微分流形初步作业

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目录

第一章　预备知识

啦啦啦啦啦 **1.1** 证:令 $V = \mathbb{R}^n$, $A = \mathbb{R}^n$, 则 $\forall x$, $y \in A$, 令 $\overrightarrow{xy} = y - x$ 则有:

① w $\forall\, x \in A$, $\overrightarrow{xx} = x - x = 0$,

② $\forall\, x \in A$, $v \in V$, 令 $y = x + v \in V$. 则 $\overrightarrow{xy} = y - x = v$. 若另有 y', $s.\,t.\ \overrightarrow{xy'} = v$, 则 $y' - x = v \Rightarrow y' = x + v = y$ 从而 y 唯一,

③ $\forall\, x, y, z \in A$. $\overrightarrow{xy} + \overrightarrow{yz} = (y - x) + (z - y) = z - y = \overrightarrow{yz}$.

所以 $\mathbb{R}^n$ 是一个n维仿射空间, 它以 $\mathbb{R}^n$ 自身为它的伴随向量空间。

1.2 证:

① $\forall\, P, Q \in E^n$, $0 \leqslant d(P, Q) = \left|\overrightarrow{PQ}\right| < +\infty$, 且 $d(P, Q) = 0 \Leftrightarrow \left|\overrightarrow{PQ}\right| = 0 \Leftrightarrow P = Q$,

② $\forall\, P, Q \in E^n$, $d(P, Q) = \left|\overrightarrow{PQ}\right| = \left|\overrightarrow{QP}\right| = d(Q, P)$,

③ $\forall\, P, Q, R \in E^n$, $d(P, R) = \left|\overrightarrow{PR}\right| = \left|\overrightarrow{PQ} + \overrightarrow{QR}\right| \leqslant \left|\overrightarrow{PQ}\right| + \left|\overrightarrow{QR}\right| \leqslant d(Q, P)$.

从而 E^n 关于距离函数 d 成为一个度量。

1.3 证:

(1) 记 E^n 的全体开子集为 τ,

① 显然 $\emptyset$, $E^n \in \tau$,

② $\forall\, A \in \tau$, $B \in \tau$, 若 $A \cap B = \emptyset$, 则 $A \cap B \in \tau$,
若 $A \cap B \neq \emptyset$, 则 $\forall\, P \in A \cap B$, 即 $P \in A$ 且 $P \in B$,
则 $\exists\, \varepsilon_1$, $\varepsilon_2 > 0$, $s.\,t.\ P \in B_{\varepsilon_1}(P) \subset A$, $P \in B_{\varepsilon_2}(P) \subset B$, 取 $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$, 则 $P \in B_\varepsilon(P) = B_{\varepsilon_1}(P) \cap B_{\varepsilon_2}(P) \subset A \cap B$, 因而 $A \cap B \in \tau$.

③ 若 $A_\alpha(\alpha \in I) \in \tau$, 则 $\forall\, P \in \bigcup\limits_{\alpha \in I} A_\alpha$, $\exists\, i \in I$, $s.\,t.\ P \in A_i \in \tau$,
则 $\exists\, \varepsilon > 0$, $s.\,t.\ P \in B_\varepsilon(P) \in A_i$, 从而 $B_\epsilon(P) \subset \bigcup\limits_{\alpha \in I} A_\alpha$, 从而 $\bigcup\limits_{\alpha \in I} A_\alpha \in \tau$.

所以, τ 为 E^n 的一个拓扑。

(2) $\forall\, P,Q \in E$ 且 $P \neq Q$. 则记 $d = d(P,Q)$, 取 $\varepsilon_1 = \varepsilon_2 = \frac{d}{3}$, 则 $P \in B_{\varepsilon_1(P)}$（开）,
$Q \in B_{\varepsilon_2}(Q)$（开）, 且 $B_{\varepsilon_1}(P) \cup B_{\varepsilon_2}(Q) = \emptyset$,
否则, 若 $\exists\, R \in B_{\varepsilon_1}(P) \cap B_{\varepsilon_2}(Q)$, 则 $d(Q,R) < \frac{d}{3}, d(Q,R) < \frac{d}{3}$,
$d = d(P,Q) \leq d(P,R) + d(Q,R) < \frac{2d}{3} < d$ 矛盾！ 所以 $B_{\varepsilon_1}(P) \cup B_{\varepsilon_2}(Q) = \emptyset$ 成立.
从而, E_n 满足 T_2 分离性公理, 为 Hausdorff 空间.

(3) 取开集族 $\mathscr{B} = \{B_\varepsilon(P) | P \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}\}$, 其中 $\mathbb{Q}$ 为有理数, 故 $\mathscr{B}$ 为可数的, 下证其为
拓扑基：
$\forall\, P \in E_n, \quad \forall\, U = B_\varepsilon(P) \in N(P)$,
$\exists\, \varepsilon' > 0$, 且 $\varepsilon' \in \mathbb{Q}$, s. t. $\frac{3\varepsilon}{4} < \varepsilon' < \varepsilon$. $\exists\, Q \in \mathbb{Q}^n$, s. t. $\left|\overrightarrow{PQ}\right| < \frac{\varepsilon'}{4}$,
令 $B = B_{\varepsilon'}(Q)$, 则 $B \subset U \in$ 且 $B \in \mathscr{B}$.
所以, $\mathscr{B}$ 为 E^n 中可数拓扑基, 从而 E^n 第二可数。

1.4 证:

(1) 任取 E_n 中直线 l, 在 l 上依次任取3个不同的点 P,Q,R, 则有 $\left|\overrightarrow{OQ}\right| = t\left|\overrightarrow{OP}\right| + \left|\overrightarrow{OR}\right|$, 其
中 $t \in (0,1)$. 记 $\sigma(P) = P' \quad (\forall\, P \in E_n)$
则 $\left|\overrightarrow{O'Q'}\right| = \left|\overrightarrow{OQ}\right| = t\left|\overrightarrow{OP}\right| + (1-t)\left|\overrightarrow{OR}\right| = t\left|\overrightarrow{O'P'}\right| + (1-t)\left|\overrightarrow{O'R'}\right| \quad (t \in (0,1))$
$\therefore\ P',Q',R'$ 三点共线且保持分比, 所以 σ 将直线映为直线.

(2) 任取 E_n 中两平行直线 l_1, l_2, 则由（1）知 l_1, l_2 在 σ 下仍为直线, 记为 l_1', l_2'. 任取不同
点 $A,B \in l_1$, 不同点 $C,D \in l_2$,
则 $\overrightarrow{AB}, \overrightarrow{CD}$ 非零, 且 $\overrightarrow{AB}//\overrightarrow{CD}$ 从而 $\exists\, \lambda \neq 0$, s. t. $\overrightarrow{AB} = \lambda\,\overrightarrow{CD}$,
而 $\left|\overrightarrow{A'B'}\right| = \left|\overrightarrow{AB}\right| = |\lambda| \cdot \left|\overrightarrow{CD}\right| = |\lambda| \cdot \left|\overrightarrow{C'D'}\right| \quad \therefore\ \overrightarrow{A'B'}//\overrightarrow{C'D'} \quad \therefore\ l_1'//l_2'$
$\therefore$ 由 l_1, l_2 任意性知 σ 把 E_n 中平行直线映为平行直线.

(3) 记 $\sigma(O) = O', \overrightarrow{OP_i} = \delta_i, \sigma(\delta_i) = \delta_i' = \overrightarrow{O'P'}, i = 1,2,...n$
则由 $\{O, \delta_i\}$ 为正交标架知

$$\left|\overrightarrow{OP_i}\right| \cdot \left|\overrightarrow{OP_j}\right| = \left\{ \begin{array}{ll} 0, & i \neq j, \\ 1, & i = j, \end{array} \right. \quad i,j = 1,2,...,n$$

$\therefore\ \{O', \delta_i\}$ 也为正交坐标系.

1.5 证: $\forall\, t \leq 0$. $\forall\, x = (x_1,...,x_n),\ y = (y_1,...,y_n) \in \mathbb{R}^n$.
由 $d(x,y) = \sqrt{\sum\limits_{i=1}^{n}(y^i - x^i)^2}$ 知 $d(tx,ty) = td(x,y)$.
又由 σ 为等距变换知 $d(\sigma(tx),\sigma(ty)) = d(tx,ty) = td(x,y) = td(\sigma(x),\sigma(y))$.
取 $y = 0 = (0,...,0)$, 则 $d(\sigma(tx),\sigma(0)) = t \cdot d(\sigma(x),\sigma(0))$,
则由 σ 保持共线性质而 $\overrightarrow{Ox}$ 与 $\overrightarrow{O(tx)}$ 共线, 知 $\overrightarrow{\sigma(0)\sigma(x)}$ 与 $\overrightarrow{\sigma(0)\sigma(tx)}$ 共线
$\therefore\ \overrightarrow{\sigma(0)\sigma(tx)} = \pm t\overrightarrow{\sigma(0)\sigma(x)}$ 不妨取 $+t, (-t$ 同理可证), 则有
$\sigma(tx) - \sigma(0) = t(\sigma(x) - \theta(0)) \Rightarrow \sigma(tx) = t\sigma(x) - (1-t)\sigma(0) \cdots\cdots\cdots (1),$

对（1）式左右两边关于 t 求导得：

$$左边 = \frac{\partial}{\partial t}\sigma_1(tx)$$

$$= (\frac{\partial}{\partial t}\sigma_1(tx),\ ...\ ,\frac{\partial}{\partial t}\sigma_n(tx))$$

$$= (x_1,...,\ x_n)\cdot\begin{pmatrix} \partial_1\sigma_1(tx) & \cdots & \partial_1\sigma_n(tx) \\ \vdots & & \vdots \\ \partial_n\sigma_1(tx) & \cdots & \partial_n\sigma_n(tx) \end{pmatrix}$$

$$左边 = \sigma(x) - \sigma(0)$$

则左边=右边，且令 t=0 后有：

$$\sigma(x) = \sigma(0) + (x_1,...,\ x_n)\cdot\begin{pmatrix} \partial_1\sigma_1(tx) & \cdots & \partial_1\sigma_n(tx) \\ \vdots & & \vdots \\ \partial_n\sigma_1(tx) & \cdots & \partial_n\sigma_n(tx) \end{pmatrix}\Bigg|_{t=0}$$

记 $a_0^j = \sigma_i(0)$ $\quad a_i^j = (\partial_i\sigma_j)|_{(t=0)}$ $\quad$ 其中 $i = 1,2,...,n.$ $\;j = 1,2,...,n.$ 有：

$$\sigma(x_1,...x_n) = (a_0^1,...,a_0^n) + (x_1,...,x_n)\cdot\begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \vdots & & \vdots \\ a_n^1 & \cdots & a_n^n \end{pmatrix}$$

取 $\varepsilon_i = (0,...,\ \overset{第i个}{\underset{\downarrow}{1}}\ ,0,...0)$, 则，

$1 = d(\varepsilon_i,0) = d(\sigma(\varepsilon_i),\sigma(0)) \Rightarrow (a_i^1)^2 + (a_i^2)^2 + \cdots + (a_i^n)^2 = 1 \ (\forall\ i = 1,2,...,n)$

$(i \neq j\ 时)$

$$\begin{aligned} 2 = 1^2 + 1^2 &= d^2(\varepsilon_i,\varepsilon_j) \\ &= d^2(\sigma(\varepsilon_i),\sigma(\varepsilon_j)) \\ &= (a_j^1 - a_i^1)^2 + \cdots + (a_j^n - a_i^n)^2 \\ &= ((a_i^1)^2 + \cdots + (a_i^n)^2) + ((a_j^1)^2 + \cdots + (a_j^n)^2) - 2(a_j^1 a_i^1 + \cdots + a_j^n a_i^n) \\ &= 1 + 1 - 2(a_j^1 a_i^1 + \cdots + a_j^n a_i^n) \end{aligned}$$

$\Rightarrow a_j^1 a_i^1 + \cdots + a_j^n a_i^n = 0$ （ $i \neq j\ 时$ ）

从而 $(a_i^j)_{n\times n}$ 为单位正交矩阵.

1.6 证: 设 Q 关于 $\{O;\delta_i\}$ 的坐标为: $x = (x^1,...\ ,x^n)$, 即 $Q - O = x^1\delta_1 + \cdots + x^n\delta_n$.

由第5题知 σ 为线性变换，而 $\sigma(O) = P,\ \sigma(\delta_i) = e_i$,

$\therefore\ \sigma(Q) - \sigma(O) = \sigma(Q - O) = \sigma(x^1\delta_1 + \cdots + x^n\delta_n) = x^1\sigma(\delta_1) + \cdots + x^n\sigma(\delta_n) = x^1 e_1 + \cdots + x^n e_n$

即点 Q' 关于 $\{P;e_i\}$ 的坐标等于点 Q 关于 $\{O;\delta_i\}$ 的坐标.

1.7 证:设 $\{O, \delta_i\}$ 为 E_n 中某一直角坐标系，f 在 $\{O, \delta_i\}$ 之下表示成

$$\overrightarrow{Of(t)} = \sum_{i=1}^{n} x^i(t)\delta_i, \ \forall \ t \in \mathbb{R}$$

$$= (\delta_1, ..., \delta_n) \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix}$$

任取另一直角坐标系 $\{P, e_i\}$ 则有唯一表示：

$$\overrightarrow{OP} = \sum_{i=1}^{n} a^i \delta_i = (\delta_1, ..., \delta_n) \begin{pmatrix} a^1(t) \\ \vdots \\ a^n(t) \end{pmatrix} \qquad e_i = \sum_{i=1}^{n} a^i \delta_i = (\delta_1, ..., \delta_n) \begin{pmatrix} a_i^1 \\ \vdots \\ a_i^n \end{pmatrix} \qquad (i = 1, 2, ...n)$$

从而

$$(e_1, ...e_n) = (\delta_1, ...\delta_n) \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_n^n \end{pmatrix}$$

记 $(a_j^i) = A$. 由 $(e_1, ...e_n)$ 与 $(\delta_1, ..., \delta_n)$ 均为正交向量组知 $|A| \neq 0$.
$\therefore A$ 可逆。记 $A^{-1} = (b_j^i)_{n \times n}$ $\therefore (\delta_1, ..., \delta_n) = (e_1, ...e_n)A^{-1}$

$$\overrightarrow{Of(t)} = (\delta_1, ..., \delta_n) \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix} = (e_1, ...e_n)A^{-1} \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix}$$

$$\overrightarrow{OP} = (\delta_1, ..., \delta_n) \begin{pmatrix} a^1(t) \\ \vdots \\ a^n(t) \end{pmatrix} = (e_1, ...e_n)A^{-1} \begin{pmatrix} a^1(t) \\ \vdots \\ a^n(t) \end{pmatrix}$$

而 $\overrightarrow{Of(t)} = \overrightarrow{OP} + \overrightarrow{Pf(t)}$ 从而

$$\overrightarrow{Pf(t)} = \overrightarrow{Of(t)} - \overrightarrow{OP}$$

$$= (e_1, ...e_n)A^{-1} \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix} - (e_1, ...e_n)A^{-1} \begin{pmatrix} a^1(t) \\ \vdots \\ a^n(t) \end{pmatrix}$$

$$= (e_1, ...e_n)A^{-1} \begin{pmatrix} x^1(t) - a^1(t) \\ \vdots \\ x^n(t) - a^n(t) \end{pmatrix}$$

$$\triangleq \sum_{k=1}^{n} e_k y^k(t) \qquad \text{其中 } y^k(t) = \sum_{j=1}^{n} b_j^k(x^j(t) - a^j(t)) \quad (k = 1, 2, ..., n)$$

由 $x^j(t)$ 连续（或r次连续可微）可得 $y^k(t)$ 连续（或r次连续可微），从而映射 $f : \mathbb{R} \to E^n$ 的连续性和r次连续可微性与 E^n 中直角坐标系的选取无关。

1.8 证: 记 $f(t)$ 在 $\{O;\delta_i\}$ 与 $\{P;e_i\}$ 下的坐标为 $(x^1(t),...,x^n(t)),(y^1(t),...,y^n(t))$, 其中

$$\overrightarrow{OP} = \sum_{i=1}^{n} a_i\delta_i, \qquad e_i = \sum_{j=1}^{n} a_i^j\delta_j \quad (i = 1,...n)$$

由的（1.14）

$$x^i(t) = a^i + \sum_{j=1}^{n} y^j(t)a_j^i$$

则等式两边同时对t求导有

$$\frac{dx^i}{dt}(t_0) = \sum_{j=1}^{n} \frac{dy^j(t)}{dt}a_j^i$$

又由（2.6）

$$f'(t_0) = \sum_{i=1}^{n} \frac{dx^i}{dt}(t_0)\delta_i$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{dy^j}{dt}a_j^i\delta_i$$
$$= \sum_{j=1}^{n} \left(\frac{y^j(t)}{dt} \sum_{i=1}^{n} a_j^i\delta_i \right)$$
$$= \sum_{j=1}^{n} \frac{dy^j}{dt}e_j$$

从而可见其形式不变，即切向量定义式（2.6）与直角坐标系的选取无关。

1.9 证:

(1)

$$D_v(g + \lambda h) = \langle \nabla(g + \lambda h)(P), v \rangle$$
$$= \langle \nabla g(P) + \lambda\nabla h(P), V \rangle$$
$$= \langle \nabla g(P), v \rangle + \lambda \langle \nabla h(P), v \rangle$$
$$= D_v(g) + \lambda D_v(h)$$

(2)

$$D_v(g \cdot h) = \langle \nabla(gh)(P), v \rangle$$
$$= \langle ((\nabla g)h)(P), v \rangle + \langle (g(\nabla h))(P), v \rangle$$
$$= h(P) \langle \nabla g(P), v \rangle + g(P) \langle \nabla h(P), v \rangle$$
$$= h(P)D_v g + g(P)D_v h$$

1.10 证: $E_n \to R$ 上的函数 $x^i : P = \lambda^1 \delta_1 + ... + \lambda^n \delta_n \to \lambda^i \quad (i = 1, 2, ..., n)$

$\forall\, P = (\lambda^1, ..., \lambda^n) \in E^n$, 在 P 的邻域内取 $Q = P + \triangle P = (\lambda^1 + \triangle\lambda^1, ..., \lambda^n + \triangle\lambda^n)$, 则

$$\lim_{\triangle\lambda^j \to 0} \frac{x^i(Q) - x^i(P)}{\triangle\lambda^j} = \lim_{\triangle\lambda^j \to 0} \frac{\triangle\lambda^i}{\triangle\lambda^j} = \delta_i^j \in C^\infty$$

从而 $x^i \in C^\infty \quad (\forall\, i = 1, 2, ..., n)$.

1.11 证:

(1) 在 E^m 中，取两新旧直角坐标分别为 $\{O; \delta_i\}, \{P; e_i\}$, 且满足

$$\begin{cases} \overrightarrow{OP} = \sum_{i=1}^m a^i \delta_i \\ e_j = \sum_{i=1}^m a_j^i \delta_i \end{cases} \quad j = 1, .., m.$$

在 E^n 中，取两新旧直角坐标分别为 $\{O; \xi_i\}, \{P; \eta_i\}$, 且满足

$$\begin{cases} \overrightarrow{OQ} = \sum_{i=1}^m b^i \xi_i \\ \eta_j = \sum_{i=1}^m b_j^i \xi_i \end{cases} \quad j = 1, .., m.$$

记原映射为 $F(\lambda^1, .., \lambda^m) = (f^1(\lambda^1, .., \lambda^m), ..., f^n(\lambda^1, .., \lambda^m))$.

设在新坐标下表示为 $G(\mu^1, ..., \mu^m) = (g^1(\mu^1, .., \mu^m), ..., g^n(\mu^1, .., \mu^m))$.

则有 $f^l = b^l + \sum_{j=1}^n b_j^l g^j \quad \lambda^k = a^k + \sum_{j=1}^m a_j^k \mu^j \ (k = 1, .., m)$. 再记 $(a_j^k)_{m \times m}$ 的逆为 $(C_k^j)_{m \times m}$

$$\therefore \frac{\partial f^l}{\partial \lambda^k} = \sum_{j=1}^n b_j^l \frac{\partial g^j}{\partial \lambda^k}$$

$$= \sum_{j=1}^n b_j^l (\sum_{i=1}^m \frac{\partial g^j}{\partial \mu^i} C_k^i)$$

$$= \sum_{j=1}^n sum_{i=1}^m b_j^l frac\partial g^j \partial \mu^i C_k^i$$

$$\therefore (\frac{\partial f^l}{\partial \lambda^k})_{m \times n} = A^{-1} (\frac{\partial g^j}{\partial \mu^i})_{m \times n} B$$

记 J_f, J_g 分别为 $F(\lambda^1, .., \lambda^m)$ 与 $G(\mu^1, .., \mu^m)$ 的Jacobi矩阵。则有 $J_f = A^{-1} J_g B$.

(2) 由于A、B可逆，所以 A^{-1}, B 均可表为若干初等行、列变换的乘积，
   $\therefore r(J_f) = r(J_g)$, 在任意点 x_0.

(3) 在 E^m 中任取点P，记 $Q = f(P)$.
   在 P 点邻域分别取两新旧曲纹坐标系 $(u^1, ..., u^m), (v^1, ..., v^m)$. 两者之间由同胚映射 g 关联： $v^i = g^i(u^1, ..., u^m) \quad i = 1, .., m.$ 记 $g^{-1} = \bar{g}$ 则 $u^i = \bar{g}^i(v^1, .., v^m) \quad i = 1, .., m.$
   同样在 Q 点邻域分别取两新旧曲纹坐标系 $(s^1, ..., s^n), (t^1, ..., t^n)$. 两者之间由同胚映射 h 关联： $t^i = h^i(s^1, ..., s^n) \quad i = 1, .., m$
   原函数 $f(u^1, .., u^m)$ 中分量记为 $f^i(u^1, .., u^m) \ (i = 1, .., n)$ 在 E^m 与 E^n 间曲纹坐标变换

下为 $\tilde{f}(v^1,..,v^m)$, 其在新坐标下分量为: $\tilde{f}^i(v^1,..,v^m) = h^i(f^1,..,f^n)$ 其中 $f^j(u^1,..,u^m) = f^j(\bar{g}(v^1,..,v^m),...,\bar{g}(v^1,..,v^m))$ $(i=1,..,n; j=1,..,n)$

从而在变换后Jacobi矩阵 $J_{n\times m}$ 为

$$\begin{aligned}
J_j^i &= \frac{\partial \tilde{f}^i}{\partial v^j} \\
&= \frac{\partial h^i}{\partial f^k}\frac{\partial f^k}{\partial u^r}\frac{\partial \bar{g}^r}{\partial v^j} \quad (i,k=1,..,n, r,j=1,..,m)
\end{aligned}$$

在 P 点，记 g 的雅克比矩阵 $(\frac{\partial g^i}{\partial u^j})_{m\times m}$ 为 G, h 的雅克比矩阵 $(\frac{\partial h^i}{\partial v^j})_{n\times n}$ 为 H, 变换前 f 的雅克比矩阵 $(\frac{\partial f^i}{\partial u^j})_{n\times m}$ 为 J_0 则有：

$$J = HJ_0G^{-1}$$

1.12 证: 记原方程组为 $x^i = f^i(u^1,..,u^n)$, $i=1,..,n$. 则 $f = f^i\delta_i$ 的雅克比矩阵的行列式为

$$\left|\frac{\partial(f^1,..,f^n)}{\partial(u^1,..,u^n)}\right| = \begin{vmatrix} -x^1 tanu^1 & -x^1 tanu^2 & \cdots & -x^1 tanu^{n-2} & -x^1 tanu^{n-1} & \frac{x^1}{u^n} \\ -x^2 tanu^1 & -x^2 tanu^2 & \cdots & -x^2 tanu^{n-2} & x^2 cotu^{n-1} & \frac{x^2}{u^n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x^{n-1} tanu^1 & x^{n-1} cotu^2 & \cdots & 0 & 0 & \frac{x^{n-1}}{u^n} \\ x^n cotu^1 & 0 & \cdots & 0 & 0 & \frac{x^n}{u^n} \end{vmatrix}$$

$$= x^1 x^2 \cdots x^n \begin{vmatrix} -tanu^1 & -tanu^2 & \cdots & -tanu^{n-2} & -tanu^{n-1} & 1 \\ -tanu^1 & -tanu^2 & \cdots & -tanu^{n-2} & cotu^{n-1} & 1 \\ -tanu^1 & -tanu^2 & \cdots & cotu^{n-2} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -tanu^1 & \cot u^2 & \cdots & 0 & 0 & 1 \\ \cot u^1 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{x^1 x^2 \cdots x^n}{u^n} \cdot \left[(-1)^{1+n}(-1)^{\frac{(n-1)(n-2)}{2}} cotu^1 \cdots cotu^{n-1} \right.$$
$$\left. + (-1)^{2+n}(-1)^{\frac{(n-1)(n-2)}{2}} cotu^1 \cdots cotu^{n-2}(-tanu^{n-1}) \right]$$

$$= (-1)^{\frac{n^2-n+4}{2}} \left(\prod_{i=1}^{n-2} r^i \prod_{j=1}^{i} cosu^j \right) x^1 x^2 (u^n)^{n-3} \frac{1}{sinu^{n-1}cosu^{n-1}}$$

当 $x^1, x^2 \neq 0$ 时，有 $u^n, cosu^1, \cdots, cosu^{n-1}, sinu^{n-1}$ 非零，又由 $r^1, \cdots r^n$ 为正数可知： $\left|\frac{\partial(f^1,..,f^n)}{\partial(u^1,..,u^n)}\right| \neq 0$, 即 f 的秩为 n . 从而 $(u^1, \cdots u^n)$ 给出 E^n 中除坐标面 $\{(0,0,x^3,\cdots,x^n): x^3,\cdots,x^n \in \mathbb{R}\}$ 以外的任意一点的邻域内的曲纹坐标系。

1.13 解: 设 (x,y,z) 是 E^3 中的直角坐标系,令

$$\begin{cases} x = \rho\cos\psi\cos\theta \\ y = \rho\cos\psi\sin\theta \qquad (0 < \rho < +\infty, -\frac{\pi}{2} < \psi < \frac{\pi}{2}, -\pi < \theta < \pi) \\ z = \rho\sin\psi \end{cases}$$

则 (ρ,ψ,θ) 给出球坐标系，记 $u^1=\rho, u^2=\psi, u^3=\theta$ 则可求其自然标架场如下：

$$\overrightarrow{r_1}=(cos\psi cos\theta, cos\psi sin\theta, sin\psi)$$

$$\overrightarrow{r_2}=(-\rho sin\psi cos\theta, -\rho sin\psi sin\theta, \rho cos\psi)$$

$$\overrightarrow{r_3}=(-\rho cos\psi sin\theta, \rho cos\psi cos\theta, 0)$$

则 $\forall\, Q=r(P), \{Q, r_i\}$ 为 E^3 中球坐标系诱导的自然标架场。

下求其度量系数：

$$g_{11}=\langle\overrightarrow{r_1},\overrightarrow{r_1}\rangle=1 \qquad\qquad g_{12}=g_{21}=\langle\overrightarrow{r_1},\overrightarrow{r_2}\rangle=0$$

$$g_{31}=g_{13}=\langle\overrightarrow{r_1},\overrightarrow{r_3}\rangle=0$$

$$g_{22}=\langle\overrightarrow{r_2},\overrightarrow{r_2}\rangle=\rho^2 \qquad\qquad g_{23}=g_{32}=\langle\overrightarrow{r_2},\overrightarrow{r_3}\rangle=0$$

$$g_{33}=\langle\overrightarrow{r_3},\overrightarrow{r_3}\rangle=\rho^2 cos^2\psi$$

$\therefore$ 度量系数为 $g_{11}=\rho^2,\ g_{33}=\rho^2 cos^2\psi,\ g_{ij}=0\ (i\ne j$ 时,其中 $i,j=1,2,3).$

则 $g^{11}=1,\ g^{22}=\frac{1}{\rho^2},\ g^{33}=\frac{1}{\rho^2 cos^2\psi},\ g^{ij}=0\ (i\ne j$ 时,其中 $i,j=1,2,3).$

且 $\frac{\partial g_{11}}{\partial u^i}=0\ (i=1,2,3)$ $\frac{\partial g_{ij}}{\partial u^k}\ (i,j,k=1,2.3)$ $\frac{\partial g_{22}}{\partial u^1}=2\rho$ $\frac{\partial g_{22}}{\partial u^2}=0$ $\frac{\partial g_{22}}{\partial u^3}=0$ $\frac{\partial g_{33}}{\partial u^1}=2\rho cos^2\psi$ $\frac{\partial g_{33}}{\partial u^2}=-\rho^2 sin2\psi$ $\frac{\partial g_{33}}{\partial u^3}=0$

从而由

$$\Gamma^k_{il}=\frac{1}{2}g^{kj}\left(\frac{\partial g_{ij}}{\partial u^l}+\frac{\partial g_{jl}}{\partial u^i}-\frac{\partial g_{il}}{\partial u^j}\right)$$

得：

$$\Gamma^1_{33}=\frac{1}{2}g^{11}\left(\frac{\partial g_{31}}{\partial u^3}+\frac{\partial g_{13}}{\partial u^1}-\frac{\partial g_{33}}{\partial u^1}\right)=\rho cos^2\psi$$

$$\Gamma^2_{33}=\frac{1}{2}g^{22}\left(\frac{\partial g_{32}}{\partial u^3}+\frac{\partial g_{32}}{\partial u^1}-\frac{\partial g_{33}}{\partial u^2}\right)=-sins\psi$$

$$\Gamma^3_{33}=\frac{1}{2}g^{33}\left(\frac{\partial g_{33}}{\partial u^3}+\frac{\partial g_{33}}{\partial u^3}-\frac{\partial g_{33}}{\partial u^3}\right)=0$$

又由

$$\frac{\partial\overrightarrow{r_1}}{\partial u^1}=0 \qquad\qquad \frac{\partial\overrightarrow{r_1}}{\partial u^2}=\frac{1}{\rho}\overrightarrow{r_2} \qquad\qquad \frac{\partial\overrightarrow{r_1}}{\partial u^3}=\frac{1}{\rho}\overrightarrow{r_3}$$

$$\frac{\partial\overrightarrow{r_2}}{\partial u^2}=-\rho\overrightarrow{r_1} \qquad\qquad \frac{\partial\overrightarrow{r_2}}{\partial u^3}=-tan\psi\overrightarrow{r_3}$$

从而 Christoffel 系数为：

$$\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{rho} \qquad\qquad \Gamma^3_{31}=\Gamma^3_{13}=\frac{1}{rho}$$

$$\Gamma^1_{22}=-rho \qquad\qquad \Gamma^3_{23}=\Gamma^3_{32}=-tan\psi$$

$$\Gamma^1_{33}=-\rho cos^2\psi \qquad\qquad \Gamma^2_{33}=-sin^2\psi$$

其余为 0.

1.14 解： 设 (x,y,z) 为 E^3 中直角坐标系，设

$$\begin{cases} x=\rho cos\theta \\ y=\rho sin\theta \qquad \text{（其中 } 0<r<+\infty,\ -\pi<\theta<\pi,\ -\infty<t<+\infty) \\ z=t \end{cases}$$

则 (ρ,θ,t) 为柱坐标系。记 $u^1=\rho,\ u^2=\theta,\ u^3=t$, 则有：

$$\overrightarrow{r_1}=(cos\theta,sin\theta,0)$$
$$\overrightarrow{r_2}=(-\rho sin\theta,\rho cos\theta,0)$$
$$\overrightarrow{r_3}=(0,0,1)$$

且 $g_{11}=1,\ g_{22}=\rho^2,\ g_{33}=1,\ g_{ij}=0\,(i\neq j时)$ 由于：

$$\frac{\partial\overrightarrow{r_1}}{\partial u^1}=0 \quad \frac{\partial\overrightarrow{r_1}}{\partial u^2}=\frac{1}{\rho}\overrightarrow{r_2} \quad \frac{\partial\overrightarrow{r_1}}{\partial u^3}=0$$
$$\frac{\partial\overrightarrow{r_2}}{\partial u^2}=-\rho\overrightarrow{r_1} \quad \frac{\partial\overrightarrow{r_2}}{\partial u^3}=0$$
$$\frac{\partial\overrightarrow{r_3}}{\partial u^3}=0$$

所以 Christoffel 系数为：

$$\Gamma_{12}^2=\Gamma_{21}^2=\frac{1}{\rho} \quad \Gamma_{22}^1=-\rho$$

其余为0

1.15 证: $\because\ v=v^i\overrightarrow{e_i}$

$$\therefore\ dv=v^j d\overrightarrow{e_j}+\overrightarrow{e_i}dv^i$$
$$=v^j(\frac{\partial\overrightarrow{e_j}}{\partial u^k}du^k)+\overrightarrow{e_i}dv^i$$
$$=v^j\Gamma_{jk}^i\overrightarrow{e_i}du^k+\overrightarrow{e_i}dv^i$$
$$=(v^j\Gamma_{jk}^i du^k+dv^i)\overrightarrow{e_i}$$

$\therefore\ v$ 为平行向量场 $\Leftrightarrow v^j\Gamma_{jk}^i+dv^i=0(\forall\ i=1,2,...,n)$

1.16 证: 定义加法 $'+':(f+g)(x)=f(x)+g(x)$, 数乘: $(\lambda f)(x)=\lambda f(x)$
先证 $(\mathscr{L}(v_1,\cdots,v_r;\mathbb{R}),+)$ 构成加法群。

(1) $\forall\,f,g\in\mathscr{L}(v_1,\cdots,v_r;\mathbb{R})\quad\forall\,i\in\{1,\cdots,r\}\quad\forall\,x_1=(x^1,\cdots,x_1^i,\cdots,x^r),x_2=(x^1,\cdots,x_2^i,\cdots,x^r)\in V_1\times\cdots\times V_r\quad\forall\,k_1,k_2\in\mathbb{R}$ 有

$$(f+g)(x^1,\cdots,k_1 x_1^i+k_2 x_2^i,\cdots x^r)$$
$$=f(x^1,\cdots,k_1 x_1^i+k_2 x_2^i,\cdots x^r)+g(x^1,\cdots,k_1 x_1^i+k_2 x_2^i,\cdots x^r)$$
$$=(k_1 f(x_1)+k_2 f(x_2))+(k_1 g(x_1)+k_2 g(x_2))$$
$$=k_1(f+g)(x_1)+k_2(f+g)(x_2)$$

$\therefore f+g\in\mathscr{L}(V_1,\cdots,V_r;\mathbb{R})$

(2) $\forall\,f,g,h\in\mathscr{L}(V_1,\cdots,V_r;\mathbb{R})\quad\forall\,x\in V_1\times,\cdots,V_r,$ 有

$$[(f+g)+h](x)=(f+g)(x)+h(x)=f(x)+g(x)+h(x)$$
$$[f+(g+h)](x)=f(x)+(g+h)(x)=f(x)+g(x)+h(x)$$

$\therefore (f+g)+h=f+(g+h)$

(3) 取 θ 为 $\forall\, x \in V_1 \times \cdots \times V_r, \theta(x) = 0$

则显然 $f + \theta = \theta + f, \forall\, f$

(4) $\forall\, f$, 取 $-f$ 为 $(-f)(x) = -f(x)$ 则 $f + (-f) = (-f) + f = \theta$

(5) $\forall\, f, g \in \mathscr{L}(V_1 \cdots V_r; \mathbb{R})$ 显然 $f + g = g + f$.

所以 $(\mathscr{L}(v_1, \cdots, v_r; \mathbb{R}), +)$ 构成Abel群。$\forall\, \lambda, \mu \in \mathbb{R}$　$\forall f, g \in \mathscr{L}(V_1 \times \cdots \times V_r; \mathbb{R})$, 又有

(6) $(\lambda, \mu)f = \lambda f + \mu f$

(7) $(\lambda\mu)g = \lambda(\mu g)$

(8) $\lambda(f + g) = \lambda f + \lambda g$

(9) $1 \cdot f = f$

所以 $\mathscr{L}(V_1, \cdots, V_r; \mathbb{R})$ 构成 $\mathbb{R}$ 上的线性空间。

1.17 证:

(1) $\forall\, k_1, k_2 \in \mathbb{R}\ u_1, u_2 \in V$, 则由 $\tilde{f}$ 为2重线性函数知

$$
\begin{aligned}
f(k_1 u_1 + k_2 u_2) &= \sum_{i=1}^{n} \tilde{f}(k_1 u_1 + k_2 u_2, \delta^i)\delta_i \\
&= \sum_{i=1}^{n} (k_1 \tilde{f}(u_1, \delta^i) + k_2 \tilde{f}(u_2, \delta^i))\delta_i \\
&= k_1 \sum_{i=1}^{n} \tilde{f}(u_1, \delta^i)\delta_i + k_2 \sum_{i=1}^{n} \tilde{f}(u_2, \delta^i)\delta_i \\
&= k_1 f(u_1) + k_2 f(u_2)
\end{aligned}
$$

所以 f 为线性变换。

(2) 若另取基底 $\{e^i\}$, 且 $e^i = a^i_j \delta_j$ 其对偶基底 $\{e_i\}$, 则有 $\delta_j = a^i_j e_i$, 其中 (b^j_i) 为 (a^j_i) 的逆矩阵。且由

$$
\delta^i(e_j) = \delta_i(a^k_j \delta_k) = a^k_j \delta^i_k = a^i_j = a^i_k e^k(e_j)
$$

知 $\delta^i = a^i_k e^k$.

$\therefore f(u) = \tilde{f}(u, \delta^i)\delta_i = \tilde{f}(u, a^i_k e^k)(b^j_i e_j) = a^i_k b^j_i \tilde{f}(u, e^k)e_j = \delta^j_k \tilde{f}(u, e^k)e_j = \tilde{f}(u, e^j)e_j$

$\therefore f$ 的定义与基底的选取无关。

1.18 证: 取定 $t, \forall\, v = (v_1, \cdots v_r) \in V \times \cdots \times V$　$v^1 \in V^\star$, 取 $F: V^\star \times V_1 \cdots \times V_r \to \mathbb{R}$ 为 $F(v^1, v_1, \cdots v_r) = v^1(t(v))$, 显然 F 是 $V^\star \times V \times \cdots V$ 上的一个 $1 + r$ 重线性函数，即 $(1, r)$ 型张量。

反之，若给定一个 $(1, r)$ 型张量 $F: V^\star \times V_1 \times \cdots V_r$, $\forall\, v = (v1, \cdots, v_r) \in V \times \cdots \times \to \mathbb{R}$, 令

$t(v) = F(\delta^i, v_1, \cdots, v_r)\delta_i$, 显然 t 为 r 重线性映射。

所以 t 等同一个 $(1, r)$ 型张量。

1.19 证:

(1) $\forall\, \alpha_1, \cdots, \alpha_2 \in \mathscr{L}(V_1, \cdots, V_p; \mathbb{R}), \quad \forall\, \beta \in \mathscr{L}(W_1, \cdots, W_q; \mathbb{R})$
   $\forall\, v \in V_1 \times \cdots \times V_p, \quad w \in W_1 \times \cdots \times W_q$

$$(\alpha_1 + \alpha_2) \otimes \beta(v, w)$$
$$= (\alpha_1 + \alpha_2)(v) \cdot \beta(w)$$
$$= [\alpha(_1(v) + \alpha_2(v)] \cdot \beta(w)$$
$$= \alpha_1(v) \cdot \beta(w) + \alpha_2(v) \cdot \beta(w)$$
$$= \alpha_1 \otimes \beta(v, w) + \alpha_2 \otimes \beta(v, w)$$
$$= (\alpha_1 \otimes \beta + \alpha_2 \otimes \beta)(v, w)$$

$\therefore\ (\alpha_1 + \alpha_2) \otimes \beta = \alpha_1 \otimes \beta + \alpha_2 \otimes \beta$ 同理 $\beta \otimes (\alpha_1 + \alpha_2) = \beta \otimes \alpha_1 + \beta \otimes \alpha_2$

(2) $\forall\, \alpha \in \mathscr{L}(V_1, ..., Vr; \mathbb{R}),\ \beta \in \mathscr{L}(W_1, ..., W_s; \mathbb{R}),\ \gamma \in \mathscr{L}(Z_1, ..., Z_t; \mathbb{R})$
   $\forall\, V_1 \times \cdots \times V_r,\ w \in W_1 \times \cdots \times W_s,\ z \in Z_1 \times \cdots \times Z_t,$ 则

$$(\alpha \otimes \beta) \otimes \gamma(v, w, z)$$
$$= (\alpha \otimes \beta)(v, w) \cdot \gamma(z)$$
$$= (\alpha(v) \cdot \beta(w)) \cdot \gamma(z)$$
$$= \alpha(v) \cdot [\beta(w)\gamma(z)]$$
$$= \alpha(v) \cdot [(\beta \otimes \gamma(w, z))]$$
$$= [\alpha \otimes (\beta \otimes \gamma)](v, w, z)$$

$\therefore (\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$

故多重线性函数的张量积服从分配率和结合律。

1.20 证: 设 $\{\xi_i\}$ 为 V 的基，$\{\eta_i\}$ 为 W 的基，记 $A = \{V \otimes W | v \in V, w \in W\}$，下证 A 不构成线性空间：

由于 $dim(V) \geqslant 2, dim(W) \geqslant 2.$ 所以分别可取 V, W 的两组基：$\xi_1, \xi_2, \eta_1, \eta_2,$则

$$\xi_1 \otimes \eta_1 \in A,\ \xi_2 \otimes \eta_2 \in A,$$

若

$$\xi_1 \otimes \eta_1 + \xi_2 \otimes \eta_2 \in A$$

则 $\exists\, a^i\ (i = 1, 2, ..., n)\ s.t.$

$$\xi_1 \otimes \eta_1 + \xi_2 \otimes \eta_2 = (a^i \xi_i) \otimes (b^j \eta_j) = a^i b^j \xi_i \otimes \eta_j$$

移项并合并后，由于 $\{\xi_i \eta_j\}$ 线性无关，所以有

$$a^i b^j = \begin{cases} 1 & i = j = 1 \text{ 以及 } i = j = 2 \text{ 时}; \\ 0 & \text{其余情况}. \end{cases}$$

$a^1 b^1 = a^2 b^2 = 1$, 从而 $a^1, b^1, a^2, b^2 \neq 0$, 但 $a^1 b^2 = 0$, 矛盾!

$\therefore \xi_1 \otimes \eta_1 + \xi_2 \otimes \eta_2 \notin A$, A 不构成线性空间。

1.21 证:

(1) 由题知 $f(\delta_i) = b_i^j \delta_i$.

若另取基底 $\{e_i\}$ 且 $e_i = a_i^j \delta_j$ 从而 $\delta_j = c_j^i e_i$ (其中 (c_i^j) 为 (a_i^j) 的逆).

则 $f(e_i) = f(a_i^j \delta_j) = a_i^j f(\delta_j) = a_i^j b_j^k \delta_k = a_i^j b_j^k c_k^l e_l$.

即 $f(e_i) = (a_i^j b_j^k c_k^l) e_l$, 记 $d_i^l = a_i^j b_j^k c_k^l$.

$\therefore f$ 在基 $\{e_i\}$ 下矩阵是 (d_i^l).

此时

$$\begin{aligned}
B_3 &= d_i^j d_k^i d_j^k \\
&= a_i^s b_s^t c_t^j a_k^e b_e^f c_f^i a_j^g b_g^h c_h^k \\
&= (a_i^s c_f^i)(a_k^e c_h^k)(a_j^g c_t^j) b_s^t b_e^f b_g^h \\
&= (\delta_f^s b_e^f)(\delta_h^e b_g^h)(\delta_t^g b_s^t) \\
&= b_e^s b_g^e b_s^g
\end{aligned}$$

$\therefore B_3$ 与基底的选取无关。

(2) 令 $F: V^\star \times V^\star \to \mathbb{R}$ 为 $F(\delta^i, u) = \delta^i(f(u))$. $(\forall u \in V)$. 则易证 F 为 $(1,1)$ 型张量，且

$$\begin{aligned}
B_3 &= b_i^j b_k^i b_j^k \\
&= F(\delta^j, \delta_i) \cdot F(\delta^i, \delta_k) \cdot F(\delta^k, \delta_j) \\
&= F(\delta^j, \delta_i) \cdot [(C_1^2 F \otimes F)(\delta^i, \delta_j)] \\
&= C_2^1 [F \otimes C_1^2 (F \otimes F)](\delta^j, \delta_j) \\
&= C_1^1 \{C_2^1 [F \otimes C_1^2 (F \otimes F)]\}.
\end{aligned}$$

1.22 解: 令 $F: V^\star \times V \to \mathbb{R}$ $\quad F(v^i, v_j) = v^i(v_j)$. 易证 F 为 $(1,1)$ 型张量. 任取 V 的基底 $\{\delta_i\}$, 其对偶基底 $\{\delta^i\}$, 则 $\{\delta_i \otimes \delta^j\}$ 为 F 的基底, 且 $\delta^i(\delta_j) = \delta_j^i$. 从而 $F(\delta^i, \delta_j) = \delta_i^j$.

$$F = F(\delta^i, \delta_j) \delta_i \otimes \delta^j = \delta_i^j \delta_i \otimes \delta^j = \sum_i \delta_i \otimes \delta^i$$

$\therefore \delta_i^j$ 为 F 的分量。

1.23 解: 若存在 (0,2) 型张量 F 满足题设条件，则任取 V 的一组基底 $\{\delta_i\}$, $F(\delta_i, \delta_j) = \delta_{ij}$. 另任取一基底 $\{e_i\}$，且 $e_i = a_i^j \delta_j$, 则

$$
\begin{aligned}
F(e_i, e_j) = \delta_{ij} &\Leftrightarrow F(a_i^k \delta_k, a_j^r \delta_r) = \delta_{ij} \\
&\Leftrightarrow a_i^k a_j^r F(\delta_k, \delta_r) = \delta_{ij} \\
&\Leftrightarrow a_i^k a_j^k = \delta_{ij} \\
&\Leftrightarrow (a_i^j) \text{ 为单位正交阵}
\end{aligned}
$$

但 a_i^j 未必为单位正交阵，矛盾！
所以不存在。

1.24 证: $\forall\, v_1, \cdots, v_q \in V$, $(\sigma(f))(v_1, \cdots, v_q) = f(v_{\sigma(1)}, \cdots, v_{\sigma(q)}) = \alpha^1(v_{\sigma(1)}) \cdots \alpha^q(v_{\sigma(q)})$.

$$
\begin{aligned}
\alpha^{\tau(1)}(v_1) \otimes \cdots \otimes \alpha^{\tau(q)}(v_1, \cdots, v_q) &= \alpha^{\tau(1)} \otimes \cdots \otimes \alpha^{\tau(q)}(v_1, \cdots, v_q) \\
(\text{由 } \tau = \sigma^{-1} \text{ 知 } i = \sigma(\tau(i))) \quad &= \alpha^{\tau(1)}(v_{\sigma(\tau(1))}) \cdots \alpha^{\tau(q)}(v_{\sigma(\tau(q))}) \\
&= \alpha^1(v_{\sigma(1)}) \cdots \alpha^q(v_{\sigma(q)})
\end{aligned}
$$

$\therefore \sigma(f) = \alpha^{\tau(1)} \otimes \cdots \otimes \alpha^{\tau(q)}$

1.25 证:

(1)

$$
\begin{aligned}
\sigma \text{ 是对称张量} &\Leftrightarrow \sigma(\xi) = \xi \\
&\Leftrightarrow \frac{1}{q!} \sum_{\sigma \in \varphi(q)} \sigma(\xi) = \xi \\
S_q(\xi) = \xi.
\end{aligned}
$$

(2)

$$
\begin{aligned}
\sigma \text{ 是反对称张量} &\Leftrightarrow \sigma(\xi) = sign(\sigma)\xi \\
&\Leftrightarrow sign(\sigma) \cdot \sigma(\xi) = \xi \\
&\Leftrightarrow \sum_{\sigma \in \varphi(q)} sign(\sigma) \cdot \sigma(\xi) = \xi \cdot q! \\
&\Leftrightarrow A_q(\xi) = \xi
\end{aligned}
$$

1.26 证: $\forall\, \xi \in V_2^0, \{\delta^i\}$ 为 V 中的基, 记 $\xi = \xi_{ij}\delta^i\delta^j$ 为二阶协变张量. 对于 $\delta \triangleq \delta^i \otimes \delta^j$.

$$S_2(\delta) = \frac{1}{2} \cdot (\delta^i \otimes \delta^j + \delta^j \otimes \delta^i)\ (\text{由24题})$$

$$A_2(\delta) = \frac{1}{2}(\delta^i \otimes \delta^j - \delta^j \otimes \delta^i)$$

$$\Rightarrow \delta = \delta^i \otimes \delta^j = S_2(\delta) + A_2(\delta)$$

$$\therefore \xi = \xi_{ij}\delta^i \otimes \delta^j$$

$$= \delta_{ij}(S_2(\delta^i \otimes \delta^j) + A_2(\delta^i \otimes \delta^j))$$

$$= S_2(\xi) + A_2(\xi)$$

其中 $S_2(\xi)$ 为对称张量，$A_2(\xi)$ 为反对称张量。

1.27 证： 取 V 中基底 $\{\delta_i\}$, 由 $\varphi \in V_2^0$, 依题意知, $\forall\, \delta_i\delta_j\delta_k$ 有

$$\varphi(\delta_i, \delta_j, \delta_k) = \varphi(\delta_j, \delta_i, \delta_k)$$

$$= -\varphi(\delta_j, \delta_k, \delta_i)$$

$$= -\varphi(\delta_k, \delta_j, \delta_i)$$

$$= \varphi(\delta_k, \delta_i, \delta_j)$$

$$= -\varphi(\delta_i, \delta_j, \delta_k)$$

$$\Rightarrow \varphi(\delta_i, \delta_j, \delta_k) = 0.$$

$$\therefore \varphi = \varphi(\delta_i, \delta_j, \delta_k)\delta^i \otimes \delta^j \otimes \delta^k = 0.$$

1.28 证： 取 V 的基 $\{\delta_i\}$, $\because a(x, x) = 0$, $\therefore a(\delta_i, \delta_j) = 0$　$(i = j)$ 时.
$\forall i \neq j$ 取 $x = \delta_i + \delta_j$ 则

$$0 = a(x, x)$$

$$= a(\delta_i + \delta_j, \delta_i + \delta_j)$$

$$= a_{ii} + a_{ij} + a_{ji} + a_{jj}$$

$$= a_{ij} + a_{ji}.$$

记 $a = a_{ij}\delta^i \otimes \delta^j$ 有

$$S_2(a) = \frac{1}{2}a_{ij}(\delta^i \otimes \delta^j + \delta^j \otimes \delta^i)$$

$$= \frac{1}{2}(a_{ij}\delta^i \otimes \delta^j + a_{ji}\delta^i \otimes \delta^j)$$

$$= \frac{1}{2}(a_{ij} + a_{ji})\delta^i \otimes \delta^j$$

$$= 0.$$

1.29 证： 由于 a 为任意对称张量，可取 a 使得 $a^{ij} \neq 0 (\forall\, i, j)$（例如，取 $a = \sum_i \sum_j \delta_i \otimes \delta_j$ ），则

$$0 = a^{ij} b_{ij} = a^{ji} b_{ji} = a^{ij} b_{ji}$$

$\therefore a^{ij}(b^{ij} + b^{ji}) = 0$, 此时 $b_{ij} + b_{ji} = 0 \Rightarrow b_{ij} = -bji,$
$\therefore b$ 为反对称张量。

1.30 证： 取 V 的基 $\{\delta_i\}$ ，由 x 的任意性，$\forall\, i, j$ 取 $x = \delta_i + \delta_j \in V$. 由 $f(x, x) = \tilde{f}(x, x)$ 知

$$f(\delta_i + \delta_j, \delta_i + \delta_j) = \tilde{f}(\delta_i + \delta_j, \delta_i + \delta_j)$$
$$\Rightarrow f(\delta_i, \delta_i) + f(\delta_i, \delta_j) + f(\delta_j, \delta_i) + f(\delta_j, \delta_j) = \tilde{f}(\delta_i, \delta_i) + \tilde{f}(\delta_i, \delta_j) + \tilde{f}(\delta_j, \delta_i) + \tilde{f}(\delta_j, \delta_j)$$
$$\Rightarrow f(\delta_i, \delta_j) + f(\delta_j, \delta_i) = \tilde{f}(\delta_i, \delta_j) + \tilde{f}(\delta_j, \delta_i)$$
$$\Rightarrow 2f(\delta_i, \delta_j) = 2\tilde{f}(\delta_i, \delta_j)$$
$$\Rightarrow f_{ij} = \tilde{f}_{ij}$$

记 $f = f_{ij}\delta^i \otimes \delta^j \quad \tilde{f} = \tilde{f}_{ij}\delta^i \otimes \delta^j$, 则 $f = \tilde{f}$.

1.31 证： 原行列式$= \sum_\sigma \delta^{\sigma(i_1),\dots,\sigma(i_r)}_{i_1,\dots,i_r} \delta^{\sigma(i_1)}_{j_1} \cdots \delta^{\sigma(i_r)}_{j_r} = \delta^{j_1,\dots,j_r}_{i_1,\dots,i_r}.$

1.32 证： (1) 记置换

$$\pi = \begin{pmatrix} i_1, \cdots, i_{r+s} \\ j_1, \cdots, j_{r+s} \end{pmatrix} \quad \pi_1 = \begin{pmatrix} k_1, \cdots, k_r \\ j_1, \cdots, j_r \end{pmatrix} \quad \pi_2 = \begin{pmatrix} i_1, \cdots, i_r, i_{r+1}, \cdots, i_{r+s} \\ k_1, \cdots, k_r, j_{r+1}, \cdots, j_{r+s} \end{pmatrix}$$

则 $\pi = \pi_1 \pi_2$, 从而 $sgn(\pi) = sgn(\pi_1) \cdot sgn(\pi_2)$.
即，取定某排列 $(l_1, \cdots, l_r)$, 有

$$\delta^{i_1,\cdots,i_{r+s}}_{j_1,\cdots,j_{r+s}} = \delta^{l_1,\cdots,l_r}_{j_1,\cdots,j_r} \cdot \delta^{i_1,\cdots,i_{r+s}}_{l_1,\cdots,l_r,j_{r+1},\cdots,j_{r+s}} \qquad \text{(注：此时不对上下指标求和)}$$

而 $(l_1, \cdots, l_r)$ 的排列共有 $r!$ 种取法.

$$\therefore \delta^{i_1,\cdots,i_{r+s}}_{j_1,\cdots,j_{r+s}} = \frac{1}{r!} \sum_{k_1,\cdots,k_r} \delta^{k_1,\cdots,k_r}_{j_1,\cdots,j_r} \delta^{i_1,\cdots,i_{r+s}}_{k_1,\cdots,k_r,j_{r+1},\cdots,j_{r+s}}.$$

(2)

$$\delta^{i_1,\cdots,i_r}_{i_1,\cdots,i_r} = A^r_n = \frac{n!}{(n-r)!}.$$

1.33 证： 记 $a = (\alpha_1, ..., \alpha_n)$, 其中 $\alpha_i = (a_i^1, ..., a_i^n)^T$, 则

$$
\begin{aligned}
\det a &= \det(\alpha_1, ..., \alpha_n) \\
&= \frac{1}{n!} \delta_{1,...,n}^{i_1,...,i_n} \det(\alpha_{i_1}, ..., \alpha_{i_n}) \\
&= \frac{1}{n!} \delta_{1,...,n}^{i_1,...,i_n} \delta_{j_1,...,j_n}^{1,...,n} a_{i_1}^{j_1} \cdots a_{j_n}^{i_n} \\
&= \frac{1}{n!} \delta_{j_1,...,j_n}^{i_1,...,i_n} a_{i_1}^{j_1} \cdots a_{j_n}^{i_n}.
\end{aligned}
$$

1.34 证：