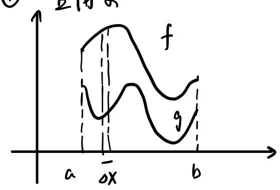
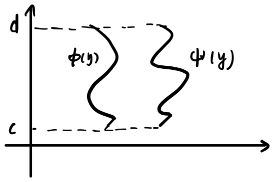
§7.1

1. 平面图形面积:

① 直角系

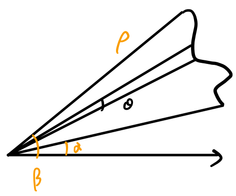




$$S=\int_a^b \left(f(x)-g(x)\right) dx \qquad s=\int_c^d \left(\psi(y)-\phi(y)\right) dy$$

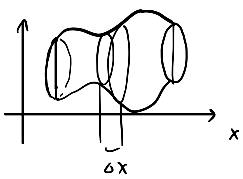
选择什么时候用 x. or y

② 极坐标



$$S=\frac{1}{2}\int_\alpha^\beta \rho^2(\theta)\, d\theta$$

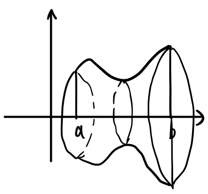
2. 已知截面的立体体积 → 切成薄片



$$V=\int_a^b S(x)\,dx$$

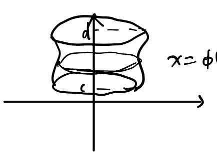
旋转体体积

① 由 $D=\{(x,y)\mid a\le x\le b,\ 0\le y\le f(x)\}$ 绕 x 轴转成.



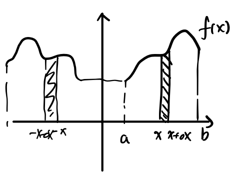
$$V=\pi\int_a^b \left[f^2(x)\right]dx$$

或由 $D=\{(x,y)\mid 0\le x\le \phi(y),\ c\le y\le d\}$ 绕 y 轴转成



$x=\phi(y)$ $\qquad V=\pi\int_c^d [\varphi(y)]^2\, dy$

③由 $D=\{(x,y)\mid a\le x\le b,\ 0\le y\le f(x)\}$ 绕 y 轴转成



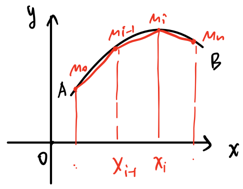
视为厚为 Δx 的圆筒相加.

$$\Delta V\approx 2\pi x f(x)$$

$$\therefore\ V=2\pi\int_a^b x f(x)\, dx$$

3. 平面曲线弧长与弧微分

$y=f(x)$ $\quad a\le x\le b$. 将 $f(x)$ 拆为折线段 $|M_iM_{i+1}|$. $\lambda=\max\{\Delta x\}$



定义弧线长

$$S=\lim_{\lambda\to 0}\sum|M_{i-1}M_i|=S$$

称 S 为 $\overset{\frown}{AB}$ 弧长

Thm.

$$s=\int_a^b \sqrt{1+[f'(x)]^2}\, dx$$

若 $\overset{\frown}{AB}$ 由极坐标定义 $\rho=\rho(\theta)$, $\theta\in[\alpha,\beta]$

$$\Rightarrow\ s=\int_\alpha^\beta \sqrt{\rho^2(\theta)+[\rho'(\theta)]^2}\, d\theta$$

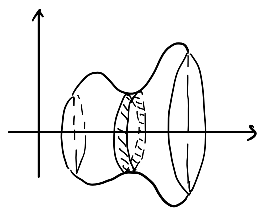
视 $\begin{cases} x=\rho(\theta)\cos\theta \\ y=\rho(\theta)\sin\theta \end{cases}$ → 当参方.

若 $\overset{\frown}{AB}$ 由参数方程定义.

$$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \quad \alpha\le t\le\beta \qquad \Rightarrow\ S=\int_\alpha^\beta \sqrt{1+\left(\frac{\psi'(t)}{\varphi'(t)}\right)^2}\cdot|\varphi'(t)|\,dt$$

$$=\int_\alpha^\beta \sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\ dt$$

4. 旋转曲面面积 → 切成圆台侧面积.



$$dS\approx 2\pi f(x)\sqrt{1+[f'(x)]^2}\ dx$$

$$S=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\ dx$$

代入时无表达式可得参方/极坐标面积

§7.4.

1. Wallis 公式:

利用 $x\in[0,\frac{\pi}{2})$. $\sin x\in(0,1)$ $\Rightarrow$ $\sin^{2n+1}x<\sin^{2n}x<\sin^{2n-1}x$

$$\Rightarrow \int \sin^{2n+1}x\, dx<\int \sin^{2n}x\, dx<\int \sin^{2n-1}x\, dx$$

$$\Rightarrow \frac{(2n)!!}{(2n+1)!!}<\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}<\frac{(2n-2)!!}{(2n-1)!!}$$

$$\Rightarrow \frac{2n}{2n+1}\frac{\pi}{2}<\frac{((2n)!!)^2}{(2n+1)((2n-1)!!)^2}<\frac{\pi}{2}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{2n+1}\cdot\frac{((2n)!!)^2}{((2n-1)!!)^2}=\frac{\pi}{2}$$

即 $$\sqrt{\pi}=\lim_{n\to\infty}\frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}}$$

2. Stirling 公式

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

即 $n! = \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} \cdot e^{\frac{\theta_n}{4n}}$, $\theta_n = 4n \cdot \ln \frac{n!}{\left(\frac{n}{e}\right)\sqrt{2\pi n}}$