

Lecture 8:

Advanced Binary Trees

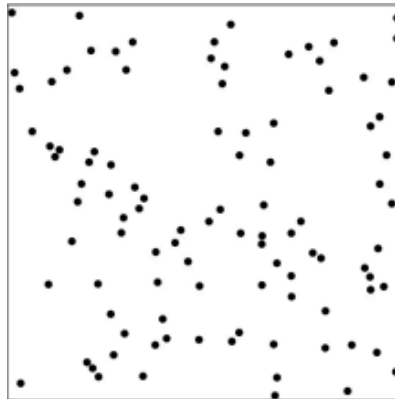
Our Roadmap

- ◆ Priority Queue (binary heap)
 - ◆ Min-heap insert / delete-min
- ◆ Binary Heaps in Dynamic Arrays
 - ◆ $O(n)$ time to build min-heap
- ◆ Binary Search Tree (BST)
 - ◆ BST operators
 - ◆ Balanced BST (AVL-tree)

Priority Queue

- ◆ A priority queue stores a set S of n integers and supports the following operations:
 - ◆ *Insert(e)*: adds a new integer to S
 - ◆ *Delete-min*: removes the smallest integer in S , and returns it.
- ◆ Priority Queue applications:
 - ◆ Artificial intelligence (A* algorithm)
 - ◆ Operating systems (load balancing)
 - ◆ Graph searching (Shortest path algorithms)

8	4	7
1	5	6
3	2	



Priority Queue Example

- ◆ Suppose that the following integers are inserted into an initially empty priority queue
 - ◆ $S = \{93, 39, 1, 26, 8, 23, 79, 54\}$
 - ◆ Perform Delete-min, the operation returns 1, and $S = \{93, 39, 26, 8, 23, 79, 54\}$
 - ◆ Perform Delete-min, the operation returns 8, and $S = \{93, 39, 26, 23, 79, 54\}$
 - ◆ Perform
- ◆ Unlike an ordinary queue (FIFO), a priority queue guarantees that the elements always leave in ascending order (or descending order with *Delete-max*), regardless of the order by which they are inserted.

Priority Queue Implementation

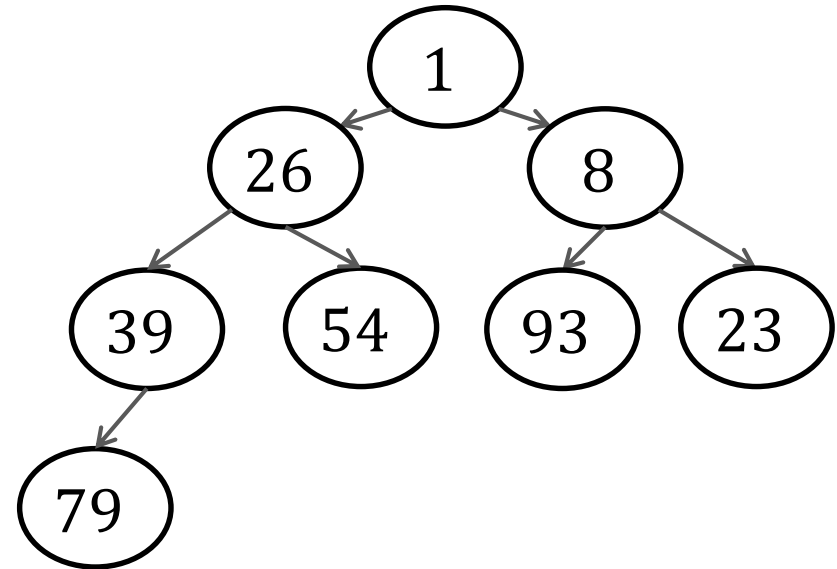
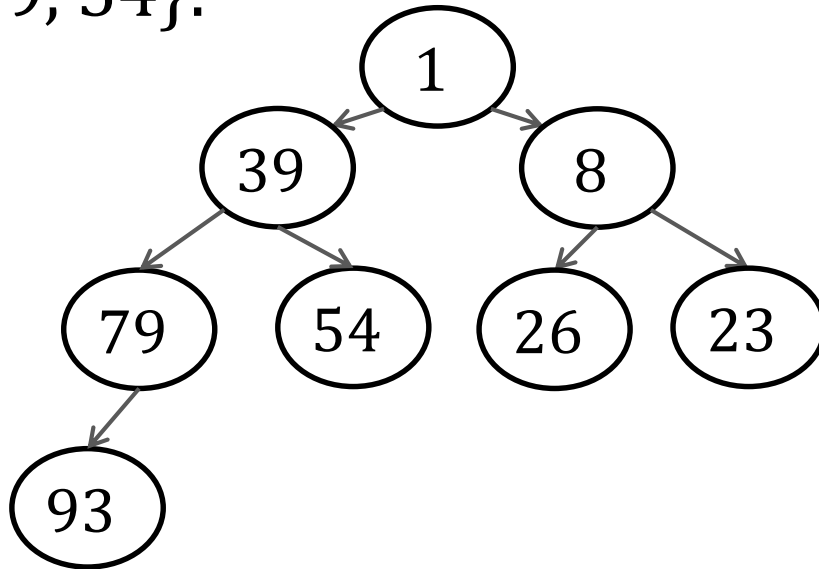
- ◆ We will implement a priority queue using a data structure called the “binary heap” to achieve the following guarantees:
 - ◆ $O(n)$ space consumption
 - ◆ $O(\log n)$ insertion time
 - ◆ $O(\log n)$ delete-min time
- ◆ The binary heap data structure is an array object that we can view as a complete binary tree.
 - ◆ Level 0 to $h-1$ are full
 - ◆ Leaf nodes in level h are “as far left as possible”

Binary Heap

- ◆ Let S be a set of n integers. A binary heap on S is a binary tree T satisfying:
 - ◆ (1) T is complete
 - ◆ (2) Every node u in T corresponds to a distinct integer in S , the integer is called the key of u (and is stored at u)
 - ◆ (3) If u is an internal node, the key of u is smaller than those of its child nodes
- ◆ Note that:
 - ◆ Condition 2 implies that T has n nodes
 - ◆ Condition 3 implies that the key of u is the smallest in the subtree of u

Binary Heap Example

- Two possible binary heaps on $S = \{93, 39, 1, 26, 8, 23, 79, 54\}$:



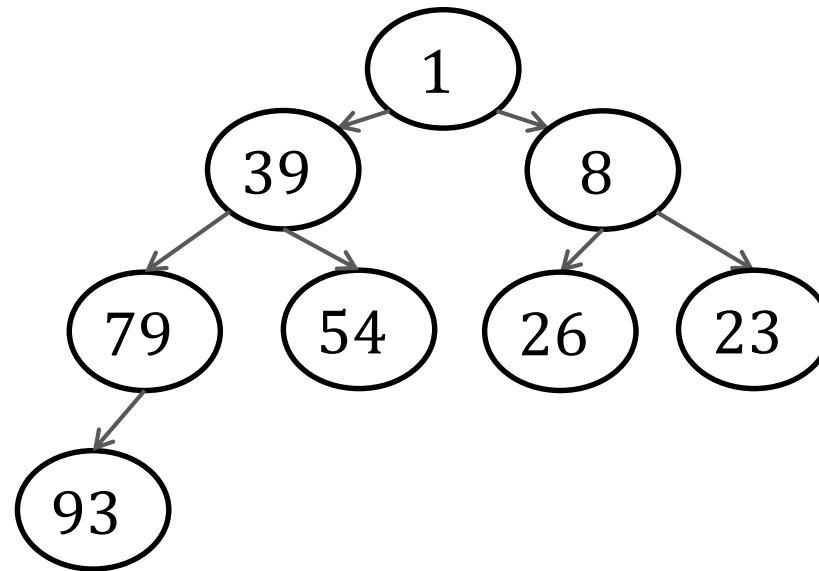
- The binary heaps of a set S is not unique.
- The smallest integer of S must be the key of the root.

Binary Heap Insertion

- ◆ We perform $\text{insert}(e)$ on a binary heap T as follows:
 - ◆ Step 1: Create a leaf node z with key e , while ensuring that T is a complete binary tree, it means there is only one place where z could be added.
 - ◆ Step 2: Set $u \leftarrow z$
 - ◆ Step 3: If u is the root, return.
 - ◆ Step 4: If the key of $u >$ the key of its parent p , return
 - ◆ Step 5: Otherwise, swap the keys of u and p . Set $u \leftarrow p$, and repeat from Step 3.

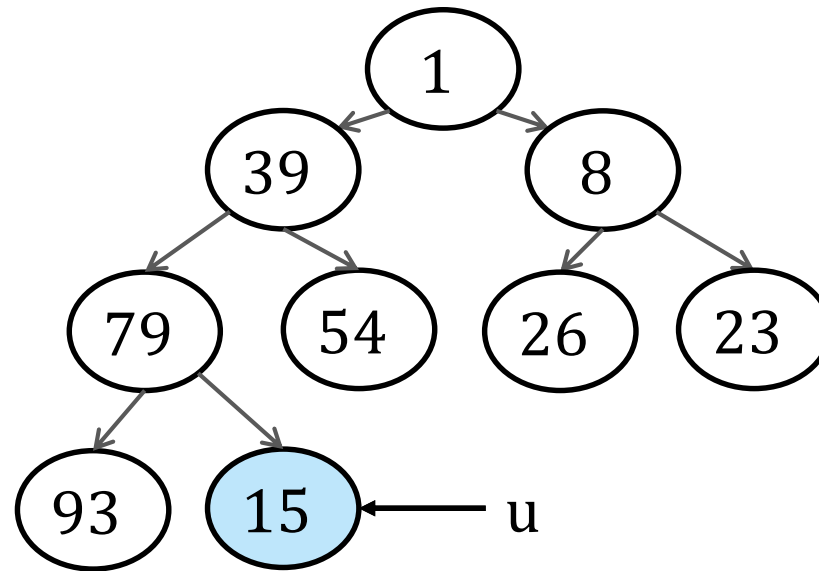
Binary Heap Insertion

- Suppose we want to insert 15 into the binary heap below:



Binary Heap Insertion

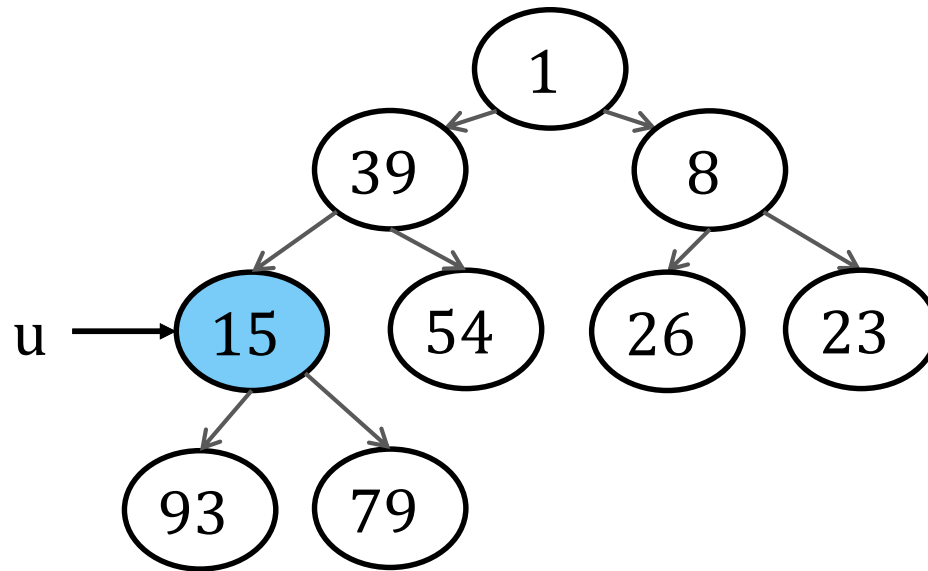
- First add 15 as a new leaf, making sure that we still have a complete binary tree.



- Step 3 is not true, go to Step 4,
- Step 4 is not true, go to step 5.

Binary Heap Insertion

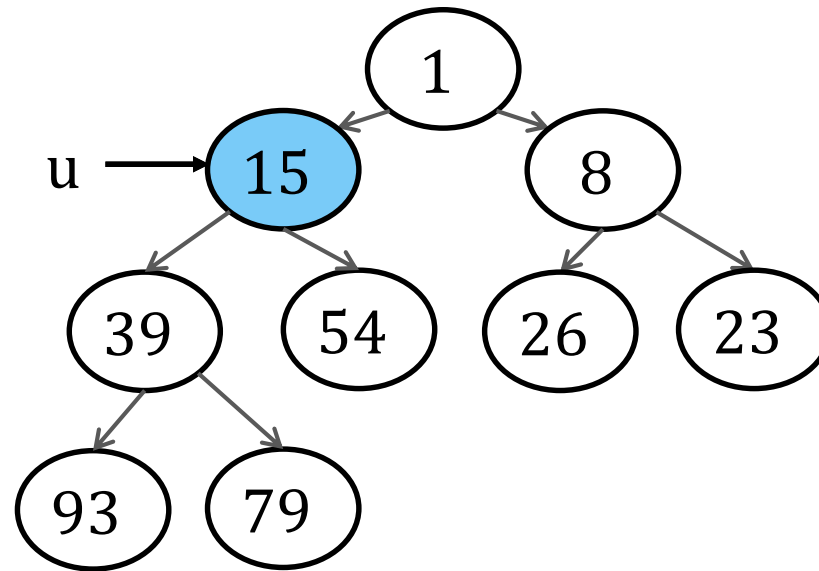
- First add 15 as a new leaf, making sure that we still have a complete binary tree.



- Swap the keys of u and its parent p
- Set $u \leftarrow p$, go back to Step 3)
- Step 3 and Step 4 are not true, go to Step 5.

Binary Heap Insertion

- First add 15 as a new leaf, making sure that we still have a complete binary tree.



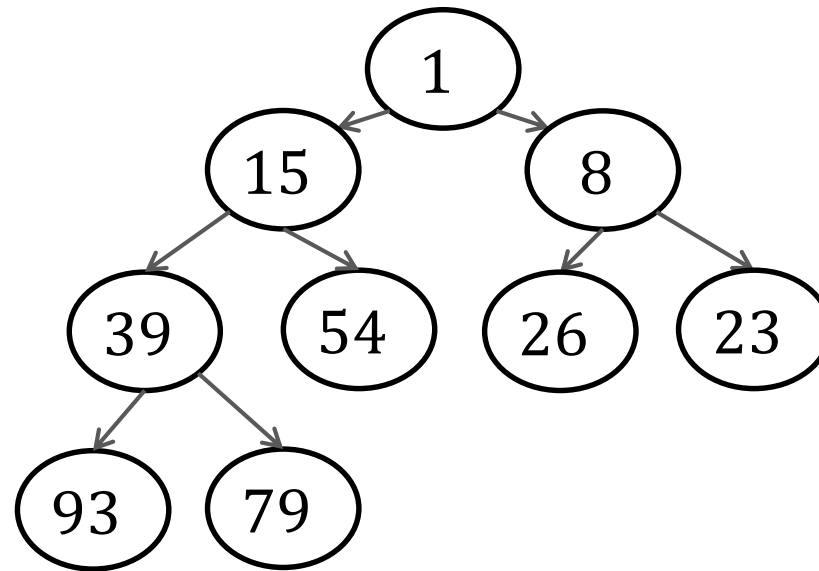
- Swap the keys of u and p
- Set $u \leftarrow p$, go back to Step 3)
- Step 3 is not true, Step 4 is true, return. Insertion complete.

Binary Heap Delete-min

- ◆ We perform delete-min on a binary heap T as follows:
 - ◆ Step 1: Report the key of the root
 - ◆ Step 2: Identify the rightmost leaf z at the bottom level of T
 - ◆ Step 3: Delete z , and store the key of z at the root
 - ◆ Step 4: Set $u \leftarrow$ the root
 - ◆ Step 5: If u is leaf, return
 - ◆ Step 6: If the key of $u <$ the keys of the children of u , return
 - ◆ Step 7: Otherwise, let v be the child of u with a smaller key
Swap the keys of u and v . Set $u \leftarrow v$, and repeat from Step 5

Binary Heap Delete-min

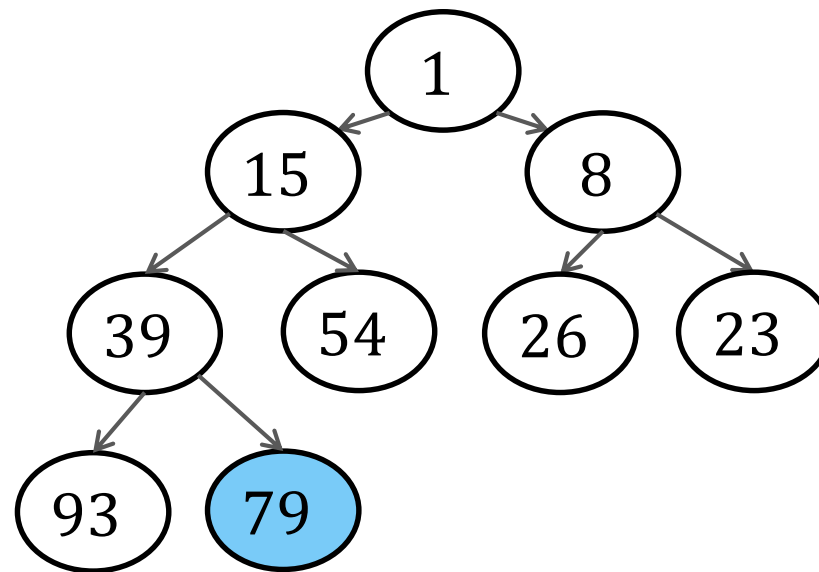
- Assume that we perform a delete-min from the binary heap below:



- Delete-min delete root node, and we should maintain the rest nodes as a complete binary tree.

Binary Heap Delete-min

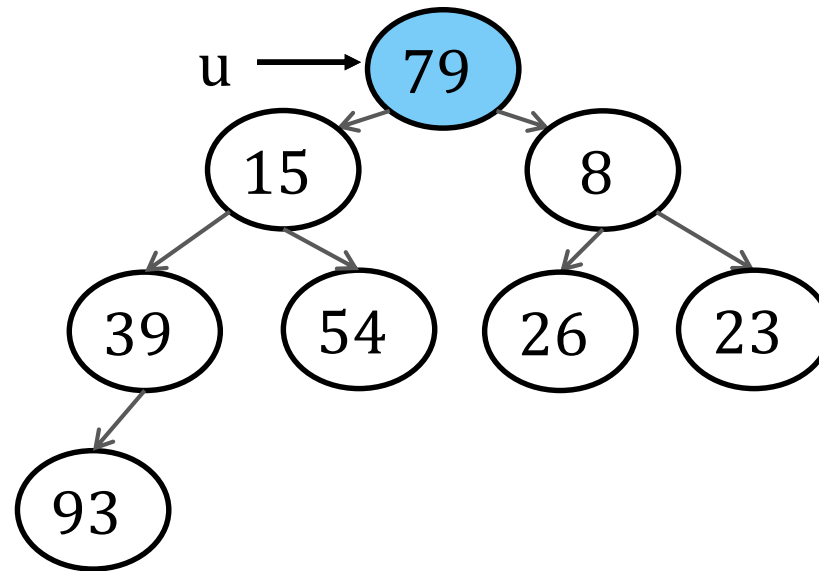
- ◆ First, find the rightmost leaf at the bottom level, it is node with key 79.



- ◆ Note that the tree is still a complete binary tree after removing this leaf.

Binary Heap Delete-min

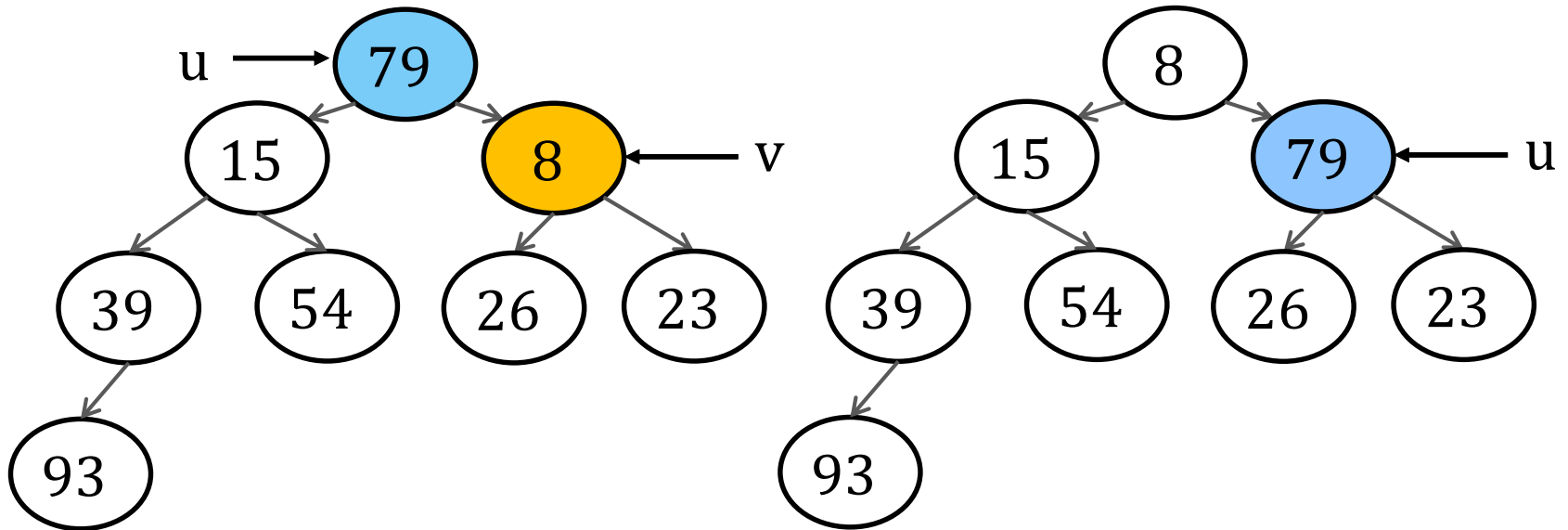
- ◆ Remove the leaf, but place the key value 79 in the root.



- ◆ Step 4: set u as the root.
- ◆ Step 5 and 6 are not true,
- ◆ Go to Step 7.

Binary Heap Delete-min

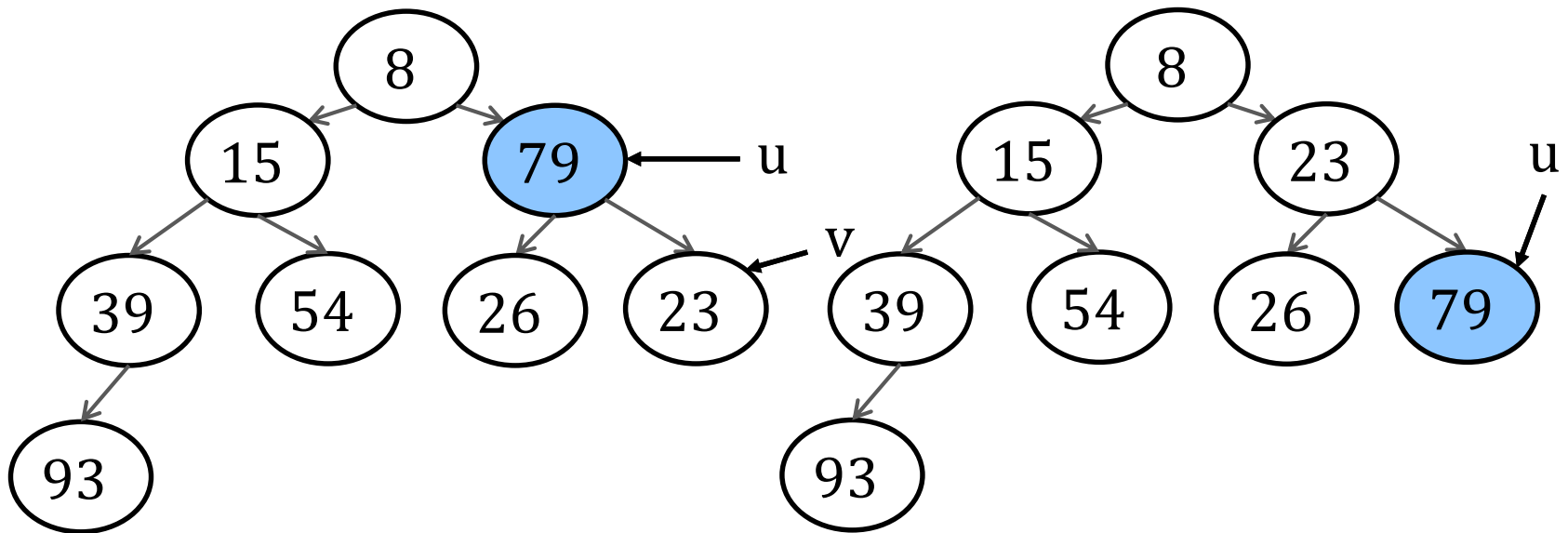
- Let v be the child of u with a smaller key.
- Swap the keys of u and v , and set $u \leftarrow v$



- Go to Step 5
- Step 5 and Step 6 are not true, go to Step 7

Binary Heap Delete-min

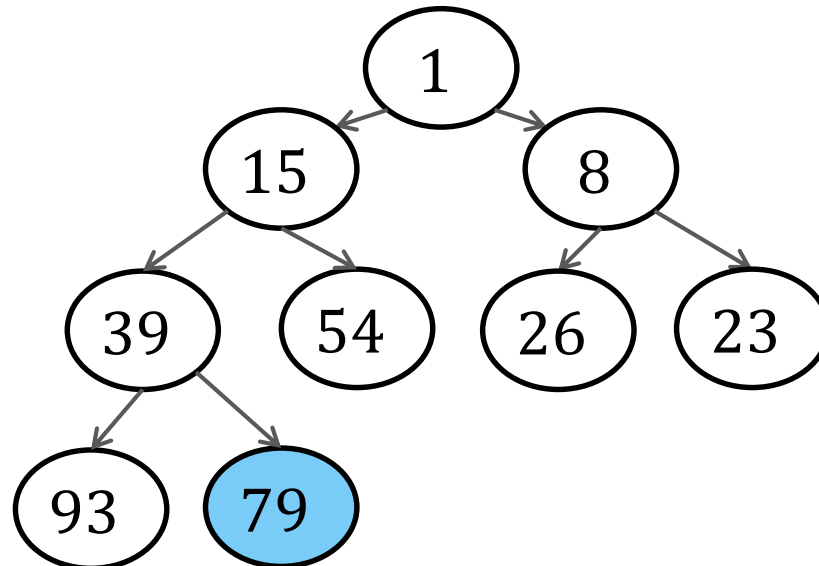
- Let v be the child of u with a smaller key.
- Swap the keys of u and v , and set $u \leftarrow v$



- Go to Step 5
- Step 5 is true, return. Delete-min complete.

How to find rightmost leaf?

- ◆ Before we analyzing the time complexity of insert and delete-min, let us first consider a sub-problem:
- ◆ Given a complete binary tree T with n nodes, how to identify quickly the rightmost leaf node at the bottom level of T (i.e., colored node in below tree).
 - ◆ It is Step 1 in insert algorithm, and Step 2 in delete-min algorithm



How to find rightmost leaf?

- ◆ We give a clever algorithm for solving the subproblem in $O(\log n)$ time.
- ◆ Write the value n in binary form. We can do that in $O(\log n)$ time.
- ◆ Skip the most significant bit. We will scan the remaining bits from left to right, start from root,
 - ◆ If the bit is 0, we go to the left child of the current node
 - ◆ Otherwise, go to right child

Find Rightmost Leaf Example

- ◆ Here $n = 9$, binary form: 1001

- ◆ Skip the first bit '1'

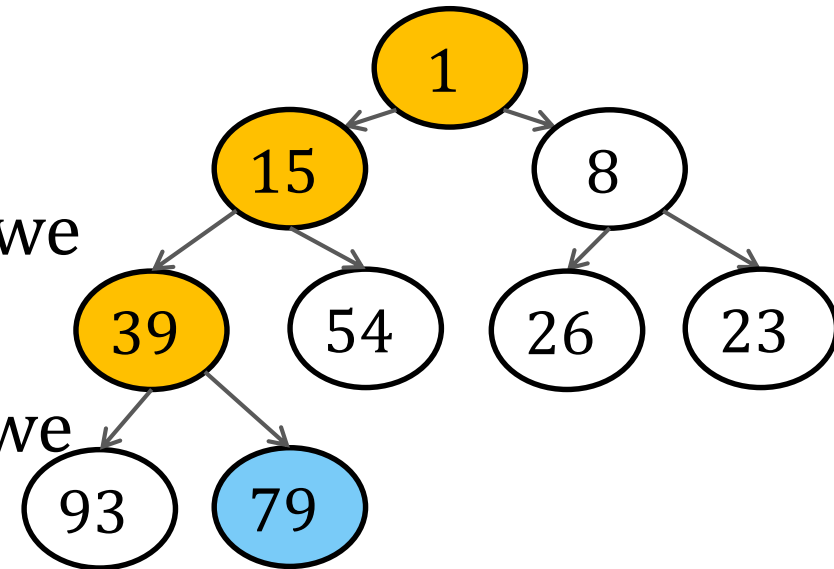
- ◆ We scan the remaining bits

- ◆ Start from root node 1.

- ◆ The 2nd leftmost bit is 0, so we visit left, and go to node 15

- ◆ The 3rd leftmost bit is 0, so we visit left, and go to node 39

- ◆ The 4th leftmost bit is 1, so we turn right, and go to node 79 (done).



Time Complexity Analysis

- ◆ We are now ready to prove that our insertion and delete-min algorithms finish in $O(\log n)$ time.
- ◆ It suffices to point out the key facts:
 - ◆ Step 1 of the insertion algorithm (page 8) and Step 2 of the delete-min algorithm (page 13) can be performed in $O(\log n)$ time, using our solution to previous sub-problem
 - ◆ The rest of insertion ascends a root-to-leaf path, while that of delete-min descends a root-to-leaf path. The time is $O(\log n)$ in both cases.
- ◆ Thus, we guarantee: (1) $O(n)$ space consumption, (2) $O(\log n)$ insertion / delete-min operations.

Our Roadmap

- ◆ Priority Queue (binary heap)
 - ◆ Min-heap insert / delete-min
- ◆ Binary Heaps in Dynamic Arrays
 - ◆ $O(n)$ time to build min-heap
- ◆ Binary Search Tree (BST)
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Binary Heaps in Dynamic Arrays

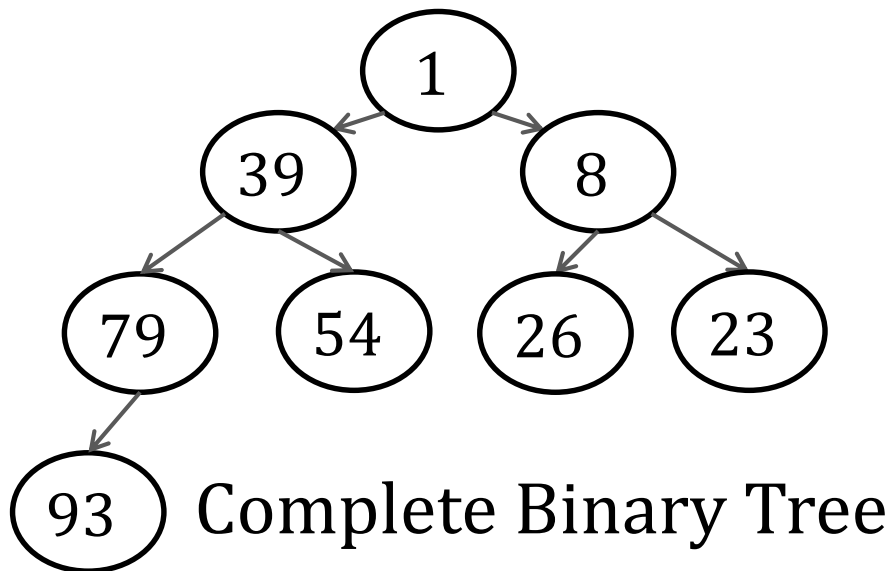
- ◆ We have already learned that the binary heap serves as an efficient implementation of a priority queue. Our previous discussion was based on pointers (for getting a parent node connected with its children). In this lecture, we will see a “pointerless” way to implement a binary heap, which in practice achieves much lower space consumption
- ◆ We will also see a way to build a heap from n integers in just $O(n)$ time, improving the obvious $O(n \log n)$ bound.

Recall

- ◆ A **priority queue** stores a set S of n integers and supports the following operations:
 - ◆ *Insert(e)*: adds a new integer to S
 - ◆ *Delete-min*: removes the smallest integer in S , and returns it.
- ◆ Let S be a set of n integers. A **binary heap** on S is a binary tree T satisfying:
 - ◆ (1) T is complete
 - ◆ (2) Every node u in T corresponds to a distinct integer in S , the integer is called the key of u (and is stored at u)
 - ◆ (3) If u is an internal node, the key of u is smaller than those of its child nodes

Storing a Complete Binary Tree

- ◆ Storing a complete binary tree using an array
- ◆ Let T be any complete binary tree with n nodes, let us linearize the nodes in the following manner:
 - ◆ Put nodes at a higher level before those at a lower level
 - ◆ Within the same level, order the nodes from left to right
- ◆ Let us store the linearized sequence of nodes in an array A of length n . Example:



1	39	8	79	54	26	23	93
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Storing in an array

Property 1

- ◆ Let us refer to the i -th element of A as $A[i]$, for simplicity, we assume the index of A starts from 1.
- ◆ Lemma: Suppose that node u of T is stored at $A[i]$. Then, the left child of u is stored at $A[2i]$, and the right child at $A[2i+1]$.
- ◆ Observe this from the example of the previous slide
- ◆ Proof leaves as your homework.
- ◆ Hints, consider the number of nodes after u , but before its left child.

More Properties

- ◆ The following is an immediate corollary of the previous lemma:
- ◆ Corollary: Suppose that node u of T is stored at $A[i]$. Then, the parent of u is stored at $A[\lfloor i/2 \rfloor]$.
- ◆ The following is a simple yet useful fact:
- ◆ Lemma: the rightmost leaf node at the bottom level is stored at $A[n]$.
- ◆ Now we have got everything we need to implement the insertion and delete-min algorithms on the array representation of a binary heap.

Insert 15

1	39	8	79	54	26	23	93
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1	39	8	79	54	26	23	93	15
---	----	---	----	----	----	----	----	----

1	39	8	15	54	26	23	93	79
---	----	---	----	----	----	----	----	----

1	15	8	39	54	26	23	93	79
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Delete-min

1	15	8	39	54	26	23	93	79
---	----	---	----	----	----	----	----	----

79	15	8	39	54	26	23	93
----	----	---	----	----	----	----	----

8	15	79	39	54	26	23	93
---	----	----	----	----	----	----	----

8	15	23	39	54	26	79	93
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Performance Guarantees

- ◆ Combining our analysis on (i) binary heaps and (ii) dynamic arrays, we obtain the following guarantees on binary heap implemented with a dynamic array:
 - ◆ Space consumption $O(n)$
 - ◆ Insertion: $O(\log n)$ time amortized
 - ◆ Delete-min: $O(\log n)$ time amortized

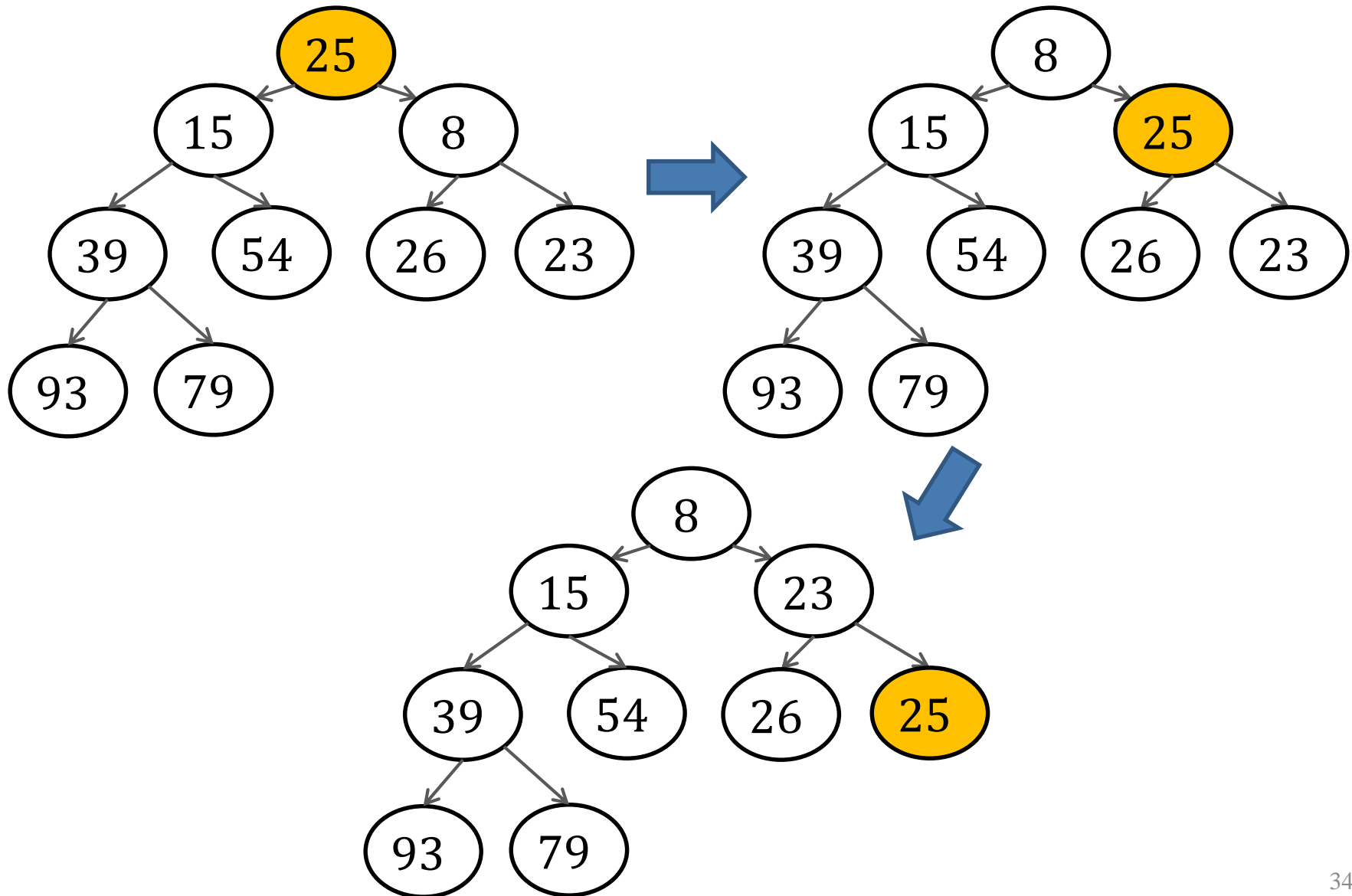
Build a binary heap in array

- ◆ Next we consider the problem of creating a binary heap on a set S of n integers. Obviously, we can do so in $O(n \log n)$ time by doing n insertions. However, this is an overall kill because the binary heap does not need to support any delete-min operations until all the n numbers have been inserted. This raises the question whether we can build the heap faster?
- ◆ The answer is positive: we will see an algorithm that does so in $O(n)$ time.

Root-fix operator

- ◆ We are given a complete binary tree T with root r . It guaranteed that:
 - ◆ The left subtree of r is binary heap
 - ◆ The right subtree of r is a binary heap
 - ◆ However, the key of r may not be smaller than the keys of its children.
- ◆ We define the root-fix operation, it fixes the issue and makes T a binary heap.
- ◆ Root-fix can be done in $O(\log n)$ time – in the same manner as the delete-min algorithm (step 4 - 7)

Root-fix Example



Building a Heap

- ◆ Create an array A that stores a set S of n integers, we can turn A into a binary heap on S using the following simple algorithm, which view A as a complete binary tree T :
- ◆ For each $i=n$ downto 1 :
 - ◆ Perform root-fix on the subtree of T rooted at $A[i]$
- ◆ Think: why are the conditions of root-fix always satisfied?

Building a Heap example

[illegible]

Root-fix

54	26	15	39	8	1	23	93
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54	26	1	39	8	15	23	93
----	----	---	----	---	----	----	----

56	8	1	39	26	15	23	93
----	---	---	----	----	----	----	----

1	8	15	39	26	54	23	93
---	---	----	----	----	----	----	----

Complexity Analysis

- ◆ Lemma: The time complexity of turn array A into a binary heap on S is $O(n)$.
- ◆ Proof as follows:
 - ◆ view A as a complete binary tree
 - ◆ The height of T is h.
 - ◆ Without loss of generality, assume that all the levels of T are full, i.e., $n=2^{h+1}-1$.
 - ◆ Why no generality is lost?
 - ◆ Analyze the total running time of Build heap algorithm
 - ◆ Proof that $\sum_{i=0}^h O(i * 2^{h-i}) = O(n)$ with $n=2^{h+1}-1$.

Our Roadmap

- ◆ Priority Queue (binary heap)
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- ◆ Binary Search Tree (BST)
 - ◆ BST operators
 - ◆ Balanced BST (AVL-tree)

Binary Search Tree (BST)

Binary Search Tree (especially, balanced BST) is the most powerful data structure of this course. This is without a doubt one of the most important data structures in computer science.

In extreme case, BST is equivalent to a linked list, thus, we guarantee the operations performance of BST by study AVL-tree.

Dynamic Predecessor Search

- ◆ Let S be a set of integers. We want to store S in a data structure to support the following operations:
 - ◆ A predecessor query: give an integer q , find its predecessor in S , which is the largest integer in S that does not exceed q .
 - ◆ Insertion: adds a new integer to S
 - ◆ Deletion: removes an integer from S
- ◆ Suppose that $S = \{3, 10, 15, 20, 30, 40, 60, 73, 80\}$
 - ◆ The predecessor of 23 is 20
 - ◆ The predecessor of 15 is 15
 - ◆ The predecessor of 2 does not exist
- ◆ Note that a predecessor query is more general than a “dictionary look-up”. Why?

Binary Search Tree (BST)

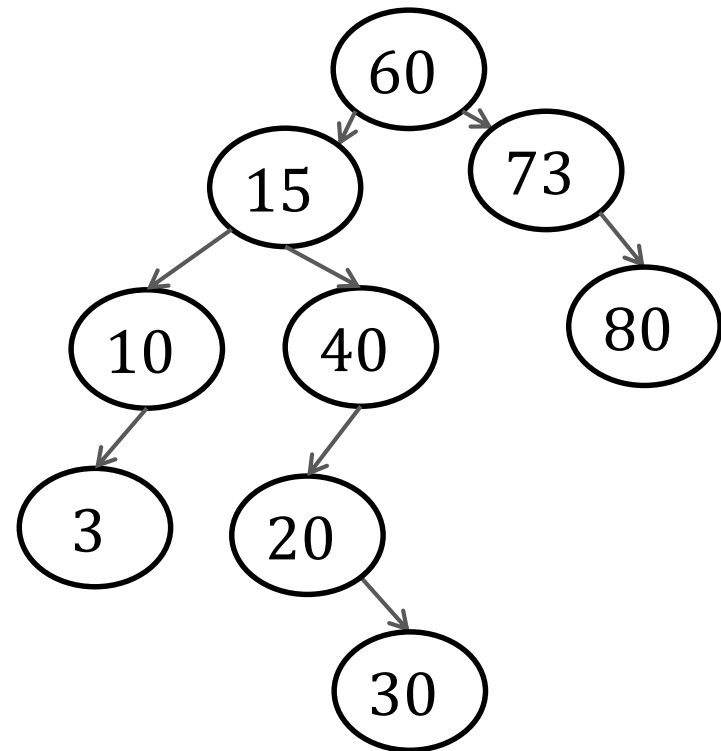
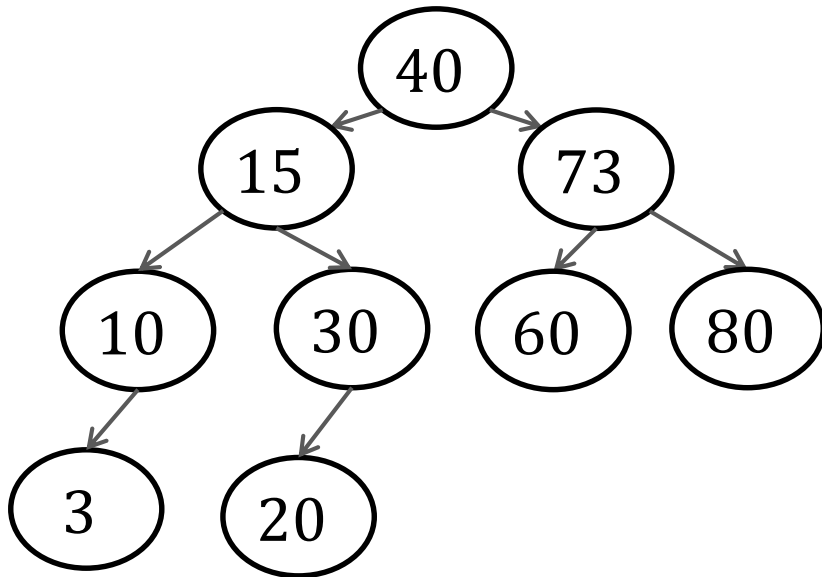
- ◆ We will learn a version of the BST that guarantees:
 - ◆ $O(n)$ space consumption
 - ◆ $O(h)$ time per predecessor query (hence, also per dictionary lookup)
 - ◆ $O(h)$ time per insertion
 - ◆ $O(h)$ time per deletion
- ◆ where $n = |S|$, h is the height of BST, Note that all the above complexities hold in the worst case.

Binary Search Tree (BST)

- ◆ A BST on a set S of n integers in a binary tree T satisfying all the following requirements:
 - ◆ T has n nodes
 - ◆ Each node u in T stores a distinct integer in S , which is called the key of u
 - ◆ For every internal u , it holds that:
 - ◆ The key of u is larger than all the keys in the left subtree of u .
 - ◆ The key of u is smaller than all the keys in the right subtree of u .

BST Example

- Two possible BSTs on $S = \{3, 10, 15, 20, 30, 40, 60, 73, 80\}$

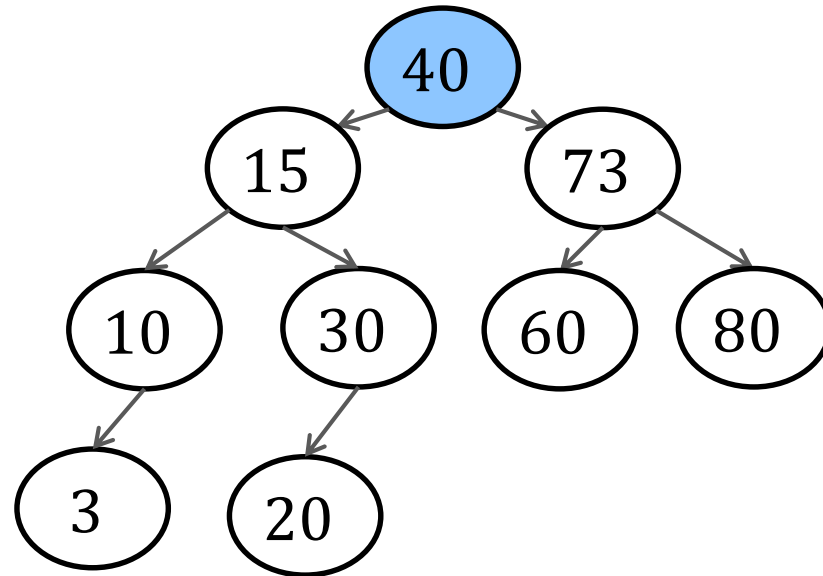


Predecessor Query

- ◆ Suppose that we have created a BST T on a set S of n integers. A predecessor query with search value q can be answered by descending a single root-to-leaf path:
 - ◆ (1) Set $p \leftarrow -\infty$ (p will contain the final answer at the end)
 - ◆ (2) Set $u \leftarrow$ the root of T
 - ◆ (3) If $u = \text{nil}$, then return p
 - ◆ (4) If key of $u = q$, then set p to q , and return p
 - ◆ (5) If key of $u > q$, then set u to the left child (now $u = \text{nil}$ if there is no left child), and repeat from Step (3)
 - ◆ (6) Otherwise, set p to the key of u and u to the right child, and repeat from Step (3)

Predecessor Query Example

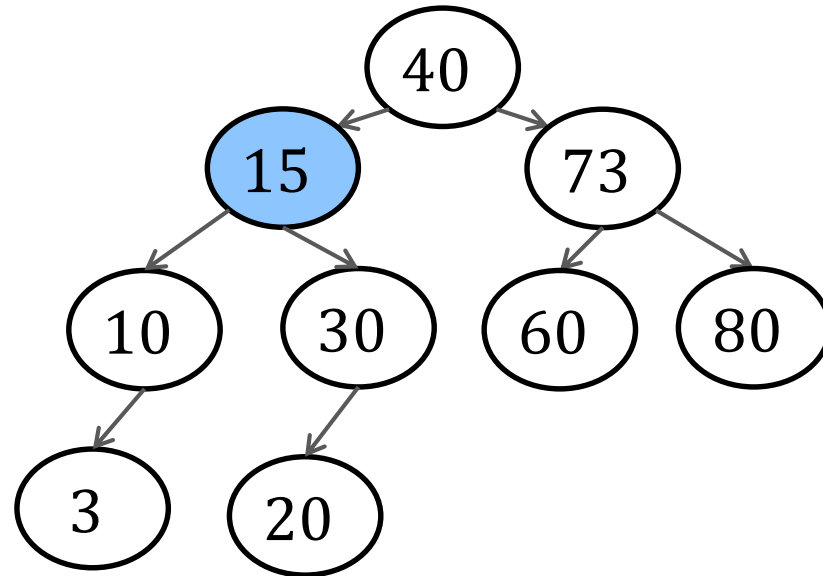
- ◆ Suppose that we want to find the predecessor of 35



- ◆ Set $p \leftarrow -\infty$, $u = \text{root } 40$
- ◆ (3) and (4) are not true, go to (5)
- ◆ Since $40 > 35$, the predecessor cannot be in the right subtree of 40, so we move to the left child of 40, now $u = \text{node } 15$.

Predecessor Query Example

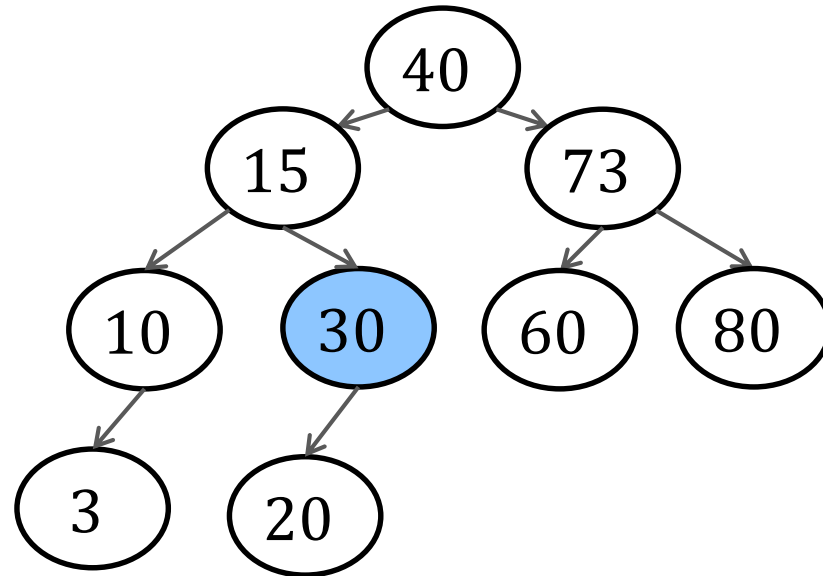
- Suppose that we want to find the predecessor of 35



- (3), (4) and (5) are not true, go to (6)
- Since $15 < 35$, $p \leftarrow 15$, since this is the predecessor of 35 so far.
- The predecessor cannot be in the left subtree of 15, so we move u to the right child, now $u = \text{node } 30$.

Predecessor Query Example

- Suppose that we want to find the predecessor of 35



- (3), (4) and (5) are not true, go to (6)
- Since $30 < 35$, $p \leftarrow 30$, since this is the predecessor of 35 so far.
- The predecessor will be in the right subtree of 30, but 30 does not have a right child. So algorithm terminates here with $p = 30$ as the final answer.

Time complexity Analysis

- ◆ Obviously, we spend $O(1)$ time at each node visited. Since the height of BST is h , therefore the total query time is $O(h)$.

Successor Query

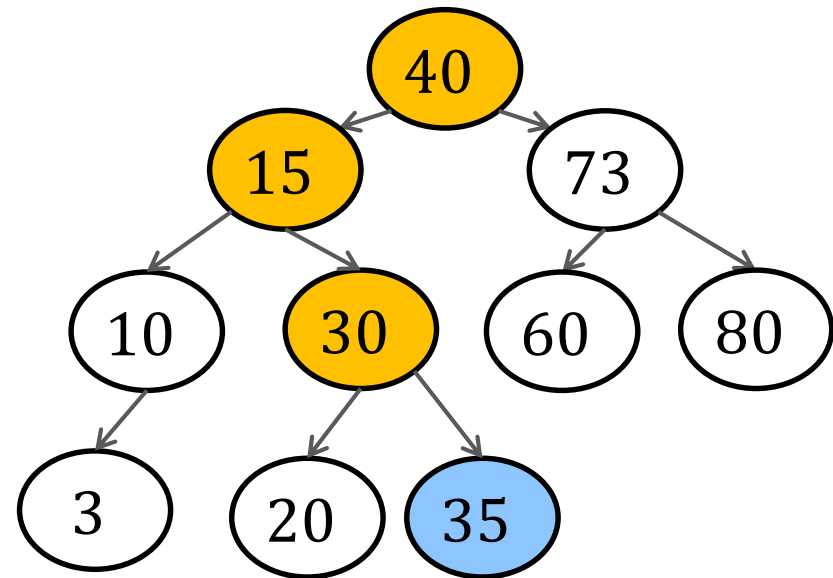
- ◆ The opposite of predecessors are successors.
- ◆ The successors of an integer q in S is the smallest integer in S that is no smaller than q .
- ◆ Suppose that $S = \{3, 10, 15, 20, 30, 40, 60, 73, 80\}$
 - ◆ The successor of 23 is 30
 - ◆ The successor of 15 is 20
 - ◆ The successor of 81 does not exist
- ◆ Given an integer q , a successor query returns the successor of q in S .
- ◆ By symmetry, we know from the earlier discussion (on predecessor queries) that a successor query can be answered using a BST in $O(h)$ time.

BST Insertion

- ◆ Suppose that we need to insert a new integer e . First create a new leaf z storing the key e . This can be done by descending a root-to-leaf path:
 - ◆ 1. Set $u \leftarrow$ the root of T
 - ◆ 2. If $e < \text{the key of } u$
 - ◆ 2.1 If u has a left child, then set u to the left child
 - ◆ 2.2 Otherwise, make z the left child of u , and done
 - ◆ 3. Otherwise:
 - ◆ 3.1 If u has a right child, then set u to the right child
 - ◆ 3.2 Otherwise, make z the right child of u , and done.
 - ◆ Repeat from Step 2.
- ◆ The total cost is proportional to the height of T , i.e., $O(h)$

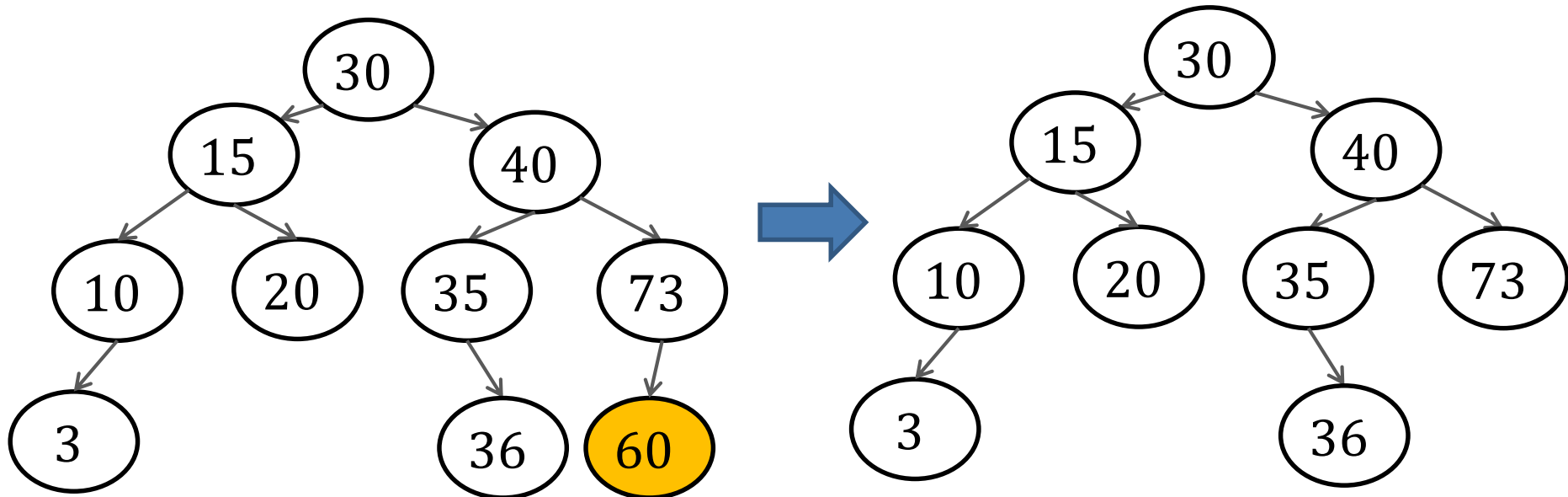
BST Insertion Example

- ◆ Inserting 35:
- ◆ u is root 40, $e < \text{the key of } u$,
 u has a left child, $u \leftarrow \text{node } 15$
- ◆ u is node 15, $e > \text{the key of } u$
 u has a right child, $u \leftarrow \text{node } 30$
- ◆ u is node 30, $e > \text{the key of } u$,
 u 's right child is nil, then set z
as the right child of u . Done.



BST Deletion

- Suppose that we want to delete an integer e . First, find the node u whose key equals to e in $O(h)$ time (through a predecessor query).
- Case 1: if u is a leaf node, simply remove it from T .
- Example: remove 60

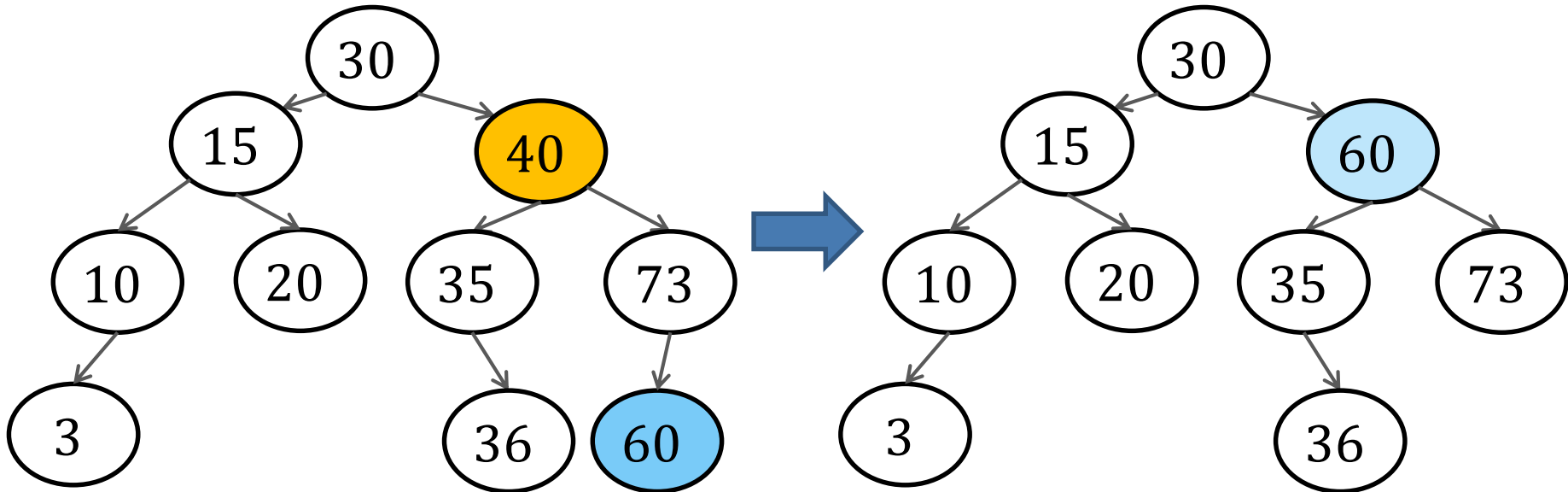


BST Deletion

- ◆ What happens if node u is not a leaf node?
- ◆ Case 2: if u has a right subtree:
 - ◆ Find the node v storing the successor s of e .
 - ◆ Set the key of u to s
 - ◆ Case 2.1: if v is a leaf node, then remove it from T
 - ◆ Case 2.2: otherwise, it must hold that v has a right child w , but not left child. Replace node v by subtree which rooted at w .
- ◆ Case 3: if u has no right subtree:
 - ◆ It must hold that u has a left child v , Replace node u by the subtree rooted at v .

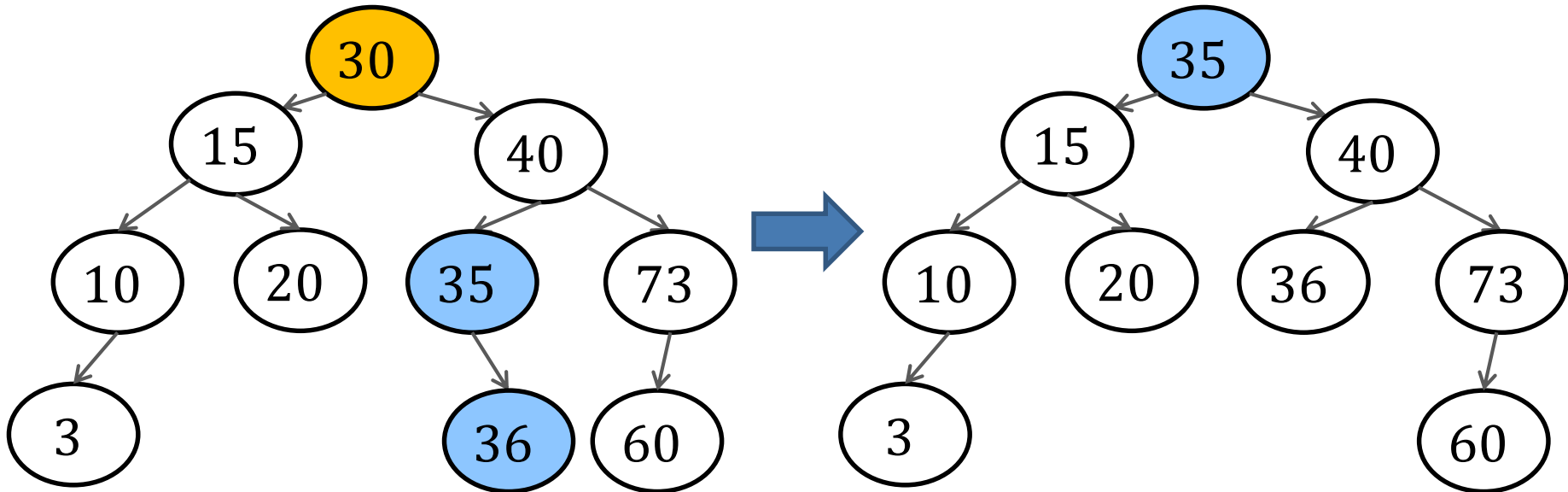
Case 2.1 Example

- ◆ Delete 40:
- ◆ u has a right subtree, node v (60) is the successor of 40.
- ◆ Set the key of u to 60
- ◆ v is a leaf, remove node v, done.



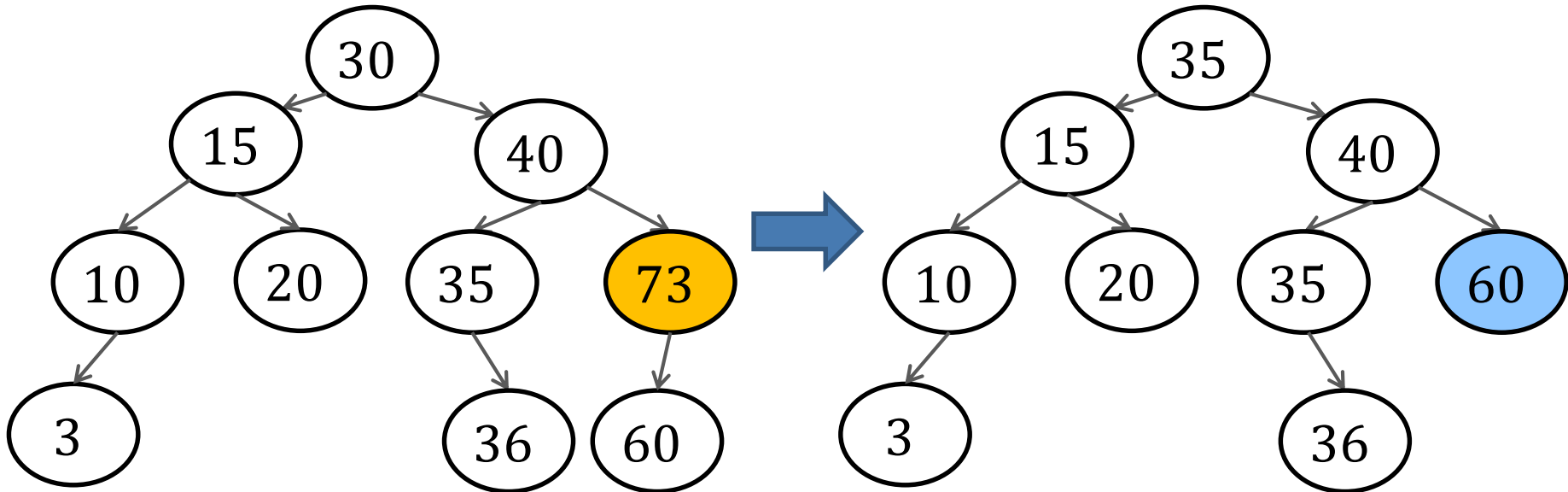
Case 2.2 Example

- ◆ Delete 30:
- ◆ u has a right subtree, node v (35) is the successor of 30.
- ◆ Set the key of u to 35
- ◆ v is not leaf node, it has right child w (36), replace node v by subtree rooted at w(36).



Case 3 Example

- ◆ Delete 73:
- ◆ u has no right subtree, and u must have a left child v (60), replace node u by node v(60).
- ◆ done.



BST Deletion

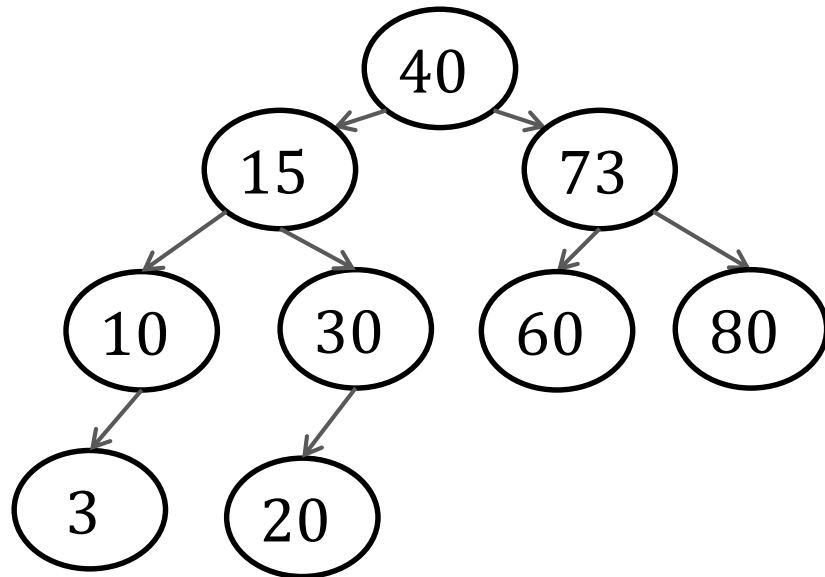
- ◆ In all above cases, we have essentially descended a root-to-leaf path (call it deletion path), and removed a leaf node.
- ◆ The cost so far is $O(h)$, recall that the successor of an integer can be found in $O(h)$ time.
- ◆ Given a set S of n integers, what is the maximum possible height of its BST?
 - ◆ $h = n$, why?
 - ◆ So what is the worst-case query cost? $O(n)$
 - ◆ However, we can guarantee $h = O(\log n)$ if the BST is balanced BST.

What is the height of tree

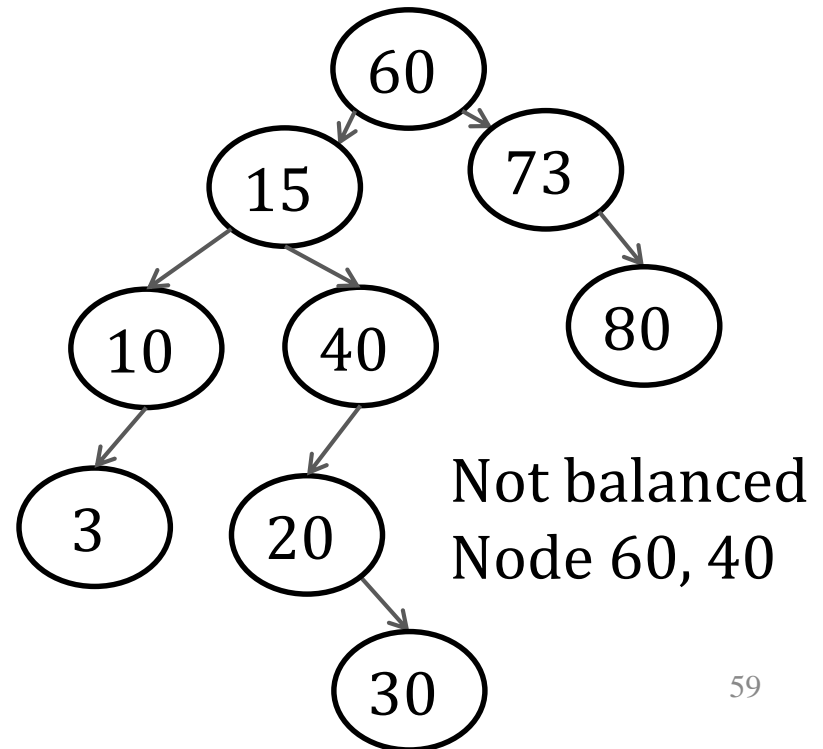
- ◆ Given a set S of n integers, what is the maximum possible height of its BST?
 - ◆ $h = n$, why?
- ◆ What is the worst-case query / insertion / deletion cost?
 - ◆ $O(n)$!!!
- ◆ How to achieve $O(\log n)$ time per operation?
 - ◆ Balanced Binary Search Tree

Balanced Binary Tree

- ◆ A binary tree T is balanced if the following holds on every internal node u of T :
 - ◆ The height of the left subtree of u differs from that the right subtree of u by at most 1.
- ◆ If u violates the above requirement, we say that u is imbalanced.



Balanced

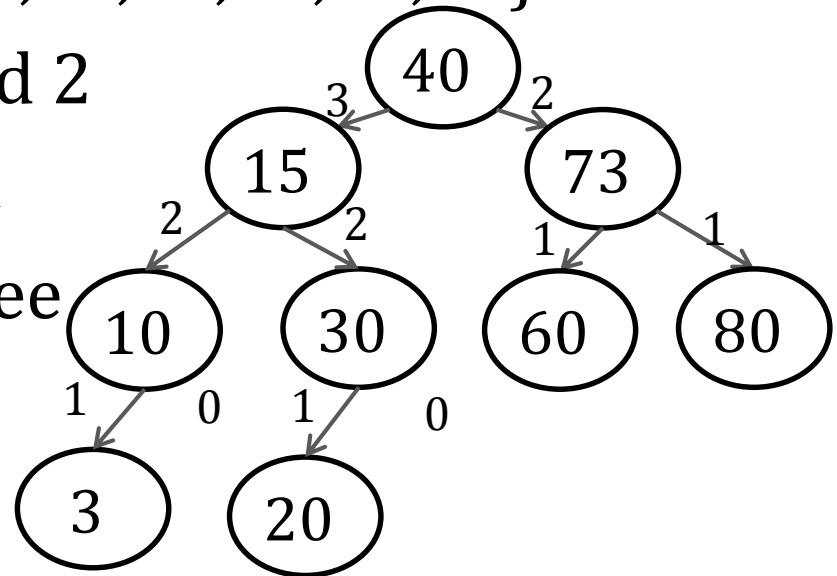


Height of a Balanced Binary Tree

- ◆ Theorem: a balanced binary tree with n nodes has height $O(\log n)$.
- ◆ Proof. (left as homework)
- ◆ Hints:
 - ◆ 1) consider minimum number of nodes in a balanced binary tree with height h
 - ◆ 2) recursive equation
 - ◆ 3) analysis two cases: case 1) h is even, case 2) h is odd.
- ◆ With the height of balanced binary tree is $O(\log n)$, we can conclude that the cost of query operation is $O(\log n)$ on a balanced binary search tree.
- ◆ How about the cost of insertion and deletion on it?

Balanced BST

- ◆ An AVL-tree on a set S of n integers is a balanced binary search tree T , where the following hold on every internal node u
 - ◆ u stores the heights of its left and right subtrees.
- ◆ An AVL-tree on $S = \{3, 10, 15, 20, 30, 40, 60, 73, 80\}$
- ◆ For example, the number 3 and 2 near root 40 indicate that its left subtree has height 3, right subtree has height 2.
- ◆ By storing the subtree heights
- an internal node know whether it has become imbalanced

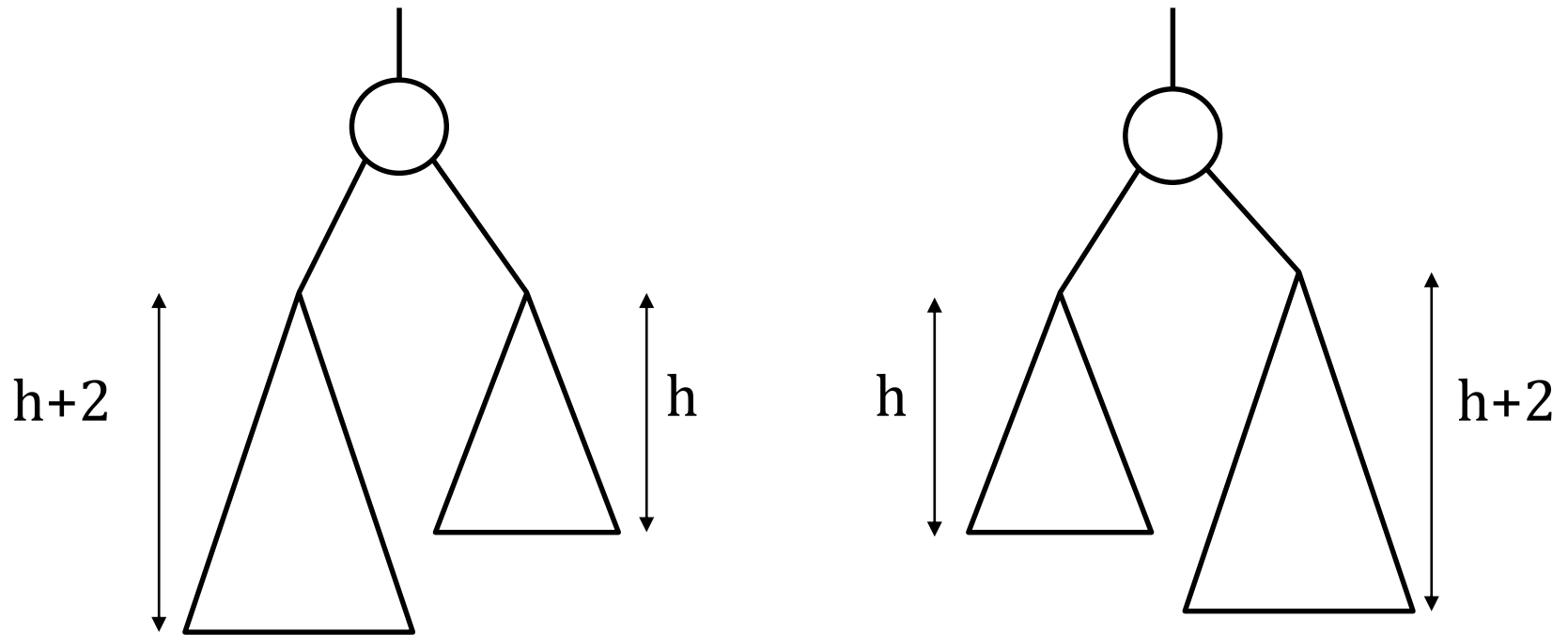


Balanced BST

- ◆ Next we will explain how to perform updates. The most important step is remedy a node u when it becomes imbalanced.
- ◆ It suffices to consider a scenario called 2-level imbalance. In this situation, two conditions apply:
 - ◆ There is a difference of 2 in the heights of the left and right subtree of u .
 - ◆ All the proper descendants of u are balanced
- ◆ We will first explain how to rebalance u in the above situation

2-level imbalance

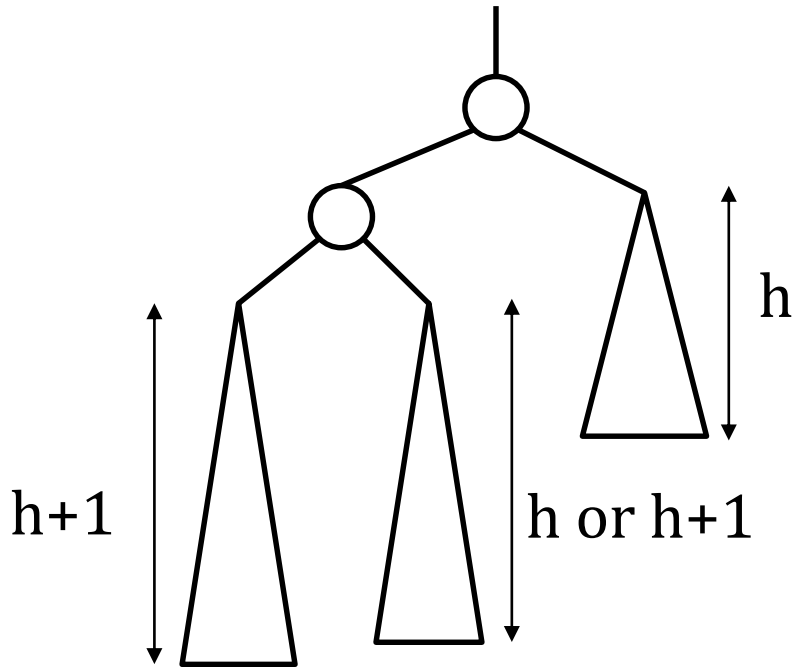
- ◆ There are two cases:



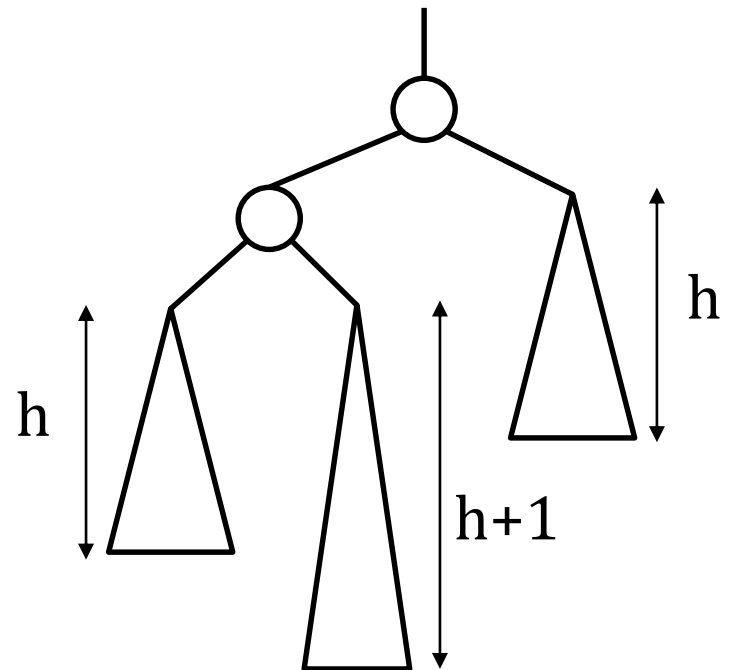
- ◆ Due to symmetry, it suffices to explain only the left case, which can be further divide to a left-left and a left-right case, as shown next.

2-level imbalance

- There are two cases:



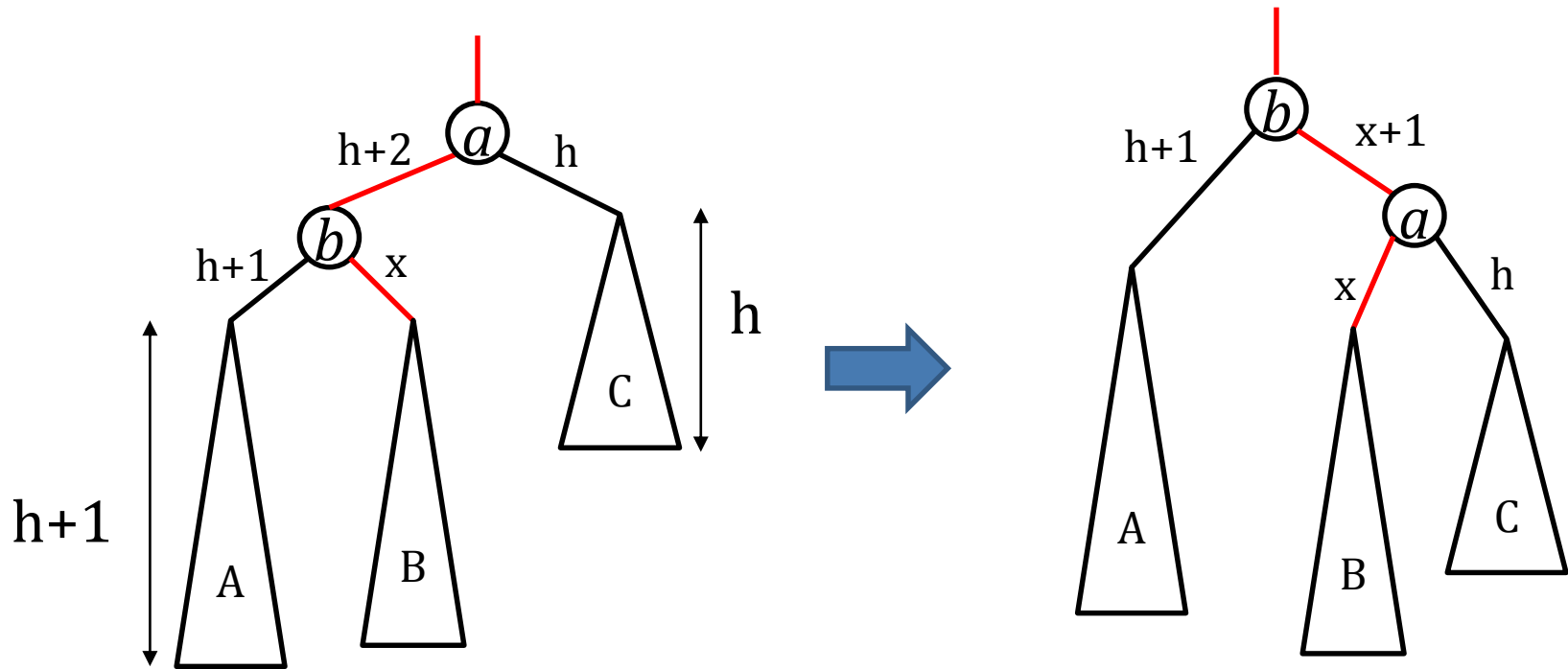
Left-Left case



Left-Right case

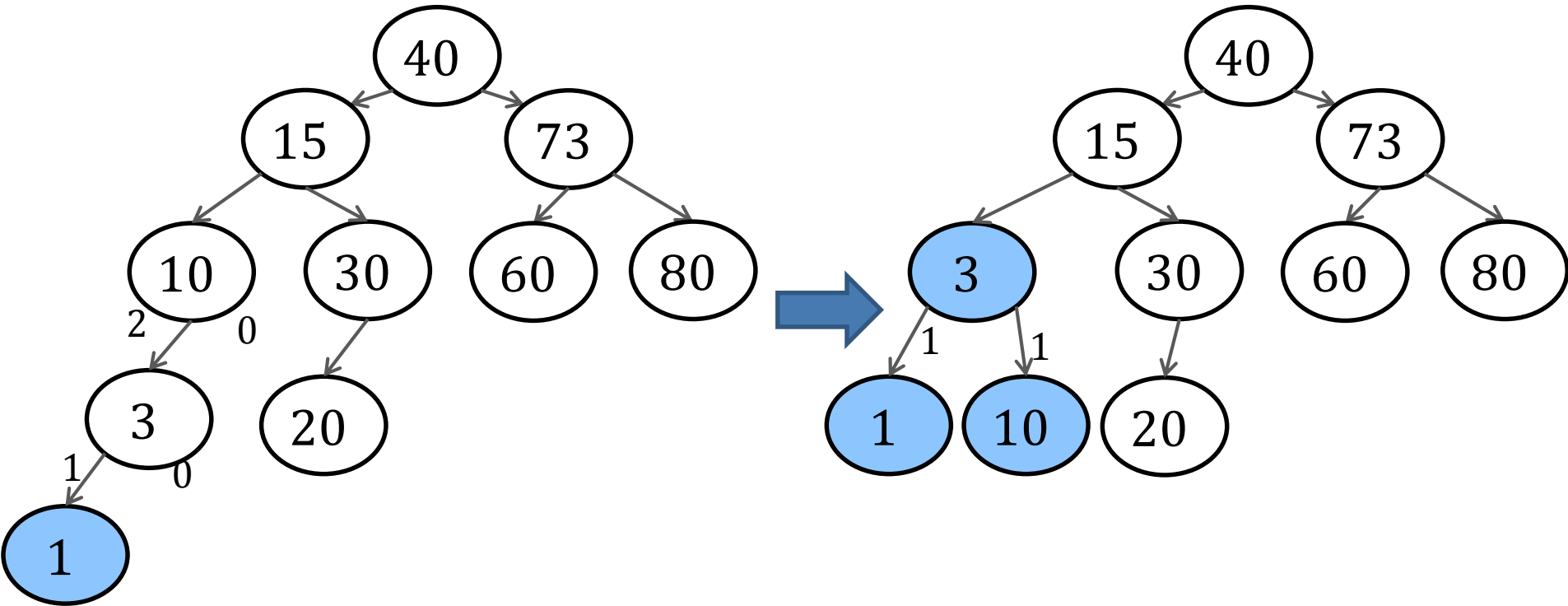
Rebalance Left-Left

- ◆ By a rotation:



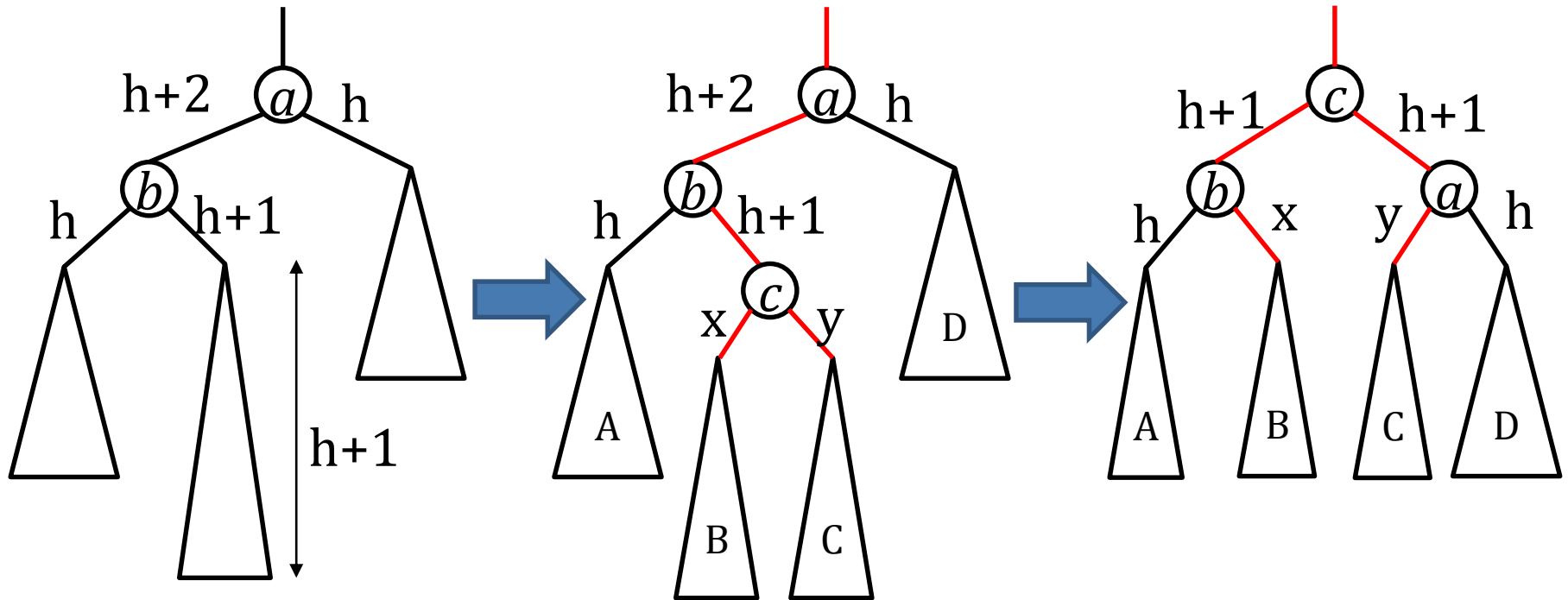
- ◆ Only 3 pointers to change (the red ones). The cost is $O(1)$.
- ◆ Recall that $x = h$ or $h+1$

Rebalance Left-Left Example



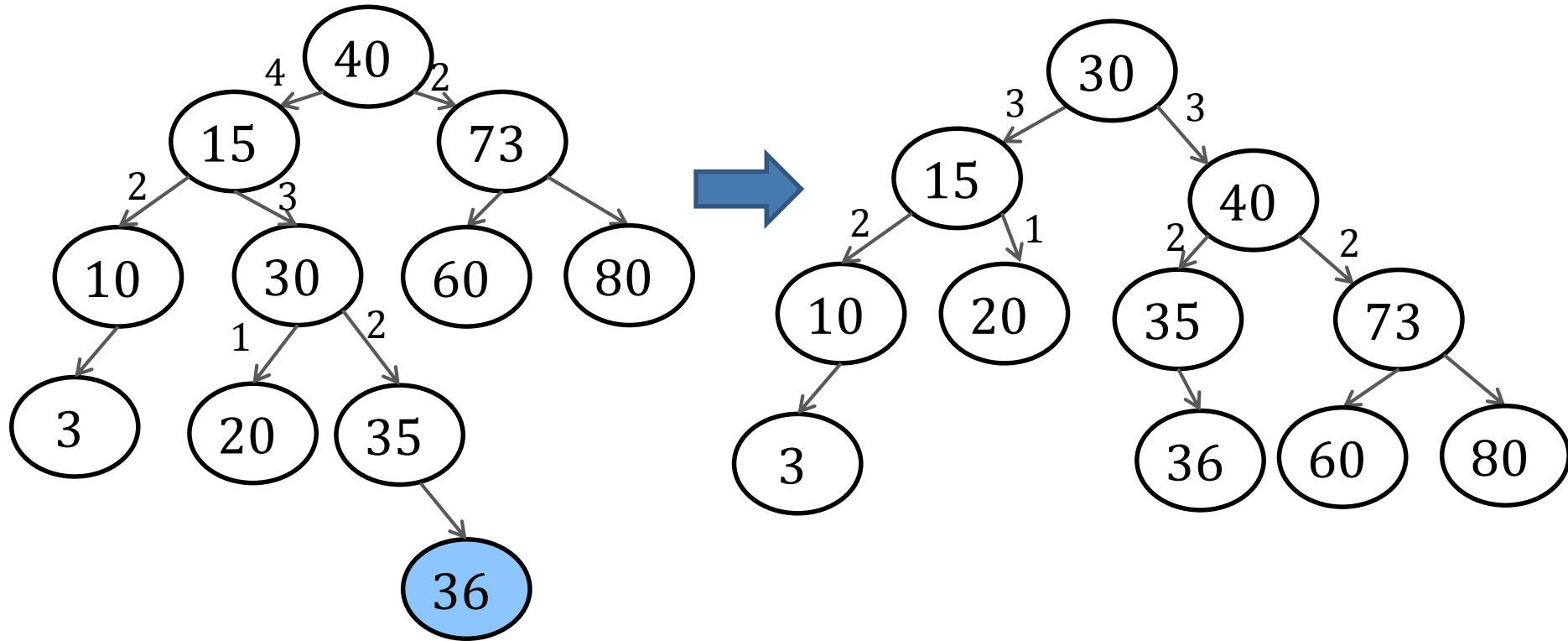
Rebalance Left-Right

- By a double rotation:



- Only 5 pointers to change (see above). Hence, the cost is $O(1)$.
- Note that x and y must be h or $h-1$. Furthermore, at least one of them must be h (why?)

Rebalance Left-Right Example



Insertion and Deletion Time

◆ Insertion time analysis

- ◆ It will be left as an exercise for you to prove
 - ◆ Only 2-level imbalance can occur in an insertion
 - ◆ Once we have remedied the lowest imbalance node, all the nodes in the tree will become balanced again
- ◆ Thus, we can conclude the insertion cost in a balanced BST is $O(\log n)$, why?

◆ Deletion time analysis

- ◆ It will be left as an exercise for you to prove
 - ◆ Only 2-level imbalance can occur after a deletion
- ◆ Thus, we can conclude the deletion cost in a balanced BST is $O(\log n)$

Balanced BST

- ◆ We now conclude our discussion on the AVL-tree, which provides the following guarantees:
 - ◆ $O(n)$ space consumption
 - ◆ $O(\log n)$ time per predecessor query (hence, also per dictionary lookup)
 - ◆ $O(\log n)$ time per insertion
 - ◆ $O(\log n)$ time per deletion
- ◆ All the above complexities hold in the worst case.

Thank You!