

# The BTL Model

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## 1 The Standard BTL Model

Suppose that there are  $K$  underlying aspects, the latent preference vector for each user is denoted as  $u \in R^{+K}$ , and  $\sum_k u_k = 1$ , latent item feature vector  $i, j \in R^{+K}$ .

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{u^T i}{u^T i + u^T j}$$

$$p(i \prec j|u) = 1 - p(i \succ j|u)$$

We use the maximal likelihood estimation. The likelihood function can be written as follows. The model parameters are denoted as  $\Theta = \{v \in V, u \in U\}$ ,  $\succ_d$  represents a pairwise ranking observation in a session  $d$ .

$$L(\Theta) = \prod_{d \in D} \prod_{i \succ_d j} \frac{\sum_{k=1}^K u_k i_k}{\sum_{k=1}^K u_k i_k + \sum_{k=1}^K u_k j_k} \quad (1)$$

Thus, the log likelihood and its lower bound is:

$$l(\Theta) = \log L(\Theta) \quad (2)$$

$$= \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} \log \frac{\sum_k u_k i_k}{\sum_k u_k i_k + \sum_k u_k j_k} \quad (3)$$

$$\geq \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} \left[ 1 - \frac{\sum_k u_k i_k + \sum_k u_k j_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t} + \log \frac{\sum_k u_k i_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t} \right]$$

where we apply

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

To maximize the lower-bound in *Equ.3* we employ conjugate gradient method.

We first fix all latent vectors for the item <sup>1</sup>, and apply  $\log \sum_k u_k i_k \geq \frac{\sum_k i_k \log u_k}{\sum_k i_k} + \log \sum_k i_k, \forall i_k \geq 0$ . Let's compute  $c^t(d, i, j) = \sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t$ ,

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<sup>1</sup>We use the supscript  $t$  for parameters in the  $t^{th}$  round

$c_k^t(d, i, j) = \frac{i_k^t + j_k^t}{c^t(d, i, j)}$ , and  $f_k^t(d, i) = \frac{i_k^t}{\sum_k i_k^t}$  for all pairwise ranking observations in  $d$  using the  $t$ -th round parameters, we have

$$\frac{\partial l}{\partial u_k} = \sum_{d \in D_u} \sum_{i \succ_d j} \left( \frac{f_k^t(d, i)}{u_k} - c_k^t(d, i, j) \right) = 0 \quad (4)$$

$$(5)$$

Here we omit the constraint that  $u_k \geq 0$  and hope that this constraint will be satisfied by the solution.

$$u_k = \frac{\sum_{d \in D_u} \sum_{i \succ_d j} \frac{i_k}{\sum_k i_k}}{\sum_{d \in D_u} \sum_{i \succ_d j} \frac{i_k + j_k}{c^t(d, i, j)}} \quad (6)$$

Next we fix all user latent vectors, and again apply the Jensen's inequality  $\log \sum_k u_k i_k \geq \frac{\sum_k u_k \log i_k}{\sum_k u_k} + \log \sum_k u_k, \forall u_k \geq 0$ . For convenience, we use  $v$  to denote a arbitral item, and  $O_u$  is the set of all observed pairwise ranking of user  $u$ , if  $v \succ j$  is the winning item then we say  $o \in W(v)$ . We first compute  $e_k^t(u) = \frac{u_k^t}{\sum_k u_k^t}$

$$\frac{\partial l}{\partial v_k} = \sum_u \sum_{o \in W(v) \& o \in O_u} \frac{e_k^t(u)}{v_k} - \sum_u \sum_{o \in O_u, i=v || j=v} \frac{u_k}{c^t(d, i, j)} = 0 \quad (7)$$

$$v_k = \frac{\sum_u \sum_{o \in W(v) \& o \in O_u} e_k^t(u)}{\sum_u \sum_{o \in O_u, i=v || j=v} \frac{u_k}{c^t(d, i, j)}} \quad (8)$$

Again we omit the constraint that  $v_k \geq 0$  because that is satisfied automatically by the solution. We have to renormalize  $\forall k, \sum_v v_k = 1$ .

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**Algorithm 1:** Standard BTL Algorithm

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**Input:** A set of pairwise rankings  $D$ , number of aspects  $K$ , maximal number of iterations  $MaxIter$

**Output:**  $U, V$

initialization: Randomize  $U, V, \forall k, u_k \geq 0, v_k \geq 0, \sum_v v_k = 1$  ;

**for**  $t = 1; t \leq MaxIter; t++$  **do**

    Randomize order of observations ;

**for**  $u \in U$  **do**

$c^t(d, i, j) = \sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t, f^t(d, i) = \sum_k i_k^t;$

**for**  $k = 1; k \leq K, k++$  **do**

$c_k^t(d, i, j) = \frac{i_k^t + j_k^t}{c^t(d, i, j)}, f_k^t(d, i) = \frac{i_k^t}{f^t(d, i)};$

**end**

$u_k^{t+1} = \frac{\sum_{d \in D_u} \sum_{i \succ_d j} f_k^t(d, i)}{\sum_{d \in D_u} \sum_{i \succ_d j} c_k^t(d, i, j)};$

**end**

**for**  $v \in V$  **do**

$e^t(u) = \sum_k u_k^t;$

**for**  $k = 1; k \leq K, k++$  **do**

$e_k^t(u) = \frac{u_k^t}{e^t(u)};$

**end**

$v_k = \frac{\sum_u \sum_{o \in W(v) \& o \in O_u} e_k^t(u)}{\sum_u \sum_{o \in O_u, i=v \parallel j=v} \frac{u_k}{c^t(i, j, d)}};$

**end**

**end**

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