Notes on Vector Fields

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1 Introduction

In honour of Wigner's great contribution to quantum field theory, we review the framework of quantum field theory and calculate the vector representation of the Poincaré group. Wigner physically proved that the projective irreducible representations of the Poincaré group classified single particle states, which is mathematically proved by George Mackey later.

Specifically, the identical representation corresponds to the scalar field, the vector representation corresponds to the vector field and the spinor representation corresponds to the spinor field. We will discuss the vector field especially.

2 Poincaré group and its representations

2.1 Poincaré group

In relativity, spacetime is described by a 4-dimensional manifold M with a Lorentz metric g_{ab}. If there is a global chart of the manifold and the components of this metric is constant in this chart, it is called the special relativity theory:

$$\phi : M \to \mathbb{R}^{1,3}, (g_{\mu\nu}) = diag(+1, -1, -1, -1)$$

Although we all know that the signature $(+, -, -, -)$ is extremely stupid and could cause several problem, we use it in order to show respect to Weinberg and other physicists.

In general relativity, the spacetime is a manifold such that it is locally Euclidean, while in this case, it is globally Euclidean. We could regard the position as a vector in a linear space rather than in a affine space. It is necessary to remember that there is no reason to regard the position as a vector in a classical theorem otherwise.

Now given a pseudo-inner product space $(\mathbb{R}^{1,3}, g_{\mu\nu})$:

$$\forall x, y \in \mathbb{R}^{1,3}, (x, y) := g_{\mu\nu} x^\mu y^\nu = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

We want to define the symmetric transformation on it:

$$\Lambda : \mathbb{R}^{1,3} \to \mathbb{R}^{1,3}, (\Lambda^\mu{}_\nu) \in M_4(\mathbb{R})$$

Λ is called symmetric with respect to g if the pseudo-inner product holds, i.e.

$$\forall x, y \in R, (x, y) = (\Lambda(x), \Lambda(y))$$

Which means in component:

$$\forall x^\mu, y^\mu \in \mathbb{R}, g_{\mu\nu} x^\mu y^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho y^\sigma$$

Such a $(\Lambda^\mu{}_\nu)$ is called a Lorentz matrix. All the Lorentz matrices form a matrix Lie group called the Lorentz group, denoted by $\mathrm{O}(1, 3) \leq \mathrm{GL}(1, 3)$. The Lie group $\mathrm{SO}(1, 3) := \mathrm{SL}(1, 3) \cap \mathrm{O}(1, 3)$ is called the proper Lorentz group.

Moreover, notice that the charts of spacetime manifold is obviously not unique, as a affine space it admits more symmetry, i.e. spacetime translation, which corresponds to the Lie group $(\mathbb{R}^{1,3}, +)$.This group is Abelian, while the Lorentz group defined above is not.

In combination, all the spacetime symmetries correspond the Lie group of:

$$\mathcal{P}(4) = \mathbb{R}^{1,3} \rtimes \mathrm{SO}(1,3)$$

It is semi-direct product rather than direct product or the free product for some physical reasons. After all, the semi-direct product is defined by:

$$(\Lambda, a) \in \mathcal{P}(4), \Lambda \in \mathrm{SO}(1,3), a \in \mathbb{R}^{1,3}$$

$$(\Lambda_1, a_1) \cdot (\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1)$$

One might need to check that $\mathcal{P}(4)$ is a Lie group, but we won't. Actually it could be regarded as a matrix Lie group (i.e. a closed (with respect to the subspace topology of $\mathbb{R}^{n\times n}$) subgroup of $\mathrm{GL}(n)$) since there is a embedding:

$$\begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 & a^0 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 & a^1 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 & a^2 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2.2 Representations and Wigner's theory

We consider the quantum symmetry. Now consider given a quantum system, which associates a complex Hilbert space $\mathcal{H}$. The state of the system associates to a element in the quotient space:

$$[|\psi\rangle] \in \mathcal{HP} := (\mathcal{H}\backslash\{0\})/\sim, |\phi\rangle \sim |\psi\rangle \iff \exists c \in \mathbb{C}\, s.t.\, |\phi\rangle = c\,|\psi\rangle$$

Since any Hilbert space is isomorphic to its dual space, one could use the Dirac notation $\langle\psi| \in \mathcal{H}^*$. Moreover, Riesz Representation Theorem suggests that:

$$\|\, \langle\psi|\, \| = \|\, |\psi\rangle\, \|$$

And the inner product could be rewritten as:

$$(|\phi\rangle\,, |\psi\rangle) = \langle\psi|\,(|\phi\rangle) =: \langle\psi|\phi\rangle$$

The inner product induces an function on $\mathcal{HP}$ by:

$$([|\phi\rangle], [|\psi\rangle]) := \frac{\langle\psi|\phi\rangle}{\|\, |\phi\rangle\, \|\|\, |\psi\rangle\, \|}$$

It is obvious that such a map is well-defined. Sometimes one could use a representative element to represent a physical state. However, the phase c is of no physical meaning.

A bijection $\mathcal{A} : \mathcal{HP} \to \mathcal{HP}$ is called symmetric if:

$$\forall [|\phi\rangle], [|\psi\rangle] \in \mathcal{HP}, ([|\phi\rangle], [|\psi\rangle]) = (\mathcal{A}[|\phi\rangle], \mathcal{A}[|\psi\rangle])$$

Such a symmetric $\mathcal{A}$ is called a Wigner's homeomorphism. All the Wigner's homeomorphisms form a group, denoted by $\mathbb{U}$. There is a famous theorem that:

Thm. 2.1 *(Wigner's Theorem) Any Wigner's homeomorphism is induced by a unitary or anti-unitary transformation $\hat{A}$ on the Hilbert space, i.e. as form:*

$$\mathcal{A}[|\psi\rangle] = [\hat{A}\,|\psi\rangle], \hat{A} \in \mathcal{L}^u(\mathcal{H})$$

The quantization condition demands that the decoherent quantum system is exactly the classical system after quantization, such that any classical symmetry (which associates to a Poincaré transformation) must relate to a quantum symmetry (which associates to a Wigner's homeomorphism, thus an unitary or anti-unitary transformation). This argument holds for both the canonical quantization and the path-integral quantization. Specifically, this suggests that the representation on a Hilbert space of the classical symmetric group is the quantum symmetric transformation. That is the reason why one needs to study the representations.

Given a group G. A projective representation of G is a group homomorphism:

$$\alpha : G \to \mathbb{U}$$

i.e. $\forall g_1, g_2 \in G, \alpha(g_1) \circ \alpha(g_2) = \alpha(g_1 g_2)$. Denote the very unitary transformation which induces α by $\hat{U}$, thus:

$$\hat{U}(g_1)\hat{U}(g_2) = \omega(g_1, g_2)\hat{U}(g_1 g_2), |\omega(g_1, g_2)| = 1, \omega(g_1, g_2) \in \mathbb{C}$$

Now consider G as a connected Lie group (since any manifold is locally path-connected, G is path-connected).

Thm. 2.2 *(Bargman Theorem)*

1. One could always choose some ω such that in some $N \in \mathcal{N}(e) \cap \mathcal{T}_G$, the map $g \mapsto \hat{U}_g$ is continuous with respcet to the norm topology.

2. If $\pi_1(G) \cong \{e\}$ (i.e. G is simply connected) or G is a affine linear Lie group, one could take $\omega = 1$ globally and the map is continuous.

However, As we discussed, the symmetric group of special relativity is the Poincaré group $\mathcal{P}(4) = \mathbb{R}^{1,3} \rtimes \mathrm{SO}(1,3)$, in which $\mathrm{SO}(1,3)$ is not even connected: $\pi_0(\mathrm{SO}(1,3)) = 2$. $\mathrm{SO}^{\uparrow}(1,3)$ stands for the connected component which contains e. $\mathrm{SO}^{\uparrow}(1,3)$ is not simply connected, thus one need to consider its topological structure.

Suppose that G is a connected Lie group and $\mathcal{H}$ is a complex Hilbert space. If G admits a universal covering $\tilde{G}$, i.e. $\pi_1(\tilde{G}) \cong \{e\}$ and $\pi : \tilde{G} \to G$ is a covering map. One could study the unitary representation of $\tilde{G}$ to generate a unitary representation of G by taking a continuous σ such that $\pi \circ \sigma = \mathrm{Id}_G$.

$$\begin{tikzcd}
\tilde{G} \arrow[d, shift right, "\pi"'] \arrow[dr, "\hat{\tilde{U}}"] & \\
G \arrow[u, shift right, "\sigma"'] \arrow[r, "\hat{U}"'] & \mathcal{L}^u(\mathcal{H})
\end{tikzcd}$$

The diagram commutes, i.e. $\hat{U}(g) = \hat{\tilde{U}}(\sigma(g))$.

Specifically consider $\mathcal{P}^{\uparrow}(4) = \mathbb{R}^{1,3} \rtimes \mathrm{SO}^{\uparrow}(1,3)$. There is a 2-sheet covering:

$$\begin{tikzcd}
\mathrm{SL}(2,\mathbb{C}) \arrow[d, "\lambda"'] \\
\mathrm{SO}^{\uparrow}(1,3)
\end{tikzcd}$$

which is defined by taking the 4-Pauli matrices:

$$(\sigma_\mu) := (I_{2\times 2}, \sigma_k)$$

$\forall x \in \mathbb{R}^{1,3}$, define $\overline{x} := x^\mu \sigma_\mu \in \mathbb{C}^{2\times 2}$:

$$\overline{x} = \begin{pmatrix} x^3 + x^0 & x^1 - ix^2 \\ ix^2 + x^1 & x^0 - x^3 \end{pmatrix}$$

Then there is some $\lambda : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^{\uparrow}(1,3)$ such that:

$$\forall x \in \mathbb{R}^{1,3}, \overline{\lambda(A)(x)} = A\overline{x}A^{\dagger}$$

Thus one could study the unitary representation of $\tilde{\mathcal{P}}^{\uparrow}(4) := \mathbb{R}^{1,3} \rtimes \mathrm{SL}(2,\mathbb{C})$.

2.3 Physical interpretation of representations

Now we turn to the physical aspect to construct such a representation of $\tilde{\mathcal{P}}^{\uparrow}(4)$. Such a process contributes to the classification of states of particles, which is physically proved by Wigner and mathematically proved by Mackey. We would not raise the full discussion since this is a note of a physical course, rather we discuss some well-chosen items .

Thm. 2.3 *(Stone-Naimark-Ambrose-Godement Theorem)*
There is an unitary representation of the elements in the subgroup $(a, I_{4\times 4}) \in \tilde{\mathcal{P}}^{\uparrow}(4)$:

$$\hat{U}(a, I_{4\times 4}) := e^{i\hat{P}_{\mu}a^{\mu}} \in \mathcal{L}^{u}(\mathcal{H})$$

where $\hat{P}^{\mu} \in \mathcal{L}^{h}(\mathcal{H})$ are hermitian and they commutes.

Since all the $\hat{P}^{\mu}$ commutes, consider their eigen-system:

$$\hat{P}_{\mu} |p, \lambda\rangle = p_{\mu} |p, \lambda\rangle$$

where λ taking some discrete value stands for the degeneracy. They satisfies the orthonormalization condition of:

$$\langle p', \lambda' | p, \lambda\rangle = \delta_{\lambda'\lambda} 2E_{\underline{p}} \delta^{(3)}(\underline{p}' - \underline{p}), p = (p_{\mu}) = (E_{\underline{p}}, -\underline{p})$$

The rest mass of any massive particle is Lorentz invariant, i.e. the group action $\mathrm{SO}(1,3)$ is transitive on the hypersurface $H_m^+ := \{p \in \mathbb{R}^{1,3} | p^2 = m^2, p^0 > 0\}$. Especially the standard momentum $p_0 = (0, \underline{0}) \in H_m^+$. Consider a map $L : H_m^+ \to \mathrm{SL}(2,\mathbb{C})$, such that

$$\lambda(L(p))p_0 = p$$

which implies:

$$L(p)\overline{p}_0 L^{\dagger}(p) = \overline{p}, L(p)L^{\dagger}(p) = \frac{\overline{p}}{m}$$

One could choose some λ such that $\forall p \in H_m^+$,

$$|p, \lambda\rangle = \hat{U}(L(p), I_{2\times 2}) |p_0, \lambda\rangle$$

That is the quantum transformation in the little group.

While if the field is massless (such as electrodynamics and gravity), the situation is different. Skip a huge amount of discussion, the whole quantum transformation group satisfies the relation (without mathematically rigorous proof):

Prop 2.4 *Consider a representation $\hat{U} : \mathcal{P}^{\uparrow}(4) \to \mathcal{L}^{u}(\mathcal{H})$ which is generated by some hermitian operators:*

$$\hat{J}^{\mu\nu} := 2i(\frac{\partial \hat{U}(a, \Lambda)}{\partial \omega_{\mu\nu}})|_{\omega_{\mu\nu}=0, \epsilon_{\mu}=0}, \hat{P}^{\mu} = -i(\frac{\partial \hat{U}(a, \lambda)}{\partial \epsilon_{\mu}})|_{\omega_{\mu\nu}=0, \epsilon_{\nu}=0}$$

where $\Lambda^{m}u_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, a^{\mu} = \epsilon^{\mu}$. The generators satisfies the commutation relations of:

$$[\hat{J}^{\mu\nu}, \hat{J}^{\rho\sigma}] = i(g^{\mu\sigma}\hat{J}^{\nu\rho} + g^{\nu\rho}\hat{J}^{\mu\sigma} - g^{\mu\rho}\hat{J}^{\nu\sigma} - g^{\nu\sigma}\hat{J}^{\mu\rho})$$

$$[\hat{J}^{\mu\nu}, \hat{P}^{\rho}] = i(g^{\nu\rho}\hat{P}^{\mu} - g^{\mu\rho}\hat{P}^{\nu})$$

$$[\hat{P}^{\mu}, \hat{P}^{\nu}] = 0$$

Especially this holds in the subgroup we discussed above.

3 Vector Fields

We discuss the vector representation and the quantum vector fields.

3.1 Vector representations and quantum transformations

Notice that if we take the matrix of:

$$(\mathcal{T}^{\mu\nu})^{\alpha}{}_{\beta} = i(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - g^{\nu\alpha}\delta^{\mu}{}_{\beta})$$

Such that $\forall\mu,\nu = 0,1,2,3$, $((\mathcal{T}^{\mu\nu})^{\alpha}{}_{\beta})_{\alpha,\beta} \in M_4(\mathbb{C})$. One could check that such a set of 4 by 4 matrices satisfies the commutation relations of the generators the Lorentz group. Specifically, it generates the Lorentz group by:

$$\Lambda := \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{T}^{\mu\nu}\right) \in \mathrm{SO}^{\uparrow}(1,3), \forall\omega_{\mu\nu} \in \mathbb{R}$$

In general, this suggests:

Prop 3.1 *Given a set of matrices $\{\mathcal{T}^{\mu\nu}\}$ which satisfies the commutation relations above, it induces an endomorphism of the proper Lorentz group $V \in \mathrm{End}(\mathrm{SO}^{\uparrow}(1,3))$ by:*

$$\forall\Lambda \in \mathrm{SO}^{\uparrow}(1,3), \Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, V(\Lambda) := \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{T}^{\mu\nu}\right) \in \mathrm{SO}^{\uparrow}(1,3)$$

Such an endomorphism is called the vector representation.

After-all V is not a quantum field, since it is not a homomorphism to $\mathcal{L}^u(\mathcal{H})$ where $\mathcal{H}$ is the Hilbert space of the quantum system. Now we consider how this process could be done.

Recall this is to construct a group homomorphism:

$$\hat{\hat{U}} : \mathcal{P}^{\uparrow} \to \mathcal{L}(\mathcal{H})$$

where $\mathcal{H}$ is the Hilbert space of the quantum system, i.e. $\hat{\hat{U}}(\Lambda, a)$ acts on the quantum states. As we discussed above, we could consider the subgroup representation:

$$\hat{U} : \mathrm{SO}^{\uparrow}(1,3) \to \mathcal{L}(\mathcal{H})$$

which is a group homomorphism:

$$\hat{U}(\Lambda_1)\hat{U}(\Lambda_2) = \hat{U}(\Lambda_1\Lambda_2), \hat{U}^{-1}(\Lambda) = \hat{U}(\Lambda^{-1})$$

Any Lorentz scalar transforms as:

$$\hat{\phi}'(x') = \hat{\phi}(x) : \hat{U}^{-1}(\Lambda)\hat{\phi}(x)\hat{U}(\Lambda) = \hat{\phi}(\Lambda^{-1}x)$$

For infinitesimal transformation (considering the Lie algebra):

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, (\Lambda^{-1})^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \omega^{\mu}{}_{\nu}$$

Define the differential operator by:

$$\hat{L}^{\mu\nu} := i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})$$

Such that:

$$[\hat{\phi}(x), \hat{J}^{\mu\nu}] = \hat{L}^{\mu\nu}\hat{\phi}(x)$$

Thus the spatial component of $\hat{J}$ acts as the orbital angular momentum.

Now we consider how the vector fields transform. Similarly write:

$$\hat{A}'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}\hat{A}^{\nu}(x)$$

That is:

$$\hat{U}^{-1}(\Lambda)\hat{A}^{\mu}(x)\hat{U}(\Lambda) = \Lambda^{\mu}{}_{\nu}\hat{A}(\Lambda^{-1}x)$$

For infinitesimal transformation:

$$\hat{A}'^{\mu}(x') = \hat{A}^{\mu}(x) - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{T}^{\rho\sigma})^{\mu}{}_{\nu}\hat{A}^{\nu}(x)$$

Thus:

$$[\hat{A}^{\mu}(x), \hat{J}^{\rho\sigma}] = \hat{L}^{\rho\sigma}\hat{A}^{\mu}(x) + (\mathcal{T}^{\rho\sigma})^{\mu}{}_{\nu}\hat{A}^{\nu}(x)$$

This suggests that the spatial component of $\hat{J}$ acts on $\hat{A}^{\mu}$ as the total angular momentum, both the orbital and the spin part. By study its universal covering as we discussed above, the spin momentum of a vector field is $s = 1$.

3.2 Interlude of quantization

We cite a profound argument by Prof. Dr. Tobias J. Osborne from Leibniz University Hannover:

Quantization is merely an educated guess.

What does it mean? In general, we believe that any classical system is described by a geometrical object. Classical mechanics is known as a geometrical physics theorem, which means that one could perceive the classical mechanics as purely an orthodox geometrical theory, i.e. one would study a manifold and some structures on it. The same argument may also holds for general relativity, but not for any orthogonal quantum theory (e.g. non-relativistic quantum mechanics and quantum field theory). In physics, sometimes we would like to classify a theory into classical if it is described by a geometrical object. It is well-known that general relativity is described by Riemann geometry, while classical mechanics is described by symplectic geometry.

Mathematically one could define the procedure of quantization strictly (well, pre-quantization exists in symplectic geometry anyway). However, when we discuss physical quantization, we do not refer to the same thing. Although it is well-known that canonical quantization (replacing the Poisson brackets by the commutators) could be regarded as a geometrical quantization of the classical system, and actually the path-integral quantization (replacing the propagators) could also be regarded as a kind of geometrical quantization. However, this is not how we define a physical quantization.

Def. 3.2 *Given a classical system $(M, \omega, \mathcal{F})$ where M is the space of states (always be a manifold), ω contains the dynamical informations (which describes how the system evolves, say, a symplectic form) and $\mathcal{F}$ is the set of classical observables, and a quantum system $(\mathcal{H}, \hat{T}, \mathcal{O})$, where $\mathcal{H}$ is the space of states (a Hilbert space), $\hat{T}$ describes how it evolves and $\mathcal{O}$ contains the quantum observables.*

The quantum system is called a quantization of the classical system if any object in the decoherent quantum system is exactly its counterpart in the very classical system.

If one is familiar with symplectic geometry, one might notice that such a definition is quite similar to Dirac's axiom of geometrical quantization, while here is two distinction. First, this is a physical, or say, philosophical postulate and is not a mathematical one. Specifically it is the consequence of physical observation. Second, the mathematical form of the quantization of a classical system is obviously not unique, they are physically equivalent as long as they give the same decoherence. Especially, canonical quantization and path-integral quantization of free fields are physical equivalent in a flat spacetime.

Inspired by the famous saying of Albert Einstein:

Education is that which remains, if one has forgotten everything he learned in school.

I personally raise my own definition of physics:

Physics is what keeps invariant, when the mathematical form varies.

Last but not least, I would like to cite a famous concept of Eugene Wigner:

I noticed the unreasonable effectiveness of mathematics in the natural Sciences.

Anyway, keep all the concepts in mind and we could talk about canonical quantization of classical solutions of field.

3.3 Massive vector fields

Consider a classical field in $(\mathbb{R}^{1,3}, g_{\mu\nu})$, which is a section of some bundles of the spacetime manifold $X \in \Gamma(\bigotimes T\mathbb{R}^{1,3})$, i.e. X is a map:

$$X : \mathbb{R}^{1,3} \to \bigotimes T\mathbb{R}^{1,3}, p \mapsto X_p \in \bigotimes T_p\mathbb{R}^{1,3}$$

Especially a Lorentz vector is an element in $\Gamma(T\mathbb{R}^{1,3})$, and a Lorentz tensor is an element in $\Gamma(\bigotimes^{2,0} T\mathbb{R}^{1,3})$. Given a chart, they are a set of covariant numbers in component.

Consider a vector field $A^a \in \Gamma(T\mathbb{R}^{1,3})$ with component A^μ. Covariance means:

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$$

It induces a 2-form by:

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu$$

Thus the according Lagrangian density is a Lorentz scalar:

$$\mathcal{L} := -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu$$

Euler-Lagrange equation donates the equation of motion:

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

which is called the Proca equation. For any massive field $m > 0$, the Lorenz condition holds:

$$\partial_\mu A^\mu = 0$$

Thus each components of A satisfies the Klein-Gordon equation. The conjugate momentum is:

$$\pi_\mu := \frac{\partial \mathcal{L}}{\partial \partial^0 A^\mu} = -F_{0\mu}$$

Notice that $\pi_0 = 0$ and the Lorenz condition holds, we consider the canonical quantization of the spatial component $A^i \to \hat{A}^i$. The according commutation relations are:

$$[\hat{A}^i(t,\underline{x}), \hat{\pi}_j(t,\underline{y})] = i\delta^i{}_j\delta^{(3)}(\underline{x}-\underline{y}), [\hat{A}^i(t,\underline{x}), \hat{A}^j(t,\underline{y})] = [\hat{\pi}_i(t,\underline{x}), \hat{\pi}_j(t,\underline{y})] = 0$$

And the Lorenz condition donates that:

$$\hat{A}^0 = -\frac{1}{m^2}\partial_i\hat{\pi}^i$$

The mathematical structure of the following analysis could be regarded as a consequence of representations of the Poincaré group, and the method is discussed above. The details seen in reference(1).

we consider how to solve the Hamiltonian. In classical theory, solving a system refers to find the physical motion function that the system obeys. This is equivalent to solve the classical equations of motion which are real partial differential equation. Now in a quantum system, one usually solves a system by means of diagonalize its Hamiltonian operator since it commutes with the evolution operator.

Consider the Fourier expansion of the classical solution of the EOM:

$$A^\mu(t,\underline{x}) = \int \frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}(C(\underline{p})\phi^\mu_{(+\underline{p})}(t,\underline{x}) + C^*(\underline{p})\phi^\mu_{(-\underline{p})}(t,\underline{x}))$$

where the positive energy solution $\phi^\mu_{(-\underline{p})}(t,\underline{x}))$ is the superposition of some discrete components, namely:

$$\psi^\mu(t,\underline{x};\underline{p},\sigma) = e^\mu(\underline{p},\sigma)e^{-ipx}$$

$\sigma = 0,1,2,3$ is the label of polarization state, and e^μ is a Lorentz vector called the polarization vector, which forms a basis of the space of Lorentz vectors. We demand that they satisfy the orthonormalization and completion condition:

$$e_\mu(\underline{p},\sigma)e^\mu(\underline{p},\sigma') = g_{\sigma\sigma'}$$

$$\sum_{\sigma=0,1,2,3} g_{\sigma\sigma}e_\mu(\underline{p},\sigma)e_\nu(\underline{p},\sigma) = g_{\mu\nu}$$

i.e. take an orthonormalized frame of the spacetime, in which any polarization vector could be decomposition as:

$$V^\mu = \sum_{\sigma=0,1,2,3} v_\sigma(\underline{p})e^\mu(\underline{p},\sigma)$$

By analysis of the canonical condition of the frame, one could take such a frame:

$$p^\mu e_\mu(\underline{p},i) = 0, i = 1,2,3$$

$$e_\mu(\underline{p},0) = \frac{p_\mu}{m}, p^\mu e_\mu(\underline{p},0) = m$$

For a massive vector field A^μ, it satisfies the transversality, thus it could only compose by $e^\mu(\underline{p},i)$, which satisfies the spacelike complete condition of:

$$\sum_{i=1,2,3} e_\mu(\underline{p},i)e_\nu(\underline{p},i) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}$$

Recall how Newman derived the Kerr-Newman solution of Einstein Equation with Newman-Penrose frame, we impose the similar process on the frame above. Consider the complexification of the spacelike components $\epsilon^\mu(\underline{p},\lambda), \lambda = +,0,-$.

$$\epsilon^\mu(\underline{p},\pm) := \frac{1}{\sqrt{2}}(e^\mu(\underline{p},1) \pm ie^\mu(\underline{p},2))$$

$$\epsilon^\mu(\underline{p},0) := e^\mu(\underline{p},3)$$

One may notice that this is precisely how Newman construct the rotation frame to derive Kerr-Newman solution. Similarly the orthonormalization and completeness condition holds:

$$\epsilon^*_\mu(\underline{p},\lambda)\epsilon^\mu(\underline{p},\lambda') = \delta_{\lambda\lambda'}$$

$$\sum_{\lambda=\pm,0} \epsilon^*_\mu(\underline{p},\lambda)\epsilon_\nu(\underline{p},\lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}$$

with ϵ^μ the Fourier expansion of a Vector field A^μ could be written as:

$$A^\mu(t,\underline{x}) = \int \frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}\sum_{\lambda=\pm,0}(\epsilon^\mu a_{\underline{p},\lambda}e^{-ipx} + \epsilon^{\mu *}a^*_{\underline{p},\lambda}e^{ipx})$$

Now apply canonical quantization:

$$\hat{A}^\mu(t,\underline{x}) = \int \frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}\sum_{\lambda=\pm,0}(\epsilon^\mu \hat{a}_{\underline{p},\lambda}e^{-ipx} + \epsilon^{\mu *}\hat{a}^\dagger_{\underline{p},\lambda}e^{ipx})$$

Take into the commutation relations:

$$[\hat{a}_{\underline{p},\lambda}, \hat{a}^\dagger_{\underline{q},\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\underline{p}-\underline{q}), [\hat{a}_{\underline{p},\lambda}, \hat{a}_{\underline{q},\lambda'}] = [\hat{a}^\dagger_{\underline{p},\lambda}, \hat{a}^\dagger_{\underline{q},\lambda'}] = 0$$

Classical Hamiltonian is:

$$\mathcal{H} := \pi_i\partial_0 A^i - \mathcal{L} = \frac{1}{2}(\underline{\pi}^2 + \frac{1}{m^2}(\nabla\cdot\underline{\pi})^2 + (\nabla\times\underline{A})^2 + m^2\underline{A}^2)$$

$$H := \int \mathrm{d}^3\underline{x}\mathcal{H} = \int \mathrm{d}^3\underline{x}\frac{1}{2}(\underline{\pi}^2 + \frac{1}{m^2}(\nabla\cdot\underline{\pi})^2 + (\nabla\times\underline{A})^2 + m^2\underline{A}^2)$$

Thus quantum Hamiltonian operator is:

$$\hat{H} = \sum_{\lambda=\pm,0}\int\frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}E_{\underline{p}}\hat{a}^\dagger_{\underline{p},\lambda}\hat{a}_{\underline{p},\lambda} + E_0, E_0 = \frac{3}{2}(2\pi)^3\delta^{(3)}(0)\int\frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}E_{\underline{p}}$$

Similarly momentum operator:

$$\hat{\underline{P}} = \sum_{\lambda=\pm,0}\int\frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}\hat{\underline{p}}\hat{a}^\dagger_{\underline{p},\lambda}\hat{a}_{\underline{p},\lambda}$$

3.4 Massless vector fields

We continue to the massless vector fields, which admits less degrees of freedom. Consider Lagrangian density of a massless vector field:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

The Equation of motion is called the Maxwell equation:

$$\partial_\mu F^{\mu\nu} = 0$$

Naively one may suppose that the degrees of freedom of A^μ is 4, but as we knew from electrodynamics, degrees of freedom of a photon is merely 2, since it admits the phase symmetry or $\mathrm{U}(1)$ symmetry, which is a local symmetry.

Consider the gauge transformation:

$$A'^\mu(x) = A^\mu(x) + \partial^\mu\chi(x)$$

where χ is a Lorentz scalar. Notice that:

$$F'^{\mu\nu}(x) = F^{\mu\nu}(x)$$

In classical electrodynamics, A^μ itself is not observable while electromagnetic fields E^i and B^i are. If one wants to calculate the fields with A^μ, one needs to impose the gauge fixing.

e.g. 3.1 (Lorenz gauge)

$$\partial_\mu A^\mu = 0$$

e.g. 3.2 (Coulomb gauge)

$$\nabla\cdot\underline{A} = 0$$

e.g. 3.3 (temporal gauge)

$$A^0 = 0$$

One could introduce a auxiliary field ξ to complete the gauge fixing, i.e. to regard ξ as a dynamical field with Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

Equation of motion of ξ is:

$$\frac{1}{2\xi^2}(\partial_\mu A^\mu)^2 = 0, \partial_\mu A^\mu = 0$$

which implies the condition of Lorenz gauge. Equation of motion of A^μ is:

$$\partial^2 A^\mu - (1 - \frac{1}{\xi})\partial^\mu \partial_\nu A^\nu = 0$$

For a classical field with Lorenz gauge, these equations describe its motion, while it is not suitable for quantization. After-all, one could demand that $\xi = 1$ and abandon ξ from being a dynamical variable, in which the Lorenz gauge would not hold. This is called the Feynman gauge. The new Lagrangian density is written as:

$$\tilde{\mathcal{L}} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu(A_\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\nu)$$

which is equivalent to:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$$

Equation of motion is called the d'Alembert equation:

$$\partial^2 A^\mu = 0$$

Canonical momentum is:

$$\pi_\mu = -\partial_0 A_\mu$$

Similarly to the massive field, the Fourier expansion is:

$$A^\mu(t, \underline{x}) = \int \frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}\sum_{\sigma=0,1,2,3} e^\mu(\underline{p},\sigma)(b_{\underline{p},\sigma}e^{-ipx} + b^*_{\underline{p},\sigma}e^{ipx})$$

Apply canonical quantization:

$$\hat{A}^\mu(t, \underline{x}) = \int \frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}\sum_{\sigma=0,1,2,3} e^\mu(\underline{p},\sigma)(\hat{b}_{\underline{p},\sigma}e^{-ipx} + \hat{b}^\dagger_{\underline{p},\sigma}e^{ipx})$$

With commutation relations:

$$[\hat{A}^\mu(t,\underline{x}), \hat{\pi}_\nu(t,\underline{y})] = i\delta^\mu{}_\nu \delta^{(3)}(\underline{x}-\underline{y}), [\hat{A}^\mu(t,\underline{x}), \hat{A}^\nu(t,\underline{y})] = [\hat{\pi}_\mu(t,\underline{x}), \hat{\pi}_\nu(t,\underline{y})] = 0$$

Commutation relations of creation and annihilation operators are:

$$[\hat{b}_{\underline{p},\sigma}, \hat{b}^\dagger_{\underline{q},\sigma'}] = -(2\pi)^3\delta_{\sigma\sigma'}\delta^{(3)}(\underline{p}-\underline{q}), [\hat{b}_{\underline{p},\sigma}, \hat{b}_{\underline{q},\sigma'}] = [\hat{b}^\dagger_{\underline{p},\sigma}, \hat{b}^\dagger_{\underline{q},\sigma'}] = 0$$

Hamiltonian operator is:

$$\hat{H} = -\sum_{\lambda=\pm,0}\int \frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}E_{\underline{p}}\sum_{\sigma,\sigma'} g_{\sigma\sigma'}\hat{b}^\dagger_{\underline{p},\sigma}\hat{b}_{\underline{p},\sigma'} + E_0, E_0 = 2(2\pi)^3\delta^{(3)}(0)\int \frac{\mathrm{d}^3\underline{p}}{(2\pi)^3}E_{\underline{p}}$$

Thus we raised the theory of free quantum vector fields, it describes the massive and massless bosons by analysis its particle states with the discussion in the initial chapter.

3.5 Feynman propagators

We end with the Feynman propagators of quantum vector fields. Expansion of interaction field operators shares the same form with the free field.

For a massive vector field, its Feynman propagator is defined by:

$$\Delta_F^{\mu\nu}(x-y) := \overbrace{\hat{A}^\mu(x)\hat{A}^\nu}(y) = \langle 0| \mathcal{T}[\hat{A}^\mu(x)\hat{A}^\nu(y)] |0\rangle$$

There is some subtle issue about the divergent behaviour, in which we could not insist to be rigorous. Anyway, one could believe that after-all:

$$\Delta_F^{\mu\nu}(x-y) \to \int \frac{\mathrm{d}^4\mathrm{p}}{(2\pi)^4} \frac{-i(g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2})}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

And in the case of massless vector field with Feynman gauge:

$$\Delta_F^{\mu\nu}(x-y) \to \int \frac{\mathrm{d}^4\mathrm{p}}{(2\pi)^4} \frac{-ig^{\mu\nu}}{p^2 + i\epsilon} e^{-ip(x-y)}$$

The according discussion will be in quantum electrodynamics, which we wouldn't show here. Finally, we could conclude that:

And QED said, Let there be light: and there was light.

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