HW 2

43) Each real sequence has a monotone subsequence.

Let x_n be a real sequence, and a set $S := \{n \mid x_m > x_n \ \forall m \geq n\}$

If S is infinite $\{n_1, n_2, \dots\}$ then the sequence

$x_{n_1} < x_{n_2} < x_{n_3} < \dots$ is a monotone subsequence.

If S is finite, there exists n such that $n < n_1 \ \forall n \in S$.

Since $n_1 \notin S$, then $\exists\, n_1$ such that $x_{n_2} \leq x_{n_1}$

Since $n_2 \notin S$, then $\exists\, n_3 > n_2$ such that $x_{n_3} \leq x_{n_2} \leq x_{n_1}$

This pattern continues, demonstrating a monotone subsequence $\boxtimes$

Let A_n be a bounded sequence. A_n must have a subsequence, for instance, A_{n_k} that is monotone.

$\Rightarrow A_{n_k}$ is bounded

45) (i) Let there exist a series $\sum_{k=1}^{\infty} a_k$. This series is summable if and only if the sequence of partial sums where $S_n = a_1 + a_2 + a_3 + \dots a_n$, is convergent. This case, $\lim_{n \to \infty} S_n$ is considered the sum of the series $\sum_{k=1}^{\infty} a_k$. $\Rightarrow S_n$ is convergent. ($\sum_{k=1}^{\infty} a_k$ is summable)

Given any $\varepsilon > 0$, $\exists$ some $\in \mathbb{N}$ such that $|S_{n+m} - S_n| < \varepsilon$, $\forall n \geq \mathbb{N}$, $\forall m \geq 1$. $S_{n-m} - S_n = a_{n+1} + \dots a_{n+2} + \dots a_{n+3} \dots$

$\sum_{k=1}^{\infty} a_k$ is summable if and only if $\varepsilon > 0$, there exists such that $|a_{n+1} + a_{n+2} + \dots a_{n+m}| < \varepsilon$, $\forall n \geq N$, $\forall m \geq 1$ $\boxed{X}$

(ii) Suppose $\sum_{k=1}^{\infty} |a_k|$ is summable $\therefore$ by (i),

Given any $\varepsilon > 0$, $\exists$ some $N \in \mathbb{N}$ such that

$|a_{n+1}| + |a_{n+2}| \dots + |a_{n+m}| < \varepsilon$, $\forall n \geq N$, $\forall m \geq 1$

$|a_{n+1} + a_{n+2} + a_{n+3} \dots a_{n+m}| \leq |a_{n+1}| + |a_{n+2}| + \dots + |a_{n+m}| \dots$

$\Rightarrow$ Given any $\varepsilon > 0$, $\exists$ some $N \in \mathbb{N}$ such that

$|a_{n+1} + a_{n+2} + \dots a_{n+m}| < \varepsilon$, $\forall n \geq N$, $\forall m \geq 1$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ is summable $\boxed{X}$

(iii) If $a_k \geq 0$, $\forall k \in \mathbb{N}$, $\Rightarrow S_{n+1} - S_n = a_{n+1} \geq 0$, $\forall n \in \mathbb{N}$

Thus $S_n \leq S_{n+1}$, $\forall n \in \mathbb{N} \therefore S_n$ is a monotonically increasing sequence. S_n is convergent iff S_n is bounded above iff $\exists$ some $k > 0$ such that $S_n \leq K$ is bounded above. (iff $\exists$ some $k > 0$ such that $S_n \leq K$, $\forall n \in \mathbb{N}$ iff S_n, is bounded above $\boxed{X}$

46) Let (a_n) be a bounded sequence. Since $\forall m \in \mathbb{N}$, the set $A_m = \{a_n : n \geq m\}$ is bounded, another sequence $b_m = \sup A_m$. $A_{m+1} \subseteq A_m \rightarrow b_{m+1} < b_m$

Thus, b_m is a monotone decreasing bounded sequence and so by monotone convergence theorem, it has a limit b.

Since $b_m \rightarrow b$, we can find $m_1 \in \mathbb{N}$ such that $|b_{m_1} - b| < \frac{1}{2}$. Since $b_{m_1} = \sup A_{m_1}$, $\exists\, n_1 \geq m_1 : b_{m_1} - \frac{1}{2} \leq a_{n_1} \leq b_{m_1} \Rightarrow |a_{n_1} - b| \leq |a_{n_1} - b_{m_1}| + |b_{m_1} - b| < \frac{1}{4}$

$\exists\, n_2 \geq m_2 : b_{m_2} - \frac{1}{4} < a_{n_2} < b_{m_2}$

We can similarly show that $|a_{n_2} - b| < \frac{1}{2}$. we can find $n_1 < n_2 < n_3 \ldots |a_{n_k} - b| < \frac{1}{k}$ through induction. This sequence converges to b.