HW 4

1) a) $X_1, X_2, X_3 \ldots \quad \mathbb{N} = \{0, 1, 2 \ldots\} \quad \lim_{i\to\infty} \mathbb{E}[X_i] = 0 \quad \lim_{i\to\infty} P[X_i = 0] = 1$

$$E[X_n] = \sum_{k=1}^{\infty} P[X_n \geq k] \geq P(X_n \geq 1) = 1 - P(X_n = 0)$$

$$E[X_n] \geq 1 - P(X_n = 0)$$

$$1 - P(X_n = 0) \leq E[X_n] \Rightarrow 1 - E[X_n] \leq P(X_n = 0) \leq 1$$

$$\lim_{n\to\infty} \left(1 - E[X_n] \leq P(X_n = 0) \leq 1\right)$$

$$1 \leq \lim_{n\to\infty} P(X_n = 0) \leq 1$$

$$\lim_{n\to\infty} P(X_n = 0) = 1$$

(Shown through Markov's Inequality)

b) $\lambda = 5, \; x = 3 \quad X \sim Exp(\lambda = 5) \quad E[X] = \frac{1}{\lambda} = \frac{1}{5}$

Markov

$$P(X \geq c) \leq \frac{E(x)}{c}$$

$$\leq \frac{\left(\frac{1}{5}\right)}{3} = \left(\frac{1}{5}\right)\left(\frac{1}{3}\right) = \frac{1}{15}$$

$$\boxed{P(X \geq 3) \leq \frac{1}{15}}$$

Chebyshev

$$\underbrace{P\left(\left|X - \tfrac{1}{5}\right| \geq c\right)}_{\frac{15}{5} - \frac{1}{5}} \leq \frac{\left(\frac{1}{25}\right)}{c^2}$$

$$P\left(X \geq \frac{14}{5}\right) \leq \frac{\left(\frac{1}{25}\right)}{\left(\frac{14}{5}\right)^2} \Rightarrow \left(\frac{1}{25}\right)\left(\frac{5}{14}\right)^2 \Rightarrow \left(\frac{1}{25}\right)\left(\frac{25}{196}\right)$$

$$= \frac{1}{196}$$

$$\boxed{P\left(X \geq \frac{14}{5}\right) \leq \frac{1}{196}}$$

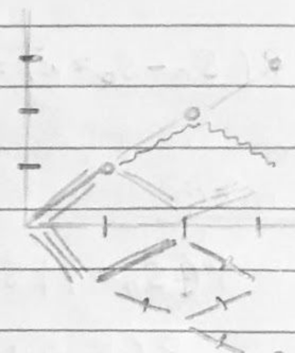
2) $(S_n)_{n\geq 0}$ $S_0 = 0$ $p = .7$ $q = 1-p = .3$

a) $P(S_3 = n)$

$S_0 + S_1 + S_2 + S_3$

$\Rightarrow X_1 + X_2 + X_3 = n$

$P(X_1 + X_2 + X_3 = n)$



$P(X_1 + X_2 + X_3 = 3) = p^3$ $n = 3$ $(.7)^3$

$P(X_1 + X_2 + X_3 = 1) = 3p^2q$ $n = 1$ $3(.7)^2(.3)$

$P(X_1 + X_2 + X_3 = -1) = 3pq^2$ $n = -1$ $3(.7)(.3)^2$

$P(X_1 + X_2 + X_3 = -3) = q^3$ $n = -3$ $(.3)^3$

$$\boxed{P(S_3 = n) = \begin{cases} .027 & \text{if } n = -3 \\ .189 & \text{if } n = -1 \\ .441 & \text{if } n = 1 \\ .343 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}}$$

b) $P_n = \exp(S_n + b_{10})$ for $n = 0, 1, 2, 3 \ldots$

$P_{10} = \exp(S_{10} + b_{10})$

$= e^{S_{10} + b_{10}}$

$= e^{S_{10}} \cdot e^{b_{10}}$

$$E[P_{10}] = E[e^{S_{10}} \cdot e^{b_{10}}]$$

$$= e^{10b} \cdot E[e^{S_{10}}] \qquad = \boxed{e^{10} \cdot (1093.415)}$$

$$= e^{10b} \cdot E[(e^{X_1})]$$

$$= e^{10b} \cdot (pe + qe^{-1})^{10}$$

$$= e^{10b} \cdot (.7e + .3e^{-1})^{10}$$

$$= e^{10} \cdot (1093.415)$$

3) $(S_n)_{n\geq 0}$ $\quad P(S_n = y \mid S_m = x)$ for $n > m$, $n < m$

$$P(S_n - S_0 = k) = \binom{n}{(n+k)(\frac{1}{2})} p^{(n+k)(\frac{1}{2})} q^{(n-k)(\frac{1}{2})}$$

For $n > m$

$$P(S_n = y \mid S_m = x)$$

$$= \frac{P(S_n = y, S_m = x)}{P(S_m = x)} = \frac{P(S_n - S_m = y - x, S_m = x)}{P(S_m = x)}$$

$$= \frac{P(S_n - S_m = y - x) \cdot \cancel{P(S_m = x)}}{\cancel{P(S_m = x)}} \rightarrow \frac{1}{2}^{n-m}$$

$$\Rightarrow \boxed{\binom{n-m}{\frac{1}{2}(y - x + n - m)} \left(\frac{1}{2}\right)^{(n-m)}}$$

For $n < m$

$$P(S_n = y \mid S_m = x)$$

$$= \frac{P(S_m = x \mid S_n = y) \cdot P(S_n = y)}{P(S_m = x)}$$

$$P(S_n = y) = \binom{n}{\frac{1}{2}(n+y)} \cdot \left(\frac{1}{2}\right)^{\frac{1}{2}(n+y)} \left(\frac{1}{2}\right)^{\frac{1}{2}(n-y)} = \left(\frac{1}{2}\right)^n \binom{n}{(n+y)(\frac{1}{2})}$$

$$P(S_m = x) = \binom{n}{\frac{1}{2}(m+x)} \left(\frac{1}{2}\right)^{(\frac{1}{2})(n+y)} \left(\frac{1}{2}\right)^{(\frac{1}{2})(m-x)} = \left(\frac{1}{2}\right)^m \binom{m}{(\frac{1}{2})(m+x)}$$

$$P(S_m = x \mid S_n = y) = \frac{P(S_m - S_n = (x-y), S_n = y)}{P(S_n = y)}$$

$$= \binom{m-n}{(x - y + m - n)(\frac{1}{2})} \left(\frac{1}{2}\right)^{m-n}$$

$$\Rightarrow \frac{\binom{m-n}{\frac{1}{2}(x - y + m - n)} \left(\frac{1}{2}\right)^{m-n} \binom{n}{\frac{1}{2}(n+y)} \left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^m \binom{m}{\frac{1}{2}(m+x)}} \Rightarrow \boxed{\frac{\binom{n}{\frac{1}{2}(y+n)} \binom{m-n}{\frac{1}{2}(x - y + m - n)}}{\binom{m}{\frac{1}{2}(m+x)}}}$$

4) $X_1, X_2 \sim \text{Pois}(\lambda)$ mean λ, $S_n = X_1 + \ldots + X_n$

$E[X_i] = \lambda$ $Var(X_i) = \lambda$

$$E[S_n] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = n\lambda$$

$$\Rightarrow E\left[\frac{S_n}{n}\right] = \left(\frac{1}{n}\right) E(S_n) = \lambda$$

$$Var\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\left(Var(S_n)\right) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{\lambda}{n}$$

$$\Rightarrow \frac{Var(S_n)}{n^2} = \frac{\lambda}{n} \to 0$$

$$\Rightarrow P\left(\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| < \varepsilon\right)$$ tends to zero $n \to \infty$

$$P\left(\left|\frac{S_n}{n} - \lambda\right| < \varepsilon\right) = \frac{\lambda}{n}$$ tends to zero $n \to \infty$

$$P\left(\frac{S_n}{n} \le t\right) = \frac{\lambda}{n}$$

$$\lim_{n \to \infty} P\left(\frac{S_n}{n} \le t\right) = \begin{cases} 1, & \lambda < t \\ 0, & \lambda > t \end{cases}$$ $\boxtimes$