Modify By Confidence Level

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1 The BTL Model —ModifyByConfidenceLevel

Suppose g is a $K-$dim vector, with one and only one component to be equal to 1, for each pair, $p(< w, v > |g_k = 1) = p^k(w \succ v) \times \Pi_{k' \neq k} p^{k'}(w \succeq v)$. then the probability of generating a session observation d given the hidden aspect a is defined as:

$$p(d|g, \theta, V, U) = p(d|\Theta^t, g) = \Pi_{w \in W^d, v \in L^d, w \succ_R v} a^K \frac{w_k}{w_k + \theta^t v_k} \Pi_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}} \tag{1}$$

where $w \succ_R v$ is the ranking pair in R relation, a is the given value.

Besides,

$$\begin{aligned} p(g|d, \Theta^t) &\propto p(g|u, \Theta^t) p(d|g, \Theta^t) \\ p(g_k = 1|d, \Theta^t) &\propto u_k^t \frac{w_k^t}{w_k^t + \theta^t l_k^t} \prod_{k' \neq k} [\frac{\theta^t w_{k'}^t}{l_{k'}^t + \theta^t w_{k'}^t}] \end{aligned} \tag{2}$$

First, let's use $\gamma(d, k, \Theta^t)$ to denote the conditional probability $p(g_k = 1|d, \Theta^t)$ given parameters in the $t-$th round, when the current session specific favorite aspect is $g_k = 1$, defined as follows

$$\begin{aligned} \gamma(d, k, \Theta^t) &= \frac{p(d, g|\Theta^t)}{\Sigma_g p(d, g|\Theta^t)} = \frac{p(g|\Theta^t) p(d|\Theta^t, g)}{\Sigma_g p(g|\Theta^t) p(d|\Theta^t, g)} \\ &= \frac{u_k \Pi_{w \in W^d, v \in L^d, w \succ_R v} a^K \frac{w_k}{w_k + \theta^t v_k} \Pi_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}}{\Sigma_{k=1}^K u_k \Pi_{w \in W^d, v \in L^d, w \succ_R v} a^K \frac{w_k}{w_k + \theta^t v_k} \Pi_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}} \end{aligned} \tag{3}$$

Note that $\forall d, \Sigma_k \gamma(d, k, \Theta^t) = 1$.

1.1 E-step

In the E-step of $t-$th EM round, compute the expectation $Q(\Theta^t) = E_G \ln p(D, G|\Theta)$

$$\begin{aligned} E_G \ln p(D,G|\Theta) &= \Sigma_d \Sigma_{k=1}^K \gamma(d,k,\Theta^t) \ln p(d,g|\Theta) \\ &= \Sigma_d \Sigma_{k=1}^K \gamma(d,k,\Theta^t) \{\ln u_k \\ &+ \Sigma_{w \in W_d, v \in V_d, w \succ_R v} a^K [\ln \frac{w_k}{w_k + \theta v_k} + \Sigma_{k' \neq k} \ln \frac{\theta w_{k'}}{v_{k'} + \theta w_{k'}}]\} \end{aligned} \tag{4}$$

1.2 M-step

1.2.1 For u

first maximize $Q(\Theta^t)$ with respect to U. For each $u \in U$, eliminating constant terms, we have:

$$\begin{aligned} \min -\Sigma_{u(d)=u} \Sigma_{k=1}^K \gamma(d,k,\Theta^t) \ln u_k \\ w.r.t \Sigma_k u_k = 1 \end{aligned} \tag{5}$$

Solving the above Lagrange function Equ. 5, we get

$$u_k = \frac{\sum_{R, w \succ_R v} a^K \Sigma_{u(d)=u} \gamma(d,k,\Theta^t)}{\sum_{R, w \succ_R v} a^K \Sigma_{s=1}^K \Sigma_{u(d)=u} \gamma(d,s,\Theta^t)} \tag{6}$$

1.2.2 For v

for the :

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, we obtain a minorization function of $\tilde{Q}(\Theta^t)$.

$$\tilde{Q}(\Theta^t) = \sum_{R, w \succ_R v} a^K \Sigma_d \Sigma_k \gamma(d,k,\Theta^t) \Sigma_{w \in W_d, v \in L_d} \{[\ln w_k + 1 - \ln(w_k^t + \theta^t v_k^t) - \frac{w_k + \theta v_k}{w_k^t + \theta^t v_k^t}] +$$

$$\Sigma_{k' \neq k} [\ln(\theta w_{k'}) + 1 - \ln(v_{k'}^t + \theta^t w_{k'}^t) - \frac{v_{k'} + \theta w_{k'}}{v_{k'}^t + \theta^t w_{k'}^t}]\} \text{..................(7)}$$

$\tilde{Q}(\Theta^t)$ can be seperated for each item v. Considering only the $k-$th component v_k, $\tilde{Q}(v_k, \Theta^t)$ involves two terms, one of which is relevant to observations $d \in W(v)$ where v acts as skyline object, the other is relevant to observations $d \in L(v)$ where v acts as comparisons, $\tilde{Q}(v_k, \Theta^t) = \tilde{Q}^1(v_k, \Theta^t) + \tilde{Q}^2(v_k, \Theta^t)$. Removing all constants and irrelevant terms for v_k, we have the following minorizing function:

$$\tilde{Q}^1(v_k,\Theta^t)=\sum_{R,w\succ_R v} a^K\{\Sigma_{d\in W(v)}|L_d|\ln v_k - v_k\Sigma_{d\in W(v)}\Sigma_{v'\in L_d}[\frac{\gamma(d,k,\Theta^t)}{\alpha(v,v',k,\Theta^t)}+\Sigma_{k'\neq k}\frac{\theta^t\gamma(d,k',\Theta^t)}{\alpha(v',v,k,\Theta^t)}]\}$$
$$\tilde{Q}^2(v_k,\Theta^t)=\sum_{R,w\succ_R v} a^K\{-v_k\Sigma_{d\in L(v)}\Sigma_{v'\in W_d}[\frac{\theta^t\gamma(d,k,\Theta^t)}{\alpha(v',v,k,\Theta^t)}+\Sigma_{k'\neq k}\frac{\gamma(d,k',\Theta^t)}{\alpha(v,v',k,\Theta^t)}]\}$$

where:

$|L_d|$ is the number of objects being dominanted in d,
$\alpha(v,v',k,\Theta^t)=v_k^t+\theta^t {v'}_k^t$.
By setting the partial derivative of $\frac{\partial\tilde{Q}(v_k,\Theta^t)}{\partial v_k}=0$, we have:

$$\begin{aligned}\frac{1}{v_k}=\sum_{R,w\succ_R v} a^K\{&\frac{\Sigma_{d\in W(v)}\Sigma_{v'\in L_d}[\frac{\gamma(d,k,\Theta^t)}{\alpha(v,v',k,\Theta^t)}+\Sigma_{k'\neq k}\frac{\theta^t\gamma(d,k',\Theta^t)}{\alpha(v',v,k,\Theta^t)}]}{\Sigma_{d\in W(v)}|L_d|}\\ &+\frac{\Sigma_{d\in L(v)}\Sigma_{v'\in W_d}[\frac{\theta^t\gamma(d,k,\Theta^t)}{\alpha(v',v,k,\Theta^t)}+\Sigma_{k'\neq k}\frac{\gamma(d,k',\Theta^t)}{\alpha(v,v',k,\Theta^t)}]}{\Sigma_{d\in W(v)}|L_d|}\}\end{aligned}$$

1.2.3 For θ

Fix u,v, update θ by:

Fix $v\in V$ and $u\in U$, rearranging $Equ.7$, we have the solution for $\frac{\partial\tilde{Q}(\Theta^t)}{\partial\theta}=0$ as:

$$\theta=\sum_{R,w\succ_R v} a^K\{\frac{(K-1)\Sigma_d|W_d||L_d|}{\Sigma_d\Sigma_k\gamma(d,k,\Theta^t)\Sigma_{w,v}[\frac{v_k}{\alpha(w,v,k,\Theta^t)}+\Sigma_{k'\neq k}\frac{w_{k'}}{\alpha(v,w,k',\Theta^t)}]}\} \qquad (7)$$