PSTAT 120B - HW #1

1) Reading Outline

- Direct Integration Method
  - Typically used for random variables that have a continuous dist
  - First, find the distribution function for function U such that $F_U(u) = P(U \le u)$. In order to do this, find the region in the $y_1, y_2 \dots y_n$ space for which $U \le u$ and then find $P(U \le u)$ by integrating $f(y_1, y_2, y_3 \dots y_n)$ over this region. The density function for U is then obtained by differentiating the distribution function, $F_U(u)$.
  - The density g of $U = h(Y)$ can be found w/ these steps:

Find event $\{U \le u\}$ — 1.) write the event $\{U \le u\}$ explicitly in terms of Y. (Find region $\{y : h(y) \le u\}$).

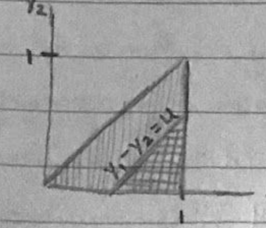
Integrate to find cdf of U. — 2.) Find the cdf of U by direct integration of $f(y)$

$$P(U \le u) = \int_{y: h(y) \le u} f(y)\,dy$$

Differentiate to find density of U — 3.) Differentiate in u to obtain density $g(u) = \frac{d}{du} P(U \le u)$

- Example 6.2

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 < 1 \\ 0, & \text{otherwise} \end{cases}$$



(Easier to integrate over lower triangular region)

For $0 \le u \le 1$, and $U = Y_1 - Y_2$

$$F_U(u) = P(Y_1 - Y_2 \le u)$$

Find event $\{U \le u\}$

Example 6.2 (Cont)

$$F_U(u) = P(U \le u) = 1 - P(U \ge u)$$

$$1 - \int_u^1 \int_0^{y_1 - u} 3y_1 \, dy_2 \, dy_1$$

$$1 - \int_u^1 3y_1 (y_1 - u) \, dy_1$$

$$1 - 3\left(\frac{y_1^3}{3} - \frac{u y_1^2}{2}\right)\Big]_u^1$$

$$1 - \left[1 - \frac{3}{2}(u) + \frac{u^3}{2}\right]$$

$$= \frac{1}{2}(3u - u^3)$$

$$F_U(u) = \begin{cases} 0 & u < 0 \\ (3u-u)^3/2 & 0 \le u \le 1 \\ 1 & u > 1 \end{cases}$$

Integrate to find cdf of U

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} \frac{3}{2}(1-u^2) & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Differentiate to find density of U

Monotone Transformation Method

- This method results in a general expression for the density of $U = h(Y)$ for an increasing or decreasing function $h(y)$. If Y_1 and Y_2 have a bivariate distribution, the univariate result can be used to find the joint density of Y_1 and $U = h(Y_1, Y_2)$. By integrating over y_1, the marginal pdf of U can be found.

MTF (Cont)

- Let $U = h(Y)$, where $h(y)$ is inc or dec. & $f_Y(y) > 0$.
  1) Find inverse function, $y = h^{-1}(u)$
  2) Evaluate $\frac{dh^{-1}}{du} = \frac{d[h^{-1}(u)]}{du}$
  3) Find $f_U(u)$ with
$$f_U(u) = f_Y[h^{-1}(u)] \left| \frac{dh^{-1}}{du} \right|$$

MGF Method

- Based on the uniqueness theorem in which, if 2 random variables have identical MGFs, they have the same probability distributions. To use this method, find the MGF of U and compare it with MGFs for common discrete and continuous random variables.
  1) Find the MGF of U, $m_U(t)$
  2) Compare $m_U(t)$ with other MGFs. They have identical distributions if $m_U(t) = m_V(t)$

Abstract

The three primary methods in this chapter are used to estimate population parameters through random variables. The application of each method varies due to there being an optimal choice for which method should be used. The method of distribution functions is used mostly for continuous distributions obtained by differentiating $F_U(u)$. The method of transformations focuses on integrating over y, to find the marginal pdf of U. The MGF method, based on uniqueness theorum, finds distributions based on traits of their MGF.

Practice Problems

6.1) a. $U_1 = 2Y-1$

$P(2Y-1 \le u)$

$P\left(Y \le \frac{u+1}{2}\right)$

$\int_0^y 2(1-u)\,du$

$\left[2u - \frac{2u^2}{2}\right]_0^y$

$= 2y - y^2$

$2\left(\frac{u+1}{2}\right) - \left(\frac{u+1}{2}\right)^2$

$u+1 - \left(\frac{u+1}{2}\right)\left(\frac{u+1}{2}\right)$

$= \frac{d}{du}\left[u+1 - \frac{u^2+2u+1}{4}\right]$

$= \frac{d}{du}\left[\frac{1}{4}(4u+4-u^2-2u-1)\right]$

$\left[\frac{1}{4}(2u+3-u^2)\right]\frac{d}{du}$

$2 \cdot \frac{1}{4}(1-u) = \boxed{\frac{1}{2}(1-u)}$

$$\boxed{f_U(u)\begin{cases}\frac{1}{2}(1-u) & -1<u<1\\ 0 & \text{otherwise}\end{cases}}$$

b. $U_2 = 1-2Y$

$P(1-2Y \le u)$

$P\left(Y \ge \frac{u-1}{-2}\right)$

$P\left(Y \ge \frac{1-u}{2}\right)$

$\left(\frac{1-u}{2}\right)\left(\frac{1-u}{2}\right)$

$1 - 2\left(\frac{1-u}{2}\right) + \left(\frac{1-u}{2}\right)^2$

$1 - \cancel{2}\left(\frac{1-u}{\cancel{2}}\right) + \frac{1-2u+u^2}{4}$

$\frac{d}{du}\left(\cancel{1-1} + u + \left(\frac{1-u}{2}\right)^2\right)$

$= 1 + \left(\frac{1}{2}\right)\left(\cancel{\frac{1}{2}}\right)\cancel{2}(1-u)(-1)$

$= \frac{1+u}{2}$

$$\boxed{f_U(u) = \begin{cases}\frac{1+u}{2} & -1<u<2\\ 0 & \text{otherwise}\end{cases}}$$

c. $U = Y^2$

$P(Y^2 \le u)$

$P(Y \le \sqrt{u})$

$\frac{d}{du}\left(2\sqrt{u} + (\sqrt{u})^2\right)$

$= \frac{1}{2}\frac{2}{\sqrt{u}} - 1$

$= \boxed{\frac{1}{\sqrt{u}} - 1}$

$$\boxed{f_U(u) = \begin{cases}\frac{1}{\sqrt{u}} - 1 & 0 \le u < 1\\ 0 & \text{otherwise}\end{cases}}$$

d. $E(U_1) = \int_{-1}^{1} u \frac{(1-u)}{2}\,du$

$= \frac{1}{2}\left[\left(\frac{u^2}{2} - \frac{u^3}{3}\right)\right]_{-1}^{1}$

$= \boxed{-1/3}$

$E(U_2) = \int_{-1}^{1} u\left(\frac{1+u}{2}\right)du$

$= \frac{1}{2}\left[\frac{u^3}{3} + \frac{u^2}{2}\right]_{-1}^{1} = \boxed{\frac{1}{3}}$

$E(U_3) = \int_0^1 u\left(\frac{1}{\sqrt{u}} - 1\right)du$

$\left[\frac{2x^{3/2}}{3} - \frac{x^2}{2}\right]_0^1$

$= \boxed{\frac{1}{6}}$

e. $E(U_1) = E(2Y-1) = 2 \times \frac{1}{3} - 1 = \boxed{-\frac{1}{3}}$

$E(U_2) = E(1-2Y) = 1 - 2\left(\frac{1}{3}\right) = \boxed{\frac{1}{3}}$

$E(U_3) = E(Y^2) = \int_0^1 2y^2(1-y)\,dy$

$\boxed{= \frac{1}{6}}$

6.7) a. $U = Z^2$

$P(Z^2 \le u)$

$P(-\sqrt{u} \le Z \le \sqrt{u})$

$F'(\sqrt{u}) - F'(-\sqrt{u})$

$= \frac{1}{2\sqrt{u}} f_z(\sqrt{u}) + \frac{1}{2\sqrt{u}} f_z(\sqrt{u})$

$= \frac{1}{\sqrt{u}} f_z(\sqrt{u}) = \boxed{\frac{1}{\sqrt{2\pi}} u^{-1/2} e^{-\frac{u}{2}}}$

b. Yes, in this case the values for $\boxed{\alpha = \frac{1}{2}}$ and $\boxed{B = \frac{1}{2}}$

c. Based on the density functions of other distributions, it is a $\boxed{\text{"}X^2\text{" distribution}}$

6.23) a. $U = 2Y - 1$

$\frac{u+1}{2} = y = h^{-1}(u)$

$\frac{d}{du}\left(\frac{u+1}{2}\right) = \frac{1}{2}$

$f_y[h^{-1}(u)] \left|\frac{dh^{-1}}{du}\right| = 2\left(1 - \frac{u+1}{2}\right)\left|\frac{1}{2}\right|$

$= \boxed{\frac{1}{2} - \frac{u}{2}}$

b. $U = 1 - 2Y$

$\frac{u-1}{-2} = y = h^{-1}(u)$

$\frac{d}{du}\left(\frac{-u-1}{2}\right) = -\frac{1}{2}$

$f_y[h^{-1}(u)] \left|\frac{dh^{-1}}{du}\right| = 2\left(1 - \frac{-1-u}{2}\right)\left|-\frac{1}{2}\right|$

$= \boxed{\frac{u}{2} + \frac{1}{2}}$

c. $U = Y^2$

$\sqrt{u} = y = h^{-1}(u)$

$\frac{d}{du}(\sqrt{u}) = \frac{1}{2\sqrt{u}}$

$f_y[h^{-1}(u)] \left|\frac{dh^{-1}}{du}\right| = 2(1 - \sqrt{u})\left|\frac{1}{2\sqrt{u}}\right|$

$= \boxed{\frac{1}{\sqrt{u}} - 1}$

6.40) Y_1 and Y_2 are independent

$U = Y_1^2 + Y_2^2$

(X^2 distribution)

$m(t) = (1-2t)^{-1/2}, \ t < \frac{1}{2}$

$m_U(t) = m_A(t)\, m_B(t)$

$= (1-2t)^{-1}$ $\lambda = 1/2$

$\boxed{f_U(u) = \frac{1}{2} e^{-u/2}}$

6.41) $Y_1, Y_2 \ldots Y_n$ independent

a_1, a_2, a_n constants.

$U = \sum_{i=1}^{n} a Y_i$

$m_Y(t) = E(e^{tY}) \Rightarrow e^{t\mu + \sigma^2 t^2/2}$

U is normally distributed with mean $\mu \sum_{i=1}^{n} a_i$ & sd. dev $\sigma\sqrt{\sum_{i=1}^{n} a_i^2}$

The density function will then be:

$= \boxed{\frac{1}{\sqrt{2\pi\sigma^2 \sum_{i=1}^{n} a_i^2}} e^{-\left(x - \mu\sum_{i=1}^{n} a_i\right)^2 / 2\sigma^2 \sum_{i=1}^{n} a_i^2}}$

6) Counter-example:

- Let $Y \sim \text{Unif}(-1,1)$ and $U = Y^2$.
- The inverse function $h^{-1}(u)$ can be either $+\sqrt{u}$ or $-\sqrt{u}$
- Being neither monotonic or one-to-one causes errors in the calculations

7.) h must be one to one on support of $y: f(y)>0$

$P(U \le u) = P(h^{-1}(U) \le h^{-1}(u))$

$$\boxed{\begin{aligned} P(U \le u) &= P(Y \le y) + P(Y \ge y_2) \\ &= P(Y \le h^{-1}(u)) + P(Y \ge h^{-1}(u)) \end{aligned}}$$

(If it was not one-to-one, there would be two points for:

$h(y_1) = u ; h(y_2) = u,$

meaning we have to consider accounting for y in new intervals where its monotone)

8) $h(Y) \Rightarrow f(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right|$

$h(Y_1, Y_2) \rightarrow f(h^{-1}(u; y_2), y_2) \left| \frac{\partial}{\partial} h^{-1}(u; y_2) \right| dy_2$

$= f_{Y_1|Y_2,Y_3}(h^{-1}(u; y_2, y_3) | y_2, y_3) \left| \frac{d}{du} h^{-1}(u; y_2, y_3) \right| f_{2,3}(y_2, y_3)\, dy_2\, dy_3$

$= f_{Y_1|Y_2,Y_3}(h^{-1}(u; y_2, y_3) | y_2, y_3) \frac{d}{du} h^{-1}(u; y_2, y_3) | f_2(y_2) f_3(y_3)\, dy_2\, dy_3$

$= \boxed{f(h^{-1}(u, y_2, y_3), y_2, y_3 \left| \frac{d}{du} h^{-1}(u; y_2; y_3) \right| dy_2 dy_3}$

(See work for number 8)

9) $g(u) = f(h^{-1}(u; y_2, \dots y_n), y_2 \dots y_n) \left| \frac{d}{du} (h^{-1}(u; y_2, \dots y_n) \right| dy_2 \dots dy_n$

Generalization of expression using random variables would be solved by using conditional chainrule in double integral

10) $Y_1, Y_2 \sim \text{Unif}(0,1)$

$U = Y_1 + Y_2$

$f(y) = \mathbb{1}\{0 < y < 1\}$

$y_1, y_2 \sim \text{Unif}(0,1)$

$f(y_1) = 1 ; \; 0 < y_1 < 1$

$f(y_2) = 1 ; \; 0 < y_2 < 1$

$f(y_1, y_2) = f(y_1) \cdot f(y_2) = 1$

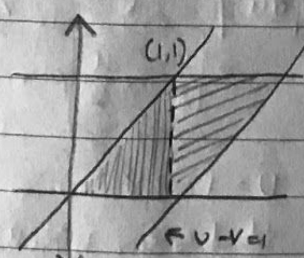
$U = Y_1 + Y_2$ where $V = Y_2$

then $Y_1 = U - V$ and $Y_2 = V$

$J = \frac{\partial(y_1, y_2)}{\partial(u, u)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$|J| = 1 \qquad 0 < Y_1 < 1 \qquad 0 < Y_2 < 1$

$\qquad 0 < U - V < 1 \qquad 0 < V < 1$



$f(u) \begin{cases} \int_0^u 1\, du & 0 < u < 1 \\ \int_{u-1}^1 du & 1 < u < 2 \end{cases}$

$$\boxed{f(u) = \begin{cases} u & 0 < u < 1 \\ 2 - u & 1 < u < 2 \end{cases}}$$