# Lower bounds on error probability of quantum channel discrimination by the Bures angle and the trace distance

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# Quantum channel discrimnation problem

**Definition (Quantum channel discrimnation problem)**

For fixed finite set of quantum channels $\{\Phi_1, \Phi_2, \ldots, \Phi_k\}$ and a probability distribution $(p_i)_{i=1,2,\ldots,k}$,

Input: An oracle $\mathcal{O} = \Phi_i$ for $i \in \{1, 2, \ldots, k\}$ drawn w.p. $p_i$

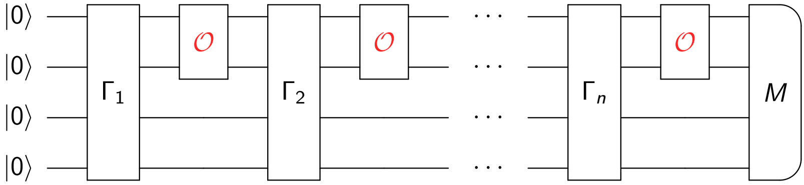
Output: $i \in \{1, 2, \ldots, k\}$ satisfying $\mathcal{O} = \Phi_i$

- A fundamental problem in quantum information science

# Algorithm for quantum channel discrimnation problem

General algorithm calling the oracle $n$ times.

- $\mathcal{O} \in \{\Phi_1, \ldots, \Phi_k\}$
- $\Gamma_1, \ldots, \Gamma_n$: Quantum channels
- $M$: Measruement



# Discrimination of quantum states

**Definition (Quantum state discrimnation problem)**

For fixed finite set of quantum states $\{\rho_1, \rho_2, \ldots, \rho_k\}$
and a probability distribution $(p_i)_{i=1,2,\ldots,k}$,

Input: A quantum state $\rho = \rho_i$ for $i \in \{1, 2, \ldots, k\}$ drawn w.p. $p_i$

Output: $i \in \{1, 2, \ldots, k\}$ satisfying $\rho = \rho_i$

**Theorem (Holevo–Helstrom theorem [Helstrom 1967], [Holevo 1972])**

*The largest success probability of discrimination of two quantum states is*

$$\frac{1 + \|p_1\rho_1 - p_2\rho_2\|_1}{2}.$$

For more than two quantum channels, the largest succes probability is a solution of semidefinite programming (SDP).

# Discrimination of two quantum channels

### Theorem (Discrimination of two quantum channels)

*The largest success probability of discrimination of two quantum channels with a single query is*

$$p_{\text{succ}} = \frac{1 + \|p_1\Phi_1 - p_2\Phi_2\|_\diamond}{2}$$

*where*

$$\|\Gamma_A\|_\diamond := \max_{\rho_{AR}} \|(\Gamma_A \otimes \mathrm{Id}_R)(\rho_{AR})\|_1.$$

### Corollary (Discrimination by non-adaptive algorithms)

*The largest success probability of discrimination of two quantum channels with $n$ non-adaptive (parallel) queries is*

$$p_{\text{succ}} = \frac{1 + \|p_1\Phi_1^{\otimes n} - p_2\Phi_2^{\otimes n}\|_\diamond}{2}.$$

# Previous qualitative result

- There are two quantum channels that cannot be distinguished exactly by any non-adaptive algorithm, but can be distinguished exactly by an addaptive algorithm with two queries [Harrow, Hassidim, Leung, Watrous, 2010].

- For any two quantum channels, either of the followings is true [Yu and Zhou 2021]
  - The quantum channels can be distinguished exactly by finite queries.
  - The error probability of quantum channel discrimination decays exponentially in the number of queries.

# Previous result and our result

| # of channels | 2 | $\geq 2$ |
|---|---|---|
| QUD | [Kawachi, Kawano, Le Gall, Tamaki 2019] | Grover search $\ell = 1$ [Zalka 1999] |
| QCD | Port-based teleportation<br>[Pirandola, Laurenza, Lupo, Pereira, 2019]<br>[This work] | [Zhuang, Pirandola, 2020] |
| QCGD | | [This work] |

# Discrimination of two quantum channels

Quantum channel discrimination of $\Phi_A^{(1)}$ and $\Phi_A^{(2)}$.

$$\rho_1 = \Phi_A^{(1)} \Gamma_{AR}^{(n)} \Phi_A^{(1)} \Gamma_{AR}^{(n-1)} \cdots \Phi_A^{(1)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n|$$
$$\rho_2 = \Phi_A^{(2)} \Gamma_{AR}^{(n)} \Phi_A^{(2)} \Gamma_{AR}^{(n-1)} \cdots \Phi_A^{(2)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n|$$

From Holevo–Helstrom theorem,

$$\frac{1 - \frac{1}{2}\|\rho_1 - \rho_2\|_1}{2}.$$

# Hybrid argument

$$\rho^{(0)} = \Phi_A^{(1)}\Gamma_{AR}^{(n)}\Phi_A^{(1)}\Gamma_{AR}^{(n-1)}\cdots\Phi_A^{(1)}\Gamma_{AR}^{(2)}\Phi_A^{(1)}\Gamma_{AR}^{(1)}\left|0^n\right\rangle\left\langle 0^n\right|$$

$$\rho^{(1)} = \Phi_A^{(1)}\Gamma_{AR}^{(n)}\Phi_A^{(1)}\Gamma_{AR}^{(n-1)}\cdots\Phi_A^{(1)}\Gamma_{AR}^{(2)}\Phi_A^{(2)}\Gamma_{AR}^{(1)}\left|0^n\right\rangle\left\langle 0^n\right|$$

$$\rho^{(2)} = \Phi_A^{(1)}\Gamma_{AR}^{(n)}\Phi_A^{(1)}\Gamma_{AR}^{(n-1)}\cdots\Phi_A^{(2)}\Gamma_{AR}^{(2)}\Phi_A^{(2)}\Gamma_{AR}^{(1)}\left|0^n\right\rangle\left\langle 0^n\right|$$

$$\vdots$$

$$\rho^{(n)} = \Phi_A^{(2)}\Gamma_{AR}^{(n)}\Phi_A^{(2)}\Gamma_{AR}^{(n-1)}\cdots\Phi_A^{(2)}\Gamma_{AR}^{(2)}\Phi_A^{(2)}\Gamma_{AR}^{(1)}\left|0^n\right\rangle\left\langle 0^n\right|$$

$$\begin{aligned}
\|\rho_1 - \rho_2\|_1 &= \|\rho^{(0)} - \rho^{(n)}\|_1 \\
&= \|\rho^{(0)} - \rho^{(1)} + \rho^{(1)} - \cdots - \rho^{(n-1)} + \rho^{(n-1)} - \rho^{(n)}\|_1 \\
&\leq \sum_{i=0}^{n-1} \|\rho^{(i)} - \rho^{(i+1)}\|_1
\end{aligned}$$

# Upper bound for the single step

$$
\begin{aligned}
\|\rho^{(i)} - \rho^{(i+1)}\|_1 &= \Big\| \Phi_A^{(1)} \Gamma_{AR}^{(n)} \cdots \Phi_A^{(1)} \Gamma_{AR}^{(i+1)} \Phi_A^{(2)} \Gamma_{AR}^{(i)} \cdots \Phi_A^{(2)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n| \\
&- \Phi_A^{(1)} \Gamma_{AR}^{(n)} \cdots \Phi_A^{(1)} \Gamma_{AR}^{(i+2)} \Phi_A^{(2)} \Gamma_{AR}^{(i+1)} \cdots \Phi_A^{(2)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n| \Big\|_1 \\
&\leq \Big\| \Phi_A^{(1)} \Gamma_{AR}^{(i+1)} \Phi_A^{(2)} \Gamma_{AR}^{(i)} \cdots \Phi_A^{(2)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n| \\
&- \Phi_A^{(2)} \Gamma_{AR}^{(i+1)} \Phi_A^{(2)} \Gamma_{AR}^{(i)} \cdots \Phi_A^{(2)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n| \Big\|_1 \\
&\leq \max_{\rho_{AR}} \Big\| \Phi_A^{(1)}(\rho_{AR}) - \Phi_A^{(2)}(\rho_{AR}) \Big\|_1 \\
&= \Big\| \Phi_A^{(1)} - \Phi_A^{(2)} \Big\|_\diamond
\end{aligned}
$$

# A simple lower bound of the error probability

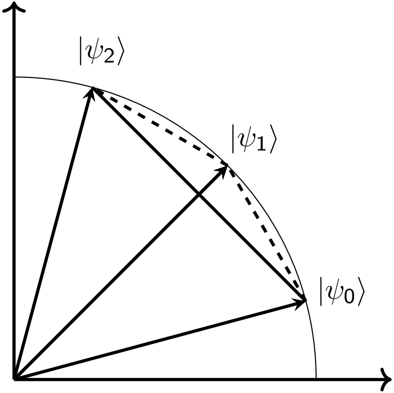
**Lemma (A simple lower bound)**

*The quantum channel discrimination of* $\Phi^{(1)}$, $\Phi^{(2)}$

$$p_{\mathrm{err}}(n) \geq \frac{1 - \frac{n}{2}\left\|\Phi_A^{(1)} - \Phi_A^{(2)}\right\|_{\diamond}}{2}$$

# Triangle inequalities are not tight

For pure states with two real state vectors $|\psi_0\rangle$ and $|\psi_2\rangle$, the trace distance is equal to the Euclidean distance up to a constant factor.




Obviously, $\| \, |\psi_0\rangle - |\psi_2\rangle \, \|_2 < \| \, |\psi_0\rangle - |\psi_1\rangle \, \|_2 + \| \, |\psi_1\rangle - |\psi_2\rangle \, \|_2$.

# Bures angle

- The fidelity: $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1$.
- The Bures angle: $A(\rho, \sigma) := \arccos F(\rho, \sigma)$.

## Lemma

$$\|p\rho - (1-p)\sigma\|_1 \leq \sqrt{1 - 4p(1-p)F(\rho, \sigma)^2}$$

*The equality is satisfied if $\rho$ and $\sigma$ are pure states.*

From the triangle ineqality of the Bures angle

$$A(\rho^{(0)}, \rho^{(n)}) \leq \sum_{i=0}^{n-1} A(\rho^{(i)}, \rho^{(i+1)})$$

# Upper bound for the single step

$$
\begin{aligned}
A(\rho^{(i)}, \rho^{(i+1)}) &= A\Big( \Phi_A^{(1)} \Gamma_{AR}^{(n)} \cdots \Phi_A^{(1)} \Gamma_{AR}^{(i+1)} \Phi_A^{(2)} \Gamma_{AR}^{(i)} \cdots \Phi_A^{(2)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n| , \\
&\qquad \Phi_A^{(1)} \Gamma_{AR}^{(n)} \cdots \Phi_A^{(1)} \Gamma_{AR}^{(i+2)} \Phi_A^{(2)} \Gamma_{AR}^{(i+1)} \cdots \Phi_A^{(2)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n| \Big) \\
&\le A\Big( \Phi_A^{(1)} \Gamma_{AR}^{(i+1)} \Phi_A^{(2)} \Gamma_{AR}^{(i)} \cdots \Phi_A^{(2)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n| , \\
&\qquad \Phi_A^{(2)} \Gamma_{AR}^{(i+1)} \cdots \Phi_A^{(2)} \Gamma_{AR}^{(1)} |0^n\rangle \langle 0^n| \Big) \\
&\le \max_{\rho_{AR}} A\Big( \Phi_A^{(1)}(\rho_{AR}), \Phi_A^{(2)}(\rho_{AR}) \Big) \\
&= \arccos \min_{\rho_{AR}} F\Big( \Phi_A^{(1)}(\rho_{AR}), \Phi_A^{(2)}(\rho_{AR}) \Big).
\end{aligned}
$$

The minimum of the fidelity can be evaluated by SDP.

# Our result using the Bures angle

## Theorem ([Ito and Mori 2021])

$$p_{\mathrm{err}}(n) \geq \frac{1}{2}\left(1 - \sqrt{1 - 4p_1 p_2 \cos^2(n\tau)}\right)$$

*where*

$$\tau := \arccos \min_{\rho_{AR}} F\left(\Phi_A^{(1)}(\rho_{AR}), \Phi_A^{(2)}(\rho_{AR})\right).$$

## Corollary (Discrimination of two amplitude damping channels)

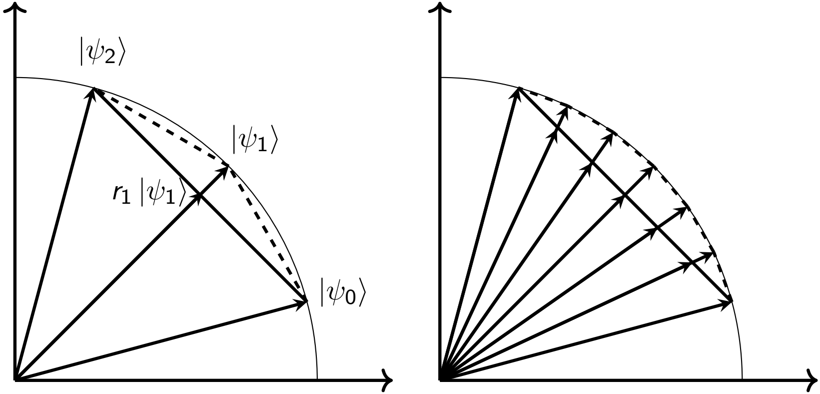
*For the discrimination of two amplitude damping channels with parameters $r_1$ and $r_2$,*

$$p_{\mathrm{err}}(n) \geq \frac{1}{2}\left(1 - \sqrt{1 - 4p_1 p_2 \cos^2(n\Delta)}\right)$$

*where* $\Delta := \arccos(\sqrt{r_1 r_2} + \sqrt{(1-r_1)(1-r_2)})$.
Especialy, when $p_1 = p_2 = 1/2$,

$$p_{\mathrm{err}}(n) \geq \frac{1}{2}\left(1 - \sin(n\Delta)\right).$$

# Improvement of the triangle inequality




Let $k \in \{0, 1, \ldots, n\}$, $\alpha \in (0, 1]$, $\beta \in (0, 1]$ satisfying $\alpha^k = \beta^{n-k}$.

The weights are chosen as
$1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^k = \beta^{n-k}, \beta^{n-k-1}, \ldots, \beta, 1.$

# Our result using the trace distance with the weights

### Theorem

*Let $k \in \{0, 1, \ldots, n\}$ be an integer. Let $\alpha$ and $\beta$ be non-negative real numbers that satisfy $p_1 \alpha^k = p_2 \beta^{n-k}$. Let $\tau_\diamond^1$ and $\tau_\diamond^2$ be*

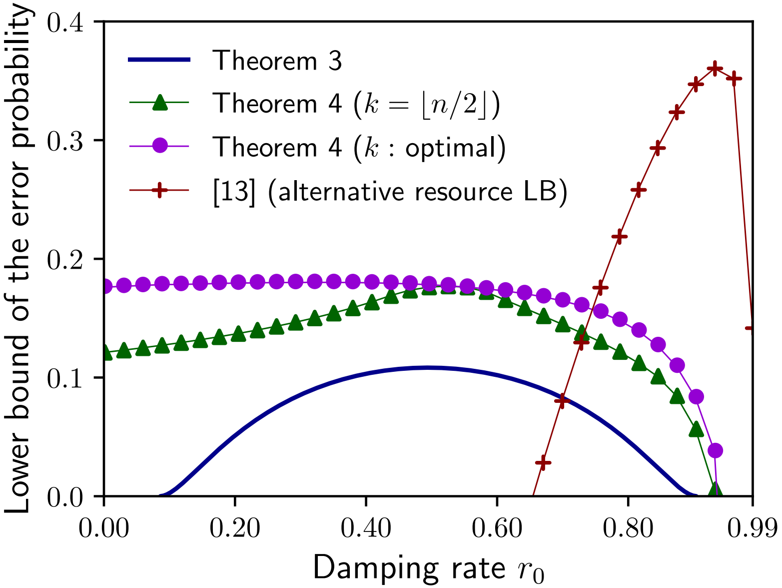
$$\tau_\diamond^1 := \max_{\rho_{AR}} \left\| \left( \Phi_A^{(1)} - \alpha \Phi_A^{(2)} \right) (\rho_{AR}) \right\|_1 ,$$
$$\tau_\diamond^2 := \max_{\rho_{AR}} \left\| \left( \beta \Phi_A^{(1)} - \Phi_A^{(2)} \right) (\rho_{AR}) \right\|_1 .$$

*The following then holds:*

$$p_{\mathrm{err}}(n) \geq \frac{1}{2} \left[ 1 - p_1 \left( \sum_{i=0}^{k-1} \alpha^i \right) \tau_\diamond^1 - p_2 \left( \sum_{i=0}^{n-k-1} \beta^i \right) \tau_\diamond^2 \right] ,$$

*for arbitrary $\alpha$, $\beta$ and, $k$.*

# Numerical results



# Quantum channel group discrimination

Quantum channel group discrimination:

Problem to find a group including the channel (group may intersect).

We also derive lower bounds of the error probability for the quantum channel group discrimination problem using the Bures angle and the trece distance with weight.

The lower bound using Bures angle proves the optimality of Grover's search (This was proven for the case with exactly one marked element by [Zalka 1999]).

The lower bound using the trace distance with weight is numerically better than previously known lower bounds for some cases.