MEM6804 Modeling and Simulation for Logistics & Supply Chain
物流与供应链建模与仿真

Theory Analysis

Lecture 2: Elements of Probability and Statistics

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上海交通大学 SHANGHAI JIAO TONG UNIVERSITY | 董浩云航运与物流研究院 CY TUNG Institute of Maritime and Logistics
中美物流研究院（工程系统管理研究院） Sino-US Global Logistics Institute (Institute of Industrial & System Engineering)

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  3. Closed under countable unions:[†] If $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$, is a **countable** sequence of sets, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

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  3. Countably additive: If $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$, is a **countable** sequence of **disjoint** sets, then $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

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  - $\Omega = \{\text{H (head)}, \text{T (tail)}\}$;
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  - A more practical and interesting $\mathcal{F}$ is the one that contains all intervals (no matter open or closed) on $[0, 1]$.

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- Sets $A_1, \ldots, A_n$ are pairwise independent if for any $i \neq j$ we have $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\,\mathbb{P}(A_j)$.
- Clearly, mutual independence implies pairwise independence, but not vice versa!

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(The First) Borel-Cantelli Lemma

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- Remark: For event A, if $\mathbb{P}(A) = 1$, then we say A happens **almost surely** (a.s.).

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  - A popular convention is to denote the RVs by upper-case letters (e.g., X and Y) and their realizations by lower-case letters (e.g., x and y).

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  - Under $(\Omega, \mathcal{F}, \mathbb{P})$, let $X(\omega) = \omega$ for $\omega \in [0, 1]$.

- The **cumulative distribution function** (CDF) of a RV X, denoted by $F : \mathbb{R} \to [0, 1]$, is defined by

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  - $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$;
  - $F(x)$ is nondecreasing in x;

- The **cumulative distribution function** (CDF) of a RV X, denoted by $F : \mathbb{R} \to [0, 1]$, is defined by

$$F(x) \coloneqq \mathbb{P}(X \le x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \le x\}), \ \forall x \in \mathbb{R},$$

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- It is easy to see that $F(x) = \sum_{y\in(-\infty,\, x]} p(y)$.

- A RV X is said to be **continuous** if there exists a **probability density function** (pdf) $f(x)$ such that

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- Observe that $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.

- The **joint** CDF of RVs X and Y, denoted by $F : \mathbb{R}\times\mathbb{R} \to [0, 1]$, is defined by

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- Observe that $\frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$.

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2. Then, $f(x|y) = \frac{\mathrm{d}}{\mathrm{d}x}F(x|Y=y) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x} f(t,y)\mathrm{d}t}{f_Y(y)} = \frac{f(x,y)}{f_Y(y)}$.

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- RVs $X_1, \ldots, X_n$ are pairwise independent if for any $i \neq j$, $X_i \perp X_j$.

1 Probability Space

2 Random Variables & Distributions

3 Expectations

4 Common Distributions

5 Useful Inequalities

6 Convergence

7 Properties of a Random Sample

Expectations

- The **expectation**, or **expected value**, or **mean**, of a RV X is defined as
$$\mathbb{E}[X] := \int_\Omega X(\omega) \mathrm{d}\,\mathbb{P}(\omega),$$
provided that $\int_\Omega |X(\omega)| \mathrm{d}\,\mathbb{P}(\omega) < \infty$ or $X \geq 0$ a.s., where the integral is the Lebesgue integral, rather than the Riemann integral.

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  - **Covariance**: $\mathrm{Cov}(X, Y) := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$.

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  - **Correlation**: $\rho(X,Y) := \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$.

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  - **Variance** (2nd central moment): $\sigma^2 := \mathrm{Var}(X) := \mathbb{E}[(X-\mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.
- Linear association:
  - **Covariance**: $\mathrm{Cov}(X,Y) := \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY]-\mathbb{E}[X]\,\mathbb{E}[Y]$.
  - **Correlation**: $\rho(X,Y) := \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$.
- In general, $X \perp Y \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \rho(X,Y)=0 \iff \mathrm{Cov}(X,Y)=0$.

Expectations

- For integer n, $\mathbb{E}[X^n]$ is called the nth **moment** of X, and $\mathbb{E}[(X-\mathbb{E}[X])^n]$ is called the nth **central moment** of X.
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- In general, $X \perp Y \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \rho(X,Y)=0 \iff \mathrm{Cov}(X,Y)=0$.
- If $(X,Y)^\intercal$ follows a bivariate normal distribution,† then $X \perp Y \iff \rho(X,Y)=0$.

† **CAUTION:** It means MORE than that X and Y both follow a normal distribution! More details latter.

Expectations

- The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X|y] := \begin{cases} \sum_{x\in\mathbb{R}} xp(x|y), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{+\infty} xf(x|y), & \text{if } X \text{ is continuous.} \end{cases}$$

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- If $X \not\perp Y$, then $\mathbb{E}[X|Y]$ and $\mathrm{Var}(X|Y)$ are also RVs, whose value depends on the value of Y.
- If $X \perp Y$, then $\mathbb{E}[X|y] = \mathbb{E}[X|Y] = \mathbb{E}[X]$, and $\mathrm{Var}(X|y) = \mathrm{Var}(X|Y) = \mathrm{Var}(X)$.

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1 Probability Space

2 Random Variables & Distributions

3 Expectations

4 Common Distributions

5 Useful Inequalities

6 Convergence

7 Properties of a Random Sample

- $X \sim \text{Bernoulli}(p)$ or $\text{Ber}(p)$, if

$$X = \begin{cases} 1, & \text{with probability } p, \\ 0, & \text{with probability } 1-p, \end{cases} \quad p \in [0,1].$$

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  - Given $X_1 + X_2 = n$, $X_1 \sim \text{B}(n, \lambda_1/(\lambda_1 + \lambda_2))$.

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  - $\mathbb{E}[X] = 1/\lambda$, $\mathrm{Var}(X) = 1/\lambda^2$.

- $X \sim \mathrm{Uniform}(a, b)$ or $\mathrm{Unif}(a, b)$ with $a < b$, if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a, b], \\ 0, & \text{otherwise.} \end{cases}$$

  - $\mathbb{E}[X] = \frac{b+a}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$.

- $X \sim \mathrm{exponential}(\lambda)$ or $\mathrm{Exp}(\lambda)$, with $\lambda > 0$, if its pdf is given by

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  - **Memoryless Property**: For $s > t \geq 0$,

$$\mathbb{P}(X > s | X > t) = \frac{\mathbb{P}(X > s, X > t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X > s)}{\mathbb{P}(X > t)} = \frac{e^{-\lambda s}}{e^{-\lambda t}} = e^{-\lambda(s-t)}$$
$$= \mathbb{P}(X > s - t).$$

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- If $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$ are independent, then $\min\{X_1, X_2\} \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$.

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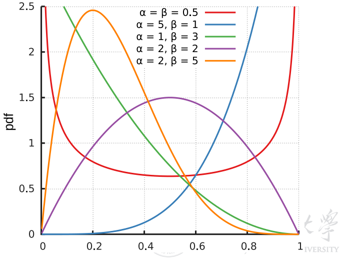
  - $\alpha = 1, \beta > 1 \Longrightarrow$ strictly decreasing
  - $\alpha < 1, \beta < 1 \Longrightarrow$ U-shaped
  - $\alpha > 1, \beta > 1 \Longrightarrow$ unimodal



- $X \sim$ Student's t distribution with p degrees of freedom, denoted as t_p, where p is an integer, if its pdf is given by

$$f(x) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1 + x^2/p)^{(p+1)/2}}, \quad x \in \mathbb{R}.$$

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Proof. Let $Y := Z^2$. For $y \in [0, \infty)$,

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If $Y \sim \chi_1^2$, i.e., $Y \sim \mathrm{Gamma}(1/2, 1/2)$, it means its pdf is

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$$\begin{aligned} f(y) &= \frac{\mathrm{d}}{\mathrm{d}y}F(y) = \phi(\sqrt{y})\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{y} - \phi(-\sqrt{y})\frac{\mathrm{d}}{\mathrm{d}y}(-\sqrt{y}) \\ &= 2\phi(\sqrt{y})\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{y} = \frac{1}{\sqrt{2\pi}}e^{-\frac{y}{2}}y^{-\frac{1}{2}}. \end{aligned}$$

If $Y \sim \chi^2_1$, i.e., $Y \sim \mathrm{Gamma}(1/2, 1/2)$, it means its pdf is

$$f(y) = \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})}y^{-\frac{1}{2}}e^{-\frac{y}{2}}.$$

The proof is completed by showing that $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}\mathrm{d}t = \sqrt{\pi}$, which can be seen if we convert to polar coordinates. ■

- If $Z \sim \mathcal{N}(0,1)$ and $V \sim \chi^2_p$ are independent, then $\frac{Z}{\sqrt{V/p}} \sim t_p$.

- If $Z \sim \mathcal{N}(0,1)$ and $V \sim \chi_p^2$ are independent, then $\frac{Z}{\sqrt{V/p}} \sim t_p$.

Proof. Since $V \sim \chi_p^2$, by definition, its pdf is

$$f_V(v) = \frac{\left(\frac{1}{2}\right)^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} v^{\frac{p}{2}-1} e^{-\frac{1}{2}v}, \quad v \in [0, \infty).$$

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Then,

$$f_T(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(T \le t) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(Z \le ty) f_Y(y)\mathrm{d}y.$$

Proof. *(Cont'd)* Note that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(Z \le ty) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{ty}\phi(z)\mathrm{d}z = y\phi(ty)$.

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Let $x := y^2$. Then, integration by substitution shows that

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$$\begin{aligned} f_T(t) &= \frac{1}{\Gamma(\frac{p}{2})}\frac{1}{(p\pi)^{1/2}} 2^{\frac{1-p}{2}} p^{\frac{p+1}{2}} \cdot \frac{1}{2}\frac{\Gamma(\frac{p+1}{2})}{(1/2)^{(p+1)/2}(t^2+p)^{(p+1)/2}} \\ &= \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{1}{(p\pi)^{1/2}}\frac{1}{(1+t^2/p)^{(p+1)/2}}. \end{aligned}$$

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- If $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is k dimensional, then
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  - $\boldsymbol{a} + \boldsymbol{B}\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{a} + \boldsymbol{B}\boldsymbol{\mu}, \boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{B}^\intercal)$.†

† The multivariate normal distribution will be degenerate if $\boldsymbol{B}$ does not have full row rank ($\boldsymbol{B}$ 不行满秩).

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- Bivariate normal distribution: $(X_1, X_2)^\mathsf{T} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = (\mu_1, \mu_2)^\mathsf{T}$, and

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- To see $\rho = 0 \Longrightarrow X_1 \perp X_2$, let $\rho = 0$, and note

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$
$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \times \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} = f_{X_1}(x_1) f_{X_2}(x_2).$$

- If $(X_1, X_2)^\mathsf{T} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $X_i \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, 2$, then $X_1 + X_2 \perp X_1 - X_2$.

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Proof. Note that

$$\boldsymbol{Y} := \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} =: \boldsymbol{B} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

- If $(X_1, X_2)^\intercal \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $X_i \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, 2$, then $X_1 + X_2 \perp X_1 - X_2$.

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$$\begin{aligned} \mathrm{Cov}(X_1 + X_2, X_1 - X_2) &= \mathrm{Cov}(X_1, X_1) - \mathrm{Cov}(X_2, X_2) \\ &= \sigma^2 - \sigma^2 = 0. \end{aligned}$$

■

- There are many other relationships among various probability distributions.
  - See, for example, Song (2005);
  - Or, Leemis & McQueston (2008) and their online interactive graph http://www.math.wm.edu/~leemis/chart/UDR/UDR.html

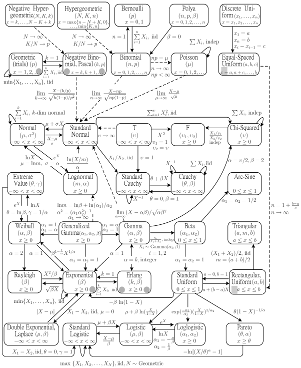

Figure: Relationships Among 35 Distributions (from Song (2005))

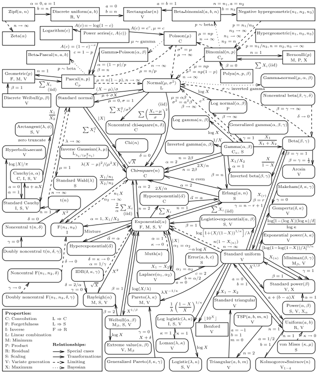

Figure: Relationships Among 76 Distributions (from Leemis & McQueston (2008))

1 Probability Space

2 Random Variables & Distributions

3 Expectations

4 Common Distributions

5 Useful Inequalities

6 Convergence

7 Properties of a Random Sample

Markov's Inequality

Let X be a RV. If $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X = 0) < 1$, then, for any $r > 0$,

$$\mathbb{P}(X \geq r) \leq \frac{\mathbb{E}[X]}{r},$$

with equality if and only if

$$X = \begin{cases} r, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$

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- Markov's Inequality has many variations, which are usually called Chebyshev's Inequality.

Chebyshev's Inequality

Let X be a RV and $g(x)$ be a nonnegative function. Then, for any $r > 0$,

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Let X be a RV and $g(x)$ be a nonnegative function. Then, for any $r > 0$,

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Chebyshev's Inequality

Let X be a RV. Then, for any $r, p > 0$,

$$\mathbb{P}(|X| \geq r) \leq \frac{\mathbb{E}[|X|^p]}{r^p},$$

$$\mathbb{P}(|X - \mu| \geq r) \leq \frac{\sigma^2}{r^2},$$

where $\mu := \mathbb{E}[X]$, and $\sigma^2 := \mathrm{Var}(X)$.

- Chebyshev's Inequality is typically very conservative.

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- If $Z \sim \mathcal{N}(0,1)$, a tighter bound is available: For any $t > 0$,

$$2\Phi(-t) = \mathbb{P}(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}}\frac{1}{t}e^{-t^2/2},$$

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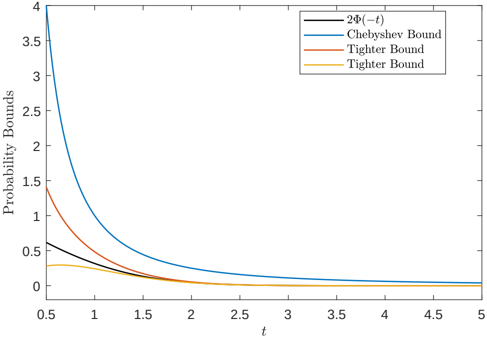

- A function $g(x)$ is **convex** if

$$g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda)g(y),$$

for all x and y, and $\lambda \in (0,1)$.

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Jensen's Inequality

Let X be a RV. If $g(x)$ is a convex function, then

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X]),$$

with equality if and only if $g(x)$ is a linear function on some set A such that $\mathbb{P}(X \in A) = 1$.

Hölder's Inequality

Let X and Y be any two RVs, and let p and q be any two positive numbers (necessarily greater than 1) satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then,

$$|\mathbb{E}[XY]| \le \mathbb{E}[|XY|] \le \{\mathbb{E}[|X|^p]\}^{1/p}\{\mathbb{E}[|Y|^q]\}^{1/q}.$$

Cauchy-Schwarz Inequality ($p = q = 2$)

Let X and Y be any two RVs, then

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Liapounov's Inequality ($Y \equiv 1$)

Let X be a RV, then for any $s > r > 1$,

$$\{\mathbb{E}[|X|^r]\}^{1/r} \leq \{\mathbb{E}[|Y|^s]\}^{1/s}.$$

Minkowski's Inequality

Let X and Y be any two RVs. Then, for $p \geq 1$,

$$\{\mathbb{E}[|X+Y|^p]\}^{1/p} \leq \{\mathbb{E}[|X|^p]\}^{1/p} + \{\mathbb{E}[|Y|^p]\}^{1/p}.$$

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- **Remark**: The preceding Hölder's Inequality (including its special cases) and Minkowski's Inequality also apply to numerical sums where there is no explicit reference to an expectation.

1. Probability Space
2. Random Variables & Distributions
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7. Properties of a Random Sample

Consider a sequence of RVs $\{X_n : n \geq 1\}$ and another RV X.

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- **Convergence in Distribution**, $X_n \xrightarrow{d} X$ or $X_n \Rightarrow X$:

$$\lim_{n\to\infty} F_n(x) = F(x), \text{ for any continuous point } x \text{ of } F(x),$$

  where F_n and F are CDF of X_n and X, respectively.

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- **Convergence in L^r Norm** ($r \in [1, \infty)$), $X_n \xrightarrow{L^r} X$:

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0,$$

  given $\mathbb{E}[|X_n|^r] < \infty$ for any $n \geq 1$ and $\mathbb{E}[|X|^r] < \infty$.

- Simple relationships:

$$
\begin{array}{ccccc}
X_n \xrightarrow{a.s.} X & \Longrightarrow & X_n \xrightarrow{p} X & \Longrightarrow & X_n \Rightarrow X \\
& & \Uparrow & & \\
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- $X_n \xrightarrow{p} X \quad \Longleftrightarrow$ For every subsequence $X_n(m)$ there is a further subsequence $X_n(m_k)$ such that $X_n(m_k) \xrightarrow{a.s.} X$.

- Question: If $X_n \Rightarrow X$ or $X_n \xrightarrow{p} X$ or $X_n \xrightarrow{a.s.} X$, does it imply $\mathbb{E}[X_n] \to \mathbb{E}[X]$?

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Fatou's Lemma

Suppose $X_n \geq Y$ a.s. for all n where $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[\liminf_{n\to\infty} X_n] \leq \liminf_{n\to\infty} \mathbb{E}[X_n]$. In particular, if $X_n \geq 0$ a.s. for all n, then the result holds.

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Monotone Convergence Theorem (MCT)

Suppose $X_n \xrightarrow{a.s.} X$, and $0 \leq X_1 \leq X_2 \leq \cdots$ a.s.. Then $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

Dominated Convergence Theorem (DCT)

Suppose $X_n \xrightarrow{a.s.} X$, $|X_n| \le Y$ a.s. for all n, and $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

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- The DCT is still true if $\xrightarrow{a.s.}$ is replaced by $\xrightarrow{p}$.
- An **even more general** result:
  Suppose $X_n \xrightarrow{p} X$, $|X_n| \le Y$ a.s. for all n, and $\mathbb{E}[|Y|^r] < \infty$ with $r \ge 1$. Then, $\mathbb{E}[|X_n|^r] < \infty$, $\mathbb{E}[|X_n|] < \infty$, and $X_n \xrightarrow{L^r} X$.

- If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then
  1. $X = Y$ a.s.;
  2. Random vector $(X_n, Y_n)^\intercal \xrightarrow{a.s.} (X, Y)^\intercal$.

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- None of the above are true for convergence in distribution.
- If $X_n \Rightarrow X$ and $Y_n \Rightarrow$ a constant c, then
  - Random vector $(X_n, Y_n)^\intercal \Rightarrow (X, c)^\intercal$.

    $\implies aX_n + bY_n \Rightarrow aX + bc$; $X_nY_n \Rightarrow cX$. (Slutsky's theorem)

Continuous Mapping Theorem

Consider a sequence of RVs $\{X_n : n \geq 1\}$ and another RV X. Suppose g is a function that has the set of discontinuity points D such that $\mathbb{P}(X \in D) = 0$. Then,

$$
\begin{aligned}
X_n \xrightarrow{a.s.} X \quad &\Longrightarrow \quad g(X_n) \xrightarrow{a.s.} g(X);\\
X_n \xrightarrow{p} X \quad &\Longrightarrow \quad g(X_n) \xrightarrow{p} g(X);\\
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1 Probability Space

2 Random Variables & Distributions

3 Expectations

4 Common Distributions

5 Useful Inequalities

6 Convergence

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Weak Law of Large Numbers (WLLN)

Suppose $X_1, \ldots, X_n$ are iid with mean μ and variance $\sigma^2 < \infty$.[†] Then, $\bar{X}_n \xrightarrow{p} \mu$.

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- Note that for **normal** distribution, $\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$, regardless of the value of n.
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Central Limit Theorem (CLT)

Suppose $X_1, \ldots, X_n$ are iid with mean μ and variance $\sigma^2 \in (0, \infty)$. Then,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \Rightarrow \mathcal{N}(0,1).$$