Assignment #1, CSC 2504

by Shenlong Wang
slwang@cs.toronto.edu

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1 Problem 1

Let T be an affine transformation matrix that is invertible. Show that T^{-1} is also affine.

Proof. Since T is an affine transformation matrix. It has the following form:

$$T = \begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

where m_{ij}, i, $j < 2$ are parameters of 2D rotation and scaling, and x_t, y_t are parameters of translation. According to linear algebra, the inverse of this 3×3 matrix can be written as:

$$T^{-1} = T = \begin{bmatrix} M^{-1} & M^{-1}\vec{b} \\ 0 & 1 \end{bmatrix} \tag{2}$$

where

$$M^{-1} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^{-1} = \frac{1}{m_{11}m_{22} - m_{12}m_{21}} \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix} \tag{3}$$

and

$$\vec{b} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{4}$$

Therefore, the inverse matrix T^{-1} is also a transformation matrix. □

2 Problem 2

Two transformations $f\ 1$ and $f\ 2$ commute when $f\ 1 \circ f\ 2 = f\ 2 \circ f\ 1$. For each pair of transformations below, specify whether or not they commute. In each case, your solution can either be a derivation that proves/disproves commutativity, or if f and g do not commute, a specific counter-example.

2.1 Problem 2.1

A translation and a uniform scaling.
$Answer$: Suppose a point $[x, y]^T$, it is first translated by $[x_t, y_t]^T$, then scaled by a factor s. We have the transformation matrix:

$$T = \begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

And the scaling matrix S:

$$S = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

We can have:

$$ST = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 & sx_t \\ 0 & s & sy_t \\ 0 & 0 & 1 \end{bmatrix}$$
$$TS = \begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 & x_t \\ 0 & s & y_t \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly $ST \neq TS$. Therefore, the two transformations cannot commute.

2.2 Problem 2.2

Two different translations.
Answer: Suppose a point $[x, y]^T$. The first translation is $[x_{t,1}, y_{t,1}]^T$, the second translation is $[x_{t,2}, y_{t,2}]^T$. We have the transformation matrix:

$$T_1 = \begin{bmatrix} 1 & 0 & x_{t,1} \\ 0 & 1 & y_{t,1} \\ 0 & 0 & 1 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 & x_{t,2} \\ 0 & 1 & y_{t,2} \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Obviously, we have

$$T_1 \cdot T_2 = T_1 \cdot T_2 = \begin{bmatrix} 1 & 0 & x_{t,1} + x_{t,2} \\ 0 & 1 & y_{t,1} + y_{t,2} \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Therefore, two translations are commutative.

2.3 Problem 2.3

A shear with respect to the x-axis and a uniform scaling.
Answer: Suppose a point $[x, y]^T$. It is shared with respect to x-axis by a factor m, and uniformly scaled by a factor s. We have the transformation matrix:

$$M = \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, S = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

By matrix multiplication, we have:

$$M \cdot S = S \cdot M = \begin{bmatrix} s & sm & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

Therefore, two translations are commutative.

2.4 Problem 2.4

A shear with respect to the x-axis and a non-uniform scaling.
Answer: Suppose a point $[x, y]^T$. It is shared with respect to x-axis by a factor m, and non-uniformly scaled by factors s_x, s_y. We have the transformation matrix:

$$M = \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

By matrix multiplication, we have:

$$M \cdot S = \begin{bmatrix} s_x & s_y m & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$S \cdot M = \begin{bmatrix} s_x & s_x m & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, two translations cannot commute.

3 Problem 3

3.1 Problem 3.1

Derive the homography that maps points $(1,0),(0,1),(0,0),(3,1)$ to points$(-1,0),(2,0),(1,2),(4,$ $2)$, respectively.
Answer: Suppose the homography matrix M is:

$$H=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{12}$$

Since the homograph matrix project the maps the four points to the other four. We have:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \tilde{x}=x'/z', \tilde{y}=y'/z' \tag{13}$$

Therefore, we have eight equations (two per each point-to-point correspondence):

$$\begin{aligned} a_1+0\cdot b_1+c_1 &=(-1)\cdot(a_3+0\cdot b_3+c_3) \\ a_2+0\cdot b_2+c_2 &=0\cdot(a_3+0\cdot b_3+c_3) \\ 0\cdot a_1+b_1+c_1 &=2\,(0\cdot a_3+b_3+c_3) \\ 0\cdot a_2+b_2+c_2 &=0\cdot(0\cdot a_3+b_3+c_3) \\ 0\cdot a_1+0\cdot b_1+c_1 &=(0\cdot a_3+0\cdot b_3+c_3) \\ 0\cdot a_2+0\cdot b_2+c_2 &=2\,(0\cdot a_3+0\cdot b_3+c_3) \\ 3\,a_1+b_1+c_1 &=4\,(3\,a_3+b_3+c_3) \\ 3\,a_2+b_2+c_2 &=2\,(3\,a_3+b_3+c_3) \end{aligned}$$

Solving this linear equation systems with SVD, we have:

$$M=\begin{bmatrix} 0.2195 & 0.7683 & -0.1098 \\ 0.2195 & 0.2195 & -0.2195 \\ 0.0000 & 0.4391 & -0.1098 \end{bmatrix} \tag{14}$$

3.2 Problem 3.2

Where does the point (1, 1) get mapped to under this homography.
Answer: Under this homography the point $[1,1]$ get mapped to $[8/3,5/3]$.

3.3 Problem 3.2

Does this homography represent an affine transformation? Explain why it does ordoes not.
Answer: No, it does not. It cannot preserve parallelism between lines.

4 Problem 4

4.1 Problem 4.1

Let f be an arbitrary homography, and let p_1, p_2, and p_3 be the vertices of a triangle. Prove that if that triangle contains a point q, the transformed triangle defined by $f(p_1)$, $f(p_2)$, $f(p_3)$ may not contain point $f(q)$. In other words, the inclusion relationship between points and triangles is not preserved under arbitrary homographies.
Answer: For points $p_1=[1,0]$, $p_2=[0,1]$, $p_3=[0,0]$, $q=[1/3,1/3]$, clearly p_1, p_2, p_3 are not collinear (Or they cannot construct a triangle), q are inside the triangle. we will construct a homography mapping f, such that:

$$f(p_1)=p_1, f(p_2)=p_2, f(p_3)=p_3, f(q)=[1,1] \tag{15}$$

The homographs matrix of f is:

$$M = \begin{bmatrix} 0.3015 & 0.0000 & 0.0000 \\ 0.0000 & 0.3015 & 0.0000 \\ 0.6030 & 0.6030 & -0.3015 \end{bmatrix} \tag{16}$$

This homography f maps q from inside the triangle to outside the triangle. Therefore, the inclusion relationship between points and triangles is not preserved under arbitrary homographies.

4.2 Problem 4.2

Is the inclusion relationship between points and triangles preserved under affine transformations? Explain why or why not.

Answer: The inclusion relationship between points and triangle is well preserved under affine transformations.

Proof. We know for any points q, as shown in the figure, if and only if $\angle\, p_1\, q\, p_2 + \angle\, p_2\, q\, p_3 + \angle\, p_3\, q\, p_1 = 2\,\pi$, q is inside the triangle $\triangle\, p_1\, p_2\, p_3$. And considering affine transformation is combined by rotation, translation and scaling. We only need to prove rotation, translation and scaling does not change the three angles.

(1) Translation. For any points a, b, c, their angle

$$\angle\, a\, b\, c = \arccos\left(\frac{<\overrightarrow{a\,b}, \overrightarrow{c\,b}>}{\|\overrightarrow{a\,b}\|\|\overrightarrow{c\,b}\|}\right) \tag{17}$$

Considering translation m,

$$a' = a + m, b' = b + m, c' = c + m \tag{18}$$

We have $\forall a, b$

$$\overrightarrow{a'\,b'} = a' - b' = a + m - (b + m) = a - b = \overrightarrow{a\,b} \tag{19}$$

Therefore, translation would not change status of whether a point is inside the triangle.

(2) Scaling. For any points a, b, c, their angle

$$\angle\, a\, b\, c = \arccos\left(\frac{<\overrightarrow{a\,b}, \overrightarrow{c\,b}>}{\|\overrightarrow{a\,b}\|\|\overrightarrow{c\,b}\|}\right) \tag{20}$$

Considering scaling s,

$$a' = s\, a, b' = s\, b, c' = s\, c \tag{21}$$

We have $\forall a, b$

$$\overrightarrow{a'\,b'} = a' - b' = s\, a - s\, b = s\, \overrightarrow{a\,b} \tag{22}$$

Therefore, for any points a, b, c, their angle

$$\angle\, a'\, b'\, c' = \arccos\left(\frac{<\overrightarrow{a'\,b'}, \overrightarrow{c'\,b'}>}{\|\overrightarrow{a'\,b'}\|\|\overrightarrow{c'\,b'}\|}\right) = \arccos\left(\frac{<s\,\overrightarrow{a\,b}, s\,\overrightarrow{c\,b}>}{\|s\,\overrightarrow{a\,b}\|\,\|s\,\overrightarrow{c\,b}\|}\right) = \arccos\left(\frac{<\overrightarrow{a\,b}, \overrightarrow{c\,b}>}{\|\overrightarrow{a\,b}\|\|\overrightarrow{c\,b}\|}\right) = \angle\, a\, b\, c \tag{23}$$

Therefore, scaling would not change status of whether a point is inside the triangle.

(3) Rotation. For any points a, b, c, their angle

$$\angle\, a\, b\, c = \arccos\left(\frac{<\overrightarrow{a\,b}, \overrightarrow{c\,b}>}{\|\overrightarrow{a\,b}\|\|\overrightarrow{c\,b}\|}\right) \tag{24}$$

Considering rotation R,

$$a' = R\, a \tag{25}$$

We have $\forall a, b$

$$\overrightarrow{a'\,b'} = a' - b' = R\, a - R\, b = R\, \overrightarrow{a\,b} \tag{26}$$

Since $R^T R = I$, $\|R\| = 1$, for any points a, b, c, their angle

$$\angle\, a'\, b'\, c' = \arccos\left(\frac{<\overrightarrow{a'\,b'}, \overrightarrow{c'\,b'}>}{\|\overrightarrow{a'\,b'}\|\|\overrightarrow{c'\,b'}\|}\right) = \arccos\left(\frac{<R\,\overrightarrow{a\,b}, R\,\overrightarrow{c\,b}>}{\|R\,\overrightarrow{a\,b}\|\,\|R\,\overrightarrow{c\,b}\|}\right) = \arccos\left(\frac{<\overrightarrow{a\,b}, \overrightarrow{c\,b}>}{\|\overrightarrow{a\,b}\|\|\overrightarrow{c\,b}\|}\right) = \angle\, a\, b\, c \tag{27}$$

Therefore, rotation would not change status of whether a point is inside the triangle.
Above all, we can conclude the affine transformation does not change the status of whether a point is inside the triangle. □

5 Problem 5

Prove that a shear in y can be created with a combination of rotations and shears in x.
$Answer$: for any shear in y, the shear matrix is:

$$M_y = \begin{bmatrix} 1 & 0 & 0 \\ m_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{28}$$

We can rewrite it as:

$$M_y = \begin{bmatrix} 1 & 0 & 0 \\ m_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & m_y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

which is equivalent to rotate $3\pi/2$, shear, and then rotate $3\pi/2$.

6 Problem 6

Consider the following 2D parametric curve:

$x(t) = t\cos(20\pi t)$
$y(t) = \sin(20\pi t)$

where $0 \leq t \leq 1$. Find the tangent vector and a normal vector to this curve as a function of t. (A normal vector is any vector perpendicular to the tangent at a curve point).
$Answer$: The tangent vector of the parametric curve is:

$$\vec{\alpha} = \frac{(x'(t), y'(t))}{\|(x'(t), y'(t)\|} = (20\pi t\sin(20\pi t) + \cos(20\pi t), -20\pi\cos(20\pi t)) \tag{30}$$

And the normal vector of the parameteric curve

$$\vec{\beta} = \frac{(-y'(t), x'(t))}{\|(-y'(t), x'(t))\|} = (20\pi\cos(20\pi t), 20\pi t\sin(20\pi t) + \cos(20\pi t)) \tag{31}$$

And we can also get the unit-length tangent vector and unit-length normal vector by dividing with its l2 norm.