STA 602. HW01

Yicheng Shen

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Q1.

From Bayes Theorem, we know that

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

In this case, we have that

$$\begin{aligned} P(Knew|Correct) &= \frac{P(Correct|Knew)P(Knew)}{P(Correct)} \\ &= \frac{1 \times p}{1 \times p + \frac{1}{m}(1-p)} \\ &= \frac{mp}{mp + 1 - p} \end{aligned}$$

Q2.

Let X and Y be the time of from 12pm to arrival of the the man and woman, respectively.

So, from the question we know that X and Y are both random variable with $X \sim Unif(0, 60)$ and $Y \sim Unif(0, 60)$.

Since their marginal pdf are both 1/60 and they are independent events, their joint pdf is

$$f(x, y) = \frac{1}{3600}$$

So we first calculate the man waits for more than 10 mins:

$$P(X < Y - 10) = \int_{10}^{60} \int_{0}^{y-10} \frac{1}{3600} \, dxdy = \frac{25}{72}$$

Similarly, we could get that $P(Y < X - 10) = \frac{25}{72}$ as well.

Therefore, $P(|X - Y| > 10) = \frac{25}{36}$.

Q3.

Since we know that $Z \sim N(0, 1)$, we can write out the pdf:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{-z^2}{2}}$$

Now by the definition of expected value (or first norm)

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xf(x)\,dx \\
&= \int_{x}^{\infty} xf(x)\,dx + \int_{-\infty}^{x} xf(x)\,dx \\
&= \int_{x}^{\infty} zf(z)\,dz + 0 \\
&= \int_{x}^{\infty} z\frac{1}{\sqrt{2\pi}}e^{-\frac{-z^2}{2}}\,dz \\
&= -\frac{1}{\sqrt{2\pi}}e^{-\frac{-z^2}{2}}|_{x}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{-x^2}{2}}
\end{aligned}$$

Q4.

Since X follows the Binomial(n, p), we can write out its pmf as

$$\binom{n}{x}p^x(1-p)^{n-x}$$

Now we add the condition that U = p

$$\begin{aligned}
f_X(x) &= \binom{n}{x}p^x(1-p)^{n-x} \\
&= \int_0^1 \binom{n}{x}p^x(1-p)^{n-x}\,dp \\
&= \binom{n}{x}\int_0^1 p^x(1-p)^{n-x}\,dp \\
&= \binom{n}{x}\frac{x!(n-x)!}{(n+1)!}
\end{aligned}$$

Q5.

(a) We can use the moment generating functions to prove

First of all, since X and Y are both Poisson r.v., we know that

$$M_X(t) = e^{\lambda_1(e^t-1)},\ M_Y(t) = e^{\lambda_2(e^t-1)}$$

To show the sum of their distribution:

$$\begin{aligned}
M_{X+Y}(t) &= E[e^{t(X+Y)}] \\
&= E[e^{t(X)}]E[e^{t(Y)}] \\
&= M_X(t)M_Y(t) \\
&= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} \\
&= e^{(\lambda_1+\lambda_2)(e^t-1)}
\end{aligned}$$

So we have shown that their sum is still a Poisson distribution with parameter $\lambda = \lambda_1 + \lambda_2$.

(b) By Bayes Theorem used below, we would found that P(X|X+Y=n) follows a Binomial distribution.

$$\begin{aligned} P(X = x|X + Y = n) &= \frac{P(X + Y = n|X = x)P(X = x)}{P(X + Y = n)} \\ &= \frac{P(Y = n - x)P(X = x)}{\frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}} \\ &= \frac{\frac{(\lambda_2)^{n-x}e^{-(\lambda_2)}}{(n-x)!}\frac{(\lambda_1)^x e^{-(\lambda_1)}}{x!}}{\frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}} \\ &= \frac{n!}{(n-x)!x!}(\lambda_2)^{n-x}(\lambda_1)^x/(\lambda_1+\lambda_2)^n \\ &= \frac{n!}{(n-x)!x!}(\frac{\lambda_2}{\lambda_1+\lambda_2})^{n-x}(\frac{\lambda_1}{\lambda_1+\lambda_2})^x \\ &= \frac{1}{n+1} \end{aligned}$$

Q6.

Since Y follows a uniform distribution with Unif(0,X)

$$E(Y|X) = \frac{X}{2}; Var(Y|X) = \frac{X^2}{12}$$

Then we want to compute the unconditional ones, by Law of total expectation:

$$E(Y) = E(E(Y|X)) = E(\frac{X}{2}) = \frac{1}{4}$$

For Variance, we can use the Law of total variance:

$$\begin{aligned} Var(Y) &= E(Var(Y|X)) + Var(E(Y|X)) \\ &= E(\frac{X^2}{12}) + Var(\frac{X}{2}) \\ &= \frac{1}{12}E(X^2) + \frac{1}{4}Var(X) \\ &= \frac{1}{12}\frac{1}{3} + \frac{1}{4}\frac{1}{12} \\ &= \frac{1}{12}\frac{1}{3} + \frac{1}{4}\frac{1}{12} \\ &= \frac{7}{144} \end{aligned}$$

Q7.

(a) The table of joint distribution is shown below.

Table	X = 1	X = 0
Y = 1	(0.5)(0.4) = 0.2	(0.5)(0.6) = 0.3
Y = 0	(0.5)(0.6) = 0.3	(0.5)(0.4) = 0.2

(b) From the table we have

$$E[Y] = E[Y|X = 1] + E[Y|X = 0] = 0.2 + 0.3 = 0.5$$

(c) It can be noticed that Var[Y] is the larger one. Intuitively, knowing X as the condition will give us more information and thus reduce the amount of variability.

$$\begin{aligned}
Var[Y|X = 0] &= E[Y^2|X = 0] - (E[Y|X = 0])^2 \\
&= 1^2 * 0.6 - (1 * 0.6)^2 = 0.24 \\
Var[Y|X = 1] &= E[Y^2|X = 1] - (E[Y|X = 1])^2 \\
&= 1^2 * 0.4 - (1 * 0.4)^2 = 0.24 \\
Var[Y] &= E[Y^2] - (E[Y])^2 \\
&= 0.5 - (1 * 0.5)^2 = 0.25
\end{aligned}$$

(d) Using Bayes Theorem:

$$P(X = 0|Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = 0.3/(0.3 + 0.2) = 0.6$$