Assignment #2, CSC 2504

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1 Problem 1

1.1 Problem 1.1

How many three dimensional points are needed in a mapping to uniquely define a homograph in 3D?

Answer: The homography matrix in 3D space should can be expressed in a 4×4 matrix with 16 elements:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ e & f & g & h \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{1}$$

Since arbitrary scaling corresponds to the same homography in 3D, it has 15 degrees of freedom. For each different point, it will introduce three equations:

$$x' = \frac{a_1 x + b_1 y + c_1 z + d_1}{ex + fy + gz + h} \tag{2}$$

$$y' = \frac{a_2 x + b_2 y + c_2 z + d_2}{ex + fy + gz + h} \tag{3}$$

$$z' = \frac{a_3 x + b_3 y + c_3 z + d_3}{ex + fy + gz + h} \tag{4}$$

Therefore, in total 5 points are enough to generate 15 equations for calculating 15 parameters of the homography matrix.

1.2 Problem 1.2

How many mappings are needed to uniquely define an affine transformation in 3D?

Answer: Here I assume the question you ask is "How many points correspondences are needed to uniquely define an affine transform in 3D?". The degree of freedom of a 3D affine matrix should be 12, which includes 3×3 matrix combines transformations of rotation, scaling and shearing, plus a translation with 3 DOF, which can be expressed as follows:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{3\times 3} & \mathbf{t}_{3\times 1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{5}$$

Each point correspondence brings 3 linear equations. Therefore, we need at least four different points in total to define an uniquely 3D affine matrix.

2 Problem 2

In general, are the operations of performing a rotation about the x-axis and performing a rotation about the y-axis commutative? Explain why or why not.

Answer:Suppose the rotation angle around x-axis is ϕ, the rotation angle around y-axis is θ, we have:

$$\mathbf{T}_{\mathrm{xy}} = \mathbf{R}_x\mathbf{R}_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin\phi\sin\theta & \cos\phi & -\sin\phi\cos\theta \\ -\cos\phi\sin\theta & \sin\phi & \cos\phi\cos\theta \end{bmatrix} \tag{6}$$

$$\mathbf{T}_{\mathrm{yx}} = \mathbf{R}_y\mathbf{R}_x = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\phi\sin\theta & \cos\phi\sin\theta \\ 0 & \cos\phi & -\sin\phi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \tag{7}$$

Therefore $\mathbf{T}_{\mathrm{xy}} \neq \mathbf{T}_{\mathrm{yx}}$. Rotation about the x-axis and rotation about the y-axis are not commutative.

3 Problem 3

Given a camera position of (1,2,3), a viewing direction of (3,2,1), and an up vector of (0,1,0), compute the world to camera transform matrix.

Answer: The camera position is denoted as $\mathbf{e}$, the view direction is denoted as $\mathbf{g}$ and the up vector is denoted as $\mathbf{t}$. I construct the camera matrix as follows:

$$\mathbf{w} = -\frac{\mathbf{g}}{\|\mathbf{g}\|} = -(\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}})^T \tag{8}$$

$$\mathbf{u} = \frac{\mathbf{t} \times \mathbf{w}}{\|\mathbf{t} \times \mathbf{w}\|} = \frac{(-\frac{1}{\sqrt{14}}, 0, \frac{3}{\sqrt{14}})^T}{\|(-\frac{1}{\sqrt{14}}, 0, \frac{3}{\sqrt{14}})\|} = (-\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}})^T \tag{9}$$

$$\mathbf{v} = \mathbf{w} \times \mathbf{u} = (-\frac{6}{\sqrt{140}}, \frac{10}{\sqrt{140}}, -\frac{2}{\sqrt{140}})^T \tag{10}$$

Therefore, the camera-to-world matrix is :

$$M_{\text{cw}} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{e} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

The world-to-camera matrix should be its inverse:

$$M_{\text{wc}} = M_{\text{cw}}^{-1} = \begin{bmatrix} \mathbf{M} & -\mathbf{Me} \\ 0 & 1 \end{bmatrix} \tag{12}$$

where $\mathbf{M} = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix}$ Therefore,

$$M_{\text{wc}} = \begin{bmatrix} -\frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} & -\frac{8}{\sqrt{10}} \\ -\frac{6}{\sqrt{140}} & \frac{10}{\sqrt{140}} & -\frac{2}{\sqrt{140}} & \frac{12}{\sqrt{140}} \\ -\frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{14}} & \frac{6}{\sqrt{14}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

4 Problem 4

Suppose a canonical view transformation is defined with parameters L=-1, R=1, B=-1, T=1, f=1, and F=1001. Calculate the pseudodepths of the points (0,0,-1), (0,0,-10), (0,0,-100), and (0,0,-1000). Compare the relative values of the depths (z-coordinates) with the relative values of the pseudodepths. Is the relationship between depth and pseudodepth linear?

Answer:

$$M_{\text{canonical}} = \begin{bmatrix} \frac{2f}{R-L} & 0 & \frac{R+L}{R-L} & 0 \\ 0 & \frac{2f}{T-B} & \frac{T+B}{T-B} & 0 \\ 0 & 0 & \left(\frac{f+F}{F-f}\right) & \frac{2fF}{F-f} \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\frac{501}{500}) & \frac{1001}{500} \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{14}$$

Therefore,

$$\hat{\mathbf{x}}_1 = M_{\text{canonical}}\mathbf{x}_1 = M_{\text{canonical}}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{15}$$

$$\hat{\mathbf{x}}_2 = M_{\text{canonical}}\mathbf{x}_2 = M_{\text{canonical}}\begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.8018 \end{bmatrix} \tag{16}$$

$$\hat{\mathbf{x}}_3 = M_{\text{canonical}}\mathbf{x}_3 = M_{\text{canonical}}\begin{bmatrix} 0 \\ 0 \\ -100 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.9820 \end{bmatrix} \tag{17}$$

$$\hat{\mathbf{x}}_4 = M_{\text{canonical}}\mathbf{x}_4 = M_{\text{canonical}}\begin{bmatrix} 0 \\ 0 \\ -1000 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{18}$$

$$\tag{19}$$

Clearly, since $\frac{\hat{\mathbf{x}}_3-\hat{\mathbf{x}}_2}{\mathbf{x}_3-\mathbf{x}_2} \neq \frac{\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_1}{\mathbf{x}_2-\mathbf{x}_1}$, it is not linear transform.

5 Problem 5

Let $p = (p_x, p_y, p_z)$ and $q = (q_x, q_y, q_z)$ be two endpoints of a line segment, defined in camera coordinates. Let m=0.5(p + q) be the midpoint of the line segment. Given a simple perspective projection with focal length $-f$, let p', q', and m' be the perspective projections of p, q, and m respectively. Use the mathematical form of perspective projection to determine whether $m' = 0.5(p' + q')$. If it is not true for all p and q, characterize the conditions under which it would be true. Is such a midpoint invariant under orthographic projection?

Answer: The perspective matrix should be:

$$\mathbf{M}_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \tag{20}$$

Therefore, we have:

$$\mathbf{p}' \sim \mathbf{M}_p\mathbf{p} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z/f \end{bmatrix} \tag{21}$$

$$\mathbf{q}' \sim \mathbf{M}_p\mathbf{q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = \begin{bmatrix} q_x \\ q_y \\ q_z/f \end{bmatrix} \tag{22}$$

$$\mathbf{m}' \sim \mathbf{M}_p\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} 0.5(p_x+q_x) \\ 0.5(p_y+q_y) \\ 0.5(p_z+q_z) \\ 1 \end{bmatrix} \tag{23}$$

That is to say:

$$\mathbf{p}' = \begin{bmatrix} p_x \frac{f}{p_z} \\ p_y \frac{f}{p_z} \end{bmatrix}, \mathbf{q}' = \begin{bmatrix} q_x \frac{f}{q_z} \\ q_y \frac{f}{q_z} \end{bmatrix}, \mathbf{m}' = \begin{bmatrix} (p_x+q_x)\frac{f}{p_z+q_z} \\ (p_y+q_y)\frac{f}{p_z+q_z} \end{bmatrix} \tag{24}$$

Clearly $0.5(\mathbf{p}'+\mathbf{q}') = \begin{bmatrix} \frac{0.5(p_xq_z+q_xp_z)}{p_zq_z} \\ \frac{0.5(p_yq_z+q_yp_z)}{p_zq_z} \end{bmatrix} \neq \mathbf{m}', \forall p_x, p_y, q_x, q_y$. However, if $p_z = q_z \neq 0$, we have:

$$0.5(\mathbf{p}'+\mathbf{q}') = \begin{bmatrix} \frac{0.5(p_xq_z+q_xp_z)}{p_zq_z} \\ \frac{0.5(p_yq_z+q_yp_z)}{p_zq_z} \end{bmatrix} = \begin{bmatrix} \frac{p_x+q_x}{2p_z} \\ \frac{p_y+q_y}{2p_zq_z} \end{bmatrix} = \begin{bmatrix} (p_x+q_x)\frac{f}{p_z+q_z} \\ (p_y+q_y)\frac{f}{p_z+q_z} \end{bmatrix} = \mathbf{m}' \tag{25}$$

6 Problem 6

An circular paraboloid surface is described by the implicit equation:

$$f(x,y,z) = x^2 + y^2 - z = 0 \tag{26}$$

6.1 Problem 6.a

Show that the parametric function $g(u,v) = (\sqrt{v}\sin u, \sqrt{v}\cos u, v)$ defines the same surface.
Answer: let us take $x = \sqrt{v}\sin u, y = \sqrt{v}\cos u, z = v$ into $f(x,y,z)$, we have:

$$f(x,y,z) = x^2 + y^2 - z = v(\cos^2 u + \sin^2 u) - v = 0 \tag{27}$$

6.2 Problem 6.b

Compute the surface normal at point p=(x,y,z) using the surface implicit equation.
Answer: the surface normal is:

$$\mathbf{n} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)^T = (2x, 2y, -1) \tag{28}$$

6.3 Problem 6.c

Compute the tangent vectors using the surface parametric form and use them to compute the surface normal. Show that the normal is a scalar multiple of the one you computed in b).
Answer: Tangent vectors are:

$$\mathbf{t}_1 = \frac{\partial g(u,v)}{\partial u} = (-\sqrt{v}\cos u, \sqrt{v}\sin u, 0)^T \tag{29}$$

$$\mathbf{t}_2 = \frac{\partial g(u,v)}{\partial u} = (\frac{1}{2\sqrt{v}}\sin u, \frac{1}{2\sqrt{v}}\cos u, 1)^T \tag{30}$$

The normal vector is the cross product of two tangent vectors:

$$\mathbf{n}' = \mathbf{t}_1 \times \mathbf{t}_2 = (\sqrt{v}\sin u, \sqrt{v}\cos u, -\frac{1}{2})^T = (x, y, -\frac{1}{2}) = \frac{1}{2}\mathbf{n} \tag{31}$$

6.4 Problem 6.d

Using the fact that z is a function of x and y, come up with a different, simple parametric description h(u, v) for the surface that does not use sin and cos. Answer: According to the implicit function $f(x,y,z) = x^2 + y^2 - z = 0$, we have $z = x^2 + y^2$. Motivated by this, we can design equivalent parameter function as follows:

$$x = u, y = v, z = u^2 + v^2, g(u,v) = (u, v, u^2 + v^2) \tag{32}$$

6.5 Problem 6.e

Compute the tangent vectors using the parametric form from part d) and use them to compute the surface normal again. Show that this normal is a scalar multiple of the one

you computed in c).
Answer: Tangent vectors of the new parameter form are:

$$\mathbf{t}_1 = \frac{\partial g(u,v)}{\partial u} = (1, 0, 2u)^T \tag{33}$$

$$\mathbf{t}_2 = \frac{\partial g(u,v)}{\partial u} = (0, 1, 2v)^T \tag{34}$$

The normal vector is the cross product of two tangent vectors:

$$\mathbf{n}'' = \mathbf{t}_1 \times \mathbf{t}_2 = (-2u, -2v, 1)^T = (-2x, -2y, 1) = -2\mathbf{n}' \tag{35}$$

7 Problem 7

The figure above shows a top-down view of a 3D scene, with segments in blue and outward normals in red; the eye location is illustrated by a black dot.

7.1 Problem 7.a

Draw a BSP tree for the above scene by adding the segments in the labeled order (i.e. from 'a' to 'h', breaking segments as necessary).
Answer: The BSP tree is shown as follows in Figure 1:

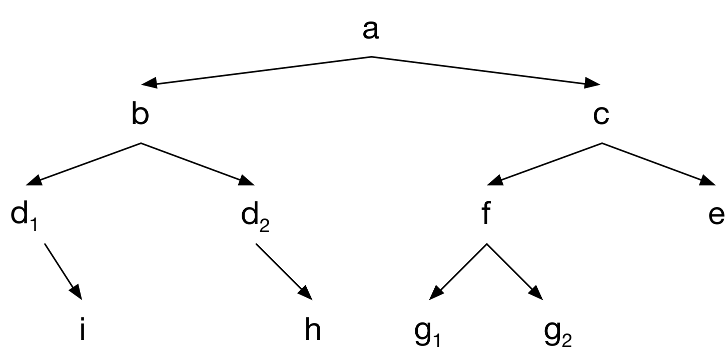

Figure 1: BSP tree

7.2 Problem 7.b

Indicate how your tree will be traversed when rendering from the eye location shown.
The traversed order should be:

$$i \to d_1 \to b \to h \to d_2 \to a \to g_2 \to f \to g_1 \to c \to e \tag{36}$$