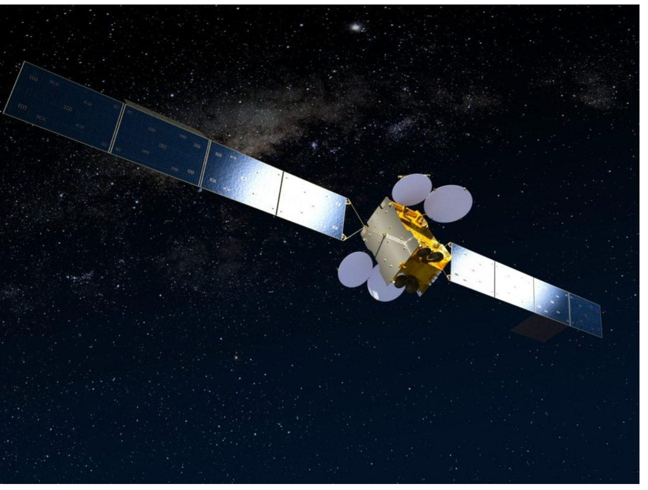
Position Fixing

How do I estimate my position using only:

- distance and
- angle

measurements?

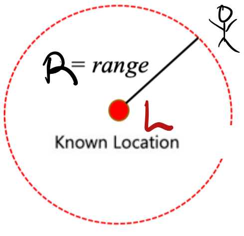


Assumptions (for now)

- Static problems (no motion)
- Noiseless measurements

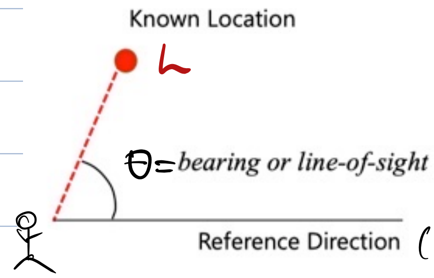
3 types of measurements:

- Range:



circular fix
(2D case)

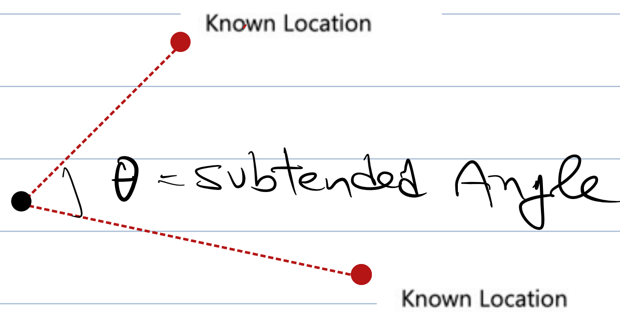
- Bearing



(assumed Known)

linear fix (2D case)

- Subtended Angle



what type of fix is this?

Settings in practice:

- 2D: ground vehicles; ships
- 3D: aerospace vehicles; submarine

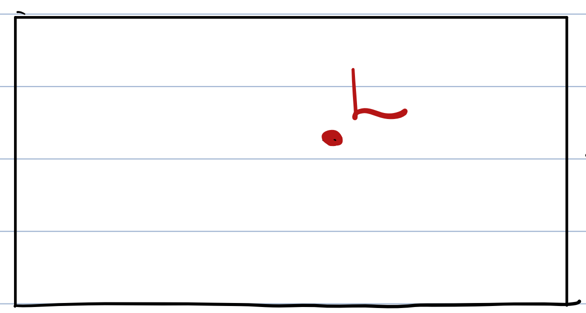
Observation

Under noiseless assumption, position

fixing is <u>geometry</u>

Let's dive into more details

Let L be a know location in the environment:



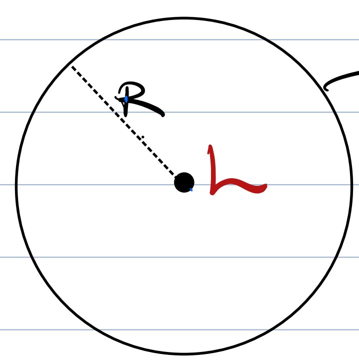
(2D) environment

Suppose you measure your <u>distance</u> from L:

Distance = R

Where could you be?

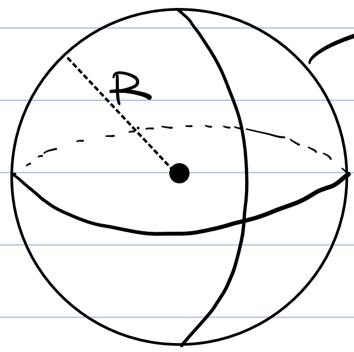
<u>2D case:</u>



anywhere on the circle

circular fix

3D case



anywhere on the sphere

spherical fix

Q1. How many distance fixes do we need in 2D to determine (unambiguisly) position?

A. 1

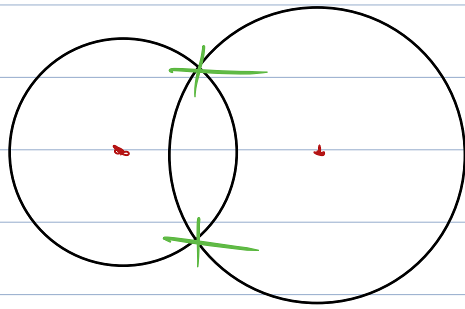
B. 2

C. 3

D. ∞

Answer:

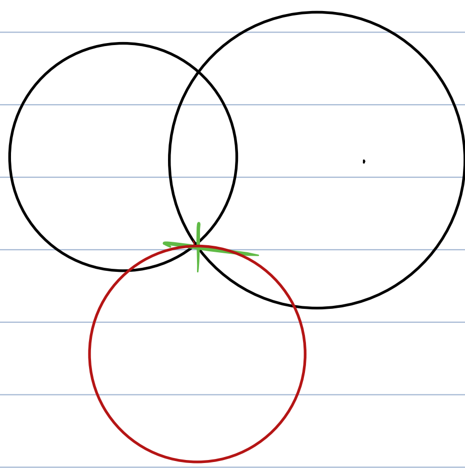
· If 2 distances:



you could be on either ×

⇒ ambiguous!

· If 3 distances



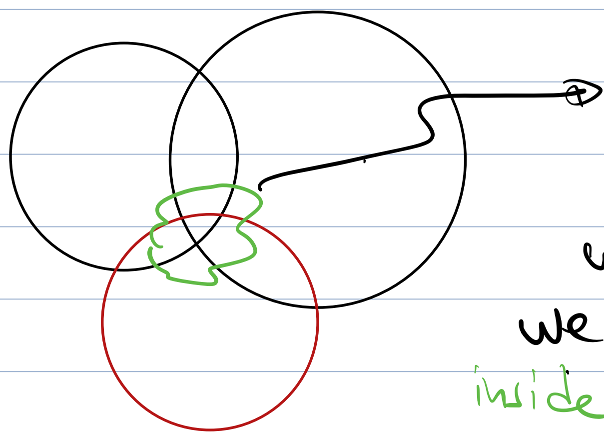
⇒ unambiguous

What happens in 3D?

· 2 spheres ⇒ you are on circle

- 3 spheres $\Rightarrow$ intersection of circle with sphere (still ambiguous)
- 4 spheres $\Rightarrow$ single point $\Rightarrow$ no ambiguity

[If there was noise, then you may have instead:



Ambiguous NOW!

Intuitively we can argue we are somewhere inside the green area.

NEED TO OPTIMIZE TO FIND AN EDUCATED GUESS.

WE NEED EQUATIONS FOR THAT (see below)]

COMPUTING CIRCULAR FIXES

For simplicity, consider 2 measurements:

· Coordinates $\left.\begin{aligned} L_1 &= (x_1, y_1), R_1 \\ L_2 &= (x_2, y_2), R_2 \end{aligned}\right\}$ Known

· Your location: (x, y) unknown

$$\Downarrow$$

$$(x - x_1)^2 + (y - y_1)^2 = R_1^2 \quad (E1)$$

$$(x - x_2)^2 + (y - y_2)^2 = R_2^2 \quad (E2)$$

$\Rightarrow$ 2 equations, 2 unknowns; how many solutions?

Q2. How many solutions (E1)-(E2) may have, assuming arbitrary (x_1, y_2), (x_2, y_2), R_1, and R_2?

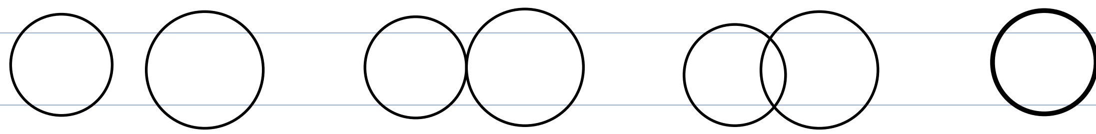
A. $\emptyset$

B. 1

C. 2

D. ∞

Answer (E1)-(E2) are non-linear:



$\emptyset$ 1 2 ∞

How to solve (E1)-(E2):

$$f(x,y) \triangleq \begin{bmatrix} (x-x_1)^2+(y-y_1)^2-R_1^2 \\ (x-x_2)^2+(y-y_2)^2-R_2^2 \end{bmatrix}$$

$$J(x,y) \triangleq \|f(x,y)\|_2^2 \quad , \text{where } \left\|\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right\|_2 \triangleq \sqrt{z_1^2+z_2^2}$$

Find (x,y) that minimizes J

$\Rightarrow$ in Homework 2 you will use MATLAB's fminunc to solve the problem

Noisy case: If we had noise, then we would solve the same optimization problem to find a "best" guess

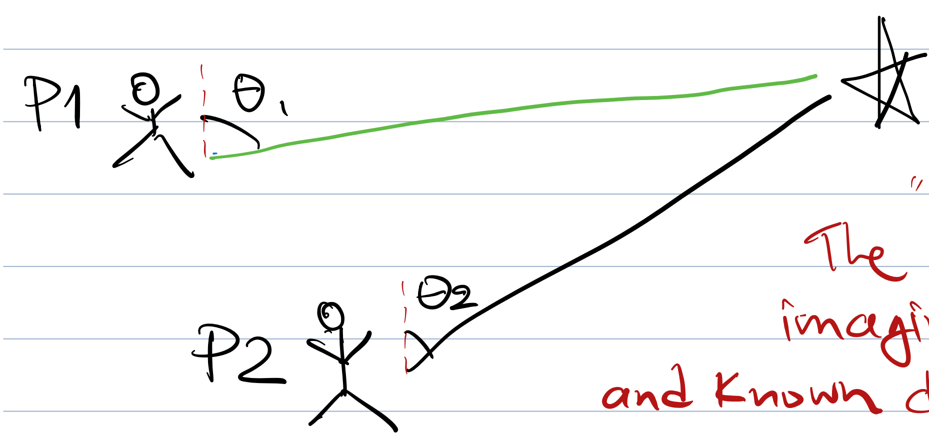
Bearing measurements

Beacons: A beacon is an object that can be used for angle measurements.

There are 2 types of beacons:

- Infinite beacon (star S ☆)
- Finite beacon (lighthouse L)

Star beacon

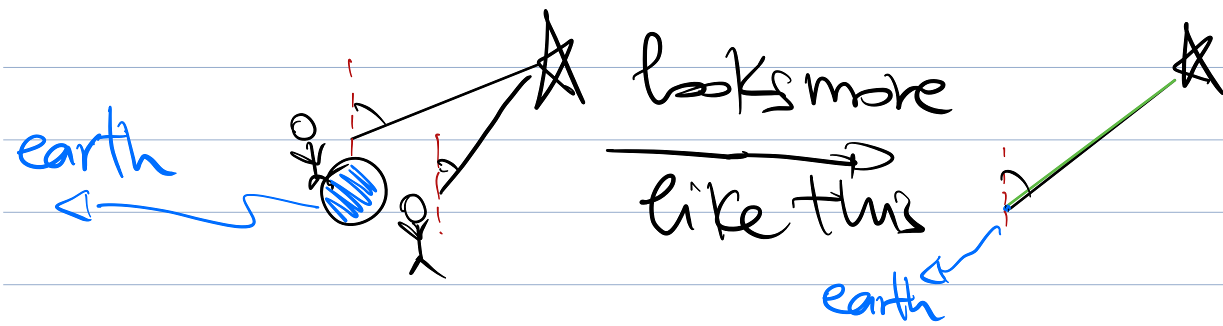


The "¦" represents imaginary common and known direction

Q3. What is true for θ_1 and θ_2?

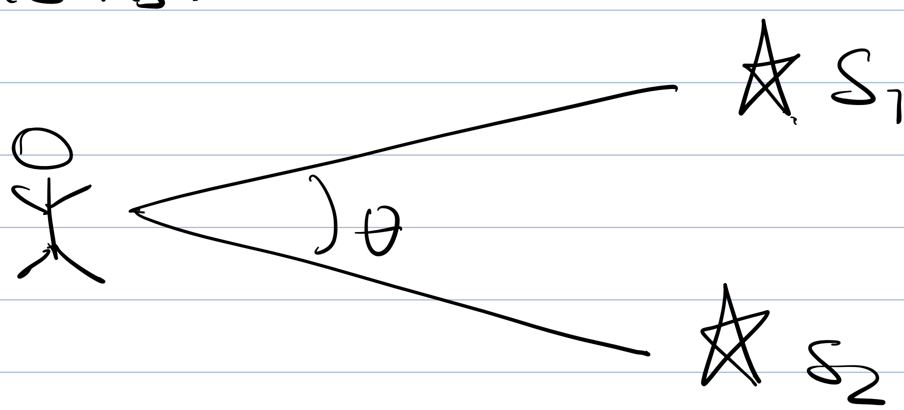
A. $\theta_1 < \theta_2$
B. $\theta_1 > \theta_2$
C. $\theta_1 = \theta_2$
D. NONE of the above

Answer



$\Rightarrow$ Infite beacon acts as common direction!

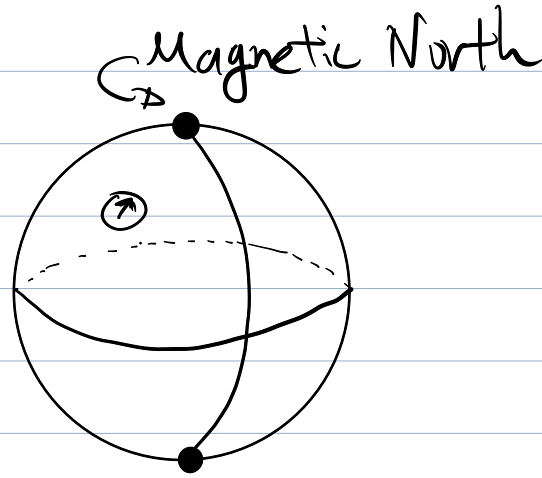
Suppose we observe 2 infite beacons:



Everyone sees the same angle!

$\Rightarrow$ cannot determine position.

[The <u>compass</u> is a device for determining the direction of an "infinite" beacon

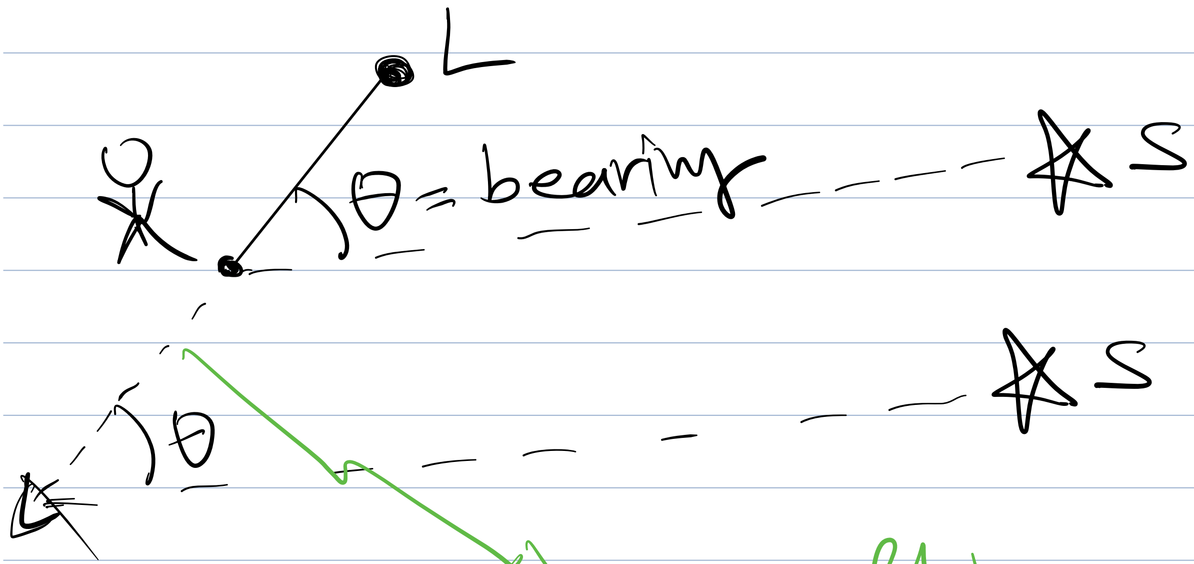


]

Bearing measurement 2D case

Suppose we can observe:

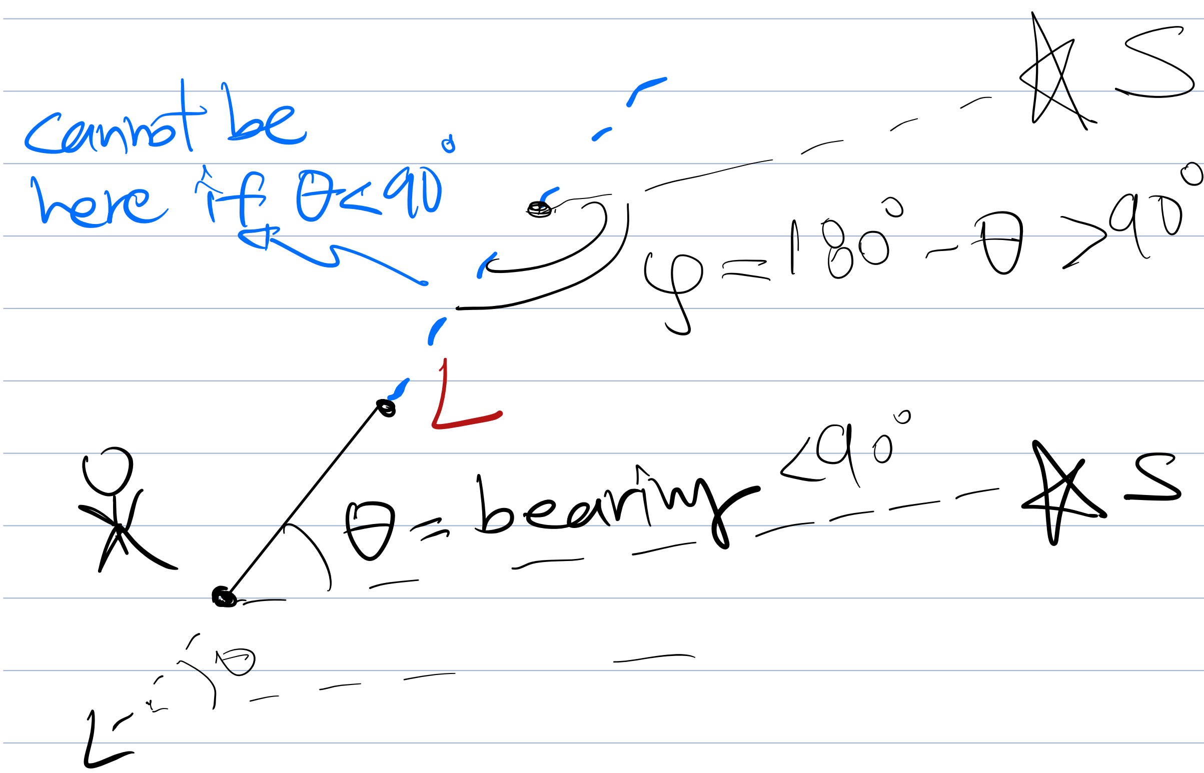
- 1 finite beacon (lighthouse L)
- 1 infinite beacon (star S)



you could be anywhere on this ray (half-line), that starts from L

⟹ Linear fix is here a ray

⟹ Bearing gives direction of ray:

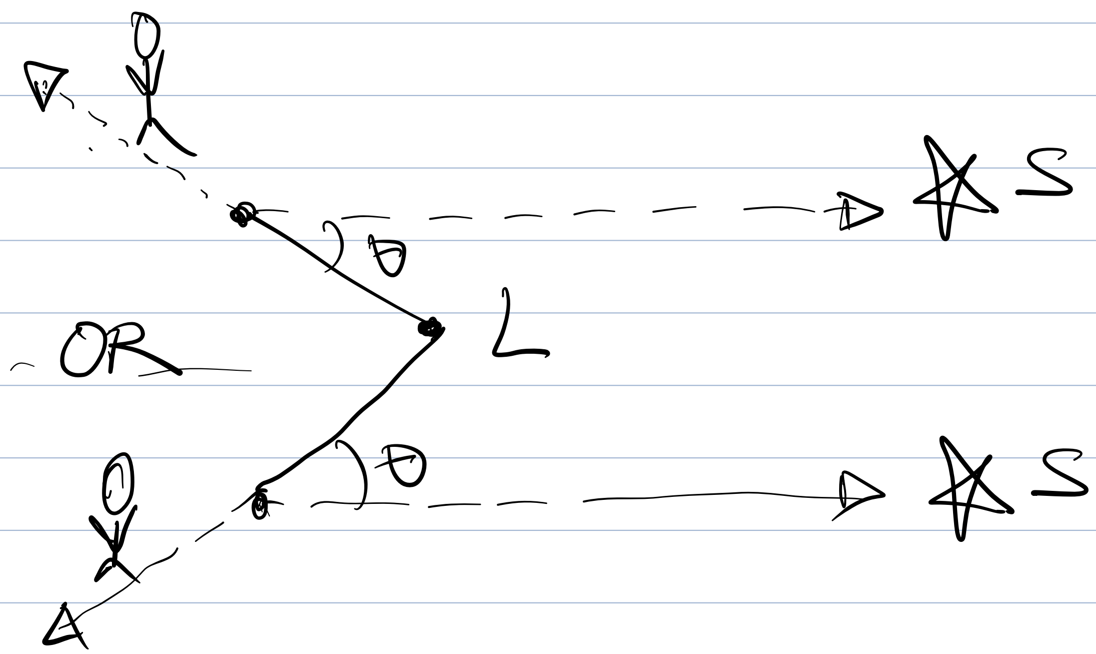


Is this the whole picture?

Or is it even more ambiguous?

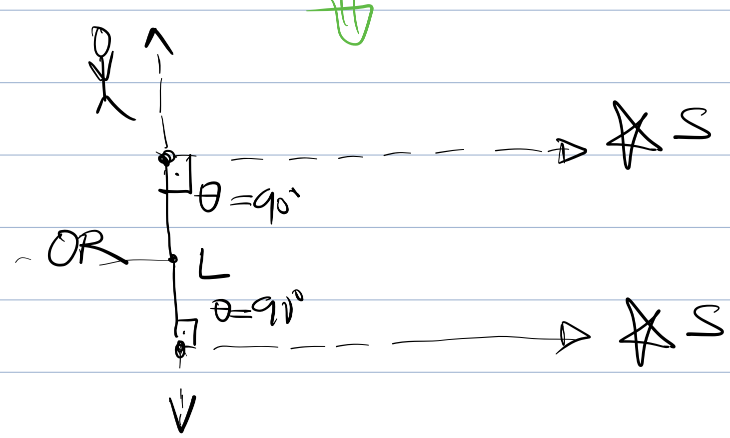
Particularly, what if we cannot recognize between "above" and "below" $\angle$?

Double-Ray Fix

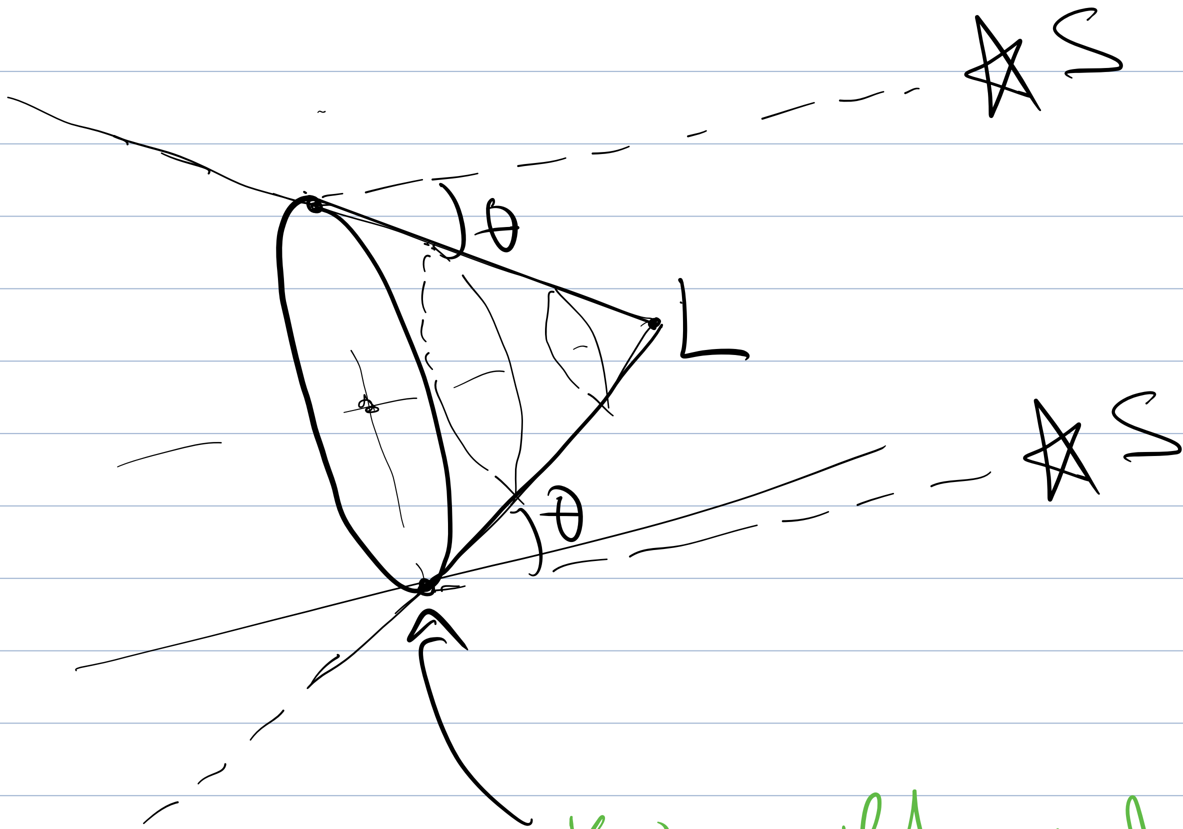


But on earth we know up/down
so angles are signed

What if $\theta = 90°$?



Bearing measurement 3D case

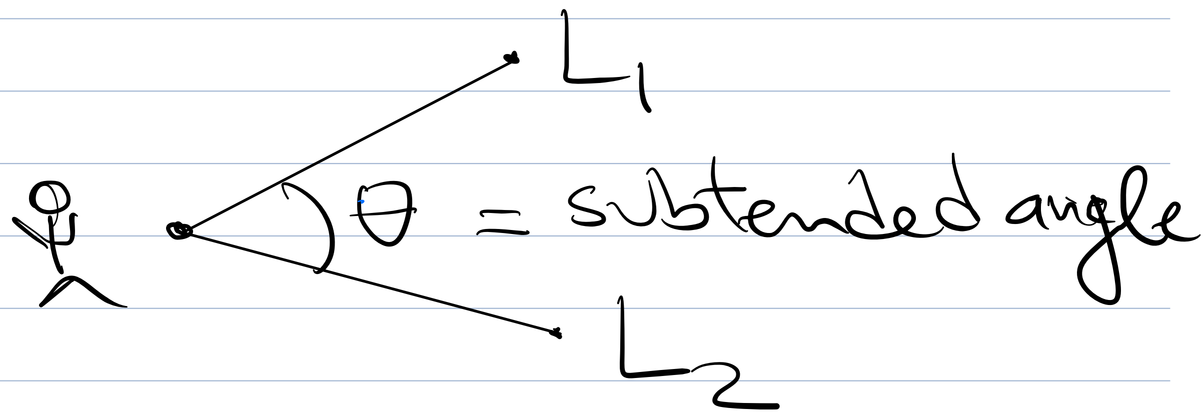


you could anywhere on this cone!

$\Longrightarrow$ conical fix

Subtended Angle

Consider 2 finite beacons

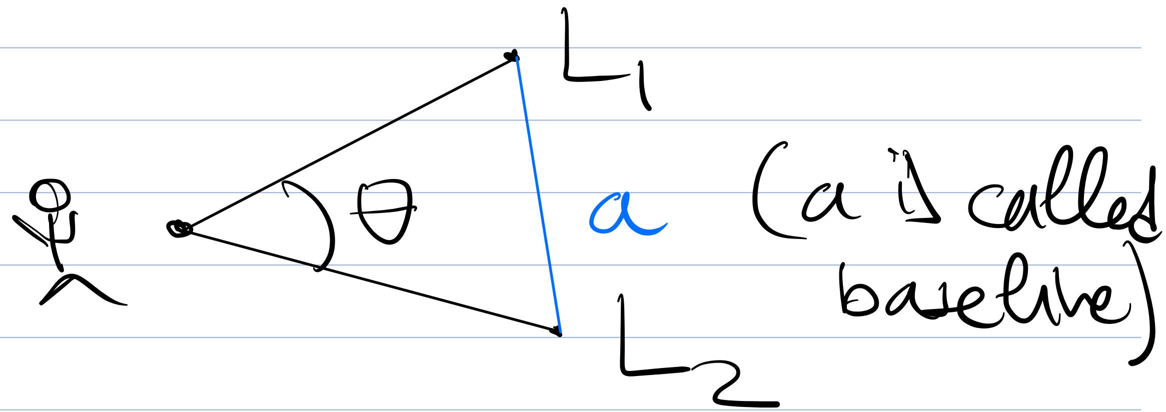


What kind of fix is this?

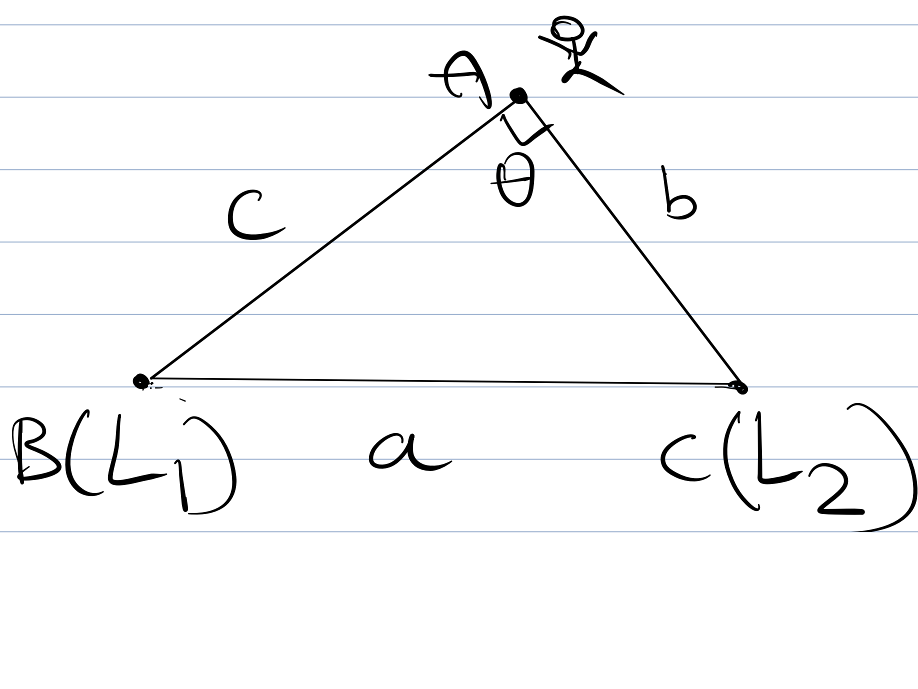
case 1) θ is acute ($<90^\circ$)

case 2) $\theta = 90^\circ$

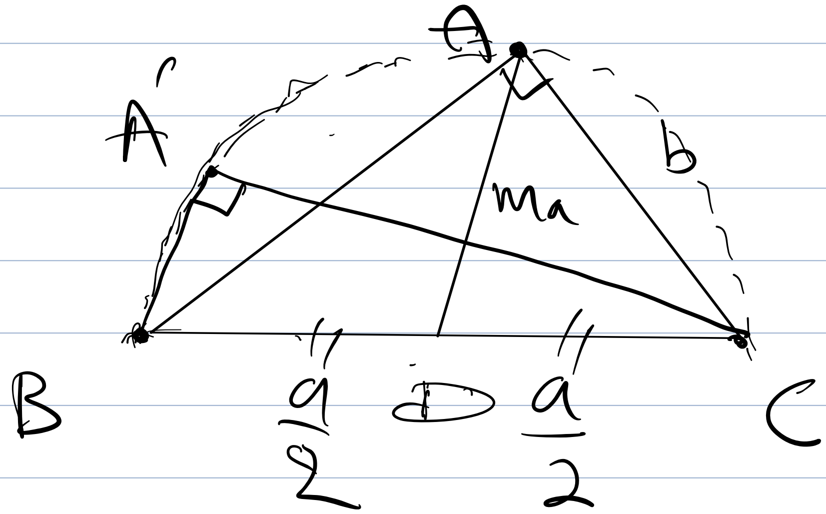
case 3) θ is abtuse ($>90^\circ$)



Let's start our analysis with Case 2 ($\theta = 90°$)



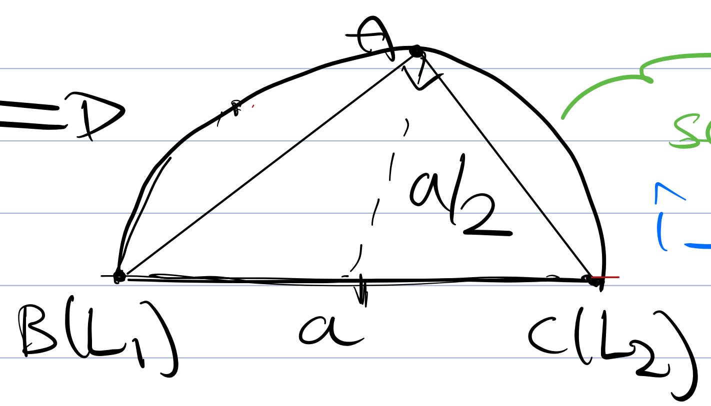
Where is A?



We can prove, using the "Median Lemma" that

$$m_a = \frac{a}{2}.$$

$\Rightarrow$



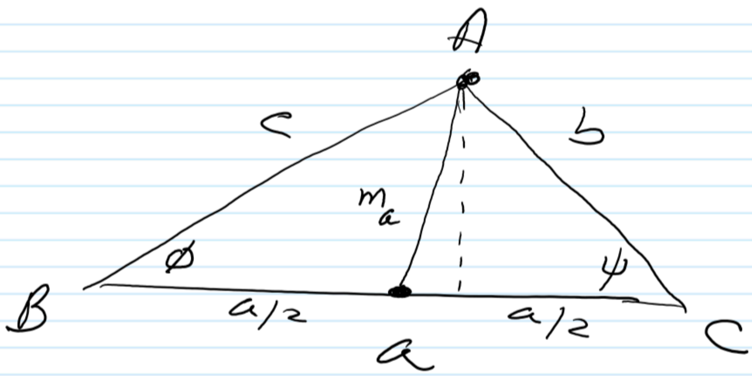
semi-circular fix, if L_1 and L_2 are distinguishable.

If L_1 and L_2 are INdistin-
guishable, then, CIRCULAR
FIX.

similarly to not knowing
up/down in the bearing case above

Median Lemma

Proof by Dennis. S. Bernstein follow
below:



m_a is the median to side a

$$m_a^2 = \frac{a^2}{4} + c^2 - ac\cos\phi$$
$$m_a^2 = \frac{a^2}{4} + b^2 - ab\cos\psi$$
cosine rule

$$2\Sigma \Rightarrow 4m_a^2 = a^2 + 2b^2 + 2c^2 - 2a(c\cos\phi + b\cos\psi)$$

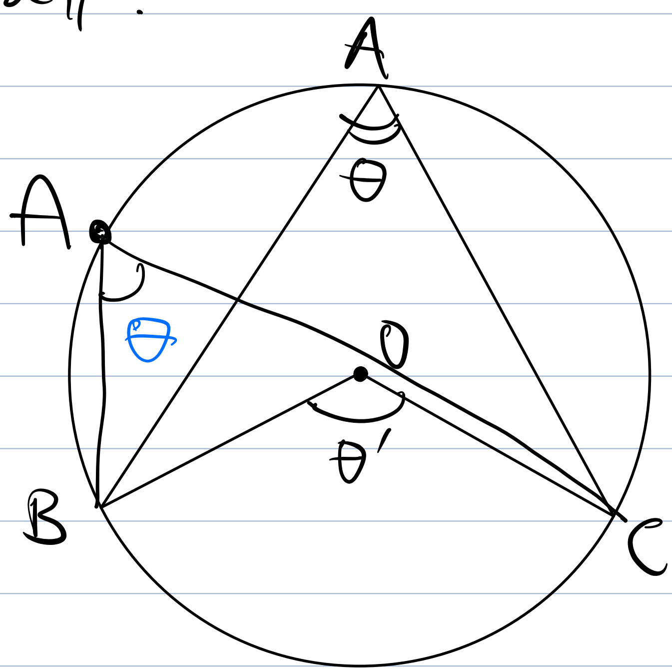
$c\cos\phi + b\cos\psi = a$ (can see this)

$$\Rightarrow 4m_a^2 = a^2 + 2b^2 + 2c^2 - 2a^2$$
$$= 2b^2 + 2c^2 - a^2$$

$$\Rightarrow \boxed{m_a^2 = \tfrac{1}{2}(b^2 + c^2) - \tfrac{1}{4}a^2} \checkmark$$

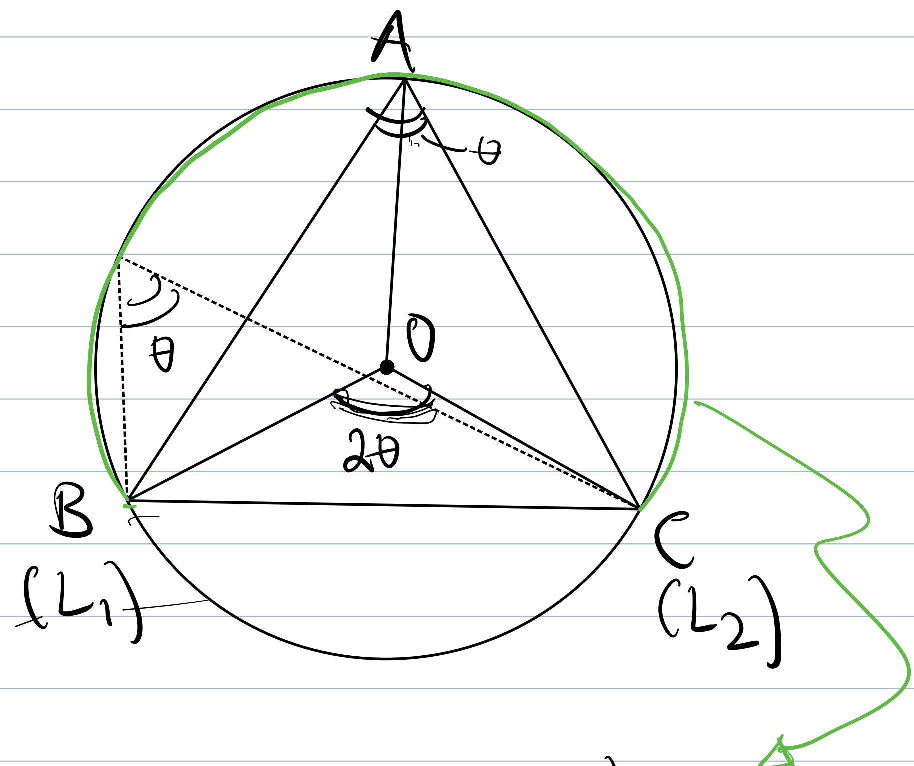
Case 1 ($\theta < 90°$)

[We will use the "Double Angle Lemma":



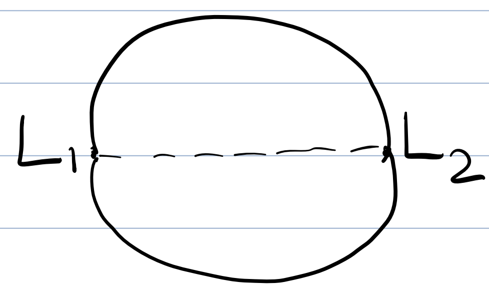
- O is center of circle
- A, B, C are on circle

Then, $\theta' = 2\theta$.]



If A is acute ($\theta < 90°$), then you could be anywhere on the green-colored arc $\overset{\frown}{BC}$

Note that if we cannot distinguish L_1 and L_2, then:

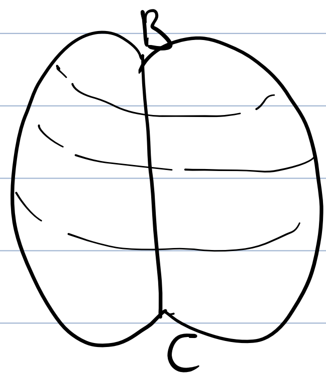


Case 3 ($\theta > 90°$): Homework 2.

Subtended Agle position Fix in 3D

In 3D we rotate around BC (L_1 - L_2)

E.g., for $\theta < 90°$:



Summary

	2D	3D
Range	Circular	Spherical
Bearing	Ray	Conical
Subtended Angle	Circular Arc	Rotated Circular Arc

Next lecture:

- Mathematical analysis of Position Fixing in 3D via 3D Geometry.