The BTL Model

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1 The Standard BTL Model –

Base on the standard BTL model:

$$
\begin{aligned}
p(i \succ j|u) &= \frac{ui}{ui+uj} \\
p(i \prec j|u) &= \frac{uj}{uj+ui}
\end{aligned}
$$

Suppose that there are K underlying aspects. for each pair, $p(<w,v>) = p^k(w \succ v)$. then the probability of generating a session observation d is defined as:

$$
p(<w,v>|V,U) = \frac{\sum_{k=1}^{K} u_k w_k}{\sum_{k=1}^{K} u_k w_k + \sum_{k=1}^{K} u_k v_k} \tag{1}
$$

The likelihood function can be written as follows. The model parameters are denoted as $\Theta = \{v \in V, u \in U)\}$

$$
L(\Theta) = p(D|\Theta) = \Pi_{d \in D} \Pi_{w \in W^d, v \in L^d} \frac{\sum_{k=1}^{K} u_k w_k}{\sum_{k=1}^{K} u_k w_k + \sum_{k=1}^{K} u_k v_k} \tag{2}
$$

Thus, the log likelihood is:

$$
\begin{aligned}
l(\Theta) &= log\ L(\Theta) & (3) \\
&= log\ \Pi_{d \in D} \Pi_{w \in W^d, v \in L^d} \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k} & (4) \\
&= \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} log\ \frac{\sum_k u_k w_k}{\sum_k u_k w_k + \sum_k u_k v_k} \\
&= \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} [log \sum_k u_k w_k - log(\sum_k u_k w_k + \sum_k u_k v_k)]
\end{aligned}
$$

we want to choose $\{U\ and\ V\}$ to maximize $l(\Theta)$ by stochastic gradient ascent(SGA).

1.1 For u

$$u_j := u_j + \alpha \frac{\partial\, l(\Theta)}{\partial\, u_j}$$

$$\frac{\partial\, l(\Theta)}{\partial\, u_j} = \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} \left(\frac{w_j}{\sum_k u_k w_k} - \frac{w_j + v_j}{\sum_k u_k w_k + \sum_k u_k v_k} \right) \tag{5}$$

1.2 For v

$$v_j := v_j + \alpha \frac{\partial\, l(\Theta)}{\partial\, v_j}$$

as for winner:

$$\frac{\partial\, l(\Theta)}{\partial\, w_j} = \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} \left(\frac{u_j}{\sum_k u_k w_k} - \frac{u_j}{\sum_k u_k w_k + \sum_k u_k v_k} \right) \tag{6}$$

as for loser:

$$\frac{\partial\, l(\Theta)}{\partial\, v_j} = \Sigma_{d \in D} \Sigma_{w \in W^d, v \in L^d} \left(- \frac{u_j}{\sum_k u_k w_k + \sum_k u_k v_k} \right) \tag{7}$$