PSTAT 120B Quiz 2

1) $Y_1, \ldots Y_n \overset{iid}{\sim} \text{Unif}(0, \theta)$

a) $\hat{\theta}_1 = Y_{(n)}$

$F_{Y_{(n)}}(y) = P(Y_{(n)} \leq y)$

$= P(Y_1 \leq y)^n$

$= \left(\int_0^y f_{Y_1}(x)\, dy\right)^n$

$= \left[\int_0^y \frac{1}{\theta}\, dy\right]^n$

Because of this, we know

cdf of $Y_{(n)}$ $\begin{cases} 0, & y \leq 0 \\ \frac{y^n}{\theta^n} & 0 < y < \theta \\ 1 & y \geq \theta \end{cases}$

and thus, we can solve for the pdf, which is: (derivative of $F_{Y_{(n)}}$)

pdf of $Y_{(n)}$ $\begin{cases} \frac{ny^{n-1}}{\theta^n}, & 0 < y < \theta \\ 0 & \text{otherwise} \end{cases}$

$E(Y_{(n)}) = \int_0^\theta y\, f_{Y_{(n)}}(y)\, dy$

$= \int_0^\theta y \frac{ny^{n-1}}{\theta^n}\, dy$

$= \frac{n}{\theta^n}\left[\frac{y^{n+1}}{n+1}\right]_0^\theta$

(Mean)

$E(Y_{(n)}) = \frac{n}{\theta^n}\left[\frac{\theta^{n+1}}{n+1} - \frac{0^{n+1}}{n+1}\right] = \boxed{\frac{n\theta}{n+1}}$

$E(Y_{(n)}^2) = \int_0^\theta y^2 f_{Y_{(n)}}(y)\, dy$

$= \int_0^\theta y^2 \frac{ny^{n-1}}{\theta^n}\, dy$

$= \frac{n}{\theta^n}\left[\frac{y^{n+2}}{n+2}\right]_0^\theta = \frac{n}{\theta^n}\left[\frac{\theta^{n+2}}{n+2} - \frac{0^{n+2}}{n+2}\right]$

$E(Y_{(n)}^2) = \frac{\theta^2 n}{n+2}$

$\text{Var}(Y_{(n)}) = \frac{\theta^2 n}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = \frac{\theta^2 n}{n+2} - \frac{n^2\theta^2}{n^2+2n+1}$

$\text{Var}(Y_{(n)}) \Rightarrow n\theta^2 \frac{(n+1)^2 - n^2 - 2n}{(n+1)^2(n+2)} = \boxed{\frac{n\theta^2}{(n+1)^2(n+2)}}$

Bias: $E(\hat{\theta}_1) - \theta = \frac{\theta n}{n+1} - \theta = \frac{\theta n}{n+1} - \frac{\theta n + \theta}{n+1}$

$\boxed{\text{Bias} = \frac{-\theta}{n+1}}$

b) $MSE(\hat{\theta}_1) = E(\hat{\theta} - \theta)^2 = Var(\hat{\theta}) + (Bias(\hat{\theta}))^2$

$= \frac{n\theta^2}{(n+1)^2(n+2)} + \frac{\theta^2}{(n+1)^2}$

$= \frac{n\theta^2}{(n+1)^2(n+2)} + \frac{n\theta^2 + 2\theta^2}{(n+1)^2(n+2)}$

$= \frac{n\theta^2 + n\theta^2 + 2\theta^2}{(n+2)(n+1)^2} = \frac{2n\theta^2 + 2\theta^2}{(n+2)(n+1)^2}$

$= \frac{2\theta^2(n+1)}{(n+2)(n+1)^2} = \frac{2\theta^2}{(n+2)(n+1)}$

$\boxed{MSE(\hat{\theta}_1) = \frac{2\theta^2}{(n+2)(n+1)}}$

Based on prev problem, we can multiply by $\frac{n+1}{n}$

2) $E(\hat{\theta}_1) = \frac{n\theta}{n+1}$ to make an unbiased estimator, $\hat{\theta}_2$.

$$\hat{\theta}_2 = \frac{n+1}{n} \times \hat{\theta}_1$$

$$E(\hat{\theta}_2) = \frac{n+1}{n} \times E[\hat{\theta}_1]$$

$$= \frac{\cancel{(n+1)}}{\cancel{n}} \times \frac{\cancel{n}}{\cancel{(n+1)}} \theta$$

(mean) $\boxed{E(\hat{\theta}_2) = \theta}$

$$Var(\hat{\theta}_2) = Var\left[\frac{n+1}{n} \hat{\theta}_1\right]$$

$$= \left(\frac{n+1}{n}\right)^2 Var(\hat{\theta}_1)$$

$$= \frac{\cancel{(n+1)^2}}{\cancel{n^2}} \cdot \frac{\cancel{n}\theta^2}{\cancel{(n+1)^2}(n+2)}$$

$$\boxed{Var(\hat{\theta}_2) = \frac{\theta^2}{n(n+2)}}$$

$$MSE(\hat{\theta}_2) = var(\hat{\theta}_2) + (Bias(\hat{\theta}_2))^2$$

Because $\hat{\theta}_2$ is unbiased,
$(Bias(\hat{\theta}_2))^2 = 0$.

Because $MSE(\hat{\theta}_2) = Var(\hat{\theta}_2)$,

$$\boxed{MSE(\hat{\theta}_2) = \frac{\theta^2}{n(n+2)}}$$

Solve when $MSE(\hat{\theta}_1) > MSE(\hat{\theta}_2)$

3) a) $$\frac{2\theta^2}{(n+2)(n+1)} > \frac{\theta^2}{n(n+2)}$$

$$\frac{2\theta^2 n(n+2)}{(n+2)(n+1)} > \theta^2$$

$$\frac{2\theta^2 n}{n+1} > \theta^2$$

$$\frac{\cancel{\theta^2} n}{\cancel{\theta^2}(n+1)} > \frac{1}{2}$$

$$2n > n+1$$

$$\boxed{MSE(\hat{\theta}_1) > MSE(\hat{\theta}_2) \text{ as long as } n > 1}$$

b) Solve when $Var(\hat{\theta}_1) > Var(\hat{\theta}_2)$

$$\frac{n\theta^2}{(n+1)^2(n+2)} > \frac{\theta^2}{n(n+2)}$$

$$\frac{\cancel{\theta^2}\cancel{(n+2)}\, n^2}{(n+1)^2\, \cancel{(n+2)}\cancel{\theta^2}} > 1$$

$$\frac{n^2}{(n+1)^2} > 1$$

$$n^2 > n^2 + 2n + 1$$

This should not be true for any values of n.

$Var(\hat{\theta}_1) < Var(\hat{\theta}_2)$, always

c) $\hat{\theta}_2$ is a more optimal estimator. It is unbiased and has a lower MSE overall. Because of this, it is a more accurate estimator.