Zhu algebras of superconformal vertex algebras

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Based on joint work with Ryo Sato (Aichi Institute of Technology)
Supported by Asahipen Hikari Foundation

Shenzhen–Nagoya Workshop on Quantum Science 2025

2025/09/26

1. Vertex algebras and Zhu algebras

Introduction and some new results on Zhu algebras of vertex algebras, partly based on

Ryo Sato (Aichi Institute of Technology) and S.Y.,
"Zhu algebras of superconformal vertex algebras",
arXiv:2509.13124, 33pp.

1. Vertex algebras and Zhu algebras [7 pages]
    1. 1.1. Vertex algebras
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2. Zhu algebras of superconformal vertex algebras
3. Zhu algebras of SUSY vertex algebras
4. Summary and open problems

A vertex algebra (VA) is an algebraic structure introduced by Borcherds[1] to formulate two-dimensional chiral conformal field theory.

(c.f. Talks of Dr. T. Iwane and Dr. Y. Nishinaka, Day 2.)

Definition (vertex algebras[2]) $a_{(j)}b \in V \rightsquigarrow a_{(j)} \in \mathrm{End}\, V$

A VA is a $\mathbb{C}$-linear space V with binary operations $\cdot_{(j)}\cdot$ on V ($j \in \mathbb{Z}$) and distinguished $|0\rangle \in V$ such that, for any $a, b \in V$, using the notation

$a(z) := \sum_{j\in\mathbb{Z}} z^{-j-1} a_{(j)} \in (\mathrm{End}\, V)[\![z^{\pm 1}]\!]$, (vertex operator of a)

(i) $a(z)b \in V(\!(z)\!)$: Laurent series of z, (quantum field)

(ii) $|0\rangle(z) = \mathrm{id}_V$, $a(z)|0\rangle = a + O(z)$, (vacuum)

(iii) $\partial \in \mathrm{End}\, V$, $\partial a := a_{(-2)}|0\rangle$ satisfies $[\partial, a(z)] = (\partial a)(z) = \partial_z a(z)$, where $[\cdot,\cdot]$ is the commutator of operators, (translation)

(iv) $\exists N_{a,b} \in \mathbb{Z}_{\geq 0}$ such that $(z-w)^{N_{a,b}}[a(z), b(w)] = 0$. (locality)

V being a $\mathbb{C}$-linear superspace, we have vertex superalgebras (VSA).

[1] R. Borchers, "Vertex algebras, Kac-Moody algebras, and the Monster", Proc. Nat. Acad. Sci., **83** (1986), no. 10, 3068–71.

[2] E. Frenkel, D. Ben-Zvi, "Vertex Algebras and Algebraic Curves", 2nd ed., AMS, 2004.

- V: VA, $V \to (\mathrm{End}\, V)[\![z^{\pm 1}]\!]$, $a \mapsto a(z) = \sum_{j\in\mathbb{Z}} z^{-j-1} a_{(j)}$,
  (i) quantum field (ii) vacuum (iii) translation (iv) locality.
- The locality axiom (iv) can be replaced by[3]

(iv-1) $a(z)b = e^{z\partial} b(-z)a$, (skew-symmetry)

(iv-2) $x^{-1}\delta(\frac{z-w}{x})a(z)b(w) - x^{-1}\delta(\frac{w-z}{-x})b(w)a(z) = w^{-1}\delta(\frac{z-x}{w})(a(x)b)(w)$,
$\delta(z) := \sum_{n\in\mathbb{Z}} z^n \in \mathbb{C}[\![z^{\pm 1}]\!]$: formal delta function, (Jacobi identity)

and by[4]

(iv-1') $a_{(j)}b = \sum_{k\geq 0} (-1)^{j+k+1} \frac{1}{k!}\partial^k (b_{(j+k)}a)$,
(skew-symmetry)

(iv-2') $(a_{(m)}b)_{(n)} = \sum_{k\geq 0}(-1)^k \binom{m}{k} (a_{(m-k)}b_{(n+k)} - (-1)^m b_{(m+n-k)}a_{(k)})$.
(Borcherds identity)

- Vertex algebras can be regarded as "Lie algebra objects" in a certain (pseudo-)tensor category.
  $\rightsquigarrow$ The theory of chiral algebras (c.f. talk of Dr. Nishinaka in 2023).

[3] I. B. Frenkel, Y.-Z. Huang, J. Lepowsky, "On axiomatic approaches to vertex operator algebras and modules", Memoirs AMS, 1993.
[4] Original definition by Borcherds (1986).

- V: VA, $V \to (\operatorname{End} V)[\![z^{\pm 1}]\!]$, $a \mapsto a(z) = \sum_{j \in \mathbb{Z}} z^{-j-1} a_{(j)}$.
- λ-bracket encodes (j)-operations: $[a_\lambda b] := \sum_{j \geq 0} \frac{1}{j!} \lambda^j a_{(j)} b \in V[\lambda]$.

Definition (conformal (= Virasoro) element)

$L \in V$ is conformal of central charge $c \in \mathbb{C}$ if

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,n},\ L_n := L_{(n+1)}$$
$$\iff [L_\lambda L] = (\partial + 2\lambda) L + \frac{c}{12}\lambda^3 \quad \text{(Virasoro relation)}.$$

Example (affine VA $V^k(\mathfrak{g})$)

$\widehat{\mathfrak{g}} := \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K$: the affine Lie algebra of a simple Lie algebra $\mathfrak{g}$.

$$[K, \widehat{\mathfrak{g}}] = 0, \quad [xt^m, yt^n] = [x,y] t^{m+n} + m\delta_{m+n,0}(x|y)K.$$

$V^k(\mathfrak{g}) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k$: vacuum module of level $k \in \mathbb{C}$.

It has a unique VA structure with $|0\rangle = 1 \otimes 1$ and vertex operators

$$xt^{-1}|0\rangle \mapsto \sum_{j \in \mathbb{Z}} z^{-j-1}(xt^j) \quad (x \in \mathfrak{g}).$$

If $k \neq -h^\vee$, Sugawara vector $L_{\mathrm{Sug}} := \frac{1}{2(k+h^\vee)} \sum_a (J_a t^{-1})(J^a t^{-1})|0\rangle$

$(\{J^a\}_a \subset \mathfrak{g}$: basis, $\{J_a\}_a$: dual basis w.r.t. Killing form on $\mathfrak{g})$

is conformal with $c = \frac{k \dim \mathfrak{g}}{k+h^\vee}$, and $V^k(\mathfrak{g})$ is a vertex operator algebra.

VAs are complicated objects. To study them, we use their invariants.

Definition/Theorem (Yongchang Zhu's C_2- and assoc. algebras[5])

- C_2-Poisson algebra $R(V)$ of any VA V.
  $R(V) := V/(V_{(-2)}V)$, $\qquad \overline{a} \cdot \overline{b} := \overline{a_{(-1)}b}$: commutative product,
  $V_{(-2)}V := \mathrm{Span}\{a_{(-2)}b \mid a, b \in V\}$, $\qquad \{\overline{a}, \overline{b}\} := \overline{a_{(0)}b}$: Poisson bracket.
- Zhu algebra $A(V)$ of a VOA $V = \bigoplus_\Delta V_\Delta$ (L_0-eigen decomposition).
  $A(V) := V/(V \underset{2}{*} V)$, $\qquad [a] * [b] := [a \underset{1}{*} b]$: associative product,
  $a \underset{n}{*} b := \sum_{j \geq 0} \binom{\Delta(a)}{j} a_{(j-n)} b$.

- $R(V)$ describes semi-classical-limit structure.
  (c.f. chiral quantization problem, S.Y.'s talk in 2023.)
- $A(V)$ for a VOA V describes the representation theory:
  Thm. (Zhu) simple $A(V)$-mods $\overset{1:1}{\longleftrightarrow}$ simple highest wt. V-mods.
- For a VOA V, $R(V) \overset{\exists}{\twoheadrightarrow} \mathrm{gr}^{L_0} A(V)$ of graded Poisson algebras.

[5] Y. Zhu, "Modular Invariance of Characters of Vertex Operator Algebras", J. Amer. Math. Soc., **9** (1996), no. 1, 237–302.

Example (affine VA)

$V^k(\mathfrak{g}) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k$, $xt^{-1}\,|0\rangle \mapsto \sum_{n\in\mathbb{Z}}(xt^n)z^{-n-1}\quad (x\in\mathfrak{g})$.

- C_2-Poisson algebra:
  $\underline{R\big(V^k(\mathfrak{g})\big) \cong \operatorname{Sym}\mathfrak{g}} = \mathbb{C}[\mathfrak{g}^*]$: Lie-Poisson algebra of $\mathfrak{g}$.
  $\overline{(x_1t^{-1})\cdots(x_rt^{-1})\,|0\rangle} \leftrightarrow x_1\cdots x_r\quad (x_i\in\mathfrak{g})$,
  $\{x_i, x_j\}_{\operatorname{Sym}\mathfrak{g}} := [x_i, x_j]_{\mathfrak{g}}$.
- Zhu algebra[6] for $k \neq -h^\vee$: $V^k(\mathfrak{g})$ is a VOA with L_{Sug}.
  $A\big(V^k(\mathfrak{g})\big) \cong U(\mathfrak{g})$: universal enveloping alg., $[x]\leftrightarrow x\quad (x\in\mathfrak{g})$.
  L_{Sug} gives $[L_{\mathrm{Sug}}] \in Z\big(A(V^k(\mathfrak{g}))\big)$.
- $R\big(V^k(\mathfrak{g})\big) \cong \operatorname{Sym}\mathfrak{g} \xrightarrow{\sim} \operatorname{gr}^{L_0} A\big(V^k(\mathfrak{g})\big) \cong \operatorname{gr}^{PBW} U(\mathfrak{g})$.

[6] I. B. Frenkel, Y. Zhu, "Vertex operator algebras associated to representations of affine and Virasoro algebras", Duke Math. J. (1992).

Example (W-algebras[7])

- $\mathfrak{g}$: simple Lie alg., $k \in \mathbb{C}$, $f \in \mathfrak{g}$: nilpotent element.
  $\rightsquigarrow$ quantum Drinfeld-Sokolov (BRST) reduction $H^{\bullet}_{DS,f}(?)$ of $V^k(\mathfrak{g})$:
  $\rightsquigarrow \mathcal{W}^k(\mathfrak{g}, f) := H^0_{DS,f}(V^k(\mathfrak{g}))$.
  C_2 alg.[8]: $R(\mathcal{W}^k(\mathfrak{g}, f)) \cong \mathbb{C}[S_f]$, $S_f := f + \mathfrak{g}^e$: Slodowy slice.
  Zhu alg.[9] for $k \neq -h^\vee$: $A(\mathcal{W}^k(\mathfrak{g}, f)) \cong U(\mathfrak{g}, f)$: finite W-algebra.
  $R(\mathcal{W}^k(\mathfrak{g}, f)) \cong \mathbb{C}[S_f] \xrightarrow{\sim} \operatorname{gr}^{L_0} A(\mathcal{W}^k(\mathfrak{g}, f)) \cong \operatorname{gr}^{L_0} U(\mathfrak{g}, f)$.
- Sub-example: $\mathcal{W}^k(\mathfrak{sl}(2), f = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]) \cong Vir_{c=1-\frac{6(k+1)^2}{k+2}}$: Virasoro VA,
  $\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$, $[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}$.
  $M_c := U(\mathcal{L}) \otimes_{U(\mathcal{L}_{\geq 0} + \mathbb{C}C)} \mathbb{C}_c$: Verma module, $c \in \mathbb{C}$.
  $Vir_c := M_c / U(\mathcal{L}) L_{-1}(1 \otimes 1) \ni |0\rangle := 1 \otimes 1$, $L := L_{-2}|0\rangle$.
  C_2 and Zhu algebras: $R(Vir_c) \xrightarrow{\sim} A(Vir_c) \cong \mathbb{C}[x]$, $x := [L]$.

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7. B. Feigin, E. Frenkel, "Quantization of the Drinfeld-Sokolov reduction", Phys. Lett. B, **246** (1990), 75–81.
8. T. Arakawa, "Associated varieties of modules over Kac-Moody algebras and C_2-cofiniteness of W-algebras", IMRN (2015), 11605–666.
9. A. De Sole, V. G. Kac, "Finite vs affine W-algebras", Jpn. J. Math., **1** (2006), 137–261.

- The original $A(V)$ requires V to have Hamiltonian L_0. Determining $A(V)$ is difficult in general.
- Yi-Zhi Huang[10] introduced a generalization $\widetilde{A}_\gamma(V)$ applicable for any V.

  $\widetilde{A}_\gamma(V) := V/(V \overset{\gamma}{\underset{2}{\bullet}} V)$, $\quad [a] \overset{\gamma}{\bullet} [b] := [a \overset{\gamma}{\underset{1}{\bullet}} b]$: associative product,

  $a \overset{\gamma}{\underset{n}{\bullet}} b := \mathrm{res}_z \big[f_n(z;\gamma) a(z) b \big] dz, \;\; f_n(z;\gamma) := \gamma^n e^{\gamma z}/(e^{\gamma z}-1)^n.$
- For a VOA V, $(\widetilde{A}_{\gamma=1}(V), \overset{\gamma=1}{\bullet}) \cong (A(V), *)$.
- We have $\widetilde{A}_{\gamma=0}(V) \cong R(V)$, where $f_n(z;0) := z^{-n}$.
- Below we consider $(\widetilde{A}(V), \bullet) := (\widetilde{A}_{\gamma=1}(V), \overset{\gamma=1}{\bullet})$ for any VA V.

10 Y.-Z. Huang, "Differential equations, duality and modular invariance", Comm. Contemp. Math., **7** (2005), no. 5, 649–706.

- The definition of $\widetilde{A}(V)$ looks more complicated than $A(V)$:
  - $A(V) \coloneqq V/(V \underset{2}{*} V), \quad [a] * [b] \coloneqq [a \underset{1}{*} b]$,
    $a \underset{n}{*} b \coloneqq \sum_{j \geq 0} \binom{\Delta(a)}{j} a_{(j-n)} b$.
  - $\widetilde{A}(V) \coloneqq V/(V \underset{2}{\bullet} V), \quad [a] \bullet [b] \coloneqq [a \underset{1}{\bullet} b]$,
    $a \underset{n}{\bullet} b \coloneqq \mathrm{res}_z \left[e^z (e^z - 1)^{-n} a(z) b \right] dz$.
- However, $\widetilde{A}(V)$ has the advantage in that
  (1) determining $\widetilde{A}(V)$ is simpler than the original $A(V)$,
  (2) it has a natural SUSY analogue.
- We demonstrate (1) for superconformal vertex algebras V in §2, and (2) by proposing the definition of Zhu algebras for SUSY vertex algebras in §3.

2. Zhu algebras of superconformal vertex algebras

SCA := superconformal vertex algebra $\qquad [a_\lambda b] := \sum_{j\geq 0} \frac{1}{j!}\lambda^j a_{(j)} b$

- $N=0$ SCA $V^{N=0}$ = Virasoro VA: generated by even L.
  - $[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3$. ($L$: conformal of central charge $c \in \mathbb{C}$)
- $N=1$ SCA $V^{N=1}$ = Neveu-Schwarz VSA: even L & odd G.
  - L: conformal of central charge c,
  - $[L_\lambda G] = (\partial + \frac{3}{2}\lambda)G$, ($G$: primary of conformal weight $\frac{3}{2}$)
  - $[G_\lambda G] = 2L + \frac{c}{3}\lambda^2$.

  L generates $V^{N=0} \subset V^{N=1}$.
- $N=2$ SCA $V^{N=2}$: generated by even L, J and odd $G^\pm$,

  (c.f. talk of Dr. X. Zhang, Day 1.)
  - L: conformal of central charge c,
  - J: even primary of conformal weight 1,
  - $G^\pm$: odd primary of conformal weight $\frac{3}{2}$,
  - $[J_\lambda J] = \frac{c}{3}\lambda$, $[J_\lambda G^\pm] = \pm G^\pm$, $[G^+{}_\lambda G^-] = L + \frac{1}{3}(\partial + 2\lambda)J + \frac{c}{6}\lambda^2$.

  L and $G^+ + G^-$ generate $V^{N=1} \subset V^{N=2}$.

2.1. Superconformal vertex algebras

- $N = 4$ SCA $V^{N=4}$: generated by even $L, J^{0,\pm}$ and odd $G^{\pm}, \overline{G}^{\pm}$,

  $[a_\lambda b] := \sum_{j\geq 0} \frac{1}{j!}\lambda^j a_{(j)} b$

  - L: conformal of central charge c,
  - $J^{0,\pm}$: primary of conformal weight 1,
  - $G^{\pm}, \overline{G}^{\pm}$: odd primary of conformal weight $\frac{3}{2}$,
  - $[J^0{}_\lambda J^{\pm}] = \pm 2J^{\pm}, \quad [J^0{}_\lambda J^0] = \frac{c}{3}\lambda, \quad [J^+{}_\lambda J^-] = J^0 + \frac{c}{6}\lambda,$
    $[J^0{}_\lambda G^{\pm}] = \pm G^{\pm}, \quad [J^0{}_\lambda \overline{G}^{\pm}] = \pm \overline{G}^{\pm},$
    $[J^{\pm}{}_\lambda G^{\mp}] = G^{\pm}, \quad [J^{\pm}{}_\lambda \overline{G}^{\mp}] = -\overline{G}^{\pm},$
    $[G^{\pm}{}_\lambda \overline{G}^{\pm}] = (\partial + 2\lambda)J^{\pm}, \quad [G^{\pm}{}_\lambda \overline{G}^{\mp}] = L \pm \frac{1}{2}(\partial + 2\lambda)J^0 + \frac{c}{6}\lambda^2.$

  $L, J^0, G^+, \overline{G}^-$ generate $V^{N=2} \subset V^{N=4}$.
- $N = 3$ SCA $V^{N=3}$ and big $N = 4$ SCA $V^{N=4,\mathrm{big}}$.

- $\widetilde{A}(V) := V/(V \underset{2}{\bullet} V)$, $\quad [a] \bullet [b] := [a \underset{1}{\bullet} b]$,
  $a \underset{n}{\bullet} b := \mathrm{res}_z \left[e^z (e^z - 1)^{-n} a(z) b \right] dz$.
- **Lemma 1** Generators a^i of V induce generators $[a^i]$ of $\widetilde{A}(V)$.
  Lemma 2 $[L]$ of the conformal element L is central.
  Lemma 3 The following holds in $\widetilde{A}(V)$ for any $a, b \in V$:
  $[\partial a] = 0, \quad [a] \bullet [b] - p(a,b)[b] \bullet [a] = [a_{(0)} b]$.
  Lem. 3 holds only for $\widetilde{A}(V)$, not for $A(V)$. $\quad p(a,b) := (-1)^{\mathrm{parity}(a) \cdot \mathrm{parity}(b)}$
  $\rightsquigarrow$ Can read off generators & relations of $\widetilde{A}(V)$ from the λ-brackets.
  Can neglect ∂- and $\lambda^{\geq 1}$-terms! $\quad [a_\lambda b] := \sum_{j \geq 0} \frac{1}{j!} \lambda^j a_{(j)} b$
- For example, from the λ-brackets of $V^{N=4}$ (the previous page), $\widetilde{A}(V^{N=4})$ is generated by $[L], [J^{0,\pm}], [G^\pm], [\overline{G}^\pm]$, and they satisfy
  $[L]$: central, $\quad [[J^0], [J^\pm]] = \pm 2 [J^\pm]$, $\quad [[J^+], [J^-]] = [J^0]$,
  $[[J^0], [G^\pm]] = \pm [G^\pm]$, $[[J^0], [\overline{G}^\pm]] = \pm [\overline{G}^\pm]$, $[[J^\pm], [G^\mp]] = [G^\pm]$,
  $[[J^\pm], [\overline{G}^\mp]] = -[\overline{G}^\pm]$, $\quad [[G^\pm], [\overline{G}^\pm]] = 0$, $\quad [[G^\pm], [\overline{G}^\mp]] = [L]$.
  $\rightsquigarrow$ (the universal enveloping algebra of) some Lie superalgebra

Theorem (Zhu algebra of $V^{N=4}$ [Sato–Y., Theorem 2.4])

$$U(\mathfrak{psl}(2|2)^{f_{\min}}) \xrightarrow{\sim} \widetilde{A}(V^{N=4})$$

$$(f_{\min}, j^{0,\pm}, g^{\pm}, \overline{g}^{\pm}) \longmapsto ([L], [J^{0,\pm}], [G^{\pm}], [\overline{G}^{\pm}])$$

- $\mathfrak{sl}(2|2) := \{x \in \mathfrak{gl}(2|2) \mid \operatorname{str} x = 0\}$: special linear Lie superalgebra.
  $\mathfrak{psl}(2|2) := \mathfrak{sl}(2|2)/\mathbb{C}I$ (I: identity supermatrix), $\operatorname{sdim} = 6|8$.
  $\mathfrak{psl}(2|2)^{f_{\min}} := \{x \in \mathfrak{psl}(2|2) \mid [x, f_{\min}] = 0\}$, $\operatorname{sdim} \mathfrak{psl}(2|2)^{f_{\min}} = 4|4$.
- $f_{\min} \in \mathfrak{psl}(2|2)$ is a minimal nilpotent element.
- Basis of $\mathfrak{psl}(2|2)^{f_{\min}}$:
  $f_{\min} := \left[\begin{array}{c|c} O & O \\ \hline O & F \end{array}\right]$, $j^0 := \left[\begin{array}{c|c} H & O \\ \hline O & O \end{array}\right]$, $j^+ := \left[\begin{array}{c|c} E & O \\ \hline O & O \end{array}\right]$, $j^- := \left[\begin{array}{c|c} F & O \\ \hline O & O \end{array}\right]$,
  $g^+ := \left[\begin{array}{c|c} O & E_{11} \\ \hline O & O \end{array}\right]$, $g^- := \left[\begin{array}{c|c} O & F \\ \hline O & O \end{array}\right]$, $\overline{g}^+ := \left[\begin{array}{c|c} O & O \\ \hline E_{22} & O \end{array}\right]$, $\overline{g}^- := \left[\begin{array}{c|c} O & O \\ \hline F & O \end{array}\right]$,
  where $E := \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $F := \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, $H := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $E_{11} := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $E_{22} := \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

Theorem (Zhu algebras of $V^{N=0,1,2,4}$ [Sato–Y., Theorems 2.1–2.4])

We constructed explicit isomorphisms

$$\widetilde{A}(V^{N=0}) \xrightarrow{\simeq} U(\mathfrak{sl}(2)^{f_{\min}}), \qquad \widetilde{A}(V^{N=1}) \xrightarrow{\simeq} U(\mathfrak{osp}(1|2)^{f_{\min}}),$$
$$\widetilde{A}(V^{N=2}) \xrightarrow{\simeq} U(\mathfrak{sl}(1|2)^{f_{\min}}), \qquad \widetilde{A}(V^{N=4}) \xrightarrow{\simeq} U(\mathfrak{psl}(2|2)^{f_{\min}}).$$

RHS are isomorphic to finite W algebras $U(\mathfrak{g}, f_{\min})$.

- SCAs are (affine) W algebras:

$$V^{N=0}_{c=1-\frac{6(k+1)^2}{k+2}} \cong \mathcal{W}^k(\mathfrak{sl}(2), f_{\min}), \quad V^{N=1}_{c=\frac{3}{2}-\frac{12(k+1)^2}{2k+3}} \cong \mathcal{W}^k(\mathfrak{osp}(1|2), f_{\min}),$$
$$V^{N=2}_{c=-3(2k+1)} \cong \mathcal{W}^k(\mathfrak{sl}(1|2), f_{\min}), \quad V^{N=4}_{c=-6(k+1)} \cong \mathcal{W}^k(\mathfrak{psl}(2|2), f_{\min}).$$

  Hence, reduction commutes with taking Zhu algebras[11].
- Similar statements hold for $V^{N=3}$ and $V^{N=4,\mathrm{big}}$.

[11] A. De Sole, V. G. Kac, "Finite vs affine W-algebras", Jpn. J. Math., **1** (2006), 137–261.

$N=3$ SCA $V^{N=3}$: generated by even $L, A^{1,2,3}$ and odd $G^{1,2,3}, \Phi$ with

- L: conformal of central charge c
- $A^{1,2,3}$: even primary of conformal weight 1
- $G^{1,2,3}$: odd primary of conformal weight $\frac{3}{2}$
- Φ: odd primary of conformal weight $\frac{1}{2}$
- the remaining non-zero λ-brackets
  $$[A^i{}_\lambda A^j] = \varepsilon_{ijk}A^k + \tfrac{c}{3}\lambda\delta_{ij}, \quad [A^i{}_\lambda G^j] = \varepsilon_{ijk}G^k + \lambda\Phi\delta_{ij},$$
  $$[G^i{}_\lambda G^j] = 2L\delta_{ij} - \varepsilon_{ijk}(\partial + 2\lambda)A^k + \tfrac{c}{3}\lambda^2\delta_{ij}, \quad [\Phi_\lambda G^i] = A^i,$$
  $$[\Phi_\lambda \Phi] = -\tfrac{c}{3}.$$

Theorem (Zhu algebras of $V^{N=3}$ [Sato–Y., Theorem 2.6])

$$\widetilde{A}(V^{N=3}) \cong U(\mathfrak{osp}(3|2)^{f_{\min}}) \otimes Cl(\mathbb{C}),$$

where $f_{\min} := \left[\begin{array}{c|c} O & O \\ \hline O & F \end{array}\right] \in \mathfrak{osp}(3|2)$, $F := \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$,

and $Cl(\mathbb{C})$ is the Clifford algebra with 1 generator.

(c.f. talk of Prof. J.-S. Huang in Day 2)

Big $N=4$ SCA $V^{N=4,\mathrm{big}}$: gen. by even $L, J^{0,\pm}, J'^{0,\pm}, \xi$; odd $G^{\pm\pm}, \sigma^{\pm\pm}$,

- L: conformal of central charge $c \in \mathbb{C}$
- $J^{0,\pm}, J'^{0,\pm}, \xi$: even primary of conformal weight 1
- $G^{\pm\pm}$: odd primary of conformal weight $\frac{3}{2}$
- $\sigma^{\pm\pm}$: odd primary of conformal weight $\frac{1}{2}$
- the remaining non-zero λ-brackets: containing $a \in \mathbb{C} \setminus \{0,-1\}$
  $[\sigma^{+\pm}{}_\lambda \sigma^{-\mp}] = -\frac{c}{6}$, $[\xi_\lambda \xi] = -\frac{c}{6}\lambda$,
  J, J' and themselves : 8 relations, J, J' and G: 16 relations,
  G and themselves: 10 rels., J, J' and σ: 16 rels., G and σ: 16 rels.

Theorem (Zhu algebras of $V^{N=4}$ [Sato–Y., Theorem 2.10])

$$\widetilde{A}(V^{N=4}) \cong U(D(2,1;a)^{f_{\min}}) \otimes Cl(\mathbb{C}^4),$$

where $Cl(\mathbb{C}^4)$ is the Clifford algebra with 4 generator.

3. Zhu algebras for SUSY vertex algebras

An $N_K = N$ supersymmetric vertex algebra[12] (SUSY VA) is an extension of VA encoding two-dimensional chiral supersymmetric CFTs.

$Z = (z, \zeta^1, \ldots, \zeta^N)$: supervariable. $\quad Z^{j|J} = z^j \zeta^J := z^j \zeta^{j_1} \cdots \zeta^{j_r}$.

$(j \in \mathbb{Z},\ J = \{j_1, \ldots, j_r\} \subset [N] := \{1, \ldots, N\})$

Definition ($N_K = N$ supersymmetric vertex algebra)

An $N_K = N$ SUSY VA is a $\mathbb{C}$-linear superspace $\mathbb{V}$ with even $|0\rangle \in \mathbb{V}$ and

$$\mathbb{V} \to (\operatorname{End} \mathbb{V})[\![Z^{\pm 1}]\!], \quad a \mapsto a(Z) = \sum_{j \in \mathbb{Z},\, J \subset [N]} Z^{-j-1|[N] \setminus J} a_{(j|J)},$$

such that, for any $a, b \in V$,

(i) $a(Z)b \in \mathbb{V}(\!(Z)\!)$, (quantum superfield)

(ii) $|0\rangle(Z) = \mathrm{id}_V$, $a(Z)|0\rangle = a + O(Z)$, (vacuum)

(iii) $\eth^i \in \operatorname{End} \mathbb{V}$, $\ a \mapsto a_{(-1|e_i)}|0\rangle \quad (e_i := \{i\} \subset [N]$ for $i = 1, \ldots, N)$,
$(\eth^i a)(Z) = (\partial_{\zeta^i} + \zeta^i \partial_z) a(Z)$ and $[\eth^i, a(Z)] = (\partial_{\zeta^i} - \zeta^i \partial_z) a(Z)$,
mixing even and odd elements! (odd translation)

(iv) $\exists N_{a,b} \in \mathbb{Z}_{\geq 0}$ such that $(z - w)^{N_{a,b}} [a(Z), b(W)] = 0$. (locality)

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[12] R. Heluani, V. Kac, "Supersymmetric Vertex Algebras", Comm. Math. Phys., **271** (2007), 103–178.

- $\mathbb{V}$: $N_K = N$ SUSY VA, $\mathbb{V} \ni a \mapsto a(Z) = \sum_{j,\,J} Z^{-j-1|[N]\setminus J} a_{(j|J)}$.
- $\partial := (\not\partial^1)^2 = \cdots = (\not\partial^N)^2$ satisfies $(\partial a)(Z) = [\partial, a(Z)] = \partial_z a(Z)$.
- Λ-bracket: $[a_\Lambda b] := \sum_{j \geq 0,\,J} \frac{(-1)^{|J|N + \binom{|J|+1}{2}}}{j!} \Lambda^{j|J} a_{(j|J)} b$.
  $\Lambda^{j|J} := \lambda^j \chi^J$, λ: even, $\chi^1, \ldots, \chi^N$: odd, $[\chi^i, \chi^j] = 2\delta_{ij}\lambda$.

Example $[a_\lambda b] := \sum_{n \geq 0} \frac{1}{n!} \lambda^n a_{(n)} b$

- $N = 1$ SCA $V^{N=1}$, generated by even L and odd G,
  $[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3$, $[L_\lambda G] = (\partial + \frac{3}{2}\lambda)G$, $[G_\lambda G] = 2G + \frac{c}{3}\lambda^2$.
  $N_K = 1$ SUSY str.: $\not\partial := G_{-\frac{1}{2}} = G_{(0)}$, $a \mapsto a(Z) := a(z) + \zeta(\not\partial a)(z)$.
  In particular, $L = 2\not\partial G$, $G(Z) = G(z) + 2\zeta L(z)$ and
  $$[G_\Lambda G] = \chi[G_\lambda G] + 2[L_\lambda G] = (2\partial + 3\lambda + \chi\not\partial)G + \tfrac{c}{3}\lambda^2\chi.$$
- $V^{N=2}$ is an $N_K = 2$ SUSY VA, and $V^{N=4,\text{big}}$ is an $N_K = 4$ SUSY VA.
  $$[G_\Lambda G] = (2\partial + (4-N)\lambda + \textstyle\sum_{i=1}^N \chi^i \not\partial^i)G + (\text{central term}).$$
- Chiral de Rham complexes of smooth/Kähler/hyperkähler manifolds are $N_K = 1/2/4$ SUSY VAs.

Huang's version $\widetilde{A}(V)$ of Zhu algebra for VA V suggests:

$$\Big(a \overset{\gamma}{\underset{n}{\bullet}} b := \mathrm{res}_z[\gamma^n e^{\gamma z}(e^{\gamma z}-1)^{-n} a(z)b]dz \text{ for non-SUSY VA}\Big)$$

Definition/Theorem (SUSY Zhu algebra [Sato–Y., §3.2])

For an $N_K = N$ SUSY VA $\mathbb{V}$,

$$\widetilde{A}_\gamma(\mathbb{V}) := \mathbb{V}/(\mathbb{V} \overset{\gamma}{\underset{2}{\bullet}} \mathbb{V}), \quad [a] \overset{\gamma}{\bullet} [b] := [a \overset{\gamma}{\underset{1}{\bullet}} b]\text{: associative product,}$$

$$a \overset{\gamma}{\underset{n}{\bullet}} b := \mathrm{sres}_Z[\gamma^n \zeta^{[N]} e^{\gamma z}(e^{\gamma z}-1)^{-n} a(Z)b]\delta Z$$

(sres: super-residue, δZ: Berezin differential)

- **Lemma** $\widetilde{A}_\gamma(\mathbb{V}) \cong \widetilde{A}_\gamma(V_{\mathrm{red}})$, where $V_{\mathrm{red}} := (\mathbb{V}, |0\rangle, a \mapsto a(z,0))$.
- **Proposition** $\not\partial^i$ induces differentials $[\not\partial^i]$ on $\widetilde{A}_\gamma(\mathbb{V})$.
- **Proposition** We can recover the SUSY C_2-Poisson algebra[13] as
  $$\widetilde{A}_{\gamma=0}(\mathbb{V}) \cong R(\mathbb{V}) := \mathbb{V}/(\mathbb{V}_{(-2|*)}\mathbb{V}).$$
  $R(\mathbb{V})$ is Poisson superalg. with $\overline{a} \cdot \overline{b} := \overline{a_{(-1|[N])}b}$, $\{\overline{a}, \overline{b}\} := \overline{a_{(0|[N])}b}$ and odd derivations $\overline{\not\partial^i}$

[13] S.Y., "Li filtrations of SUSY vertex algebras", Lett. Math. Phys., **112** (2022), Article no. 103, 77pp.

Example (SUSY Zhu algebras of $N=1$ SCA [Sato–Y., §3.2])

- $\mathbb{V}^{N=1}$: $N=1$ SCA as $N_K=1$ SUSY VA.
  Generated by odd G with $[G_\Lambda G] = (2\partial + \chi \not\partial + 3\lambda)G + \frac{c}{3}\lambda^2$.
- Zhu algebra:
  $\widetilde{A}(\mathbb{V}) \cong U(\mathfrak{osp}(1|2)^{f_{\min}})$, $[G] \mapsto g$, $[L] = 2[\not\partial G] \mapsto f$,
  $U = \langle f, g \mid f : \text{even}, g : \text{odd}, [f,f] = [g,g] = 0, [g,g] = 2f \rangle_{\text{alg}}$,
  with differential $[\not\partial]$.
- SUSY C_2-Poisson algebra:
  $R(\mathbb{V}) \cong \mathbb{C}[\overline{G}, \overline{\not\partial G}] = \mathbb{C}[\overline{G}, \overline{L}]$, $\{\overline{G}, \overline{G}\} = 2\overline{L}$
  with odd derivation $\overline{\not\partial}$.

We have similar results for $V^{N=2,3,4}$ and $V^{N=4,\text{big}}$.

Summary:

(1) We determined $\widetilde{A}(V)$ for superconformal vertex algebras V, which is is simpler than the original $A(V)$.

(2) We proposed Zhu algebras $\widetilde{A}(\mathbb{V})$ of SUSY vertex algebras $\mathbb{V}$, which is a natural SUSY extension of $\widetilde{A}(V)$.

Open problems

- Classification of simple modules of $N = 3$ and big $N = 4$ SCAs. (Simple modules of $N = 1, 2, 4$ SCAs are known.)
- SUSY analogue of chiral homology $\mathrm{CH}_*(\Sigma, V)$[14]
- SUSY analogue of $\mathrm{HH}^1(A(V)) \cong \mathrm{CH}_0(\text{nodal curve}, V)$[15].
- Relation to (Yangian limit of) quantum toroidal $\mathfrak{gl}_{1|1}$ and super-Macdonald polynomials. (c.f. talk of Prof. H. Kanno, Day 1)
- $\cdots$

Thank you.

[14] A. Beilinson, V. Drinfeld, "Chiral Algebras", AMS Colloquium Publ., **51**, Amer. Math. Soc., Providence, RI, 2004.

[15] J. van Ekeren, R. Heluani, "Chiral homology of elliptic curves and the Zhu algebra", Comm. Math. Phys., **386** (2021), 495–550.