

The BTL Model

Xiaolin Shen

August 5, 2018

1 The BTL Model –

The winning item $w \in W^d$ is a skyline object. To define the probability of skyline events, use the BTL model.

$$p(i \succ j|u) = \frac{u_i}{u_i + u_j}$$

$$p(i \prec j|u) = \frac{u_j}{u_j + u_i}$$

If the favorite aspect in current session is k , then following the definition of skyline object, $w \in W^d$ is superior in the favorite aspect, and is not worse than other items in other aspects. $w \in W^d$ ranks higher than item $v \in L^d$ in the aspect k is

$$p^k(w \succ v) = \frac{u_k w_k}{u_k w_k + u_k v_k}$$

the probability of item $w \in W^d$ ranks not lower than item $v \in L^d$ in the aspect $k' = k$. Therefore,

$$p^{k'}(w \succeq v) = 1 - p^{k'}(v \succ w) = 1 - \frac{u_k v_k}{u_k w_k + u_k v_k} = \frac{u_k w_k}{u_k w_k + u_k v_k}$$

Suppose g is a K -dim vector, with one and only one component to be equal to 1, for each pair, $p(< w, v > | g_k = 1) = p^k(w \succ v) \times \prod_{k' \neq k} p^{k'}(w \succeq v)$. then the probability of generating a session observation d given the hidden aspect a is defined as:

$$p(d|g, V, U) = \prod_{w \in W^d, v \in L^d} \prod_{k=1}^K \left[\frac{u_k w_k}{u_k w_k + u_k v_k}^{g_k} \frac{u_k w_k}{u_k w_k + u_k v_k}^{1-g_k} \right] \quad (1)$$

$$= \prod_{w \in W^d, v \in L^d} \prod_{k=1}^K \left[\frac{u_k w_k}{u_k w_k + u_k v_k} \right] \quad (2)$$

First, let's use $\gamma(d, k, \Theta^t)$ to denote the conditional probability $p(g_k = 1|d, \Theta^t)$ given parameters in the t -th round, when the current session specific favorite aspect is $g_k = 1$, defined as follows

$$\begin{aligned}\gamma(d, k, \Theta^t) &= \frac{p(d, g|\Theta^t)}{\sum_g p(d, g|\Theta^t)} = \frac{p(g|\Theta^t)p(d|\Theta^t, g)}{\sum_g p(g|\Theta^t)p(d|\Theta^t, g)} \\ &= \frac{u_{g^k} \prod_{w \in W^d, v \in L^d} \prod_{k=1}^K \frac{u_k w_k}{u_k w_k + u_k v_k}}{\sum_{k=1}^K u_{g^k} \prod_{w \in W^d, v \in L^d} \prod_{k=1}^K \frac{u_k w_k}{u_k w_k + u_k v_k}}\end{aligned}\quad (3)$$

Note that $\forall d, \sum_k \gamma(d, k, \Theta^t) = 1$, g^k denote u 's favorite aspect.

1.1 E-step

In the E-step of t -th EM round, compute the expectation $Q(\Theta^t) = E_G \ln p(D, G|\Theta)$

$$\begin{aligned}E_G \ln p(D, G|\Theta) &= \sum_d \sum_{k=1}^K \gamma(d, k, \Theta^t) \ln p(d, g|\Theta) \\ &= \sum_d \sum_{k=1}^K \gamma(d, k, \Theta^t) \{ \ln u_{g^k} \\ &\quad + \sum_{w \in W^d, v \in V^d} [\ln \frac{u_k w_k}{u_k w_k + u_k v_k}] \}\end{aligned}\quad (4)$$

1.2 M-step

1.2.1 For u

firstly, maximize $Q(\Theta^t)$ with respect to U . For each $u \in U$, eliminating constant terms, we have:

By setting the partial derivative of $\frac{\partial \tilde{Q}(u_k, \Theta^t)}{\partial u_k} = 0$, we have:

$$\sum_d \sum_{k=1}^K \sum_{w \in W^d, v \in V^d} \left(\frac{1}{u_k} + \frac{w_k}{u_k w_k} - \frac{v_k + w_k}{u_k w_k + u_k v_k} \right) = 0 \quad (5)$$

$$u_k = \sum_{u(d)=u} \gamma(d, k, \Theta^t) \quad (6)$$

1.2.2 For v

secondly, maximize $Q(\Theta^t)$ with respect to V . for the :

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, we obtain a minorization function of $\tilde{Q}(\Theta^t)$.

$$\tilde{Q}(\Theta^t) = \sum_d \sum_k \gamma(d, k, \Theta^t) \sum_{w \in W^d, v \in L^d} \{ [\ln(u_k w_k) + 1 - \ln(u_k^t w_k^t + u_k^t v_k^t) - \frac{u_k w_k + u_k v_k}{u_k^t w_k^t + u_k^t v_k^t}] \} \quad (7)$$

$\tilde{Q}(\Theta^t)$ can be separated for each item v . Considering only the k -th component v_k , $\tilde{Q}(v_k, \Theta^t)$ involves two terms, one of which is relevant to observations $d \in W(v)$ where v acts as skyline object, the other is relevant to observations $d \in L(v)$ where v acts as comparisons, $\tilde{Q}(v_k, \Theta^t) = \tilde{Q}^1(v_k, \Theta^t) + \tilde{Q}^2(v_k, \Theta^t)$. Removing all constants and irrelevant terms for v_k , we have the following minorizing function:

$$\begin{aligned}\tilde{Q}^1(v_k, \Theta^t) &= \sum_{d \in W(v)} |L_d| \ln u_k v_k - v_k \sum_{d \in W(v)} \sum_{v' \in L_d} \left[\frac{u_k \gamma(d, k, \Theta^t)}{\alpha(u, w, v, k)} \right] \\ \tilde{Q}^2(v_k, \Theta^t) &= -v_k \sum_{d \in L(v)} \sum_{v' \in W_d} \left[\frac{u_k \gamma(d, k, \Theta^t)}{\alpha(u, w, v, k)} \right]\end{aligned}$$

where:

$|L_d|$ is the number of objects being dominated in d ,
 $\alpha(u, w, v, k) = u_k^t w_k^t + u_k^t v_k^t$.

By setting the partial derivative of $\frac{\partial \tilde{Q}(v_k, \Theta^t)}{\partial v_k} = 0$, we have:

$$\begin{aligned}\frac{1}{v_k} &= \frac{\sum_{d \in W(v)} \sum_{v' \in L_d} \left[\frac{u_k \gamma(d, k, \Theta^t)}{\alpha(u, w, v, k)} \right]}{\sum_{d \in W(v)} |L_d|} \\ &\quad + \frac{\sum_{d \in L(v)} \sum_{v' \in W_d} \left[\frac{u_k \gamma(d, k, \Theta^t)}{\alpha(u, w, v, k)} \right]}{\sum_{d \in W(v)} |L_d|}\end{aligned}$$