Instructions: All assignments are due by **midnight** on the due date specified. Assignments should be typed and submitted as a PDF. Every student must write up their own solutions in their own manner.

You should complete all problems, but only a subset will be graded (which will be graded is not known to you ahead of time).

1. (6 points) **Graded (all)** Prove: the product of two odd integers is odd.

   Proof: Assume there are two odd integers a and b. Then, by definition of odd there exists two integers k_a and k_b such that $a = 2k_a + 1$ and $b = 2k_b + 1$. Compute $a \cdot b$:

   $$a \cdot b = (2k_a + 1) \cdot (2k_b + 1) = 4k_ak_b + 2k_a + 2k_b + 1 = 2(2k_ak_b + k_a + k_b) + 1,$$

   this shows the product of the two odd integers is the form of an odd number. Consequently, the product of two odd numbers is odd.

2. (6 points) **Graded (all)** Prove: For all natural numbers m and n, if m is divisible by 5 and n is divisible by 4, then $m \cdot n$ is divisible by 10.

   Proof: Assume m is divisible by 5 and also assume n is divisible by 4. By definition of divisibility, then there exists a integer k_m and a integer k_n such that $m = 5k_m$ and $n = 4k_n$. Then, $m \cdot n$ is:

   $$m \cdot n = 5k_m \cdot 4k_n = 20k_mk_n = 10(2k_mk_n).$$

   From this expression, you can see $m \cdot n$ is divisible by 10. Therefore, if m is divisible by 5 and n is divisible by 4, then $m \cdot n$ is divisible by 10.

3. **Ungraded** Prove: For any three consecutive natural numbers, the sum of the consecutive numbers is divisible by 3.

   Proof: Assume you have three consecutive natural numbers, with values n, $n + 1$ and $n + 2$. Compute the sum:

   $$n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1) = 3k.$$

   The sum is shown to be divisible by 3. Consequently, the sum of three consecutive natural numbers is divisible by 3.

4. **Ungraded** Prove: If a is an even integer and b is divisible by 3, then ab is divisible by 6.

   Proof: Assume a is even and b is divisible by 3. Then by definition, $a = 2m$ and $b = 2n$ for some integers m and n. Consider ab:

   $$ab = (2m)(3n) = 6mn = 6(mn),$$

   this shows that ab is divisible by 6. Therefore, if a is even and b is divisible by 3, then ab is divisible by 6.

5. (6 points) **Graded (a)** Prove that if n is an integer and $n^2 - 2n + 1$ is odd, then n is even using: (a) proof by contraposition and (b) proof by contradiction.

Proof by contraposition: Assume n is odd. Then by definition of odd, there exists an integer k such that $n = 2k + 1$. Compute $n^2 - 2n + 1$:

$$n^2 - 2n + 1 = (2k + 1)^2 - 2(2k + 1) + 1 = 4k^2 + 4k + 1 - 4k - 2 + 1 = 4k^2 = 2(2k^2).$$

Here, $n^2 - 2n + 1$ is in the form of an even number. Therefore, by contraposition, if n is an integer and $n^2 - 2n + 1$ is odd, then n is even.

Proof by contradiction: Assume n is odd and $n^2 - 2n + 1$ is odd. By definition of odd, there there exists an integer k such that $n = 2k + 1$. Compute $n^2 - 2n + 1$:

$$n^2 - 2n + 1 = (2k + 1)^2 - 2(2k + 1) + 1 = 4k^2 + 4k + 1 - 4k - 2 + 1 = 4k^2 = 2(2k^2).$$

Here, $n^2 - 2n + 1$ is in the form of an even number, however, we assumed $n^2 - 2n + 1$ is odd giving a contradiction. Therefore, by contradiction, it must be that if n is an integer and $n^2 - 2n + 1$ is odd, then n is even.

6. (6 points) **Graded (b)** Prove that if n is an integer and $n^3 - 1$ is even, then n is odd using: (a) proof by contraposition and (b) proof by contradiction.

   Proof by contraposition: Assume n is even. Then by definition of even, there exists an integer k such that $n = 2k$. Compute $n^3 - 1$:

   $$n^3 - 1 = (2k)^3 - 1 = 8k^3 - 1 = 8k^3 - 2 + 1 = 2(4k^3 - 1) + 1 = 2k' + 1.$$

   Here $n^3 - 1$ is of the form of an odd integer. Therefore, by contraposition, if $n^3 - 1$ is even, then n is odd.

   Proof by contractiction: Assume $n^3 - 1$ is even and n is even. By definition of even, there exists an integer k such that $n = 2k$. Compute $n^3 - 1$:

   $$n^3 - 1 = (2k)^3 - 1 = 8k^3 - 1 = 8k^3 - 2 + 1 = 2(4k^3 - 1) + 1 = 2k' + 1.$$

   Here $n^3 - 1$ is of the form of an odd integer, however, we assumed $n^3 - 1$ is even giving a contradiction. Therefore, by contradiction, if $n^3 - 1$ is even, then n is odd.

7. **Ungraded** Prove for all natural numbers n and m, nm is odd if and only if n and m are both odd.

   This proof is for a theorem using "if and only if" therefore, it must be proved in both directions. **Prove "if p, then q" by contradiction:** Assume nm is odd and n and m are not both odd. Then at least one of n and m are even. Suppose n is the one that is even. Then $n = 2k$ for some natural number k. Then $nm = 2km$ and so nm is even. This contradicts the assumption that nm is odd. Similarly, if we assume instead that m is the one that is even, we reach the same contradiction. If both m and n are both even we reach the same result. Thus, it must be that n and m are both odd.

   Prove "if q then p" directly: Assume n and m are both odd. Then, there are natural numbers k and j such that $n = 2k + 1$ and $m = 2j + 1$. Then,

   $$nm = (2k + 1)(2j + 1) = 4kj + 2(k + j) + 1 = 2(2kj + k + j) + 1$$

   So, nm is odd by definition. Therefore, if n and m are odd, nm is odd.

Combining both parts, we have shown for any natural number n and m, nm is odd if and only if n and m are both odd.

8. **Ungraded** Prove that there is a positive integer that equals the sum of the positive integers not exceeding it.

   This is an existence proof. 3 is an example of such a positive integer, $3 = 1 + 2$.

9. **Ungraded** Prove or disprove: If a and b are rational numbers, then a^b is also rational.

   Disprove: Let $a = 2$ and $b = \frac{1}{2}$, which are both rational numbers. Then $a^b = \sqrt{2}$ which is irrational.

10. **Ungraded** Prove or disprove: The sum of four consecutive integers is divisible by 4.

    Disprove: Let the consecutive integers be 1, 2, 3, 4; the sum is 10 which is not divisible by 4.

11. **Ungraded** Prove that if n is an integer that $n^3 - n$ is even.

    Proof: Let n be an integer.

    Case (i): Let n be even. By definition, there exists an integer k s.t. $n = 2k$.

    $$n^3 - n = (2k)^3 - 2k = 8k^3 - 2k = 2(4k^3 - k)$$

    This is of the form of an even number.

    Case (ii): Let n be odd. By definition, there exists an integer k s.t. $n = 2k + 1$.

    $$\begin{aligned} n^3 - n &= (2k+1)^3 - (2k+1) = (4k^2 + 4k + 1)(2k+1) - (2k+1) \\ &= 8k^3 + 12k^2 + 6k + 1 - 2k - 1 = 8k^3 + 12k^2 + 4k = 2(8k^3 + 6k^2 + 2k) \end{aligned}$$

    This is of the form of an even number.

    Therefore, because $n^3 - n$ is even in all cases, it holds that for any integer n, $n^3 - n$ is even.

12. (6 points) **Graded (all)** Prove: Suppose a and b are integers. If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

    Proof: This proof technique using proof by contraposition and proof by cases.

    Suppose it is not the case that a and b are odd. Then, at least one of a and b is even. Suppose:

    - Case 1. Suppose a is even. Then, $a = 2c$ for some integer c. Thus, $a^2(b^2 - 2b) = (2c)^2(b^2 - 2b) = 2(2c(b^2 - 2b)) = 2k$, which is of the form of an even integer.
    - Case 2. Suppose b is even. Then, $b = 2c$ for some integer c. Thus, $a^2(b^2 - 2b) = a^2((2c)^2 - 2(2c)) = 2(a^2(2c^2 - 2c)) = 2k'$, which is of the form of an even integer.

    For each case, we have shown that $a^2(b^2 - 2b)$ is even. Therefore, by contraposition, if $a^2(b^2 - 2b)$ is odd, then a and b are odd.

Sequences

13. **Ungraded** What are the first four terms of each sequence:

(a) $a_n = 4 - 2n \ \ \forall n \geq 0$
(b) $b_n = 6 - 3 \cdot 2^n \ \ \forall n \geq 0$
(c) $c_1 = 4, c_n = 3 \cdot c_{n-1} - 2 \ \ \forall n \geq 2$
(d) $d_1 = -1, d_n = 5 \cdot d_{n-1} + n \ \ \forall n \geq 2$
(e) $e_0 = 1, e_1 = 1, \ e_n = ne_{n-1} + n^2 e_{n-2} + 1 \ \forall n \geq 2$

(a) $a_0 = 4, \ a_1 = 2, \ a_2 = 0, \ a_3 = -2$
(b) $b_0 = 3, \ b_1 = 0, \ b_2 = -12, \ b_3 = -48$
(c) $c_1 = 4, \ c_2 = 10, \ c_3 = 28, \ c_4 = 82$
(d) $d_1 = -1, \ d_2 = -3, \ d_3 = -12, \ d_4 = -46$
(e) $e_0 = 1, \ e_1 = 1, \ e_2 = 6, \ e_3 = 27$

14. (6 points) **Graded (a-b,d)** Find a closed formula for each sequence; assume the sequence starts with $n = 0, 1, 2, \ldots$.

(a) $1, -4, 9, -16, 25, \ldots$
(b) $8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$
(c) $3, 4, 7, 12, 19, 28, 39 \ldots$
(d) $6, 1, -4, -9, -14, -19 \ldots$
(e) $3, 6, 12, 24, 48, \ldots$
(f) $1, 0, 1, -4, 9, -16, 25, -36 \ldots$

(a) $a_n = (n+1)^2(-1)^n$
(b) $a_n = 8 \cdot (\frac{1}{2})^n$
(c) $a_n = n^2 + 3$
(d) $a_n = 6 - 5n$
(e) $a_n = 3 \cdot 2^n$
(f) $a_n = (n-1)^2(-1)^n$

15. (4 points) **Graded (b-c)** Find a recursive formula for each sequence; assume the sequence with $n = 0, 1, 2, \ldots$.

(a) $3, 6, 12, 24, 48, \ldots$
(b) $7, 10, 15, 22, 31, 42, 55, 70, \ldots$
(c) $9, 4, -1, -6, -11, -16, \ldots$
(d) $2, 5, 13, 42, 171, 858, 5151, \ldots$

(a) $a_0 = 3, \ a_n = 2 \cdot a_{n-1}$
(b) $a_0 = 7, \ a_n = a_{n-1} + 2n + 1$
(c) $a_0 = 9, \ a_n = a_{n-1} + -5$
(d) $a_0 = 2, \ a_n = n \cdot a_{n-1} + 3$

16. **Ungraded** Determine whether each answer is a solution to the recurrence relation,

$$a_n = a_{n-1} + 2a_{n-2} + 2n - 9$$

(a) $a_n = 0$
(b) $a_n = -n + 2$

(a) Not a solution.

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} + 2n - 9 \\ &= (0) + 2 \cdot 0 + 2n - 9 \\ &= 2n - 9 \\ 0 &\neq 2n - 9 \end{aligned}$$

(b) Yes, $a_n = -n + 2$ is a solution.

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} + 2n - 9 \\ &= (-(n-1) + 2) + 2(-(n-2) + 2) + 2n - 9 \\ &= -n + 1 + 2 - 2n + 4 + 4 + 2n - 9 \\ &= -n + 2 \end{aligned}$$

Bonus Questions

17. (6 points (bonus)) Prove: For an integer a if $7|4a$, then $7|a$.
Hint: you may want to use the definition of even and knowledge of the products of even and odd integers.

Proof: Assume a is an integer and $7|4a$. By definition of divisibility, there is some integer c such that $4a = 7c$. We know that $4a$ is even because it can be written as $2(2a)$. Also, because $4a = 7c$ we know that $7c$ is even. If $7c$ is even, then c must be even (odd*odd = odd and odd*even = even). Then, we can replace c with $2d$ for some integer d.

From the initial equation, replace $c = 2d$, we get the following:

$$\begin{aligned} 4a &= 7c \\ &= 7(2d) = 14d \\ 2a &= 7d && \text{(divide by 2)} \end{aligned}$$

Since $7d$ is equal to $2a$ if follows that $7d$ is even. Using the same reasoning as above d must be even (odd*odd = odd and odd*even = even). With d as an even number, then it can be respresented as $d = 2e$ for some integer e.

Replacing d in the equation above, we get the following:

$$\begin{aligned} 2a &= 7d \\ &= 7(2e) = 14e \\ a &= 7e && \text{(divide by 2)} \end{aligned}$$

At this point we have shown, $a = 7e$ which means that $7|a$ by definition of divisibility.

Therefore, for an integer a if $7|4a$, then $7|a$.

18. (4 points (bonus)) Prove the statement: For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even.

Proof by contradiction: Assume there are integers a, b, and c such that $a^2 + b^2 = c^2$ and a and b are both odd. By definition, there exists and integer k_a and integer k_b such that $a = 2k_a + 1$ and $b = 2k_b + 1$. Consider the expression:

$$\begin{aligned} a^2 + b^2 = (2k_a + 1)^2 + (2k_b + 1)^2 &= 4k_a^2 + 4k_a + 1 + 4k_b^2 + 4k_b + 1 \\ &= 4(k_a^2 + k_b^2 + k_a + k_b) + 2, \end{aligned}$$

where $c^2 = 4(k_a^2 + k_b^2 + k_a + k_b) + 2$, this shows c^2 is even, $c^2 = 2(2k_a^2 + 2k_b^2 + 2k_a + 2k_b + 1) = 2k'$. With c^2 even, this means that c is even. But, then c^2 must be a multiple of 4. However, this is a contradiction because $4(k_a^2 + k_b^2 + k_a + k_b) + 2$ is not a multiple of 4.

Therefore, by contradiction, if $a^2 + b^2 = c^2$, then a or b is even.