The BTL Model

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August 10, 2018

1 The Standard BTL Model

Suppose that there are K underlying aspects, the latent preference vector for each user is denoted as $u \in R+^K$, and $\sum_k u_k = 1$, latent item feature vector $i, j \in R+^K$.

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{u^T i}{u^T i + u^T j}$$
$$p(i \prec j|u) = 1 - p(i \succ j|u)$$

We use the maximal likelihood estimation. The likelihood function can be written as follows. The model parameters are denoted as $\Theta = \{v \in V, u \in U\}$, $\succ_d$ represents a pairwise ranking observation in a session d.

$$L(\Theta) = \Pi_{d \in D} \Pi_{i \succ_d j} \frac{\sum_{k=1}^K u_k i_k}{\sum_{k=1}^K u_k i_k + \sum_{k=1}^K u_k j_k} \tag{1}$$

Thus, the log likelihood and its lower bound is:

$$\begin{aligned} l(\Theta) &= \log\ L(\Theta) && (2) \\ &= \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} \log\ \frac{\sum_k u_k i_k}{\sum_k u_k i_k + \sum_k u_k j_k} && (3) \\ &\geq \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} [1 - \frac{\sum_k u_k i_k + \sum_k u_k j_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t} + \log \frac{\sum_k u_k i_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t}] \end{aligned}$$

where we apply

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

To maximize the lower-bound in $Equ.3$ we employ conjugate gradient method.

We first fix all latent vectors for the item [1], and apply $\log \sum_k u_k i_k \geq \frac{\sum_k i_k \log u_k}{\sum_k i_k} + \log \sum_k i_k, \forall i_k \geq 0$. denote $c^t(i, j, d) = \sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t$ in the

[1] We use the supscript t for parameters in the t^{th} round

t-th round of the iteration, with respect to $\sum_k u_k = 1$,by applying the Lagrange, we have

$$\frac{\partial\, l(\Theta)}{\partial\, u_k} = \sum_{d \in D_u} \sum_{i \succ_d j} (\frac{i_k}{u_k \sum_k i_k} - \frac{i_k + j_k}{c^t(i,j,d)}) + \lambda = 0 \tag{4}$$

$$\frac{\partial\, l(\Theta)}{\partial \lambda} = \sum_k u_k - 1 = 0 \tag{5}$$

$$u_k = \frac{\sum_{d \in D_u} \sum_{i \succ_d j} \frac{i_k c^t(i,j,d)}{(i_k + j_k) \sum_k i_k}}{\sum_{k=1}^{K} \sum_{d \in D_u} \sum_{i \succ_d j} \frac{i_k c^t(i,j,d)}{(i_k + j_k) \sum_k i_k}} \tag{6}$$

Next we fix all us, and again apply the Jensen's inequality $\log \sum_k u_k i_k \geq \sum_k u_k \log i_k, \forall u_k \geq 0, \sum_k u_k = 1$. For convenience, we use v to denote a arbitral item, and O_u is the set of all observed pairwise ranking of user u, if $v \succ j$is the winning item then we say $o \in W(v)$.

$$\frac{\partial\, l(\Theta)}{\partial\, v_k} = \sum_u \sum_{o \in W(v) \&\& o \in O_u} \frac{u_k}{i_k} - \sum_u \sum_{o \in O_u} \frac{u_k}{c^t(i,j,d)} = 0 \tag{7}$$

$$i_k = \frac{\sum_u \sum_{o \in O_u} \frac{u_k}{c^t(i,j,d)}}{\sum_u \sum_{o \in W(v) \&\& o \in O_u} u_k} \tag{8}$$