38) (i) Let a be a cluster point such that $a < b$ since $\lim_{k\to\infty} \inf\{a_n\}_{n \geq k}\} = b$

$\Rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N} : \forall n > n_0$

$b - \epsilon < a_n$

Let $(a-\epsilon, a+\epsilon)$ contain a finite subsequence of $\{a_n\}_{n=1}$, $\Rightarrow$

all, some, or none of $\{a_n\}_{n=1}^{n_0} \Rightarrow a$ is not a cluster point

(ii) Let $\lim \sup\{a_n\}$ be a cluster such that for each n we can find a_{k_n} with

$k_n \geq n$ such that $\{\sup\{a_n, a_{n+1}, a_{n+2}\}\} < a_k + 1/n$ thus

$\lim_{n\to\infty} a_{k_n} = \lim_{n\to\infty} \{\sup\{a_n\}\}$

Let n be a cluster point of $\{a_n\}$, such that a subsequence $\{a_{k_n}\}$ exists where $\lim_{n\to\infty} a_{k_n} = n$, for each $\epsilon > 0$ there exists some N such that $n \geq N \Rightarrow a_{k_n} > n - \epsilon$,

Hence $\lim \sup\{a_n\} \geq n - \epsilon \;\forall\; \epsilon > 0$, so $\boxtimes$

39) (i) $\{a_n\}$ is bounded $\Rightarrow$ $\lim\sup\{a_n\}$ is not infinite.

Let $\epsilon > 0 \Rightarrow \bar{a} = \overline{\lim}\, a_n = \inf \bar{A}_n \Leftrightarrow \bar{A}_n > \bar{a}$ for all n

$\Leftrightarrow \exists\, m \in N$ such that $\bar{A}_n < \bar{a} + \epsilon$

$\Leftrightarrow \sup\{a_n, a_{n+1}, \dots\} \geq \bar{a} \quad \forall n \; \exists\, m \in N$ such that

$\sup\{a_m, a_{m+1}, \dots\} < \bar{a} + \epsilon$

$\Leftrightarrow a_n > \bar{a} + \epsilon$ for infinitely many values of n

$\exists\, m \in N$ such that $a_n < \bar{a} + \epsilon, \; \forall n > m$ $\boxtimes$

(ii) Let $\bar{A}_n = \sup\{a_m, a_{m+1}, \dots\}$

$\overline{\lim}\, a_n = +\infty \quad \Leftrightarrow \inf\{A_1, A_2 \dots A_n\} = \infty$

$\Leftrightarrow \bar{A}_n = \infty \;\; \forall \infty \;\; \forall n \in N$

$\Leftrightarrow \{a_n\}$ is not bounded above $\boxtimes$

(iii) Let $b_n = -a_n, n \in N$ then it follows $B_n = \inf\{b_n, b_{n+1} \dots\}$

$= -\sup\{a_n, a_{n+1} \dots\} = -\bar{A}_n$

$\Rightarrow \underline{\lim}(-a_n) = \underline{\lim}\, b_n = \sup\{B_1, B_2 \dots\}$

$= \sup\{-\bar{A}_1, -\bar{A}_2, \dots\}$

$= -\inf\{\bar{A}_1, \bar{A}_2 \dots\}$

$= -\inf \bar{A}_n = -\overline{\lim}\, a_n$

$\Rightarrow \underline{\lim}(-a_n) = -\overline{\lim}(a_n)$

$\Rightarrow \lim\inf\{-a_n\} = \lim\sup\{a_n\}$

(iv) If $\{a_n\}$ converges to a, then a is the unique limit point of $\{a_n\} \Rightarrow$ The upper (superior) and lower (inferior) limits both equal a

The condition is necessary.

Let $\{a_n\}$ be a bounded sequence such that $\overline{\lim}\, a_n = \underline{\lim}\, a_n = a$

$\Rightarrow$ a is the unique limit point of the bounded sequence $\{a_n\}$.

Since limit upper and lower (inferior, superior) are smallest and greatest limit points $\Rightarrow a_n$ converges to a $\boxtimes$

41) Let $z_n = \inf \{a_k \mid k \geq n\}$ and $y_n = \sup \{a_k \mid k \geq n\}$

then we can conclude $z_n \leq y_n$, and since $\lim z_n \leq \lim y_n$

This implies that $\lim \inf a_n \leq \lim \sup a_n$

$$\lim \inf a_n \leq \lim \sup a_n \quad \boxtimes$$