The BTL Model

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August 7, 2018

1 The Standard BTL Model -

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{ui}{ui + uj}$$
$$p(i \prec j|u) = \frac{uj}{uj + ui}$$

Suppose that there are K underlying aspects. for each pair, $p(\langle w, v \rangle) = p^k(w \succ v)$. then the probability of generating a session observation d is defined as:

$$p(\langle w, v \rangle | V, U) = \frac{\sum_{k=1}^{K} u_k w_k}{\sum_{k=1}^{K} u_k w_k + \sum_{k=1}^{K} u_k v_k}$$
(1)

The likelihood function can be written as follows. The model parameters are denoted as $\Theta = \{v \in V, u \in U)\}$

$$L(\Theta) = p(D|\Theta) = \prod_{d \in D} \prod_{w \in W^d, v \in L^d} \frac{\sum_{k=1}^K u_k w_k}{\sum_{k=1}^K u_k w_k + \sum_{k=1}^K u_k v_k}$$
(2)

Thus, the log likelihood is:

$$l(\Theta) = \log L(\Theta)$$

$$= \log \Pi_{d \in D} \Pi_{w \in W^d, v \in L^d} \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k}$$

$$= \sum_{d \in D} \sum_{w \in W^d, v \in L^d} \log \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k}$$

$$= \sum_{d \in D} \sum_{w \in W^d, v \in L^d} [\log \sum_k u_k w_k - \log(\sum_k u_k w_k + \sum_k u_k v_k)]$$

$$(4)$$

for the :

$$\ln \frac{y}{x} \ge 1 - \frac{x}{y}$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, rewrite the log likelihood as follow

$$l(\Theta) = \sum_{d \in D} \sum_{w \in W^d, v \in L^d} \left[1 - \frac{\sum_k u_k w_k + \sum_k u_k v_k}{\sum_k u_k^t w_k^t + \sum_k u_k^t v_k^t} + \log \frac{\sum_k u_k w_k}{\sum_k u_k^t w_k^t + \sum_k u_k^t v_k^t} \right]$$

$$= \sum_{d \in D} \sum_{w \in W^d, v \in L^d} \left[1 - \frac{\sum_k u_k (w_k + v_k)}{\sum_k u_k^t (w_k^t + v_k^t)} + \log \sum_k u_k w_k - \log \sum_k u_k^t (w_k^t + v_k^t) \right]$$
(6)

Using the Jensen's inequality:

$$\log \sum_{k} u_k w_k \ge \sum_{k} u_k \log w_k, if \sum_{k} u_k = 1.$$

 $l(\Theta)$ can be:

$$l(\Theta) \ge \sum_{d \in D} \sum_{w \in W^d, v \in L^d} \left[1 - \frac{\sum_k u_k(w_k + v_k)}{\sum_k u_k^t(w_k^t + v_k^t)} + \sum_k u_k \log w_k - \log \sum_k u_k^t(w_k^t + v_k^t) \right]$$
(7)

we want to choose $\{U \text{ and } V\}$ to maximize $l(\Theta)$.

1.1 For u

By setting the partial derivative of $\frac{l(\Theta)}{\partial u_k} = 0$, we have:

$$\frac{\partial l(\Theta)}{\partial u_k} = \sum_{d \in D} \sum_{w \in W^d, v \in L^d} \left(\frac{\sum_k w_k}{\sum_k u_k w_k} - \frac{\sum_k (w_k + v_k)}{\sum_k u_k^t (w_k^t + v_k^t)} \right) = 0$$

$$\frac{\sum_k w_k}{\sum_k u_k w_k} = \frac{\sum_k (w_k + v_k)}{\sum_k u_k^t (w_k^t + v_k^t)}$$

$$\sum_k u_k w_k = \frac{\sum_k w_k \sum_k u_k^t (w_k^t + v_k^t)}{\sum_k (w_k + v_k)}$$

$$u_k w_k = \frac{\sum_k w_k \sum_k u_k^t (w_k^t + v_k^t)}{\sum_k (w_k + v_k)} - \sum_{k' \neq k} u_{k'} w_{k'}$$

$$u_k = \frac{\sum_k u_k^t (w_k^t + v_k^t)}{\sum_k (w_k + v_k)} - \frac{\sum_{k' \neq k} u_{k'} w_{k'}}{w_k}$$

Eventually,

$$u_k = \sum_{d \in D} \sum_{w \in W^d, v \in L^d} \left(\frac{\sum_k u_k^t(w_k^t + v_k^t)}{\sum_k (w_k + v_k)} - \frac{\sum_{k' \neq k} u_{k'} w_{k'}}{w_k} \right)$$

Note that $\sum_{k} u_k = 1$

1.2 For v

As for winner:

By setting the partial derivative of $\frac{l(\Theta)}{\partial w_k} = 0$, we have:

$$\frac{\partial l(\Theta)}{\partial w_k} = \sum_{d \in W(v)} \sum_{v' \in L_d} \left(\frac{\sum_k u_k}{w_k} - \frac{\sum_k u_k}{\sum_k u_k^t (w_k^t + v_k'^t)} \right) = 0$$

$$w_k = \sum_{d \in W(v)} \sum_{v' \in L_d} \sum_k u_k^t (w_k^t + v_k'^t)$$
(9)

As for loser

By setting the partial derivative of $\frac{l(\Theta)}{\partial v_k} = 0$, we have:

$$\frac{\partial l(\Theta)}{\partial v_k} = \sum_{d \in L(v)} \sum_{v' \in W_d} \left(-\frac{\sum_k u_k}{\sum_k u_k^t (v_k'^t + v_k^t)} \right) = 0 \tag{10}$$

Eventually,

$$v_{k} = \sum_{d \in W(v)} \sum_{v' \in L_{d}} \sum_{k} u_{k}^{t} (w_{k}^{t} + v_{k}^{'t}) + \sum_{d \in L(v)} \sum_{v' \in W_{d}} ()$$