PSTAT 120B HW #3

Reading Outline

1) - An estimator is a rule, often expressed as a formula, that tells how to calculate the value of an estimate based on the measurements contained in the sample
   - An experimenter who wants an interval estimate of a parameter must use the sample data to calculate two values, chosen so that the interval formed by the two values includes the target parameter with a specified probability

2) - A point estimation produces a guess for the exact value of the target parameter.
   - An interval estimation produces a range of possible values for the target parameter.
   - One way to view this is that a point estimate involves a single numerical value, while an interval estimate is a range of numbers.

3) - Bias is what quantifies how far an estimator is expected to be from the target parameter.
   - The bias of an estimator is $B(\hat{\theta}) = E(\hat{\theta} - \theta)$, which shows the difference between the estimator's expected value and the true value of the target parameter

4) - The mean square error is a function of both its variance and its bias. The MSE quantifies the spread of an estimator about the target parameter
   - $MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2 = \frac{\text{var } Y}{n}$
   - $MSE = var(\hat{\theta}) + (B(\hat{\theta}))^2$

5) - An unbiased estimator for the population mean of a random sample is $\hat{\mu}$ where $\hat{\mu} = \bar{Y}$ if ($Y_1 \ldots Y_n \overset{iid}{\sim} f(y)$)

6) - An unbiased estimator for the population variance of random sample is $\hat{\theta}^2$ where $\hat{\theta}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$ if ($Y_1 \ldots Y_n \overset{iid}{\sim} f(y)$)

1. a) $\beta e^{-\beta y}, y \geq 0$

$$E(Y) = \beta \int_0^\infty e^{-\beta y} y \, dy = \beta \cdot \frac{\Gamma 2}{\beta^2} = \frac{1}{\beta}$$

$$E(Y^2) = \beta \int_0^\infty e^{-\beta y} y^2 \, dy = \beta \cdot \frac{\Gamma 3}{\beta^3} = \frac{2!}{\beta^2} = \frac{2}{\beta}$$

$$Var(Y) = E(Y^2) - E^2(Y) = \frac{2}{\beta^2} - \frac{1}{\beta^2} = \frac{1}{\beta^2}$$

$$\frac{\sqrt{n}\,(\bar{Y} - \frac{1}{\beta})}{\sqrt{1/\beta^2}} \sim N(0,1).$$

$$\bar{Y} - \frac{1}{\beta} \sim N\left(0, \frac{1}{n\beta^2}\right)$$

$$\boxed{\bar{Y} \sim N\left(\frac{1}{\beta}, \frac{1}{n\beta^2}\right)}$$

b.) $f(y) = \frac{1}{\Gamma\frac{v}{2}\, 2^{v/2}} e^{-\frac{y}{2}} y^{\frac{v}{2}-1}$

$$E(Y) = \frac{1}{\Gamma\frac{v}{2}\, 2^{v/2}} \cdot \int_0^\infty e^{-\frac{y}{2}} y^{\frac{v}{2}} dy$$

$$= \frac{1}{\Gamma\frac{v}{2}\, 2^{v/2}} \times \frac{\Gamma\left(\frac{v}{2}+1\right)}{(1/2)^{\frac{v}{2}+1}} = \frac{\frac{v}{2}\Gamma\frac{v}{2}}{\Gamma\frac{v}{2}\, 2^{v/2}} \times 2^{\frac{v}{2}+1} = \boxed{v}$$

$$E(Y^2) = \frac{1}{\Gamma\frac{v}{2}\, 2^{v/2}} \int_0^\infty e^{-\frac{y}{2}} y^{\frac{v}{2}+1} dy = \frac{\left(\frac{v}{2}+1\right)\frac{v}{2}\,\Gamma\frac{v}{2} \times 2^2}{\Gamma\frac{v}{2}} = \boxed{2v\left(\frac{v}{2}+1\right)}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = 2v\left(\frac{v}{2}+1\right) - v^2 = v^2 + 2v - v^2 = \boxed{2v}$$

$$\bar{Y} \sim N(v, 2v/n)$$

c) $E(Y) = \sum_{y=0}^\infty \frac{e^{-\lambda}\lambda^y y}{y!} = \lambda \sum_{y=1}^\infty \frac{e^{-\lambda}\lambda^{y-1}}{(y-1)!} = \lambda$

$$E(Y(Y-1)) = \sum_{y=0}^\infty \frac{e^{-\lambda}\lambda^y y(y-1)}{y!} = \lambda^2 \sum_{y=2}^\infty \frac{e^{-\lambda}\lambda^{y-2}}{(y-2)!} = \lambda^2$$

$$E(Y^2) - E(Y) = \lambda^2$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\frac{\sqrt{n}\,(\bar{Y}-\lambda)}{\sqrt{\lambda}} \sim N(0,1)$$

$$\boxed{\bar{Y} \sim N\left(\lambda, \frac{\lambda}{n}\right)}$$

d) $f_1(y) = N(\mu_1, \sigma_1^2),\ f_2(y) = N(\mu_2, \sigma_2^2)$

$$c f_1(y) = N(c\mu_1, c^2\sigma_1^2)$$

$$(1-c) f_2(y) = N((1-c)\mu_2, (1-c)^2\sigma_2^2)$$

$$E(Y) = c\mu_1 + (1-c)\mu_2$$

$$Var(Y) = c^2\sigma_1^2 + (1-c)^2\sigma_1^2$$

$$\bar{Y} - (c\mu_1 + (1-c)\mu_2) \sim N\left(0, \frac{c^2\sigma_1^2 + (1-c)^2\sigma_2^2}{n}\right)$$

$$\boxed{\bar{Y} \sim N\left(c\mu_1 + (1-c)\mu_2, \frac{c^2\sigma_1^2 + (1-c)^2\sigma_2^2}{n}\right)}$$

2.) a) $\bar{X} = \frac{1}{m}\sum_{i=1}^{m} x_i$

$$E[\bar{X}] = \frac{1}{m}\sum_{i=1}^{m} E[X_i]$$
$$= \frac{1}{m}\sum_{i=1}^{m} \mu_1$$
$$= \mu_1$$

$$V[\bar{X}] = V\left(\frac{1}{m}\sum_{i=1}^{m} X_i\right)$$
$$= \frac{1}{m^2}\sum_{i=1}^{m} V[X_i]$$
$$= \frac{1}{m^2} \times m\sigma_1^2$$
$$= \sigma_1^2/m$$

~~Knowing this,~~

$E[\bar{Y}] = \mu_2$, $V[\bar{Y}] = \frac{\sigma_2^2}{n}$

$$E[\bar{X}-\bar{Y}] = E[\bar{X}] - E[\bar{Y}]$$
$$= \boxed{\mu_1 - \mu_2}$$

b) $V[\bar{X}-\bar{Y}] = V[\bar{X}] + V[\bar{Y}] - 2\operatorname{cov}[\bar{X},\bar{Y}]$
$$= \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n} - 0$$

$$\boxed{V[\bar{X}-\bar{Y}] = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}$$

c) $\sigma_1^2 = 2$, $\sigma_2^2 = 2.5$, $m = n$

$$P(|\bar{X}-\bar{Y}| - (\mu_1-\mu_2)| \le 1) = .95$$
$$P(-1 \le \{(\bar{X}-\bar{Y}) - (\mu_1-\mu_2)\} \le 1) = .95$$

$$V[\bar{X}-\bar{Y}] = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$$
$$= \frac{2+2.5}{n}$$
$$= 4.5/n$$

$$P\left(\frac{-1}{\sqrt{4.5/n}} \le Z \le \frac{1}{\sqrt{4.5/n}}\right) = .95$$

$$2 \times P(Z \le .47\sqrt{n}) - 1 = .95$$
$$P(Z \le .47\sqrt{n}) = .975$$
$$.47\sqrt{n} = 1.96$$
$$n = 17.39$$

$\boxed{\text{At least 18}}$

3.) $E(X_i) = \mu_1$, $E(Y_i) = \mu_2$

$V(X_i) = \sigma_1^2$, $V(Y_i) = \sigma_2^2$

$W_i = X_i - Y_i$

for $i = 1, 2 \ldots n$

Let $\mu = E(W_i)$

$$E(W_i) = E(X_i) - E(Y_i)$$
$$\boxed{E(W_i) = \mu_1 - \mu_2}$$

$$Var(W_i) = Var(X_i - Y_i)$$
$$= V(X_i) + V(Y_i) - 2\operatorname{Cov}(X_i, Y_i)$$
$$\boxed{Var(W_i) = \sigma_1^2 + \sigma_2^2 - 0}$$

$$U_n = \frac{\sum_{i=1}^{n} W_i - n\mu}{\sigma/\sqrt{n}} = \frac{\bar{W} - \mu}{\sigma/\sqrt{n}}$$

$$= \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}}$$

converges to std. normal as $n \to \infty$.

4) $\hat{\theta}-\theta = \hat{\theta}-E(\hat{\theta})+E(\hat{\theta})-\theta$

$= [\hat{\theta}-E(\hat{\theta})]+[E(\hat{\theta})-\theta]$

$= [\hat{\theta}-E(\hat{\theta})]+B[\hat{\theta}]$

$MSE(\hat{\theta}) = E(\hat{\theta}-\theta)^2$

$= E((\hat{\theta}-E(\hat{\theta}))+B(\hat{\theta}))^2$

$= E((\hat{\theta}-E(\hat{\theta}))^2+2(\hat{\theta}-E(\hat{\theta}))B(\hat{\theta})+[B(\hat{\theta})]^2)$

$= V(\hat{\theta})+2[E(\hat{\theta})-E(\hat{\theta})]B(\hat{\theta})+[B(\hat{\theta})]^2$

$= V(\hat{\theta})+2(0)\cdot B(\hat{\theta})+[B(\hat{\theta})]^2$

$= V(\hat{\theta})+0+[B(\hat{\theta})]^2$

$= \underline{\underline{V(\hat{\theta})+[B(\hat{\theta})]^2}}$

5.) a) Given $\hat{\theta}$ is unbiased,

$E(\hat{\theta}) = \theta$

$B(\hat{\theta}) = E(\hat{\theta})-\theta$

$= \theta-\theta$

$\boxed{B(\hat{\theta}) = 0}$

b) $B(\hat{\theta}) = 5$

$B(\hat{\theta}) = E(\hat{\theta})-\theta$

$\Rightarrow E(\hat{\theta})-\theta = 5$

$\boxed{E(\hat{\theta}) = \theta+5}$

6.) a) $E(\hat{\theta}) = a\theta+b$

$B(\hat{\theta}) = E(\hat{\theta})-\theta$

$= a\theta+b-\theta$

$= (a-1)\theta+b$

$\boxed{B(\hat{\theta}) = b+\theta(a-1)}$

b) $E(\hat{\theta}) = a\cdot\theta+b$

$E(\hat{\theta}^*) = E\left(\frac{\hat{\theta}-b}{a}\right)$

$= \frac{E(\hat{\theta})-b}{a}$

$= \frac{a\theta+b-b}{a}$

$= \underline{\underline{\theta}}$

$\boxed{\hat{\theta}^* = \frac{\hat{\theta}-b}{a} \text{ is unbiased.}}$

7.) $MSE(\hat{\theta}) = E[(\hat{\theta}-\theta)^2]$

$= V(\hat{\theta})+(B(\hat{\theta}))^2$

a) $B(\hat{\theta}) = E(\hat{\theta})-\theta$

$= \theta-\theta$

$= \underline{\underline{0}}$

$MSE(\hat{\theta}) = V(\hat{\theta})+(B(\theta)^2)$

$= V(\hat{\theta})+0$

$= \boxed{V(\hat{\theta})}$

$MSE(\hat{\theta}) = V(\hat{\theta})$

b) $MSE(\hat{\theta}) = V(\hat{\theta})+B(\hat{\theta})^2$

$= V(\hat{\theta})+$ some value

$MSE(\hat{\theta}) \geq V(\hat{\theta})$