Homework 2

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1 HW 1

1.1 Problem Retatement

Consider a linear programming

$$\begin{aligned} &\min c^T(x) \\ &s.t. \quad Ax = b \\ &\qquad x \succeq 0 \end{aligned}$$

(i)show its Lagrange dual problem

(ii)Using its KKT conditions to show that the strong duality holds.

1.2 Answer

(i):

The primal problem is equal to

$$\begin{aligned} &\min c^T(x) \\ &s.t. \quad Ax = b \\ &\qquad - x \preceq 0 \end{aligned}$$

So,the Lagrange Function:

$$L(x, \lambda, v) = c^T + \lambda^T(-x) + v^T(Ax - b)$$

Lagrange Dual Function:

$$\begin{aligned} g(\lambda, v) = \inf_x L(x, \lambda, v) &= \inf_x \{c^T x + v^T(Ax - b) - \lambda^T x\} \\ &= \inf_x \{(c^T + v^T A - \lambda)x - v^T b\} \\ &= \begin{cases} -v^T b & , c + A^T v - \lambda = 0 \\ -\infty & , otherwise \end{cases} \end{aligned}$$

Lagrange Dual Problem:

$$\begin{aligned} &\max_x -v^T b \\ &s.t. \;\; c + A^T v - \lambda = 0 \\ &\qquad\;\; \lambda \succeq 0 \end{aligned}$$

(2)

KKT condition is:

$$\begin{cases} c - \tilde{\lambda} + A^T \tilde{v} = 0 \\ A\tilde{x} = b\tilde{x} \succeq 0 \\ \tilde{x} - b = 0 \\ \tilde{\lambda}_i \tilde{x}_i = 0 \\ \tilde{\lambda} \succeq 0 \end{cases}$$

Let $\tilde{x}, \tilde{\lambda}, \tilde{v}$ satisfy the KKT condition

Consider $L(x, \lambda,) = c^T x - \lambda^T x + v^T (Ax - b)$ is affine to x,so the $\tilde{\lambda}\tilde{v}$ make L touch the minmimum, so $\tilde{\lambda}, \tilde{v}$ satisfy

$$g(\tilde{\lambda}, \tilde{v}) = \inf_x L(x, \tilde{\lambda}, \tilde{v})$$

If choose $x = \tilde{x}$ then $g(\tilde{\lambda}, \tilde{v}) = c^T \tilde{x} - \tilde{\lambda}^T \tilde{x} + \tilde{v}^T (A\tilde{x} - b)$

consider $\tilde{\lambda}_i \tilde{x}_i = 0$,so

$$\Rightarrow g(\tilde{\lambda}, \tilde{v}) = c^T \tilde{x} f_0(\tilde{x})$$

At this point, the primal problem and the dual problem have the same solution , consider $q^* - p^* \geq 0$,so the dual gap is 0

2 HW 2

2.1 Problem Restatement

Consider the following linear programming

$$\begin{aligned} &\min -5x_1 - x_2 \\ &s.t. \quad x_1 + x_2 \le 5, \\ &\qquad 2x_1 + x_2/2 \le 8, \\ &\qquad x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

(i)Add slack variables x_3 and x_4 to convert this problem to standard form

(ii)Implement the simplex method to solve this problem

(iii)Implement the interior point method to solve this problem

2.2 Answer

(1) The primal problem is equal to

$$\begin{aligned} &\min -5x_1 - x_2 + 0x_3 + 0x_4 \\ &s.t. \quad x_1 + x_2 + x_3 = 5, \\ &\qquad 2x_1 + x_2/2 + x_4 = 8, \\ &\qquad x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0 \end{aligned}$$

this same is as the standard form:

$$\begin{aligned} &\min c^T x \\ &s.t. \quad Ax = b, \\ &\qquad x \succeq 0 \end{aligned}$$

In which, the $c = (-5, -1, 0, 0)^T$, $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & \frac{1}{2} & 0 & 4 \end{pmatrix}$,$b = (5, 8)$

(2) Apply the Simplex Method to solve this problem ,the detail of the code is available at the appendix, the answer is in the Figure 1 :

```
[3]:
#求一个初值x0
x0=Find_x0(A=A,c=c,b=b)

#使用单纯形法计算
res=SimplexMethod(c=c,A=A,b=b,x0=x0)
print("最优值: ",res[0])
print("点x: ",res[1])

最优值:  -20.0
点x:  [4. 0. 1. 0.]
```

Figure 1: The answer using simplex method

as Figure 1 shows, the optimal point is $(4, 0, 1, 0)^T$,and the solution is -20

(3) Apply the Interior Point Method to solve this problem ,the detail of the code is available at the appendix, the answer is as the Figure 2

```
[6]:
## 用内点法计算
# 先求最优值函数初值
x0,lambda0,v0,u0=FindInitialPoint(A,b,c)

#利用内点法计算
optval,x=InteriorPointMethod(A,b,c,x0,lambda0,v0,u0,epsilon=1e-11)
print("最优值：",optval)
print("点x:",x)

最优值：  -19.999999999989644
点x: [4.00000000e+00 2.09335659e-11 1.00000000e+00 2.04908898e-12]
```

Figure 2: The answer using interior method

as Figure 2 shows, the optimal point is about $(4, 0, 1, 0)^T$,and the solution is about -20

3 HW 3

3.1 Problem Restatement

First,recall the support vector machine in the lecture notes.Then consider the following relaxed linear SVM model

$$\min_{w,b,\xi} \frac{\|w\|^2}{2} + C\sum_{i=1}^{n}\xi_i$$
$$s.t.\ \ y_i(< w, x_i > +b) \geq 1 - \xi_i$$
$$\xi_i \geq 0, i = 1, ..., n$$

where C is a constant
Please show its Lagrange dual problem

3.2 Answer

Lagrange Function:

$$L(w, b, \xi, \alpha, \lambda) = f_0(x) - \sum \alpha_i(\xi_i - 1 + y_i(< w, x_i > +b)) - \sum \lambda_i \xi_i$$

KKT condition:

$$\begin{cases} \nabla_w L = w - \sum \alpha_i y_i x_i = 0 \\ \nabla_\xi L = c\mathbf{1} - \sum \alpha_i - \sum \lambda_i \\ \nabla_b L = \sum \alpha_i y_i = 0 \\ \alpha_i \geq 0 \\ \lambda_i \geq 0 \\ \alpha_i \left[y_i \left(\langle w, x_i \rangle + b \right) + \xi_i - 1 \right] = 0 \\ \xi_i \lambda_i = 0 \\ y_i \left(\langle w, x_i \rangle + b \right) + \xi_i - 1 \geq 0 \end{cases}$$

Take $w = \sum \alpha_i y_i x_i$,We can get lagrang dual problem：

$$max\frac{\|w\|^2}{2} + C\sum_{i=1}^{n}\xi_i - \sum \alpha_i(\xi_i - 1 + y_i(< \sum \alpha_i y_i x_i, x_i > +b)) - \sum \lambda_i \xi_i$$
$$= \frac{\| \sum \alpha_i x_i y_i \|^2}{2} + C\sum_{i=1}^{n}\xi_i - \sum \alpha_i(\xi_i - 1 + y_i(< \sum \alpha_i y_i x_i, x_i > +b)) - \sum \lambda_i \xi_i$$
$$= \sum_i \sum_j -\frac{1}{2}\alpha_i \alpha_j y_i y_j < x_i, x_j > + \sum \alpha_i - b\sum_{i=1}^{n}\alpha_i y_i + \sum_{i=1}^{n}(c - \alpha_i - \lambda_i)\xi_i$$

for $\alpha_i y_i = 0$ and $c - \alpha_i - \lambda_i = 0$ the Lagrange Dual Problem:

$$\begin{aligned}
&\max_{\alpha} \sum_i \sum_j -\frac{1}{2}\alpha_i \alpha_j y_i y_j < x_i, x_j > + \sum \alpha_i \\
&s.t \quad \sum \alpha_i y_i = 0 \\
&\qquad \alpha_i \geq 0
\end{aligned}$$

4 HW 4

4.1 Problem Restatement

Implement the interior point method to solve

$$
\begin{aligned}
\min\ & x_1^2 + 2x_2^2 - 2x_1 - 6x_2 - 2x_1x_2 \\
s.t.\ & \frac{x_1}{2} + \frac{x_2}{2} \le 1 \\
& -x_1 + 2x_2 \le 2 \\
& x_1 \ge 0, x_2 \ge 0
\end{aligned}
$$

4.2 Answer

Wether the primal problem is convex or not is difficult to find out, but the primal problem can be changed to the standard QP form, then implement the standard interior point method to solve this problem .

take $x_1^{'} = x_1 - x_2$,write the problem as

$$
\begin{aligned}
\min\ & x_1^{'2} + x_2^2 - 2x_1^{'} - 8x_2 \\
s.t.\ & \frac{x_1^{'}}{2} + x_2 \le 1 \\
& -x_1^{'} + x_2 \le 2 \\
& x_1 - x_2 = x_1^{'} \\
& x_1 \ge 0, x_2 \ge 0
\end{aligned}
$$

add slack constraints s_1, s_2, and let $s_3 = x_1, s_4 = x_2$ the problem can be written as

$$
\begin{aligned}
\min\ & x_1^{'2} + x_2^2 - 2x_1^{'} - 8x_2 - u\left(\sum_i^4 logs_i\right) \\
s.t.\ & \frac{x_1^{'}}{2} + x_2 + s_1 = 1 \\
& -x_1^{'} + x_2 + s_2 = 2 \\
& -x_1^{'} - x_2 + s_3 = 0 \\
& -x_2 + s_4 = 0
\end{aligned}
$$

The objective function can be written as

$$
\min \left(x_1^{'} - 1\right)^2 + (x_2 - 4)^2 - 17 - u\left(\sum_i^4 logs_i\right)
$$

So, I get the standard form

$$
\begin{aligned}
&\min \ \|Ax-b\|^2 - u\left(\sum_{i}^{4} log s_i\right) - 17 \\
&s.t. \ Cx + S = d
\end{aligned}
$$

in which

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & 1 \\ -1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$d = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

Apply the Interior Point Method to solve this problem ,the detail of the code is available at the appendix, the answer is in the Figure 3

```
[9]:
# 求一个初值
x0,s0,v0,u0=FindInitialPoint(A,b,C,d)

#求一个最优值
res=InteriorPointMethod(A,b,C,d,s0,v0,x0,u0,epsilon=1e-5)
print("最优值: ",res[0]-17)
print("x: ",res[1])
print("s:",res[2])
```

```
最优值:  -7.19999287434786
x:  [-0.39998831  1.19999288]
s: [1.27240443e-06 4.00018809e-01 8.00004573e-01 1.19999288e+00]
```

Figure 3: The answer using Interior Point Method

As the answer shows, the solution is -7.199999, the $x_1^{'} = -0.399$,the $x_2 = 1.999$,the $x_1 = s_3 = 0.8$

5 HW 5

5.1 Problem Restatement

The problem of finding the shortest distance from a point x_0 to the hyperplane $x|Ax = b$,where A has full row rank,can be formulated as the quadratic program

$$\min_x \ \frac{1}{2}\|x_0 - x\|^2$$
$$s.t. \ \ Ax = b$$

(i)Show that the optimal Lagrange multiplier is

$$v^* = (AA^T)^{-1}(Ax_0 - b)$$

(ii)Show that optimal solution is

$$x^* = x_0 - A^T(AA^T)^{-1}(Ax_0 - b)$$

5.2 Answer

(1)

Lagrange:

$$L(x, v) = \frac{1}{2}\|x_0 - x\|^2 + v^T(Ax - b)$$

Lagrange Dual Function:

$$g(v) = \inf_x L(x, v)$$
$$\nabla_x L = -(x_0 - x) + A^T v = 0$$
$$\Rightarrow x = x_0 - A^T v$$

get :

$$\begin{aligned} g(v) &= \inf_x L(x, v) \\ &= \frac{1}{2}\|A^T v\|^2 + v^T(A(x_0 - A^T v) - b) \\ &= \frac{1}{2}\|A^T v\|^2 + v^T Ax_0 - v^T AA^T v - v^T b \\ &= -\frac{1}{2}\|A^T v\|^2 + v^T Ax_0 - v^T b \end{aligned}$$

The Lagrange dual problem:

$$\max_v -\frac{1}{2}\|A^T v\|^2 + v^T Ax_0 - v^T b$$

for:

$$\nabla_v g = -AA^T v + Ax_0 - b = 0$$
$$\Rightarrow v = (AA^T)^{-1}(Ax_0 - b)$$

So $v = v^* = (AA^T)^{-1}(Ax_0 - b)$

(ii):

In the primal problem ,the $f_0 = \frac{1}{2}\|x_0 - x\|^2$ the $\nabla^2 f_0 = 1 > 0$,and the constraint $Ax = b$ is affine . So The problem is convex.

KKT condition:

$$\begin{cases} x^* - x_0 + A^T v = 0 \\ Ax = b \end{cases}$$

Take the $x^* = x_0 - A^T v^*$,satisfy the KKT condition ,the optimal solution check.

参考文献