

MEM6810 Engineering Systems Modeling and Simulation

工程系统建模与仿真

Theory Analysis

Lecture 3: Queueing Models

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(Sino-US Global Logistics Institute)



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- Queues are an unavoidable component of modern life.
 - E.g., in hospital, stores, bank, call center (online service), etc.



Figure: Queues in Hospital



Figure: Queues in Store (from [The Sun](#))



Figure: Queues in Campus (for COVID-19 Nucleic Acid Test)



Figure: Queues in Bank



Figure: Queues in Bank (No requirement to *stand physically* in queues)



Figure: Queue in Online Service

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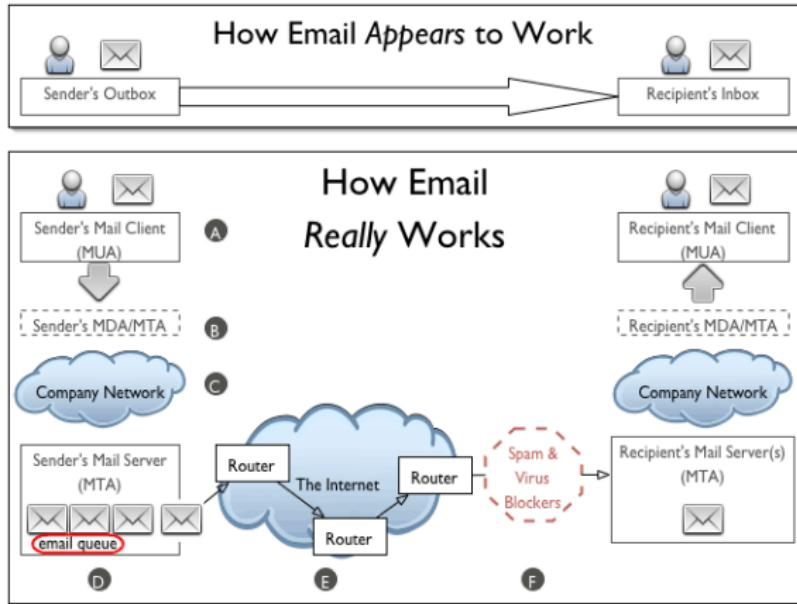


Figure: Queue in Mail Server (*from OASIS*)

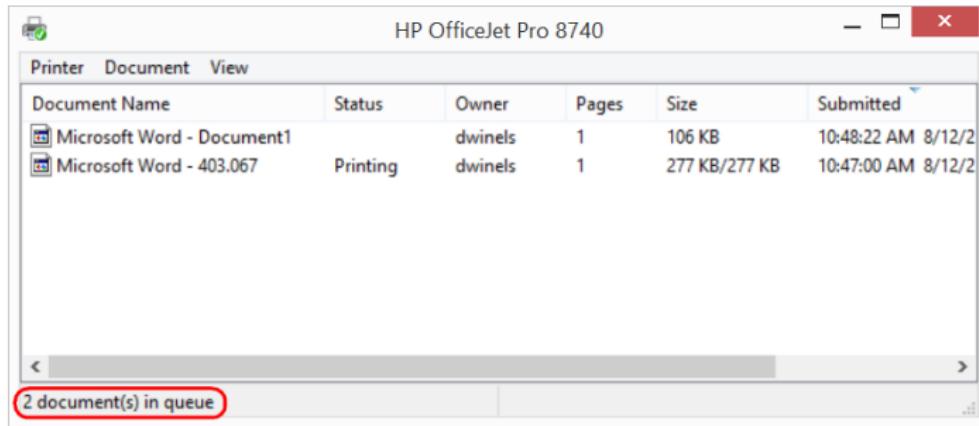


Figure: Queue in Printer

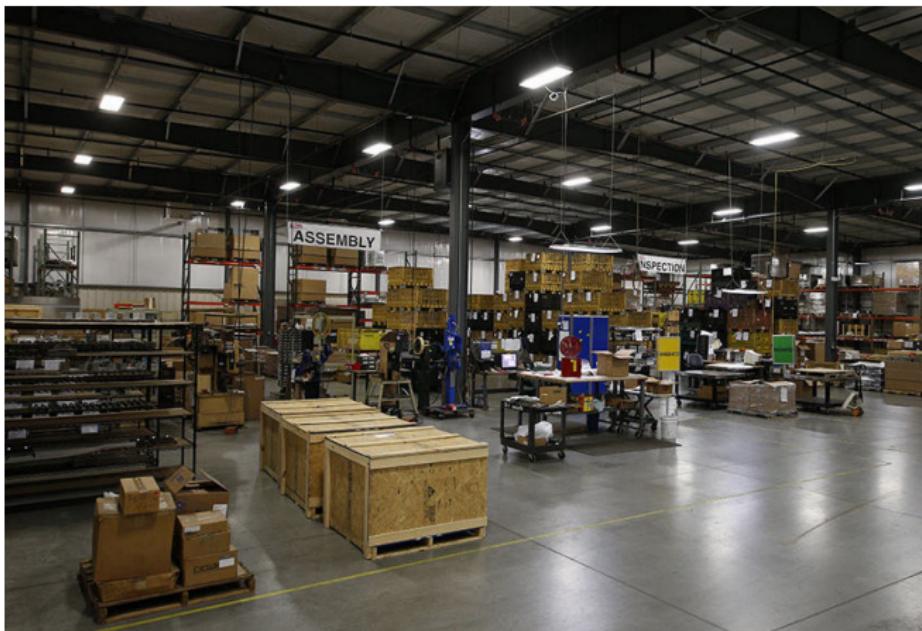


Figure: Queues (Inventories) in Manufacturing Line (*from Estes*)

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- Queues are not just for humans, however.
 - E.g., email system, printer, manufacturing line, etc.
 - Manufacturing systems maintain queues (called inventories) of raw materials, partly finished goods, and finished goods via the manufacturing process.

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- Queueing models are mathematical representation of queueing systems.

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- Studied in either way, queueing models provide us a powerful tool for designing and evaluating the performance of queueing systems.
- They help us do this by answering the following questions (and many others):
 - ① How many customers are there in the queue (or station) on average?
 - ② How long does a typical customer spend in the queue (or station) on average?
 - ③ How busy are the servers on average?



- *Simple queueing models solved analytically:*
 - Get rough estimates of system performance with negligible time and expense.
 - *More importantly, understand the dynamic behavior of the queueing systems and the relationships between various performance measures.*
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- This lecture focuses on the classical analytically solvable queueing models.

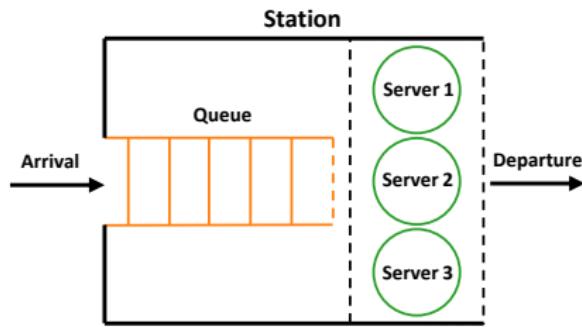
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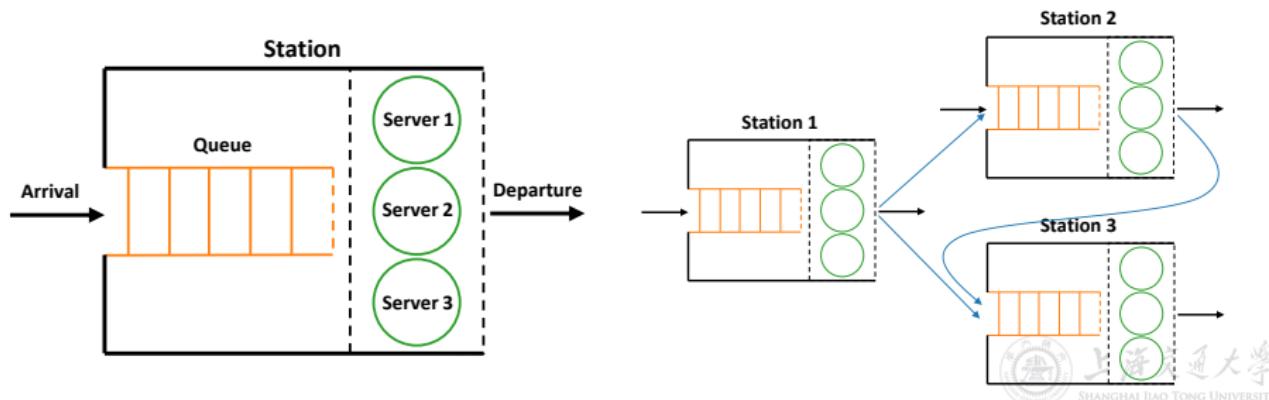
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- Suppose that there is only **one queue** in one station.
- **Capacity** is the maximal number of customers allowed in the station.
 - Number waiting in queue + number having service.
 - Finite or infinite.

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 - Customers simply leave after service.
 - E.g., customers arrive to buy coffee and then leave.



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- Multiple-station queueing system (queueing network).
 - Customers can move from one station to another (for different service), before leaving the system.
 - E.g., patients wait and get service at several different units inside a hospital.



- The **arrival process** describes how the customers come.
 - Arrivals may occur at *scheduled* times or *random* times.
 - When at random times, the **interarrival times** are usually characterized by a probability distribution.
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- An customer arriving at a station:
 - if the station capacity is full:
 - the external arrival will leave immediately (called **lost**);
 - the internal arrival may wait in the previous station (may **block** the previous server).
 - if the station capacity is not full, enter the station:
 - if there is idle server in the station, get service immediately;
 - if all servers are busy, wait in the **queue**.

- Queue discipline: Which customer to serve first.
 - First-in-first-out (FIFO), or first-come-first-served (FCFS).
 - Last-in-first-out (LIFO), or last-come-first-served (LCFS).
 - Shortest processing time first.
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- **Service time** is the duration of service in a server.
 - *Constant* or *random* duration.
 - May depend on the customer type.
 - May depend on the time of day or the queue length.

- When without specification, the queueing models considered in this lecture shall satisfy the following:
 - One customer type.
 - Random arrivals (i.e., random interarrival times, iid.).
 - No batch (or say, batch size is 1).[†]
 - One queue in one station.
 - First-come-first-served (FCFS).
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- Even so, it is not that easy to analyze the queueing models!

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- Examples: $M/M/1$, $M/G/1$, $M/M/s/K$.

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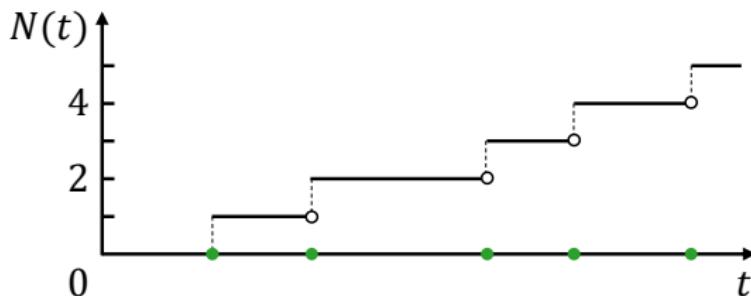
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4 Queueing Networks

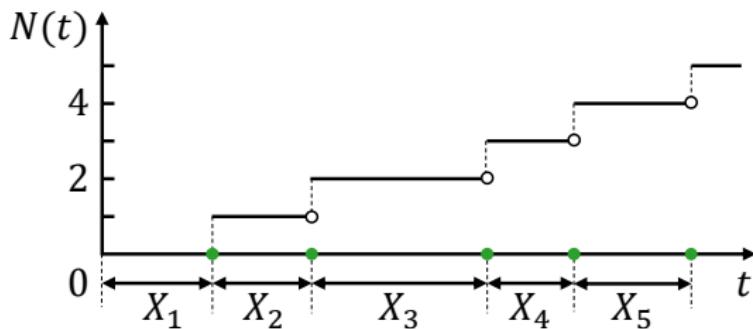
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- Let $\{X_n, n \geq 1\}$ denote the *interarrival times*:
 - X_1 denotes the time of the first arrival;
 - For $n \geq 2$, X_n denotes the time between the $(n - 1)$ st and the n th arrivals.

- **Definition 1.** The counting process $\{N(t), t \geq 0\}$ is called a **Poisson process** with rate λ , $\lambda > 0$, if:
 - $N(0) = 0$;
 - The process has **independent** and **stationary** increments;
 - For $t > 0$, $N(t) \sim \text{Pois}(\lambda t)$, i.e.,

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- Independent Increments: The numbers of arrivals in disjoint time intervals are independent.
- Stationary Increments: The distribution of number of arrivals in any time interval depends only on the length of time interval, i.e., for $s < t$, the distribution of $N(t) - N(s)$ depends only on $t - s$.

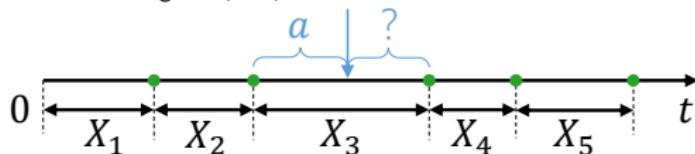
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- **Definition 1, Definition 2 and Definition 3** are equivalent.

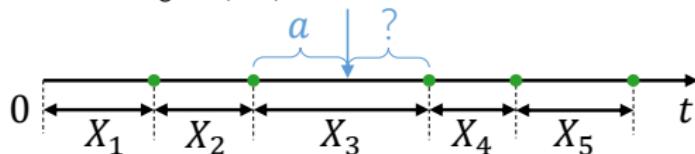
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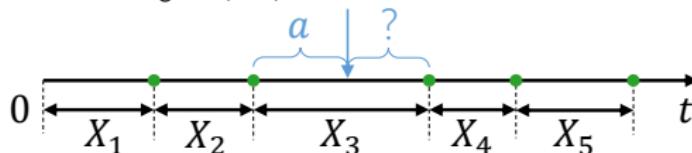
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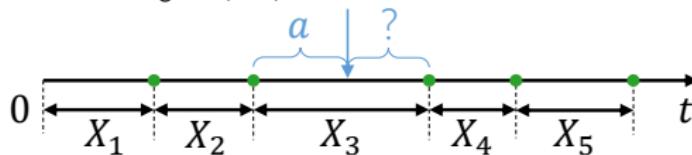
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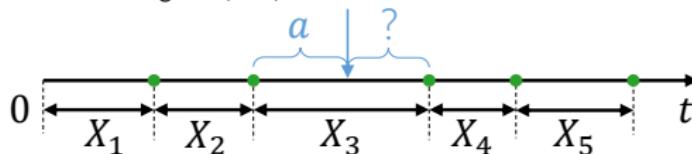
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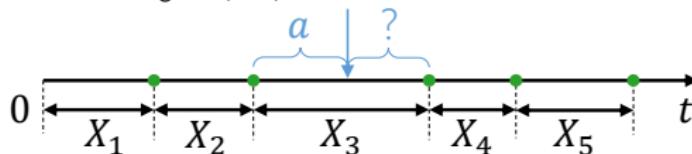
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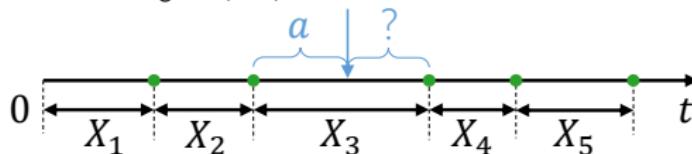
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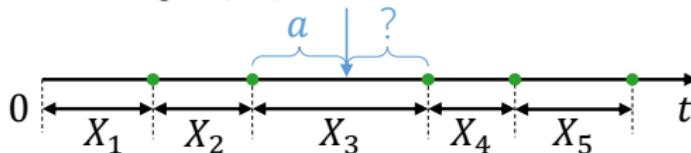
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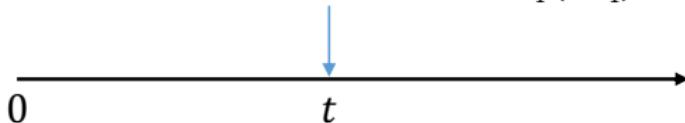
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- The Poisson process has no memory! (equivalent to the independent and stationary increments assumption)

- Let $S_n = X_1 + X_2 + \cdots + X_n$ be the arrival time of the n th arrival.
- **Question 2:** If I only know there are n arrivals up to time t , what can I say about the n arrival times S_1, \dots, S_n ?

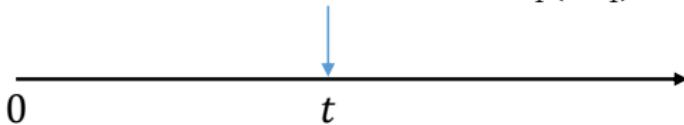
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- Intuition:
 - Since Poisson process possesses independent and stationary increments, each interval of equal length in $[0, t]$ should have the same probability of containing the arrival.
 - Hence, the arrival time should be uniformly distributed on $[0, t]$.

Proof.

$$\mathbb{P}\{X_1 < s | N(t) = 1\}$$

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■

- Remark: This result can be generalized to n arrivals.

Property (Conditional Distribution of Arrival Times)

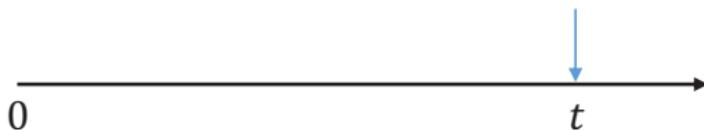
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Given $N(t) = n$, how can I generate a sample of $\{S_1, S_2, \dots, S_n\}$?

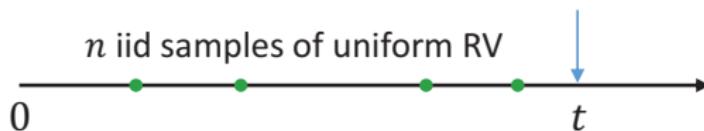


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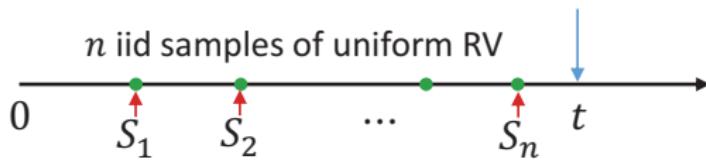
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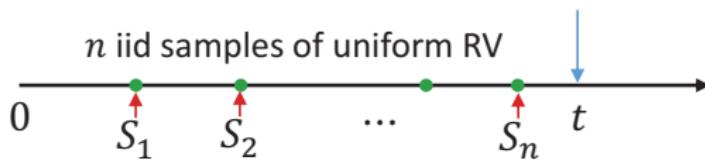
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• This is very nice for simulation!

1 Queueing Systems and Models

- ▶ Introduction
- ▶ Characteristics & Terminology
- ▶ Kendall Notation

2 Poisson Process

- ▶ Definition
- ▶ Properties

3 Single-Station Queues

- ▶ Notations
- ▶ General Results
- ▶ Little's Law
- ▶ $M/M/1$ Queue
- ▶ $M/M/s$ Queue
- ▶ $M/M/\infty$ Queue
- ▶ $M/M/1/K$ Queue
- ▶ $M/M/s/K$ Queue
- ▶ $M/G/1$ Queue

4 Queueing Networks

- ▶ Jackson Networks



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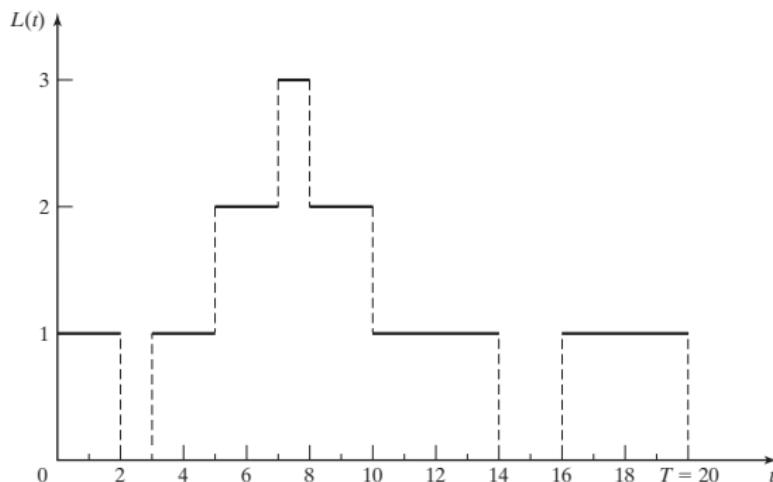


Figure: Illustration of $L(t)$ (from Banks et al. (2010))

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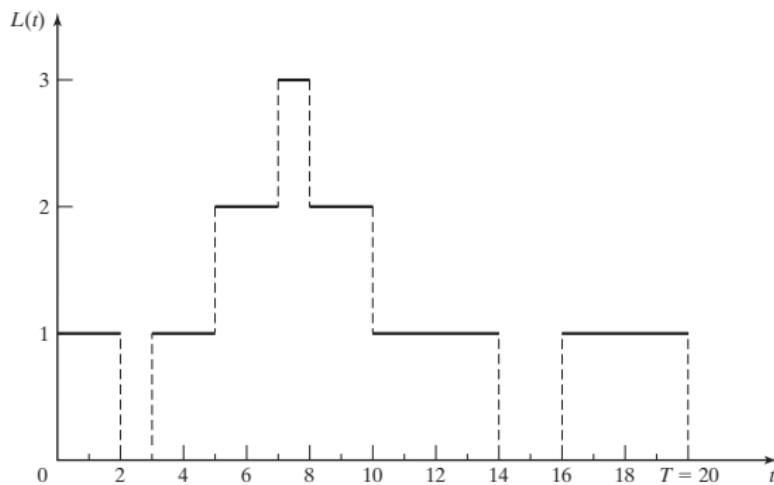


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- Let $\widehat{L}(T)$ denote the (time-weighted) average number of customers in the station up to time T :

$$\widehat{L}(T) := \frac{1}{T} \int_0^T L(t) dt.$$

- Another expression of $\widehat{L}(T)$: Let T_n denote the total time during $[0, T]$ in which the station contains exactly n customers.

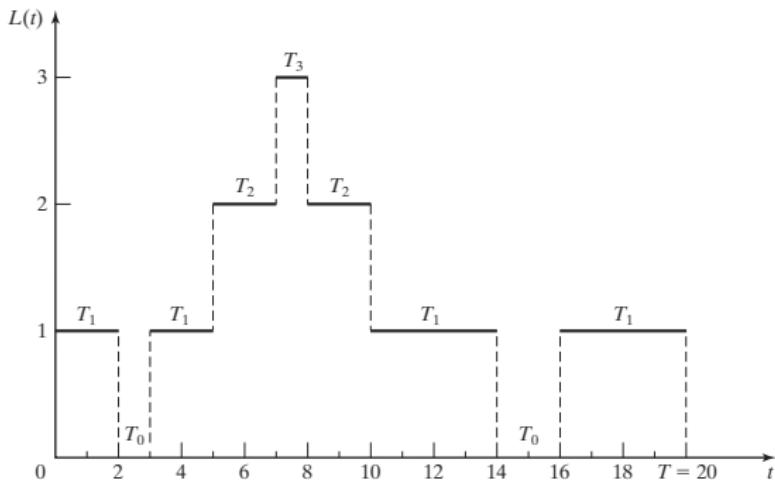


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- $$\widehat{L}(T) := \frac{1}{T} \int_0^T L(t) dt = \frac{1}{T} \sum_{n=0}^{\infty} n T_n = \sum_{n=0}^{\infty} n \left(\frac{T_n}{T} \right).$$

- Suppose during time $[0, T]$, totally $N(T)$ customers have entered the station, and let $W_1, W_2, \dots, W_{N(T)}$ denote the time each customer spends in the station up to time T .[†]

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- In a similar way, we can also define
 - $\widehat{L}_Q(T)$ – The average number of customers in the *queue* up to time T .
 - $\widehat{W}_Q(T)$ – The average *waiting* time in the *queue* up to time T .

[†]The time includes both the waiting time in queue and the time in server. The part after T is not counted.

- Now we consider the long-run measures.

- L – The long-run average number of customers in the station:

$$L := \lim_{T \rightarrow \infty} \widehat{L}(T).$$

- W – The long-run average sojourn time in the station:

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- Question: When will L , W , L_Q and W_Q exist (and $< \infty$)?

- We also define the *limiting probability* that there will be exactly n customers in the station as time goes to infinity:

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- Question: When will P_n exist?
- Moreover, for an arbitrary $X/Y/s/K$ queue
 - Let λ denote the arrival rate, i.e.,

$$\mathbb{E}[\text{interarrival time}] = \frac{1}{\lambda}.$$

- Let μ denote the service rate in one server, i.e.,

$$\mathbb{E}[\text{service time}] = \frac{1}{\mu}.$$

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Theorem 1 (Condition of Stability)

For an $X/Y/s/\infty$ queue (i.e., infinite capacity) with arrival rate λ and service rate μ , it is stable if

$$\lambda < s\mu.$$

And, an $X/Y/s/K$ queue (i.e., finite capacity) will always be stable.

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 - Since the system is stable and run for infinitely long time, it should enter some steady state (i.e., has nothing to do with the initial state).
- L can also be written as $L := \sum_{n=0}^{\infty} n P_n$ (see next slide).
 - L is also called the expected number of customers in the station in steady state;
 - W is also called the expected sojourn time in the station in steady state;
 - L_Q is also called the expected number of customers in the queue in steady state;
 - W_Q is also called the expected waiting time in the queue in steady state.

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$$P_n = \lim_{T \rightarrow \infty} \frac{\text{amount of time during } [0, T] \text{ that station contains } n \text{ customers}}{T}.$$

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Theorem 2 (Little's Law – Empirical Version)

Define the observed entering rate $\widehat{\lambda} := N(T)/T$, then

$$\widehat{L}(T) = \widehat{\lambda} \widehat{W}(T), \quad \widehat{L}_Q(T) = \widehat{\lambda} \widehat{W}_Q(T).$$

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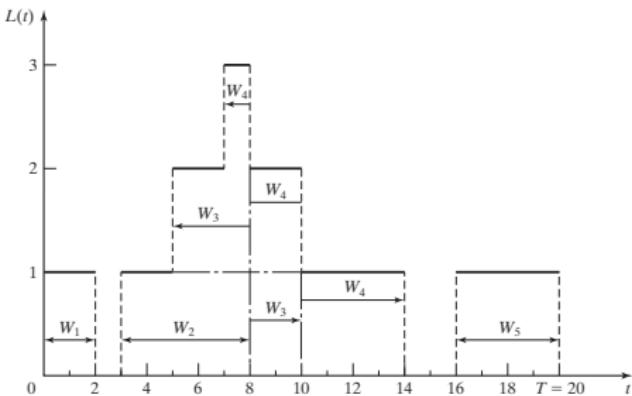
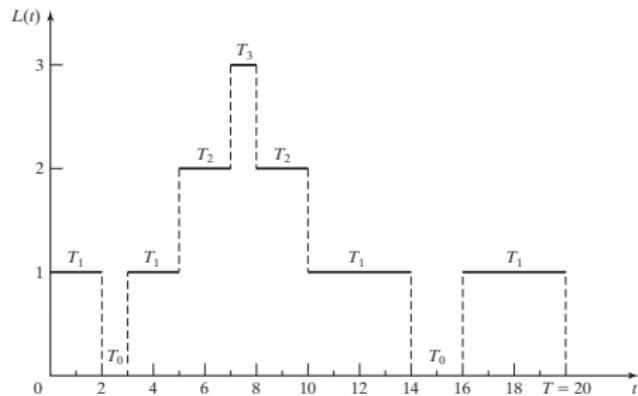


Figure: Illustration of $L(t)$ and W_i (from Banks et al. (2010))

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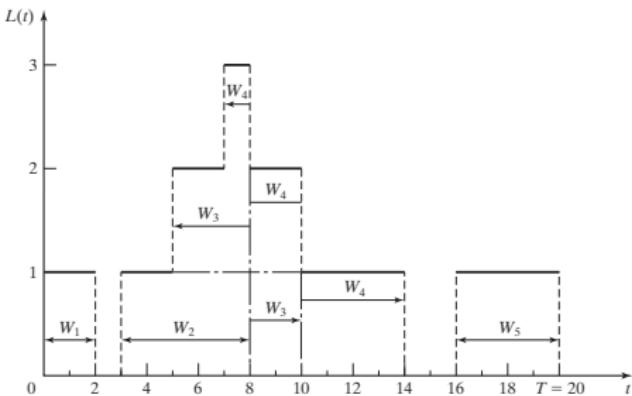
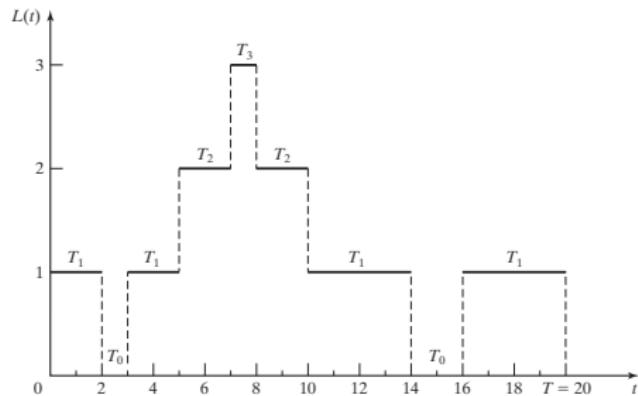


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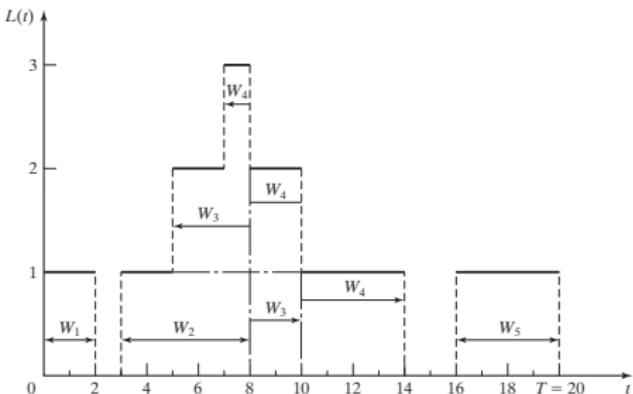
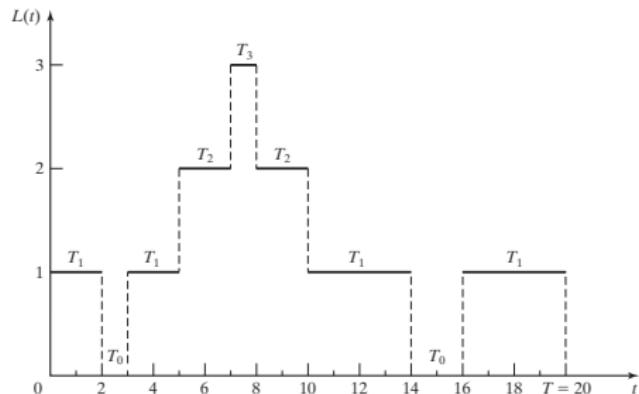


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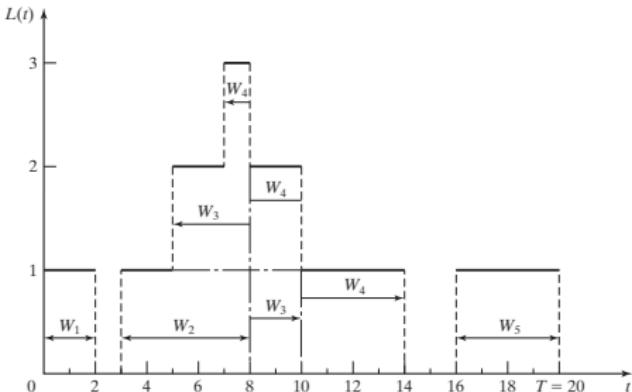
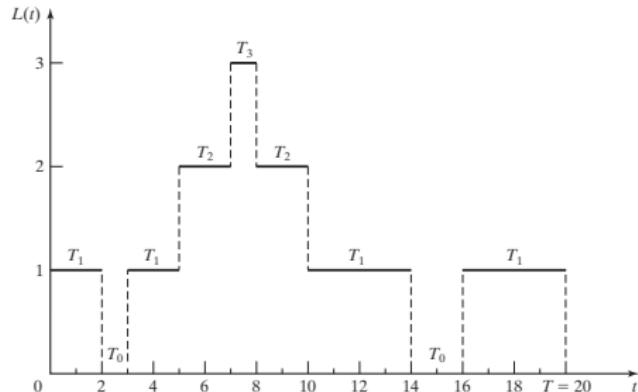


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$$\widehat{L}(T) = \frac{1}{T} \sum_{n=0}^{\infty} n T_n = \frac{1}{20}(0 \times 3 + 1 \times 12 + 2 \times 4 + 3 \times 1) = \frac{23}{20} = 1.15.$$

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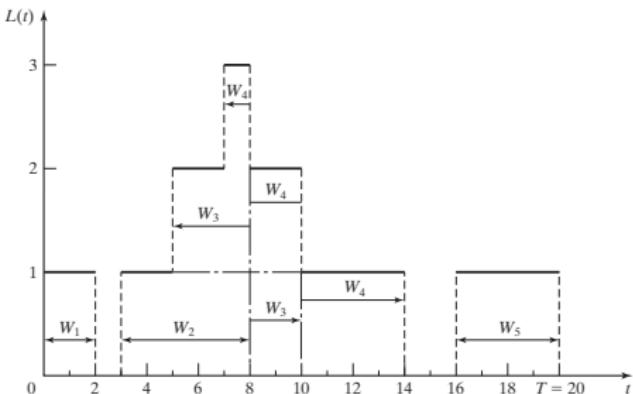
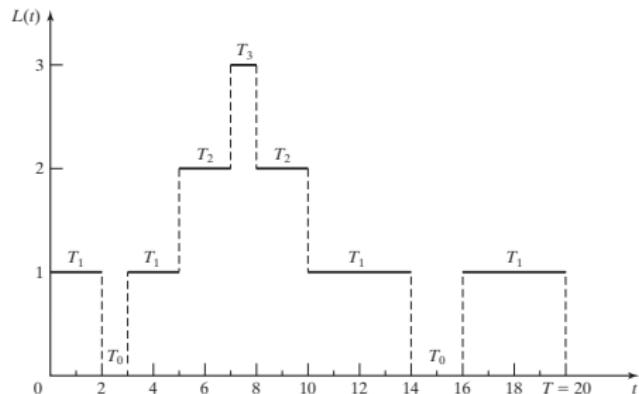


Figure: Illustration of $L(t)$ and W_i (from Banks et al. (2010))

$$\widehat{\lambda} = N(T)/T = 5/20 = 0.25.$$

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$$\text{So, } \widehat{\lambda} \widehat{W}(T) = 0.25 \times 4.6 = 1.15 = \widehat{L}(T).$$

- Verify Little's Law.

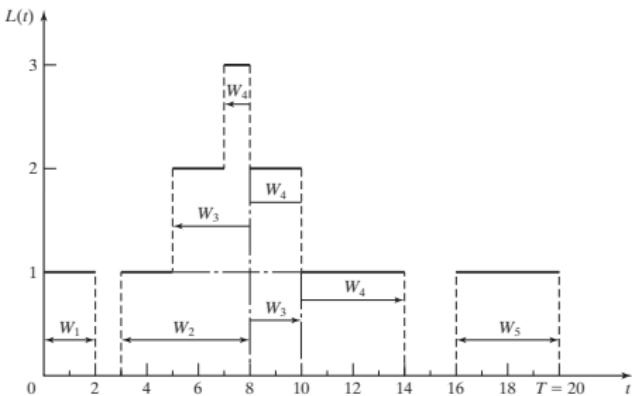
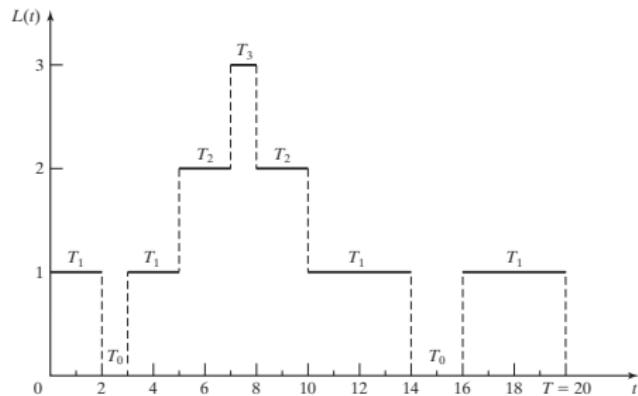


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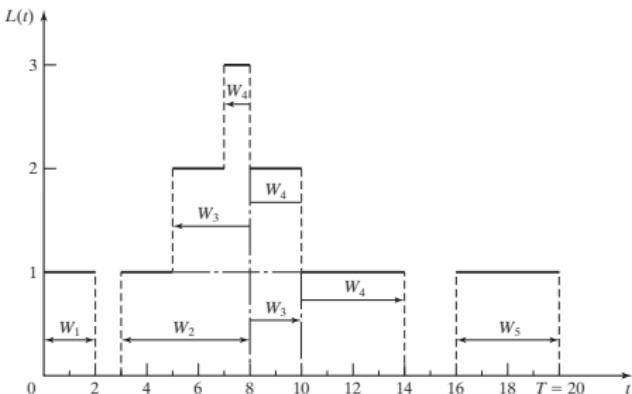
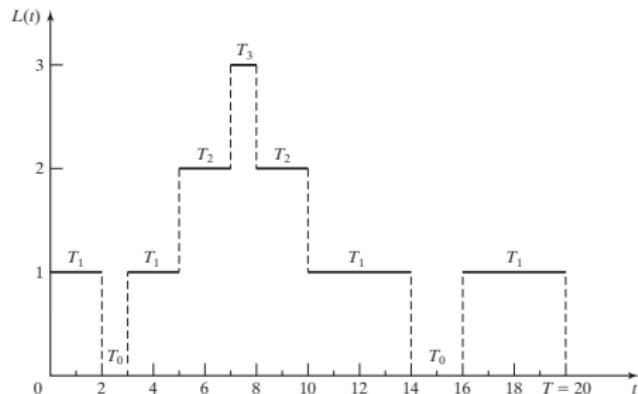


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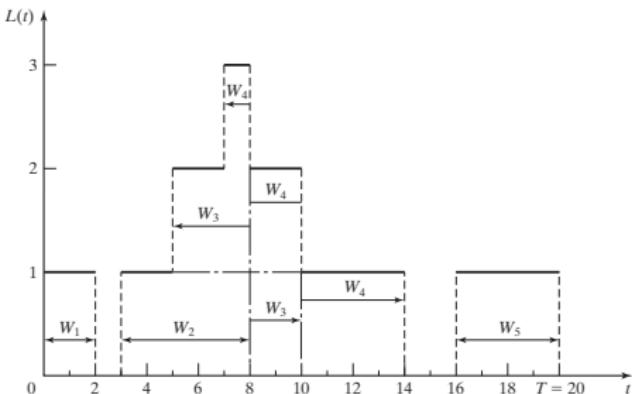
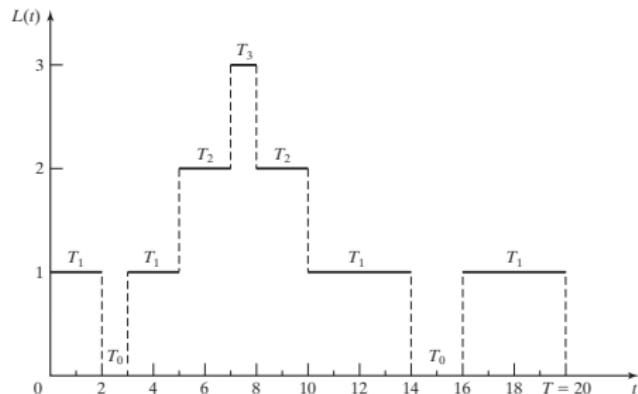


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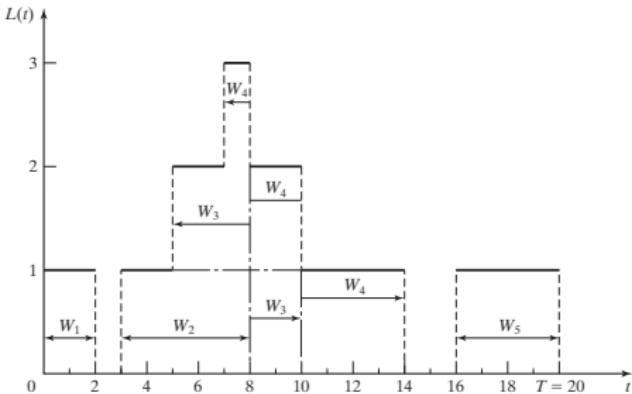
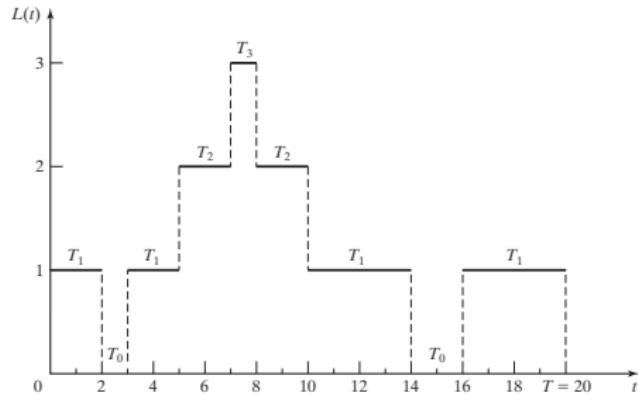


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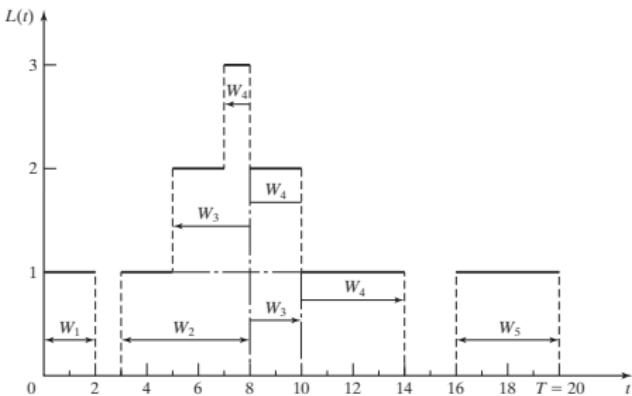
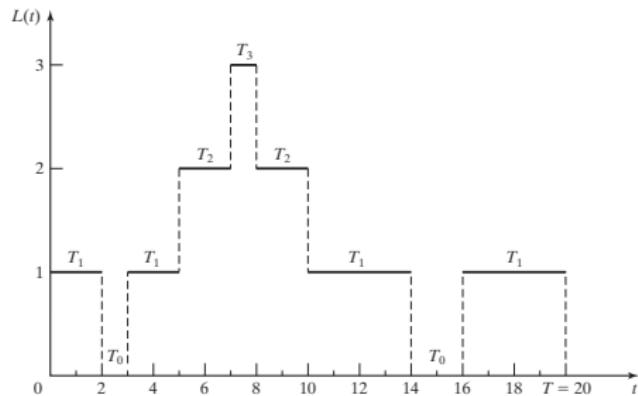


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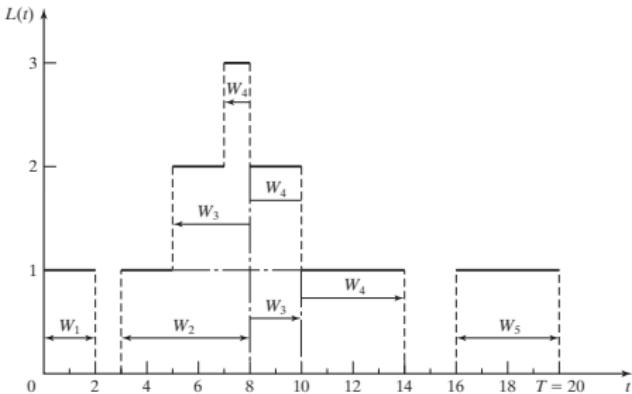
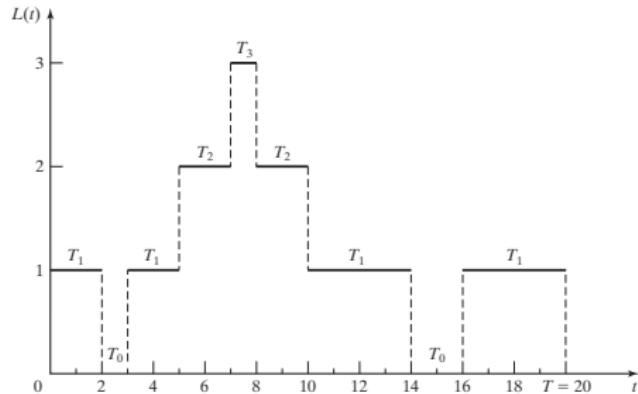


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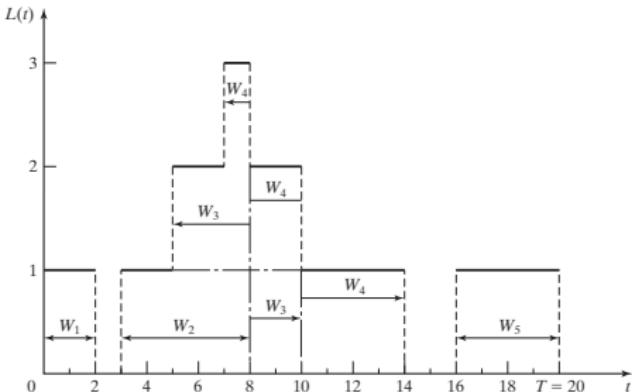
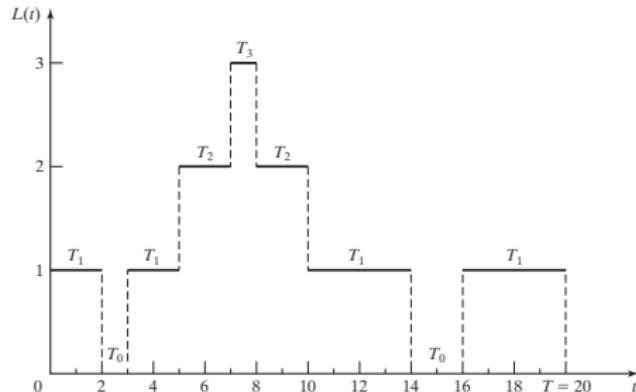


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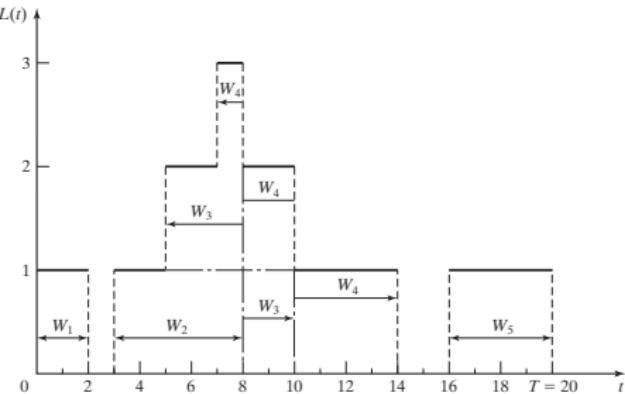
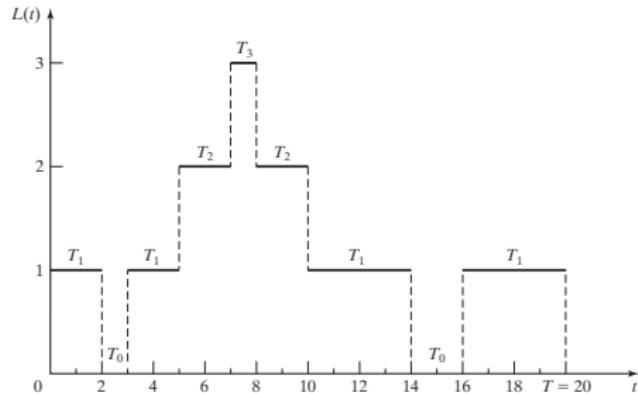


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So, $\widehat{L}(T) = \widehat{\lambda} \widehat{W}(T)$ always holds.

- The same argument for $\widehat{L}_Q(T) = \widehat{\lambda} \widehat{W}_Q(T)$.

Theorem 3 (Little's Law – Limit/Expectation Version)

For a stable queue, let λ^* denote the arrival rate or entering rate, then

$$L = \lambda^* W, \quad L_Q = \lambda^* W_Q.$$

Caution: When λ^* is the arrival rate, the time average (W, W_Q) is based on all customers (who enter the station or are lost); When λ^* is the entering rate, the time average is only based on the customers who enters the station.

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- Some Remarks:
 - For a customer who is lost (due to the finite capacity), he spends 0 amount of time in the station (or queue).
 - Once we know anyone of L, W, L_Q and W_Q , we can compute the rest using Little's Law.

• M/M/1 Queue[†]

- The interarrival times are iid random variables with $\text{Exp}(\lambda)$ distribution, that is to say, *customers arrive according to a Poisson process with rate λ .*
- The service times are iid random variables with $\text{Exp}(\mu)$ distribution.
- The customers are served in an FCFS fashion by a *single* server.
- The capacity is unlimited, i.e., waiting space is unlimited.
- M/M/1 queue is stable **if and only if** $\lambda < \mu$.
- Due to unlimited capacity, arrival rate = entering rate.

[†] M/M/1 Queue ⊂ Birth and Death Process with Infinite Capacity ⊂ Continuous-Time Markov Chain.



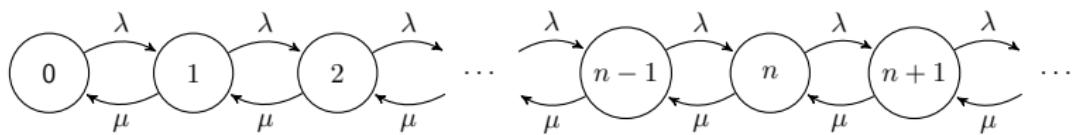
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 - $M/M/1$ queue is stable **if and only if** $\lambda < \mu$.
 - Due to unlimited capacity, arrival rate = entering rate.
- We now want to compute all the measures P_n , L , W , L_Q and W_Q .

[†] $M/M/1$ Queue \subset Birth and Death Process with Infinite Capacity \subset Continuous-Time Markov Chain.

- Recall that L can be computed via $L = \sum_{n=0}^{\infty} nP_n$, where P_n has two interpretations:
 - Long-run proportion of time that the station contains exactly n customers;
 - Probability that there are exactly n customers in the station as time goes to infinity (or equivalently, in the steady state).

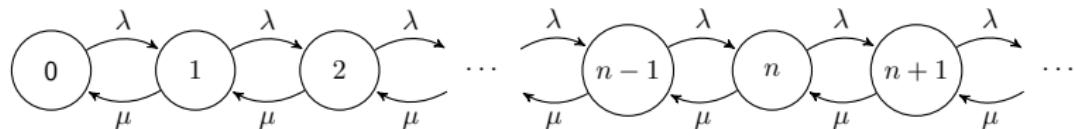
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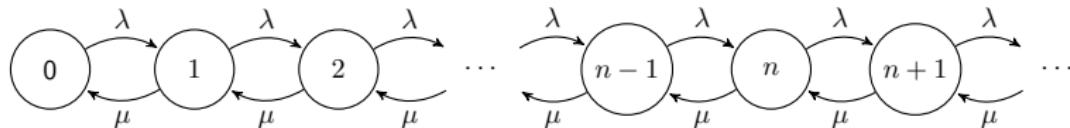
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- The state space diagram is as follows:



Single-Station Queues

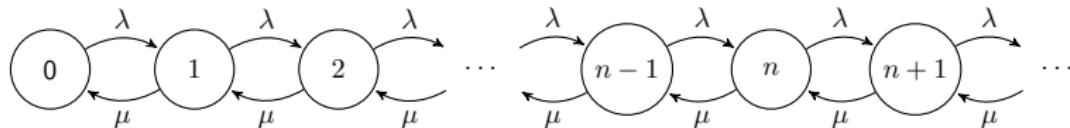
► $M/M/1$ Queue





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Rate at which the process leaves state n
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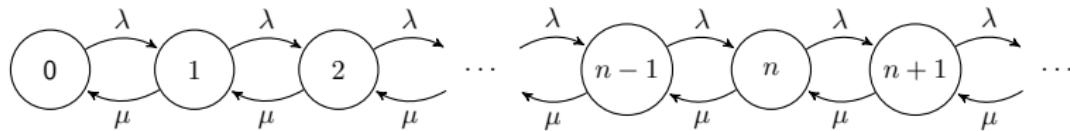
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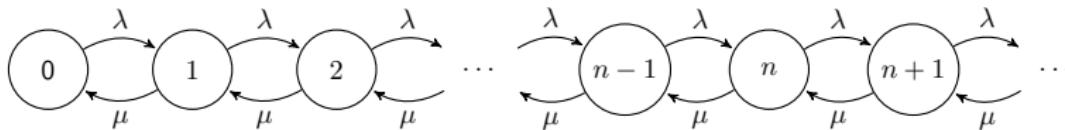
Heuristic Proof.

- In any time interval, the number of transitions into state n must equal to within 1 the number of transitions out of state n . (Why?)
- Hence, in the long run, the rate into state n must equal the rate out of state n .

Single-Station Queues

► $M/M/1$ Queue





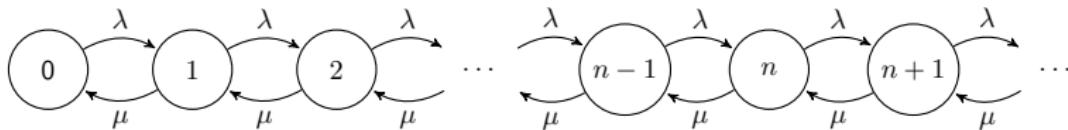
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Rate at which the process leaves state 0 = $P_0\lambda$;

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Fact

If X_1, \dots, X_n are independent random variables, and $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, \dots, n$, then

$$\min\{X_1, \dots, X_n\} \sim \text{Exp}(\lambda_1 + \dots + \lambda_n).$$

Theorem 4 (Limiting Distribution of M/M/1 Queue)

For an $M/M/1$ queue, when it is stable ($\lambda < \mu$), its limiting (steady-state) distribution is given by

$$P_n = (1 - \rho)\rho^n, \quad n \geq 0,$$

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Rewriting these equations gives

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$$P_0 = 1 - \rho, \quad \text{and} \quad P_n = (1 - \rho)\rho^n, \quad n \geq 1.$$



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• $M/M/s$ Queue[†]

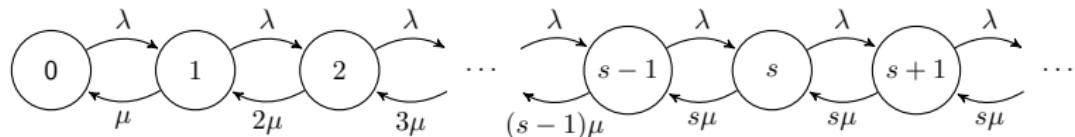
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- There are s parallel servers.
- The customers form a single queue and get served by the next available server in an FCFS fashion.
- The capacity is unlimited, i.e., waiting space is unlimited.
- $M/M/s$ queue is stable **if and only if** $\lambda < s\mu$.
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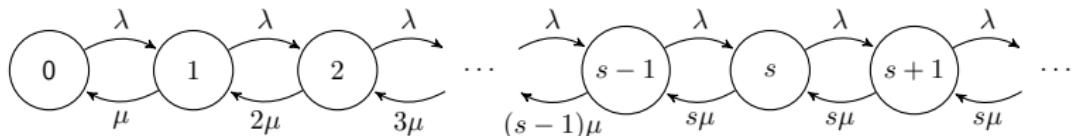
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- $M/M/s$ queue is a generalized version of $M/M/1$ queue. Let $s = 1$, all results should degenerate to those of $M/M/1$.

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Theorem 5 (Limiting Distribution of M/M/s Queue)

For an $M/M/s$ queue, when it is stable ($\lambda < s\mu$), its limiting (steady-state) distribution is given by

$$P_n = \left[\sum_{i=0}^s \frac{1}{i!} \left(\frac{\lambda}{\mu} \right)^i + \frac{s^s}{s!} \frac{\rho^{s+1}}{1-\rho} \right]^{-1} \rho_n, \quad n \geq 0,$$

where the *server utilization* $\rho := \lambda/(s\mu) < 1$, and

$$\rho_n := \begin{cases} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n, & \text{if } 0 \leq n \leq s, \\ \frac{s^s}{s!} \rho^n, & \text{if } n \geq s+1. \end{cases}$$

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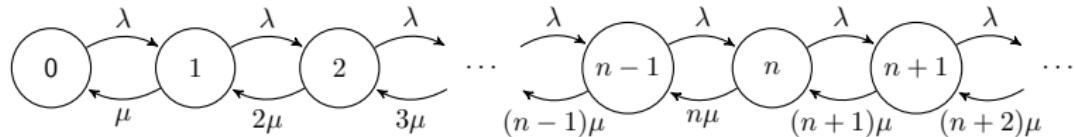
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For an $M/M/\infty$ queue, its limiting (steady-state) distribution is given by

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- $L_Q = 0$, $W_Q = 0$.

• $M/M/1/K$ Queue[†]

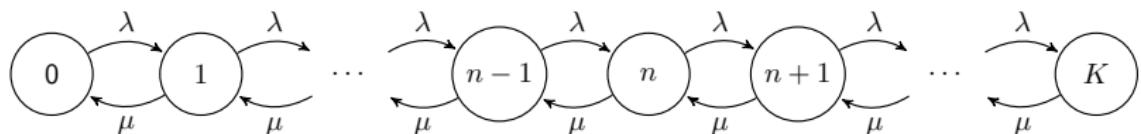
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- A customer who finds the station is full (K customers there) leaves immediately (lost).
- The entering rate, denoted as λ_e , is smaller than the arrival rate λ .
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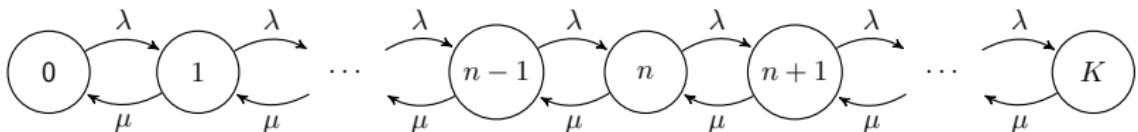
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- In steady state
 - $\mathbb{P}(\text{station is full}) = P_K$.
 - Entering rate $\lambda_e = \lambda(1 - P_K)$.

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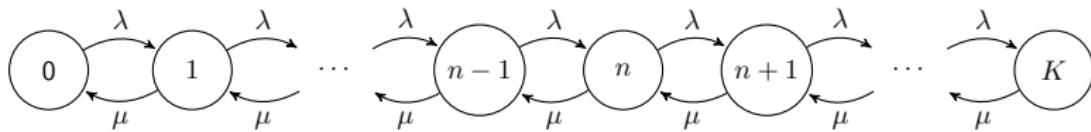


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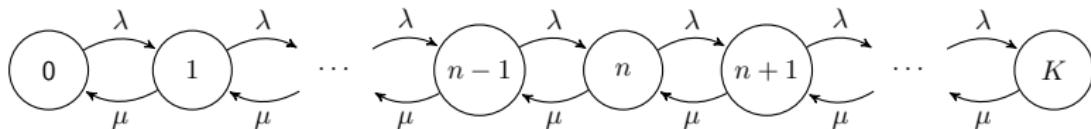
For an $M/M/1/K$ queue, its limiting (steady-state) distribution is given by

$$P_n = \begin{cases} \frac{(1-\rho)\rho^n}{1-\rho^{K+1}}, & \text{if } \rho \neq 1, \\ \frac{1}{K+1}, & \text{if } \rho = 1, \end{cases} \quad 0 \leq n \leq K,$$

where $\rho := \lambda/\mu$. (ρ is NOT the *server utilization!*)

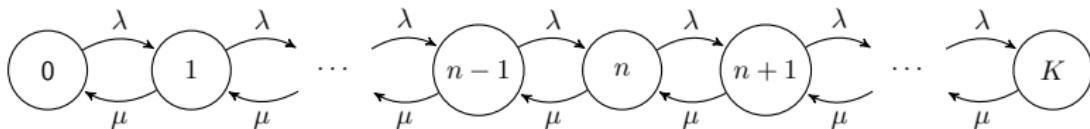


Proof.



Proof. Due to Observations 1 & 2,

State	Rate Process Leaves	=	Rate Process Enters
0	$P_0\lambda$	=	$P_1\mu$
$n, 1 \leq n \leq K - 1$	$P_n(\mu + \lambda)$	=	$P_{n-1}\lambda + P_{n+1}\mu$
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Rewriting these equations gives

$$P_0\lambda = P_1\mu,$$

$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad 1 \leq n \leq K-1,$$

$$P_K\mu = P_{K-1}\lambda.$$

Or, equivalently,

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Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

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Or, equivalently,

$$P_0\lambda = P_1\mu,$$

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Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

$$P_1 = P_0\rho,$$

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- As $\rho \rightarrow \infty$, $L \rightarrow K$, $1 - P_K \rightarrow 0$, $\rho(1 - P_K) \rightarrow 1$.

- For those entered the station
 - The expected sojourn time $W = L/\lambda_e = \frac{L}{\lambda(1-P_K)}$.
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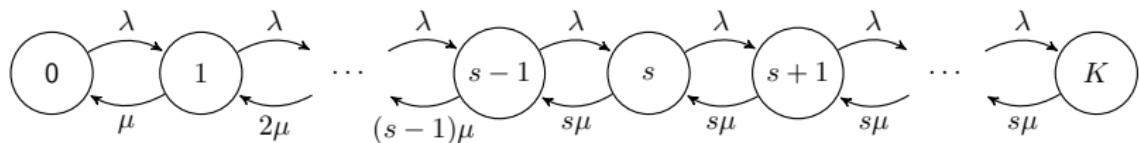
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- $M/M/s/K$ queue[†] is a generalized version of $M/M/1/K$ queue. ($K \geq s$)
- The state space diagram is as follows:



- Let $s = 1$, it becomes the $M/M/1/K$ queue.
- Let $s = K$, it becomes the $M/M/K/K$ queue.
- There is no $M/M/\infty/K$ queue!

[†] $M/M/1/K$ Queue \subset $M/M/s/K$ Queue \subset Birth and Death Process with Finite Capacity \subset CTMC

Theorem 8 (Limiting Distribution of M/M/s/K Queue)

For an $M/M/s/K$ queue, its limiting (steady-state) distribution is given by

$$P_n = \left[\sum_{i=0}^s \frac{1}{i!} \left(\frac{\lambda}{\mu} \right)^i + \varrho \right]^{-1} \rho_n, \quad 0 \leq n \leq K,$$

where $\varrho := \lambda/(s\mu)$, (ρ is NOT the *server utilization!*) and

$$\varrho := \begin{cases} \frac{s^s}{s!} \frac{\rho^{s+1}(1-\rho^{K-s})}{1-\rho}, & \text{if } \rho \neq 1, \\ \frac{s^s}{s!}(K-s), & \text{if } \rho = 1, \end{cases}$$

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- The *server utilization* $= \lambda_e/(s\mu) = \rho(1 - P_K)$.

• $M/G/1$ Queue[†]

- Customers arrive according to a Poisson process with rate λ .
- The service times are iid random variables with **arbitrary** distribution (mean: $\frac{1}{\mu}$, variance: σ^2).
- The customers are served in an FCFS fashion by a *single* server.
- The capacity is unlimited, i.e., waiting space is unlimited.
- $M/G/1$ queue is stable **if and only if** $\lambda < \mu$.

[†] $M/G/1$ queue has an embedded discrete-time Markov chain.

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- For $M/G/\infty$, the measures are the same as those in $M/M/\infty$.

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1 Queueing Systems and Models

- ▶ Introduction
- ▶ Characteristics & Terminology
- ▶ Kendall Notation

2 Poisson Process

- ▶ Definition
- ▶ Properties

3 Single-Station Queues

- ▶ Notations
- ▶ General Results
- ▶ Little's Law
- ▶ $M/M/1$ Queue
- ▶ $M/M/s$ Queue
- ▶ $M/M/\infty$ Queue
- ▶ $M/M/1/K$ Queue
- ▶ $M/M/s/K$ Queue
- ▶ $M/G/1$ Queue

4 Queueing Networks

- ▶ Jackson Networks

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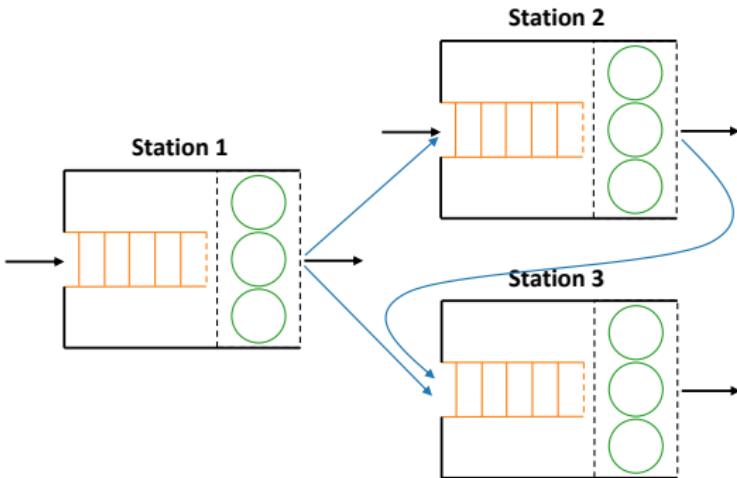


Figure: Illustration of Queueing Networks

- Jackson Queueing Network (first identified by [Jackson \(1963\)](#))[†]
 - ① The network has J single-station queues.
 - ② The j th station has s_j servers and a *single* queue.
 - ③ There is unlimited waiting space at each station (infinite capacity).
 - ④ Customers arrive at station j from outside according to a Poisson process with rate λ_j ; all arrival processes are independent of each other.
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 - ⑦ A customer finishing service may be routed to the same station (i.e., re-enter).

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- The routing probabilities p_{ij} can be put in a matrix form as follows:

$$\mathbf{P} := \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1J} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2J} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{J1} & p_{J2} & p_{J3} & \cdots & p_{JJ} \end{bmatrix}.$$

- The matrix \mathbf{P} is called the **routing matrix**.

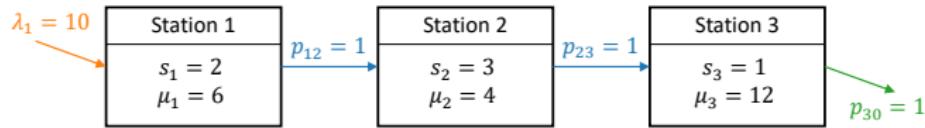
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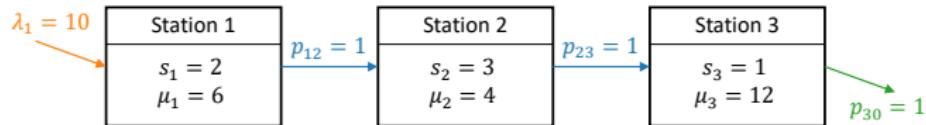
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- Since a customer leaving station i either joints some other station, or leaves, we must have

$$\sum_{j=1}^J p_{ij} + p_{i0} = 1, \quad 1 \leq i \leq J.$$

- Example 1: Tandem Queue

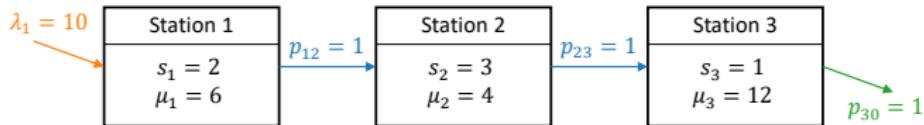


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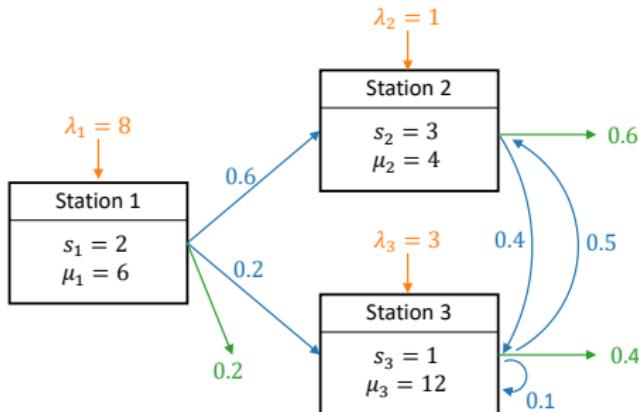
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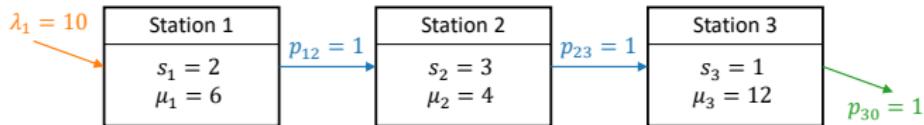


$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Example 2: General Network

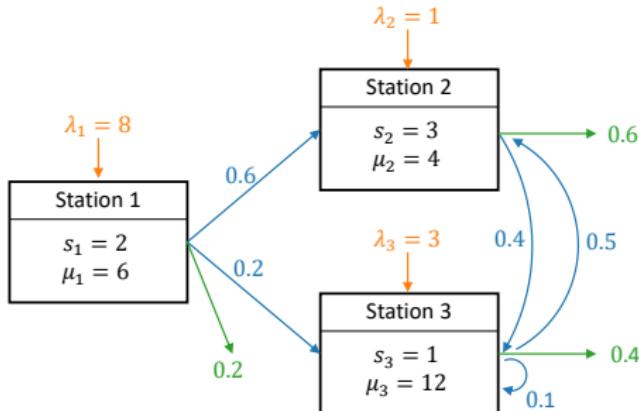


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$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Example 2: General Network



$$\mathbf{P} = \begin{bmatrix} 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 \\ 0 & 0.5 & 0.1 \end{bmatrix}.$$

- Recall that customers arrive at station j from outside with rate λ_j .
- Let b_j be the rate of internal arrivals to station j .
- Then the total arrival rate to station j , denoted as a_j , is given by

$$a_j = \lambda_j + b_j, \quad 1 \leq j \leq J.$$

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- Hence, $b_j = \sum_{i=1}^J a_i p_{ij}, \quad 1 \leq j \leq J.$
- Substituting in the previous equation, we get the **traffic equations**:

$$a_j = \lambda_j + \sum_{i=1}^J a_i p_{ij}, \quad 1 \leq j \leq J.$$

- Let $\mathbf{a}^\top = [a_1 \ a_2 \ \cdots \ a_J]$ and $\boldsymbol{\lambda}^\top = [\lambda_1 \ \lambda_2 \ \cdots \ \lambda_J]$, the traffic equations can be written in matrix form as

$$\mathbf{a}^\top = \boldsymbol{\lambda}^\top + \mathbf{a}^\top \mathbf{P},$$

or

$$\mathbf{a}^\top (\mathbf{I} - \mathbf{P}) = \boldsymbol{\lambda}^\top,$$

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- The next theorem states the stability condition for Jackson networks in terms of the above solution.

Theorem 9 (Stability of Jackson Networks)

A Jackson network with external arrival rate vector λ and routing matrix P is stable if:

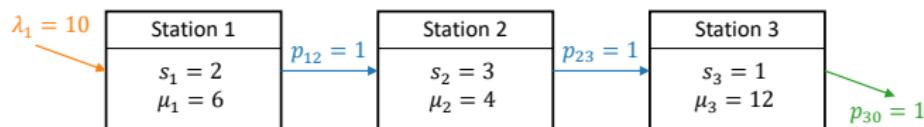
- (1) $I - P$ is invertible; and
- (2) $a_i < s_i \mu_i$ for all $i = 1, 2, \dots, J$, where a_i is given by the traffic equations.

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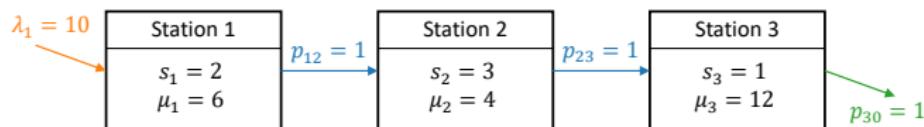
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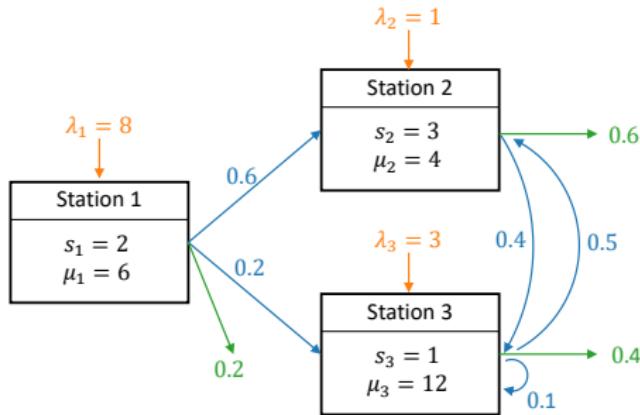
- Example 1: Tandem Queue



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}, \quad a^\top = \lambda^\top (I - P)^{-1} = [10 \ 10 \ 10].$$

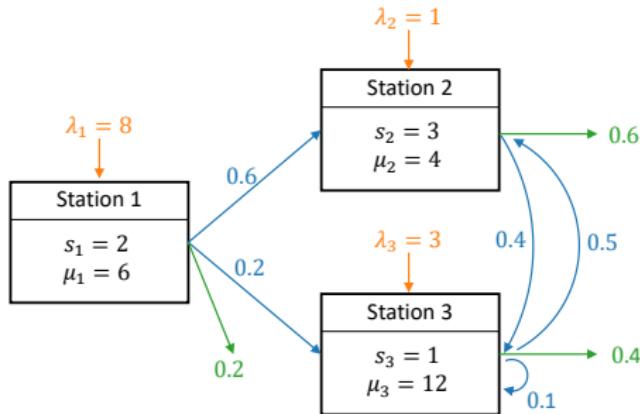
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• Example 2: General Network



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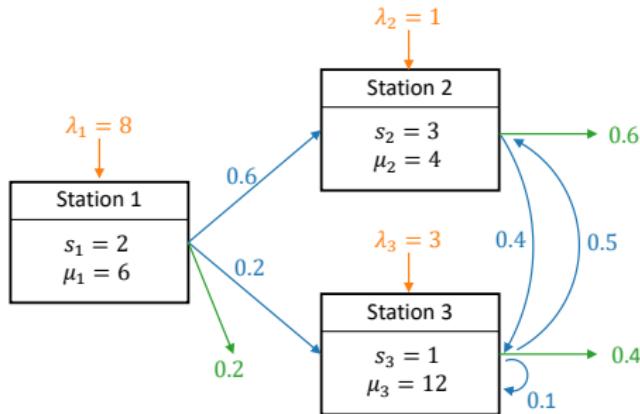
- Example 2: General Network



$$\mathbf{P} = \begin{bmatrix} 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 \\ 0 & 0.5 & 0.1 \end{bmatrix}.$$

$$\boldsymbol{\lambda} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{a}^\top = \boldsymbol{\lambda}^\top (\mathbf{I} - \mathbf{P})^{-1} = [8 \ 10.7 \ 9.9] \Rightarrow \text{Stable.}$$

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If λ_2 is increased to 4,

$$\boldsymbol{\lambda} = \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{a}^\top = \boldsymbol{\lambda}^\top (\mathbf{I} - \mathbf{P})^{-1} = [8 \ 14.6 \ 11.6] \Rightarrow \text{Unstable.}$$

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$$\begin{aligned} P(n_1, n_2, \dots, n_J) \\ = \lim_{t \rightarrow \infty} \mathbb{P}\{L_1(t) = n_1, L_2(t) = n_2, \dots, L_J(t) = n_J\}. \end{aligned}$$

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- It is a joint probability.

Theorem 10 (Limiting Distribution of Jackson Network)

For a stable Jackson network, its limiting (steady-state) distribution is given by

$$P(n_1, n_2, \dots, n_J) = P_1(n_1)P_2(n_2) \cdots P_J(n_J),$$

for $n_j = 0, 1, 2, \dots$ and $j = 1, 2, \dots, J$, where $P_j(n)$ is the limiting probability that there are n customers in an $M/M/s_j$ queue with arrival rate a_j and service rate μ_j .

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⇒ Limiting behavior of all stations are independent of each other.
- The limiting distribution of station j is the same as that in an **isolated** $M/M/s_j$ queue with arrival rate a_j and service rate μ_j . (a_j 's are solved from the **traffic equations**.)