicit reference to an expectation.



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- **Convergence in L^r Norm** ($r \in [1, \infty)$), $X_n \xrightarrow{L^r} X$:

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given $\mathbb{E}[|X_n|^r] < \infty$ for any $n \geq 1$ and $\mathbb{E}[|X|^r] < \infty$.



- Simple relationships:

$$\begin{array}{ccccc}
 X_n \xrightarrow{a.s.} X & \implies & X_n \xrightarrow{p} X & \implies & X_n \Rightarrow X \\
 & & \uparrow & & \\
 X_n \xrightarrow{L^s} X & \xRightarrow{s > r \geq 1} & X_n \xrightarrow{L^r} X & \implies & \mathbb{E}[|X_n|^r] \rightarrow \mathbb{E}[|X|^r]
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- $X_n \xrightarrow{p} X \iff$ For every subsequence $X_n(m)$ there is a further subsequence $X_n(m_k)$ such that $X_n(m_k) \xrightarrow{a.s.} X.$

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Fatou's Lemma

Suppose $X_n \geq Y$ a.s. for all n where $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[\liminf_{n \rightarrow \infty} X_n] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n]$. In particular, if $X_n \geq 0$ a.s. for all n , then the result holds.



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Monotone Convergence Theorem (MCT)

Suppose $X_n \xrightarrow{a.s.} X$, and $0 \leq X_1 \leq X_2 \leq \dots$ a.s.. Then $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$.



Dominated Convergence Theorem (DCT)

Suppose $X_n \xrightarrow{a.s.} X$, $|X_n| \leq Y$ a.s. for all n , and $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$.



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- An **even more general** result:
Suppose $X_n \xrightarrow{p} X$, $|X_n| \leq Y$ a.s. for all n , and $\mathbb{E}[|Y|^r] < \infty$ with $r \geq 1$. Then, $\mathbb{E}[|X_n|^r] < \infty$, $\mathbb{E}[|X|^r] < \infty$, and $X_n \xrightarrow{L^r} X$.

- If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then
 - ① $X = Y$ a.s.;
 - ② Random vector $(X_n, Y_n)^\top \xrightarrow{a.s.} (X, Y)^\top$.



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- If $X_n \Rightarrow X$ and $Y_n \Rightarrow$ a constant c , then
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 $\implies aX_n + bY_n \Rightarrow aX + bc$; $X_n Y_n \Rightarrow cX$. (Slutsky's theorem)

Continuous Mapping Theorem

Consider a sequence of RVs $\{X_n : n \geq 1\}$ and another RV X . Suppose g is a function that has the set of discontinuity points D such that $\mathbb{P}(X \in D) = 0$. Then,

$$X_n \xrightarrow{a.s.} X \implies g(X_n) \xrightarrow{a.s.} g(X);$$

$$X_n \xrightarrow{p} X \implies g(X_n) \xrightarrow{p} g(X);$$

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Properties of a Random Sample

- Let X_1, \dots, X_n be a **random sample** from a distribution with mean μ and variance σ^2 , i.e., X_1, \dots, X_n are iid, and $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$, $i = 1, \dots, n$.

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Properties of a Random Sample

- Let X_1, \dots, X_n be a **random sample** from a distribution with mean μ and variance σ^2 , i.e., X_1, \dots, X_n are iid, and $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$, $i = 1, \dots, n$.

- Define

$$\bar{X} := \frac{1}{n} \sum_{i=1}^n X_i, \text{ and } S^2 := \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}.$$

- For a **general** distribution, the following is true:

- ① \bar{X} is an **unbiased** estimator of μ , i.e., $\mathbb{E}[\bar{X}] = \mu$;
- ② S^2 is an **unbiased** estimator of σ^2 , i.e., $\mathbb{E}[S^2] = \sigma^2$;
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- If the distribution is $\mathcal{N}(\mu, \sigma^2)$, we further have:

- ① $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$, i.e., $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$;
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Weak Law of Large Numbers (WLLN)

Suppose X_1, \dots, X_n are iid with mean μ and variance $\sigma^2 < \infty$.[†] Then, $\bar{X}_n \xrightarrow{p} \mu$.

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Strong Law of Large Numbers (SLLN)

Suppose X_1, \dots, X_n are iid with mean μ and variance $\sigma^2 < \infty$.[†] Then, $\bar{X}_n \xrightarrow{a.s.} \mu$.

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- Note that for **normal** distribution, $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$, regardless of the value of n .
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Central Limit Theorem (CLT)

Suppose X_1, \dots, X_n are iid with mean μ and variance $\sigma^2 \in (0, \infty)$. Then,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \Rightarrow \mathcal{N}(0, 1).$$