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The minimum feedback arc set problem and the acyclic disconnection for graphs[☆]

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ABSTRACT

A minimum feedback arc set of a digraph D is a minimum set of arcs which removal leaves the resultant graph free of directed cycles; its cardinality is denoted by $\tau_1(D)$. The acyclic disconnection of D, $\overrightarrow{\omega}(D)$, is defined as the maximum number of colors in a vertex coloring of D such that every directed cycle of D contains at least one monochromatic arc. In this article we study the relationship between the minimum feedback arc set and the acyclic disconnection of a digraph, we prove that the acyclic disconnection problem is $\mathcal{NP}$-complete. We define the acyclic disconnection and the minimum feedback for graphs. We also prove that $\overrightarrow{\omega}(G) + \tau_1(G) = |V(G)|$ if G is a wheel, a grid or an outerplanar graph.

1. Introduction and primary results

A *minimum feedback arc set* of a digraph D is a minimum set of arcs which, if removed, leaves the resultant graph free of directed cycles; its cardinality is denoted by $MFAS(D)$ or $\tau_1(D)$. Finding a minimum feedback arc set is classical problem in combinatorial optimization [1,3–5,7,8]. Due to its many applications, efficient heuristics have been proposed to solve this problem, cf. [3,5]. However, it is not an easy task to calculate $\tau_1(D)$, the minimum feedback arc set problem is one of Karp's original 21 $\mathcal{NP}$-complete problems, and it remains $\mathcal{NP}$-hard even for tournaments and bipartite tournaments [1,4].

The *acyclic disconnection*, $\overrightarrow{\omega}(D)$, is defined as the maximum number of colors in a vertex coloring of D for which every directed cycle of D contains at least one monochromatic arc. The definition and first results about the behavior of this parameter in general digraphs were introduced by Neumann-Lara in [12]. It is worth noticing that this parameter is related with the dichromatic number, also introduced by Neumann-Lara in [11] and independently by Mohar in [10], in the following way. While for the dichromatic number we look for colorings such that every chromatic class is acyclic, for the acyclic disconnection we look for colorings such that the set of arcs not contained in a chromatic class induces an acyclic digraph.

The acyclic disconnection has been studied for special families of digraphs, such as tournaments [9,12] and multipartite tournaments [6]. There are not many articles relating the acyclic disconnection with other parameters, one example is [2], where its relation with the directed girth is explored. Although the acyclic disconnection was introduced in 1999, the computational complexity of the associated decision problem was unknown until now. In Section 3 we prove that the acyclic disconnection problem is $\mathcal{NP}$-complete. In Section 4 we discuss ways to relate $\overrightarrow{\omega}(D)$ with $\tau_1(D)$.

We can define these two parameters for graphs as follows: Let G be a graph and $\mathcal{O}(D)$ the set of its orientations. Define $\tau_1(G)$ and $\overrightarrow{\omega}(G)$ as $\tau_1(G) = \max_{D\in\mathcal{O}(D)}\tau_1(D)$ and $\overrightarrow{\omega}(D) = \min_{D\in\mathcal{O}(D)}\overrightarrow{\omega}(D)$. In general there is not an obvious relation between

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these two parameters, but, in Sections 5 and 6, we prove that $\tau_1(G) + \overrightarrow{\omega}(G) = |V(G)|$ for three families of planar graphs: wheels, grids and outerplanar graphs.

2. Notation and preliminaries

We use the notation of [3]. In particular, $(X_1, X_2)_D$ denotes the set of arcs with tail in X_1 and head in X_2. The number of weak components of a digraph D (the number of connected components of the underlying graph) is denoted by $\omega(D)$. If D_1 and D_2 are digraphs, $D_1 \Box D_2$ denotes the usual cartesian product of digraphs. If u is a vertex of D_1 then D_2^u is the u-fiber of $D_1 \Box D_2$; analogously for v in $V(D_2)$.

Let Γ_s denote the set of colors $\{1, 2, \ldots, s\}$. If D is a digraph and $\varphi : V(D) \to \Gamma_s$ is a vertex coloring of D, then the coloring φ induces two natural spanning subdigraphs of D. The *monochromatic digraph* of D, $M_\varphi(D)$, is the spanning subdigraph of D with arc set $\{uv \in A(D) : \varphi(u) = \varphi(v)\}$ and the *heterochromatic digraph* of D, $H_\varphi(D)$, the spanning subdigraph of D with arc set $\{uv \in A(D) : \varphi(u) \neq \varphi(v)\}$. We say that a coloring $\varphi : V(D) \to \Gamma_s$ is *externally acyclic* if $H_\varphi(D)$ is an acyclic digraph. The following observation follows from Proposition 5 of [6] and the definition of the acyclic disconnection of D.

Proposition 1. *Let D be a digraph. Then $\overrightarrow{\omega}(D) \geq s$ if and only if there is a proper coloring φ of D with s colors such that $H_\gamma(D)$ is acyclic, i.e., φ is an externally acyclic coloring of D.*

It is well known that D is acyclic if and only if there exists an ordering of its vertex set $V(D) = \{v_0, v_1, \ldots, v_n\}$ such that v_0 is a source of D and for every $1 \leq k < n$, v_k is a source of $D \setminus \{v_0, v_1, \ldots, v_{k-1}\}$. Such an order is called a *source ordering* of D.

3. $\mathcal{NP}$-completeness

In this section we prove that the acyclic disconnection problem is $\mathcal{NP}$-hard by reduction from the minimum feedback arc set problem. For this purpose we state the following two problems.

Minimum Feedback Arc Set (FAS)
Instance: A digraph D and a nonnegative integer k.
Question: Is $\tau_1(D) \leq k$?

Acyclic Disconnection (AD)
Instance: A digraph D and a nonnegative integer s.
Question: Is $\overrightarrow{\omega}(D) \geq s$?

Theorem 2. *The acyclic disconnection problem is $\mathcal{NP}$-complete, even for bipartite oriented graphs.*

Proof. Problem AD is in $\mathcal{NP}$ since a short certificate for a digraph D with a coloring φ with s colors is the digraph $H_\varphi(D)$. We can verify in polynomial time whether this digraph is acyclic, which implies by Proposition 1 that $\overrightarrow{\omega}(D) \geq s$.

To show completeness, we present a polynomial reduction from FAS. Given an instance (D, k) of FAS, we construct, in polynomial time, an instance (D', s), with $s = n - k$, of AD such that $\tau_1(D) \leq k$ if and only if $\overrightarrow{\omega}(D') \geq s$.

Let D be a digraph with $V(D) = \{v_1, v_2, \ldots, v_n\}$. Without loss of generality, we can assume that D is a tournament, [4]. Let D' be the digraph obtained by subdividing every arc of D. That is, $V(D') = V(D) \cup \{v_{xy} : xy \in A(D)\}$ and we replace every arc xy of D with the path $xv_{xy}y$.

Given an optimal externally acyclic coloring of D', $\varphi : V(D') \longrightarrow \Gamma_s$, we construct a feedback arc set S of D with $n - s$ arcs in order to show that $\tau_1(D) \leq n - s = k$.

If a monochromatic set X of an externally acyclic coloring induces a disconnected graph, then one of the connected components of $D'[X]$ can be colored with a new color without losing the property of being externally acyclic. Since φ is an optimal externally acyclic coloring, every subdigraph induced by a monochromatic set of $V(D')$ is connected.

Suppose, for a contradiction, that there exists a monochromatic path $xv_{xy}y$ in D'. Define $N = \{v \in N^-_{D'}[y] \mid \varphi(v) = \varphi(y)\}$. Then, we can define a new coloring $\sigma : V(D') \longrightarrow \Gamma_{s+1}$ such that

$$\sigma(u) = \begin{cases} s+1 & \text{if } u \in N \\ \varphi(u) & \text{otherwise.} \end{cases}$$

Observe that σ is onto because $\sigma(x) = \varphi(y)$ and $x \notin N$. The new coloring σ cannot be externally acyclic because φ is optimal. Therefore, there exists a directed cycle C of $H_\sigma(D')$ which is not a cycle of $H_\varphi(D')$. Then, there exists an arc a of C such that $a \in A(M_\varphi(D')) \cap A(H_\sigma(D'))$, thus $a = wv_{wy}$, for some $w \in N^-_D(y)$, because these are the only arcs with that property. So, $v_{wy}y \in A(M_\sigma(D')) \cap A(C)$ because y is the only out-neighbor of v_{wy}, which contradicts that C is a cycle of $H_\sigma(D')$. Hence, we conclude that the three vertices of a subdivision of an arc of D use at least two colors in D'.

It follows that every nontrivial subdigraph induced by a monochromatic class of φ is a star with a vertex of D as the center. Let S be the set of arcs of D whose subdivision has a monochromatic arc of D' with the coloring φ. Since the subdivision of every directed cycle of D is a directed cycle of D', every directed cycle of D has an arc in S and, therefore S is a feedback arc set of D. Observe that $|V(D')|$ and $|A(D')|$ are linear in terms of $|V(D)|$ and $|A(D)|$.

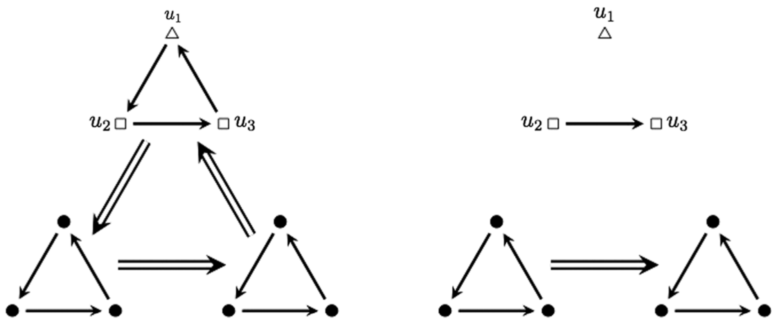

Fig. 1. An optimal coloring of T_9 and the feedback arc set induced by the coloring.

It only remains to prove that S has at most k arcs. As we proved, $M_\varphi(D')$ has s connected components and each component is a star (possibly K_1). Therefore, the number of monochromatic arcs is at most $n-s$. Since the three vertices of a subdivision of an arc of D use at least two colors in D', there is a bijective correspondence between the arcs in S and the monochromatic arcs of D'. Thus, S has at most $n-s=k$ arcs.

Conversely, if D has a minimum feedback arc set S with k arcs, we define an externally acyclic coloring, φ of D' in the following way. Assign different colors to every vertex in $V(D)$. For every arc $xy \in S$, define $\varphi(v_{xy}) = \varphi(x)$. Use a new color for each of the remaining vertices.

Recall that every directed cycle of D' is a subdivision of a directed cycle of D. Hence, since every directed cycle of D contains an arc in S, then every directed cycle of D' contains a subdivision of an arc in S, which is monochromatic by the definition of φ. Therefore, φ is an externally acyclic coloring using $n-k$ colors. Thus, $\overrightarrow{\omega}(D') \geq n-k = s$. $\square$

4. Feedback arc set vs acyclic disconnection

It is straightforward to notice that $\overrightarrow{\omega}(\overleftrightarrow{G}) = \omega(G)$ where $\overleftrightarrow{G}$ is the digraph obtained from G by replacing each edge uv by two arcs uv and vu. And also that every externally acyclic coloring of D is an externally acyclic coloring of the converse digraph D^{op}.

It is important to stress that, while the minimum feedback arc set minimizes the cardinality of the arcs in a cycle transversal, the acyclic disconnection maximizes the number of weak components of the subdigraph generated by the arcs in a cycle transversal.

If S is a feedback arc set of a digraph D, we can define the *S-coloring of D*, $\varphi(S)$, as a vertex coloring of D in which the monochromatic classes are the weak connected components of the subdigraph of D generated by the arcs of S.

Remark 3. For every feedback arc set S of a digraph D, the S-coloring of D is an externally acyclic coloring of D. And $S \subseteq A(M_{\varphi(S)}(D))$.

This remark follows from the fact that $H_{\varphi(S)}(D)$ is a subdigraph of $D - A(S)$. The following example shows that this S-coloring may not be optimal.

Example 4. Consider the Paley tournament $D_7 = \overrightarrow{C}_7(1,2,4)$. Notice that $S = \{(6,0),(6,1),(6,3),(5,0),(5,2),(4,1),(3,0)\}$ is a feedback arc set of D_7. Since D_7 has 7 edge disjoint triangles, S is a minimum feedback arc set. The S-coloring has only one color, though it is well known that $\overrightarrow{\omega}(D_7) = 2$, [12]. Therefore, the S-coloring is not optimal.

Remark 5. For every externally acyclic coloring $\varphi(D)$, $A(M_\varphi(D))$ is a feedback arc set of D.

Consider the tournament T_9 depicted in Fig. 1, where the double arcs mean that every arc in the corresponding direction is present. It is not difficult to prove that the vertex coloring φ of T_9 defined as follows: $\varphi(u_1) = r$, $\varphi(u_2) = \varphi(u_3) = b$, and the rest of the vertices have color g, is an optimal coloring. Notice that $|A(M_\varphi(T_9))| = 16$ while the minimum feedback arc set has only 12 arcs.

Therefore, related to φ, we can define a better feedback arc set of D by applying the following dismantling process.

Let φ be an externally acyclic coloring of a digraph D and $F \subseteq A(M_\varphi(D))$. We say that $a \in F$ is *essential for F* if there exist a directed cycle C of D such that $A(C) \cap F = a$. Let $A_0 = A(M_\varphi(D))$ and F_0 be the set of essential arcs for A_0. If $A_0 = F_0$ we stop the process. Else, let $a_0 \in A_0 \setminus F_0$ and $A_1 = A_0 \setminus \{a_0\}$. Define F_1 as the set of essential arcs for A_1. If $A_1 = F_1$ we stop the process. Eventually, $A_n = F_n$ for some nonnegative integer n because A_0 is a finite set. In this case, we say that F_n is a *dismantling set* of A_0.

Observe that every dismantling set of $A(M_\varphi(T_9))$ of Fig. 1 is a minimum feedback arc set of T_9.

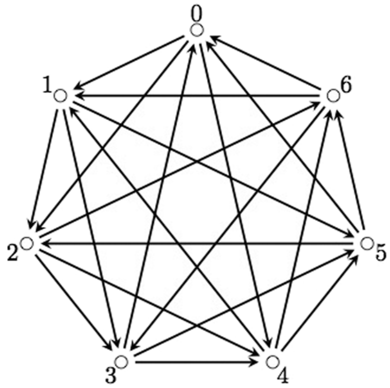

Fig. 2. The Payley tournament $C_7(1, 2, 4)$.

In general, it is not true that with the dismantling process we obtain a minimum feedback arc set. To show this, consider the Paley tournament on 7 vertices depicted in Fig. 2 and the optimal externally acyclic coloring

$$\varphi(x) = \begin{cases} r & \text{if} \quad x = 0 \\ b & \text{if} \quad x \neq 0. \end{cases}$$

We start a dismantling process by taking $A_0 = A(M_\varphi(D_7))$ and $F_0 = \{(1, 3), (1, 5), (2, 3), (2, 6), (4, 5), (4, 6)\}$. Notice that $(1, 2, 4, 1)$ and $(3, 5, 6, 3)$ are the only cycles without essential edges. Therefore, the dismantling process must add two more arcs to the set of essential arcs. Thus, every dismantling set have cardinality at least 8, even though, every minimum feedback arc set of D_7 has cardinality 7 as noticed in Example 4.

Proposition 6. *Let φ be an optimal externally acyclic coloring of a digraph D and let F be a dismantling set of $A(M\varphi(D))$. If the underlying graph of $D[F]$ is acyclic, then F is a minimum feedback arc set.*

Proof. Let $A_0 = A(M_\varphi(T))$, and $F = A_n$ be the dismantling set of A_0 for some nonnegative integer n. By definition, $A_{i+1} = A_i \setminus \{a_i\}$ for some non-essential arc a_i for A_i, hence, every directed cycle with the arc a_i has another arc in A_i for every $0 \leq i \leq n-1$. So, if the set A_i is a transversal of the directed cycles of D, then A_{i+1} is also a transversal of the directed cycles of D for every $0 \leq i \leq n-1$. From Remark 5, $A_0 = A(M_\varphi(T))$ is a transversal of the directed cycles of D, thus $A_n = F$ is a feedback arc set of D.

Now we show that F is minimum. Let S be a minimum feedback arc set, and suppose that $\overrightarrow{\omega}(D) = s$. By Remark 3, $M_{\varphi(S)}(D)$ has at least s connected components. Since $S \subseteq A(M_{\varphi(S)}(D))$, $|S| \geq |V(D)| - s$. On the other hand, the underlying graph of $D[F]$ is acyclic and $F \subseteq A(M_{\varphi(S)}(D))$, then $|F| \leq |V(D)| - s$, therefore F is minimum. □

5. Wheels and grids

In this section we will prove that a nice relationship exists between $\overrightarrow{\omega}$ and τ_1 for grid and wheel graphs. We begin with the analysis of wheel graphs. As usual, W_n will denote the wheel with n spokes.

Theorem 7. *If $n \geq 3$ is an integer, then*

$$\tau_1(W_n) + \overrightarrow{\omega}(W_n) = |V(W_n)|.$$

Proof. We begin the proof with the following claim.

Claim 1. *Let D be any orientation of W_n, then*

$$\tau_1(D) \leq \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad \overrightarrow{\omega}(D) \geq \left\lceil \frac{n}{2} \right\rceil + 1.$$

Proof of Claim 1. Let W_n be the join of $C_n = (x_1, \dots, x_n, x_1)$ and the vertex x. Assume without loss of generality that $d^-(x) \leq \lfloor \frac{n}{2} \rfloor$.

If C_n is not a directed cycle in D, then every cycle of D must contain the vertex x. Let $F = (N^-(x), x)_D$ which is a feedback arc sets of D. Hence, $\tau_1(D) \leq \lfloor \frac{n}{2} \rfloor$. For the second statement, consider the F-coloring of D. Such coloring is an externally acyclic coloring with at least $\lceil \frac{n}{2} \rceil + 1$ colors. Hence, $\overrightarrow{\omega}(D) \geq \lceil \frac{n}{2} \rceil + 1$.

If C_n is a directed cycle of D. Let F be the set of arcs (u, v) of C_n, such that $v \in N^-(x)$. Clearly F is a feedback arc set of D and $|F| \le d^-(x)$, thus, $\tau_1(D) \le \lfloor \frac{n}{2} \rfloor$. Again, the F-coloring in D uses at least $\lceil \frac{n}{2} \rceil + 1$ colors. Thus, $\overrightarrow{\omega}(D) \ge \lceil \frac{n}{2} \rceil + 1$. $\square$

Now, let D be the orientation of W_n such that every interior face is a directed triangle, if n is even; or every interior face but one is a directed triangle, if n is odd. It is direct to verify that $\tau_1(D) = \lfloor \frac{n}{2} \rfloor$ and $\overrightarrow{\omega}(D) = \lceil \frac{n}{2} \rceil + 1$.

It follows from Claim 1 that $\tau_1(W_n) = \lfloor \frac{n}{2} \rfloor$ and $\overrightarrow{\omega}(W_n) = \lceil \frac{n}{2} \rceil + 1$. Therefore $\tau_1(W_n) + \overrightarrow{\omega}(W_n) = n + 1 = |V(W_n)|$. $\square$

The analogous result for grids is a consequence of the following more general theorem.

Theorem 8. *Let $n, m \ge 2$ be integers, let T_n and T_m be trees on n and m vertices, respectively, and let D be an orientation of $G = T_n \square T_m$, then*

$$\overrightarrow{\omega}(D) \ge \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor \text{ and } \tau_1(D) \le mn - \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor.$$

Proof. We give a coloring φ of $V(D)$ with $\lfloor \frac{mn+m+n-1}{2} \rfloor$ colors and prove by induction on $m + n$ that it is externally acyclic. For the base case $(m + n = 4)$ it suffices to show that D admits an externally acyclic coloring with $\lceil \frac{3m}{2} \rceil$ colors. Let $V(P_2)$ be defined as $V(P_2) = \{u, v\}$ and assume without loss of generality that $d^+(T_m^u) \ge \lceil \frac{m}{2} \rceil$. Define the coloring φ of D as follows: assign the same color to both ends of each arc in $(T_m^v, T_m^u)_D$ (using different colors for distinct arcs). Assign a new color to every other vertex in D. The coloring φ uses $m + d^+(T_m^u) \ge \lceil \frac{3m}{2} \rceil$ colors and it is externally acyclic, because every cycle of D must have an arc in $(T_m^v, T_m^u)_D$. Therefore, $\overrightarrow{\omega}(D) \ge \lceil \frac{3m}{2} \rceil$. Since $A(M_\varphi(D))$ is a feedback arc set and $|A(M_\varphi(D))| = |(T_m^v, T_m^u)_D| \le \lfloor \frac{m}{2} \rfloor$, the result follows.

Now assume $n, m > 2$. Let u be a leaf of T_n with support vertex v and $T_{n-1} = T_n - u$. Let D' be the subdigraph of D induced by $V(T_{n-1} \square T_m)$. By induction hypothesis there exists an externally acyclic coloring φ' of D' with $\lfloor \frac{m(n-1)+(n-1)+m-1}{2} \rfloor$ colors. Assume without loss of generality that $d^+(T_m^u) \ge \lceil \frac{m}{2} \rceil$. Define $\varphi : V(D) \longrightarrow \Gamma_s$ as follows:

- Color the vertices of D' with φ',
- $\varphi((x, u)) = \varphi'((x, v))$ if $(x, v) \in N^-(T_m^u)$,
- add a new color for every other vertex in T_m^u.

Notice that φ uses s colors where $s = \lfloor \frac{m(n-1)+(n-1)+m-1}{2} \rfloor + \lceil \frac{m}{2} \rceil = \lfloor \frac{mn+m+n-1}{2} \rfloor$.

Finally, the coloring φ is externally acyclic because every cycle of D, which is not a cycle of D', must have an arc in $(T_m^v, T_m^u)_D$.

The proof of $\tau_1(D) \le mn - \lfloor \frac{mn+m+n-1}{2} \rfloor$ is analogous. Proceed by induction to construct a feedback arc set of $T_n \square T_m$ as the union of the feedback arc set of $T_{n-1} \square T_m$ and the edge set $M = \{(x, u)(x, v) \mid (x, v) \in N^-(T_m^u)\}$. Since $|M| \le \lfloor \frac{m}{2} \rfloor$, the result follows. $\square$

Corollary 9. *If D is a strongly connected orientation of a grid $G = T_n \square T_m$, then*

$$\left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor \le \overrightarrow{\omega}(G) \text{ and } \tau_1(G) \le mn - \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor.$$

Let G be a planar graph and D be an orientation of G. We say that D is a *whirlpool orientation* of G if there exists an embedding of D in which every internal face is bounded by a directed cycle.

Let $P_n = 0, 1, \dots, n-1$ and $P_m = 0, 1, \dots, m-1$ be paths of length at least one. Let $D^{@}_{nm}$ be the digraph defined by the following orientation of the grid $G = P_n \square P_m$.

$$\text{Orient} \begin{cases} (i,j)(i,j+1) \text{ as: } \begin{cases} ((i,j),(i,j+1)) & \text{if } i \overset{2}{\equiv} j \\ ((i,j+1),(i,j)) & \text{otherwise.} \end{cases} \\ (i,j)(i+1,j) \text{ as: } \begin{cases} ((i+1,j),(i,j)) & \text{if } i \overset{2}{\equiv} j \\ ((i,j),(i+1,j)) & \text{otherwise.} \end{cases} \end{cases}$$

It is clear that $D^{@}_{nm}$ is a whirlpool orientation of the grid. We will prove that this orientation achieves the bounds of Corollary 9 using the following observations.

Let Φ be the set of all externally acyclic colorings of a digraph D, for $\phi \in \Phi$ we define the *chromatic excess* of ϕ, $\rho(\phi)$, as the sum $\sum_{i=1}^{t}(c_i - 1)$, where $c_i = |\{x : \phi(x) = i\}|$.

Remark 10. For any digraph D, $\overrightarrow{\omega}(D) = max_{\varphi \in \Phi}\{|V(D)| - \rho(\varphi)\}$. Hence, if $\varphi \in \Phi$ has $\overrightarrow{\omega}(D)$ chromatic classes, then $|V(D)| - \rho(\varphi)$.

Lemma 11. *Let $D^{@}_{nm}$ be the whirpool orientation of the grid G_{nm}, if φ is an optimal externally acyclic coloring of $D^{@}_{nm}$. Then,*

$$\overrightarrow{\omega}(D^{@}_{nm}) \leq \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor \text{ and}$$

$$\tau_1(D^{@}_{nm}) \geq mn - \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor .$$

Proof. Let φ be an optimal externally acyclic coloring of $D^{@}_{nm}$ with k colors. Let s_i, $0 \leq i \leq \overrightarrow{\omega}(D^{@}_{nm})$ be the number of internal faces of $D^{@}_{nm}$ having a monochromatic arc of color i. Notice that every internal face of $D^{@}_{nm}$ induce a directed cycle, so every internal face of $D^{@}_{nm}$ must have at least one monochromatic arc. Since $D^{@}_{nm}$ has $(n-1)(m-1)$ internal faces,

$$\sum_{i=1}^{k} s_i \geq (n-1)(m-1).$$

Let $C_1, C_2, \ldots, C_k$ be the chromatic classes of W_{nm} and $c_i = |C_i|$ for every $1 \leq i \leq k$. For each $1 \leq i \leq k$, it is straightforward to prove by induction on c_i, that $s_i \leq 2(c_i - 1)$. Therefore, if $\rho(\varphi)$ is the chromatic excess,

$$2\rho(\varphi) = 2\sum_{i=1}^{k} 2(c_i-1) \geq \sum_{i=1}^{k} s_i \geq (m-1)(n-1)$$

which implies that $\rho(\varphi) \geq \frac{mn-m-n+1}{2} \geq mn - \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor$. By Remark 10, the bound of the acyclic disconnection holds. Analogously, every internal face needs to be covered by at least one edge of any feedback arc set. Since every edge covers at most two internal faces, the minimum feedback arc set has at least $\frac{(m-1)(n-1)}{2}$ edges, which means that

$$\tau_1(D^{@}_{nm}) \geq mn - \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor . \qquad \square$$

The following result is a direct consequence of Theorem 8 and the previous lemma.

Corollary 12. *Let G_{nm} be a grid and $D^{@}_{nm}$ its whirlpool orientation. Then,*

$$\overrightarrow{\omega}(D^{@}_{nm}) + \tau_1(D^{@}_{nm}) = n \quad \text{and} \quad \overrightarrow{\omega}(G_{nm}) + \tau_1(G_{nm}) = n.$$

6. Outerplanar graphs

In this section we prove that $\overrightarrow{\omega}(G) + \tau_1(G) = |V(G)|$ for a larger family of planar graphs: outerplanar graphs.

It is not difficult to see that if G is a graph and D is an orientation of G, every minimum feedback arc set of D is a union of minimum feedback arc sets of the corresponding orientation of every block of G. It also happens that every optimal externally acyclic coloring of D can be built from optimal external acyclic colorings of the orientations of the blocks of G.

Let $\beta'(G)$ be the edge covering number of a graph G.

Lemma 13. *Let G be an outerplanar 2-connected graph, and let T be the interior dual of G. If $|V(T)| = 1$, then $\tau_1(G) = 1$. Otherwise,*

$$\tau_1(G) = \beta'(T).$$

Proof. Recall that G is 2-connected, hence T is a tree.

If $|V(T)| = 1$, then T is an isolated vertex and hence G is a cycle. Clearly $\tau_1(D) \leq 1$ for any orientation D of G.

So, we will suppose that $|V(T)| = 2$ and proceed by induction on $|V(T)|$ to prove that, for every orientation D of G, $\tau_1(D) \leq \beta'(T)$. Consider an orientation D of G. We assume that D is strong, otherwise D has at most one directed cycle and $\tau_1(D) \leq 1$. Hence, D is either a directed cycle with a diagonal, or it consists of two directed cycles sharing a single arc. In either case it is direct to verify that $\tau_1(D) = 1 = \beta'(T)$.

Suppose that $|V(T)| = n > 2$ and let D be an orientation of G. Consider a longest path $(v_0, \ldots, v_k)$ of T. Since T is a tree, v_0 is a leaf and at most one neighbor of v_1 is not a leaf. Moreover, v_0 and v_1 correspond to cycles C_0 and C_1 in G, respectively, sharing the edge xy in G. Let $C_1 = P \cup (y, x)$, where P is an xy-path in G. If G' is the graph obtained from G by deleting the interior vertices of P, then G' is a 2-connected outerplanar graph and the interior dual of G' is $T' = T - v_0$. Thus, if D' is the subdigraph of D induced by the vertices of G', then, by induction hypothesis, $\tau_1(D') \leq \beta'(T')$.

There are two cases. If $\beta'(T) = \beta'(T') + 1$, then consider F, a minimum feedback arc set of D', and add an arc from the orientation of P to F. This is a (not necessarily minimum) feedback arc set of D with $\tau_1(D') + 1 \leq \beta'(T') + 1 = \beta'(T)$ arcs. Therefore, $\tau_1(D) \leq \beta'(T)$.

Otherwise, $\beta'(T) = \beta'(T')$. Observe that this case cannot occur unless $d(v_1) = 2$. Thus, v_1 is a leaf in T' and, analogously, there exists a cycle C_2 in G, corresponding to v_2, such that C_1 and C_2 share the edge wz. Let Q_1 and Q_2 be an xw-path and a zy-path in G, respectively, such that $C_1 = Q_1 \cup (w, z) \cup Q_2 \cup (y, x)$. Note that Q_1 or Q_2 may be a single vertex.

Let G'' be the graph obtained from G' by deleting the vertices in $Q_1 \cup Q_2$ other than w and z. Again, G'' is a 2-connected outerplanar graph with interior dual $T'' = T' - v_1$. Since $\beta'(T) = \beta'(T')$, we have that $\beta'(T) = \beta'(T'') + 1$. If D'' is the subdigraph of D induced by $V(G'')$, then by induction hypothesis we have $\tau_1(D'') \leq \beta'(T'')$. Let F be a minimum feedback arc set of G'' and assume without loss of generality that xy is oriented as (x, y).

As a first case, suppose that $D[V(C_1)]$ and $D[V(C_2)]$ are directed cycles of D. It is easy to observe that $F \cup \{(x, y)\}$ is a feedback arc set of G.

For the second case, suppose that $D[V(C_1)]$ is a directed cycle but $D[V(C_2)]$ is not. If a is any arc in $D[V(C_1)]$ different from (x, y), then it is direct to verify that $F \cup \{a\}$ is a feedback arc set of G.

Finally, suppose that $D[V(C_1)]$ is not a directed cycle. Observe that C_2 is a cycle of length at least 3, and it shares one edge with C_1 and one edge with C_2. Hence, there is at least one edge that C_1 shares with the exterior face of G. Let a be the orientation of such arc in D. It is not hard to observe that $F \cup \{a\}$ is a feedback arc set of G.

Since in every case we obtained a feedback arc set of G with $\tau_1(D'') + 1$ arcs, and $\tau_1(D'') + 1 \leq \beta'(T'') + 1 = \beta'(T)$, the first inequality follows by induction.

For the second inequality, let D be an orientation of G such that every interior face is a directed cycle. Clearly, the removal of one arc of D can break at most two of such (adjacent) cycles. Hence, at least $\beta'(T)$ arcs are needed to break all the directed cycles corresponding to the interior faces of G. Therefore, $\tau_1(D) \geq \beta'(T)$. □

An argument analogous to the one used to prove Lemma 13 can be used to prove a lower bound for the acyclic disconnection of any orientation of a 2-connected outerplanar graph. Due to its similarity, we will only sketch the proof of the following lemma.

Lemma 14. *Let G be an outerplanar 2-connected graph, and let T be the interior dual of G. If $|V(T)| = 1$, then $\overrightarrow{\omega}(G) = n - 1$. Otherwise,*

$$\overrightarrow{\omega}(G) = |V(G)| - \beta'(T).$$

Sketch of proof. As in the previous lemma, we begin proving by induction on $|V(T)|$ that, if $|V(T)| = 1$, then $\overrightarrow{\omega}(D) \geq n - 1$ for every orientation D of G. Otherwise, for every orientation D of G, $\overrightarrow{\omega}(D) \geq |V(G)| - \beta'(T)$.

If $|V(T)| = 1$, then G is a cycle, which is either directed in D, and $\overrightarrow{\omega}(D) = n - 1$, or not, and $\overrightarrow{\omega}(D) = n$. If $|V(T)| \geq 2$, we consider a longest path $(v_0, \ldots, v_k)$ in T, and remove from G the vertices of the cycle C_0 corresponding to v_0, except for the arc a, which C_0 shares with C_1, the cycle corresponding to v_1 in G. If G' is the resulting graph, then the interior dual of G' is $T' = T - v_0$. Also, G' is a 2-connected outerplanar graph. Let D' be defined as $D' = D[V(G')]$. Hence, by induction hypothesis, $\overrightarrow{\omega}(D') \geq |V(G')| - \beta'(T')$.

Suppose that $\beta'(T) = \beta'(T') + 1$. Consider a $\overrightarrow{\omega}(D')$-externally acyclic coloring of D', and extend it to D by using the same color for both extremes of an arc in C_0, other than a, and assigning new colors to each of the remaining vertices. It is direct to verify that such coloring is externally acyclic, since no directed cycle in D' is properly colored, and any possible new directed cycle will not be properly colored because one arc of C_0 is not properly colored. Since $\overrightarrow{\omega}(D') \geq |V(G')| - \beta'(T')$, and the proposed coloring repeats precisely one color for the remaining vertices of D, we conclude that $\overrightarrow{\omega}(D) \geq \overrightarrow{\omega}(D') + |V(G)| - |V(G')| - 1 \geq |V(G)| - (\beta'(T') + 1) = |V(G)| - \beta'(T)$.

The remaining case is dealt analogously.

To conclude, observe that the orientation D proposed in the proof of Lemma 13 has acyclic disconnection equal to $|V(G)| - \beta'(T)$. □

Theorem 15. *If G is an outerplanar graph, then*

$$\overrightarrow{\omega}(G) + \tau_1(G) = |V(G)|.$$

Proof. Since both the acyclic disconnection and the size of a minimum feedback arc set are additive with respect to the blocks of the graph, it suffices to prove the result for 2-connected graphs. Now, the result follows directly from Lemmas 13 and 14. □

7. Conclusions

Although the acyclic disconnection of a digraph was introduced in 1999, it has received little attention in the past 15 years. Probably this phenomenon can be attributed to the lack of results relating it to more classical parameters of directed graphs. Nonetheless, the present work succeeds connecting this parameter with the size of a minimum feedback arc set.

Even though, as we mentioned earlier, these two parameters are essentially different, but both of them are closely related to the cyclic structure of a digraph. Moreover, the differences between them seem to dilute when we restrict ourselves to the family of planar graphs (where the cyclic structure is better known). In this context we propose the following problem.

Problem 1. Characterize the graphs G such that, $\overrightarrow{\omega}(G) + \tau_1(G) = |V(G)|$.

Based on the results obtained in the present work, it is natural to ask whether the equality in Problem 1 holds for every planar graph.

Notice that the three families (wheels, grids, outerplanar graphs) analyzed here received a similar treatment. First, we proved an upper (lower) bound for $\tau_1(G)$ $(\overrightarrow{\omega}(G))$, for every G in the corresponding family, and then exhibited an orientation of G reaching the given bound. The reader may have noticed that we used the same orientation for both parameters, i.e., we found an orientation D of every graph G in our families such that $\overrightarrow{\omega}(D) + \tau_1(D) = |V(D)|$. Again, this is not true for every oriented graph, so it is natural to propose the following problem.

Problem 2. Characterize the oriented graphs D such that, $\overrightarrow{\omega}(D) + \tau_1(D) = |V(D)|$.

Every orientation we considered of every graph from the families analyzed in this work has the above property, nonetheless, we were unable to find a general proof for this fact. From the evidence collected while working on this subject, we think that probably the equality in Problem 2 holds for every planar oriented graph.

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