HW 14

19. Prove $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}\dots\frac{1}{n^2}\le 2-\frac{1}{n}$ for every $n\in\mathbb{N}$

<u>Case 1:</u> $n=1$

$$\frac{1}{n^2}=\frac{1}{1^2}\le 2-\frac{1}{n}$$

$$\frac{1}{1}\le 2-\frac{1}{1}$$

$$1\le 1 \quad ✓$$

Case 2: $n\ge 1$

$$=\sum_{k=1}^{n+1}\frac{1}{k^2}=\sum_{k=1}^{n}\frac{1}{k^2}+\frac{1}{(n+1)^2}\le 2-\frac{1}{n}+\frac{1}{(n+1)^2}$$

$$=2-\frac{(n+1)^2-n}{n(n+1)^2}=2-\frac{n^2+n+1}{n(n+1)^2}$$

$$=2-\frac{n^2+n+1}{n(n+1)^2}<2-\frac{n^2 n}{n(n+1)^2}=2-\frac{1}{n+1} \quad \boxtimes$$

20. Prove that $(1+2+3\dots+n)^2=1^3+2^3\dots+n^3$ for every $n\in\mathbb{N}$.

For $n=1$, $1^2=1=1^3=1$. Assuming this is true for $n=k$, then $(1+2\dots+k)^3=1^3+2^3\dots+k^3$.

For $n=k+1$,

$$(1+2+3+k+k+1)^2=\left[(1+2+3+\dots+k)+(k+1)\right]$$

$$=(1+2+3\dots+k)^2+2(1+2+3\dots+k)(k+1)+(k+1)^2$$

$$=1^3+2^3+3^3+k^3+(k+1)\left[2(1+2+3\dots k)+k+1\right]$$

$$=1^3+2^3\dots k^3+(k+1)^2(k+1)$$

$$=1^3+2^3+3^3\dots k^3+(k+1)^3.$$

Therefore by mathematical induction,

$$(1+2+3+k)^2=1^3+2^3+3^3+\dots+k^3$$ for every int. $n\in\mathbb{N}$ $\boxtimes$

22. If $n \in \mathbb{N}$, then $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}$

For $n=k$

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{2^k}\right) \geq \frac{1}{4}+\frac{1}{2^{k+1}}$$

$n=k+1$

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right)$$

$$= \left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^k}\right)\right]\left(1-\frac{1}{2^{k+1}}\right)$$

$$\geq \left[\frac{1}{4}+\frac{1}{2^{k+1}}\right]\left(1-\frac{1}{2^{k+1}}\right) = \frac{1}{4}+\frac{1}{2^{k+1}}-\frac{1}{4}\cdot\frac{1}{2^{k+1}}-\frac{1}{2^{k+1}}\cdot\frac{1}{2^{k+1}}$$

$$= \frac{1}{4}+\frac{1}{2^{k+1}}\left[1-\frac{1}{4}-\frac{1}{2^{k+1}}\right] = \frac{1}{4}+\frac{1}{2^{k+1}}\left[\frac{3}{4}-\frac{1}{2^{k+1}}\right]$$

Since $k \geq 1$ and $2^{1+1} \leq 2^{k+1}$ then $\frac{1}{2^{k+1}} \leq \frac{1}{2^{1+1}} = \frac{1}{4}$ so $-\frac{1}{2^{k+1}} \geq -\frac{1}{4}$

So, $\frac{1}{4}+\frac{1}{2^{k+1}}\left[\frac{3}{4}-\frac{1}{4}\right] = \frac{1}{4}+\frac{1}{2^{k+1}}\left(\frac{1}{2}\right)$

$$= \frac{1}{4}+\frac{1}{2^{(k+1)+1}} = \frac{1}{4}+\frac{1}{2^{k+2}}$$

So we know that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right) \geq \frac{1}{4}+\frac{1}{2^{(k+1)+1}}$

therefore, $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}$ $\boxtimes$

24. Prove that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for each natural number n $\boxtimes$

$$\sum_{k=1}^{n+1} k\binom{n+1}{k} = \sum_{k=1}^{n+1} k\frac{(n+1)!}{k!(n+1-k)!} = \sum_{k=1}^{n+1} k\frac{n!}{(k-1)!(n-k)!}\left(\frac{n+1}{k(n-k+1)}\right)$$

$$= \sum_{k=1}^{n+1} k\frac{n!}{(k-1)!(n-k)!}\left(\frac{k+n-k+1}{k(n-k+1)}\right)$$

$$= \sum_{k=1}^{n+1} k\frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k}+\frac{1}{n-k+1}\right) = \sum_{k=1}^{n+1} k\frac{n!}{k!(n-k)!}+\sum_{k=1}^{n+1} k\frac{n!}{(k-1)!(n-(k-1))!}$$

$$= \sum_{k=1}^{n+1} k\binom{n}{k}+\sum_{k=1}^{n+1} k\binom{n}{k-1} = \sum_{k=1}^{n+1} k\binom{n}{k}+\sum_{k=0}^{n}(k+1)\binom{n}{k}$$

$$= \sum_{k=1}^{n} k\binom{n}{k}+\sum_{k=1}^{n} k\binom{n}{k}+\sum_{k=0}^{n}\binom{n}{k} = 2\sum_{k=1}^{n} k\binom{n}{k}+\sum_{k=0}^{n}\binom{n}{k}1^k 1^{n-k}$$

$$= 2\cdot n2^{n-1}+2^n = (n+1)2^n$$

$\boxtimes$