The BTL Model

Xiaolin Shen

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1 The Standard BTL Model –

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{ui}{ui + uj}$$
$$p(i \prec j|u) = \frac{uj}{uj + ui}$$

Suppose that there are K underlying aspects. for each pair, $p(< w, v >) = p^k(w \succ v)$. then the probability of generating a session observation d is defined as:

$$p(< w, v > |V, U) = \frac{\sum_{k=1}^{K} u_k w_k}{\sum_{k=1}^{K} u_k w_k + \sum_{k=1}^{K} u_k v_k} \tag{1}$$

The likelihood function can be written as follows. The model parameters are denoted as $\Theta = \{v \in V, u \in U)\}$

$$L(\Theta) = p(D|\Theta) = \Pi_{d \in D} \Pi_{w \in W^d, v \in L^d} \frac{\sum_{k=1}^{K} u_k w_k}{\sum_{k=1}^{K} u_k w_k + \sum_{k=1}^{K} u_k v_k} \tag{2}$$

Thus, the log likelihood is:

$$\begin{aligned} l(\Theta) &= \log\ L(\Theta) & (3) \\ &= \log\ \Pi_{d \in D} \Pi_{w \in W^d, v \in L^d} \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k} & (4) \\ &= \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} \log\ \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k} \\ &= \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} [\log \sum_k u_k w_k - \log(\sum_k u_k w_k + \sum_k u_k v_k)] \end{aligned}$$

for the :

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, rewrite the log likelihood as follow.

$$l(\Theta) = \Sigma_{d\in D}\Sigma_{w\in W^d,v\in L^d}[1 - \frac{\sum_k u_k w_k + \sum_k u_k v_k}{\sum_k u_k^t w_k^t + \sum_k u_k^t v_k^t} + \log \frac{\sum_k u_k w_k}{\sum_k u_k^t w_k^t + \sum_k u_k^t v_k^t}] \tag{5}$$

$$= \Sigma_{d\in D}\Sigma_{w\in W^d,v\in L^d}[1 - \frac{\sum_k u_k(w_k + v_k)}{\sum_k u_k^t(w_k^t + v_k^t)} + \log \sum_k u_k w_k - \log \sum_k u_k^t(w_k^t + v_k^t)] \tag{6}$$

Using the Jensen's inequality:

$$\log \sum_k u_k w_k \geq \sum_k u_k \log w_k, if \sum_k u_k = 1.$$

$l(\Theta)$ can be :

$$l(\Theta) \geq \Sigma_{d\in D}\Sigma_{w\in W^d,v\in L^d}[1 - \frac{\sum_k u_k(w_k + v_k)}{\sum_k u_k^t(w_k^t + v_k^t)} + \sum_k u_k \log w_k - \log \sum_k u_k^t(w_k^t + v_k^t)] \tag{7}$$

we want to choose $\{U \; and \; V\}$ to maximize $l(\Theta)$.

1.1 For u

By setting the partial derivative of $\frac{l(\Theta)}{\partial u_k} = 0$, we have:

$$\frac{\partial \, l(\Theta)}{\partial \, u_k} = \Sigma_{d\in D}\Sigma_{w\in W^d,v\in L^d}(\frac{\sum_k w_k}{\sum_k u_k w_k} - \frac{\sum_k(w_k + v_k)}{\sum_k u_k^t(w_k^t + v_k^t)}) = 0 \tag{8}$$

$$\frac{\sum_k w_k}{\sum_k u_k w_k} = \frac{\sum_k(w_k + v_k)}{\sum_k u_k^t(w_k^t + v_k^t)}$$

$$\sum_k u_k w_k = \frac{\sum_k w_k \sum_k u_k^t(w_k^t + v_k^t)}{\sum_k(w_k + v_k)}$$

$$u_k w_k = \frac{\sum_k w_k \sum_k u_k^t(w_k^t + v_k^t)}{\sum_k(w_k + v_k)} - \sum_{k' \neq k} u_{k'} w_{k'}$$

$$u_k = \frac{\sum_k u_k^t(w_k^t + v_k^t)}{\sum_k(w_k + v_k)} - \frac{\sum_{k' \neq k} u_{k'} w_{k'}}{w_k}$$

Eventually,

$$u_k = \Sigma_{d\in D}\Sigma_{w\in W^d,v\in L^d}(\frac{\sum_k u_k^t(w_k^t + v_k^t)}{\sum_k(w_k + v_k)} - \frac{\sum_{k' \neq k} u_{k'} w_{k'}}{w_k})$$

Note that $\sum_k u_k = 1$

1.2 For v

As for winner:
By setting the partial derivative of $\frac{l(\Theta)}{\partial w_k} = 0$, we have:

$$\frac{\partial\, l(\Theta)}{\partial\, w_k} = \Sigma_{d \in W(v)} \Sigma_{v' \in L^d} \left(\frac{\sum_k u_k}{w_k} - \frac{\sum_k u_k}{\sum_k u_k^t (w_k^t + v_k^{'t})} \right) = 0 \tag{9}$$

$$w_k = \Sigma_{d \in W(v)} \Sigma_{v' \in L^d} \sum_k u_k^t (w_k^t + v_k^{'t})$$

As for loser:
By setting the partial derivative of $\frac{l(\Theta)}{\partial v_k} = 0$, we have:

$$\frac{\partial\, l(\Theta)}{\partial\, v_k} = \Sigma_{d \in L(v)} \Sigma_{v' \in W^d} \left(- \frac{\sum_k u_k}{\sum_k u_k^t (v_k^{'t} + v_k^t)} \right) = 0 \tag{10}$$

$$v_k =$$

Eventually,

$$v_k = \Sigma_{d \in W(v)} \Sigma_{v' \in L^d} \sum_k u_k^t (w_k^t + v_k^{'t})$$