MEM6810 Engineering Systems Modeling and Simulation

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

Spring 2022 (full-time)

Assignment 1

Due Date: March 16 (in class)

Instruction

(a) You can answer in English or Chinese or both.
(b) Show **enough** intermediate steps.
(c) Write your answers **independently**.

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Question 1 (10 points)

Suppose that $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,1)$, and $\mathrm{Cov}(X,Y) = 0$. Give an example that X is not independent of Y.

Question 2 ($10 + 10 = 20$ points)

Suppose that $X_1 \sim \mathrm{Pois}(\lambda_1)$, $X_2 \sim \mathrm{Pois}(\lambda_2)$, and $X_1 \perp X_2$.

(1) Prove that $X_1 + X_2 \sim \mathrm{Pois}(\lambda_1 + \lambda_2)$.
(2) Prove that given $X_1 + X_2 = n$, $X_1 \sim \mathrm{B}(n, \lambda_1/(\lambda_1 + \lambda_2))$.

Question 3 ($20 + 20 = 40$ points)

Suppose that $X_1 \sim \mathrm{Gamma}(\alpha_1, \lambda)$, $X_2 \sim \mathrm{Gamma}(\alpha_2, \lambda)$, and $X_1 \perp X_2$.

(1) Prove that $X_1 + X_2 \sim \mathrm{Gamma}(\alpha_1 + \alpha_2, \lambda)$ using the definition.
(2) Prove that $X_1 + X_2 \sim \mathrm{Gamma}(\alpha_1 + \alpha_2, \lambda)$ using the bivariate transformation.

Question 4 (10 points)

Recall the Numerical Integration example in Lec 1 page 28/33. Suppose that $f(x)$ is continuous on $[a, b]$. Let

$$Y_N := \frac{b-a}{N}[f(X_1) + \cdots + f(X_N)].$$

Prove that $Y_N \xrightarrow{a.s.} \int_a^b f(x)\mathrm{d}x$ as $N \to \infty$.

Question 5 $(10 + 10 = 20 \text{ points})$

Recall the Numerical Integration example in Lec 1 page 28/33. Now we consider the case of higher dimensionality. Suppose $\boldsymbol{x} \in \mathbb{R}^d$ and $f(\boldsymbol{x})$ is continuous on $[a, b]^d$. We want to compute $\int_{[a,\,b]^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$.

(1) Describe the Monte Carlo method and give the final estimator Y_N, where N denotes the number of randomly sampled points. [Hint: Similar to the 1D case.]

(2) **Definition.** *Let $\{b_n\}$ be a sequence of real numbers and $\{W_n\}$ a sequence of random variables. We say $\{W_n\}$ is* at most of order b_n in probability*, denoted $W_n = \mathcal{O}_p(b_n)$, if for every $\epsilon > 0$ there exist a finite $\Delta_\epsilon > 0$ and $N_\epsilon \in \mathbb{N}$, such that*

$$\mathbb{P}\left\{\omega : \left|\frac{W_n(\omega)}{b_n}\right| > \Delta_\epsilon\right\} < \epsilon \textit{ for all } n > N_\epsilon.$$

Prove that $Y_N - \int_{[a,\,b]^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \mathcal{O}_p(N^{-1/2})$, which does not depend on the dimensionality d.