14) Let every bounded subset of E have an outer measure zero. Let $I_k = [k, k+1]$ be a countable collection of disjoint bound intervals that decompose $\mathbb{R}$. E can then be decomposed as a countable union of bounded subsets of E

$$E = \bigcup_{k \in \mathbb{Z}} E \cap I_k$$

By hypothesis $m^*(E \cap I_k) = 0$. Due to the subadditivity and finite quality of m^*, then:

$$0 < m^*(E) = m^*\left(\bigcup_{k \in \mathbb{Z}} E \cap I_k\right) \leq \sum_{k \in \mathbb{Z}} m^*(E \cap I_k) = 0$$

This would be a contradiction. Thus if a set E has positive outer measure there is a bounded subset of E that also has positive outer measure ⊠

positive outer measure $\boxtimes$

17) Let $\varepsilon > 0$. Suppose there exists a closed set F and an open set O such that $F \subset E \subset O$ and $m(O/E) < \frac{\varepsilon}{2}$ and $m(E/F) < \frac{\varepsilon}{2}$. Since $O/F = (O/E) \cup (E/F)$ it then follows from countable subadditivity that

$$m(O/F) = m((O/E) \cup (E/F)) < m(O/E) + m(E/F)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Suppose that for all $\varepsilon > 0$ there exists a closed set F and an open set O such that $F \subset E \subset O$ and $m(O|F) < \varepsilon$. Let $\varepsilon > 0$ and F be a closed set and A be an open set such that $F \subset E \subset O$ and $m(O/F) < \varepsilon$. Since $O/E \subset O/F$, then $m^*(O|E) < m(O|F) < \varepsilon$. Thus, E is measurable

20) Suppose that $m^*((a,b)) = m^*((a,b) \cap E) + m^*((a,b) \sim E)$ is correct for every bound interval (a,b). Let $\epsilon > 0$ since E has finite outer measure there is a countable collection of bounded open intervals such that $\sum_{k=1}^{\infty} m^*(I_k) < m^*(E) + \epsilon$,

Since I_k is open and bounded, $m^* I_k = m^*(I_k \cap E) + m^*(I_k \sim E)$

$$\sum_{k=1}^{\infty} m^*(I_k) = \sum_{k=1}^{\infty} m^*(I_k \cap E) + \sum_{k=1}^{\infty} m^*(I_k \sim E)$$

$$\geq m^*\left(\bigcup_{k=1}^{\infty} (I_k \cap E)\right) + m^*\left(\bigcup_{k=1}^{\infty} I_k \sim E\right).$$ Since $O = \bigcup_{k=1}^{\infty} I_k$

and $E = \bigcup_{k=1}^{\infty} I_k \cap E$ and $\bigcup_{k=1}^{\infty} (I_k \sim E) = O \sim E$. and

$$m^*(E) + \epsilon > \sum_{k=1}^{\infty} m^*(I_k) \geq m^*(E) + m^*(O \sim E)$$

Thus $m^*(O \sim E) < \epsilon$ which is the equivalent to measurability of E.