1) a) $f_X(x) = \begin{cases} \frac{\theta}{x^{1+\theta}} & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \int_1^\infty \frac{\theta}{x^{1+\theta}}\,dx$$

$$= \theta \cdot \left[\frac{-1}{\theta(x^\theta)}\right]_1^\infty$$

$$\frac{-1}{\infty^\theta} + \frac{1}{1^\theta} = 0 + 1$$

$$\int_{-\infty}^{\infty} f_x(x)\,dx \Rightarrow \int_1^\infty f_x(x)\,dx = \underline{\underline{1}}$$

As a result, we know our $f_X(x)$ is a proper density function

b) $f_X(x) = \frac{\theta}{x^{1+\theta}} \Rightarrow F_X(x) = \int_1^x \frac{\theta}{t^{1+\theta}}\,dt$ (CDF)

$$F_X(x) = \theta\left[\frac{-1}{-\theta t^\theta}\right]_1^x$$

$$F_X(x) = \frac{-1}{x^\theta} + \frac{1}{1^\theta}$$

$$F_X(x) = \frac{-1}{x^\theta} + 1$$

c) We can draw samples by first finding the cdf as done in 1b and then generating a probability in (0,1) with Unif (0,1). We know that Unif (0,1) is a one to one continuous distribution that will be used as Y. From here we can use an inverse function for the cdf to get y such that $F_Y(y) = -\frac{1}{y^\theta} + 1$. Using such an inversed function and solving for it will allow us to find the random samples

2) a)

$X \backslash Y$	0	1	2
0			
1	$\frac{1}{8}$		$\frac{1}{8}$

(i) Given $X = 1$, Y is uniformly distributed

(ii) $P_{X|Y}(0|0) = 2/3$

(iii) $E[Y | X = 0] = 4/5$

We can confirm the following since given $X=1$, Y is uniformly distributed

$$P_{X|Y}(1,1) = \frac{1}{8}$$

$$P_{X|Y}(0,0) = 2/3 = \frac{P_{(X|Y)}(0,0)}{P_Y(0)} = \frac{P_{(X|Y)}(0,0)}{P_{X|Y}(0,0) + P_{X|Y}(1,0)}$$

$$= \frac{P(X|Y)(0,0)}{P_{(X|Y)}(0,0) + \frac{1}{8}}$$

$$P(X|Y)(0,0) = \left(\frac{2}{3}\right)\left(P_{X|Y}(0,0) + \frac{1}{8}\right)$$

$$= \left(\frac{2}{3}\right)\left(P_{X|Y}(0,0)\right) + \left(\frac{2}{3}\right)\left(\frac{1}{8}\right)$$

$$\left(P(X|Y)(0,0)\right) - \left(\frac{2}{3}\right)\left(P_{(X|Y)}(0,0)\right) = \left(\frac{2}{24}\right)$$

$$\frac{1}{3}\left(P_{(X|Y)}(0,0)\right) = \left(\frac{2}{24}\right)$$

$$P(X|Y)(0,0) = 1/4$$

$$E(Y|X=0) = \frac{4}{5} = 0 \cdot \left(P_{Y|X}(0,0)\right) + 1 \cdot \left(P_{Y|X}(1,0)\right) + 2 \cdot \left(P_{Y|X}(2,0)\right)$$

$$= \frac{P_{X|Y}(0,1) + 2\left(P_{X|Y}(0,2)\right)}{P_X(0)} = \frac{P_{X|Y}(0,1) + 2\left(P_{X|Y}(0,2)\right)}{P_{X|Y}(0,0) + P_{X|Y}(0,1) + P_{X|Y}(0,2)}$$

$$= \frac{P_{X|Y}(0,1) + 2\left(\frac{3}{8} - P_{X|Y}(0,1)\right)}{\frac{2}{8} + P_{X|Y}(0,1) + \frac{3}{8} - P_{X|Y}(0,1)}$$

$$\Rightarrow P_{X|Y}(0,1) + 2\left(\frac{3}{8} - P_{X|Y}(0,1)\right) = \left(\frac{4}{5}\right)\left(\frac{5}{8}\right)$$

$$\Rightarrow P_{X|Y}(0,1) + 2\left(\frac{3}{8}\right) - 2P_{X|Y}(0,1) = \frac{20}{40}$$

$$\Rightarrow P_{X|Y}(0,1) = 2/8 = 1/4$$

$$P_{X|Y}(0,2) = \frac{3}{8} - P_{X|Y}(0,1)$$
$$= \frac{3}{8} - \frac{1}{4}$$
$$= \frac{1}{8}$$

We can fill out the table as follows:

X \ Y	0	1	2
0	1/4	1/4	1/8
1	1/8	1/8	1/8

3) a) $$P(A=k) = \sum_{n=0}^{\infty} P(B=n) \cdot P(A=k/B=n)$$
$$= \sum_{n=0}^{\infty} \left(\frac{e^{-\lambda}\lambda^n}{n!}\right)\left(\binom{n}{k}p^k(1-p)^{n-k}\right)$$
$$= e^{-\lambda}\lambda^k p^k \sum_{n=k}^{\infty} \frac{\lambda^{n-k}(1-p)^{n-k}}{n!}\binom{n}{k}$$
$$= e^{-\lambda}(\lambda p)^k \sum_{n=k}^{\infty} \frac{(\lambda(1-p))^{n-k}}{n!}\left(\frac{n!}{(n-k)!k!}\right)$$
$$= \frac{e^{-\lambda}(\lambda p)^k}{k!} \sum_{n=k}^{\infty} \frac{(\lambda(1-p))^{n-k}}{(n-k)!}$$
$u=n-k$ $$= \frac{e^{-\lambda}(\lambda p)^k}{k!} \sum_{u=0}^{\infty} \frac{(\lambda(1-p))^u}{u!} = \frac{e^{-\lambda}(\lambda p)^k}{k!} e^{\lambda(1-p)}$$
$$= \frac{e^{-\lambda p}(\lambda p)^k}{k!}$$ $\boxed{\text{It follows poisson distribution}}$

b) $P(B=n \mid A=k) = \dfrac{P(B=n)\,P(A=k \mid B=n)}{P(A=k)}$

$$= \frac{\frac{e^{-\lambda}\lambda^n}{n!}\binom{n}{k}p^k(1-p)^{n-k}}{\frac{e^{-\lambda p}(\lambda p)^k}{k!}}$$

$$= \left(\frac{k!}{n!}\right)\frac{e^{-\lambda}\cdot e^{\lambda p}\cdot\lambda^n \frac{n!}{(n-p)!k!}p^k(1-p)^{n-k}}{\lambda^k p^k}$$

$$= \frac{e^{-\lambda(1-p)}\lambda^{n-k}(1-p)^{n-k}}{(n-k)!}$$

$$= \boxed{\frac{e^{-\lambda(1-p)}(\lambda(1-p))^{n-k}}{(n-k)!}}$$

4) a) $f(x,y) = \begin{cases} \frac{x+y}{4}, & 0<x<y<2 \\ 0 & \text{otherwise} \end{cases}$

marginal y pdf

$f(y) = \int_0^y \frac{1}{4}(x+y)\,dx$

$= \frac{1}{4}\int_0^y (x+y)\,dx$

$= \frac{1}{4}\left[\frac{x^2}{2}+yx\right]_0^y$

$= \frac{1}{4}\left[\frac{y^2}{2}+y^2\right]$

$= \frac{3y^2}{8}$

conditional pdf $x \mid y=y$

$f_{x|y}(x,y) = \dfrac{\left(\frac{x+y}{4}\right)}{\left(\frac{3}{8}y^2\right)} = \boxed{\left(\frac{2}{3}\right)\left(\frac{x+y}{y^2}\right) \text{ when } 0<x<y}$

b) $P(X<\frac{1}{2} \mid Y=1)$

$= \int_0^{1/2} \frac{2}{3}(x+1)\,dx$

$= \frac{2}{3}\left[\frac{x^2}{2}+x\right]_0^{1/2}$

$= \frac{2}{3}\left[\frac{1}{8}+\frac{1}{2}\right]$

$\boxed{P(X<\frac{1}{2} \mid Y=1) = \frac{5}{12}}$

$P(X<\frac{3}{2} \mid Y=1)$

$= \int_0^1 \frac{2}{3}(x+1)\,dx$

$\frac{2}{3}\left[\frac{x^2}{2}+x\right]_0^1 = \frac{2}{3}\left(\frac{1}{2}+1\right)$

$= \frac{2}{3}\cdot\frac{3}{2}$

$\boxed{P(X<\frac{3}{2} \mid Y=1) = 1}$

4) c) $E[X^2 | Y=y] = \int_0^y x^2 f(x|y)\,dx$

$$= \int_0^y x^2 \left(\frac{2}{3}\right)\left(\frac{x+y}{y^2}\right) dx$$

$$= \frac{2}{3y^2} \int_0^y x^3 + x^2 y\, dx$$

$$= \frac{2}{3y^2} \left[\frac{x^4}{4} + \frac{x^3 y}{3} \right]_0^y$$

$$= \frac{2}{3y^2} \left[\frac{y^4}{4} + \frac{y^3 y}{3} \right] = \frac{2}{3y^2} \left(\frac{7}{12} y^4 \right)$$

Marginal pdf of x

$$f(x) = \int_x^2 \frac{x+y}{4}\, dy = \frac{1}{4} \left[xy + \frac{y^2}{2} \right]_x^2$$

$$= \frac{1}{4} \left[2x + 2 - \frac{3x^2}{2} \right]$$

$$= \frac{1}{8} (4x + 4 - 3x^2) \quad \text{for} \quad 0 < x < 2$$

$$E[X^2] = \int_0^2 \frac{1}{8} (4x^3 + 4x^2 - 3x^4)\, dx$$

$$= \frac{1}{8} \left[\frac{4x^4}{4} + \frac{4x^3}{3} - \frac{3x^5}{5} \right]_0^2$$

$$= \frac{1}{8} \left[16 + \frac{32}{3} - \frac{96}{5} \right]$$

$$= \frac{1}{8} \left[\frac{80}{3} - \frac{96}{5} \right] = \frac{1}{8} \left[\frac{400 - 288}{15} \right]$$

$$= \frac{112}{120} = \underline{\frac{14}{15}}$$