HW 3

48) A function f is continuous on irrationals, discontinues on rationals

$$\lim_{x \to 0} \frac{\sin(a)}{a} = \lim_{q \to \infty} \frac{\sin\left(\frac{1}{q}\right)}{1/q} = 1$$

$\Rightarrow \forall \epsilon > 0$, $\exists$ an M that has a large value where $q > M$

so that $\left| \frac{\sin \frac{1}{q}}{1/q} - 1 \right| < \epsilon$

Let x be an irrational number for $\forall \epsilon > 0$ such that $\delta_1 = \epsilon$ and $\delta_2 = \min_{q \le M} | x - \frac{p}{q} |$. Let $\delta = \min | \delta_1, \delta_2 |$. For any rational $y = \frac{p}{q}$ with $|x - y| < \delta$, it follows that

$$|f(x) - f(y)| = |x - p(\sin(\tfrac{1}{q}))| = x - \tfrac{p}{q} + \tfrac{p}{q} - p\sin(\tfrac{1}{q}) \le$$
$$|x - \tfrac{p}{q}| + |\tfrac{p}{q} - p\sin\tfrac{1}{q}| \le 2\epsilon.$$

For $\forall x \in [0,1]$, and $\forall \epsilon > 0$ such that $\delta = \epsilon > 0$ for any irrational y with $|x-y| < \delta \Rightarrow |f(x) - f(y)| = |x-y| < \epsilon$

From this, we can conclude $f(x)$ is continuous on $[0,1]$

$\Rightarrow$ This is true for $[N, N+1]$ where N is an integer

49) Let f & g be continuous functions such that they exist in E.

(i) We know that $(f+g)(x) = f(x) + g(x)$

Any point $a \in E$, f & g are continuous for any ϵ there exists δ_1, δ_2 such that $|f(x)-f(a)| < \epsilon$ & $|g(x)-g(a)| < \epsilon$ such that $|x-a| < \delta_1$ and $|x-a| < \delta_2$ respectively.

$$|(f+g)(x) - (f+g)(a)| = |f(x) + g(x) - f(a) - g(a)|$$
$$\leq |f(x) - f(a)| + |g(x) + g(a)|$$
$$= 2\epsilon \quad , |x-a| < \delta$$

$\Rightarrow$ $f+g$ is continuous at $x=a$, $f+g$ is continuous on E $\boxtimes$

(ii) f and h are continuous functions

For any $\epsilon > 0$ there exists δ such that $|f(y) - f(b)| < \epsilon$, $|y-b| < \delta$,

there exists $\delta > 0$ such that $|h(x) - h(a)| < \delta'$, $|x-a| < \delta$

$$\Rightarrow |f \circ h(x) - f \circ h(a)| = |f(h(x)) - f(h(a))|$$
$$= |f(y) - f(b)|$$
$$< \epsilon \quad \Rightarrow f \circ h \text{ is continuous } \boxtimes$$

(iii) $$\text{Max}[f,g] = h(x) = \begin{cases} f(x), & f(x) \geq g(x) \\ g(x), & f(x) \leq g(x) \end{cases}$$

$d(x) = f(x) - g(x)$ is continuous

$$h(x) = \begin{cases} f(x) & \text{if } d(x) \geq 0 \\ g(x) & \text{if } d(x) \leq 0 \end{cases}$$

$\Rightarrow$ h is continuous on $d^{-1}([-\infty, 0])$ & continuous on $d^{-1}([0, \infty])$

f & g are continuous function $\Rightarrow$ Both sets are closed

$\Rightarrow \max(f, g)$ is continuous

(iv) Let f be a real continuous function.

For any point $a \in E$, $||f(x)| - f(a)|| \leq |f(x) - f(a)|$

$< \epsilon$

$\Rightarrow$ $|f|$ is continuous @ $x = a$

$\Rightarrow$ $|f|$ is continuous on E

Since f is a Lipschitz function on I, $\exists$ a positive real number k

such that $|f(x_1) - f(x_2)| \leq k|x_1 - x_2|$ for all $x_1, x_2 \in I$.

... such that $|x_1 - x_2| <$ ...

50) Since f is a Lipschitz function on I, $\exists$ a positive real number k such that $|f(x_1) - f(x_2)| \le k|x_1 - x_2|$ for all $x_1, x_2 \in I$.

Let $\epsilon > 0$, then for all points $x_1, x_2 \in I$ such that $|x_1 - x_2| < \frac{\epsilon}{k}$

$$\Rightarrow |f(x_1) - f(x_2)| < k \cdot \frac{\epsilon}{k}$$

$$\Rightarrow |f(x_1) - f(x_2)| < k \cdot \frac{\epsilon}{k} = \epsilon$$

$$|f(x_1) - f(x_2)| < \epsilon$$

$\Rightarrow$ f is uniform & continuous function on I

Let $f = \sqrt{x}$ within $[0,1]$. Because $f = \sqrt{x}$ is continuous on a bound interval, it is a uniform continuous function.

If $f(x) = \sqrt{x}$ is lipschitz function then there should exist a positive real number, $k > 0$

$$\Rightarrow |\sqrt{x} - \sqrt{y}| \le k|x-y| \text{ for all } x, y \in [0,1]$$

$$|\sqrt{x} - \sqrt{y}| = \frac{1}{2k} > \frac{1}{4k} = k|x-y|$$

$\Rightarrow$ $f(x) = \sqrt{x}$ is not a lipshitz function $\boxtimes$