HW 12

24. Prove that $\bigcap_{x \in \mathbb{R}} [3-x^2, 5+x^2] = [3,5]$

1) Let $a \in \bigcap_{x \in \mathbb{R}} [3-x^2, 5+x^2]$. Thus, $\forall x \in \mathbb{R}$, $a \in [3-x^2, 5+x^2]$. Because of this, $0 \in \mathbb{R}$, so $a \in [3,5]$. Hence, $\bigcap_{x \in \mathbb{R}} [3-x^2, 5+x^2] \subseteq [3,5]$

2) Let $b \in [3,5]$. Then, $3 \leq b \leq 5$. Suppose $x \in \mathbb{R}$. Then $x^2 \geq 0$. Because of this, $3-x^2 \leq 3$ and $5+x^2 \geq 5$. So $3-x^2 \leq 3 \leq b \leq 5 \leq 5+x^2$. Thus, $b \in [3-x^2, 5+x^2]$. Because $\forall x \in \mathbb{R}$, $b \in [3-x^2, 5+x^2] \Rightarrow b \in \bigcap_{x \in \mathbb{R}} [3-x^2, 5+x^2]$. Then, $[3,5] \subseteq \bigcap_{x \in \mathbb{R}} [3-x^2, 5+x^2]$, therefore $\bigcap_{x \in \mathbb{R}} [3-x^2, 5+x^2] = [3,5]$ $\boxtimes$

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1. If A, B, C are sets, then $(A \cap B) - C = (A-C) \cap (B-C)$.

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\begin{aligned}
(A \cap B) - C &= \{x : x \in (A \cap B) \cap (x \notin C)\} \\
&= \{x : (x \in A \cap x \in B) \cap (x \notin C)\} \\
&= \{x : (x \in A \cap x \notin C) \cap (x \in B \cap x \notin C)\} \\
&= \{x : (x \in A \cap \sim C) \cap (x \in (B-C))\} \\
&= \{x : x \in (A-C) \cap x \in (B-C)\} = (A-C) \cap (B-C) \; \boxtimes
\end{aligned}
$$

2. $\{x \mid \exists y \in \mathbb{R} : x = y^2 - 2y\} = \mathbb{R}$.

Disproof: Let $x = -2$. Then $x = y^2 - 2y \Rightarrow 0 = y^2 - 2y - x$. Then, $y = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm i2}{2} = 1$. So, $y \notin \mathbb{R}$. there are no real roots of $y^2 - 2y - x = 0$. by the fundamental thm of Algebra, there are no real y such that $-2 = y^2 - 2y$.

Therefore, $\{x \mid \exists y \in \mathbb{R} : x = y^2 - 2y\} \neq \mathbb{R}$, since $-2 \notin \{x \mid \exists y \in \mathbb{R} : x = y^2 - 2y\}$ $\blacksquare$