PH340: Quantum Statistical Field Theory

Centre for Condensed Matter Theory, Physics Department, IISc Bangalore
Semester II, 2019–2020

Examination, 19 May 2020

Instructions

- This question paper will be available on the course web page and will be sent by email to all those who have responded with their phone numbers.
- As soon as you download the paper/or receive the email, kindly reply to the email that you have received the paper and are attempting it. Your return email will be treated as signing of the attendance sheet.
- You may use up to 180 minutes to produce answers. You must time yourself.
- You must adopt the highest standards of academic integrity. You should neither consult the internet, or any of your friends, or not-so-friends.
- Your completed scripts should be scanned in **PDF format**, and emailed to `ph340.qsft@gmail.com` with a copy to `shenoy@iisc.ac.in`. You email must reach these addresses **NO LATER THAN 17:00HRS on 19 May 2020**.
- Please ensure that your script contains your name and roll number.
- You really do not need to refer to any text, notes etc. as most of the questions can be answered with a clear conceptual understanding.
- The grading of your work will be done by you in the regular class meeting on 17:30 on the 19th.
- Note that all questions are not of equal difficulty; some are routine, some not.

EX/1. **"Majorana Boson":** You may have heard a lot about "Majorana fermions" – particularly the statement that a *Majorana fermion is a fermionic particle that is its own antiparticle*! In the problem we will explore the possibility a "Majorana Boson"

(a) Let a be a *bosonic operator*. Define an operator $\Phi = a + a^\dagger$. Free marks: Show that $\Phi^\dagger = \Phi$. This means that Φ is its own antiparticle, and hence a candidate "Majorana boson"!

(b) What are the possible eigenvalues ϕ of Φ that satisfy the equation

$$\Phi|\phi\rangle = \phi|\phi\rangle$$

The space spanned by these eigenstates , i. e., $\text{span}\{|\phi\rangle\}$, is the Hilbert space of the Majorana boson.

(c) Write a(n) (over)completeness relation using the eigenstates $|\phi\rangle$.

(d) Suppose A is the operator acting on span$\{|\phi\rangle\}$, the space spanned by the states $|\phi\rangle$, write an expression for the the trace of A using the (over)completeness relation developed in the previous part.

(e) Consider a one-dimensional chain of Majorana boson system, where sites are dnoted by i, j etc. The Majorana boson operators satisfy

$$[\Phi_i, \Phi_j] = 0$$

Let the Hamiltonian of this system be

$$H = \sum_i V(\Phi_{i+1}, \Phi_i)$$

where V a function (that is as yet unspecified) that determines the interaction between the Majorana boson at site i and the one at site $i+1$. Assuming the system to be N site chain with periodic boundary conditions, obtain a path integral formulation for the partition function of the system at temperature T. Comment on what "classical problem" does this correspond to.

(f) Suppose you were writing a paper summarizing the findings of this exercise. Find an approriate title with not more than ten words, and write an abstract no longer than one hundred words. (The shorter and crisper the abstract, the better!)

EX/2. **Two sites, again!** Consider *spinless* fermions that populate *two sites* labelled 1 and 2. The hamiltonian of the system is

$$H = -t\left(c_1^\dagger c_2 + c_2^\dagger c_1\right) + V c_2^\dagger c_1^\dagger c_1 c_2 - \mu(c_1^\dagger c_1 + c_2^\dagger c_2)$$

(a) What is the dimension of the Hilbert-Fock space of the system? Write out the basis states.

(b) By explicit diagonalization of the Hamiltonian expressed in the above basis, find the exact eigenstates and eigenvalues.

(c) Comment on the physics of the system for $V > 0$ and $V < 0$.

(d) Use the exact eigensystem to compute the free energy of the system at temperature T.

(e) Apply an "external stimulus potential" ϕ at site 1, what is the frequency dependent "density response" (described by operator $n_1 = c_1^\dagger c_1$) at site 1. (Hint: Use Lehman representation.) For the same stimulus, what is the density response at site 2. Does your answer make sense?

(f) Using Grassmann calculus, obtain an expression for the free energy. Your answer must agree with what you found above.

(g) Find a path integral expression for the partition function.

(h) Treating V as a perturbation, set up a diagrammatic expansion for the free energy. Identify all connected diagrams at order 2.

(i) Set up the diagrammatic expansion for the onsite Green's functoin $\mathcal{G}_{11}(\tau) = -\langle T_\tau c_1(\tau) c_1^\dagger(0)\rangle$, and obtain all self energy terms up to order V^2 (it will be simpler to work with $\mathcal{G}_{11}(i\omega_n)$).

EX/3. **Infrared Symmetries** Consider a tight binding chain of spinless fermions

$$H = -t \sum_i \left(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} \right) - \mu \sum_i c_i^\dagger c_i$$

the coordinate of the site i is ia where a is the lattice spacing.

(a) What the symmetries of this system? Is it Galielean invariant?

(b) Consider $\mu = -2t + t/10$. What is the ground state of the system? What is the Fermi momentum k_F?

(c) Now we will consider *low energy* physics and develop a *continuum* (in real space) field theory for this problem. Choose a momentum a cut off momentum $\Lambda \ll k_F$. (Λ could be, for example, set by the maximum temperature that you will wish to investigate.) Consider, the fourier transform

$$c_i = \sum_k e^{ikx_i} c_k$$

and look for a *low energy* expansion keeping only the "modes" close to the Fermi surface. Also, we define a continuum field

$$\begin{aligned}\psi(x) = \frac{1}{\sqrt{a}} c_i &= \sum_{k\in[-\Lambda,+\Lambda]} e^{i(-k_F+k)x} c_{-k_F+k} + \frac{1}{\sqrt{a}} \sum_{k\in[-\Lambda,\Lambda]} e^{i(k_F+k)x} c_{k_F+k} \\ &= e^{-ik_F x}\psi_L(x) + e^{ik_F x}\psi_R(x)\end{aligned}$$

where $\psi_L(x)$ is the "left moving" field ad $\psi_L(x)$ is the right moving field. This last equation can be written even nicely

$$\psi(x) = \sum_{\nu=\pm 1} e^{i\nu k_F x}\psi_\nu(x)$$

where $\nu = -1$ is the left mover, and $\nu = 1$ is the right mover. Find an expression for the "continuum Hamiltonian"

$$H = \int \mathrm{d}x \; (.....) \,,$$

you have to find (.....).

(d) Write down a path integral for the parition function in terms of Grassman fields $\psi_\nu(x,\tau)$, and obtain the "classical action" of this system.

(e) Show that the system has an emergent Lorentz invariance (!); what is the speed of light?