Argue $\frac{\sqrt{p(1-p)}}{\sqrt{\bar{Y}(1-\bar{Y})}} \xrightarrow{P} 1$

a)

1) If $Y_1, \ldots Y_n \overset{iid}{\sim}$ Bernoulli (p) + MLE of $p = \bar{Y}$

+ MLE regularity conditions hold

- We know that if MLE of $p = \bar{Y}$, then MLE of $g(p) = g(\bar{Y})$
- This implies $g(p) = \sqrt{p(1-p)}$ and $g(\bar{Y}) = \sqrt{\bar{Y}(1-\bar{Y})}$
- $\hat{\theta}_{MLE} \xrightarrow{P} \theta$, and because MLE regularity conditions hold, this implies $\sqrt{\bar{Y}(1-\bar{Y})} \xrightarrow{P} \sqrt{p(1-p)}$
- As a result,

$$\frac{(\sqrt{p(1-p)})}{(\sqrt{\bar{Y}(1-\bar{Y})})} \Rightarrow \frac{\cancel{(\sqrt{p(1-p)})}}{\cancel{(\sqrt{\bar{Y}(1-\bar{Y})})}} = 1 .$$

1 b) $\frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{p(1-p)}} \xrightarrow{d} N(0,1)$

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N\left(0, \frac{1}{I(\theta)}\right) \text{ if } n \to \infty$$

$\hookrightarrow$ From Fisher info, we know $I(\theta) = E\left(-\frac{d^2}{d^2\theta} l(\theta;Y)\right)$

A few important notes to make.

$\hookrightarrow I(\theta) = \frac{1}{\theta(1-\theta)}$ for Bernoulli (θ)

$\hookrightarrow \sqrt{n - I(\hat{\theta}_n)}(\hat{\theta}-\theta) \xrightarrow{d} N(0,1)$ } ~~Lecture PDF +~~ HW

$\hookrightarrow \bar{Y} = \hat{\theta}$ & $\theta = p$

$$\sqrt{n - I(\theta)} \cdot (\bar{Y}-\theta) \xrightarrow{d} N(0,1)$$

$$= \sqrt{n - \frac{1}{\theta(1-\theta)}} = \frac{\sqrt{n}}{\sqrt{\theta(1-\theta)}} = \frac{\sqrt{n}}{\sqrt{p(1-p)}}$$

$$\frac{\sqrt{n}}{\sqrt{\theta(1-\theta)}}(\hat{\theta}-\theta) \Rightarrow \frac{\sqrt{n}}{\sqrt{p(1-p)}}(\bar{Y}-p) \xrightarrow{d} N(0,1)$$

a)

2) • As shown in number 1, we know $\frac{\sqrt{p(1-p)}}{\sqrt{\bar{Y}(1-\bar{Y})}} \xrightarrow{p} 1$. and $\frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{p(1-p)}} \xrightarrow{d} N(0,1)$

• In addition to this, if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$,

we know that $X_n \cdot Y_n \xrightarrow{d} X_n \cdot Y_n \xrightarrow{d} X \cdot Y$.

Because $$\frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{\bar{Y}(1-\bar{Y})}} = \frac{\sqrt{p(1-p)}}{\sqrt{\bar{Y}(1-\bar{Y})}} \times \frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{p(1-p)}}$$

$$\frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{\bar{Y}(1-\bar{Y})}} = 1 \times \frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{p(1-p)}}$$

$$\frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{(\bar{Y}(1-\bar{Y})}} = 1 \times N(0,1)$$

$$\frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{(\bar{Y}(1-\bar{Y})}} \xrightarrow{d} N(0,1)$$

2 b) $$P\left(-Z_{\alpha/2} \le \frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{\bar{Y}(1-\bar{Y})}} \le Z_{\alpha/2}\right) \approx 1-\alpha$$

$$= P\left(-Z_{\alpha/2} \cdot \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}} \le (\bar{Y}-p) \le Z_{\alpha/2} \cdot \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}}\right) \approx 1-\alpha$$

$$= P\left(Z_{\alpha/2} \cdot \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}} \ge p-\bar{Y} \ge -Z_{\alpha/2} \cdot \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}}\right) \approx 1-\alpha$$

$$= P\left(\bar{Y} + Z_{\alpha/2} \cdot \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}} \ge p \ge -Z_{\alpha/2} \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}} + \bar{Y}\right) \approx 1-\alpha$$

$$= P\left(-Z_{\alpha/2} \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}} + \bar{Y} \le p \le \bar{Y} + Z_{\alpha/2} \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}}\right) \approx 1-\alpha$$

CI Interval $\Rightarrow$ $$\left(\bar{Y} - Z_{\alpha/2} \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}},\ \bar{Y} + Z_{\alpha/2} \frac{\sqrt{\bar{Y}(1-\bar{Y})}}{\sqrt{n}}\right)$$

3 a) $Y_i = \mathbb{1}$, $Y_1 \ldots Y_n \overset{iid}{\sim}$ Bernoulli (p), $n = 95$, $\bar{y} = \frac{5}{95}$
$= .0526$

$Z_{.025} = 1.96$, with this info, we can sub values from 2b.

$$\left(.0526 - \frac{1.96\sqrt{.0526(1-.0526)}}{\sqrt{95}} , .0526 + \frac{1.96\sqrt{.0526(1-.0526)}}{\sqrt{95}} \right)$$

Computing this, we get:

95% CI $\Rightarrow$ $(.00770956, .09749044)$

b) point estimate for efficacy $\Rightarrow 1 - \frac{p}{1-p}$ for specific monotone function

$$(\hat{p}_L, \hat{p}_U) \Rightarrow \left(1 - \frac{\hat{p}_L}{1-\hat{p}_L}, 1 + \frac{\hat{p}_U}{1-\hat{p}_U}\right)$$

$$\left(1 - \frac{.00770956}{1-.00770956}, 1 + \frac{.09749044}{1-.09749044}\right)$$

$= (.99223054 , .89197849)$

95% CI $\Rightarrow$ $(.89197849, .99223054)$