PSTAT 120B HW 6

- Raw Blackwell Thm - Let θ be an unbiased estimator for θ such that $V(\hat{\theta}) < \infty$. If U is sufficient for θ, then for all θ, $E(\hat{\theta}^*) = \theta$ and $V(\hat{\theta}^*) \leq V(\hat{\theta})$. Thm 5.15 implies that
$$V(\hat{\theta}) = V[E(\hat{\theta}|U)] + E[V(\hat{\theta}|U)] = V(\hat{\theta}^*) + E[V(\hat{\theta}|U)]$$
Because $V(\hat{\theta}|U=u) \geq 0$ for all u, it follows that $E[V(\hat{\theta}|U)] \geq 0$, and therefore $V(\hat{\theta}) \geq V(\hat{\theta}^*)$. This thm can be used to find good estimators and prove optimality in certain situations. In addition, it provides a means to construct a better estimator from crude one
- The moment estimator of θ in terms of sample moments m_k and population moments $\mu_k(\theta)$ are:
  The k^{th} population moment $(\mu_k = E\,Y_i^k)$
  The k^{th} sample moment $(m_k = \frac{1}{n}\sum_{i=1}^{n} Y_i^k)$
- The maximum likelihood estimator of θ is $\hat{\theta} = \text{argmax}_{\theta \in \Theta}\, \ell(\theta, y)$ where $f(y,\theta)$ is a joint density function for data y with parameter space Θ and we write the likelihood as a function of the parameters $\mathcal{L}(\theta; y) = f(y, \theta)$, $\theta \in \Theta$. Maximum likelihood estimators are those parameters values that maximize the likelihood of data, and the likelihood is just another name for the joint density. When we maximize the likelihood with respect to parameters, the data argument (y_i's, x_i's, etc) is treated as fixed
- Fisher information defined in the sample about θ is written as:
$$E\left[\frac{-d^2}{d^2\theta} \ell(\theta; y)\right] \in (0, \infty)$$

- The invariance property of MLEs is of $\hat{\theta}$ is the MLE of θ, and t is any function w/ a twice differential inverse on θ, then $t(\hat{\theta})$ is the MLE of $t(\theta)$
- The asymptotic distribution of $\sqrt{n}(\hat{\theta}-\theta)$ is:
  $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d}$ as $n \to \infty$. In addition, the limiting distribution of the MLE we state the uniparameter case without proof. Under many regulatory conditions,

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, I(\theta)^{-1})$$

1) a) Poisson : $f(y;\theta) = \dfrac{\theta^{y} \exp\{-\theta\}}{y!}$

✓ Sufficient Stat: $\dfrac{\theta^{\sum_{i=1}^{n} y_i}\, n \cdot \exp\{-\theta\}}{y_1! \cdot y_2! \cdot y_n!}$

$$= \underbrace{\theta^{\sum_{i=1}^{n} y_i}\, n \exp\{-\theta\}}_{g(u \sum y_i \,;\, \theta)} \;\; \frac{1}{y_1!\, y_2!\, y_n!}$$

Moment Estimator of θ:

$$M = E Y_1 = \sum_{i=0}^{\infty} y_1 f(y_1, \theta)$$

$$= \theta \exp\{-\theta\} \sum_{i=0}^{\infty} \frac{\theta^{y-1}}{(y-1)!}$$

$$= \theta \exp\{-\theta\} \exp\{\theta\} = \theta$$

$$m_i = \frac{1}{n} \sum_{i=1}^{n} Y_i = \bar{Y}$$ $\hat{\theta}$ is a momen estimator

MLE

$$\hat{\theta} = \arg\max_{\theta} (y_1, y_2 \cdots y_n ; \theta)$$

$$\frac{d}{d\theta} \ell(y_1 y_2 y_3 \ldots y_n \theta) = \frac{d}{d\theta}\left(-n\theta - \sum_{i=1}^{n} \ln(y!) + \ln(\theta) \sum_{i=1}^{n} y_i\right)$$

$$= -n + \frac{1}{\theta} \sum_{i=1}^{n} y_i$$

Fisher Info

$$\theta = 1/n \sum_{i=1}^{n} Y_i$$

$$f(Y, \theta) = \frac{\theta^{y} \exp\{-\theta\}}{y!}$$

$$\frac{d}{d\theta} \log f(Y; \theta)$$

$$= (y \log \theta - \theta - \log(y!)) \frac{d}{d\theta}$$

$$= \frac{y}{\theta} - 1$$

$$E\left(\frac{Y}{\theta^2}\right) = \frac{1}{\theta}$$

2. MLE for θ; Y_1, Y_2, Y_3, Y_n with an exp distribution with mean θ

results in $g(y) = \theta^{-n} e^{\sum -\frac{y_i}{\theta}}$ as a joint density

likelihood $\ell(y) = \ln(g(y)) = \ln(\theta^{-n} \cdot e^{\sum -\frac{y_i}{\theta}}$

$= \ln(\theta^{-n}) - \sum \frac{y_i}{\theta}$

$= -n \ln(\theta) - \frac{1}{\theta} \sum y_i$

$$\frac{d}{d\theta} \ell(y) = -\frac{n}{\theta} + \frac{\sum y_i}{\theta^2} = \frac{-n\theta + n\bar{Y}}{\theta^2} = \frac{-n(\theta - \bar{Y})}{\theta^2}$$

$\Rightarrow 0$, which implies $\hat{\theta} = \bar{Y}$, thus the MLE is $e^{-t/\hat{\theta}}$ or $e^{-t/\bar{Y}}$

3. $f_x(x) \begin{cases} \frac{1}{N} & x = 1, 2, 3 \dots N \\ 0 & \text{otherwise} \end{cases}$

a) $\mu_1 = E[X] = \sum_{x=1}^{N} x f(x) = \frac{1}{N} \sum_{x=1}^{N} x$

$= \frac{1}{N} \left(\frac{N(N+1)}{2} \right) = \frac{N+1}{2}$

$m_1 = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$

$\mu_1 = m_1$ $\qquad \hat{N}_1 = 2\bar{X} - 1$

b) $\hat{N}_1 = 2\bar{X} - 1$

$E(\hat{N}_1) = E(2\bar{X} - 1) = 2E(\bar{X}) - 1$

$= 2 \cdot \frac{1}{N} \sum_{i=1}^{N} X_i - 1$

$= \frac{2}{N} \sum_{i=1}^{N} X_i - 1$

$= \frac{2}{N} \left(\frac{N(N+1)}{2} \right) - 1$

$= N + 1 - 1 = \underline{\underline{N}}$

$E[X_i^2] = \sum_{X_i=1}^{N} x_i^2$

$= \frac{1}{N} \sum_{X_i=1}^{N} x_i^2$

$= \frac{1}{N} \left(\frac{N(N+1)(2N+1)}{6} \right)$

$= \frac{(N+1)(2N+1)}{6}$

$V(\hat{N}_1) = V(2\bar{X} - 1)$

$= 4 Var(\bar{X})$

$V(X_i) = E(X_i^2) - [E(X_i)]^2$

$= \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4}$

$= \frac{2N^2 + N + 2N + 1}{6} - \frac{N^2 + 2N + 1}{4}$

$= \frac{2(2N^2 + 3N + 1) - 3(N^2 + 2N + 1)}{12}$

$= \frac{4N^2 + 6N + 2 - 3N^2 - 6N - 3}{12}$

$= \frac{N^2 - 1}{12}$

4. Poisson Case:

$$f(y,\theta) = \frac{\theta^y \exp(-\theta)}{y!}$$

$\log f(y,\theta) = y\log\theta - \theta \log y!$

$$\frac{d}{d\theta} \log f(y,\theta) = \frac{d}{d\theta}[y\log\theta - \theta \log y!]$$

$$= \frac{y}{\theta} - 1 \Rightarrow -\frac{y}{\theta}$$

$$\frac{d^2}{d\theta^2} \log f(y,\theta) = \frac{-y}{\theta^2}$$

$$I(\theta) = -nE\left(\frac{-y}{\theta^2}\right)$$

$$= \frac{nE(y)}{\theta^2}$$

for $E(y) = \theta$,

$$I(\theta) = \frac{n\theta}{\theta^2}$$

$$I(\theta) = \frac{n}{\theta}$$

$\bar{Y} = \frac{1}{n}\sum_{i=1}$ & $g(\theta) = \frac{1}{\theta}$

If $\theta > 0$, $g(\bar{y}) = \frac{1}{\bar{y}}$ is a consistent estimator of $g(\theta)$

$$g(\theta) = \frac{1}{\theta} = I(\theta)$$

With CLT, $\frac{\sqrt{n}(\bar{y}-\theta)}{\sqrt{\bar{y}}} \sim N(0,1)$

$\sqrt{\bar{y}} \to \sqrt{\theta}$ $\quad \frac{\sqrt{n}(\bar{y}-\theta)}{\sqrt{\theta}} \sim N(0,1)$

(Consistent)

5. $\log f(y,\theta) = y\log\theta - \theta\log y!$

$$\frac{d^2}{d\theta^2} \log f(y,\theta) = -\frac{y}{\theta^2}$$

$$I(\theta) = \frac{n}{\theta}$$

$\bar{y} = \frac{1}{n}\sum_{i=1}$ consistent for θ

$$g(\theta) = \frac{1}{\theta}$$

$$g(\bar{y}) = \frac{1}{\bar{y}} = \frac{1}{\theta} = I(\theta)$$

$$\frac{\sqrt{n}(\bar{y}-\theta)}{\sqrt{\theta}} \sim N(0,1)$$

$$\Rightarrow \frac{\sqrt{n}(\bar{Y}-\theta)}{\sqrt{\bar{Y}}} \overset{d}{\sim} N(0,1)$$