CSC236H, Summer 2019
Assignment 1
Due June 09th, 11:00 p.m.

- You may work in groups of no more than **two** students, and you should produce a single solution in a PDF file named `a1.pdf`, submitted to MarkUs. Submissions must be **typed**.
- Please refer to the course information sheet for the **late submission policy**.
- For each question, please write up detailed answers carefully. Make sure that you use notation and terminology correctly, and that you explain and justify what you are doing. Marks will be deducted for incorrect or ambiguous use of notation and terminology, and for making incorrect, unjustified, ambiguous, or vague claims in your solutions.
  In particular, keep in mind that one of the objectives of this assignment is to get you to practice writing proofs by induction. This means that a good amount of marks are allocated to the structure of your proofs and how they were written up.

1. It is well-known that every set of size $n \in \mathbb{N}$ has exactly 2^n subsets. We can classify these subsets based on their number of elements. For example, if S is a set of size n, then S contains exactly one subset of size 0 (the empty set), n subsets of size 1, and $\frac{n(n-1)}{2}$ subsets of size 2 (you may use these facts without proof in your solution).

   Use **simple induction** to prove that for all $n \in \mathbb{N}$, every set of size n has $\frac{n(n-1)(n-2)}{6}$ subsets of size 3.

   Do NOT appeal to binomial coefficients in your solution!

2. For each $n \in \mathbb{N}$, we define the set $B_n = \{2^i : i \in \mathbb{N} \text{ and } 0 \leq i < n\}$. (Notice that $B_0 = \emptyset$).
   Consider the following claim:
   For any $n \in \mathbb{N}$ and any $s \in \mathbb{N}$ with $0 \leq s < 2^n$, B_n contains a subset A such that the sum of all elements in A is equal to s; that is:
   $$\sum_{x \in A} x = s$$

   (a) Use **simple or complete induction** to prove this claim.

   (b) Use the **Principle of Well-Ordering** to prove this claim. (Your proof must directly uses the Principle of Well-Ordering)

3. Consider the set of binary strings S defined recursively as follows:

   - $1 \in S$;
   - if $w_1, w_2 \in S$, then $0 \cdot w_1 \cdot w_2 \in S$ ($a \cdot b$ denotes concatenation of two strings a and b).

   Let $Z(v)$ denote the number of occurrences of 0 in the binary string v and $O(v)$ denote the number of occurrences of 1 in v.

   Use **structural induction** to prove that, for every string $w \in S$ and every proper prefix u of w, $O(u) \leq Z(u)$.
   (The string u is a prefix of the string w if there exists a string v such that $w = u \cdot v$. It is a proper prefix of w if and only if neither u nor v is the empty string).