MEM6810 Engineering Systems Modeling and Simulation

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

Spring 2023 (full-time)

Assignment 1

Due Date: March 28 (in class)

Instruction

(a) You can answer in English or Chinese or both.
(b) Show **enough** intermediate steps.
(c) Write your answers **independently**.
(d) If you copy the solutions from somewhere, you must **indicate the source**.

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Question 1 ($5 + 5 = 10$ points)

(1) Prove the result in Buffon's Needle: $\mathbb{P}(\text{needle crosses a line}) = \dfrac{2l}{\pi d}$. (Lec 1 page 25/33)
(2) If the straight needle is bent to V shape, and let X denote the number of intersection points between a needle and the lines, prove that $\mathbb{E}[X] = \dfrac{2l}{\pi d}$.

Question 2 (10 points)

Prove that $\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y])$.

Question 3 (10 points)

Suppose that $X_1 \sim \text{Gamma}(\alpha_1, \lambda)$, $X_2 \sim \text{Gamma}(\alpha_2, \lambda)$, and $X_1 \perp X_2$. Prove that $X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \lambda)$.

Question 4 (10 points)

Prove that $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Question 5 (10 points)

Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Question 6 (10 points)

For *positive* real numbers a_i, $i = 1, 2, \ldots, n$, define

$$\begin{aligned}
\text{arithmetic mean: } a_A &= \frac{1}{n}(a_1 + \cdots + a_n), \\
\text{geometric mean: } a_G &= (a_1 \times \cdots \times a_n)^{1/n}, \\
\text{harmonic mean: } a_H &= \frac{1}{\frac{1}{n}(\frac{1}{a_1} + \cdots + \frac{1}{a_n})}.
\end{aligned}$$

Use Jensen's Inequality to prove that $a_H \le a_G \le a_A$. (Hint: Use the log() function.)

Question 7 (10 points)

Prove that, if $X_n \xrightarrow{L^s} X$ and $s > r \ge 1$, then $X_n \xrightarrow{L^r} X$. (Hint: Use Hölder's Inequality.)

Question 8 (10 points)

Prove that, if $X_n \xrightarrow{L^r} X$ for $r \ge 1$, then $\mathbb{E}[|X_n|^r] \to \mathbb{E}[|X|^r]$. (Hint: Use Minkowski's Inequality. Also note that for a sequence of numbers $a_1, a_2, \ldots$, and a continuous function $f()$, as $n \to \infty$, $a_n \to a$ implies that $f(a_n) \to f(a)$.)

Question 9 (10 points)

Prove that S^2 is an unbiased estimator of σ^2, i.e., $\mathbb{E}[S^2] = \sigma^2$, but S is a biased estimator of σ, i.e., $\mathbb{E}[S] \ne \sigma$. (Hint: Use Jensen's Inequality.)

Question 10 (10 points)

Prove the Weak Law of Large Numbers with iid assumption and $\sigma^2 < \infty$. (Hint: Use Chebyshev's Inequality.)