Week 2 HW 1

1b) - Let $f(n)$ be a function from $\mathbb{N}$ to $\mathbb{Z}$ where

$f(n) = \frac{n}{2}$ if n is even

$f(n) = \frac{-(n-1)}{2}$ if n is odd

The function is a bijection.

- Let k and n be 2 even natural numbers

$f(k) = k/2$, $f(n) = n/2$

$f(k) = f(n) \Rightarrow k = n$

- Let k and n be 2 odd natural numbers

$f(k) = -\frac{(k-1)}{2}$, $f(n) = -\frac{(n-1)}{2}$

$f(k) = f(n) \Rightarrow k = n$

The function is injective.

- Let q be even, appearing in the function like so

$f(2k) = \frac{2k}{2} = q$, such that $q = k$

For every q in $\mathbb{Z}$ there is a naturally occurring $2k$

- Let q be odd, appearing in the function like so

$f(2k-1) = \frac{(-2k-1-1)}{2} = q$, such that $q = k$

For every q in $\mathbb{Z}$ there exists a $2k-1$

The function is surjective. $\boxtimes$

Prove.

18) $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \ldots \mathbb{N}}_{k \text{ times}}$, Let $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ for $k = 1, 2$

Assume $\mathbb{N}^m = \underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \mathbb{N}}_{m \text{ times}}$ is countably infinite for $m = k$

If A and B are countably infinite, $A \times B$ is countably infinite

For $k = m+1$, $\mathbb{N}^{m+1} = \underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \ldots \times \mathbb{N}}_{m+1 \text{ times}}$

Let $A = \underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \ldots \times \mathbb{N}}_{m \text{ times}}$, $B = \mathbb{N}$

A & B are countably infinite, and $A \times B$ are countably infinite

The same is true for $k = m+1$ when it is true for $k = m$ and $k = 1, 2$

Thus by induction, $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \ldots \times \mathbb{N}}_{k \text{ times}}$ is countably infinite for all $k \in \mathbb{N}$ $\boxtimes$

20) Suppose $f: A \to B$, $g: B \to C$ are one to one

Let $a_1, a_2 \in A$ such that $g \circ f(a_1) = g \circ f(a_2)$

$\Rightarrow g[f(a_1)] = g[f(a_2)]$

$f(a_1) = f(a_2)$ because g is one to one

$\Rightarrow a_1 = a_2$ because f is one to one.

$\Rightarrow$ for every $a_1, a_2 \in A$, $g \circ f(a_1) = g \circ f(a_2) \Rightarrow a_1 = a_2$, $g \circ f: A \to C$ is one to one ✓

Let $c \in C$ & $g: B \to C$ is onto. There exists $b \in B$ such that $g(b) = c$

Let $b \in B$ & $f: A \to B$ is onto. There exists $a \in A$ such that $f(a) = b$

$\Rightarrow c = g(b) \Rightarrow g[f(a)] = g \circ f(a)$. Hence for each $c \in C$, there exists $a \in A$ where $g \circ f(a) = c \Rightarrow g \circ f$ is onto ✓

Assume $f: A \to B$ is onto and one to one. Let $b_1, b_2 \in B$ such that $f^{-1}(b_1) = f^{-1}(b_2) = a$

$b_1 = b_2 = f(a)$, $b_1 = b_2$; Hence $f^{-1}: B \to A$ is one to one ✓

$f: A \to B$ is onto, thus for each $b \in B$ $\exists$ $a \in A$ s.t. $f(a) = b$

for each $a \in A$, $\exists$ $b \in B$ s.t. $a = f^{-1}(b) \Rightarrow f^{-1}: B \to A$ is onto ✓