The BTL Model

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August 6, 2018

1   The Standard BTL Model $-$

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{ui}{ui + uj}$$

$$p(i \prec j|u) = \frac{uj}{uj + ui}$$

Suppose that there are K underlying aspects. for each pair, $p(<w,v>) = p^k(w \succ v)$. then the probability of generating a session observation d is defined as:

$$p(<w,v>|V,U) = \frac{\sum_{k=1}^{K} u_k w_k}{\sum_{k=1}^{K} u_k w_k + \sum_{k=1}^{K} u_k v_k} \tag{1}$$

The likelihood function can be written as follows. The model parameters are denoted as $\Theta = \{v \in V, u \in U)\}$

$$L(\Theta) = p(D|\Theta) = \Pi_{d \in D} \Pi_{w \in W^d, v \in L^d} \frac{\sum_{k=1}^{K} u_k w_k}{\sum_{k=1}^{K} u_k w_k + \sum_{k=1}^{K} u_k v_k} \tag{2}$$

Thus, the log likelihood is:

$$l(\Theta) = log\, L(\Theta) \tag{3}$$

$$= log\, \Pi_{d \in D} \Pi_{w \in W^d, v \in L^d} \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k} \tag{4}$$

$$= \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} log\, \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k}$$

$$= \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} [log \sum_k u_k w_k - log(\sum_k u_k w_k + \sum_k u_k v_k)]$$

for the :

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, rewrite the log likelihood as follow.

$$l(\Theta) \geq \Sigma_{d\in D}\Sigma_{w\in W^d, v\in L^d}[1 - \frac{\sum_k u_k w_k + \sum_k u_k v_k}{\sum_k u_k^t w_k^t + \sum_k u_k^t v_k^t} + log\frac{\sum_k u_k w_k}{\sum_k u_k^t w_k^t + \sum_k u_k^t v_k^t}] \tag{5}$$

Using the Jensen's inequality again, $l(\Theta)$ can be :

$$l(\Theta) \geq \Sigma_{d\in D}\Sigma_{w\in W^d, v\in L^d}[1 - \frac{\sum_k u_k(w_k + v_k)}{\sum_k u_k^t(w_k^t + v_k^t)} + 1 - \frac{\sum_k u_k^t(w_k^t + v_k^t)}{\sum_k u_k^t(w_k^t + v_k^t)} + log\frac{\sum_k u_k w_k}{\sum_k u_k^t(w_k^t + v_k^t)}] \tag{6}$$

$$= \Sigma_{d\in D}\Sigma_{w\in W^d, v\in L^d}[1 - \frac{\sum_k u_k(w_k + v_k)}{\sum_k u_k^t(w_k^t + v_k^t)} + log\sum_k u_k w_k - log\sum_k u_k^t(w_k^t + v_k^t)] \tag{7}$$

we want to choose $\{U\ and\ V\}$ to maximize $l(\Theta)$.

1.1 For u

By setting the partial derivative of $\frac{l(\Theta)}{\partial u_k} = 0$, we have:

$$\frac{\partial\, l(\Theta)}{\partial\, u_k} = \Sigma_{d\in D}\Sigma_{w\in W^d, v\in L^d}(\frac{w_k}{\sum_k u_k w_k} - \frac{w_k + v_k}{\sum_k u_k^t(w_k^t + v_k^t)}) = 0 \tag{8}$$

$$\frac{w_k}{\sum_k u_k w_k} = \frac{w_k + v_k}{\sum_k u_k^t(w_k^t + v_k^t)}$$

$$\sum_k u_k w_k = \frac{w_k\sum_k u_k^t(w_k^t + v_k^t)}{(w_k + v_k)}$$

$$u_k w_k = \frac{w_k\sum_k u_k^t(w_k^t + v_k^t)}{(w_k + v_k)} - \sum_{k'\neq k} u_{k'} w_{k'}$$

$$u_k = \frac{\sum_k u_k^t(w_k^t + v_k^t)}{w_k + v_k} - \frac{\sum_{k'\neq k} u_{k'} w_{k'}}{w_k}$$

1.2 For v

As for winner:
By setting the partial derivative of $\frac{l(\Theta)}{\partial w_k} = 0$, we have:

$$\frac{\partial\, l(\Theta)}{\partial\, w_k} = \Sigma_{d\in D}\Sigma_{w\in W^d, v\in L^d}(\frac{u_k}{\sum_k u_k w_k} - \frac{u_k}{\sum_k u_k^t(w_k^t + v_k^t)}) = 0 \tag{9}$$

$$w_k = \sum_k u_k^t (w_k^t + v_k^t)$$

As for loser:

By setting the partial derivative of $\frac{l(\Theta)}{\partial v_k} = 0$, we have:

$$\frac{\partial\, l(\Theta)}{\partial\, v_k} = \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} \left(- \frac{u_k}{\sum_k u_k^t (w_k^t + v_k^t)} \right) = 0 \tag{10}$$

$$v_k =$$