Modify By Confidence Level

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1 The BTL Model —ModifyByConfidenceLevel

Suppose g is a $K-$dim vector, with one and only one component to be equal to 1, for each pair, $p(<w,v>|g_k=1) = p^k(w \succ v) \times \Pi_{k' \neq k} p^{k'}(w \succeq v)$. then the probability of generating a session observation d given the hidden aspect a is defined as:

$$p(d|g,\theta,V,U) = p(d|\Theta^t,g) = \Pi_{w \in W^d, v \in L^d, w \succ_R v} a^K \frac{w_k}{w_k + \theta^t v_k} \Pi_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}} \tag{1}$$

where $w \succ_R v$ is the ranking pair in R relation, a is the given value.

Besides,

$$\begin{aligned} p(g|d,\Theta^t) &\propto p(g|u,\Theta^t)p(d|g,\Theta^t) \\ p(g_k=1|d,\Theta^t) &\propto u_k^t \frac{w_k^t}{w_k^t + \theta^t l_k^t} \prod_{k' \neq k} [\frac{\theta^t w_{k'}^t}{l_{k'}^t + \theta^t w_{k'}^t}] \end{aligned} \tag{2}$$

First, let's use $\gamma(d,k,\Theta^t)$ to denote the conditional probability $p(g_k = 1|d,\Theta^t)$ given parameters in the $t-$th round, when the current session specific favorite aspect is $g_k = 1$, defined as follows

$$\begin{aligned} \gamma(d,k,\Theta^t) &= \frac{p(d,g|\Theta^t)}{\Sigma_g p(d,g|\Theta^t)} = \frac{p(g|\Theta^t)p(d|\Theta^t,g)}{\Sigma_g p(g|\Theta^t)p(d|\Theta^t,g)} \\ &= \frac{u_k \Pi_{w \in W^d, v \in L^d, w \succ_R v} a^K \frac{w_k}{w_k + \theta^t v_k} \Pi_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}}{\Sigma_{k=1}^K u_k \Pi_{w \in W^d, v \in L^d, w \succ_R v} a^K \frac{w_k}{w_k + \theta^t v_k} \Pi_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}} \end{aligned} \tag{3}$$

Note that $\forall d, \Sigma_k \gamma(d,k,\Theta^t) = 1$.

1.1 E-step

In the E-step of $t-$th EM round, compute the expectation $Q(\Theta^t) = E_G \ln p(D,G|\Theta)$

$$
\begin{aligned}
E_G \ln p(D,G|\Theta) &= \sum_{R,w\succ_R v} a^K \Sigma_d \Sigma_{k=1}^K \gamma(d,k,\Theta^t) \ln p(d,g|\Theta) \qquad (4)\\
&= \sum_{R,w\succ_R v} a^K \Sigma_d \Sigma_{k=1}^K \gamma(d,k,\Theta^t)\{\ln u_k\\
&+ \Sigma_{w\in W_d, v\in V_d}[\ln \frac{w_k}{w_k+\theta v_k} + \Sigma_{k'\neq k} \ln \frac{\theta w_{k'}}{v_{k'}+\theta w_{k'}}]\}
\end{aligned}
$$

1.2 M-step

1.2.1 For u

first maximize $Q(\Theta^t)$ with respect to U. For each $u \in U$, eliminating constant terms, we have:

$$
\min - \sum_{R,w\succ_R v} a^K \Sigma_{u(d)=u} \Sigma_{k=1}^K \gamma(d,k,\Theta^t) \ln u_k \qquad (5)
$$

$$
w.r.t \Sigma_k u_k = 1
$$

Solving the above Lagrange function Equ. 5, we get

$$
u_k = \frac{\sum_{R,w\succ_R v} a^K \Sigma_{u(d)=u} \gamma(d,k,\Theta^t)}{\sum_{R,w\succ_R v} a^K \Sigma_{s=1}^K \Sigma_{u(d)=u} \gamma(d,s,\Theta^t)} \qquad (6)
$$

1.2.2 For v

for the :

$$
\ln \frac{y}{x} \geq 1 - \frac{x}{y}
$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, we obtain a minorization function of $\tilde{Q}(\Theta^t)$.

$$
\tilde{Q}(\Theta^t) = \sum_{R,w\succ_R v} a^K \Sigma_d \Sigma_k \gamma(d,k,\Theta^t) \Sigma_{w\in W_d, v\in L_d} \{[\ln w_k + 1 - \ln(w_k^t + \theta^t v_k^t) - \frac{w_k+\theta v_k}{w_k^t+\theta^t v_k^t}] +
$$

$$
\Sigma_{k'\neq k}[\ln(\theta w_{k'}) + 1 - \ln(v_{k'}^t + \theta^t w_{k'}^t) - \frac{v_{k'}+\theta w_{k'}}{v_{k'}^t+\theta^t w_{k'}^t}]\} \ldots\ldots\ldots\ldots\ldots\ldots (7)
$$

$\tilde{Q}(\Theta^t)$ can be seperated for each item v. Considering only the $k-$th component v_k, $\tilde{Q}(v_k,\Theta^t)$ involves two terms, one of which is relevant to observations $d \in W(v)$ where v acts as skyline object, the other is relevant to observations $d \in L(v)$ where v acts as comparisons, $\tilde{Q}(v_k,\Theta^t) = \tilde{Q}^1(v_k,\Theta^t) + \tilde{Q}^2(v_k,\Theta^t)$. Removing all constants and irrelevant terms for v_k, we have the following minorizing function:

$$\tilde{Q}^1(v_k, \Theta^t) = \sum_{R, w \succ_R v} a^K \{\Sigma_{d \in W(v)} |L_d| \ln v_k - v_k \Sigma_{d \in W(v)} \Sigma_{v' \in L_d} [\frac{\gamma(d,k,\Theta^t)}{\alpha(v,v',k,\Theta^t)} + \Sigma_{k' \neq k} \frac{\theta^t \gamma(d,k',\Theta^t)}{\alpha(v',v,k,\Theta^t)}]\}$$

$$\tilde{Q}^2(v_k, \Theta^t) = \sum_{R, w \succ_R v} a^K \{-v_k \Sigma_{d \in L(v)} \Sigma_{v' \in W_d} [\frac{\theta^t \gamma(d,k,\Theta^t)}{\alpha(v',v,k,\Theta^t)} + \Sigma_{k' \neq k} \frac{\gamma(d,k',\Theta^t)}{\alpha(v,v',k,\Theta^t)}]\}$$

where:

$|L_d|$ is the number of objects being dominanted in d,
$\alpha(v, v', k, \Theta^t) = v_k^t + \theta^t {v'}_k^t$.
By setting the partial derivative of $\frac{\partial \tilde{Q}(v_k,\Theta^t)}{\partial v_k} = 0$, we have:

$$\frac{1}{v_k} = \sum_{R, w \succ_R v} a^K \{ \frac{\Sigma_{d \in W(v)} \Sigma_{v' \in L_d} [\frac{\gamma(d,k,\Theta^t)}{\alpha(v,v',k,\Theta^t)} + \Sigma_{k' \neq k} \frac{\theta^t \gamma(d,k',\Theta^t)}{\alpha(v',v,k,\Theta^t)}]}{\Sigma_{d \in W(v)} |L_d|} + \frac{\Sigma_{d \in L(v)} \Sigma_{v' \in W_d} [\frac{\theta^t \gamma(d,k,\Theta^t)}{\alpha(v',v,k,\Theta^t)} + \Sigma_{k' \neq k} \frac{\gamma(d,k',\Theta^t)}{\alpha(v,v',k,\Theta^t)}]}{\Sigma_{d \in W(v)} |L_d|} \}$$

1.2.3 For θ

Fix u, v, update θ by:

Fix $v \in V$ and $u \in U$, rearranging $Equ.7$, we have the solution for $\frac{\partial \tilde{Q}(\Theta^t)}{\partial \theta} = 0$ as:

$$\theta = \sum_{R, w \succ_R v} a^K \{ \frac{(K-1)\Sigma_d |W_d||L_d|}{\Sigma_d \Sigma_k \gamma(d,k,\Theta^t) \Sigma_{w,v} [\frac{v_k}{\alpha(w,v,k,\Theta^t)} + \Sigma_{k' \neq k} \frac{w_{k'}}{\alpha(v,w,k',\Theta^t)}]} \} \quad (7)$$