

Modify By Confidence Level

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July 5, 2018

1 The BTL Model

1.1 E-step

$$\begin{aligned} p(g|d, \Theta^t) &\propto p(g|u, \Theta^t)p(d|g, \Theta^t) \\ p(g_k = 1|d, \Theta^t) &\propto u_k^t \frac{w_k^t}{w_k^t + \theta^t l_k^t} \prod_{k' \neq k} \left[\frac{\theta^t w_{k'}^t}{l_{k'}^t + \theta^t w_{k'}^t} \right] \end{aligned} \quad (1)$$

First, let's use $\gamma(d, k, \Theta^t)$ to denote the conditional probability $p(g_k = 1|d, \Theta^t)$ given parameters in the t -th round, when the current session specific favorite aspect is $g_k = 1$, defined as follows

$$\begin{aligned} \gamma(d, k, \Theta^t) &= \frac{p(d, g|\Theta^t)}{\sum_g p(d, g|\Theta^t)} = \frac{p(g|\Theta^t)p(d|\Theta^t, g)}{\sum_g p(g|\Theta^t)p(d|\Theta^t, g)} \\ &= \frac{u_k \prod_{w \in W^d, v \in L^d} \frac{w_k}{w_k + \theta^t v_k} \prod_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}}{\sum_{k=1}^K u_k \prod_{w \in W^d, v \in L^d} \frac{w_k}{w_k + \theta^t v_k} \prod_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}} \end{aligned} \quad (2)$$

Note that $\forall d, \sum_k \gamma(d, k, \Theta^t) = 1$.

1.2 S-step

then simply add an S-step after the E-step, the value of g for each session d is

$$k \sim u_k^t \frac{w_k^t}{w_k^t + \theta^t l_k^t} \prod_{k' \neq k} \left[\frac{\theta^t w_{k'}^t}{l_{k'}^t + \theta^t w_{k'}^t} \right] \quad (3)$$

In the E-step of t -th EM round, compute the expectation $Q(\Theta^t) = E_G \ln p(D, G|\Theta)$

$$\begin{aligned} E_G \ln p(D, G|\Theta) &= \sum_d \gamma(d, k, \Theta^t) \ln p(d, g|\Theta) \\ &= \sum_d \gamma(d, k, \Theta^t) \left\{ \ln u_k + \sum_{w \in W^d, v \in V_d} \left[\ln \frac{w_k}{w_k + \theta v_k} + \sum_{k' \neq k} \ln \frac{\theta w_{k'}}{v_{k'} + \theta w_{k'}} \right] \right\} \end{aligned}$$

1.3 M-step

1.3.1 For u

first maximize $Q(\Theta^t)$ with respect to U . For each $u \in U$, eliminating constant terms, we have:

$$\min -\sum_{u(d)=u} \sum_{k=1}^K \gamma(d, k, \Theta^t) \ln u_k \quad (4)$$

$$w.r.t \sum_k u_k = 1$$

Solving the above Lagrange function Equ. 4, we get

$$u_k = \frac{\sum_{u(d)=u} \gamma(d, k, \Theta^t)}{\sum_{s=1}^K \sum_{u(d)=u} \gamma(d, s, \Theta^t)} \quad (5)$$

1.3.2 For v

for the :

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, we obtain a minorization function of $\tilde{Q}(\Theta^t)$.

$$\tilde{Q}(\Theta^t) = \sum_d \sum_k \gamma(d, k, \Theta^t) \sum_{w \in W_d, v \in L_d} \{ [\ln w_k + 1 - \ln(w_k^t + \theta^t v_k^t) - \frac{w_k + \theta v_k}{w_k^t + \theta^t v_k^t}] +$$

$$\sum_{k' \neq k} [\ln(\theta w_{k'}) + 1 - \ln(v_{k'}^t + \theta^t w_{k'}^t) - \frac{v_{k'} + \theta w_{k'}}{v_{k'}^t + \theta^t w_{k'}^t}] \}$$

$\tilde{Q}(\Theta^t)$ can be separated for each item v . Considering only the k -th component v_k , $\tilde{Q}(v_k, \Theta^t)$ involves two terms, one of which is relevant to observations $d \in W(v)$ where v acts as skyline object, the other is relevant to observations $d \in L(v)$ where v acts as comparisons, $\tilde{Q}(v_k, \Theta^t) = \tilde{Q}^1(v_k, \Theta^t) + \tilde{Q}^2(v_k, \Theta^t)$. Removing all constants and irrelevant terms for v_k , we have the following minorizing function:

$$\tilde{Q}^1(v_k, \Theta^t) = \sum_{d \in W(v)} |L_d| \ln v_k - v_k \sum_{d \in W(v)} \sum_{v' \in L_d} \left[\frac{\gamma(d, k, \Theta^t)}{\alpha(v, v', k, \Theta^t)} + \sum_{k' \neq k} \frac{\theta^t \gamma(d, k', \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]$$

$$\tilde{Q}^2(v_k, \Theta^t) = -v_k \sum_{d \in L(v)} \sum_{v' \in W_d} \left[\frac{\theta^t \gamma(d, k, \Theta^t)}{\alpha(v', v, k, \Theta^t)} + \sum_{k' \neq k} \frac{\gamma(d, k', \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]$$

where:

$$|L_d| \text{ is the number of objects being dominated in } d,$$

$$\alpha(v, v', k, \Theta^t) = v_k^t + \theta^t v_{k'}^t.$$

By setting the partial derivative of $\frac{\partial \tilde{Q}(v_k, \Theta^t)}{\partial v_k} = 0$, we have:

$$\frac{1}{v_k} = \frac{\sum_{d \in W(v)} \sum_{v' \in L_d} \left[\frac{\gamma(d, k, \Theta^t)}{\alpha(v, v', k, \Theta^t)} + \sum_{k' \neq k} \frac{\theta^t \gamma(d, k', \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]}{\sum_{d \in W(v)} |L_d|} + \frac{\sum_{d \in L(v)} \sum_{v' \in W_d} \left[\frac{\theta^t \gamma(d, k, \Theta^t)}{\alpha(v', v, k, \Theta^t)} + \sum_{k' \neq k} \frac{\gamma(d, k', \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]}{\sum_{d \in W(v)} |L_d|}$$

More concrete:
if $\gamma(d, k, \Theta^t)$'s k equals to v_k 's k:

$$\frac{1}{v_k} = \frac{\sum_{d \in W(v)} \sum_{v' \in L_d} \left[\frac{\gamma(d, k, \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]}{\sum_{d \in W(v)} |L_d|} + \frac{\sum_{d \in L(v)} \sum_{v' \in W_d} \left[\frac{\theta^t \gamma(d, k, \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]}{\sum_{d \in W(v)} |L_d|}$$

else:

$$\frac{1}{v_k} = \frac{\sum_{d \in W(v)} \sum_{v' \in L_d} \left[\frac{\theta^t \gamma(d, k', \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]_{k'=k}}{\sum_{d \in W(v)} |L_d|} + \frac{\sum_{d \in L(v)} \sum_{v' \in W_d} \left[\frac{\gamma(d, k', \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]_{k'=k}}{\sum_{d \in W(v)} |L_d|}$$

1.3.3 For θ

Fix u, v , update θ by:

Fix $v \in V$ and $u \in U$, rearranging *Equ.9*, we have the solution for $\frac{\partial \tilde{Q}(\Theta^t)}{\partial \theta} = 0$ as:

$$\theta = \frac{(K-1) \sum_d |W_d| |L_d|}{\sum_d \sum_k \gamma(d, k, \Theta^t) \sum_{w, v} \left[\frac{v_k}{\alpha(w, v, k, \Theta^t)} + \sum_{k' \neq k} \frac{w_{k'}}{\alpha(v, w, k', \Theta^t)} \right]} \quad (6)$$

Note that :

in the $\sum_d \sum_k \gamma(d, k, \Theta^t)$ only summing these d which $\gamma(d, k, \Theta^t)$'s k equals to d's k.