Modify By Confidence Level

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1 The BTL Model —ModifyByConfidenceLevel

Suppose g is a $K-$dim vector, with one and only one component to be equal to 1, for each pair, $p(<w,v>|g_k=1) = p^k(w \succ v) \times \Pi_{k' \neq k} p^{k'}(w \succeq v)$. then the probability of generating a session observation d given the hidden aspect a is defined as:

$$p(d|g,\theta,V,U) = p(d|\Theta^t, g) = \Pi_{w \in W^d, v \in L^d} \Pi_{k=1}^K \left[\frac{w_k}{w_k + \theta v_k}^{g_k} \frac{\theta w_k}{v_k + \theta w_k}^{1-g_k} \right] \quad (1)$$

First, let's use $\gamma(d,k,\Theta^t)$ to denote the conditional probability $p(g_k=1|d,\Theta^t)$ given parameters in the $t-$th round, when the current session specific favorite aspect is $g_k=1$, defined as follows

$$\gamma(d,k,\Theta^t) = \frac{p(d,g|\Theta^t)}{\Sigma_g p(d,g|\Theta^t)} = \frac{p(g|\Theta^t)p(d|\Theta^t,g)}{\Sigma_g p(g|\Theta^t)p(d|\Theta^t,g)} \quad (2)$$

$$= \frac{u_k \Pi_{w \in W^d, v \in L^d} \frac{w_k}{w_k + \theta^t v_k} \Pi_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}}{\Sigma_{k=1}^K u_k \Pi_{w \in W^d, v \in L^d} \frac{w_k}{w_k + \theta^t v_k} \Pi_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}}$$

Note that $\forall d, \Sigma_k \gamma(d,k,\Theta^t) = 1$.

1.1 E-step

$$p(g|d,\Theta^t) \propto p(g|u,\Theta^t)p(d|g,\Theta^t) \quad (3)$$

$$p(g_k=1|d,\Theta^t) \propto u_k^t \frac{w_k^t}{w_k^t + \theta^t l_k^t} \Pi_{k' \neq k} \left[\frac{\theta^t w_{k'}^t}{l_{k'}^t + \theta^t w_{k'}^t} \right]$$

1.2 S-step

then simply add an S-step after the E-step,the value of g for each session d is

$$k \sim u_k^t \frac{w_k^t}{w_k^t + \theta^t l_k^t} \prod_{k' \neq k} \left[\frac{\theta^t w_{k'}^t}{l_{k'}^t + \theta^t w_{k'}^t} \right] \quad (4)$$

In the E-step of $t-$th EM round, compute the expectation $Q(\Theta^t) = E_G \ln p(D, G|\Theta)$

$$E_G \ln p(D, G|\Theta) = \Sigma_d \gamma(d, k, \Theta^t) \ln p(d, g|\Theta)$$
$$= \Sigma_d \gamma(d, k, \Theta^t)\{\ln u_k + \Sigma_{w \in W_d, v \in V_d}[\ln \frac{w_k}{w_k + \theta v_k} + \Sigma_{k' \neq k} \ln \frac{\theta w_{k'}}{v_{k'} + \theta w_{k'}}]\}$$

1.3   M-step

1.3.1   For u

first maximize $Q(\Theta^t)$ with respect to U. For each $u \in U$, eliminating constant terms, we have:

$$\min -\Sigma_{u(d)=u}\Sigma_{k=1}^{K}\gamma(d, k, \Theta^t) \ln u_k \tag{5}$$
$$w.r.t \Sigma_k u_k = 1$$

Solving the above Lagrange function Equ. 5, we get

$$u_k = \frac{\Sigma_{u(d)=u}\gamma(d, k, \Theta^t)}{\Sigma_{s=1}^{K}\Sigma_{u(d)=u}\gamma(d, s, \Theta^t)} \tag{6}$$

1.3.2   For v

for the :
$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, we obtain a minorization function of $\tilde{Q}(\Theta^t)$.

$$\tilde{Q}(\Theta^t) = \Sigma_d \Sigma_k \gamma(d, k, \Theta^t)\Sigma_{w \in W_d, v \in L_d}\{[\ln w_k + 1 - \ln(w_k^t + \theta^t v_k^t) - \frac{w_k + \theta v_k}{w_k^t + \theta^t v_k^t}]+$$

$$\Sigma_{k' \neq k}[\ln(\theta w_{k'}) + 1 - \ln(v_{k'}^t + \theta^t w_{k'}^t) - \frac{v_{k'} + \theta w_{k'}}{v_{k'}^t + \theta^t w_{k'}^t}]\}$$

$\tilde{Q}(\Theta^t)$ can be seperated for each item v. Considering only the $k-$th component v_k, $\tilde{Q}(v_k, \Theta^t)$ involves two terms, one of which is relevant to observations $d \in W(v)$ where v acts as skyline object, the other is relevant to observations $d \in L(v)$ where v acts as comparisons, $\tilde{Q}(v_k, \Theta^t) = \tilde{Q}^1(v_k, \Theta^t) + \tilde{Q}^2(v_k, \Theta^t)$. Removing all constants and irrelevant terms for v_k, we have the following minorizing function:

$$\tilde{Q}^1(v_k, \Theta^t) = \Sigma_{d \in W(v)}|L_d| \ln v_k - v_k \Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \left[\frac{\gamma(d, k, \Theta^t)}{\alpha(v, v', k, \Theta^t)} + \Sigma_{k' \neq k} \frac{\theta^t \gamma(d, k', \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]$$

$$\tilde{Q}^2(v_k, \Theta^t) = -v_k \Sigma_{d \in L(v)} \Sigma_{v' \in W_d} \left[\frac{\theta^t \gamma(d, k, \Theta^t)}{\alpha(v', v, k, \Theta^t)} + \Sigma_{k' \neq k} \frac{\gamma(d, k', \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]$$

where:

$|L_d|$ is the number of objects being dominanted in d,
$\alpha(v, v', k, \Theta^t) = v_k^t + \theta^t v'^t_k$.
By setting the partial derivative of $\frac{\partial \tilde{Q}(v_k, \Theta^t)}{\partial v_k} = 0$, we have:

$$\frac{1}{v_k} = \frac{\Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \left[\frac{\gamma(d, k, \Theta^t)}{\alpha(v, v', k, \Theta^t)} + \Sigma_{k' \neq k} \frac{\theta^t \gamma(d, k', \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]}{\Sigma_{d \in W(v)}|L_d|}$$

$$+ \frac{\Sigma_{d \in L(v)} \Sigma_{v' \in W_d} \left[\frac{\theta^t \gamma(d, k, \Theta^t)}{\alpha(v', v, k, \Theta^t)} + \Sigma_{k' \neq k} \frac{\gamma(d, k', \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]}{\Sigma_{d \in W(v)}|L_d|}$$

More concrete:
if $\gamma(d, k, \Theta^t)$'s k equals to v_k's k:

$$\frac{1}{v_k} = \frac{\Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \left[\frac{\gamma(d, k, \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]}{\Sigma_{d \in W(v)}|L_d|} + \frac{\Sigma_{d \in L(v)} \Sigma_{v' \in W_d} \left[\frac{\theta^t \gamma(d, k, \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]}{\Sigma_{d \in W(v)}|L_d|}$$

else:

$$\frac{1}{v_k} = \frac{\Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \left[\frac{\theta^t \gamma(d, k', \Theta^t)}{\alpha(v', v, k, \Theta^t)} |k'=k \right]}{\Sigma_{d \in W(v)}|L_d|} + \frac{\Sigma_{d \in L(v)} \Sigma_{v' \in W_d} \left[\frac{\gamma(d, k', \Theta^t)}{\alpha(v, v', k, \Theta^t)} |k'=k \right]}{\Sigma_{d \in W(v)}|L_d|}$$

1.3.3 For θ

Fix u, v, update θ by:

Fix $v \in V$ and $u \in U$, rearranging $Equ.9$, we have the solution for $\frac{\partial \tilde{Q}(\Theta^t)}{\partial \theta} = 0$ as:

$$\theta = \frac{(K-1)\Sigma_d |W_d||L_d|}{\Sigma_d \Sigma_k \gamma(d, k, \Theta^t) \Sigma_{w,v} \left[\frac{v_k}{\alpha(w, v, k, \Theta^t)} + \Sigma_{k' \neq k} \frac{w_{k'}}{\alpha(v, w, k', \Theta^t)} \right]} \tag{7}$$

Note that :
in the $\Sigma_d \Sigma_k \gamma(d, k, \Theta^t)$ only summing these d which $\gamma(d, k, \Theta^t)$'s k equals to d's k.