STA 602 - Intro to Bayesian Statistics

Lecture 15

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Two-sample comparison

- In many inference problems we are interested in comparing multiple samples of data to identify difference among them.
- The most typical problem is the two-sample problem that compares two-groups of observations
  - E.g., patients vs healthy, treatment vs control, etc.
- The most classical version of the two-sample problem focuses on comparing the mean of some measurement between the groups.
- The modern version of this problem is generalized to identifying a variety of differences in the underlying distributions (e.g., mean, variance, tail, local, ...)
- We will look at the Bayesian approach to comparing the mean.

The two-sample problem

- Suppose there are two samples of data $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_m)$ each can be considered i.i.d. from their respective sampling distribution

$$X_i \overset{\text{iid}}{\sim} F_1 \quad \text{and} \quad Y_j \overset{\text{iid}}{\sim} F_2.$$

- In the most simple version, we assume that F_1 and F_2 are both Gaussian with equal variance

$$F_1 = \mathrm{N}(\theta_1, \sigma^2) \quad \text{and} \quad F_2 = \mathrm{N}(\theta_2, \sigma^2).$$

- A generalization allows the variance to be different for the two groups σ_1^2 and σ_2^2.
- The interest is in the difference between the two means $\theta_1 - \theta_2$. For example, one might be interested in testing the null hypothesis

$$H_0 : \theta_1 - \theta_2 = 0 \quad \text{vs} \quad H_1 : \theta_1 - \theta_2 \neq 0.$$

The two-sample t-test

- The classical test for testing H_0 is the *t-test*.

$$t_{pool} = \frac{\bar{x} - \bar{y}}{s_{pool}\sqrt{1/n + 1/m}}$$

  where s^2_{pool} is the sample variance estimate based on the pooled sample combining the two groups of observations:

$$s^2_{pool} = \frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2} = \frac{\sum_i (x_i - \bar{x})^2 + \sum_j (y_j - \bar{y})^2}{n+m-2}.$$

- The pooled sample is meaningful only due to our assumption that the variance is equal for the two groups.
- t_{pool} has a t-distribution with $n+m-2$ degrees of freedom under H_0.
- Equivalent, a test for H_0 can be achieved by estimating $\theta_1 - \theta_2$ using $\bar{x} - \bar{y}$ and construct a (frequentist) confidence interval:

$$\left[\bar{x} - \bar{y} - t_{1-\alpha/2} \cdot s_{pool}\sqrt{1/n + 1/m}, \bar{x} - \bar{y} + t_{1-\alpha/2} \cdot s_{pool}\sqrt{1/n + 1/m}\right].$$

- A p-value can be computed under H_0.

Two-sample t-test with unequal variances

- If one suspects that the variance is not equal, this test cannot be used. A modified version when the variance is unequal is called Welch's t-test
$$t_{welch} = \frac{\bar{x} - \bar{y}}{\sqrt{s_x^2/n + s_y^2/m}},$$
which has approximately (not exactly) a t-distribution.

The Bayesian approach to two-sample comparison

- Let's now try to quantify our uncertainty about the underlying parameter of interest $\theta_1 - \theta_2$ using probability distribution.
- We could place priors on θ_1 and θ_2 and find the induced posterior on $\theta_1 - \theta_2$.
- What prior would be appropriate?

Prior specification

- How about a prior with independence

$$p(\theta_1, \theta_2) = p(\theta_1)p(\theta_2).$$

- Such a prior is usually unreasonable in two-sample problems because the fact that we are comparing the two samples in the first place is because we *a priori* conjectured that the two samples might be similar to each other.
- For example, if we are comparing the survival rate of cancer patients with or without a treatment at a hospital, or the SAT scores of two groups of students from two classes in the same high school.
- In other words, an appropriate prior usually should induce a positive correlation between θ_1 and θ_2.
- But incorporating prior belief about such dependence appears hard.

A helpful reparameterization

- Consider a reparameterization of the model

$$\theta_1 = \mu + \delta \quad \text{and} \quad \theta_2 = \mu - \delta.$$

- We replaced two parameters (θ_1, θ_2) with two new parameters (μ, δ).
- The sampling model is completely equivalent!
- But specifying a prior on (μ, δ) is easier now:
  - μ represents the overall average level.
  - δ represents difference between the two groups.
- It is now much more reasonable to assume prior independence between (μ, δ), if we are comfortable assuming that the level of difference doesn't depend on the overall level.
- Question: What if we want to assume that the difference is proportional to the overall level?
  - In that case, we may want to reparametrize as $\theta_1 = \mu(1 + \delta)$ and $\theta_2 = \mu(1 - \delta)$.

Prior specification

- So we adopt the following prior

$$p(\mu, \delta, \sigma^2) = p(\mu)p(\delta)p(\sigma^2)$$

where

$$\mu \sim \mathrm{N}(\mu_0, \lambda_0^2),$$
$$\delta \sim \mathrm{N}(\delta_0, \tau_0^2),$$

and

$$\sigma^2 \sim \mathrm{IG}(\nu_0/2, \nu_0\sigma_0^2/2).$$

Bayesian inference on δ

- Now we can proceed as usual by first finding the posterior

$$p(\mu,\delta,\gamma|\mathbf{x},\mathbf{y}) \propto p(\mathbf{x},\mathbf{y}|\mu,\delta,\gamma)p(\mu,\delta,\gamma)$$

  where we let $\gamma = 1/\sigma^2$ for notational simplicity.
- The likelihood

$$\begin{aligned} p(\mathbf{x},\mathbf{y}|\mu,\delta,\gamma) &\propto p(\mathbf{x}|\mu,\delta,\gamma)p(\mathbf{y}|\mu,\delta,\gamma) \\ &\propto \gamma^{n/2}e^{-\frac{\gamma}{2}\sum_i(x_i-\mu-\delta)^2} \cdot \gamma^{m/2}e^{-\frac{\gamma}{2}\sum_j(y_j-\mu+\delta)^2}. \\ &\propto \gamma^{(n+m)/2}e^{-\frac{\gamma}{2}\left[\sum_i(x_i-\mu-\delta)^2+\sum_j(y_j-\mu+\delta)^2\right]}. \end{aligned}$$

The joint probability

- By Bayes theorem

$$
\begin{aligned}
& p(\mu, \delta, \gamma | \mathbf{x}, \mathbf{y}) \\
\propto & p(\mathbf{x}, \mathbf{y} | \mu, \delta, \gamma) p(\mu) p(\delta) p(\gamma) \\
\propto & \gamma^{\frac{n+m}{2}} e^{-\frac{\gamma}{2}\left[\sum_i (x_i-\mu-\delta)^2+\sum_j (y_j-\mu+\delta)^2\right]} \cdot e^{-\frac{(\mu-\mu_0)^2}{2\lambda_0^2}} \cdot e^{-\frac{(\delta-\delta_0)^2}{2\tau_0^2}} \cdot \gamma^{\frac{\nu_0}{2}-1} e^{-\frac{\gamma}{2}\cdot \nu_0 \sigma_0^2}
\end{aligned}
$$

The full conditional of σ^2

- The full conditional of $\gamma = 1/\sigma^2$ is

$$p(\gamma|\mu,\delta,\mathbf{x},\mathbf{y}) \propto \gamma^{\frac{\nu_0+n+m}{2}-1} e^{-\frac{\gamma}{2}\left[\nu_0\sigma_0^2+\sum_i(x_i-\mu-\delta)^2+\sum_j(y_j-\mu+\delta)^2\right]}.$$

That is,

$$\gamma\,|\,\mu,\delta,\mathbf{x},\mathbf{y} \sim \text{Gamma}\left(\frac{\nu_{n,m}}{2},\frac{\nu_{n,m}\sigma_{n,m}^2}{2}\right)$$

or

$$\sigma^2\,|\,\mu,\delta,\mathbf{x},\mathbf{y} \sim \text{IG}\left(\frac{\nu_{n,m}}{2},\frac{\nu_{n,m}\sigma_{n,m}^2}{2}\right).$$

where $\nu_{n,m} = \nu_0 + n + m$ and
$\nu_{n,m}\sigma_{n,m}^2 = \nu_0\sigma_0^2 + \sum_i(x_i-\mu-\delta)^2 + \sum_j(y_j-\mu+\delta)^2$.

The full conditional of μ

- The full conditional of μ is

$$p(\mu|\delta,\gamma,\mathbf{x},\mathbf{y}) \propto e^{-\frac{\gamma}{2}\left[\sum_i (x_i-\mu-\delta)^2+\sum_j (y_j-\mu+\delta)^2\right]} \cdot e^{-\frac{(\mu-\mu_0)^2}{2\lambda_0^2}}$$
$$\propto e^{-\frac{\gamma}{2}\left[\sum_i (\tilde{x}_i-\mu)^2+\sum_j (\tilde{y}_j-\mu)^2\right]} \cdot e^{-\frac{(\mu-\mu_0)^2}{2\lambda_0^2}}$$

  where $\tilde{x}_i = x_i - \delta$ and $\tilde{y}_j = y_j + \delta$.
- Now from our earlier results for a single Gaussian sample, we know the full conditional is given by

$$\mu \,|\, \delta,\gamma,\mathbf{x},\mathbf{y} \sim \mathrm{N}(\mu_{n,m},\lambda_{n,m}^2)$$

  where

$$\mu_{n,m} = \lambda_{n,m}^2 \left(\frac{\sum_i \tilde{x}_i + \sum_j \tilde{y}_j}{n+m} \cdot \frac{n+m}{\sigma^2} + \frac{\mu_0}{\lambda_0^2} \right)$$
$$\frac{1}{\lambda_{n,m}^2} = (n+m)\gamma + \frac{1}{\lambda_0^2} = \frac{n+m}{\sigma^2} + \frac{1}{\lambda_0^2}.$$

The full conditional of δ

- Finally, the full conditional of δ is

$$p(\delta|\mu,\gamma,\mathbf{x},by) \propto e^{-\frac{\gamma}{2}\left[\sum_i (x_i-\mu-\delta)^2+\sum_j (y_j-\mu+\delta)^2\right]} \cdot e^{-\frac{(\delta-\delta_0)^2}{2\tau_0^2}}$$
$$\propto e^{-\frac{\gamma}{2}\left[\sum_i (\hat{x}_i-\delta)^2+\sum_j (\hat{y}_j-\delta)^2\right]} \cdot e^{-\frac{(\delta-\delta_0)^2}{2\tau_0^2}}$$

  where

$$\hat{x}_i = x_i - \mu \quad \text{and} \quad \hat{y}_j = \mu - y_j.$$

- Thus we can again draw from our earlier results for a single Gaussian sample,

$$\delta \,|\, \mu,\gamma,\mathbf{x},\mathbf{y} \sim \mathrm{N}(\delta_{n,m}, \tau_{n,m}^2)$$

  where

$$\delta_{n,m} = \tau_{n,m}^2 \left(\frac{\sum_i \hat{x}_i + \sum_j \hat{y}_j}{n+m} \cdot \frac{n+m}{\sigma^2} + \frac{\delta_0}{\tau_0^2} \right)$$
$$\frac{1}{\tau_{n,m}^2} = (n+m)\gamma + \frac{1}{\tau_0^2} = \frac{n+m}{\sigma^2} + \frac{1}{\tau_0^2}.$$

Gibbs sampling

- Initialize $(\mu^{(0)}, \delta^{(0)}, \sigma^{2\,(0)})$
- For $t = 1, 2, \ldots$
  - Update μ:
    - Compute $\lambda_{n,m}^{2\,(t)}$ and $\mu_{n,m}^{(t)}$.
    - Draw
$$\mu^{(t)} \sim \mathrm{N}(\mu_{n,m}^{(t)}, \lambda_{n,m}^{2\,(t)}).$$
  - Update δ
    - Compute $\tau_{n,m}^{2\,(t)}$ and $\delta_{n,m}^{(t)}$.
    - Draw
$$\delta^{(t)} \sim \mathrm{N}(\delta_{n,m}^{(t)}, \tau_{n,m}^{2\,(t)}).$$
  - Update σ^2
    - Compute $\nu_{n,m}\sigma_{n,m}^{2\,(t)}$.
    - Draw
$$\sigma^2 \sim \mathrm{IG}(\nu_{n,m}/2, \nu_{n,m}\sigma_{n,m}^{2\,(t)}/2).$$

Bayesian hypothesis testing

- It is tempting but *wrong* to “reject” H_0 at level α, say, if $\mathrm{P}(\delta > 0|\mathbf{x},\mathbf{y}) > 1-\alpha$.
- A test like this will not provide nearly comparable inference under frequentest criteria (e.g., Type I error) compared to a frequentest test, e.g., a one-sided t-test, that rejects at level α.
- In one extreme, notice that $P(\delta = 0|\mathbf{x},\mathbf{y}) = 0$ always since δ is continuous!

Predictive inference

- Sometimes people are interested in predictive quantities such as

$$
\begin{aligned}
&\mathrm{P}(X_{n+1} - Y_{m+1} > 0 \,|\, \mathbf{x}, \mathbf{y}) \\
&= \int \mathrm{P}(X_{n+1} - Y_{m+1} > 0 | \mu, \delta, \sigma^2) p(\mu, \delta, \sigma^2 \,|\, \mathbf{x}, \mathbf{y}) d\mu d\delta d\sigma^2,
\end{aligned}
$$

  which can also be evaluated through MCMC with a Gibbs sampler.

Example: Air pollutant measurements

- Suppose we took 7 measurements in the morning and 9 measurements in the afternoon.

$$\mathbf{x} = (104, 105, 103, 102, 105, 107, 106)$$
$$\mathbf{y} = (104, 103, 106, 105, 102, 102, 108, 105, 104)$$

  and we are interested in the change in the pollution level from morning to afternoon.

- Suppose we are using the same device, and so σ^2 is assumed to be the same in the morning and afternoon.

Prior specification

- Based on historical data, *a priori* we think the average pollution level

$$\mu \sim \mathrm{N}(100, 25)$$

That is, $\mu_0 = 100$ and $\lambda_0 = 5$.

- We think that the difference between morning and afternoon is typically close to 0, with standard deviation about 2,

$$\delta \sim \mathrm{N}(0, 4)$$

That is, $\delta_0 = 0$ and $\tau_0 = 2$.

- We again adopt a weak prior on σ^2

$$\sigma^2 \sim \mathrm{IG}(\nu_0/2, \nu_0\sigma_0^2/2)$$

where $\nu_0 = 1$ and $\sigma_0^2 = 4$.

Example: Air pollutant measurements

```
x <- c(104,105,103,102,105,107,106) # the data
y <- c(104,103,106,105,102,102,108,105,104)
n <- length(x) # sample size
m <- length(y)

# Prior specification
mu.0 <- 100; lambda2.0 <- 25;
delta.0 <- 0; tau2.0 <- 4;
nu.0 <- 1; sigma2.0 <- 4

# Initialization
niter <- 10000
nburnin <- 1000

xbar <- mean(x); ybar <- mean(y)
sx2 <- var(x); sy2 <- var(y)
s2.pool <- ((n-1)*sx2 + (m-1)*sy2)/(n+m-2)

mu.curr <- (xbar+ybar)/2
delta.curr <- (xbar-ybar)/2
sigma2.curr <- s2.pool

THETA <- matrix(NA,nrow=niter,ncol=3,dimnames=list(1:niter,c("mu","delta","sigma2")))
```

Start Gibbs sampling

```
for (t in 1:niter) {

  ## Update mu
  x.tilde <- x - delta.curr
  y.tilde <- y + delta.curr
  lambda2.n.m <- 1/((n+m)/sigma2.curr+1/lambda2.0)
  mu.n.m <- lambda2.n.m*(mean(c(x.tilde,y.tilde))*(n+m)/sigma2.curr + mu.0/lambda2.0)
  mu.curr <- rnorm(1,mean=mu.n.m,sd=sqrt(lambda2.n.m))

  ## Update delta
  x.hat <- x - mu.curr
  y.hat <- mu.curr - y
  tau2.n.m <- 1/((n+m)/sigma2.curr+1/tau2.0)
  delta.n.m <- tau2.n.m*(mean(c(x.hat,y.hat))*(n+m)/sigma2.curr + delta.0/tau2.0)
  delta.curr <- rnorm(1,mean=delta.n.m,sd=sqrt(tau2.n.m))

  ## Update sigma2
  sigma2.curr <-
    1/rgamma(1, shape=(nu.0+n+m)/2,
               rate=1/2*(nu.0*sigma2.0+sum((x-mu.curr-delta.curr)^2)+
                                        sum((y-mu.curr+delta.curr)^2)))

  ## Save the current iteration
  THETA[t,] <- c(mu.curr,delta.curr,sigma2.curr)
}
```

MCMC diagnostics

```
library(coda)
THETA.coda <- mcmc(THETA[-(1:nburnin),], start = 1+nburnin) # no burn-in steps
options(digits=3)
summary(THETA.coda)
```

```
##
## Iterations = 1001:10000
## Thinning interval = 1
## Number of chains = 1
## Sample size per chain = 9000
##
## 1. Empirical mean and standard deviation for each variable,
##    plus standard error of the mean:
##
##           Mean    SD Naive SE Time-series SE
## mu     104.413 0.495  0.00522        0.00522
## delta    0.104 0.486  0.00512        0.00512
## sigma2   3.963 1.667  0.01758        0.01937
##
## 2. Quantiles for each variable:
##
##           2.5%     25%     50%     75%  97.5%
## mu     103.447 104.091 104.401 104.731 105.40
## delta   -0.854  -0.208   0.105   0.412   1.07
## sigma2   1.865   2.856   3.606   4.643   8.25
```

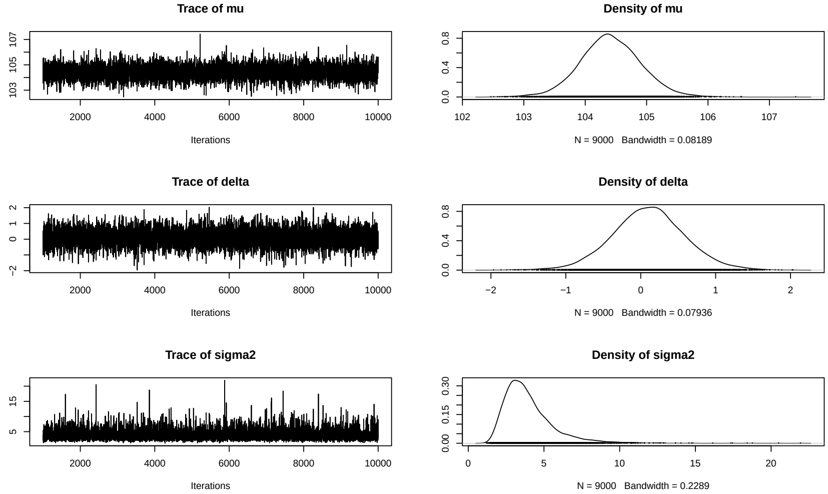
Autocorrelation and ESS

```
effectiveSize(THETA.coda)
```

```
##     mu  delta sigma2
##   9000   9000   7411
```

Trace plots

```
plot(THETA.coda)
```



Autocorrelation plots

```
autocorr.plot(THETA.coda,lag.max=100)
```