Instructions: All assignments are due by **midnight** on the due date specified. Assignments should be typed and submitted as a PDF in Canvas.

Every student must write up their own solutions in their own manner.

You should complete all problems, but only a subset will be graded (which will be graded is not known to you ahead of time).

Relations

1. (3 points) **Graded (b-d)** Let $X = \{a, b\}$ and $Y = \{1, 2\}$.
   (a) Give the sets $X \times Y$ and $\mathcal{P}(X \times Y)$.
   (b) How many possible relations exist from X to Y?
   (c) What does $\mathcal{P}(X \times Y)$ represent with respect to relations?
   (d) How many binary relations exist on the set $C = \{1, 2, 3, 4\}$?
   You do **not** need to list all such relations

   (a) $X \times Y = \{(a, 1), (b, 1), (a, 2), (b, 2)\}$
   $\mathcal{P}(X \times Y) = \{\ \emptyset, \{(a, 1)\}, \{(a, 2)\}, \{(b, 1)\}, \{(b, 2)\}$
   $\{(a, 1), (1, 2)\}, \{(a, 1), (b, 1)\}, \{(a, 1), (b, 2)\}, \{(a, 2), (b, 1)\}, \{(a, 2), (b, 2)\}, \{(b, 1), (b, 2)\},$
   $\{(a, 1), (a, 2), (b, 1)\}, \{(a, 1), (a, 2), (b, 2)\}, \{(a, 1), (b, 1), (b, 2)\}, \{(a, 2), (b, 1), (b, 2)\},$
   $\{(a, 1), (a, 2), (b, 1), (b, 2)\}\ \}$

   (b) $2^{|X \times Y|} = 2^{2 \cdot 2} = 2^4 = 16$

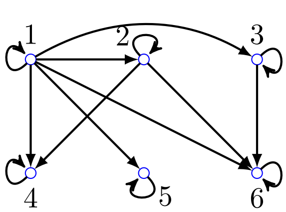
   (c) Each element of $\mathcal{P}(X \times Y)$ is a possible relation.

   (d) $2^{|C \times C|} = 2^{4 \cdot 4} = 2^{16} = 65536$

2. (3 points) **Graded (a(i),b(i),c)** For each part, describe a relation using the different representations asked for.
   (a) Let $A = \{1, 2, 3, 4, 5, 6\}$.
      (i) Write out the relation R on A that expresses $x \mid y$ (divides), that is if $x \mid y$ then $(x, y) \in R$, that is describe the relation using the set enumeration methods (list all elements of the set).
      (ii) Draw the relation as a digraph.
      (iii) Describe the relation as a zero-one matrix (assume rows/columns are ordered numerically).

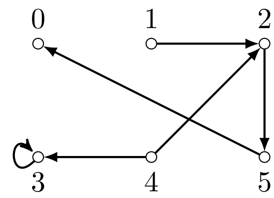
      (a).(i) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6),$
      $(3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

      (a).(ii) & (a).(iii)



$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Let R be a relation on a set A, illustrated below.



(i) Write out the sets A and R.

(ii) Describe the relation as a zero-one matrix (assume rows/columns are ordered numerically).

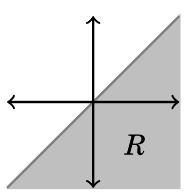
(b).(i) $A = \{0, 1, 2, 3, 4, 5\}$, $R = \{(1, 2), (2, 5), (3, 3), (4, 2), (4, 3), (5, 0)\}$

(b).(ii) $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

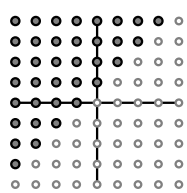
(c) Congruence modulo 5 is a relation, R, on $\mathbb{Z}$, where $(x, y) \in R$ means $x \equiv y \ (mod\ 5)$. Write out the set R in set-builder notation.

(c) $R = \{(x, y) \mid x, y \in \mathbb{Z} \text{ and } 5 \mid (x - y)\}$

3. **Ungraded** In the following figures relations R are indicated by gray shading. In figure (a), the relation is on $\mathbb{R}$ in (b) the relation is on $\mathbb{Z}$. State what familiar relation is being represented.



(a) $>$



(b) $<$

4. (16 points) **Graded (b,d-f,i-j,l-m)** Consider the following relations, determine whether the relation described is reflexive (R), symmetric (S), antisymmetric (AS), and transitive (T). Report the results in a table, for example,

Relation	**R?**	**S?**	**AS?**	**T?**
(a) R_a	Yes	Yes	No	Yes
(b) R_b	No	Yes	No	Yes
(c) Rosen 9.1.4(a)	No	No	Yes	Yes
(d) Rosen 9.1.4(c)	Yes	Yes	No	Yes
(e) Rosen 9.1.6(d)	Yes	Yes	No	No
(f) R_f	No	No	Yes	Yes
(g) $>$ on $\mathbb{Z}$	No	No	Yes	Yes
(h) $\leq$ on $\mathbb{Z}$	Yes	No	Yes	Yes
(i) $\neq$ on $\mathbb{Z}$	No	Yes	No	No
(j) $\mid$ on $\mathbb{Z}$	**Yes**	No	Yes	**Yes**
(k) Rosen 9.1.6(b)	Yes	Yes	No	Yes
(l) Rosen 9.1.6(c)	Yes	Yes	No	Yes
(m) Rosen 9.1.6(e)	Yes	Yes	No	No
(n) Rosen 9.1.6(f)	No	Yes	No	No

Also, briefly explain for any "no" answer, why the relation does not have a given property.

(a) R_a is not antisymmetric, it has both (a, b) and (b, a) where $a \neq b$

(b) R_b is not reflixive, e.g., missing (1,1). R_b is not antisymmetric, it has both $(2, 0)$ and $(0, 2)$, but $0 \neq 2$

(c) R_c is not reflexive because no one can be taller than themselves. R_c is not symmetric if a is taller then b it can not be the opposite.

(d) R_d is not antisymmetric, person a may have the same name as person b, $(a, b) \in R_d$ and $(b, a) \in R_d$, but a and b are not the same person.

(e) R_e is not antisymmetric, if a and b share a common grandparent, this can be expressed as both $(a, b) \in R_e$ and $(b, a) \in R_e$. R_e is not transitive. Let b have grandparents b_1 and b_2, a has b_1 as a grandparent (so, $(a, b) \in R_e$), and c has b_2 as a grandparent (so, $(b, c) in R_e$), but a and c do not share a common grandparent.

(f) R_f is not reflexive because, for example, it does not have (Elon Musk, Elon Musk) in the relation. R_f is not symmetric it is missing (Mark Z, Bill Gates).

(g) $>$ is not reflexive, e.g., missing $(0, 0)$. $>$ is not symmetric, e.g., both $0 > 2$ and $2 > 0$ can not be in the relation.

(h) The $\leq$ relation on $\mathbb{Z}$ is not symmetric, e.g., both $0 > 2$ and $2 > 0$ can not be in the relation.

(i) The $\neq$ relation on $\mathbb{Z}$ is not reflexive, e.g., you can not state $1 \neq 1$. $\neq$ is not antisymmetric, e.g., if $1 \neq 2$, then $2 \neq 1$ breaks the definition of antisymmetry. $\neq$ is not transitive, e.g., if $1 \neq 2$ and $2 \neq 1$, then $1 = 1$ breaking the property of transitive.

(j) The $\mid$ relation on $\mathbb{Z}$ is not symmetric, e.g., if 2 divides 4, then 4 can not divide 2.

(k) The relation is not antisymmetric, because both (1,-1) and (-1,1) are in the relation.

(l) The relation is not antisymmetric, because both (1,-1) and (-1,1) are in the relation.

(m) The relation is not antisymmetric, because both (2,3) and (3,2) are in the relation. It is not transitive, e.g., both (1,0) and (0, -2) are in the relation, but (1,-2) is not.

(n) The relation is not reflexive, because (1,1) not in the relation. It is not antisymmetric, since (2,0) and (0,2) are both in the relation. It is not transitive, e.g., both (1,0) and (0, -2) are in the relation, but (1,-2) is not.

Consider the relations R, S, T, U on the set $\{a, b, c, d\}$. Use the definitions and properties discussed in class and chapter 9.1 and the properties or operations of **irreflexive**, **asymmetric**, **inverse relations**, and **complementary relations** mentioned before Rosen 9.1 # 11 (p. 581), # 18 (p. 582), and # 26 (p. 582).

Let $R = \{(a, a), (b, c), (c, b), (c, d), (d, c), (d, d)\}$,
$S = \{(a, a), (a, d), (b, a), (b, b), (b, d), (c, a), (c, c), (d, c), (d, d)\}$,
$T = \{(a, a), (a, b), (b, c), (b, d), (c, d), (d, a), (d, b)\}$, and
$U = \{(a, a), (a, d), (b, c), (b, d), (c, a), (d, d)\}$

5. (3 points) **Graded** (S) Determine (Yes/No) whether R and S have each of the following properties: reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and transitive.

Relation	Reflexive?	Irreflexive?	Symmetric?	AntiSym.?	Asym.?	Transitive?
R	No	No	Yes	No	No	No
S	Yes	No	No	Yes	No	No

6. (15 points) **Graded (d-e,h-j,l)** Find the following expressions:

a) (1pt) $R \cup S$ b) (1pt) $R \cap S$ c) (1pt) $R - S$ d) (1pt) $S - R$
e) (1pt) $\overline{S}$ f) (1pt) S^{-1} g) (2pt) $T \circ T$ h) (2pt) $U \circ T$
i) (2pt) $T \circ U$ j) (4pt) R^3 k) (4pt) U^3 l) (5pt) $R \circ S \circ T$

(a) $R \cup S = \{(a,a),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),(d,c),(d,d)\}$

(b) $R \cap S = \{(a,a),(d,c),(d,d)\}$

(c) $R - S = \{(b,c),(c,b),(c,d)\}$

(d) $S - R = \{(a,d),(b,a),(b,b),(b,d),(c,a),(c,c)\}$

(e) $\overline{S} = \{(a,b),(a,c),(b,c),(c,b),(c,d),(d,a),(d,b)\}$

(f) $S^{-1} = \{(a,a),(a,b),(a,c),(b,b),(c,c),(c,d),(d,a),(d,b),(d,d)\}$

(g) $T^2 = \{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,d),(c,a),(c,b),(d,a),(d,b),(d,c),(d,d)\}$

(h) $U \circ T = \{(a,a),(a,c),(a,d),(b,a),(b,d),(c,d),(d,a)(d,c),(d,d)\}$

(i) $T \circ U = \{(a,a),(a,b),(b,a),(b,b),(b,d),(c,a),(c,b),(d,a),(d,b)\}$

(j) $R^2 = \{(a,a),(b,b),(b,d),(c,c),(c,d),(d,b),(d,c),(d,d)\}$
$R^3 = R^2 \circ R = \{(a,a),(b,c),(b,d),(c,b),(c,c),(c,d),(d,b),(d,c),(d,d)\}$

(k) $U^2 = \{(a,a),(a,d),(b,a),(b,d),(c,a),(c,d),(d,d)\}$
$U^3 = U^2 \circ U = \{(a,a),(a,d),(b,a),(b,d),(c,a),(c,d),(d,d)\}$

(l) $R \circ S \circ T = \{(a,a),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,b),(c,c),(c,d),(d,a),(d,c),(d,d)\}$

Bonus

7. (2 points (bonus)) There are 16 possible relations R on the set $A = \{a,b\}$. Describe all of them as directed graphs (be sure to label the nodes in the graph). Which relations are reflexive? symmetric? transitive?