

Modify By Confidence Level

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1 The BTL Model —ModifyByConfidenceLevel

Suppose g is a K -dim vector, with one and only one component to be equal to 1, for each pair, $p(< w, v > | g_k = 1) = p^k(w \succ v) \times \prod_{k' \neq k} p^{k'}(w \succeq v)$. then the probability of generating a session observation d given the hidden aspect a is defined as:

$$p(d|g, \theta, V, U) = p(d|\Theta^t, g) = \prod_{w \in W^d, v \in L^d} \frac{w_k}{w_k + \theta^t v_k} \prod_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}} \quad (1)$$

First, let's use $\gamma(d, k, \Theta^t)$ to denote the conditional probability $p(g_k = 1 | d, \Theta^t)$ given parameters in the t -th round, when the current session specific favorite aspect is $g_k = 1$, defined as follows

$$\begin{aligned} \gamma(d, k, \Theta^t) &= \frac{p(d, g | \Theta^t)}{\sum_g p(d, g | \Theta^t)} = \frac{p(g | \Theta^t) p(d | \Theta^t, g)}{\sum_g p(g | \Theta^t) p(d | \Theta^t, g)} \\ &= \frac{u_k \prod_{w \in W^d, v \in L^d} \frac{w_k}{w_k + \theta^t v_k} \prod_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}}{\sum_{k=1}^K u_k \prod_{w \in W^d, v \in L^d} \frac{w_k}{w_k + \theta^t v_k} \prod_{k' \neq k} \frac{\theta^t w_{k'}}{v_{k'} + \theta^t w_{k'}}} \end{aligned} \quad (2)$$

Note that $\forall d, \sum_k \gamma(d, k, \Theta^t) = 1$.

1.1 E-step

$$\begin{aligned} p(g|d, \Theta^t) &\propto p(g|u, \Theta^t) p(d|g, \Theta^t) \\ p(g_k = 1 | d, \Theta^t) &\propto u_k^t \frac{w_k^t}{w_k^t + \theta^t l_k^t} \prod_{k' \neq k} \left[\frac{\theta^t w_{k'}^t}{l_{k'}^t + \theta^t w_{k'}^t} \right] \end{aligned} \quad (3)$$

1.2 S-step

then simply add an S-step after the E-step, the value of g for each session d is

$$k \sim u_k^t \frac{w_k^t}{w_k^t + \theta^t l_k^t} \prod_{k' \neq k} \left[\frac{\theta^t w_{k'}^t}{l_{k'}^t + \theta^t w_{k'}^t} \right] \quad (4)$$

In the E-step of t -th EM round, compute the expectation $Q(\Theta^t) = E_G \ln p(D, G|\Theta)$

$$\begin{aligned} E_G \ln p(D, G|\Theta) &= \sum_d \gamma(d, k, \Theta^t) \ln p(d, g|\Theta) \\ &= \sum_d \gamma(d, k, \Theta^t) \{ \ln u_k + \sum_{w \in W_d, v \in V_d} [\ln \frac{w_k}{w_k + \theta v_k} + \sum_{k' \neq k} \ln \frac{\theta w_{k'}}{v_{k'} + \theta w_{k'}}] \} \end{aligned}$$

1.3 M-step

1.3.1 For u

first maximize $Q(\Theta^t)$ with respect to U . For each $u \in U$, eliminating constant terms, we have:

$$\begin{aligned} \min -\sum_{u(d)=u} \sum_{k=1}^K \gamma(d, k, \Theta^t) \ln u_k \\ w.r.t. \sum_k u_k = 1 \end{aligned} \quad (5)$$

Solving the above Lagrange function Equ. 5, we get

$$u_k = \frac{\sum_{u(d)=u} \gamma(d, k, \Theta^t)}{\sum_{s=1}^K \sum_{u(d)=u} \gamma(d, s, \Theta^t)} \quad (6)$$

1.3.2 For v

for the :

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

we can derive a lower bound for the log-likelihood over the complete data, given the parameters learnt from previous round. hence, we obtain a minorization function of $\tilde{Q}(\Theta^t)$.

$$\begin{aligned} \tilde{Q}(\Theta^t) &= \sum_d \sum_k \gamma(d, k, \Theta^t) \sum_{w \in W_d, v \in L_d} \{ [\ln w_k + 1 - \ln(w_k^t + \theta^t v_k^t) - \frac{w_k + \theta v_k}{w_k^t + \theta^t v_k^t}] + \\ &\quad \sum_{k' \neq k} [\ln(\theta w_{k'}) + 1 - \ln(v_{k'}^t + \theta^t w_{k'}^t) - \frac{v_{k'} + \theta w_{k'}}{v_{k'}^t + \theta^t w_{k'}^t}] \} \end{aligned}$$

$\tilde{Q}(\Theta^t)$ can be separated for each item v . Considering only the k -th component v_k , $\tilde{Q}(v_k, \Theta^t)$ involves two terms, one of which is relevant to observations $d \in W(v)$ where v acts as skyline object, the other is relevant to observations $d \in L(v)$ where v acts as comparisons, $\tilde{Q}(v_k, \Theta^t) = \tilde{Q}^1(v_k, \Theta^t) + \tilde{Q}^2(v_k, \Theta^t)$. Removing all constants and irrelevant terms for v_k , we have the following minorizing function:

$$\begin{aligned}\tilde{Q}^1(v_k, \Theta^t) &= \Sigma_{d \in W(v)} |L_d| \ln v_k - v_k \Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \left[\frac{\gamma(d, k, \Theta^t)}{\alpha(v, v', k, \Theta^t)} + \Sigma_{k' \neq k} \frac{\theta^t \gamma(d, k', \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right] \\ \tilde{Q}^2(v_k, \Theta^t) &= -v_k \Sigma_{d \in L(v)} \Sigma_{v' \in W_d} \left[\frac{\theta^t \gamma(d, k, \Theta^t)}{\alpha(v', v, k, \Theta^t)} + \Sigma_{k' \neq k} \frac{\gamma(d, k', \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]\end{aligned}$$

where:

$|L_d|$ is the number of objects being dominated in d ,
 $\alpha(v, v', k, \Theta^t) = v_k^t + \theta^t v_k'^t$.

By setting the partial derivative of $\frac{\partial \tilde{Q}(v_k, \Theta^t)}{\partial v_k} = 0$, we have:

$$\begin{aligned}\frac{1}{v_k} &= \frac{\Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \left[\frac{\gamma(d, k, \Theta^t)}{\alpha(v, v', k, \Theta^t)} + \Sigma_{k' \neq k} \frac{\theta^t \gamma(d, k', \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]}{\Sigma_{d \in W(v)} |L_d|} \\ &\quad + \frac{\Sigma_{d \in L(v)} \Sigma_{v' \in W_d} \left[\frac{\theta^t \gamma(d, k, \Theta^t)}{\alpha(v', v, k, \Theta^t)} + \Sigma_{k' \neq k} \frac{\gamma(d, k', \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]}{\Sigma_{d \in W(v)} |L_d|}\end{aligned}$$

More concrete:
if $\gamma(d, k, \Theta^t)$'s k equals to v_k 's k :

$$\frac{1}{v_k} = \frac{\Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \left[\frac{\gamma(d, k, \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]}{\Sigma_{d \in W(v)} |L_d|} + \frac{\Sigma_{d \in L(v)} \Sigma_{v' \in W_d} \left[\frac{\theta^t \gamma(d, k, \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]}{\Sigma_{d \in W(v)} |L_d|}$$

else:

$$\frac{1}{v_k} = \frac{\Sigma_{d \in W(v)} \Sigma_{v' \in L_d} \left[\frac{\theta^t \gamma(d, k', \Theta^t)}{\alpha(v', v, k, \Theta^t)} \right]_{k'=k}}{\Sigma_{d \in W(v)} |L_d|} + \frac{\Sigma_{d \in L(v)} \Sigma_{v' \in W_d} \left[\frac{\gamma(d, k', \Theta^t)}{\alpha(v, v', k, \Theta^t)} \right]_{k'=k}}{\Sigma_{d \in W(v)} |L_d|}$$

1.3.3 For θ

Fix u, v , update θ by:

Fix $v \in V$ and $u \in U$, rearranging *Equ.9*, we have the solution for $\frac{\partial \tilde{Q}(\Theta^t)}{\partial \theta} = 0$ as:

$$\theta = \frac{(K-1) \Sigma_d |W_d| |L_d|}{\Sigma_d \Sigma_k \gamma(d, k, \Theta^t) \Sigma_{w, v} \left[\frac{v_k}{\alpha(w, v, k, \Theta^t)} + \Sigma_{k' \neq k} \frac{w_{k'}}{\alpha(w, v, k', \Theta^t)} \right]} \quad (7)$$

Note that :

in the $\Sigma_d \Sigma_k \gamma(d, k, \Theta^t)$ only summing these d which $\gamma(d, k, \Theta^t)$'s k equals to d 's k .