1. define $\vec{x}(t)\big|_A = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix}$

$${}^{A}\dot{\vec{x}}(t)\big|_A = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{bmatrix}$$

$\because$ $|\vec{x}(t)|$ is constant

$\therefore$ $r_1^2 + r_2^2 + r_3^2 = T$

define $f(t) = r_1^2 + r_2^2 + r_3^2$ $f(t)$ is constant

$\Rightarrow$ $\nabla f(t) = 0$

$$\Rightarrow \begin{bmatrix} 2r_1\dot{r}_1 \\ 2r_2\dot{r}_2 \\ 2r_3\dot{r}_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} r_1\dot{r}_1 \\ r_2\dot{r}_2 \\ r_3\dot{r}_3 \end{bmatrix} = 0$$

$$\vec{x}(t) \cdot {}^{A}\dot{\vec{x}}(t) = [r_1\ r_2\ r_3]\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{bmatrix} = \begin{bmatrix} r_1\dot{r}_1 \\ r_2\dot{r}_2 \\ r_3\dot{r}_3 \end{bmatrix} = 0$$

$\Rightarrow$ $\vec{x}(t)$ and ${}^{A}\dot{\vec{x}}(t)$ are mutually orthogonal

2.

$${}^{A}\dot{\vec{x}} = {}^{B}\dot{\vec{x}} + \vec{\omega}_{B|A} \times \vec{x} \quad ①$$

$${}^{A}\ddot{\vec{x}} = {}^{A}\dot{\left({}^{B}\dot{\vec{x}}\right)} + {}^{A}\dot{\vec{\omega}}_{B|A} \times \vec{x} + \vec{\omega}_{B|A} \times {}^{A}\dot{\vec{x}} \quad ②$$

from ①, replace $\vec{x}$ with ${}^{B}\dot{\vec{x}}$

$${}^{A}\dot{\left({}^{B}\dot{\vec{x}}\right)} = {}^{B}\ddot{\vec{x}} + \vec{\omega}_{B|A} \times {}^{B}\dot{\vec{x}} \quad ③$$

put ①, ③ to ②

$${}^{A}\ddot{\vec{x}} = {}^{B}\ddot{\vec{x}} + \vec{\omega}_{B|A} \times {}^{B}\dot{\vec{x}} + {}^{B}\dot{\vec{\omega}}_{B|A} \times \vec{x} + \vec{\omega}_{B|A} \times \left({}^{B}\dot{\vec{x}} + \vec{\omega}_{B|A} \times \vec{x}\right)$$

$${}^{A}\ddot{\vec{x}} = {}^{B}\ddot{\vec{x}} + 2\vec{\omega}_{B|A} \times {}^{B}\dot{\vec{x}} + {}^{B}\dot{\vec{\omega}}_{B|A} \times \vec{x} + \vec{\omega}_{B|A} \times \left(\vec{\omega}_{B|A} \times \vec{x}\right)$$

3.

1) $W_{D/A|D} = \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & \sin\Phi\cos\Theta \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = S(\Phi,\Theta) \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix}$

$$W_{D/A|D} = \underbrace{\dot{\Phi}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mu_1} + \underbrace{\dot{\Theta}\begin{bmatrix} 0 \\ \cos\Phi \\ -\sin\Phi \end{bmatrix}}_{\mu_2} + \underbrace{\dot{\Psi}\begin{bmatrix} -\sin\Theta \\ \sin\Phi\cos\Theta \\ (\cos\Phi)\cos\Theta \end{bmatrix}}_{\mu_3}$$

if μ_1, μ_2, μ_3 are linear independent, then $W_{D/A|D} = \text{Span}\{\mu_1, \mu_2, \mu_3\}$

$\Rightarrow \forall\, w \in W_{D/A|D}$, $\exists\, \dot{\Phi}, \dot{\Theta}, \dot{\Psi} \in \mathbb{R}$ s.t. $w = \dot{\Phi}\mu_1 + \dot{\Theta}\mu_2 + \dot{\Psi}\mu_3$

otherwise, $\exists\, w \in W_{D/A|D}$, $\forall\, \dot{\Phi}, \dot{\Theta}, \dot{\Psi} \in \mathbb{R}$ s.t. $w \neq \dot{\Phi}\mu_1 + \dot{\Theta}\mu_2 + \dot{\Psi}\mu_3$

if $\Theta = 0$

$A = [\mu_1 \,\vdots\, \mu_2 \,\vdots\, \mu_3] = \begin{bmatrix} 1 & 0 & \mp 1 \\ 0 & \cos\Phi & 0 \\ 0 & -\sin\Phi & 0 \end{bmatrix}$ $\quad \det(A) = 0$

Thus, μ_1, μ_2, μ_3 is linear independent. not all $W_{D/A|D}$ are attainable

if $W_{D/A|D}$ are not all attainable, μ_1, μ_2, μ_3 are linear dependent

$$\det\left(\begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & \sin\Phi\cos\Theta \\ 0 & -\sin\Phi & \cos\Phi\cos\Theta \end{bmatrix}\right) = 0 \Rightarrow \cos^2\Phi\cos\Theta + \sin^2\Phi\cos\Theta = 0 \qquad \Theta = \pm\frac{\pi}{2}$$

Thus not all $W_{D/A|D}$ can be attainable if and only if $\Theta = \pm\frac{\pi}{2}$

2) when $\Phi = \pm\frac{\pi}{2}$,

$W_{D/A|D} = \begin{bmatrix} \dot{\Phi} \mp \dot{\Psi} \\ 0 \\ -\dot{\Theta} \end{bmatrix}$ $\quad \forall\, w' \in W\hat{k}_D$, $\exists\, \dot{\Phi}, \dot{\Psi}, \dot{\Theta}$, s.t. $W_{D/A|D} = W'$

thus, $W_{D/A|D} = W\hat{k}_D$ is attainable

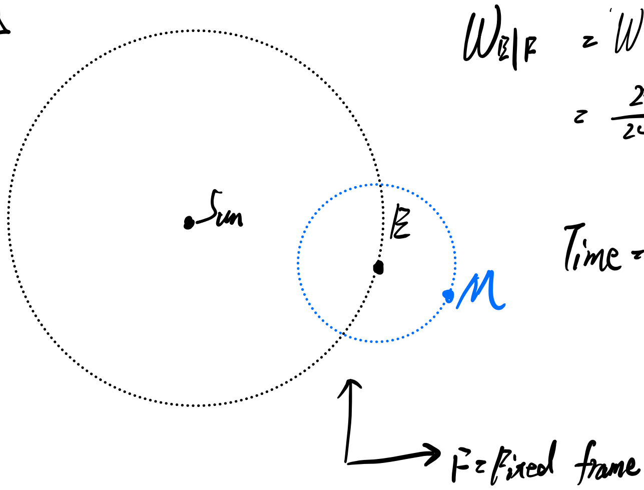
When $\vec{W}_{D/A} = W\vec{k}_D$ is attainable

$\exists.\ \dot{\Phi}, \dot{\Theta}, \dot{\Psi}$, s.t. $\dot{\Phi}\mu_1 + \dot{\Theta}\mu_2 + \dot{\Psi}\mu_3 = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}$

$$\begin{cases}\dot{\Phi} \mp \dot{\psi} = 0\\ \cos\Phi\,\dot{\theta} = 0 \\ -\sin\Phi\,\dot{\theta} = \omega\end{cases} \Rightarrow \begin{cases}\cos\Phi = 0\\ \dot{\Phi} = \pm\dot{\psi} \\ \omega = -\sin\Phi\,\dot{\theta}\end{cases} \Rightarrow \Phi = \pm\frac{\pi}{2}$$

$\Rightarrow \vec{\omega}_{B|A} = \omega\hat{k}_B$ is attainable by Euler-angle derivatives iff $\Phi = \pm\frac{\pi}{2}$
when $\theta = \pm\frac{\pi}{2}$

4



$$\omega_{E|F} = \omega_1 + \omega_2 = 2\pi/24h + 2\pi/365.25\,day$$
$$= \frac{2\pi}{24} + \frac{2\pi}{365.25\times 24} = \frac{1465}{17532}\pi \quad rad/h$$

$$Time = \frac{2\pi}{\omega_{E|F}} = \frac{24\cdot 365.25}{366.25} = 23.93h$$
$$= 23h\,56min$$

(ii) $\omega_{S|F} = \frac{2\pi}{27\cdot 24}$

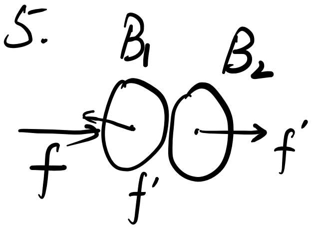
$$\omega_{E|S} = \omega_{E|F} - \omega_{S|F} = \frac{1465\pi}{17532} - \frac{2\pi}{27\cdot 24} = \frac{6349}{78894}\pi \ rad/h$$

$$Time' = \frac{2\pi}{\omega_{E|S}} = 24.852h = 24h\,51min$$

(iii) $\omega_{M|F} = \frac{2\pi}{27.3\cdot 24}$

$$\frac{2\pi}{\omega_{M|F} - \omega_2} = \frac{2\pi}{\frac{2\pi}{27.3\cdot 24} - \frac{2\pi}{365.25\cdot 24}} = 708.13h = 708h\,7min$$

5.



$$\begin{cases} m_2 a_2 = f' \\ m_1 a_1 = f - f' \\ a_1 = a_2 \end{cases} \Rightarrow \quad (m_1+m_2)a = f, \quad a = \frac{f}{m_1+m_2}, \quad f' = \frac{m_2}{m_1+m_2} f$$

extend:

f $n \geq 3$

$$f' = \frac{\sum_{i=j}^{n} m_{i+1}}{\sum_{i=1}^{n} m_i} f$$

j means the jth item

6. let the principal axis be the $\hat{k}$ of frame B, frame A is the world frame. frame B is fixed on the body

$$\vec{M}_{B|W} = \vec{J}_{B|W} \, {}^{B}\dot{\vec{w}}_{B|A} + \vec{w}_{B|A} \times \vec{J}_{B|W} \vec{w}_{B|A}$$

$$\vec{w}_{B|A} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}$$ $\vec{J}_{B|W}$ is a diagonal matrix

Thus $\vec{w}_{B|A} \times \vec{J}_{B|W} \vec{w}_{B|A} = \vec{w}_{B|A} \times \begin{bmatrix} 0 \\ 0 \\ I_{zz} w \end{bmatrix} = 0$

$\because$ there is no applied moments

$\therefore$ $\vec{M}_{B|W} = 0$

$\Rightarrow$ $\vec{J}_{B|W} \cdot {}^{B}\dot{\vec{w}}_{B|A} = 0$

$\Rightarrow$ ${}^{B}\dot{\vec{w}}_{B|A} = 0$

with no deceleration, the body will spins indefinitely

7.

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & \sin\Phi \tan\theta & \cos\Phi \tan\theta \\ 0 & \cos\Phi & -\sin\Phi \end{bmatrix} \begin{bmatrix} \cos 2t \\ \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ 0 & \sin\Phi \sec\Theta & \cos\Phi \sec\Theta \end{bmatrix} \begin{bmatrix} \vdots \\ 0.025t \end{bmatrix}$$

For 7(a) we use following codes. And the code includes a fuction “vdp1”

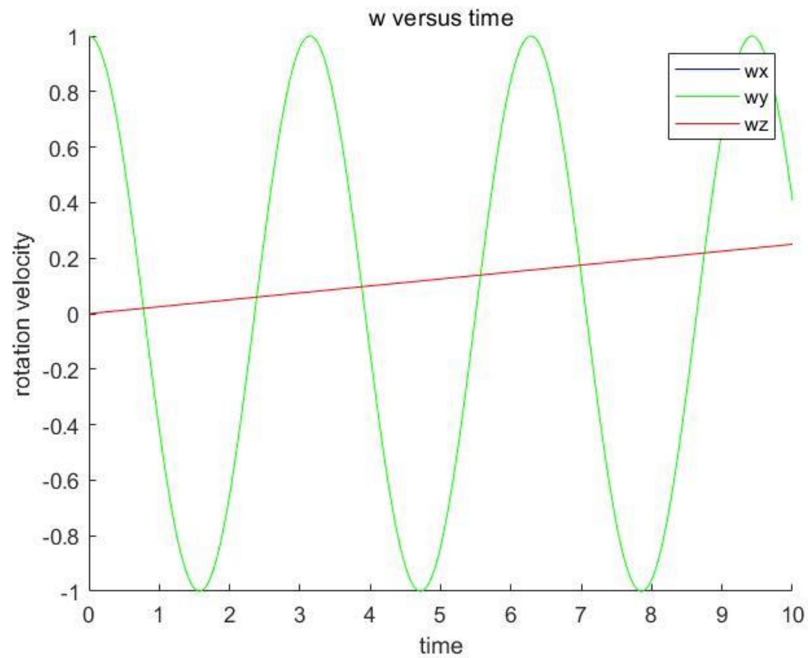
```
clc;
clear all;
time = 0:0.01:10;
[t,y] = ode45(@vdp1,time,[0;0;0]);
%y(1) = phi, y(2) = theta, y(3) = psi
w = [cos(2*t),cos(2*t),0.025*t];

figure(1)
title('w versus time')
hold on
plot(t,w(:,1),'b');
plot(t,w(:,2),'g');
plot(t,w(:,3),'r');
hold off
legend('wx','wy','wz');
xlabel('time')
ylabel('rotation velocity')

figure(2)
title('angles versus time')
hold on
plot(t,y(:,1),'b');
plot(t,y(:,2),'g');
plot(t,y(:,3),'r');
hold off
legend('phi','theta','psi');
xlabel('time')
ylabel('angles in radian')

O_matrix_1 = zeros(3,3,1001);
```

```
o_indexs = zeros(9,length(t));
for i=1:length(t)
    ang = y(i,:);
    o_matrix_t = angle(ang(1),ang(2),ang(3))';
    o_matrix =   o_matrix_t';
    O_matrix_1(:,:,i) = o_matrix;
    for j=1:9
        o_indexs(j,i) = o_matrix_t(j);
    end
end

figure(3)
hold on
for j=1:9
    subplot(3,3,j);
    plot(t,o_indexs(j,:));
    txt = [int2str(j),'th value of O-matrix'];
    title(txt);
    xlabel('time(s)')
end
hold off
save("O_matrix.mat","o_indexs","O_matrix_1");
```

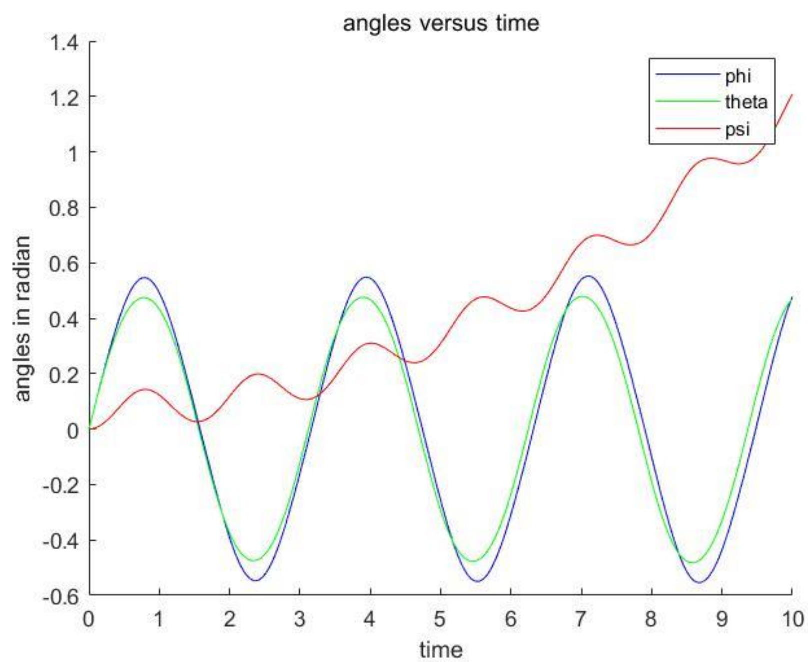
Following is function vdp1:

```
function dydt = vdp1(t,y)
dydt = [cos(2*t)*(1+sin(y(1))*tan(y(2)))+cos(y(1))*tan(y(2))*0.025*t;...
    cos(2*t)*cos(y(1))-sin(y(1))*0.025*t;...
    cos(2*t)*sin(y(1))*sec(y(2))+0.025*t*cos(y(1))*sec(y(2))];
end
```

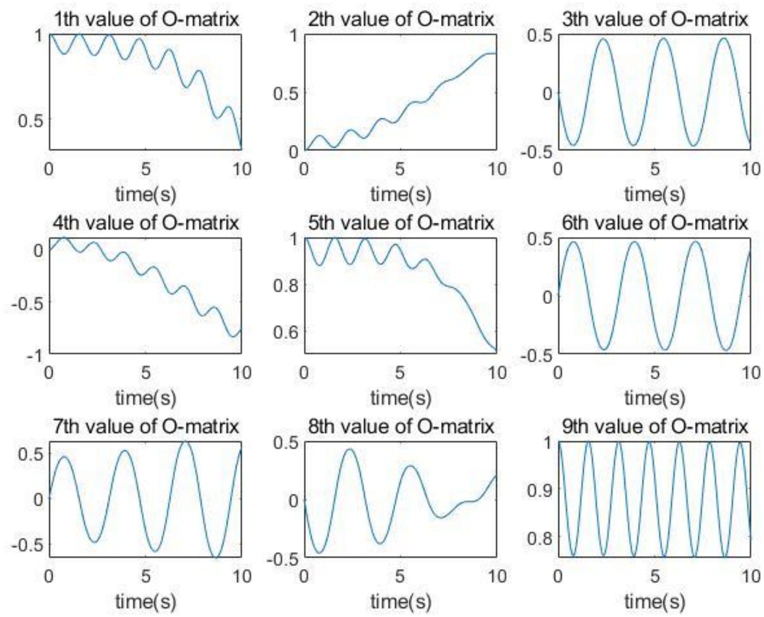
and the pictures look like below:



P1: w versus time



P2: angles versus time



P3: values in orientation matrix versus time

For 7(b) we use the following codes. And the code includes a fuction "vec_to_mat", "angle"

```
clc;
clear all;
w0 = [1,1,0]';
%A = vec_to_mat(w0);   % Some arbitrary matrix we will use
F0 = angle(0,0,0);   % matrix initial value
F1=[1,2,3;4,5,6;7,8,9];
odefun = @(t,y) deriv(t,y);   % Anonymous derivative function with A
tspan = 0:0.01:10;
f0 = reshape(F0,[1,9])';
f1 = reshape(F1,[1,9])';
[T,F] = ode45(odefun,tspan,f0);   % Pass in column vector initial value
%T = F';
F = reshape(F.',3,3,[]);   % Reshape the output as a sequence of 3x3
matrices
```

```
o_indexs_2 = zeros(9,length(T));
for i=1:length(T)
    o_matrix_t_2 = F(:,:,i)';
    for j=1:9
        o_indexs_2(j,i) = o_matrix_t_2(j);
    end
end

last_method = load("O_matrix.mat");
o_indexs_1 = last_method.o_indexs;

figure(1)
hold on
for j=1:9
    subplot(3,3,j);
    hold on

    plot(T,o_indexs_2(j,:),'b','LineWidth',2);
    plot(T,o_indexs_1(j,:),'color','#D95319','LineStyle','--','LineWidth',2);
    hold off
    legend('in b','in a');
    %plot(T,o_indexs_2(j,:),'b');
    txt = [int2str(j),'th value of O-matrix'];
    title(txt);
    xlabel('time(s)')
end
hold off
save("O_matrix_2.mat","F");

function dy = deriv(t,y)
A = vec_to_mat([cos(2*t),cos(2*t),0.025*t]);
```

```
F = reshape(y,size(A));   % Reshape input y into matrix
FA = -A*F;   % Do the matrix multiply
dy = reshape(FA,[1,9])';   % Reshape output as a column vector
end
```

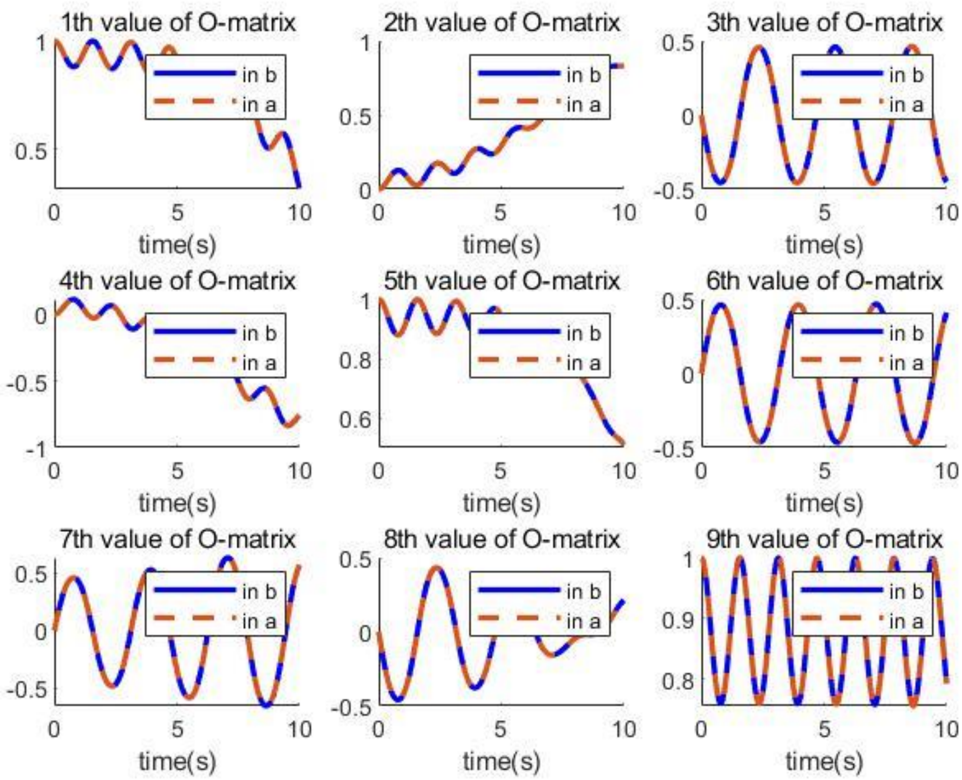
Following is function angle:

```
function o_matrix = angle(a,b,c)
o_matrix = [cos(b)*cos(c), cos(b)*sin(c), -sin(b);...
     cos(c)*sin(a)*sin(b)-cos(a)*sin(c),        cos(a)*cos(c)+sin(a)*sin(b)*sin(c), 
cos(b)*sin(a);...
     sin(a)*sin(c)+cos(a)*cos(c)*sin(b),        cos(a)*sin(b)*sin(c)-cos(c)*sin(a), 
cos(a)*cos(b)];
end
```

Following is function vec_to_mat：

```
function matrix = vec_to_mat(w)
wx = w(1);
wy = w(2);
wz = w(3);
matrix = [0,-wz,wy;wz,0,-wx;-wy,wx,0];
end
```

and the pictures look like below:



P4: values in 2 orientation matrix

The values concide

For 7(c) we use the following codes. And the code includes a fuction "cal_Eular"

```
clc;
clear all;
%from a and b import Orientation matrix
O_1 = load("O_matrix.mat");
O_2 = load("O_matrix_2.mat");
w_1 = O_1.y;
O_matrix_1 = O_1.O_matrix_1;
O_matrix_2 = O_2.F;
%then use cal_Eular fuction to calculate the angles
O1_solutions = zeros(6,length(O_matrix_1(1,1,:)));
O2_solutions = zeros(6,length(O_matrix_2(1,1,:)));
```

```
for i=1:length(O_matrix_1)
    [O1_solutions(1:3,i),O1_solutions(4:6,i)] = cal_Eular(O_matrix_1(:,:,i));
    [O2_solutions(1:3,i),O2_solutions(4:6,i)] = cal_Eular(O_matrix_2(:,:,i));
end

T = 0:0.01:10;



figure(1)
subplot(1,3,1);
hold on

plot(T,w_1(:,1),'LineWidth',3);
plot(T,O1_solutions(1,:),'color','#D95319','LineStyle','--','LineWidth',2);
plot(T,O2_solutions(1,:),'LineWidth',2,'color','y','LineStyle','-.');
title('phi versus time');
xlabel('time(s)');
legend('angle','angle from O matrix in a','angle from O matrix in b');
hold off

subplot(1,3,2);
hold on
plot(T,w_1(:,2),'LineWidth',3);
plot(T,O1_solutions(2,:),'color','#D95319','LineStyle','--','LineWidth',2);
plot(T,O2_solutions(2,:),'LineWidth',2,'color','y','LineStyle','-.');
title('theta versus time');
xlabel('time(s)')
legend('angle','angle from O matrix in a','angle from O matrix in b');
hold off

subplot(1,3,3);
```

```
hold on
plot(T,w_1(:,3),'LineWidth',3);
plot(T,O1_solutions(3,:),'color','#D95319','LineStyle','--','LineWidth',2);
plot(T,O2_solutions(3,:),'LineWidth',2,'color','y','LineStyle','-.');
title('psi versus time');
xlabel('time(s)')
legend('angle','angle from O matrix in a','angle from O matrix in b');
hold off


%--------------------------------------
figure(2)
subplot(1,3,1);
hold on
plot(T,w_1(:,1),'LineWidth',2);
plot(T,O1_solutions(4,:),'color','#D95319','LineStyle','--','LineWidth',2);
plot(T,O2_solutions(4,:),'LineWidth',2,'color','y','LineStyle','-.');
title('phi versus time');
xlabel('time(s)');
legend('angle','angle from O matrix in a','angle from O matrix in b');
hold off

subplot(1,3,2);
hold on
plot(T,w_1(:,2),'LineWidth',2);
plot(T,O1_solutions(5,:),'color','#D95319','LineStyle','--','LineWidth',2);
plot(T,O2_solutions(5,:),'LineWidth',2,'color','y','LineStyle','-.');
title('theta versus time');
xlabel('time(s)')
legend('angle','angle from O matrix in a','angle from O matrix in b');
hold off
```

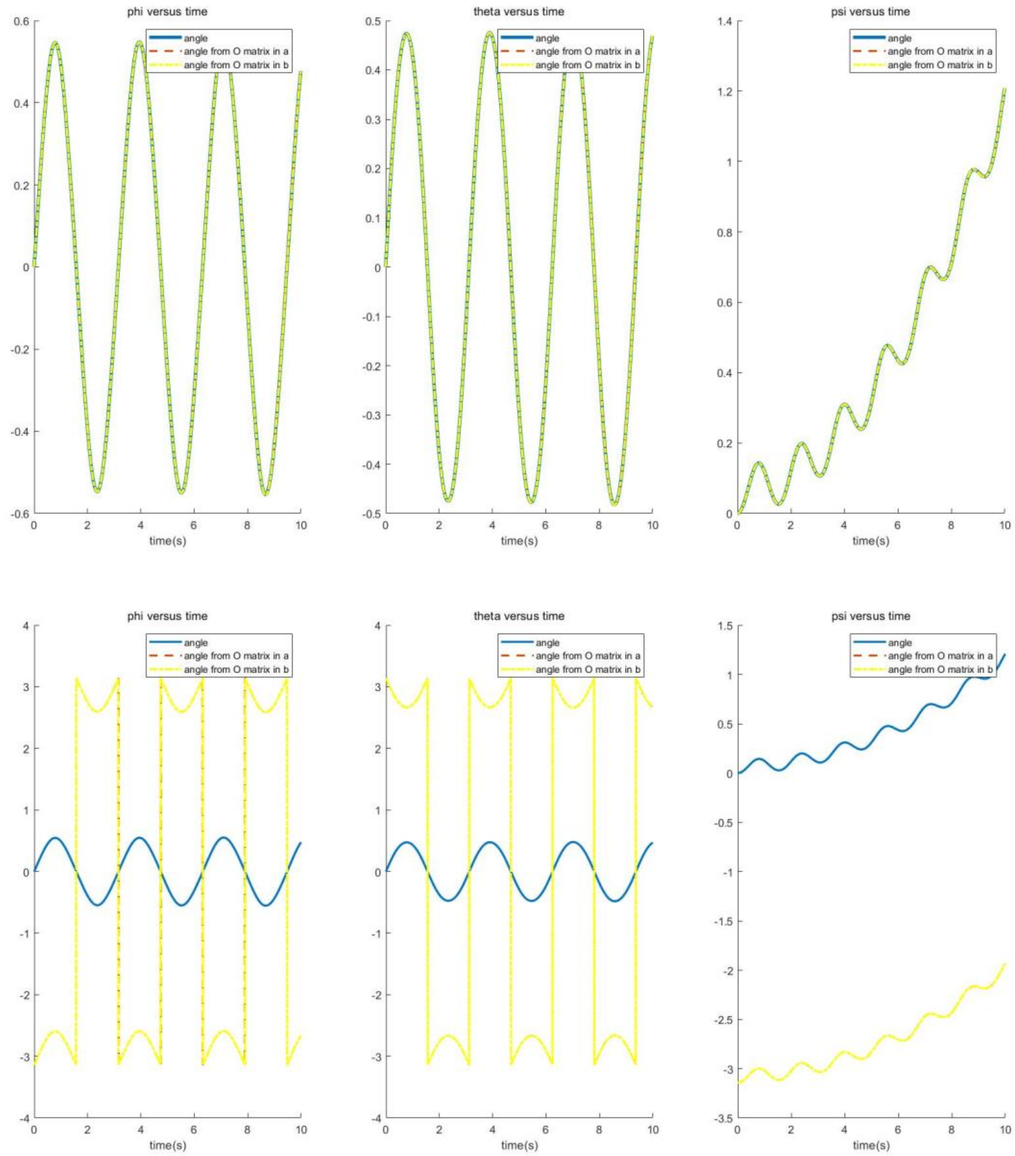
```
subplot(1,3,3);
hold on
plot(T,w_1(:,3),'LineWidth',2);
plot(T,O1_solutions(6,:),'color','#D95319','LineStyle','--','LineWidth',2);
plot(T,O2_solutions(6,:),'LineWidth',2,'color','y','LineStyle','-.');
title('psi versus time');
xlabel('time(s)')
legend('angle','angle from O matrix in a','angle from O matrix in b');
hold off
%then plot
```

Following is function cal_Eular：

```
function [solution1, solution2] = cal_Eular(o_matrix)
%get orientation_matrix in, Eular angles out
    theta_1 = -asin(o_matrix(1,3));
    theta_2 = pi-theta_1;
    if(theta_1<0)
        theta_2 = -pi-theta_1;
    end

    Psi_1 = atan2(o_matrix(1,2)/cos(theta_1),...
        o_matrix(1,1)/cos(theta_1));
    Psi_2 = atan2(o_matrix(1,2)/cos(theta_2),...
        o_matrix(1,1)/cos(theta_2));

    Phi_1 = atan2(o_matrix(2,3)/cos(theta_1),...
        o_matrix(3,3)/cos(theta_1));
    Phi_2 = atan2(o_matrix(2,3)/cos(theta_2),...
        o_matrix(3,3)/cos(theta_2));

    solution1 = [Phi_1,theta_1,Psi_1];
    solution2 = [Phi_2,theta_2,Psi_2];
```

end

and the pictures look like below:



P5: Eular Angles gotten by different methods

Eular Angles have two different solutions, therefore, there are two pictures. However, these two solutions coincide in different methods and one of the solutions coincides with the Eular Angle we integraled in 7(a)