We have done position fixing BUT it's only about estimating quantities at a fixed time instant.

Dead reckoning is the next step forward, toward estimating quantities over time

Note: Applying the methods of position fixing over time is also an option (i.e., at each time instant solve the position fixing equations).

Inertial navigation is Dead reckoning using only:

- initial position/attitude
- acceleration
- angular velocity
- time

Example

Most engineers have heard of it.
Few know what it is.
Many subtleties and intricacies.

It took a long time to understand that Inertial Navigation was even possible. We will see why as we examine how the accelerometer and the gyro work, and as we read the relevant resources I have posted on the syllabus (please read them ALL)

All in all, Inertial Navigation is a system technology that depends on many technological advances.

Note: Has the GPS made IN obsolete?

The short answer is No:

- GPS doesn't work everywhere:
  - under water / underground
  - indoors
  - outside Earth :)

- GPS can be jammed and spoofed:

GPS is a public signal:

  - jamming: stop reception of signal (Denial-of-service attack)

  - spoofing: corruption of the signal (deception attack)

(typically, one first jams and then

spoofs by replacing the jammed signal with a desired, deceiving signal)

- GPS has bandwidth ~10Hz. But control of agile vehicles requires control at frequences ≥ 1-10kHz! GPS cannot keep up with that frequency of updating measurements.

Instead, IN sensor have typically bandwidth ≥ 10kHz!

Overall, GPS cannot replace IN!

Instead, it can only support it.

How specifically?

IN drifts, due to bias and noise (recall our discussions about increasing error and/or increasing variance of error in the presence of bias and/or noise). That is, even minuscule errors blow up over time (that is, IN drifts!)

If every once in a while we take into account of GPS measurements (e.g., via Kalman filtering), this drift is (largely) corrected!

SLAM is another way to (largely)

correct INS's drift. We will study this later in this course.

Note (inside the note): GPS also suffers from inaccuracies. E.g., due errors from numerical computations, and atmospheric effects.

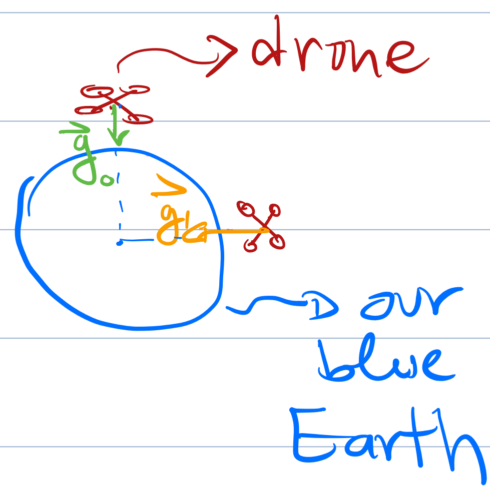
We dive into IN by first examining how an accelerometer works.

We will see that an accelerometer needs to know what is the direction of the gravity ($\vec{g}$). If $\vec{g}$ is unknown, an accelerometer is useless.

This necessity to know $\vec{g}$, naturally leads to pairing accelerometers and gyros so the change in $\vec{g}$'s direction is detected.

Note: $\vec{g}$ changes with time (i.e, $\vec{g} = \vec{g}(t)$) when we travel great distances!

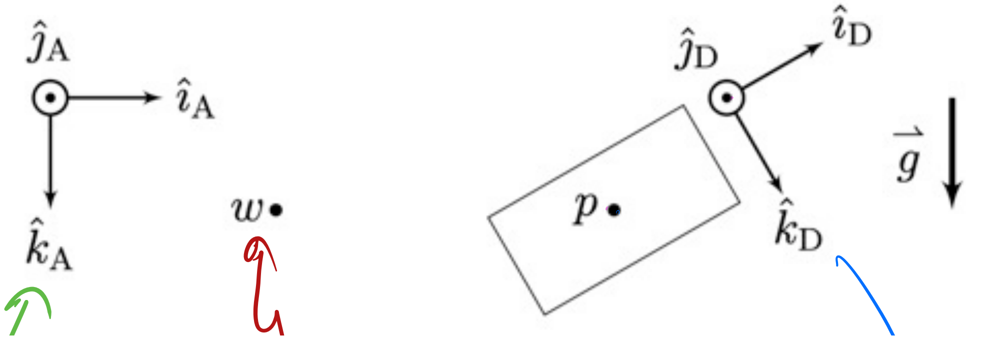
Example:



Note: Tracking $\vec{g}(t)$ accurately connects with the "Schuler pendulum" and the latter's role in "Inertially Stabilized Platforms".

Accelerometer

What an accelerometer measures?



inertial frame

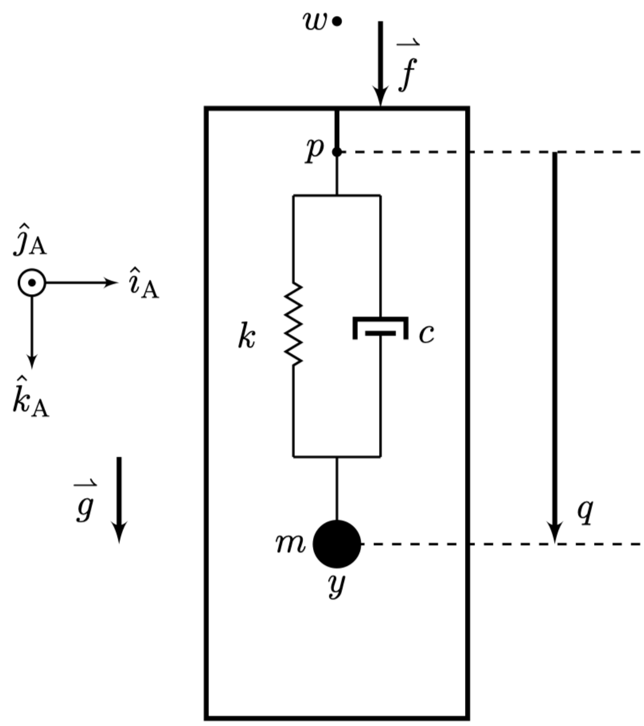
a static point

frame aligned with the directions of acceleration the accelerometer is measuring along

The accelerometer measures:

$$a_{meas}(t) \triangleq \vec{a}_{P/W/A}(t)\Big|_D + \vec{g}\Big|_D$$

To demonstrate the above, consider the 1-axis accelerometer below:



- k is the stiffness of the spring
- c is the damping coefficient of a dashpot
- m is the mass of the proofmass

- spring and dashpot are aligned with gravity
- spring and dashpot are massless

- $\vec{f}$ is external force (along $\hat{k}_A$ axis)

- y is location of proofmass m
- q is its displacement from p

- M is the mass of the accelerometer's box, and such that center of mass of accelerometer is at p:

$$\vec{r}_{cm/p} = \frac{M}{M+m}\vec{r}_{p/p} + \frac{m}{M+m}\vec{r}_{y/p} = \frac{m}{M+m} q \hat{k}_A \quad (7)$$

It is: $\vec{g} = g\hat{k}_A$

$$\vec{f} = f\hat{k}_A$$

$$\vec{a}_{p/w/A} = a\hat{k}_A$$

Differentiating

$$\vec{r}_{\mathrm{cm}/w} = \vec{r}_{\mathrm{cm}/p} + \vec{r}_{p/w}, \tag{8}$$

twice with respect to $\mathrm{F_A}$ yields

$$\vec{a}_{\mathrm{cm}/w/\mathrm{A}} = \vec{a}_{\mathrm{cm}/p/\mathrm{A}} + \vec{a}_{p/w/\mathrm{A}}. \tag{9}$$

Now, it follows from Euler's first law applied to the center of mass of the accelerometer [32, p. 340] that

$$(M+m)\vec{a}_{\mathrm{cm}/w/\mathrm{A}} = \vec{f} + (M+m)\vec{g}, \tag{10}$$

and thus (9) and (10) imply that

$$(M+m)\vec{a}_{\mathrm{cm}/p/\mathrm{A}} + (M+m)\vec{a}_{p/w/\mathrm{A}} = \vec{f} + (m+M)\vec{g}. \tag{11}$$

(7) above

Next, note that (7) implies that

$$\vec{a}_{\mathrm{cm}/p/\mathrm{A}} = \frac{m}{M+m}\ddot{q}\hat{k}_{\mathrm{A}}. \tag{12}$$

Hence, resolving (11) in $\mathrm{F_A}$ and using (7) yields

$$m\ddot{q} + (M+m)a = f + (m+M)g. \tag{13}$$

The acceleration to be measured is thus given by

$$a = \frac{1}{M+m}f + g - \frac{m}{M+m}\ddot{q}. \tag{14}$$

In practice, f is usually unknown, and thus (14) is not useful.

Now, letting $\vec{f}_{\mathrm{proofmass}} \triangleq -(c\dot{q}+kq)\hat{k}_{\mathrm{A}}$ denote the force applied to the proofmass by the spring and dashpot, Newton's second law implies that

$$m\vec{a}_{y/w/\mathrm{A}} = \vec{f}_{\mathrm{proofmass}} + m\vec{g}. \tag{15}$$

Therefore,

$$m\vec{a}_{y/p/\mathrm{A}} + m\vec{a}_{p/w/\mathrm{A}} = \vec{f}_{\mathrm{proofmass}} + m\vec{g}, \tag{16}$$

which implies that

$$m\ddot{q} + ma = -(c\dot{q}+kq) + mg, \tag{17}$$

that is,

$$m\ddot{q} + c\dot{q} + kq = m(g - a). \tag{18}$$

In the case where the box has zero inertial acceleration, that is, $a \equiv 0$, it follows that

$$m\ddot{q} + c\dot{q} + kq = mg, \tag{19}$$

and thus $q(t) \to mg/k$ as $t \to \infty$, whereas, in the case where the box is freefalling, that is, $a \equiv g$, it follows that

$$m\ddot{q} + c\dot{q} + kq = 0, \tag{20}$$

and thus $q(t) \to 0$ as $t \to \infty$.

Next, it follows from (18) that the inertial acceleration is

$$a = g - \ddot{q} - \frac{c}{m}\dot{q} - \frac{k}{m}q. \tag{21}$$

Assuming that m, c, k, and g are known and that q, $\dot{q}$, and $\ddot{q}$ are measured, (21) provides the measurement

$$a_{\rm meas} = g - \ddot{q} - \frac{c}{m}\dot{q} - \frac{k}{m}q \tag{22}$$

of the inertial acceleration. When the box has zero inertial acceleration, $a(t) = 0$, and thus $q(t) \to mg/k$ as $t \to \infty$, it follows that $a_{\rm meas}(t) \to 0$ as $t \to \infty$, and, when the box is freefalling, $a(t) = g$,, and thus $q(t) \to 0$ as $t \to \infty$, it follows that $a_{\rm meas}(t) \to g$ as $t \to \infty$.

In practice, however, it is typically difficult to measure or estimate $\dot{q}$ and $\ddot{q}$, and thus a practical measurement $\hat{a}_{\rm meas}$ is

$$\hat{a}_{\rm meas} = g - \frac{k}{m}q. \tag{23}$$

Hence, when $\dot{q}(t) \approx 0$ and $\ddot{q}(t) \approx 0$, it follows that $\hat{a}_{\rm meas}$ provides a useful estimate of the inertial acceleration (21). The accuracy of (23) depends on the settling time of the proofmass when a varies slowly over time.

Another impediment to using (21) to estimate the inertial acceleration of the box arises from the fact that the box may be rotated by the angle θ relative to the vertical direction. In this case, (21) becomes

* (see below)

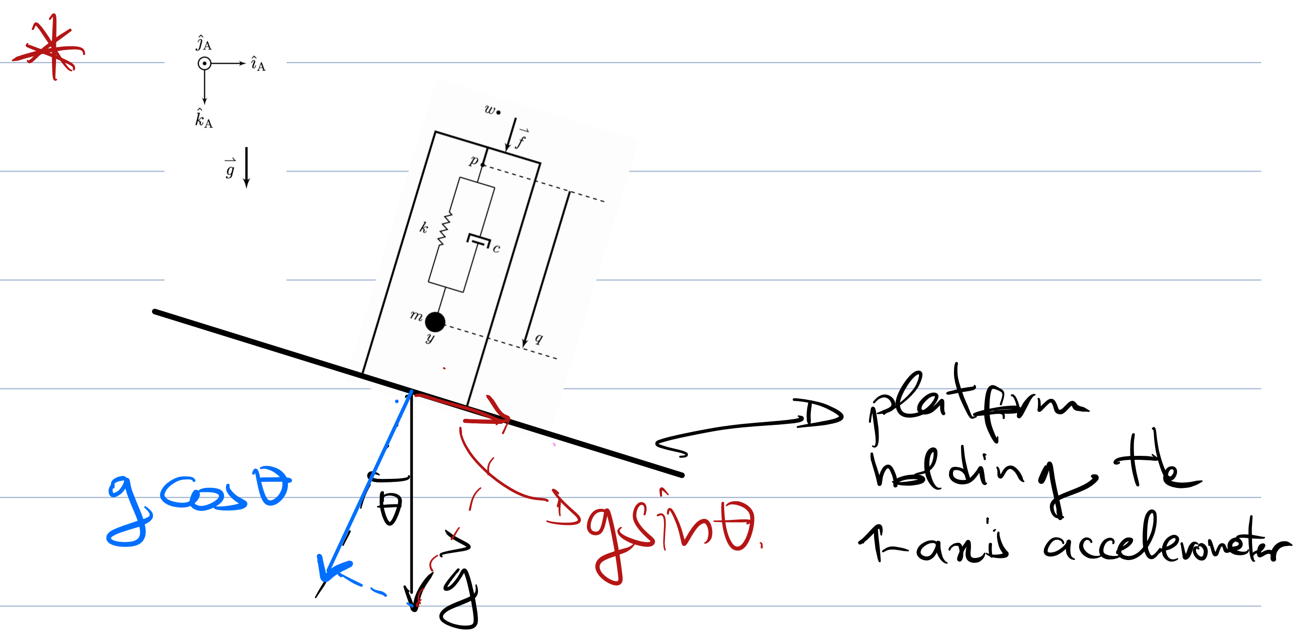
$$a = (\cos\theta)g - \ddot{q} - \frac{c}{m}\dot{q} - \frac{k}{m}q. \tag{24}$$

Note that (24) involves θ, which must be know in order to correctly determine a. If, however the gravity term is ignored along with the $\dot{q}$ and $\ddot{q}$ terms, then the resulting measurement is

$$\hat{a}_{\rm meas} = -\frac{k}{m}q \approx a - (\cos\theta)g. \tag{25}$$

In this case, when the box has zero inertial acceleration, it follows that $\hat{a}_{\text{meas}}(t) \to -(\cos\theta)g$ as $t \to \infty$, and, when the box is freefalling, it follows that $\hat{a}_{\text{meas}}(t) \to 0$ as $t \to \infty$. Both of these measurements are erroneous, however, which demonstrates the need to know the direction θ of gravity relative to the direction of the accelerometer in order to obtain the correct offset measurement

$$\hat{a}_{\text{meas,g}} = (\cos\theta)g - \frac{k}{m}q. \tag{26}$$



All in all, in (2b):

$$\hat{a}_{meas} = (\cos\theta)\, g - \frac{k}{m}\, g$$

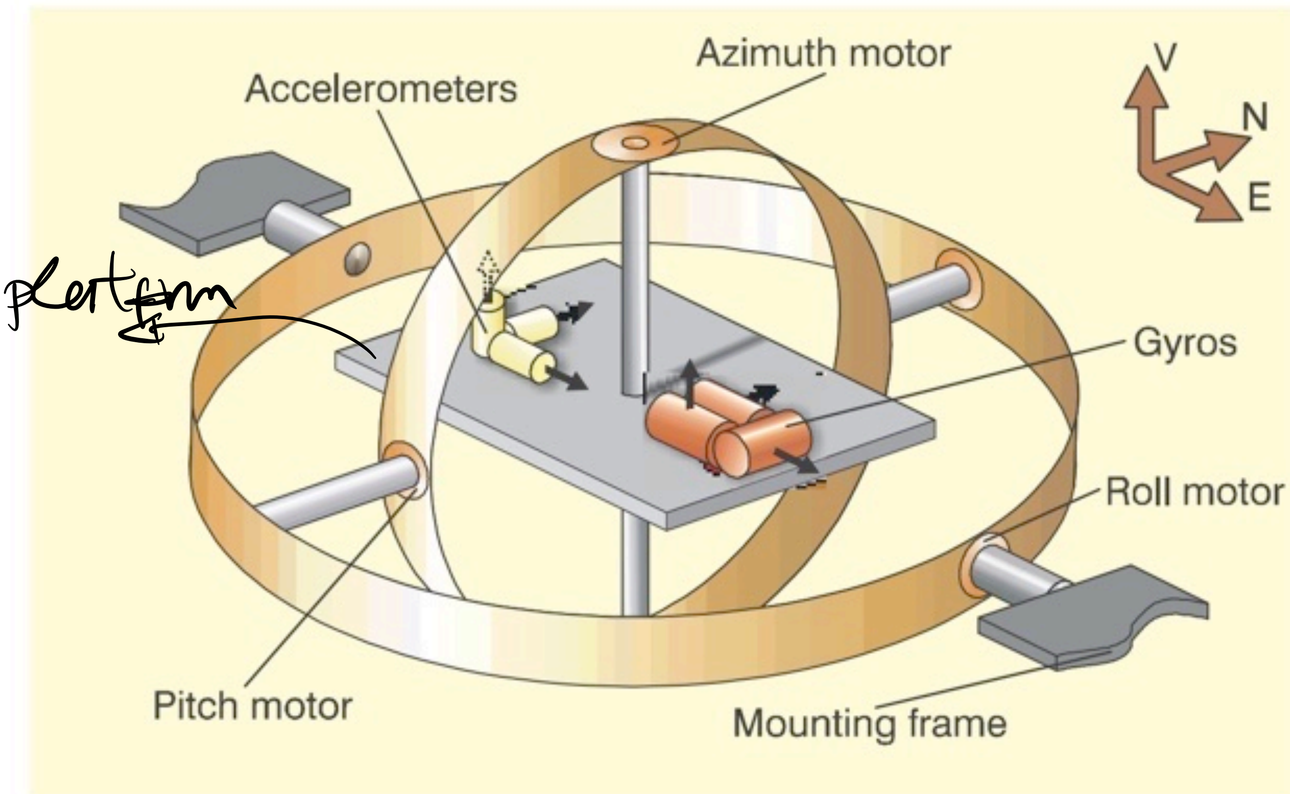
need to know θ

$\Downarrow$ known/calibrated

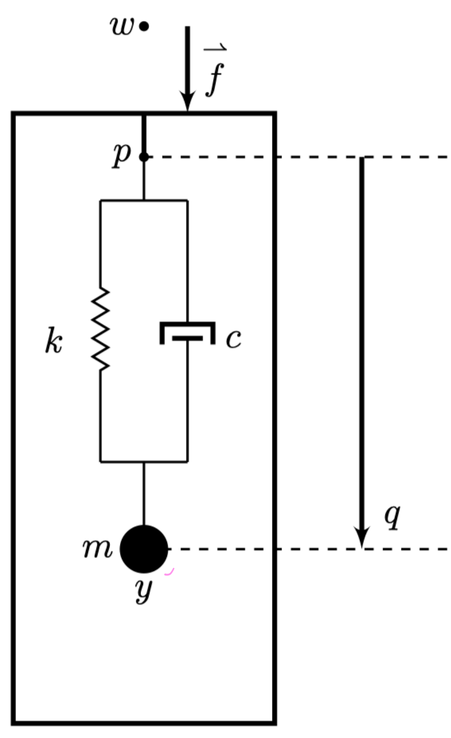
$\Longrightarrow$ We need to know attitude to measure inertial acceleration!

$\Longrightarrow$ pair 1-axis accelerometer with 1-axis gyro!

In 3D problems:



Note 1: In practice, we make the accelerometers such that the mass m actully doesn't move freely, in contrast to what we depicted before (repeated below):



Instead, m can move only a tiny bit but these small motions are detected and translated to electic signals which hugely amplified!
(recall when we talked about sensor curves, that the input y to a sensor was mapped via an $f(y)$ to some voltage V).

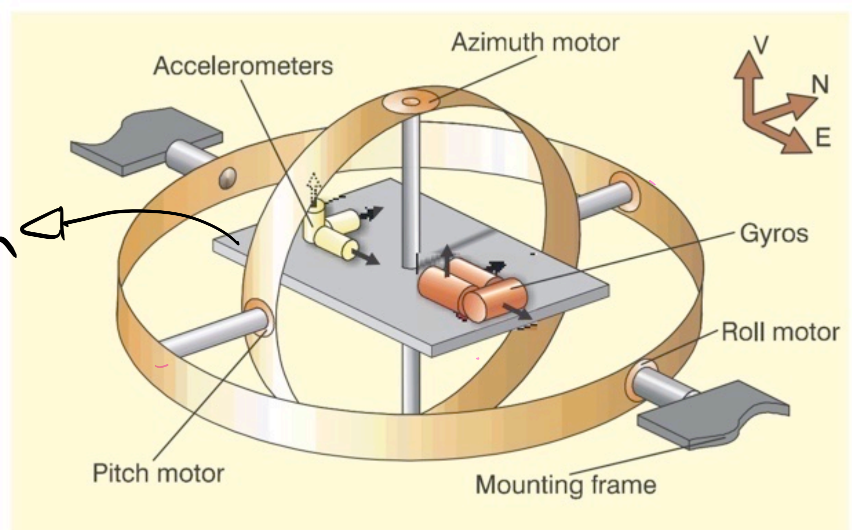
That way,

- the sensivity of the sensor is increased (smaller motion of m are detected)

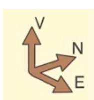
• the life of the sensor is (potentially) prolonged.

Note 2. Similarly, when we have the platform below,

platform



in practice we don't allow it to move freely, but when a certain motion is detected is cancelled out by feedback control via rotors that resist the detected motion. And the motion is similarly translated to voltages and amplified.

This results to the platform to be "inertialy stabilized" with respect to 

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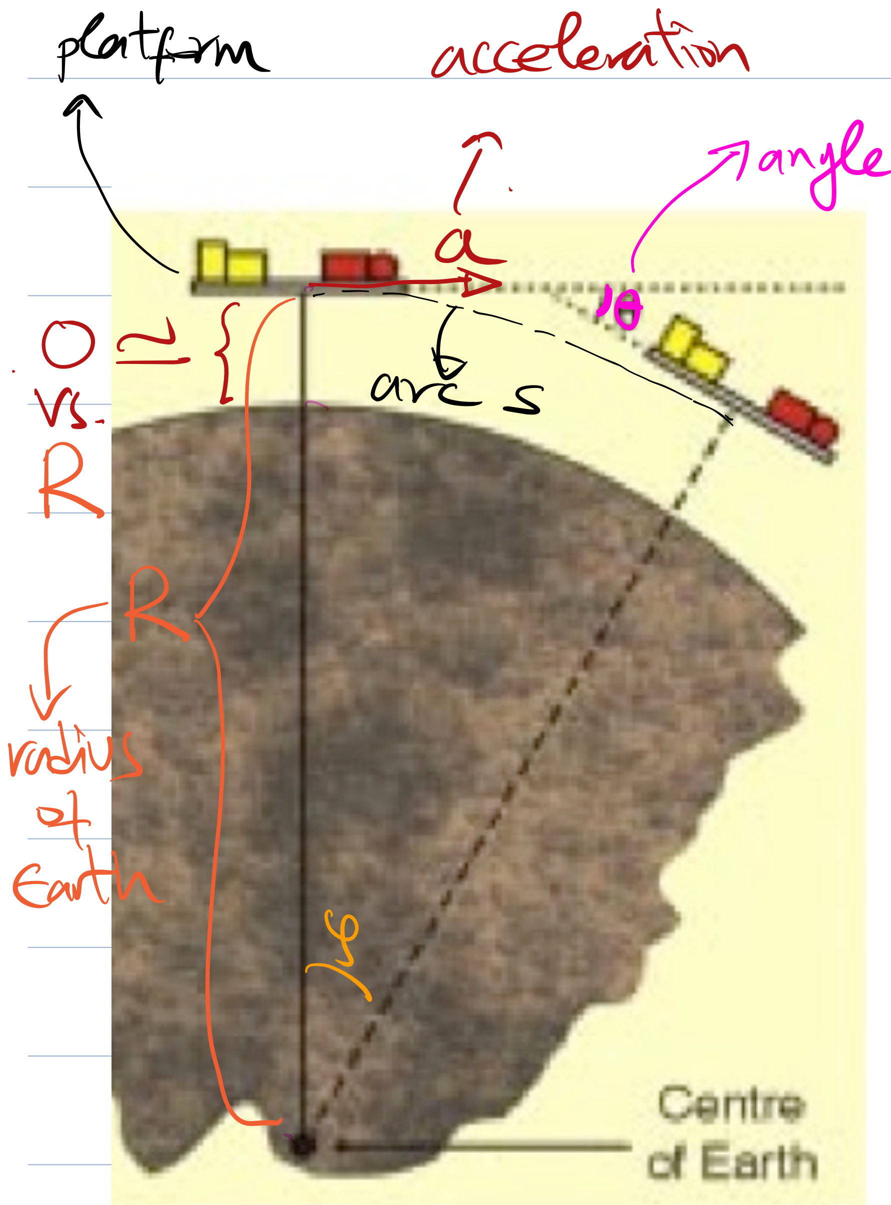
Note 3 But having an inertially stabilized platform is not always correct:



$$s = \varphi R$$
$$\Rightarrow \dot{s} = \dot{\varphi} R$$
$$\Rightarrow \ddot{s} = \ddot{\varphi} R$$
$$\Rightarrow a = \ddot{\varphi} R$$

acceleration measured by accelerometer

<u>Fact</u> Platform maintains local vertical if and only if $\theta = \varphi$

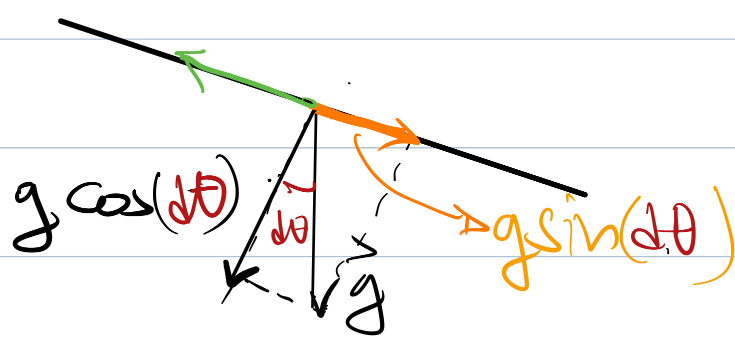
$\Rightarrow$ platform need to be rotated as it travels are s.

Note 4 ("Schuler Tuning").

But the above is not enough!

An error $d\theta$ may always exist:

inertially stabilized platform

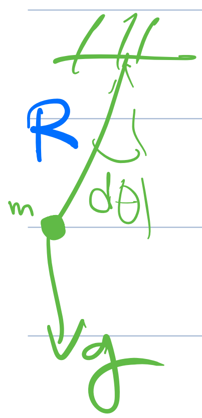


The error $d\theta$ makes the platform think there is an accelation $g\sin(d\theta)$ which needs to be resisted, i.e., corrected by an acceleration $-g\sin(d\theta)$. That is, the platform will need to move per the

$$-g\sin(d\theta) = R(\ddot{d\theta})$$

$$\Longrightarrow R(\ddot{d\theta}) + g\sin d\theta = 0 \qquad (*)$$

$\Longrightarrow$ same equation as that of a pendulum of length R! That's the "Schuler Pendulum" (1923)



$\Longrightarrow$ periodic motion with 84-min period for R equal to Earth's radius.

Note 5 Notes 3-4 essentially say that the platform should be stabilized such that:

$$\theta = \varphi + d\theta .$$

Where:

- $a = \ddot{\varphi} R$ (a and R known)
- $-g\sin(d\theta) = R\, d\ddot{\theta}$ (g and R known)

Intro to Gyros

Let's watch together:

- https://youtu.be/ekzwbt3hu2k
- https://youtu.be/ty9QSiVC2g0

And see these at home:

- https://youtu.be/KToggTKa9Lk
- https://youtu.be/HmmbOVfHqcg