PSTAT 120B - Quiz 1 (Shravan Shenoy)

1.) $X = -n \log W$

$U_i = -\log Y_i$

a) where $Y_1 \dots Y_n \overset{iid}{\sim} Unif(0,1)$

$\Rightarrow \boxed{U_i \sim Exp(1)}$

$\Rightarrow \boxed{\mathbb{E}(U_i) = 1}$

$\Rightarrow \boxed{Var(U_i) = 1}$

$\Rightarrow \boxed{m_U(t) = \frac{1}{1-t}}$

b) $X = -n \log W$

$$= \sum_{i=1}^{n} U_i$$

$$M_X(t) = \mathbb{E}(e^{Xt})$$

$$= \mathbb{E}\left(e^{t \cdot \sum_{i=1}^{n} U_i}\right)$$

$$= \prod_{i=1}^{n} \mathbb{E}(e^{U_i \cdot t})$$

$$= \prod_{i=1}^{n} m_U(t)$$

$$= \prod_{i=1}^{n} \frac{1}{1-t}$$

$$\boxed{M_X(t) = \frac{1}{(1-t)^n}}$$

Gamma Distribution has a similar MGF with the same functional form

c) MGF from previous problem (1b) shows properties of gamma distribution, in which

$$\boxed{X \sim Gamma(n, 1)}$$

Because of this, we can find the density to be:

$$\boxed{f_X(x) = \begin{cases} \frac{x^{n-1} e^{-x}}{\Gamma(n)} & x > 0,\ n \,\&\, 1 > 0 \\ 0 & \text{otherwise} \end{cases}}$$

Note: A special property of

$$\Gamma(n) = (n-1)!$$

Because of this, the density can also be written as:

$$\boxed{f_X(x) \begin{cases} \frac{x^{n-1} e^{-x}}{(n-1)!} & x > 0,\ n \,\&\, 1 > 0 \\ 0 & \text{otherwise} \end{cases}}$$

2.) W is strictly monotone function of $X = -n \log W$

a) $$f(w) = \frac{d}{dw} \cdot P(x \leq h^{-1}(w))$$
$$= \frac{d}{dw} \cdot F(h^{-1}(w))$$
$$= f(h^{-1}(w)) \left| \frac{d}{dw} h^{-1}(w) \right|$$
$$= \frac{e^{n \cdot \log w} \cdot (-n \log w)^{n-1}}{\Gamma(n)} \cdot \left| \frac{-n}{w} \right|$$

$$\boxed{f(w) = \begin{cases} \dfrac{(w^n) n \cdot (-n \log w)^{n-1}}{\Gamma(n) \cdot (w)} & w \leq 1 \\ 0 & \text{otherwise} \end{cases}}$$

b) $$f(w) = \int f(h^{-1}(w)) \left| \frac{d}{dw} h^{-1}(w) \right| \cdot dw$$
$$\Rightarrow \int \frac{(-n \log w)^{n-1} (w^n) n}{w \cdot \Gamma(n)}$$

(The quiz says to not compute this integral)

3.) $-\log \omega = \bar{U}$

$\bar{U} \sim N(\mu, \frac{\sigma^2}{n})$

so $-\log \omega \sim N(\mu, \frac{\sigma^2}{n})$

$\cancel{-}\ln \omega = \frac{1}{n} \sum_{i=1}^{n} \ln Y_i$

since $\boxed{\omega = \left(\prod_{i=1}^{n} Y_i \right)^{1/n}}$

For personal reference

a) $n \cdot -\ln W = -\sum_{i=1}^{n} \ln Y_i$

$n \cdot -\ln \omega = \text{Gamma}(n, 1)$

$-\ln \omega = \text{Gamma}(n, 1/n)$

$\mathbb{E}[-\ln \omega] = \frac{1}{n} \times n = 1$

$\text{Var}[-\ln \omega] = (\frac{1}{n})^2 \times n = 1/n$

Using CLT $\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)$:

$$\frac{\bar{U} - \mu}{\sigma/\sqrt{n}} \Rightarrow \frac{-\ln W - \mathbb{E}[\ln \omega]}{\sigma/\sqrt{n}}$$

$$= \frac{-\ln \omega - 1}{\sqrt{1}/\sqrt{n}} \sim N(0,1)$$

$$\Rightarrow -\ln \omega \sim N(1, \tfrac{1}{n})$$

Now compare

$N(1, \frac{1}{n})$ and $N(\mu, \frac{\sigma^2}{n})$

$\boxed{\mu = 1, \sigma^2 = 1}$

b) $\sqrt{n}(-\log \omega - 1)$

We already solved: $-\overset{\log}{\ln}\, \omega \sim N(1, \frac{1}{n})$

So, we can now do:

$(-\log \omega - 1) \sim N(0, \frac{1}{n})$

$\boxed{\sqrt{n}(-\log \omega - 1) \sim N(0,1)}$

c) $P(\omega \leq e^{-1}) = P(-\log \omega - 1 \geq 0)$

$= P(\sqrt{n}(-\log \omega - 1) \geq 0)$

We know:

$\sqrt{n}(-\ln \omega - 1) \sim N(0,1)$ & $\lim_{n\to\infty} \frac{1}{e} \to 0$

Thus, $\boxed{\Phi(0) \Rightarrow 1/2}$