

# The BTL Model

Xiaolin Shen

August 8, 2018

## 1 The Standard BTL Model

Suppose that there are  $K$  underlying aspects, the latent preference vector for each user is denoted as  $u \in R^{+K}$ , and  $\sum_k u_k = 1$ , latent item feature vector  $i, j \in R^{+K}$ .

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{u^T i}{u^T i + u^T j}$$

$$p(i \prec j|u) = 1 - p(i \succ j|u)$$

We use the maximal likelihood estimation. The likelihood function can be written as follows. The model parameters are denoted as  $\Theta = \{v \in V, u \in U\}$ ,  $\succ_d$  represents a pairwise ranking observation in a session  $d$ .

$$L(\Theta) = \prod_{d \in D} \prod_{i \succ_d j} \frac{\sum_{k=1}^K u_k i_k}{\sum_{k=1}^K u_k i_k + \sum_{k=1}^K u_k j_k} \quad (1)$$

Thus, the log likelihood and its lower bound is:

$$l(\Theta) = \log L(\Theta) \quad (2)$$

$$= \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} \log \frac{\sum_k u_k i_k}{\sum_k u_k i_k + \sum_k u_k j_k} \quad (3)$$

$$\geq \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} \left[ 1 - \frac{\sum_k u_k i_k + \sum_k u_k j_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t} + \log \frac{\sum_k u_k i_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t} \right]$$

where we apply

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

To maximize the lower-bound in *Equ.3* we employ conjugate gradient method.

We first fix all  $vs$ , and apply  $\log \sum_k u_k i_k \geq \frac{\sum_k i_k \log u_k}{\sum_k i_k} + \log \sum_k i_k, \forall i_k \geq 0$ . denote  $c^t(i, j, d) = \sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t$  in the  $t$ -th round of the iteration, with respect to  $\sum_k u_k = 1$ , by applying the Lagrange, we have

$$\frac{\partial l(\Theta)}{\partial u_k} = \sum_{d \in D_u} \sum_{i \succ_d j} \left( \frac{i_k}{u_k \sum_k i_k} - \frac{i_k + j_k}{c^t(i, j, d)} \right) + \lambda = 0 \quad (4)$$

$$\frac{\partial l(\Theta)}{\partial \lambda} = \sum_k u_k - 1 = 0 \quad (5)$$

$$u_k = \frac{\sum_{d \in D_u} \sum_{i \succ_d j} \frac{i_k c^t(i, j, d)}{(i_k + j_k) \sum_k i_k}}{\sum_{k=1}^K \sum_{d \in D_u} \sum_{i \succ_d j} \frac{i_k c^t(i, j, d)}{(i_k + j_k) \sum_k i_k}} \quad (6)$$

Next we fix all  $u$ s, and again apply the Jensen's inequality  $\log \sum_k u_k i_k \geq \sum_k u_k \log i_k, \forall u_k \geq 0, \sum_k u_k = 1$ . For convenience, we use  $v$  to denote a arbitral item, and  $O_u$  is the set of all observed pairwise ranking of user  $u$ , if  $v \succ j$  is the winning item then we say  $o \in W(v)$ .

$$\frac{\partial l(\Theta)}{\partial v_k} = \sum_u \sum_{o \in W(v) \& o \in O_u} \frac{u_k}{i_k} - \sum_u \sum_{o \in O_u} \frac{u_k}{c^t(i, j, d)} = 0 \quad (7)$$

$$i_k = \frac{\sum_u \sum_{o \in O_u} \frac{u_k}{c^t(i, j, d)}}{\sum_u \sum_{o \in W(v) \& o \in O_u} u_k} \quad (8)$$