The BTL Model

Xiaolin Shen

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1 The Standard BTL Model

Suppose that there are K underlying aspects, the latent preference vector for each user is denoted as $u \in R+^K$, and $\sum_k u_k = 1$, latent item feature vector $i, j \in R+^K$.

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{u^T i}{u^T i + u^T j}$$
$$p(i \prec j|u) = 1 - p(i \succ j|u)$$

We use the maximal likelihood estimation. The likelihood function can be written as follows. The model parameters are denoted as $\Theta = \{v \in V, u \in U)\}$, $\succ_d$ represents a pairwise ranking observation in a session d.

$$L(\Theta) = \Pi_{d \in D} \Pi_{i \succ_d j} \frac{\sum_{k=1}^K u_k i_k}{\sum_{k=1}^K u_k i_k + \sum_{k=1}^K u_k j_k} \tag{1}$$

Thus, the log likelihood and its lower bound is:

$$l(\Theta) = \log\ L(\Theta) \tag{2}$$
$$= \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} \log\ \frac{\sum_k u_k i_k}{\sum_k u_k i_k + \sum_k u_k j_k} \tag{3}$$
$$\geq \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} [1 - \frac{\sum_k u_k i_k + \sum_k u_k j_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t} + \log \frac{\sum_k u_k i_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t}]$$

where we apply

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

To maximize the lower-bound in $Equ.3$ we employ conjugate gradient method.

We first fix all vs, and apply $\log \sum_k u_k i_k \geq \frac{\sum_k i_k \log u_k}{\sum_k i_k} + \log \sum_k i_k, \forall i_k \geq 0$. denote $c^t(i,j,d) = \sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t$ in the t-th round of the iteration, with respect to $\sum_k u_k = 1$,we have

$$\frac{\partial\, l(\Theta)}{\partial\, u_k} = \sum_{d \in D_u} \sum_{i \succ_d j} \left(\frac{i_k}{u_k \sum_k i_k} - \frac{i_k + j_k}{c^t(i,j,d)} \right) = 0 \tag{4}$$

1.1 For u

By setting the partial derivative of $\frac{l(\Theta)}{\partial u_k} = 0$, we have:

$$\frac{\partial\, l(\Theta)}{\partial\, u_k} = \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} \left(\frac{\sum_k w_k}{\sum_k u_k w_k} - \frac{\sum_k (w_k + v_k)}{\sum_k u_k^t (w_k^t + v_k^t)} \right) = 0 \tag{5}$$

$$\frac{\sum_k w_k}{\sum_k u_k w_k} = \frac{\sum_k (w_k + v_k)}{\sum_k u_k^t (w_k^t + v_k^t)}$$

$$\sum_k u_k w_k = \frac{\sum_k w_k \sum_k u_k^t (w_k^t + v_k^t)}{\sum_k (w_k + v_k)}$$

$$u_k w_k = \frac{\sum_k w_k \sum_k u_k^t (w_k^t + v_k^t)}{\sum_k (w_k + v_k)} - \sum_{k' \neq k} u_{k'} w_{k'}$$

$$u_k = \frac{\sum_k u_k^t (w_k^t + v_k^t)}{\sum_k (w_k + v_k)} - \frac{\sum_{k' \neq k} u_{k'} w_{k'}}{w_k}$$

Eventually,

$$u_k = \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} \left(\frac{\sum_k u_k^t (w_k^t + v_k^t)}{\sum_k (w_k + v_k)} - \frac{\sum_{k' \neq k} u_{k'} w_{k'}}{w_k} \right)$$

Note that $\sum_k u_k = 1$

1.2 For v

As for winner:
By setting the partial derivative of $\frac{l(\Theta)}{\partial w_k} = 0$, we have:

$$\frac{\partial\, l(\Theta)}{\partial\, w_k} = \Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \left(\frac{\sum_k u_k}{w_k} - \frac{\sum_k u_k}{\sum_k u_k^t (w_k^t + v_k'^t)} \right) = 0 \tag{6}$$

$$w_k = \Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \sum_k u_k^t (w_k^t + v_k'^t)$$

As for loser:
By setting the partial derivative of $\frac{l(\Theta)}{\partial v_k} = 0$, we have:

$$\frac{\partial\, l(\Theta)}{\partial\, v_k} = \Sigma_{d \in L(v)} \Sigma_{v' \in W_d} \left(- \frac{\sum_k u_k}{\sum_k u_k^t (v_k'^t + v_k^t)} \right) = 0 \tag{7}$$

$$v_k =$$

Eventually,

$$v_k = \Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \sum_k u_k^t (w_k^t + v_k^{'t}) + \Sigma_{d \in L(v)} \Sigma_{v' \in W_d} ()$$