Topic 23

Floating-Point Representation

Normalization

Base 10:

$$26.73_{10} = 2 \times 10^{1} + 6 \times 10^{0} + 7 \times 10^{-1} + 3 \times 10^{-2}$$

$$26.73_{10} = 20 + 6 + \frac{7}{10} + \frac{3}{100}$$

Normalized:

$$2.673_{10} \times 10^{1}$$

Base 2:

$$8.25_{10}$$

Normalized:

$$1000.01_{2} = 1.00001_{2} \times 2^{3}$$

Floating Point

Representation for non-integer numbers

Including very small and very large numbers

Like scientific notation

-2.34×10^{56} — Normalized

$+0.002 \times 10^{-4}$

$+987.02 \times 10^{9}$ — Not normalized

In binary

$\pm 1.xxxxxxx_2 \times 2^{yyyy}$

Types float and double in C

Floating Point Standard

Defined by IEEE Std 754-1985

Developed in response to divergence of representations

- Portability issues for scientific code

Now almost universally adopted

Two representations

- Single precision (32-bit)
- Double precision (64-bit)

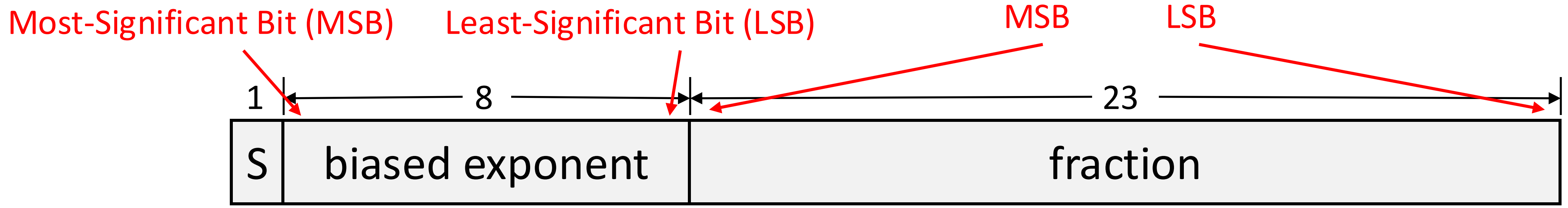
Floating Point Representation

The IEEE Std 754-1985 32-bit floating point representation uses:

1 bit for the sign (positive or negative)

8 bits for the range (exponent field)

23 bits for the precision (fraction field)



$N = \{($

Floating Point Representation (example)

1	8	23
S	biased exponent	fraction

$$N = \begin{cases} (-1)^S \times 1.fraction \times 2^{biasedexponent-127}, & 1 \le biasedexponent \le 254 \\ (-1)^S \times 0.fraction \times 2^{-126}, & biasedexponent = 0 \end{cases}$$

How is the number $-6\frac{5}{8}$ represented in floating point?

$$-6\frac{5}{8} = -\left(4 + 2 + \frac{4}{8} + \frac{1}{8}\right) = -\left(4 + 2 + \frac{1}{2} + \frac{1}{8}\right)$$

$$-(1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3})$$

$$-(110.101_2) = -(1.10101_2 \times 2^2)$$

The biasedexponent is given by: biasedexponent = actualexponent + 127

biasedexponent = 2 + 127 = 129

1 10000001 10101000000000000000000

Floating Point Representation (example)

1	8	23
S	biasedexponent	fraction

$$N = \begin{cases} (-1)^S \times 1.fraction \times 2^{biasedexponent-127}, & 1 \le biasedexponent \le 254 \\ (-1)^S \times 0.fraction \times 2^{-126}, & biasedexponent = 0 \end{cases}$$

What is the decimal value of the following floating-point number?

$$0\underbrace{011\ 1101\ 1}_{\text{biasedexponent}}000\ 0000\ 0000\ 0000\ 0000\ 0000$$

$$biasedexponent = 2^6 + 2^5 + 2^4 + 2^3 + 2^1 + 2^0 = 64 + 32 + 16 + 8 + 2 + 1 = 123$$

$$biasedexponent = 2^7 - 2^3 + 2^1 + 2^0 = 128 - 8 + 3 = 123$$

$$actualexponent = biasedexponent - 127 = 123 - 127 = -4$$

$$N = (-1)^0 \times 1.0_2 \times 2^{123-127} = 1.0 \times 2^{-4} = \frac{1}{16}$$

Answering Dylan Ashley's Question (Fall 2014)

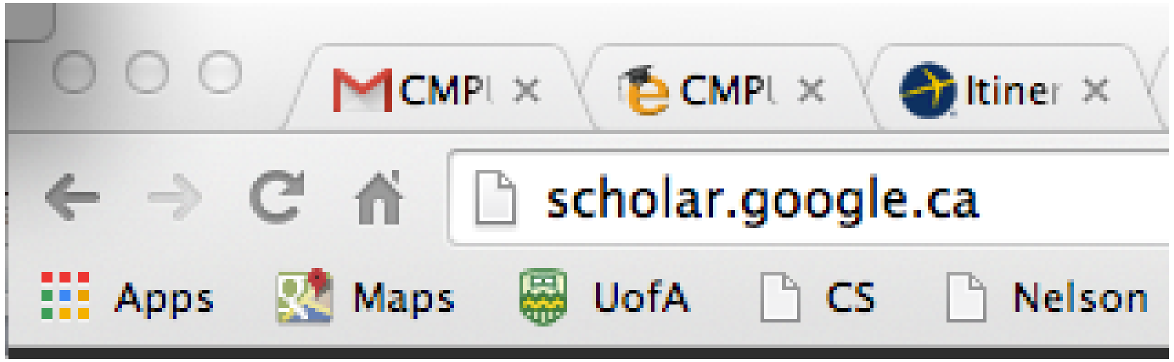
Dylan Ashley (He/Him) · 2nd

PhD student studying reinforcement learning with Jürgen Schmidhuber. MSc with Richard Sutton at the University of Alberta. Sometimes amateur photographer.

Lugano, Ticino, Switzerland · Contact info

Why does the IEEE standard use the bias format for the exponent instead of a two's-complement representation?

How to Find the Answer



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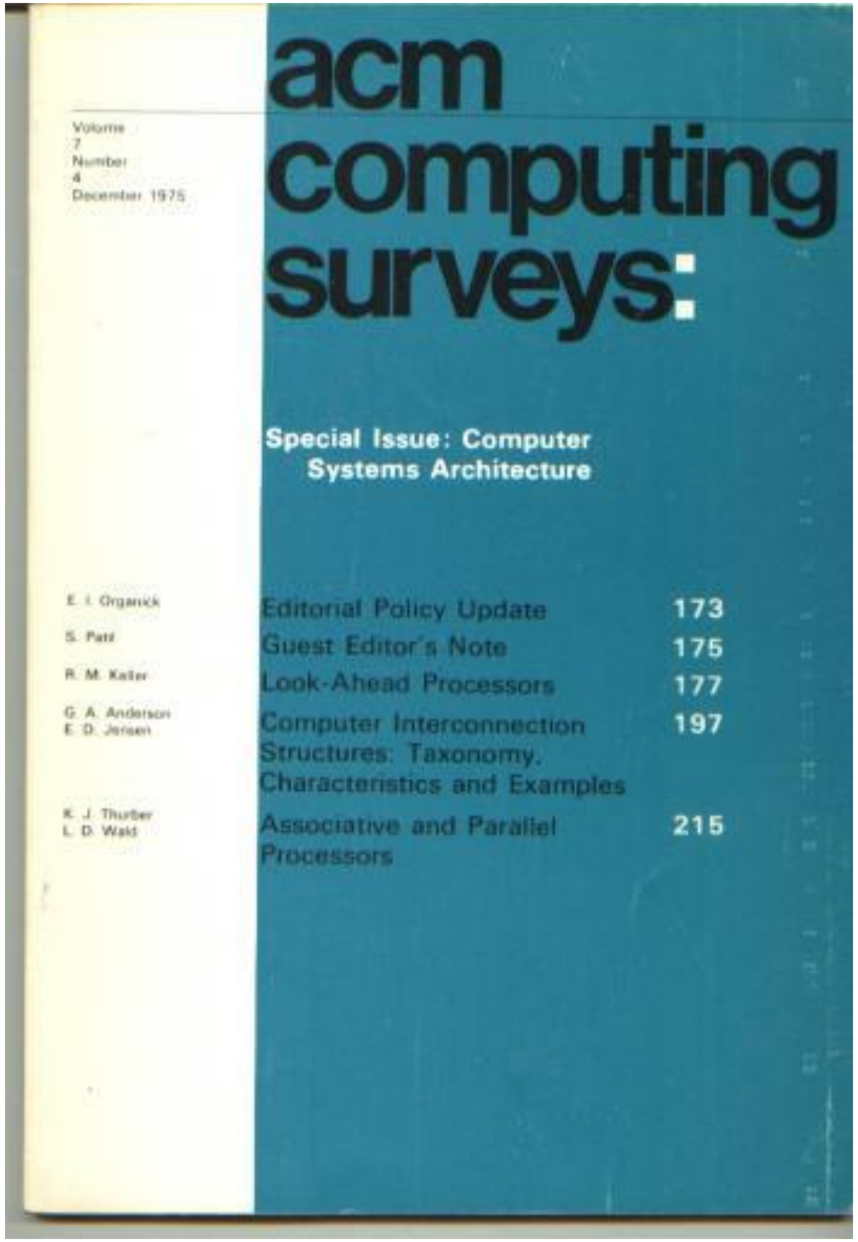

ACM Computing Surveys, Vol 23, No 1, March 1991

What Every Computer Scientist Should Know About Floating-Point Arithmetic

DAVID GOLDBERG

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Floating-point arithmetic is considered an esotoric subject by many people. This is rather surprising, because floating-point is ubiquitous in computer systems: Almost every language has a floating-point datatype; computers from PCs to supercomputers have floating-point accelerators; most compilers will be called upon to compile floating-point algorithms from time to time; and virtually every operating system must respond to floating-point exceptions such as overflow This paper presents a tutorial on the aspects of floating-point that have a direct impact on designers of computer systems. It begins with background on floating-point representation and rounding error, continues with a discussion of the IEEE floating-point standard, and concludes with examples of how computer system builders can better support floating point.

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   Example: “bias”

representation. In the case of single precision, where the exponent is stored in 8 bits, the bias is 127 (for double precision it is 1023). What this means is that if $\overline{k}$ is the value of the exponent bits interpreted as an unsigned integer, then the exponent of the floating-point number is $\overline{k} - 127$. This is often called the *biased exponent* to distinguish from the unbiased exponent $\overline{k}$. An advantage of biased representation is that nonnegative flouting-point numbers can be treated as integers for comparison purposes.

The Problem (example)

Compare 2.5 and 0.25. Which is larger?

$$2.5 = 2^1 + 2^{-1} = 10.1 \times 2^0 = 1.01 \times 2^1$$

$$0.25 = 2^{-2} = 0.01 \times 2^0 = 1.0 \times 2^{-2}$$

Representation with exponent in two's-complement format

```
2.5 = 0 00000001 01000000000000000000000
 -2 = 11111110
0.25 = 0 11111110 00000000000000000000000
```

Representation with exponent in bias format

biasedexponent – 127 = 1 → biasedexponent = 128

```
2.5 = 0 10000000 01000000000000000000000
```

biasedexponent – 127 = -2 → biasedexponent = 125

```
0.25 = 0 01111101 00000000000000000000000
```

The Problem (example)

Compare 2.5 and 0.25. Which is larger?

$$2.5 = 2^1 + 2^{-1} = 10.1 \times 2^0 = 1.01 \times 2^1$$

$$0.25 = 2^{-2} = 0.01 \times 2^0 = 1.0 \times 2^{-2}$$

Representation with exponent in two's-complement format

```
 2.5 = 0 00000001 01000000000000000000000
```

Using integer comparison: 2.5 < 0.25 ✗

```
0.25 = 0 11111110 00000000000000000000000
```

Representation with exponent in biased format

```
 2.5 = 0 10000000 01000000000000000000000
```

Using integer comparison: 0.25 < 2.5 ✓

```
0.25 = 0 01111101 00000000000000000000000
```

Floating Point Representation (example)

1	8	23
S	biasedexponent	fraction

$$N = \begin{cases} (-1)^S \times 1.fraction \times 2^{biasedexponent-127}, & 1 \le biasedexponent \le 254 \\ (-1)^S \times 0.fraction \times 2^{-126}, & biasedexponent = 0 \end{cases}$$

What is the decimal value of the following floating-point number?

$$0\underbrace{100\ 0001\ 1}_{\text{biasedexponent}}001\ 0100\ 0000\ 0000\ 0000\ 0000$$

$$biasedexponent = 2^7 + 2^1 + 2^0 = 128 + 2 + 1 = 131$$

$$actualexponent = biasedexponent - 127 = 131 - 127 = 4$$

$$N = (-1)^0 \times 1.00101_2 \times 2^{131-127} = 1.00101_2 \times 2^4 = 10010.1_2$$

$$N = 2^4 + 2^1 + 2^{-1} = 16 + 2 + \frac{1}{2} = 18.5$$

Floating Point Representation (example)

1	8	23
S	biasedexponent	fraction

$$N = \begin{cases} (-1)^S \times 1.fraction \times 2^{biasedexponent-127}, & 1 \le biasedexponent \le 254 \\ (-1)^S \times 0.fraction \times 2^{-126}, & biasedexponent = 0 \end{cases}$$

What is the largest number that can be represented in 32-bit floating point using the IEEE 754 format above?

$$\mathtt{0}\underbrace{\mathtt{111\ 1111\ 0}}_{\text{biasedexponent}}\mathtt{111\ 1111\ 1111\ 1111\ 1111\ 1111}$$

$$biasedexponent = 2^8 - 2^1 = 256 - 2 = 254$$

$$fraction = 1 \times 2^{-1} + 1 \times 2^{-2} + \cdots + 1 \times 2^{-22} + 1 \times 2^{-23}$$

$$fraction = 1 \times 2^0 - 1 \times 2^{-23} = 1 - \frac{1}{2^{23}} = 1 - \frac{1}{1024 \times 1024 \times 8} = 0.99999988079$$

Floating Point Representation (example)

1	8	23
S	biasedexponent	fraction

$$N = \begin{cases} (-1)^S \times 1.fraction \times 2^{biasedexponent-127}, & 1 \leq biasedexponent \leq 254 \\ (-1)^S \times 0.fraction \times 2^{-126}, & biasedexponent = 0 \end{cases}$$

What is the largest number that can be represented in 32-bit floating point using the IEEE 754 format above?

$$0\underbrace{111\ 1111\ 0}_{\text{biasedexponent}}111\ 1111\ 1111\ 1111\ 1111\ 1111$$

$$actualexponent = 254 - 127 = 127$$

$$fraction = 0.99999988079$$

$$N = (-1)^0 \times 1.99999988079_{10} \times 2^{127} \approx 2^{128}$$

Floating Point Representation (example)

1	8	23
S	biasedexponent	fraction

$$N = \begin{cases} (-1)^S \times 1.fraction \times 2^{biasedexponent-127}, & 1 \le biasedexponent \le 254 \\ (-1)^S \times 0.fraction \times 2^{-126}, & biasedexponent = 0 \end{cases}$$

What is the smallest non-negative number that can be represented in 32-bit floating point using the IEEE 754 format above?

$$0\underbrace{000\ 0000\ 0}_{biasedexponent}000\ 0000\ 0000\ 0000\ 0000\ 0001$$

$$actualexponent = 0 - 126 = -126$$ - note the special rule above for biasedexponent = 0

$$fraction = 1 \times 2^{-23}$$

$$N = (-1)^0 \times 2^{-23} \times 2^{-126} \approx 2^{-149}$$

Special Floating Point Representation

The 8-bit field of the biasedexponent can represent numbers from 0 to 255

What is the value represented when the exponent is 255 (ie. 11111111_2)?

biasedexponent = 255 = 11111111_2 indicates a special value

- biasedexponent = 255 and fraction = 0:
  - the value represented is ±infinity
- biasedexponent = 255 and fraction ≠ 0:
  - the value represented is Not a Number (NaN)

Double Precision

32-bit floating-point representation is usually called single-precision representation

A double-precision floating-point representation requires 64 bits:

- 1 sign bit
- 11 bits for exponent
- 52 bits for fraction (also called significand)

A half-precision floating-point representation requires 16 bits:

- 1 sign bit
- 5 bits for exponent
- 10 bits for fraction (also called significand)

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Floating-Point Formats in the World of Machine Learning

Sept. 9, 2022

Different floating-point formats allow machine-learning systems to operate more efficiently and use less space.

Cabe Atwell

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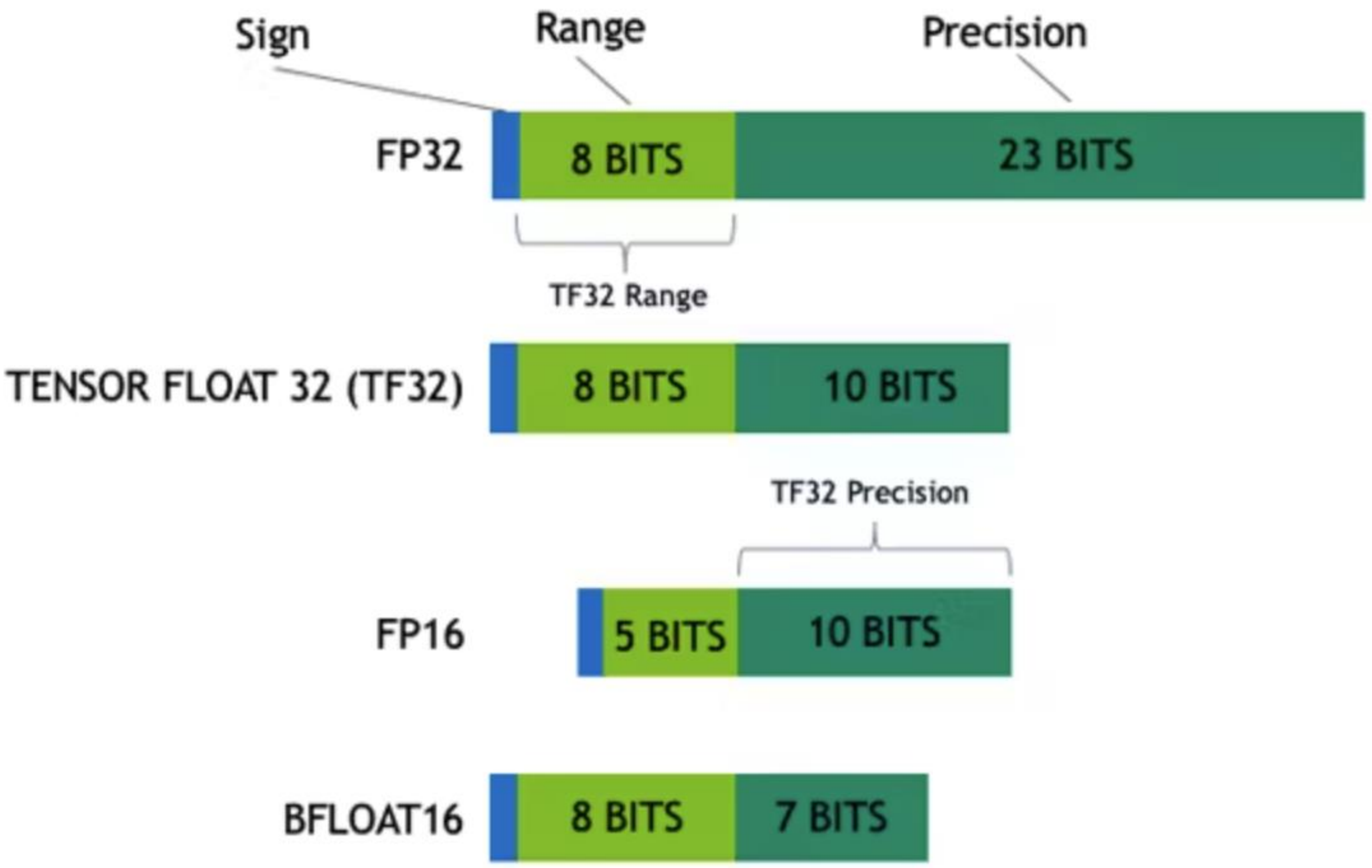
Floating Point Formats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FP8 FORMATS FOR DEEP LEARNING

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Abstract

FP8 is a natural progression for accelerating deep learning training inference beyond the 16-bit formats common in modern processors. In this paper we propose an 8-bit floating point (FP8) binary interchange format consisting of two encodings - E4M3 (4-bit exponent and 3-bit mantissa) and E5M2 (5-bit exponent and 2-bit mantissa). While E5M2 follows IEEE 754 conventions for representatio of special values, E4M3's dynamic range is extended by not representing infinities and having only one mantissa bit-pattern for NaNs. We demonstrate the efficacy of the FP8 format on a variety of image and language tasks, effectively matching the result quality achieved by 16-bit training sessions. Our study covers the main modern neural network architectures - CNNs, RNNs, and Transformer-based models, leaving all the hyperparameters unchanged from the 16-bit baseline training sessions. Our training experiments include large, up to 175B parameter, language models. We also examine FP8 post-training-quantization of language models trained using 16-bit formats that resisted fixed point int8 quantization.

Table 1: Details of FP8 Binary Formats

	E4M3	E5M2
Exponent bias	7	15