Instructions: All assignments are due by **midnight** on the due date specified. Assignments should be typed and submitted as a PDF. Every student must write up their own solutions in their own manner.

You should complete all problems, but only a subset will be graded (which will be graded is not known to you ahead of time).

Sequences

1. (6 points) **Graded (e)** Rosen Ch 2.4 # 16(c,e), p. 168.

$$
\begin{aligned}
(c) \qquad a_n &= a_{n-1} - n \\
&= a_{n-2} - (n-1) - n = a_{n-1} - (n + (n-1)) \\
&= a_{n-3} - (n-2) - (n + (n-1)) = a_{n-2} - (n + (n-1) + (n-2)) \\
&= \cdots \\
&= a_{n-n} - (n + (n-1) + (n-2) + \ldots + (n-(n-1)) \\
&= a_0 - (n + (n-1) + (n-2) + \ldots + 1) \\
&= 4 - (n + (n-1) + (n-2) + \ldots + 1) = 4 - \frac{n(n+1)}{2}
\end{aligned}
$$

$$
\begin{aligned}
(e) \qquad a_n &= (n+1)a_{n-1} \\
&= (n+1)na_{n-2} \\
&= (n+1)n(n-1)a_{n-3} \\
&= \cdots \\
&= (n+1)n(n-1)\cdots(n-(n-2))a_{n-n} \\
&= (n+1)n(n-1)\cdots 2a_0 \\
&= 2(n+1)!
\end{aligned}
$$

2. **Ungraded** A mortage loan is paid off in periodic (monthly) installments, while interest is also charged each period. A mortage with an annual interest rate of r has a monthly interest rate of $i = \frac{r}{12}$. A mortage of M dollars at monthly interest rate i has payments of P dollars. At the end of the month, interest is added to the previous balance, and then the payment P is subtracted from the result. Let m_n be the balance due after n months, where $m_0 = M$, $m_1 = M(1+i)-P, \ldots$
   (a) Let $M = 10,000$, $i = 0.03$ and $P = 105.13$, determine m_2 and m_3.
   (b) Find a recursive formula for the mortage balance, m_n

   (a) $m_0 = 10,000$, $m_1 = 10194.87$, $m_2 = 10395.59$, $m_3 = 10602.32$
   (b) $m_n = m_{n-1}(1+i) - P$, $m_o = M$

Summations

3. **Ungraded** Write out in sigma notation the sum of the first 40 terms of the series $3+6+9+12+\ldots$.

The sum can take many form depending on index of summation and limits. Here are two samples

$$\sum_{i=1}^{40} 3 \cdot i \qquad \sum_{j=1}^{40} 3 \cdot j \qquad \sum_{k=0}^{39} 3 \cdot (k+1)$$

4. (8 points) **Graded ((a,c-e))** What are the values of the sums:

(a) $\sum_{j \in S}(2j-1)$, where $S = \{1,2,4,6\}$ (b) $\sum_{i=1}^{8} 4$ (c) $\sum_{k=0}^{4}(-3)^k$

(d) $\sum_{j=0}^{7}(2^{j+1} - 2^{j-1})$ (e) $\sum_{i=1}^{4}\sum_{j=1}^{3}(2i+3j)$ (f) $\sum_{j=0}^{3}\sum_{k=1}^{3} j * 2^k$

(a) $(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 4 - 1) + (2 \cdot 6 - 1) = 1 + 3 + 7 + 11 = 22$

(b) $\sum_{i=1}^{8} 4 = 8 \cdot 4 = 32$

(c) $(-3^0) + (-3^1) + (-3^2) + (-3^3) + (-3^4) = 1 - 3 + 9 - 27 + 81 = 61$

(d) $\sum_{j=0}^{7} 2^{j+1} - \sum_{j=0}^{7} 2^{j-1} = 510 - 127.5 = 382.5$

(e) $\sum_{i=1}^{4}\left(\sum_{j=1}^{3} 2i + \sum_{j=1}^{3} 3j\right) = 132$

(f) $\sum_{j=0}^{3}\sum_{k=1}^{3} j * 2^k = 84$

Arithmetic Properties of Summations

Fact 1:	$\sum_{i=1}^{n} c = nc$ when c is a constant	Fact 2:	$\sum_{i=j}^{n} c = (n-j+1)c$ when c is a constant
Fact 3:	$\sum_{i=j}^{n}(f(i) \pm g(i)) = \sum_{i=j}^{n} f(i) \pm \sum_{i=j}^{n} g(i)$	Fact 4:	$\sum_{i=j}^{n} cf(i) = c\sum_{i=j}^{n} f(i)$ when c is a constant

Closed-form Summation Formulae

Name	Sum	Closed Form	Name	Sum	Closed Form
Table 2.1	$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r-1}$, if $r \neq 1$	Table 2.2	$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
Table 2.3	$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$	Table 2.4	$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
Table 2.5	$\sum_{k=0}^{\infty} x^k$, $\lvert x\rvert < 1$	$\frac{1}{1-x}$	Table 2.6	$\sum_{k=1}^{\infty} kx^{k-1}$, $\lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

5. (8 points) **Graded (all)** Find a closed form for the summation $\sum_{i=0}^{n}(2 \cdot 3^i - 3 \cdot 2^i)$. Show how you find the closed form solution; justify each step using the four facts of the arithmetic properties of summations and Table 2 above (p. 166 in the book).

$$\begin{aligned}
&\sum_{i=0}^{n}(2 \cdot 3^i - 3 \cdot 2^i) \\
&\quad = \sum_{i=0}^{n} 2 \cdot 3^i - \sum_{i=0}^{n} 3 \cdot 2^i && \text{(Fact 3)} \\
&\quad = 2\sum_{i=0}^{n} 3^i - 3\sum_{i=0}^{n} 2^i && \text{(Fact 4)} \\
&\quad = 2\left(\frac{3^{n+1}-1}{2}\right) - 3\left(\frac{2^{n+1}-1}{1}\right) && \text{(Table 2.1)} \\
&\quad = 3^{n+1} - 1 - 3(2^{n+1} - 1) && \text{(algebra)} \\
&\quad = 3^{n+1} - 3 \cdot 2^{n+1} + 2 && \text{(algebra)}
\end{aligned}$$

6. **Ungraded** Find a closed form for the summation $\sum_{i=4}^{n} 7 \cdot 5^i$. Show how you find the closed form solution; justify each step using the four facts of the arithmetic properties of summations and Table 2 above (p. 166 in the book).

$$\begin{aligned}
\sum_{i=4}^{n} 7 \cdot 5^i &= \sum_{j=0}^{n-4} 7 \cdot 5^{j+4} && \text{(change of index)} \\
&= \sum_{j=0}^{n-4} 7 \cdot 5^j 5^4 && \text{(algebra)} \\
&= 7 \cdot 5^4 \sum_{j=0}^{n-4} 5^j && \text{(Fact 4)} \\
&= 7 \cdot 5^4 \left(\frac{5^{n-3}-1}{4}\right) && \text{(Table 2.1)} \\
&= \frac{7 \cdot 5^4(5^{n-3}-1)}{4} && \text{(algebra)}
\end{aligned}$$

or,

$$\begin{aligned}
\sum_{i=4}^{n} 7 \cdot 5^i &= \sum_{j=0}^{n-4} 7 \cdot 5^{j+4} && \text{(change of index)} \\
&= \sum_{j=0}^{n-4} 7 \cdot 5^j 5^4 && \text{(algebra)} \\
&= \frac{7 \cdot 5^4 \cdot 5^{n-3} - 7 \cdot 5^4}{4} && \text{(Table 2.1)} \\
&= \frac{7 \cdot 5^4(5^{n-3}-1)}{4} && \text{(algebra)}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\sum_{i=4}^{n} 7\cdot 5^i &= \sum_{i=0}^{n} 7\cdot 5^i - \sum_{i=0}^{3} 7\cdot 5^i && \text{(prop. of sum)}\\
&= \frac{7\cdot 5^{n+1}-7}{4} - \frac{7\cdot 5^4 - 7}{4} && \text{(Table 2.1)}\\
&= \frac{7\cdot 5^{n+1} - 7\cdot 5^4}{4} && \text{(algebra)}\\
&= \frac{7\cdot 5^4\cdot 5^{n-3} - 7\cdot 5^4}{4} && \text{(algebra)}\\
&= \frac{7\cdot 5^4(5^{n-3}-1)}{4}
\end{aligned}$$

Induction

Follow the template for inductive proofs given on p. 329 of the book.

7. (10 points) **Graded (all)** Let $P(n)$ be the statement $\sum_{j=1}^{n} 2j = n + n^2$ for $n \geq 1$.
   (a) (1 pt) What is the statement $P(1)$?
   (b) (1 pt) Show that P(1) is true, completing the basis step of the proof.
   (c) (1.5 pts) What is the inductive hypothesis?
   (d) (1.5 pts) What do you need to prove in the inductive step?
   (e) (4 pts) Complete the inductive step.
   (f) (1 pt) Explain why these steps show that this formula is true whenever n is a positive integer.

   (a) (1 pt) What is the statement $P(1)$?
   $P(1)$ is the statement $\sum_{j=1}^{1} 2j = 1 + 1^2$.
   (b) (1 pt) Show that P(1) is true, completing the basis step of the proof.
   $\sum_{j=1}^{1} 2j = 2\cdot 1 = 2 = 1 + 1^2$.
   (c) (1.5 pts) What is the inductive hypothesis?
   The inductive hypothesis is to assume $P(k)$ is true for an arbitrary, fixed integer $k \geq 1$, that is

$$\sum_{j=1}^{k} 2j = k + k^2$$

   (d) (1.5 pts) What do you need to prove in the inductive step?
   For the inductive step, show for each $k \geq 1$ that $P(k)$ implies $P(k+1)$.
   That is, show $P(k+1)$:

$$\sum_{j=1}^{k+1} 2j = (k+1) + (k+1)^2 = k^2 + 3k + 2$$

   (e) (4 pts) Complete the inductive step. Start with $P(k+1)$

$$\begin{aligned}
\sum_{j=1}^{k+1} 2j &= \sum_{j=1}^{k} 2j + 2(k+1)\\
&= k + k^2 + 2(k+1) && \text{(IH)}\\
&= k^2 + k + 2k + 2\\
&= k^2 + 3k + 2
\end{aligned}$$

   This shows $P(k+1)$ is true, assuming $P(k)$ is true, completing the inductive step.

(f) (1 pt) Explain why these steps show that this formula is true whenever n is a positive integer.
The basis step and inductive step are completed. Therefore by principle of mathematical induction, the statement, $P(n)$, is true for every positive integer n.

8. **Ungraded** Prove using mathematical induction that

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^n = \frac{5^{n+1} - 1}{4} \text{ for all } n \geq 0.$$

Let $P(n)$ be $1 + 5 + 5^2 + 5^3 + \cdots + 5^n = \frac{5^{n+1}-1}{4}$.

Show, for all $n \geq 0, P(n)$.

Basis Step:
Show $n = 0$, $1 = \frac{5^{0+1}-1}{4} = \frac{5-1}{4} = 1$.
Therefore, $P(0)$ is true.

Inductive Step:
Assume $P(k)$ is true, for some arbitrary, fixed integer $k \geq 0$,

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^k = \frac{5^{k+1} - 1}{4}$$

Show $P(k+1)$ is true,

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^{k+1} = \frac{5^{k+2} - 1}{4}$$

Begin with $P(k)$ and add the next term, 5^{k+1} to both sides.

$$\begin{aligned}
1 + 5 + 5^2 + 5^3 + \cdots + 5^k &= \frac{5^{k+1} - 1}{4} \\
1 + 5 + 5^2 + 5^3 + \cdots + 5^k + 5^{k+1} &= \frac{5^{k+1} - 1}{4} + 5^{k+1} \\
&= \frac{5^{k+1} - 1 + 4 \cdot 5^{k+1}}{4} \\
&= \frac{5 \cdot 5^{k+1} - 1}{4} = \frac{5^{k+2} - 1}{4}
\end{aligned}$$

This is the form of $P(k+1)$, thus completing the inductive step.

Therefore, by mathematical induction $P(n)$ is true for all $n \geq 0$.

9. (8 points) **Graded (all)** Rosen Ch 5.1 # 32, p. 330

Let $P(n)$ be the proposition that $3 \mid n^3 + 2n$ for positive integers n.

Basis Step: Show $P(1)$ is true, $3 \mid n^3 + 2n$ or $3 \mid 1 + 2$ or $3 \mid 3$ which is true.

Inductive Step:

Assume $P(k)$ is true for an arbitrary, fixed integer $k \geq 1$, that is,

$$3 \mid k^3 + 2k \tag{IH}$$

Show $P(k+1)$ is true, that is,

$$3 \mid (k+1)^3 + 2(k+1)$$

Start with the expression used in $P(k+1)$, we want to show this is divisible by 3.

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= (k^3 + 2k) + (3k^2 + 3k + 3)\end{aligned}$$

Each of the terms in parenthesis are divisible by 3. The first term $k^3 + 2k$ is divisible by 3 using the Inductive Hypothesis, the second term has a 3 pulled from the expression, $3 \cdot (k^2 + k + 1)$ so it is also divisible by 3.

This shows $P(k+1)$ is true when $P(k)$ is true, completing the inductive step.

Hence, the basis step and inductive step are completed, by mathematical induction $P(n)$ is true for all n such that $n \geq 1$.

Bonus Questions

10. (4 points (bonus)) Find a closed form for the summation $\sum_{i=1}^{n}\sum_{j=1}^{n}(6i^2 - 2j)$. Show how you find the closed form solution; justify each step using the four facts of the arithmetic properties of summations and Table 2, p. 166 in the book.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}(6i^2 - 2j) &= \sum_{i=1}^{n}(\sum_{j=1}^{n}(6i^2 - 2j)) && \text{(implied parentheses)} \\
&= \sum_{i=1}^{n}(\sum_{j=1}^{n}6i^2 - \sum_{j=1}^{n}2j) && \text{(Fact 3)} \\
&= \sum_{i=1}^{n}(6i^2\sum_{j=1}^{n}1 - 2\sum_{j=1}^{n}j) && \text{(Fact 4, twice)} \\
&= \sum_{i=1}^{n}(6i^2 n - 2\sum_{j=1}^{n}j) && \text{(Fact 1)} \\
&= \sum_{i=1}^{n}(6i^2 n - 2\frac{n(n+1)}{2}) && \text{(Table 2)} \\
&= \sum_{i=1}^{n}6i^2 n - \sum_{i=1}^{n}n(n+1) && \text{(Fact 3)} \\
&= 6n\sum_{i=1}^{n}i^2 - n(n+1)\sum_{i=1}^{n}1 && \text{(Fact 4, twice)} \\
&= 6n(\frac{n(n+1)(2n+1)}{6}) - n(n+1)\sum_{i=1}^{n}1 && \text{(Table 2)} \\
&= n^2(n+1)(2n+1) - n^2(n+1) && \text{(algebra)} \\
&= n^2(n+1)(2n+1-1) \\
&= 2n^3(n+1)
\end{aligned}$$

11. (4 points (bonus)) Let $\{s_n\}$ be the sequence defined as,

$$s_1 = 4 \quad \text{and} \quad s_n = 3s_{n-1} - 2, \forall n \geq 2.$$

Show $\forall n \geq 1, s_n = 3^n + 1$.

Proof: Let $P(n)$ be that the nth term of the sequence is found as $s_n = 3^n + 1$ for $n \geq 1$

Basis Step: Show $P(1)$ is true, $s_1 = 3^1 + 1 = 4$

Inductive Step:

Assume $P(k)$ is ture for an arbitrary fixed integer $k \geq 1$, that is,

$$s_k = 3^k + 1 \tag{IH}$$

Show $P(k+1)$ is true, that is,

$$s_{k+1} = 3^{k+1} + 1$$

Start with $P(k+1)$ and the definition of the sequence,

$$\begin{aligned} s_{k+1} &= 3s_k - 2 \\ &= 3(3^k + 1) - 2 && \text{(IH)} \\ &= 3^{k+1} + 3 - 2 = 3^{k+1} + 1 \end{aligned}$$

This shows $P(k+1)$ is true, assuming $P(k)$, completing the inductive step.

Therefore, we have shown by mathematical induction, $P(n)$ is true for all $n \geq 1$.