HW 3

1.) $X \sim Unif(0,1)$

$Y \mid X = x \sim U(0, x)$

$= \int_0^x y f_{Y|X}(y|x)\,dy = \mathbb{E}[Y|X]$

$= \int_0^x y \left(\frac{1}{x}\right) dy$

$= \frac{1}{x}\left[\frac{y^2}{2}\right]_0^x$

$\mathbb{E}[Y|X] = \frac{X}{2}$

$P(Z \leq z)$ Let $Z = \frac{X}{2}$

$P(\frac{X}{2} < z)$

$P(X < 2z)$

$= \int_{-\infty}^{2z} f_X(x)\,dx$

$[x]_{-\infty}^{2z} = 2z$

$$F_Z(z) = \begin{cases} 0, & z \leq 0 \\ 2z, & 0 \leq z < 1/2 \\ 1, & z \geq 1/2 \end{cases}$$

2.) $\mathbb{E}[S_N] = \mathbb{E}[N] \cdot \mathbb{E}[X_1]$ Need to prove

$\mathbb{E}[S_N] = \mathbb{E}[\mathbb{E}[S_N | N]]$

$E[S_N | N = n] = E[X_1 + X_2 + X_3 + \ldots + X_N | N = n]$

$= E[X_1 + X_2 + X_3 + \ldots + X_n]$

$= E[X_1] + E[X_2] + E[X_3] \ldots + E[X_n]$

$= nE[X_1]$

$E[S_N | N] = N(E[X_1])$

$E[S_N] = E[E[S_N | N]]$

$= E[N \cdot E[X_1]]$

$= E[N] \cdot E[X_1]$

3) a)

$m_x(t) = \frac{1}{3}e^{4t} + \frac{1}{6}e^{-5t} + \frac{1}{3}e^{-4t} + \frac{1}{6}e^{5t}$

$m_x'(t) = \frac{4}{3}e^{4t} - \frac{5}{6}e^{-5t} - \frac{4}{3}e^{-4t} + \frac{5}{6}e^{5t}$

$= \frac{4}{3} - \frac{5}{6} - \frac{4}{3} + \frac{5}{6}$ (since $e^0 = 1$)

$M_x'(0) = E(X)$

$= \mu$

$\boxed{E(X) = \mu = 0}$

$Var(X) = E(X^2) - E(X)^2$

$m_x''(t) = \frac{16}{3}e^{4t} + \frac{25}{6}e^{-5t} + \frac{16}{3}e^{-4t} + \frac{25}{6}e^{5t}$

$= 19$ (since $e^0 = 1$)

$\boxed{Var(X) = 19}$

b)

$f_x(t) = \begin{cases} \frac{1}{6} & x = -5 \\ \frac{1}{3} & x = -4 \\ \frac{1}{3} & x = 4 \\ \frac{1}{6} & x = 5 \end{cases}$

$E[X] = \sum_{k \in S_x}$

$= -5\left(\frac{1}{6}\right) - 4\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) + 5\left(\frac{1}{6}\right)$

$= 0$

$\underline{\underline{\mu = 0}}$

$E[X^2] = \sum_{k \in S_x} k^2 \cdot p_x(k)$

$= (-5)^2\left(\frac{1}{6}\right) + (-4)^2\left(\frac{1}{3}\right) + (4)^2\left(\frac{1}{3}\right) + 5^2\left(\frac{1}{6}\right)$

$= \frac{25}{6} + \frac{16}{3} + \frac{16}{3} + \frac{25}{6} = \underline{\underline{19}}$

$\underline{\underline{Var(X) = 19}}$

a)

4) $X_n \sim Pois(n),\ n \geq 1$

$$E\left[\frac{X_n}{n}\right] = \frac{E[X_n]}{n} = E[X_n] \times \frac{1}{n}$$

$$= n \times \frac{1}{n} = 1$$

As $n \to \infty$, $E\left[\frac{X_n}{n}\right] \Rightarrow 1$

Yes, $\left(\frac{X_n}{n}\right), n \geq 1$ converges as $n \to \infty$ with probability 1

$$Var\left[\frac{X_n}{n}\right] = \left(\frac{1}{n}\right)^2 \cdot Var(X_n)$$

$$= \left(\frac{1}{n}\right)^2 \cdot n$$

$$= \frac{1}{n} \quad \text{as } n \to \infty$$

b) Use CLT. Mean $E(X_n) = n$

Variance $Var(X_n) = n$

$$\frac{X_n - E(X_n)}{\sqrt{Var(X_n)}} \xrightarrow{L} N(0,1),\ n \to \infty$$

$$\frac{X_n - n}{\sqrt{n}} \xrightarrow{L} N(0,1),\ n \to \infty$$

$$\lim_{n\to\infty} P\left[\frac{X_n - n}{\sqrt{n}} \leq x\right] = P[X \leq x]$$

$$\lim_{n\to\infty} P\left[\frac{X_n - n}{\sqrt{n}} \leq x\right] = P[X \leq x] \text{ for all } x \in \mathbb{R}$$

c) $\lim_{n\to\infty} e^{-n}\left(1 + n + \frac{n^2}{2!} + \ldots + \frac{n^n}{n!}\right) = \frac{1}{2}$

$X_n \sim Pois(n)$

$$f_{X_n}(k) = \begin{cases} \frac{e^{-n} n^k}{k!} & , k = 0, 1, \ldots n \\ 0 & , \text{otherwise} \end{cases}$$

$$\lim_{n\to\infty} \left(P(X_n = 0) + P(X_n = 1) + \ldots P(X_n = n)\right)$$

$$\Rightarrow \lim_{n\to\infty} P(X_n \leq n) = \lim_{n\to\infty} P\left(\frac{X_n - n}{\sqrt{n}} \leq 0\right)$$

$$= P(Z < 0) = \Phi(0) = 1/2$$

$$\boxed{\lim_{n\to\infty} e^{-n}\left(1 + n + \frac{n^2}{2!} + \ldots + \frac{n^n}{n!}\right) = \frac{1}{2}}$$