The BTL Model

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1 The Standard BTL Model

Suppose that there are K underlying aspects, the latent preference vector for each user is denoted as $u \in R+^K$, and $\sum_k u_k = 1$, latent item feature vector $i, j \in R+^K$.

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{u^T i}{u^T i + u^T j}$$
$$p(i \prec j|u) = 1 - p(i \succ j|u)$$

We use the maximal likelihood estimation. The likelihood function can be written as follows. The model parameters are denoted as $\Theta = \{v \in V, u \in U\}$, $\succ_d$ represents a pairwise ranking observation in a session d.

$$L(\Theta) = \Pi_{d\in D} \Pi_{i\succ_d j} \frac{\sum_{k=1}^K u_k i_k}{\sum_{k=1}^K u_k i_k + \sum_{k=1}^K u_k j_k} \tag{1}$$

Thus, the log likelihood and its lower bound is:

$$\begin{aligned} l(\Theta) &= \log\ L(\Theta) &(2)\\ &= \sum_u \sum_{d\in D_u} \sum_{i\succ_d j} \log\ \frac{\sum_k u_k i_k}{\sum_k u_k i_k + \sum_k u_k j_k} &(3)\\ &\geq \sum_u \sum_{d\in D_u} \sum_{i\succ_d j} [1 - \frac{\sum_k u_k i_k + \sum_k u_k j_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t} + \log \frac{\sum_k u_k i_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t}] \end{aligned}$$

where we apply

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

To maximize the lower-bound in $Equ.3$ we employ conjugate gradient method.

We first fix all latent vectors for the item [1], and apply $\log \sum_k u_k i_k \geq \frac{\sum_k i_k \log u_k}{\sum_k i_k} + \log \sum_k i_k, \forall i_k \geq 0$. Let's compute $c^t(d,i,j) = \sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t$,$c_k^t(d,i,j) =$

[1] We use the supscript t for parameters in the t^{th} round

$\frac{i_k^t+j_k^t}{c^t(d,i,j)}$ denote $c^t(i,j,d)=\sum_k u_k^t i_k^t+\sum_k u_k^t j_k^t$ in the t-th round of the iteration, with respect to $\sum_k u_k=1$,by applying the Lagrange, we have

$$\frac{\partial\, l(\Theta)}{\partial\, u_k}=\sum_{d\in D_u}\sum_{i\succ_d j}\left(\frac{i_k}{u_k\sum_k i_k}-\frac{i_k+j_k}{c^t(i,j,d)}\right)+\lambda=0 \tag{4}$$

$$\frac{\partial\, l(\Theta)}{\partial\lambda}=\sum_k u_k-1=0 \tag{5}$$

$$u_k=\frac{\sum_{d\in D_u}\sum_{i\succ_d j}\frac{i_k c^t(i,j,d)}{(i_k+j_k)\sum_k i_k}}{\sum_{k=1}^K\sum_{d\in D_u}\sum_{i\succ_d j}\frac{i_k c^t(i,j,d)}{(i_k+j_k)\sum_k i_k}} \tag{6}$$

Next we fix all us, and again apply the Jensen's inequality $\log\sum_k u_k i_k\geq\sum_k u_k\log i_k,\forall u_k\geq 0,\sum_k u_k=1$. For convenience, we use v to denote a arbitral item, and O_u is the set of all observed pairwise ranking of user u, if $v\succ j$is the winning item then we say $o\in W(v)$.

$$\frac{\partial\, l(\Theta)}{\partial\, v_k}=\sum_u\sum_{o\in W(v)\&\&o\in O_u}\frac{u_k}{i_k}-\sum_u\sum_{o\in O_u}\frac{u_k}{c^t(i,j,d)}=0 \tag{7}$$

$$i_k=\frac{\sum_u\sum_{o\in O_u}\frac{u_k}{c^t(i,j,d)}}{\sum_u\sum_{o\in W(v)\&\&o\in O_u}u_k} \tag{8}$$