584 Midterm Homework

Due Sat, **November** 5 **by** 11:59 **PM uploaded to Canvas as a single PDF file**.

Note: Canvas will not accept late uploads. If you miss the deadline, your score will be zero. Therefore, I suggest you upload a partial version a few hours before the deadline to make sure you have something in case you miss the deadline.

Since this is a midterm, you must be sure to NOT miss the deadline.

Instructions:

i) For this midterm, **you are not allowed to discuss the exam with anyone except of the GSI and Instructor,** and you cannot use any materials other than what is posted on Canvas. Therefore, all detailed work must be your own. No use of solutions from prior offerings of this course is allowed.

ii) Your work must be neat and professional in appearance. You may type it in Word or Latex if you wish Use a ruler to draw all lines and diagrams. No crossouts of any kind may appear anywhere.

iii) Put a box around your final answer to help the grader.

iv) Your submission should include a report pdf file, that includes a copy-paste of all your Matlab code, and your Matlab .m files. All files should be bundled together to a zip file and uploaded to Canvas.

v) Label your report file as: SmithAE584F22Midterm.pdf

vi) Label your zip file as: SmithAE584F22Midterm_CodeFiles.zip

vii) The midterm involves many steps, so be sure to start soon and leave enough time to do careful work.

v*iii*) I suggest that you read through each problem before you start doing any work to make sure you understand what is being asked. If anything is not clear, please email Hongyu, and he will post clarifications. I do not want anyone to lose points due to any questions not being clear.

You will have to use fminunc and ode45 in this midterm. Examples have been posted in the course Canvas page.

There are four problems. Each problem is worth 25 points.

Problem 1.

Let radian (rad) be the unit of angle. Let $\mathrm{F_A} = \begin{bmatrix}\hat{\imath}_\mathrm{A} & \hat{\jmath}_\mathrm{A} & \hat{k}_\mathrm{A}\end{bmatrix}$ be a frame, such that $\hat{\imath}_\mathrm{A}$ points toward the East, $\hat{\jmath}_\mathrm{A}$ points toward the North, and $\hat{k}_\mathrm{A} = -\hat{\imath}_\mathrm{A} \times \hat{\jmath}_\mathrm{A}$. Let w be a point, L_1, L_2, and L_3 be points representing the locations of 3 lighthouses and let P be a point that represents your position. For all $i \in \{1, 2, 3\}$, define the bearing measurements from the lighthouses to your position using a star in the North direction as

$$\theta_i \triangleq \theta_{\vec{r}_i/\hat{\jmath}_\mathrm{A}/\hat{k}_\mathrm{A}}, \tag{1}$$

where $\vec{r}_i \triangleq \vec{r}_{L_i/P}$, and define the noisy bearing measurements from the lighthouses to your position as

$$\hat{\theta}_i \triangleq \theta_i + n_i, \tag{2}$$

where $n_i \in \mathcal{N}(0, \sigma^2)$ (that is, the Gaussian density with mean 0 and variance σ^2) and $\sigma > 0$. Furthermore, for all distinct $i, j \in \{1, 2, 3\}$, let $\hat{P}_{i,j}$ be a point that represents the position of P obtained from noisy bearing measurements $\hat{\theta}_i$ and $\hat{\theta}_j$. For a single run, points $\hat{P}_{1,2}, \hat{P}_{1,3}$, and $\hat{P}_{2,3}$ form a triangle with centroid $\hat{c}$. Suppose that

$$\vec{r}_{L_1/w}\big|_\mathrm{A} = \begin{bmatrix}0\\0\\0\end{bmatrix} \mathrm{m}, \quad \vec{r}_{L_2/w}\big|_\mathrm{A} = \begin{bmatrix}4\\2\\0\end{bmatrix} \mathrm{m}, \quad \vec{r}_{L_3/w}\big|_\mathrm{A} = \begin{bmatrix}1\\4\\0\end{bmatrix} \mathrm{m}, \quad \vec{r}_{P/w}\big|_\mathrm{A} = \begin{bmatrix}2.5\\2\\0\end{bmatrix} \mathrm{m}. \tag{3}$$

Determine θ_1, θ_2, and θ_3. Then, for all $\sigma \in \{0.1, 0.2, \ldots, 0.9, 1\}$, determine the probability that P is located inside the triangle formed by $\hat{P}_{1,2}, \hat{P}_{1,3}$, and $\hat{P}_{2,3}$, and determine the average distance from P to $\hat{c}$ (use the 2-norm). To do this, for each $\sigma \in \{0.1, 0.2, \ldots, 0.9, 1\}$, run an algorithm to determine $\hat{P}_{1,2}, \hat{P}_{1,3}$, and $\hat{P}_{2,3}$ 100,000 times and determine the percentage of times that P was inside the triangle and the average distance to $\hat{c}$. In a figure, plot the percentage of times that P was inside the triangle versus σ. In another figure, plot the average distance of P to $\hat{c}$ versus σ. Is there a trend in these two figures? Can you explain why?

HINT 1: To obtain the position from bearings, you can use the formulas from Problem 1 in Homework 3.

HINT 2: To determine whether a point is inside a triangle, you can use the algorithm given in `https://www.geeksforgeeks.org/check-whether-a-given-point-lies-inside-a-triangle-or-not/`. To calculate the area of a triangle, you may use Heron's formula.

HINT 3: In the previous method, instead of the equality comparison, obtain the absolute value of the difference between the left- and right-hand sides of the equality and determine whether this value is lower that $\varepsilon = 10^{-6}$.

HINT 4: As a clarification, the structure of the code after determining θ_1, θ_2, and θ_3 should be similar to the following snippet

```
for jj = 1:10 % For each sigma value
    % Do some setup stuff (if needed)
    for ii = 1:100000
        % Calculate 3 estimated positions based on noisy bearing measurements.

        % Determine whether actual position is inside the triangle formed by estimated
        % positions

        % Determine distance from triangle centroid to actual position

        % Save these results somewhere for use outside current loop
    end
    % Use data obtained in inner loop to calculate average distance from
    % triangle centroid to actual position and the percentage of times the
    % actual position was inside the triangle formed by the estimated
    % positions.
end
```

Problem 2.

Let w be a point on the Sun with zero inertial acceleration, let F_A be an inertial frame, let Astronomical Unit (AU) be the unit of length, and let radian (rad) be the unit of angle. Let L_1 and L_2 be points representing the locations of Mars and the Sun, respectively, such that

$$\vec{r}_{L_1/w}\big|_{\mathrm{A}} = \begin{bmatrix} 1.52 \\ 0 \\ 0 \end{bmatrix} \text{AU}, \quad \vec{r}_{L_2/w}\big|_{\mathrm{A}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{AU}. \tag{4}$$

Let the unit vectors $\hat{s}_1, \hat{s}_2$ be such that

$$\hat{s}_1\big|_{\mathrm{A}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{s}_2\big|_{\mathrm{A}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5}$$

For all $k \in \{0, \ldots, 50\}$, $i \in \{1, 2\}$, let P_k denote the location of a spacecraft at step k, let $\psi_{2,\hat{s}_i,k}$ be the bearing measurement of the spacecraft location P_k from point L_2 relative to a star in the direction $\hat{s}_i$, and let $\theta_{L_1/L_2,k}$ be the subtended angle between the points L_1, L_2 from the spacecraft location P_k. Suppose that $\vec{r}_{P_0/w}\big|_{\mathrm{A}} = \begin{bmatrix} 0.52 & 0 & -1 \end{bmatrix}^{\mathrm{T}}$ AU, and that, for all $k \in \{1, \ldots, 50\}$, the bearings and subtended angles are given in the file AE584_Midterm_P2.mat, such that $\psi_{2,\hat{s}_1,k} = \text{bearingL2St1}(k), \psi_{2,\hat{s}_2,k} = \text{bearingL2St2}(k)$, and $\theta_{L_1/L_2,k} = \text{subAngL1L2}(k)$.

For all $k \in \{1, \ldots, 50\}$, determine $\vec{r}_{P_k/w}\big|_{\mathrm{A}}$ from the given bearings and subtended angles. Then, for all $k \in \{0, \ldots, 50\}$, plot each component of $\vec{r}_{P_k/w}\big|_{\mathrm{A}}$ versus step k in a 3-by-1 subfigure. Finally, in a 3D plot, use scatter3 to place points on the locations of the Sun and Mars in yellow and red, respectively, and plot the 3D trajectory of the spacecraft.

HINT: In order to determine the current position, you can use the previous position as the initial guess.

Problem 3.
Let $\mathrm{F_I}$ be an inertial frame, and let the Earth frame $\mathrm{F_E}$ be obtained by applying a 3-1-3 sequence of Euler-angle rotations to $\mathrm{F_I}$, where Φ_E, Θ_E, and Ψ_E denote the precession, nutation, and spin angles, respectively. The frames $\mathrm{F_E}$ and $\mathrm{F_I}$ are thus related by

$$\mathrm{F_I} \xrightarrow[3]{\Phi_\mathrm{E}} \mathrm{F_{E'}} \xrightarrow[1]{\Theta_\mathrm{E}} \mathrm{F_{E''}} \xrightarrow[3]{\Psi_\mathrm{E}} \mathrm{F_E}. \tag{6}$$

Furthermore, let $\mathrm{F_B}$ be a body-fixed frame obtained by applying a 3-2-1 sequence of Euler-angle rotations to $\mathrm{F_E}$, where Φ, Θ, and Ψ denote the bank, elevation, and azimuth angles, respectively. The frames $\mathrm{F_B}$ and $\mathrm{F_E}$ are thus related by

$$\mathrm{F_E} \xrightarrow[3]{\Psi} \mathrm{F_{B'}} \xrightarrow[2]{\Theta} \mathrm{F_{B''}} \xrightarrow[1]{\Phi} \mathrm{F_B}. \tag{7}$$

Assume that, for all $t \geq 0$,

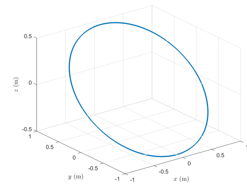
$$\vec{\omega}_{\mathrm{B/I}}\big|_\mathrm{B}(t) = \begin{bmatrix} \cos 2t \\ \cos 2t \\ 0.025t \end{bmatrix}, \tag{8}$$

and that $\mathcal{O}_{\mathrm{B/I}}\big|_\mathrm{I}(0) = I_3$, where I_3 is the 3-by-3 identity matrix. Furthermore, assume that, for all $t \geq 0$,

$$\dot{\Phi}_\mathrm{E}(t) = \sin 0.05t, \quad \dot{\Theta}_\mathrm{E}(t) = 0.3\cos 0.01t, \quad \dot{\Psi}_\mathrm{E}(t) = 0.5 \sin 0.01t, \tag{9}$$

and that $\Phi_\mathrm{E}(0) = 0$ rad, $\Theta_\mathrm{E}(0) = \pi/6$ rad, and $\Psi_\mathrm{E}(0) = 0$ rad. For all $t \in \{0, 0.01, \ldots, 9.99, 10\}$ s, determine $\mathcal{O}_{\mathrm{B/E}}(t)$. Plot all components of $\mathcal{O}_{\mathrm{B/E}}$ versus time in a 3-by-3 figure grid using the subplot function. Also, plot Φ, Θ, and Ψ versus time in a 3-by-1 figure grid using the subplot function.

i.e., the trajectory is slopped with respect to the horizontal plane; e.g., as in the plot to the right:



Problem 4.

Let $g = 9.80665$ m/s^2 be the acceleration due to gravity, and let $\phi = \pi/6$ rad. Suppose that a 3-axis accelerometer and a 3-axis rate gyro are attached to a quadcopter following an <u>inclined, circular trajectory.</u> Let $\mathrm{F_A}$ be an inertial frame, let $\mathrm{F_B}$ be a frame fixed to the quadcopter, and suppose that the axes of both the rate gyro and the accelerometer are aligned with $\mathrm{F_B}$. Let c be the center of mass of the quadcopter, let w be a point with zero inertial acceleration, and let $\vec{r}_{c/w}(t)$ and $\mathcal{O}_{\mathrm{B/A}}(t)$ be the position vector of the quadcopter center of mass and the orientation matrix of $\mathrm{F_B}$ relative to $\mathrm{F_A}$ at time t, respectively. Furthermore, suppose that, for all $t \in [0, 20]$ s, the measurements from the sensors are given by

$$\vec{\omega}_{\mathrm{B/A}}\big|_{\mathrm{B}}(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ rad/s}, \tag{10}$$

$$\vec{a}_{c/w/A}\big|_{\mathrm{B}}(t) + \vec{g}\big|_{\mathrm{B}}(t) = \vec{a}_{c/w/A}\big|_{\mathrm{B}}(t) + \mathcal{O}_{\mathrm{B/A}}(t) \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} = \begin{bmatrix} -1 - g\sin\phi\sin t \\ -g\sin\phi\cos t \\ -g\cos\phi \end{bmatrix} \text{ m/s}^2, \tag{11}$$

and that

$$\vec{r}_{c/w}\big|_{\mathrm{A}}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ m}, \tag{12}$$

$$\overset{\mathrm{A}\bullet}{\vec{r}}_{c/w}\big|_{\mathrm{A}}(0) = \begin{bmatrix} 0 \\ \cos\phi \\ \sin\phi \end{bmatrix} \text{ m/s}, \quad \mathcal{O}_{\mathrm{B/A}}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}. \tag{13}$$

For all $t \in \{0, 0.01, \ldots, 19.99, 20\}$ s, determine $\vec{r}_{c/w}\big|_{\mathrm{A}}(t)$ and $\mathcal{O}_{\mathrm{B/A}}(t)$. In a figure, plot all components of $\mathcal{O}_{\mathrm{B/A}}$ versus time in a 3-by-3 figure grid using the subplot function. In another figure, plot the 3 components of $\vec{r}_{c/w}\big|_{\mathrm{A}}$ versus time in a 3-by-1 figure grid using the subplot function. Finally, in a 3D figure, plot the 3D trajectory of the center of mass of the quadcopter.

HINT: Use a single ode45 call to integrate the rate-gyro and accelerometer measurements.