- In Ch. 8, we are introduced to point and interval estimation - specifically, how they can be used for target parameters.
- Through this method, we can account for several target parameters, which will typically involve population mean, variance, and standard deviation.
- Point estimation is used for accounting for bias, MSE, and the error of estimation of the point estimators used.
- Interval estimators calculate two numbers that form endpoints of an interval based confidence limit
- Confidence Intervals can often-times be used to calculate population mean + variance
- Point estimators + interval estimators can also involve topics of relative efficiency, consistancy, and minimum variance unbiased estimation.
- Such methods are typically used to find the most accurate estimate for distinct target parameters.

1.) $Y_i \overset{indep}{\sim} N(\beta_0 + \beta_1 x_i, \sigma^2)$ for $i, \ldots, n$, where x_i are fixed covariates

$$f(y_i; \beta_0 \beta_1 \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right)$$

$$L(\beta_0 \beta_1 \sigma^2; y_i) = \prod_{i=1}^{n} f(y_i; \beta_0 \beta_1 \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2\right)$$

a) $\ln L(\beta_0 \beta_1 \sigma^2; y_i) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$

$\Rightarrow \ln \ell(\beta_0 \beta_1 \sigma^2; y_i) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$

b) $\frac{\partial}{\partial \beta_0}\ell(\beta_0, \beta_1, \sigma^2; y_i) = \frac{1}{-2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) \times -2$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)$$

c) $\frac{\partial}{\partial \beta_1}\ell(\beta_0, \beta_1, \sigma^2; y_i) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) x_i \times \cdot -2$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i x_i - \beta_0 x_i - \beta_1 x_i^2)$$

d) $\frac{\partial}{\partial \sigma^2}\ell(\beta_0, \beta_1, \sigma^2, y_i) = \frac{\partial}{\partial\sigma^2}\left(-\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2\right)$

$$= -\frac{n}{2\sigma^2} - \frac{1}{2}\left(\frac{\sigma^2 \times 0 - \sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i^2)^2}{\sigma^4}\right)$$

$$= \frac{-n}{2\sigma^2} + \frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^4}$$

e) $\frac{\partial}{\partial \beta_0}\ell(\beta_0, \beta_1, \sigma^2, y_i) = 0$, $\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) = 0$

$$\beta_0 = \frac{\left(\sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i\right)}{n} = \frac{\sum_{i=1}^{n} y_i}{n} - \beta_1 \frac{\sum_{i=1}^{n} x_i}{n} \Rightarrow \hat{\beta}_0 = \bar{y} - \beta_1 \bar{x}$$

$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i x_i - \hat{\beta}_0 x_i - \beta_1 x_i^2)$

$= \sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n}(\bar{y} - \beta_1 \bar{x}) x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$

$= \sum_{i=1}^{n} y_i x_i - n\bar{y}\bar{x} - \beta_1\left(\sum_{i=1}^{n} x_i^2 - \bar{x}\sum_{i=1}^{n} x_i\right)$

$\Rightarrow \beta_1\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right) = \sum_{i=1}^{n} y_i x_i - n\bar{y}\bar{x}$

$$\boxed{\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x y_i - n\bar{y}\bar{x}}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 \qquad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}}$$