Instructions: All assignments are due by **midnight** on the due date specified. Assignments should be typed and submitted as a PDF in Canvas.

Every student must write up their own solutions in their own manner.

You should complete all problems, but only a subset will be graded (which will be graded is not known to you ahead of time).

Relations

1. (10 points) **Graded (all)** Let R on $\{a, b, c, d, e\}$ be $\{(a, e), (b, a), (c, d), (d, b), (e, e)\}$. Determine the following:
   (a) Δ and the reflexive closure of R.
   (b) R^{-1} and the symmetric closure of R.
   (c) R^2
   (d) R^3
   (e) R^* and the transitive closure of R

   (a) $\Delta = \{(a, a), (b, b), (c, c), (d, d), (e, e)\}$
   reflexive closure is $\{(a, a), (a, e), (b, a), (b, b), (c, c), (c, d), (d, b), (d, d), (e, e)\}$

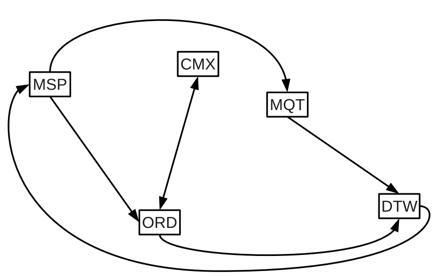
   (b) $R^{-1} = \{(a, b), (b, d), (d, c), (e, a), (e, e)\}$
   symmetric closure is $\{(a, b), (a, e), (b, a), (b, d), (c, d), (d, b), (d, c), (e, a), (e, e)\}$

   (c) $R \circ R = \{(a, e), (b, e), (c, b), (d, a), (e, e)\}$

   (d) $R^3 = R^2 \circ R = \{(a, e), (b, e), (c, a), (d, e), (e, e)\}$

   (e) $R^* = R \cup R^2 \cup R^3 \cup R^4 \cup R^5 =$
   $\{(a, e), (b, a), (b, e), (c, a), (c, b), (c, d), (c, e), (d, a), (d, b), (d, e), (e, e)\}$
   $R^4 = \{(a, e), (b, e), (c, e), (d, e), (e, e)\}$
   $R^5 = \{(a, e), (b, e), (c, e), (d, e), (e, e)\}$

2. **Ungraded** Given the following directed graph showing a relation R of flights between cities. Answer the following questions.



   (a) (1pt) Give the set A the relation R is on.
   (b) (1pt) Give the relation R (that is, list the ordered pairs explicitly).
   (c) (2pt) Express the relation R as a zero-one matrix (present cities in alphabetic order, e.g., CMX, DTW, MQT, ...)
   (d) (2pt) Find the diagonal relation on A (list all ordered pairs explicitly).
   (e) (2pt) Find the reflexive closure of R (list all ordered pairs explicitly).

(f) (2pt) Find R^{-1} (list all ordered pairs explicitly).
(g) (2pt) Find the symmetric closure of R.
(h) (1pt) Describe what the pairs in $R \circ R$ represent.
(i) (2pt) Find $R \circ R$ (list all ordered pairs explicitly).
(j) (1pt) Describe what the pairs in R^3 represent.
(k) (2pt) Find R^3 (list all ordered pairs explicitly).
(l) (1pt) Describe what the transitive closure, R^* represents.
(m) (3pt) Find R^* (list all ordered pairs explicitly).

(a) $\{\text{CMX}, \text{DTW}, \text{MQT}, \text{MSP}, \text{ORD}\}$

(b) $\{(\text{CMX}, \text{ORD}), (\text{DTW}, \text{MSP}), (\text{MQT}, \text{DTW}), (\text{MSP}, \text{MQT}), (\text{MSP}, \text{ORD}), (\text{ORD}, \text{CMX}), (\text{ORD}, \text{DTW})\}$

(c) $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$

(d) $\{(\text{CMX}, \text{CMX}), (\text{DTW}, \text{DTW}), (\text{MQT}, \text{MQT}), (\text{MSP}, \text{MSP}), (\text{ORD}, \text{ORD})\}$

(e) $\{(\text{CMX}, \text{CMX}), (\text{CMX}, \text{ORD}), (\text{DTW}, \text{DTW}), (\text{DTW}, \text{MSP}), (\text{MQT}, \text{DTW}), (\text{MQT}, \text{MQT}), (\text{MSP}, \text{MQT}), (\text{MSP}, \text{MSP}), (\text{MSP}, \text{ORD}), (\text{ORD}, \text{CMX}), (\text{ORD}, \text{DTW}), (\text{ORD}, \text{ORD})\}$

(f) $\{(\text{CMX}, \text{ORD}), (\text{DTW}, \text{MQT}), (\text{DTW}, \text{ORD}), (\text{MQT}, \text{MSP}), (\text{MSP}, \text{DTW}), (\text{ORD}, \text{CMX}), (\text{ORD}, \text{MSP})\}$

(g) $\{(\text{CMX}, \text{ORD}), (\text{DTW}, \text{MQT}), (\text{DTW}, \text{MSP}), (\text{DTW}, \text{ORD}), (\text{MQT}, \text{DTW}), (\text{MQT}, \text{MSP}), (\text{MSP}, \text{DTW}), (\text{MSP}, \text{MQT}), (\text{MSP}, \text{ORD}), (\text{ORD}, \text{CMX}), (\text{ORD}, \text{DTW}), (\text{ORD}, \text{MSP})\}$

(h) the cities for which you can travel between in two flights

(i) $\{(\text{CMX}, \text{CMX}), (\text{CMX}, \text{DTW}), (\text{DTW}, \text{MQT}), (\text{DTW}, \text{ORD}), (\text{MQT}, \text{MSP}), (\text{MSP}, \text{CMX}), (\text{MSP}, \text{DTW}), (\text{ORD}, \text{MSP}), (\text{ORD}, \text{ORD})\}$

(j) the cities for which you can travel between in three flights

(k) $\{(\text{CMX}, \text{MSP}), (\text{CMX}, \text{ORD}), (\text{DTW}, \text{CMX}), (\text{DTW}, \text{DTW}), (\text{MQT}, \text{MQT}), (\text{MQT}, \text{ORD}), (\text{MSP}, \text{MSP}), (\text{MSP}, \text{ORD}), (\text{ORD}, \text{CMX}), (\text{ORD}, \text{DTW}), (\text{ORD}, \text{MQT}), (\text{ORD}, \text{ORD})\}$

(l) the cities for which you can travel between in taking n number of flights

(m) $\{(\text{CMX}, \text{CMX}), (\text{CMX}, \text{DTW}), (\text{CMX}, \text{MQT}), (\text{CMX}, \text{MSP}), (\text{CMX}, \text{ORD}), (\text{DTW}, \text{CMX}), (\text{DTW}, \text{DTW}), (\text{DTW}, \text{MQT}), (\text{DTW}, \text{MSP}), (\text{DTW}, \text{ORD}), (\text{MQT}, \text{CMX}), (\text{MQT}, \text{DTW}), (\text{MQT}, \text{MQT}), (\text{MQT}, \text{MSP}), (\text{MQT}, \text{ORD}), (\text{MSP}, \text{CMX}), (\text{MSP}, \text{DTW}), (\text{MSP}, \text{MQT}), (\text{MSP}, \text{MSP}), (\text{MSP}, \text{ORD}), (\text{ORD}, \text{CMX}), (\text{ORD}, \text{DTW}), (\text{ORD}, \text{MQT}), (\text{ORD}, \text{MSP}), (\text{ORD}, \text{ORD})\}$

3. (6 points) **Graded (a-b,d(i)-(iii))** Let the sets *Director*, D, be a set of movie directors, *Movie*, M, be a set of movies, and *Actor*, A, be a set of actors. We define the following relations:

- $appearedIn \subseteq$ *Actor* $\times$ *Movie* or $appearedIn \subseteq A \times M$, where a $appearedIn$ m if a was an actor who appeared in movie m
- $directedFilm \subseteq$*Director* $\times$ *Movie* or $directedFilm \subseteq D \times M$, where d $directedFilm$ m if d was the director of movie m

- $starredIn \subseteq Actor \times Movie$ of $starredIn \subseteq A \times M$, where $a\ starredIn\ m$ if a was an actor with top 3 billing in movie m

Express the following using only set operations and relations:

(a) There must be actors who have both appeared in and starred in a movie.
(b) Some people appear in movies they direct.
(c) Some movies are directed by people who do not star in them.

Take the following relations and express what they mean in English. Think about what sets define the elements the relations is made up of.

(d) $appearedIn \circ appearedIn^{-1}$
Is this relation (i) reflexive? (ii) symmetric? and (iii) transitive?
(e) $directedFilm \circ starredIn^{-1}$
Is this relation (i) reflexive? (ii) symmetric? and (iii) transitive?

(a) $appearedIn \cap starredIn \neq \emptyset$
(b) $directedFilm \cap appearIn \neq \emptyset$
(c) $directedFilm \not\subseteq starredIn$
(d) This relation is defined as $\subseteq M \times M$, where (x, y) in the relation if movie x shared actors with movie y. Many of this properties can be seen with the following example.

(HarrisonFord, Star Wars), (HarrisonFord, Blade Runner 2049), (Robin Wright, Blade Runner 2049), (Robin Wright, The Princess Bride) $\in appearedIn$

So, $appearedIn \circ appearedIn^{-1}$ would have the following elements:

$(m, a) \in appearedIn^{-1}$	$(a, m) \in appearedIn$	$appearedIn \circ appearedIn^{-1}$
(Star Wars, HarrisonFord)	(HarrisonFord, Blade Runner 2049)	(Star Wars, Blade Runner 2049)
(Blade Runner 2049, HarrisonFord)	(HarrisonFord, Blade Runner 2049)	(Blade Runner 2049, Blade Runner 2049)
(Blade Runner 2049, Robin Wright)	(Robin Wright, The Princess Bride)	(Blade Runner 2049, The Princess Bride)
...		

(i) Yes, every movie shares an actor with itself
(ii) Yes, if movie a shares an actor with movie b, movie b will also share an actor with a
(iii) No, e.g, Star Wars and Blade Runner are connected via Harrison Ford and Blade Runner and The Princess Bride are connected via Robin Wright, but that doesn't mean Star Wars and The Princess Bride are paired.

(e) This relation is defined as $\subseteq M \times M$, where (x, y) in relation if x was directed by the actor who starred in y. Note, this assumes there is an overlap between actors and directors which there is in practice.
(i) No
(ii) No
(iii) No

4. (2 points) **Graded (e)** Rosen Ch 9.5 # 2(a,c-e) p. 615

Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(a) $\{(a, b) \mid a$ and b are the same age $\}$. Equivalence relation
(c) $\{(a, b) \mid a$ and b share a common parent $\}$. Not an equivalence relation, not transitive
(d) $\{(a, b) \mid a$ and b have met $\}$. No, It is not transitive.
(e) $\{(a, b) \mid a$ and b speak a common language $\}$. Not an equivalence relation, not transitive

5. **Ungraded** Let R_m be the relation on $\mathbb{Z}^+$ where $R_m = \{(a, b) \mid a \equiv b \ (mod\ m)\}$.
   (a) Give the four members of $[4]_{R_2}$ with the smallest values.
   (b) Give the four members of $[4]_{R_6}$ with the smallest values.
   (c) Give the four members of $[5]_{R_3}$ with the smallest values.
   (d) How many unique equivalence classes for R_7 exist on the members of $\mathbb{Z}^+$.

   (a) $[4]_{R_2} = \{2, 4, 6, 8, \dots\}$
   (b) $[4]_{R_6} = \{4, 10, 16, 22, \dots\}$
   (c) $[5]_{R_3} = \{2, 5, 8, 11, \dots\}$
   (d) There are 7 unique equivalence classes for R_7

6. (2 points) **Graded ((a,b))** Rosen Ch 9.5 # 24, p. 616.
   If it is an equivalence relation, list the equivalence classes assuming the elements are ordered $x_1, x_2, x_3, x_4, \dots$.
   (a) The relation is not symmetric, therefore, it is not an equivalence relation.
   (b) The relation is an equivalence relation.
   The equivalence classes are $\{x_1, x_3\}, \{x_2, x_4\}$
   (c) The relation is an equivalence relation.
   The equivalence classes are $\{x_1, x_2, x_3\}, \{x_4\}$

7. **Ungraded** Rosen Ch 9.5 # 56(a,b), p.617
   (a) **No, not necessarily**.
   Let R_1 and R_2 be defined over $\{0, 1, 2\}$. R_1 has equivalence classes $\{0\}$ and $\{1, 2\}$; R_2 has equivalence classes $\{0, 2\}$ and $\{1\}$. Then $R_1 \cup R_2$ is not transitive, since $(0, 2) \in R_2$ (and therefore $R_1 \cup R_2$) and $(2, 1) \in R_1$ (and therefore $R_1 \cup R_2$), but $(0, 1) \notin R_1 \cup R_2$.
   (b) **Yes** $R_1 \cap R_2$ must be an equivalence relation.
   If R_1 and R_2 are reflexive, then the intersection must also then be reflexive.

   If R_1 and R_2 are both symmetric, then the intersection of R and R_2 must also be symmetric.

   If R_1 and R_2 are both transitive, then the intersection of R_1 and R_2 is also transitive.

8. (2 points) **Graded (all)** Rosen Ch 9.6 #4(a,c), p. 630.
   (a) There should be unequal people of the same height, therefore, the relation is not antisymmetric and (S, R) is not a poset.
   (c) This is a poset, R is reflexive, antisymmetric, and transitive.

9. (3 points) **Graded (c-f)** Which of the following are posets?
   (a) $(\mathbb{Z}, =)$, Yes
   (b) $(\mathbb{Z}, \neq)$, No, it is not reflexive.
   (c) $(\mathbb{Z}, \geq)$, Yes
   (d) $(\mathbb{R}, =)$, Yes
   (e) $(\mathbb{R}, <)$, No, it is not reflexive.
   (f) $(\mathbb{R}, \geq)$, Yes

10. **Ungraded** Rosen Ch 9.6 #32(a-f), p. 631.

(a) maximal elements: $\{l, m\}$
(b) minimal elements: $\{a, b, c\}$
(c) there is no greatest element
(d) there is no least element
(e) $\{k, l, m\}$
(f) $\{k\}$

Graphs

11. **Ungraded** Use a graph to model the following problems:

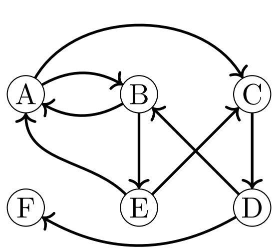
    (a) Twitter "follows" - Construct a graph of who follows who on Twitter on a set of people. Ann follows Bob, Ann also follows Charlie, Bob follows Ann, Bob also follows Ed, Charlie follows Dana, Dana follows Bob, Dana also follows Flora, Ed follows Ann, Ed also follows Charlie.

    (b) Integer division -Construct a graph that models whether for a pair of integers, one integer even divides the other. Consider the set of integers $V = \{2, 3, 4, 6, 8, 9, 12, 18, 27\}$; two vertices are adjacent if one evenly divides the other.

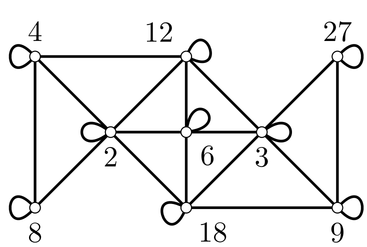
    (c) Airline flights
    An airline has the following direct flights on their system, where each direct route between city a and b and back is listed below. Model the all the routes on the system as a graph.

City a	City b	# of flights	City a	City b	# of flights
Atlanta	Chicago	1	Chicago	San Francisco	2
Atlanta	Miami	1	Denver	Los Angeles	2
Atlanta	New York	1	Denver	San Francisco	1
Boston	New York	2	Los Angeles	New York	3
Boston	Chicago	1	Los Angeles	San Francisco	2
Chicago	Denver	1	Miami	New York	1
Chicago	New York	2	New York	San Francisco	2

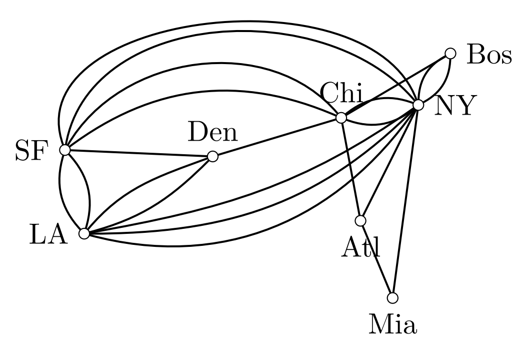
(a) follows graph



(b) integer division graph



(c) airline flights graph



12. (2 points) **Graded (a-b)** Determine the type of graphs depicted in each part of Problem 11: simple graph? multigraph? pseudograph? directed graph? directed multigraph?

(a) directed graph
(b) pseudograph
(c) multigraph

13. (3 points) **Graded (all)** For the graphs in Problem 11(a-c), determine the number of vertices and edges.

(a) $|V| = 6$, $|E| = 9$

(b) $|V| = 9$, $|E| = 25$

(c) $|V| = 8$, $|E| = 22$

14. (2 points) **Graded (a)** For the graphs in Problem 11(a-c), determine degree of each vertex (for directed graphs, indicate both the in-degree and out-degree of each vertex).

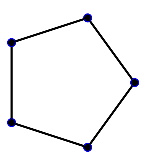
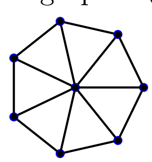
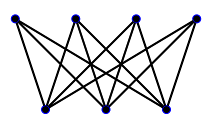
(a)

	A	B	C	D	E	F
in-degree	2	2	2	1	1	1
out-degree	2	2	1	2	2	0

(b) deg(2)=7, deg(3)=7, deg(4)=5, deg(6)=6, deg(8)=4, deg(9)=5, deg(12)=6, deg(18)=6, deg(27)=4

(c) deg(Atl) = 3, deg(Bos) = 3, deg(Chi) = 7, deg(Den) = 4, deg(Mia) = 2, deg(LA) = 7, deg(NY) = 11, deg(SF) = 7

15. **Ungraded** Draw these graphs. **(a)** C_5 **(b)** W_7 **(c)** $K_{3,4}$

(a)  (b)  (c) 

16. (4 points) **Graded (a-b)** For which values of n are (a) C_n, (b) K_n and (c) W_n bipartite? Explain your answer.

(a) C_n: if n is even, the cycle graph is bipartite, if n is odd, the graph is not bipartite

(b) K_n: for $n = 2$, the complete graph is bipartite, for $n \geq 3$, the graphs are not bipartite (contain triangles)

(c) W_n: there are no values of n where the wheel graph is bipartite; every graph contains triangle.

17. (4 points) **Graded (all)**

(a) A simple graph with $|V| = n$ and $|E| = 17$ has one vertex of degree 1, two vertices of degree 2, one vertex of degree 3, and two vertices of degree 5. The remaining vertices of the graph have degree 4. What is n?

(b) A graph with $|V| = 25$ and $|E| = 62$ has vertices of degree 3, 4, 5, or 6. There are two vertices of degree 4 and 11 vertices of degree 6. How many vertices of the graph have degree 3?

Use the handshaking theorem:

(a) $1 + 2 + 1 + x + 2 = n$ Sum of vertices = n
$1 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + x \cdot 4 + 2 \cdot 5 = 2 \cdot 17$ Handshaking theorem,
$\rightarrow x = 4$, $\therefore \mathbf{n = 10}$

(b) $x + 2 + y + 11 = 25$ Sum of # of vertices, where x is # with degree 3
$3 \cdot x + 4 \cdot 2 + 5 \cdot y + 6 \cdot 11 = 2 \cdot 62$ Handshaking theorem,
2 equations, 2 unknowns, solve for x, $\mathbf{x = 5}$

Bonus

18. (2 points (bonus)) List all the equivalence relations on the set $\{a, b, c\}$. (Describe the relation be describing the partition on the set).

    There are five such relations

    $$\{\{a\},\{b\},\{c\}\};\{\{a,b\},\{c\}\};\{\{a,c\},\{b\}\};\{\{b,c\},\{a\}\};\{\{a,b,c\}\}$$