AEROSP 584
Homework 3

Due Friday, October 21, by 11:59 PM.

Instructions:

1. Please check out and follow the homework preparation and uploading guidelines posted in the Section "Assignment Policies and Specification" of the syllabus, especially with regard to how to submit if you are working on a team (optional for Homework 2).

   Extra credit will occasionally be given for especially neat work.

2. Attach all your Matlab and Simulink code in a single zip file, along with a single PDF file with your answers to the homework problems; please include in the PDF a copy-paste of your code.

3. In this homework, you will be using the MATLAB function *fminunc*. A tutorial has been posted on Canvas; please follow this link to find it. The tutorial has code that will be useful for the assignment.

4. Please read and follow the honor code guidelines in the Section "Additional Policies" of the syllabus.

Problem 1. Consider a map in 2D where points have coordinates (x, y), where x increases along East and y increases along North. Bearing measurements are relative to North, where positive angles represent clockwise rotations as seen from above. Let L_1 and L_2 be points on the map representing the locations of lighthouses, and let P denote your location. The coordinates of L_1 are (x_1, y_1), the coordinates of L_2 are (x_2, y_2), and your coordinates are (x_0, y_0). For $i = 1, 2$, denote the bearing of L_i relative to North by θ_i. For $i = 1, 2$, denote the ray determined by P and L_i by $y = m_i x + b_i$.

a) Derive the expressions for m_i and b_i given by

$$m_i = \frac{y_0 - y_i}{x_0 - x_i} = \tan\left(\frac{\pi}{2} - \theta_i\right), \tag{1}$$

$$b_i = \frac{x_0 y_i - x_i y_0}{x_0 - x_i}. \tag{2}$$

As a check, derive

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} m_i & -1 \\ b_i - y_i & x_i \end{bmatrix}^{-1} \begin{bmatrix} m_i x_i - y_i \\ b_i x_i \end{bmatrix}. \tag{3}$$

b) Derive the expression for (x_0, y_0) in terms of the lighthouse locations and bearing measurements given by

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} T_1 & -1 \\ T_2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} T_1 x_1 - y_1 \\ T_2 x_2 - y_2 \end{bmatrix}, \tag{4}$$

where $T_i \triangleq \tan\left(\frac{\pi}{2} - \theta_i\right)$.

c) Let $(x_1, y_1) = (0, 0)$ m, $(x_2, y_2) = (4, 2)$ m, and let $\theta_1 = -165^\text{o}, \theta_2 = 150^\text{o}$. Write a Matlab program that obtains (x_0, y_0) using the derived equations. In a plot, place green dots on the locations of the lighthouses and place a red dot on the obtained location.

d) Suppose there is a third lighthouse L_3 located at (x_3, y_3) m that yields a bearing θ_3. Let $(x_1, y_1) = (0, 0)$ m, $(x_2, y_2) = (4, 2)$ m, $(x_3, y_3) = (1, 4)$ m, and let $\theta_1 = -140^\text{o}, \theta_2 = 90^\text{o}$, $\theta_3 = -30^\text{o}$. Obtain the 3 position fixes that each pair of lighthouses yield using your code. In a plot, place green dots on the locations of the lighthouses, place blue dots on the locations of the obtained position fixes and place a red dot at the center of the resultant "three-cornered hat" (the mean of the three vertices you obtained).

Problem 2. Reconsider Problem 1, but now we will use the equations I derived in the notes for determining the position fix (2D Position Fixing with 1 Subtended Angle and 1 Bearing section in the lecture 4 notes). We'll use fminunc for this problem.

a) Let $(x_1, y_1) = (0, 0)$ m, $(x_2, y_2) = (4, 2)$ m, and let $\theta_1 = -165^\text{o}, \theta_2 = 150^\text{o}$. Write a Matlab program that obtains (x_0, y_0) by solving the two equations in the lecture notes using numerical optimization (fminunc). To do this, use a grid of initial points $p_\text{init} \in \{-2, 0, \ldots, 8, 10\} \times \{-2, 0, \ldots, 8, 10\}$ and choose the result that yields the lowest cost function. In a plot, place green dots on the locations of the lighthouses and place a red dot on the obtained location.

NOTE 1: The equations in the lecture notes are derived such that the bearing used corresponds to lighthouse 2. The subtended angle can be obtained from the pair of bearings. For example, in this case, the subtended angle can be chosen as 45^o. Finally, remember that the star used for the bearings is in the direction of the North.

NOTE 2: Due to the structure of your optimization cost function, the positions of the lighthouses may yield a cost function value close to 0. You can assume that the position fix cannot be within $\varepsilon = 0.01$ m of both lighthouses.

b) Suppose there is a third lighthouse L_3 located at $(x_3, y_3)m$ that yields a bearing θ_3. Let $(x_1, y_1) = (0, 0)$ m, $(x_2, y_2) = (4, 2)$ m, $(x_3, y_3) = (1, 4)$ m, and let $\theta_1 = -140^o, \theta_2 = 90^o$, $\theta_3 = -30^o$. Obtain the 3 position fixes that each pair of lighthouses yield via numerical optimization. To do this, for each position fix, use a grid of initial points $p_{\text{init}} \in \{-2, 0, \ldots, 8, 10\} \times \{-2, 0, \ldots, 8, 10\}$ and choose the result that yields the lowest cost function. In a plot, place green dots on the locations of the lighthouses, place blue dots on the locations of the obtained position fixes and place a red dot at the center of the three-cornered hat (the mean of the three vertices you obtained).

NOTE: As in a), for every pair of lighthouses, you'll have to determine the subtended angle and use the bearing of a single lighthouse. Take into account the notes of a) as well.

Problem 3. Now consider the 3D case with 2 lighthouses and 1 star (1 subtended angle and 1 bearing angle) and consider that all points have coordinates (x, y, z). Let L_1 and L_2 be points representing the locations of the lighthouses and let P denote your location, with coordinates denoted by (x_1, y_1, z_1) m, (x_2, y_2, z_2) m, and (x_0, y_0, z_0) m respectively. Let $\psi_{2,\hat{r}}$ be the bearing of L_2 relative to a star in the direction of the $\hat{r}$ unit vector, and let θ be the subtended angle.

In this case, suppose that $(x_1, y_1, z_1) = (0, 0, 0)$ m, $(x_2, y_2, z_2) = (5, 0, 0)$ m, $\hat{r}$ points in the direction of the positive y-axis, $\psi_{2,\hat{r}} = 135^o$, and $\theta = 90^o$.

Use fminunc to find possible position fixes. To do this, use a grid of initial points $p_{\text{init}} \in \{-5, -3, \ldots, 3, 5\} \times \{-5, -3, \ldots, 3, 5\} \times \{-5, -3, \ldots, 3, 5\}$ and save all obtained solutions.

In a figure, plot the 2D projections (example in course files) of your obtained solutions using blue points and plot the projections of your actual location $(x_0, y_0, z_0) = (2.5, 2.5, 0)$ m as a red point. What can you say about the distribution of the obtained solutions?

Then, in another figure, plot a sphere using the command [Xsp,Ysp,Zsp] = sphere followed by surf(2.5.*Xsp + 2.5,2.5.*Ysp,2.5.*Zsp,'FaceAlpha',0.25), and plot all of your obtained solutions as blue points using the scatter3 function. What do you observe? Can you explain this using the theory discussed in class?

Problem 4. Now consider the 3D case with 2 lighthouses and 2 stars (1 subtended angle and 2 bearing angles). Following from the setup of Problem 3, let $\psi_{2,\hat{v}}$ be the bearing of L_2 relative to a star in the direction of the $\hat{v}$ unit vector.

In this case, suppose that $(x_1, y_1, z_1) = (0, 0, 0)$ m, $(x_2, y_2, z_2) = (5, 0, 0)$ m, $\hat{r}$ points in the direction of the positive y-axis, $\psi_{2,\hat{r}} = 135^o$, $\hat{v}$ points in the direction of the positive z-axis, $\psi_{2,\hat{v}} = 90^o$, and $\theta = 90^o$.

Use fminunc to find your location. To do this, use a grid of initial points $p_{\text{init}} \in \{-5, -3, \ldots, 3, 5\} \times \{-5, -3, \ldots, 3, 5\} \times \{-5, -3, \ldots, 3, 5\}$ and choose the result that yields the lowest cost function. In a figure, plot the 2D projections (example in course files) of the lighthouses using green points and plot the projections of your obtained location as a red point. Is it the same location as the actual location mentioned in Problem 3?

NOTE: As in Problem 2, the positions of the lighthouses may yield a cost function value close to 0. You can assume that the position fix cannot be within $\varepsilon = 0.01$ m of both lighthouses.

Problem 5. Reconsider Problem 4 but now assume that that all angle measurements $(\theta, \psi_{2,\hat{r}}, \psi_{2,\hat{v}})$ are corrupted by Gaussian noise with mean 0^o and standard deviation 2^o (add 2*randn(1) to your measurements in degrees).

To do this, use a grid of initial points $p_{\text{init}} \in \{-1, 2, 5\} \times \{-1, 2, 5\} \times \{-1, 2, 5\}$ and choose the result that yields the lowest cost function. Do this 200 times with different sensor noise for each measurement each time.

In a figure, plot the 2D projections (example in course files) of the lighthouses using green points, plot the projections of your obtained locations as blue points, and plot the projection of your actual location (x_0, y_0, z_0) = $(2.5, 2.5, 0)$ m as a red point.

In another figure, plot a histogram of the position fix errors (2-norm of difference between obtained location and actual location), and compute the mean and standard deviation of the error. For the histogram, use 20 bins.