

# The BTL Model

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## 1 The Standard BTL Model –

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{u_i}{u_i + u_j}$$

$$p(i \prec j|u) = \frac{u_j}{u_j + u_i}$$

Suppose that there are  $K$  underlying aspects. for each pair,  $p(< w, v >) = p^k(w \succ v)$ . then the probability of generating a session observation  $d$  is defined as:

$$p(< w, v > |V, U) = \frac{\sum_{k=1}^K u_k w_k}{\sum_{k=1}^K u_k w_k + \sum_{k=1}^K u_k v_k} \quad (1)$$

The likelihood function can be written as follows. The model parameters are denoted as  $\Theta = \{v \in V, u \in U\}$

$$L(\Theta) = p(D|\Theta) = \prod_{d \in D} \prod_{w \in W^d, v \in L^d} \frac{\sum_{k=1}^K u_k w_k}{\sum_{k=1}^K u_k w_k + \sum_{k=1}^K u_k v_k} \quad (2)$$

Thus, the log likelihood is:

$$l(\Theta) = \log L(\Theta) \quad (3)$$

$$\begin{aligned} &= \log \prod_{d \in D} \prod_{w \in W^d, v \in L^d} \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k} \quad (4) \\ &= \sum_{d \in D} \sum_{w \in W^d, v \in L^d} \log \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k} \\ &= \sum_{d \in D} \sum_{w \in W^d, v \in L^d} [\log \sum_k u_k w_k - \log (\sum_k u_k w_k + \sum_k u_k v_k)] \end{aligned}$$

for the :

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, rewrite the log likelihood as follow.

$$l(\Theta) = \sum_{d \in D} \sum_{w \in W^d, v \in L^d} [1 - \frac{\sum_k u_k w_k + \sum_k u_k v_k}{\sum_k u_k^t w_k^t + \sum_k u_k^t v_k^t} + \log \frac{\sum_k u_k w_k}{\sum_k u_k^t w_k^t + \sum_k u_k^t v_k^t}] \quad (5)$$

$$= \sum_{d \in D} \sum_{w \in W^d, v \in L^d} [1 - \frac{\sum_k u_k (w_k + v_k)}{\sum_k u_k^t (w_k^t + v_k^t)} + \log \sum_k u_k w_k - \log \sum_k u_k^t (w_k^t + v_k^t)] \quad (6)$$

Using the Jensen's inequality:

$$\log \sum u_k w_k \geq \sum u_k \log w_k, \text{ if } \sum u_k = 1.$$

$l(\Theta)$  can be :

$$l(\Theta) \geq \sum_{d \in D} \sum_{w \in W^d, v \in L^d} [1 - \frac{\sum_k u_k (w_k + v_k)}{\sum_k u_k^t (w_k^t + v_k^t)} + \sum_k u_k \log w_k - \log \sum_k u_k^t (w_k^t + v_k^t)] \quad (7)$$

we want to choose  $\{U \text{ and } V\}$  to maximize  $l(\Theta)$ .

### 1.1 For u

By setting the partial derivative of  $\frac{l(\Theta)}{\partial u_k} = 0$ , we have:

$$\frac{\partial l(\Theta)}{\partial u_k} = \sum_{d \in D} \sum_{w \in W^d, v \in L^d} (\frac{w_k}{\sum_k u_k w_k} - \frac{w_k + v_k}{\sum_k u_k^t (w_k^t + v_k^t)}) = 0 \quad (8)$$

$$\begin{aligned} \frac{w_k}{\sum_k u_k w_k} &= \frac{w_k + v_k}{\sum_k u_k^t (w_k^t + v_k^t)} \\ \sum_k u_k w_k &= \frac{w_k \sum_k u_k^t (w_k^t + v_k^t)}{(w_k + v_k)} \\ u_k w_k &= \frac{w_k \sum_k u_k^t (w_k^t + v_k^t)}{(w_k + v_k)} - \sum_{k' \neq k} u_{k'} w_{k'} \\ u_k &= \frac{\sum_k u_k^t (w_k^t + v_k^t)}{w_k + v_k} - \frac{\sum_{k' \neq k} u_{k'} w_{k'}}{w_k} \end{aligned}$$

Eventually,

$$u_k = \sum_{d \in D} \sum_{w \in W^d, v \in L^d} (\frac{\sum_k u_k^t (w_k^t + v_k^t)}{w_k + v_k} - \frac{\sum_{k' \neq k} u_{k'} w_{k'}}{w_k})$$

## 1.2 For v

As for winner:

By setting the partial derivative of  $\frac{l(\Theta)}{\partial w_k} = 0$ , we have:

$$\frac{\partial l(\Theta)}{\partial w_k} = \sum_{d \in D} \sum_{w \in W^d, v \in L^d} \left( \frac{u_k}{\sum_k u_k w_k} - \frac{u_k}{\sum_k u_k^t (w_k^t + v_k^t)} \right) = 0 \quad (9)$$

$$w_k = \sum_k u_k^t (w_k^t + v_k^t)$$

As for loser:

By setting the partial derivative of  $\frac{l(\Theta)}{\partial v_k} = 0$ , we have:

$$\frac{\partial l(\Theta)}{\partial v_k} = \sum_{d \in D} \sum_{w \in W^d, v \in L^d} \left( -\frac{u_k}{\sum_k u_k^t (w_k^t + v_k^t)} \right) = 0 \quad (10)$$

$$v_k =$$