微分流形初步作业

sher

2020 年 6 月 18 日

目录

¡¡¡¡¡¡¡ HEAD

第一章　预备知识

1.1 证:令$V=\mathbb{R}^n,\ A=\mathbb{R}^n$, 则 $\forall x,\ y\in A$, 令 $\overrightarrow{xy}=y-x$ 则有：

① w $\forall\ x\in A,\ \overrightarrow{xx}=x-x=0$,

② $\forall\ x\in A,\ v\in V$, 令 $y=x+v\in V$. 则 $\overrightarrow{xy}=y-x=v$. 若另有 y', $s.\ t.\ \overrightarrow{xy'}=v$, 则 $y'-x=v\Rightarrow y'=x+v=y$ 从而 y 唯一,

③ $\forall\ x,y,z\in A.\ \overrightarrow{xy}+\overrightarrow{yz}=(y-x)+(z-y)=z-y=\overrightarrow{yz}$.

所以 $\mathbb{R}^n$ 是一个n维仿射空间，它以 $\mathbb{R}^n$ 自身为它的伴随向量空间。

1.2 证:

① $\forall\ P,Q\in E^n,\ 0\leqslant d(P,Q)=\left|\overrightarrow{PQ}\right|<+\infty$, 且$d(P,Q)=0\Leftrightarrow\left|\overrightarrow{PQ}\right|=0\Leftrightarrow P=Q$,

② $\forall\ P,Q\in E^n,\ d(P,Q)=\left|\overrightarrow{PQ}\right|=\left|\overrightarrow{QP}\right|=d(Q,P)$,

③ $\forall\ P,Q,R\in E^n,\ d(P,R)=\left|\overrightarrow{PR}\right|=\left|\overrightarrow{PQ}+\overrightarrow{QR}\right|\leqslant\left|\overrightarrow{PQ}\right|+\left|\overrightarrow{QR}\right|\leqslant d(Q,P)$.

从而 E^n 关于距离函数 d 成为一个度量。

1.3 证:

(1) 记 E^n 的全体开子集为 τ,

① 显然 $\emptyset,\ E^n\in\tau$,

② $\forall\ A\in\tau,\ B\in\tau$, 若 $A\cap B=\emptyset$, 则 $A\cap B\in\tau$,
若 $A\cap B\neq\emptyset$, 则 $\forall\ P\in A\cap B$, 即 $P\in A$ 且 $P\in B$,
则 $\exists\ \varepsilon_1,\ \varepsilon_2>0,\ s.\ t.\ P\in B_{\varepsilon_1}(P)\subset A,\ P\in B_{\varepsilon_2}(P)\subset B$, 取 $\varepsilon=min\{\varepsilon_1,\varepsilon_2\}$, 则 $P\in B_\varepsilon(P)=B_{\varepsilon_1}(P)\cap B_{\varepsilon_2}(P)\subset A\cap B$, 因而 $A\cap B\in\tau$.

③ 若 $A_\alpha(\alpha\in I)\in\tau$, 则 $\forall\ P\in\bigcup\limits_{\alpha\in I}A_\alpha,\ \exists\ i\in I,\ s.\ t.\ P\in A_i\in\tau$,
则 $\exists\ \varepsilon>0,\ s.\ t.\ P\in B_\varepsilon(P)\in A_i$, 从而 $B_\epsilon(P)\subset\bigcup\limits_{\alpha\in I}A_\alpha$, 从而 $\bigcup\limits_{\alpha\in I}A_\alpha\in\tau$.

所以, τ 为 E^n 的一个拓扑。

(2) $\forall\ P,Q\in E$ 且 $P\neq Q$. 则记 $d=d(P,Q)$, 取 $\varepsilon_1=\varepsilon_2=\frac{d}{3}$, 则 $P\in B_{\varepsilon_1(P)}$（开），$Q\in B_{\varepsilon_2}(Q)$（开），且 $B_{\varepsilon_1}(P)\cup B_{\varepsilon_2}(Q)=\emptyset$,
否则, 若 $\exists\ R\in B_{\varepsilon_1}(P)\cap B_{\varepsilon_2}(Q)$, 则 $d(Q,R)<\frac{d}{3}, d(Q,R)<\frac{d}{3}$,
$d=d(P,Q)\leq d(P,R)+d(Q,R)<\frac{2d}{3}<d$ 矛盾！ 所以 $B_{\varepsilon_1}(P)\cup B_{\varepsilon_2}(Q)=\emptyset$ 成立.
从而, E_n 满足 T_2 分离性公理, 为 Hausdorff 空间.

(3) 取开集族 $\mathscr{B}=\{B_\varepsilon(P)|P\in\mathbb{Q}^n,\varepsilon\in\mathbb{Q}\}$, 其中 $\mathbb{Q}$ 为有理数，故 $\mathscr{B}$ 为可数的，下证其为拓扑基：
$\forall\ P\in E_n,\quad \forall\ U=B_\varepsilon(P)\in N(P)$,
$\exists\ \varepsilon'>0$, 且 $\varepsilon'\in\mathbb{Q},\ s.\ t.\ \frac{3\varepsilon}{4}<\varepsilon'<\varepsilon.\quad \exists\ Q\in\mathbb{Q}^n,\ s.\ t.\ \left|\overrightarrow{PQ}\right|<\frac{\varepsilon'}{4}$,
令 $B=B_{\varepsilon'}(Q)$, 则 $B\subset U\in$ 且 $B\in\mathscr{B}$.
所以, $\mathscr{B}$ 为 E^n 中可数拓扑基, 从而 E^n 第二可数。

1.4 证:

(1) 任取 E_n 中直线 l, 在 l 上依次任取3个不同的点 P,Q,R, 则有 $\left|\overrightarrow{OQ}\right|=t\left|\overrightarrow{OP}\right|+\left|\overrightarrow{OR}\right|$, 其中 $t\in(0,1)$. 记 $\sigma(P)=P'\quad(\forall\ P\in E_n)$
则 $\left|\overrightarrow{O'Q'}\right|=\left|\overrightarrow{OQ}\right|=t\left|\overrightarrow{OP}\right|+(1-t)\left|\overrightarrow{OR}\right|=t\left|\overrightarrow{O'P'}\right|+(1-t)\left|\overrightarrow{O'R'}\right|\quad(t\in(0,1))$
$\therefore\ P',Q',R'$ 三点共线且保持分比，所以 σ 将直线映为直线.

(2) 任取 E_n 中两平行直线 l_1,l_2, 则由（1）知 l_1,l_2 在 σ 下仍为直线，记为 l_1',l_2'. 任取不同点 $A,B\in l_1$, 不同点 $C,D\in l_2$,
则 $\overrightarrow{AB},\overrightarrow{CD}$ 非零，且 $\overrightarrow{AB}//\overrightarrow{CD}$ 从而 $\exists\ \lambda\neq 0,\ s.\ t.\ \overrightarrow{AB}=\lambda\ \overrightarrow{CD}$,
而 $\left|\overrightarrow{A'B'}\right|=\left|\overrightarrow{AB}\right|=|\lambda|\cdot\left|\overrightarrow{CD}\right|=|\lambda|\cdot\left|\overrightarrow{C'D'}\right|\quad\therefore\overrightarrow{A'B'}//\overrightarrow{C'D'}\quad\therefore\ l_1'//l_2'$
$\therefore$ 由 l_1,l_2 任意性知 σ 把 E_n 中平行直线映为平行直线.

(3) 记 $\sigma(O)=O',\overrightarrow{OP_i}=\delta_i,\sigma(\delta_i)=\delta_i'=\overrightarrow{O'P'},i=1,2,...n$
则由 $\{O,\delta_i\}$ 为正交标架知

$$\left|\overrightarrow{OP_i}\right|\cdot\left|\overrightarrow{OP_j}\right|=\begin{cases}0, & i\neq j,\\ 1, & i=j,\end{cases}\qquad i,j=1,2,...,n$$

$\therefore\ \{O',\delta_i\}$ 也为正交坐标系.

1.5 证: $\forall\ t\leq 0.\ \forall\ x=(x_1,...,x_n),\ y=(y_1,...,y_n)\in\mathbb{R}^n$.
由 $d(x,y)=\sqrt{\sum\limits_{i=1}^{n}(y^i-x^i)^2}$ 知 $d(tx,ty)=td(x,y)$.
又由 σ 为等距变换知 $d(\sigma(tx),\sigma(ty)=d(tx,ty)=td(x,y)=td(\sigma(x),\sigma(y))$.
取 $y=0=(0,...,0)$, 则 $d(\sigma(tx),\sigma(0)=t\cdot d(\sigma(x),\sigma(0))$,
则由 σ 保持共线性质而 $\overrightarrow{Ox}$ 与 $\overrightarrow{O(tx)}$ 共线, 知 $\overrightarrow{\sigma(0)\sigma(x)}$ 与 $\overrightarrow{\sigma(0)}\sigma(tx)$ 共线
$\therefore\ \overrightarrow{\sigma(0)\sigma(tx)}=\pm t\overrightarrow{\sigma(0)\sigma(x)}$ 不妨取 $+t,(-t$ 同理可证), 则有
$\sigma(tx)-\sigma(0)=t(\sigma(x)-\theta(0))\Rightarrow\sigma(tx)=t\sigma(x)-(1-t)\sigma(0)\cdots\cdots\cdots(1)$,

对（1）式左右两边关于 t 求导得：

$$\begin{aligned}
\text{左边} &= \frac{\partial}{\partial t}\sigma_1(tx) \\
&= (\frac{\partial}{\partial t}\sigma_1(tx), \dots, \frac{\partial}{\partial t}\sigma_n(tx)) \\
&= (x_1, \dots,\ x_n) \cdot \begin{pmatrix} \partial_1\sigma_1(tx) & \cdots & \partial_1\sigma_n(tx) \\ \vdots & & \vdots \\ \partial_n\sigma_1(tx) & \cdots & \partial_n\sigma_n(tx) \end{pmatrix} \\
\text{左边} &= \sigma(x) - \sigma(0)
\end{aligned}$$

则左边=右边，且令 t=0 后有：

$$\sigma(x) = \sigma(0) + (x_1, \dots,\ x_n) \cdot \left.\begin{pmatrix} \partial_1\sigma_1(tx) & \cdots & \partial_1\sigma_n(tx) \\ \vdots & & \vdots \\ \partial_n\sigma_1(tx) & \cdots & \partial_n\sigma_n(tx) \end{pmatrix}\right|_{t=0}$$

记 $a_0^j = \sigma_i(0) \qquad a_i^j = (\partial_i\sigma_j)|_(t=0) \qquad$ 其中 $i = 1, 2, \dots, n. \quad j = 1, 2, \dots, n.$ 有:

$$\sigma(x_1, \dots x_n) = (a_0^1, \dots, a_0^n) + (x_1, \dots, x_n) \cdot \begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \vdots & & \vdots \\ a_n^1 & \cdots & a_n^n \end{pmatrix}$$

取 $\varepsilon_i = (0, \dots, \overset{\text{第i个}}{\overset{\downarrow}{1}}, 0, \dots 0)$, 则,
$1 = d(\varepsilon_i, 0) = d(\sigma(\varepsilon_i), \sigma(0)) \Rightarrow (a_i^1)^2 + (a_i^2)^2 + \cdots + (a_i^n)^2 = 1 \ (\forall\ i = 1, 2, \dots, n)$
$(i \neq j$ 时)

$$\begin{aligned}
2 = 1^2 + 1^2 &= d^2(\varepsilon_i, \varepsilon_j) \\
&= d^2(\sigma(\varepsilon_i), \sigma(\varepsilon_j)) \\
&= (a_j^1 - a_i^1)^2 + \cdots + (a_j^n - a_i^n)^2 \\
&= ((a_i^1)^2 + \cdots + (a_i^n)^2) + ((a_j^1)^2 + \cdots + (a_j^n)^2) - 2(a_j^1 a_i^1 + \cdots + a_j^n a_i^n) \\
&= 1 + 1 - 2(a_j^1 a_i^1 + \cdots + a_j^n a_i^n)
\end{aligned}$$

$\Rightarrow a_j^1 a_i^1 + \cdots + a_j^n a_i^n = 0$ ($i \neq j$ 时)

从而 $(a_i^j)_{n\times n}$为单位正交矩阵.

1.6 证: 设 Q 关于 $\{O; \delta_i\}$ 的坐标为: $x = (x^1, \dots, x^n)$, 即 $Q - O = x^1\delta_1 + \cdots + x^n\delta_n$.
由第5题知 σ 为线性变换，而 $\sigma(O) = P,\ \sigma(\delta_i) = e_i$,
$\therefore\ \sigma(Q) - \sigma(O) = \sigma(Q - O) = \sigma(x^1\delta_1 + \cdots + x^n\delta_n) = x^1\sigma(\delta_1) + \cdots + x^n\sigma(\delta_n) = x^1 e_1 + \cdots + x^n e_n$
即点 Q' 关于 $\{P; e_i\}$ 的坐标等于点 Q 关于 $\{O; \delta_i\}$ 的坐标.

1.7 证:设 $\{O,\delta_i\}$ 为 E_n 中某一直角坐标系， f 在 $\{O,\delta_i\}$ 之下表示成

$$\begin{aligned}\overrightarrow{Of(t)} &= \sum_{i=1}^{n} x^i(t)\delta_i,\ \forall\ t\in\mathbb{R}\\ &= (\delta_1,...,\delta_n)\begin{pmatrix} x^1(t)\\ :\\ x^n(t)\end{pmatrix}\end{aligned}$$

任取另一直角坐标系 $\{P,e_i\}$ 则有唯一表示:

$$\overrightarrow{OP}=\sum_{i=1}^{n}a^i\delta_i=(\delta_1,...,\delta_n)\begin{pmatrix} a^1(t)\\ :\\ a^n(t)\end{pmatrix}\qquad e_i=\sum_{i=1}^{n}a^i\delta_i=(\delta_1,...,\delta_n)\begin{pmatrix} a_i^1\\ :\\ a_i^n\end{pmatrix}\quad (i=1,2,...n)$$

从而

$$(e_1,...e_n)=(\delta_1,...\delta_n)\begin{pmatrix} a_1^1 & \cdots & a_n^1\\ : & & :\\ a_1^n & \cdots & a_n^n\end{pmatrix}$$

记 $(a_j^i)=A.$ 由 $(e_1,...e_n)$ 与 $(\delta_1,...,\delta_n)$ 均为正交向量组知 $|A|\neq 0.$
$\therefore A$ 可逆。记 $A^{-1}=(b_j^i)_{n\times n}$ $\therefore (\delta_1,...,\delta_n)=(e_1,...e_n)A^{-1}$

$$\overrightarrow{Of(t)}=(\delta_1,...,\delta_n)\begin{pmatrix} x^1(t)\\ :\\ x^n(t)\end{pmatrix}=(e_1,...e_n)A^{-1}\begin{pmatrix} x^1(t)\\ :\\ x^n(t)\end{pmatrix}$$

$$\overrightarrow{OP}=(\delta_1,...,\delta_n)\begin{pmatrix} a^1(t)\\ :\\ a^n(t)\end{pmatrix}=(e_1,...e_n)A^{-1}\begin{pmatrix} a^1(t)\\ :\\ a^n(t)\end{pmatrix}$$

而 $\overrightarrow{Of(t)}=\overrightarrow{OP}+\overrightarrow{Pf(t)}$ 从而

$$\begin{aligned}\overrightarrow{Pf(t)} &= \overrightarrow{Of(t)}-\overrightarrow{OP}\\ &= (e_1,...e_n)A^{-1}\begin{pmatrix} x^1(t)\\ :\\ x^n(t)\end{pmatrix}-(e_1,...e_n)A^{-1}\begin{pmatrix} a^1(t)\\ :\\ a^n(t)\end{pmatrix}\\ &= (e_1,...e_n)A^{-1}\begin{pmatrix} x^1(t)-a^1(t)\\ :\\ x^n(t)-a^n(t)\end{pmatrix}\\ &\triangleq \sum_{k=1}^{n}e_ky^k(t)\qquad \text{其中 } y^k(t)=\sum_{j=1}^{n}b_j^k(x^j(t)-a^j(t))\quad (k=1,2,...,n)\end{aligned}$$

由 $x^j(t)$ 连续（或r次连续可微）可得 $y^k(t)$ 连续（或r次连续可微），从而映射 $f:\mathbb{R}\to E^n$ 的连续性和r次连续可微性与 E^n 中直角坐标系的选取无关。

1.8 证: 记 $f(t)$ 在 $\{O;\delta_i\}$ 与 $\{P;e_i\}$ 下的坐标为 $(x^1(t),...,x^n(t)),(y^1(t),...,y^n(t))$, 其中

$$\overrightarrow{OP}=\sum_{i=1}^n a_i\delta_i,\qquad e_i=\sum_{j=1}^n a_i^j\delta_j\quad(i=1,...n)$$

由的（1.14）

$$x^i(t)=a^i+\sum_{j=1}^n y^j(t)a_j^i$$

则等式两边同时对t求导有

$$\frac{dx^i}{dt}(t_0)=\sum_{j=1}^n\frac{dy^j(t)}{dt}a_j^i$$

又由（2.6）

$$\begin{aligned}f'(t_0)&=\sum_{i=1}^n\frac{dx^i}{dt}(t_0)\delta_i\\&=\sum_{i=1}^n\sum_{j=1}^n\frac{dy^j}{dt}a_j^i\delta_i\\&=\sum_{j=1}^n\left(\frac{y^j(t)}{dt}\sum_{i=1}^n a_j^i\delta_i\right)\\&=\sum_{j=1}^n\frac{dy^j}{dt}e_j\end{aligned}$$

从而可见其形式不变，即切向量定义式（2.6）与直角坐标系的选取无关。

1.9 证:

(1)

$$\begin{aligned}D_v(g+\lambda h)&=\langle\nabla(g+\lambda h)(P),v\rangle\\&=\langle\nabla g(P)+\lambda\nabla h(P),V\rangle\\&=\langle\nabla g(P),v\rangle+\lambda\langle\nabla h(P),v\rangle\\&=D_v(g)+\lambda D_v(h)\end{aligned}$$

(2)

$$\begin{aligned}D_v(g\cdot h)&=\langle\nabla(gh)(P),v\rangle\\&=\langle((\nabla g)h)(P),v\rangle+\langle(g(\nabla h))(P),v\rangle\\&=h(P)\langle\nabla g(P),v\rangle+g(P)\langle\nabla h(P),v\rangle\\&=h(P)D_vg+g(P)D_vh\end{aligned}$$

1.10 证: $E_n \to R$ 上的函数 $x^i : P = \lambda^1\delta_1 + ... + \lambda^n\delta_n \to \lambda^i \quad (i = 1, 2, ..., n)$
$\forall\ P = (\lambda^1, ..., \lambda^n) \in E^n$, 在 P 的邻域内取 $Q = P + \triangle P = (\lambda^1 + \triangle\lambda^1, ..., \lambda^n + \triangle\lambda^n)$, 则

$$\lim_{\triangle\lambda^j \to 0} \frac{x^i(Q) - x^i(P)}{\triangle\lambda^j} = \lim_{\triangle\lambda^j \to 0} \frac{\triangle\lambda^i}{\triangle\lambda^j} = \delta_i^j \in C^\infty$$

从而 $x^i \in C^\infty \quad (\forall\ i = 1, 2, ..., n)$.

1.11 证:

(1) 在 E^m 中，取两新旧直角坐标分别为 $\{O; \delta_i\}, \{P; e_i\}$, 且满足

$$\begin{cases} \overrightarrow{OP} = \sum_{i=1}^m a^i\delta_i \\ e_j = \sum_{i=1}^m a_j^i\delta_i \end{cases} \qquad j = 1, .., m.$$

在 E^n 中，取两新旧直角坐标分别为 $\{O; \xi_i\}, \{P; \eta_i\}$, 且满足

$$\begin{cases} \overrightarrow{OQ} = \sum_{i=1}^m b^i\xi_i \\ \eta_j = \sum_{i=1}^m b_j^i\xi_i \end{cases} \qquad j = 1, .., m.$$

记原映射为 $F(\lambda^1, .., \lambda^m) = (f^1(\lambda^1, .., \lambda^m), ..., f^n(\lambda^1, .., \lambda^m))$.
设在新坐标下表示为 $G(\mu^1, ..., \mu^m) = (g^1(\mu^1, .., \mu^m), ..., g^n(\mu^1, .., \mu^m))$.
则有 $f^l = b^l + \sum_{j=1}^n b_j^l g^j \quad \lambda^k = a^k + \sum_{j=1}^m a_j^k \mu^j \ (k = 1, .., m)$. 再记 $(a_j^k)_{m\times m}$ 的逆为 $(C_k^j)_{m\times m}$

$$\begin{aligned} \therefore \frac{\partial f^l}{\partial\lambda^k} &= \sum_{j=1}^n b_j^l \frac{\partial g^j}{\partial\lambda^k} \\ &= \sum_{j=1}^n b_j^l (\sum_{i=1}^m \frac{\partial g^j}{\partial\mu^i} C_k^i) \\ &= \sum_{j=1}^n sum_{i=1}^m b_j^l frac\partial g^j \partial\mu^i C_k^i \end{aligned}$$

$$\therefore (\frac{\partial f^l}{\partial\lambda^k})_{m\times n} = A^{-1}(\frac{\partial g^j}{\partial\mu^i})_{m\times n} B$$

记 J_f, J_g 分别为 $F(\lambda^1, .., \lambda^m)$ 与 $G(\mu^1, .., \mu^m)$ 的Jacobi矩阵。则有 $J_f = A^{-1}J_gB$.

(2) 由于A、B可逆，所以 A^{-1}, B 均可表为若干初等行、列变换的乘积，
$\therefore r(J_f) = r(J_g)$, 在任意点 x_0.

(3) 在 E^m 中任取点P，记 $Q = f(P)$.
在 P 点邻域分别取两新旧曲纹坐标系 $(u^1, ..., u^m), (v^1, ..., v^m)$. 两者之间由同胚映射 g 关联: $v^i = g^i(u^1, ..., u^m) \quad i = 1, .., m.$ 记 $g^{-1} = \bar{g}$ 则 $u^i = \bar{g}^i(v^1, .., v^m) \quad i = 1, .., m.$
同样在 Q 点邻域分别取两新旧曲纹坐标系 $(s^1, ..., s^n), (t^1, ..., t^n)$. 两者之间由同胚映射 h 关联: $t^i = h^i(s^1, ..., s^n) \quad i = 1, .., m$
原函数 $f(u^1, .., u^m)$ 中分量记为 $f^i(u^1, .., u^m)$ $(i = 1, .., n)$ 在 E^m 与 E^n 间曲纹坐标变换

下为 $\tilde{f}(v^1,..,v^m)$, 其在新坐标下分量为: $\tilde{f}^i(v^1,..,v^m)=h^i(f^1,..,f^n)$ 其中 $f^j(u^1,..,u^m)=f^j(\bar{g}(v^1,..,v^m),...,\bar{g}(v^1,..,v^m))$ $(i=1,..,n;j=1,..,n)$

从而在变换后Jacobi矩阵 $J_{n\times m}$ 为

$$\begin{aligned}J^i_j&=\frac{\partial\tilde{f}^i}{\partial v^j}\\&=\frac{\partial h^i}{\partial f^k}\frac{\partial f^k}{\partial u^r}\frac{\partial\bar{g}^r}{\partial v^j}\quad(i,k=1,..,n,r,j=1,..,m)\end{aligned}$$

在 P 点，记 g 的雅克比矩阵 $(\frac{\partial g^i}{\partial u^j})_{m\times m}$ 为 G, h 的雅克比矩阵 $(\frac{\partial h^i}{\partial v^j})_{n\times n}$ 为 H, 变换前 f 的雅克比矩阵 $(\frac{\partial f^i}{\partial u^j})_{n\times m}$ 为 J_0 则有：

$$J=HJ_0G^{-1}$$

1.12 证: 记原方程组为$x^i=f^i(u^1,..,u^n),\ i=1,..,n$. 则 $f=f^i\delta_i$ 的雅克比矩阵的行列式为

$$\begin{aligned}\left|\frac{\partial(f^1,..,f^n)}{\partial(u^1,..,u^n)}\right|&=\begin{vmatrix}-x^1tanu^1&-x^1tanu^2&\cdots&-x^1tanu^{n-2}&-x^1tanu^{n-1}&\frac{x^1}{u^n}\\-x^2tanu^1&-x^2tanu^2&\cdots&-x^2tanu^{n-2}&x^2cotu^{n-1}&\frac{x^2}{u^n}\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\-x^{n-1}tanu^1&x^{n-1}cotu^2&\cdots&0&0&\frac{x^{n-1}}{u^n}\\x^ncotu^1&0&\cdots&0&0&\frac{x^n}{u^n}\end{vmatrix}\\&=x^1x^2\cdots x^n\begin{vmatrix}-tanu^1&-tanu^2&\cdots&-tanu^{n-2}&-tanu^{n-1}&1\\-tanu^1&-tanu^2&\cdots&-tanu^{n-2}&cotu^{n-1}&1\\-tanu^1&-tanu^2&\cdots&cotu^{n-2}&0&1\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\-tanu^1&\cot u^2&\cdots&0&0&1\\\cot u^1&0&\cdots&0&0&1\end{vmatrix}\\&=\frac{x^1x^2\cdots x^n}{u^n}\cdot\Big[(-1)^{1+n}(-1)^{\frac{(n-1)(n-2)}{2}}cotu^1\cdots cotu^{n-1}\\&\qquad+(-1)^{2+n}(-1)^{\frac{(n-1)(n-2)}{2}}cotu^1\cdots cotu^{n-2}(-tanu^{n-1})\Big]\\&=(-1)^{\frac{n^2-n+4}{2}}\left(\prod_{i=1}^{n-2}r^i\prod_{j=1}^{i}cosu^j\right)x^1x^2(u^n)^{n-3}\frac{1}{sinu^{n-1}cosu^{n-1}}\end{aligned}$$

当 $x^1,x^2\neq 0$ 时，有 $u^n,cosu^1,\cdots,cosu^{n-1},sinu^{n-1}$ 非零，又由 $r^1,\cdots r^n$ 为正数可知: $\left|\frac{\partial(f^1,..,f^n)}{\partial(u^1,..,u^n)}\right|\neq 0$, 即 f 的秩为 n . 从而 $(u^1,\cdots u^n)$ 给出 E^n 中除坐标面 $\{(0,0,x^3,\cdots,x^n):x^3,\cdots,x^n\in\mathbb{R}\}$ 以外的任意一点的邻域内的曲纹坐标系。

1.13 解: 设 (x,y,z) 是 E^3 中的直角坐标系,令

$$\begin{cases}x=\rho\cos\psi\cos\theta\\y=\rho\cos\psi\sin\theta\qquad(0<\rho<+\infty,-\frac{\pi}{2}<\psi<\frac{\pi}{2},-\pi<\theta<\pi)\\z=\rho\sin\psi\end{cases}$$

则 (ρ,ψ,θ) 给出球坐标系，记 $u^1=\rho,u^2=\psi,u^3=\theta$ 则可求其自然标架场如下：

$$\overrightarrow{r_1}=(cos\psi cos\theta,cos\psi sin\theta,sin\psi)$$
$$\overrightarrow{r_2}=(-\rho sin\psi cos\theta,-\rho sin\psi sin\theta,\rho cos\psi)$$
$$\overrightarrow{r_3}=(-\rho cos\psi sin\theta,\rho cos\psi cos\theta,0)$$

则 $\forall\ Q=r(P),\{Q,r_i\}$ 为 E^3 中球坐标系诱导的自然标架场。
下求其度量系数：

$$g_{11}=\langle\overrightarrow{r_1},\overrightarrow{r_1}\rangle=1 \qquad g_{12}=g_{21}=\langle\overrightarrow{r_1},\overrightarrow{r_2}\rangle=0$$
$$g_{31}=g_{13}=\langle\overrightarrow{r_1},\overrightarrow{r_3}\rangle=0$$
$$g_{22}=\langle\overrightarrow{r_2},\overrightarrow{r_2}\rangle=\rho^2 \qquad g_{23}=g_{32}=\langle\overrightarrow{r_2},\overrightarrow{r_3}\rangle=0$$
$$g_{33}=\langle\overrightarrow{r_3},\overrightarrow{r_3}\rangle=\rho^2cos^2\psi$$

$\therefore$ 度量系数为 $g_{11}=\rho^2,\ g_{33}=\rho^2cos^2\psi,\ g_{ij}=0\ (i\neq j$ 时,其中 $i,j=1,2,3)$.
则 $g^{11}=1,\ g^{22}=\frac{1}{\rho^2},\ g^{33}=\frac{1}{\rho^2cos^2\psi},\ g^{ij}=0\ (i\neq j$ 时,其中 $i,j=1,2,3)$.
且 $\frac{\partial g_{11}}{\partial u^i}=0\ (i=1,2,3)$ $\frac{\partial g_{ij}}{\partial u^k}\ (i,j,k=1,2.3)$ $\frac{\partial g_{22}}{\partial u^1}=2\rho$ $\frac{\partial g_{22}}{\partial u^2}=0$ $\frac{\partial g_{22}}{\partial u^3}=0$ $\frac{\partial g_{33}}{\partial u^1}=2\rho cos^2\psi$ $\frac{\partial g_{33}}{\partial u^2}=-\rho^2sin2\psi$ $\frac{\partial g_{33}}{\partial u^3}=0$
从而由

$$\varGamma_{il}^k=\frac{1}{2}g^{kj}(\frac{\partial g_{ij}}{\partial u^l}+\frac{\partial g_{jl}}{\partial u^i}-\frac{\partial g_{il}}{\partial u^j})$$

得：

$$\varGamma_{33}^1=\frac{1}{2}g^{11}(\frac{\partial g_{31}}{\partial u^3}+\frac{\partial g_{13}}{\partial u^1}-\frac{\partial g_{33}}{\partial u^1})=\rho cos^2\psi$$
$$\varGamma_{33}^2=\frac{1}{2}g^{22}(\frac{\partial g_{32}}{\partial u^3}+\frac{\partial g_{32}}{\partial u^1}-\frac{\partial g_{33}}{\partial u^2})=-sins\psi$$
$$\varGamma_{33}^3=\frac{1}{2}g^{33}(\frac{\partial g_{33}}{\partial u^3}+\frac{\partial g_{33}}{\partial u^3}-\frac{\partial g_{33}}{\partial u^3})=0$$

又由

$$\frac{\partial\overrightarrow{r_1}}{\partial u^1}=0 \qquad \frac{\partial\overrightarrow{r_1}}{\partial u^2}=\frac{1}{\rho}\overrightarrow{r_2} \qquad \frac{\partial\overrightarrow{r_1}}{\partial u^3}=\frac{1}{\rho}\overrightarrow{r_3}$$
$$\frac{\partial\overrightarrow{r_2}}{\partial u^2}=-\rho\overrightarrow{r_1} \qquad \frac{\partial\overrightarrow{r_2}}{\partial u^3}=-tan\psi\overrightarrow{r_3}$$

从而 Christoffel 系数为：

$$\varGamma_{12}^2=\varGamma_{21}^2=\frac{1}{rho} \qquad \varGamma_{31}^3=\varGamma_{13}^3=\frac{1}{rho}$$
$$\varGamma_{22}^1=-rho \qquad \varGamma_{23}^3=\varGamma_{32}^3=-tan\psi$$
$$\varGamma_{33}^1=-\rho cos^2\psi \qquad \varGamma_{33}^2=-sin^2\psi$$

其余为 0.

1.14 解: 设 (x,y,z) 为 E^3 中直角坐标系，设

$$\begin{cases} x=\rho cos\theta \\ y=\rho sin\theta \\ z=t \end{cases} \qquad (其中\ 0<r<+\infty,\ -\pi<\theta<\pi,\ -\infty<t<+\infty)$$

则 (ρ,θ,t) 为柱坐标系。记 $u^1=\rho,\ u^2=\theta,\ u^3=t$, 则有：

$$\begin{aligned}\overrightarrow{r_1}&=(cos\theta,sin\theta,0)\\ \overrightarrow{r_2}&=(-\rho sin\theta,\rho cos\theta,0)\\ \overrightarrow{r_3}&=(0,0,1)\end{aligned}$$

且 $g_{11}=1,\ g_{22}=\rho^2,\ g_{33}=1,\ g_{ij}=0\,(i\neq j\text{时})$ 由于：

$$\begin{array}{rrr}\frac{\partial\overrightarrow{r_1}}{\partial u^1}=0 & \frac{\partial\overrightarrow{r_1}}{\partial u^2}=\frac{1}{\rho}\overrightarrow{r_2} & \frac{\partial\overrightarrow{r_1}}{\partial u^3}=0\\ & \frac{\partial\overrightarrow{r_2}}{\partial u^2}=-\rho\overrightarrow{r_1} & \frac{\partial\overrightarrow{r_2}}{\partial u^3}=0\\ & & \frac{\partial\overrightarrow{r_3}}{\partial u^3}=0\end{array}$$

所以 Christoffel 系数为：

$$\varGamma_{12}^2=\varGamma_{21}^2=\frac{1}{\rho}\quad \varGamma_{22}^1=-\rho$$

其余为0

1.15 证: $\because\ v=v^i\overrightarrow{e_i}$

$$\begin{aligned}\therefore\ dv&=v^jd\overrightarrow{e_j}+\overrightarrow{e_i}dv^i\\ &=v^j(\frac{\partial\overrightarrow{e_j}}{\partial u^k}du^k)+\overrightarrow{e_i}dv^i\\ &=v^j\varGamma_{jk}^i\overrightarrow{e_i}du^k+\overrightarrow{e_i}dv^i\\ &=(v^j\varGamma_{jk}^idu^k+dv^i)\overrightarrow{e_i}\end{aligned}$$

$\therefore\ v$为平行向量场 $\Leftrightarrow v^j\varGamma_{jk}^i+dv^i=0(\forall\ i=1,2,...,n)$

1.16 证: 定义加法 $'+':(f+g)(x)=f(x)+g(x)$, 数乘: $(\lambda f)(x)=\lambda f(x)$
先证 $(\mathscr{L}(v_1,\cdots,v_r;\mathbb{R}),+)$ 构成加法群。

(1) $\forall\,f,g\in\mathscr{L}(v_1,\cdots,v_r;\mathbb{R})\quad\forall\,i\in\{1,\cdots,r\}\quad\forall\,x_1=(x^1,\cdots,x_1^i,\cdots,x^r),x_2=(x^1,\cdots,x_2^i,\cdots,x^r)\in V_1\times\cdots\times V_r\quad\forall\,k_1,k_2\in\mathbb{R}$ 有

$$\begin{aligned}&(f+g)(x^1,\cdots,k_1x_1^i+k_2x_2^i,\cdots x^r)\\ =&f(x^1,\cdots,k_1x_1^i+k_2x_2^i,\cdots x^r)+g(x^1,\cdots,k_1x_1^i+k_2x_2^i,\cdots x^r)\\ =&(k_1f(x_1)+k_2f(x_2))+(k_1g(x_1)+k_2g(x_2))\\ =&k_1(f+g)(x_1)+k_2(f+g)(x_2)\end{aligned}$$

$\therefore f+g\in\mathscr{L}(V_1,\cdots,V_r;\mathbb{R})$

(2) $\forall\ f,g,h\in\mathscr{L}(V_1,\cdots,V_r;\mathbb{R})\quad\forall\ x\in V_1\times,\cdots,V_r$, 有

$$[(f+g)+h](x)=(f+g)(x)+h(x)=f(x)+g(x)+h(x)$$

$$[f+(g+h)](x)=f(x)+(g+h)(x)=f(x)+g(x)+h(x)$$

$\therefore(f+g)+h=f+(g+h)$

(3) 取 θ 为 $\forall\, x \in V_1 \times \cdots \times V_r, \theta(x) = 0$
则显然 $f + \theta = \theta + f, \forall\, f$

(4) $\forall\, f$, 取 $-f$ 为 $(-f)(x) = -f(x)$ 则 $f + (-f) = (-f) + f = \theta$

(5) $\forall\, f, g \in \mathscr{L}(V_1 \cdots V_r; \mathbb{R})$ 显然 $f + g = g + f$.

所以 $(\mathscr{L}(v_1, \cdots, v_r; \mathbb{R}), +)$ 构成Abel群。$\forall\, \lambda, \mu \in \mathbb{R} \quad \forall f, g \in \mathscr{L}(V_1 \times \cdots \times V_r; \mathbb{R})$, 又有

(6) $(\lambda, \mu)f = \lambda f + \mu f$

(7) $(\lambda\mu)g = \lambda(\mu g)$

(8) $\lambda(f + g) = \lambda f + \lambda g$

(9) $1 \cdot f = f$

所以 $\mathscr{L}(V_1, \cdots, V_r; \mathbb{R})$ 构成 $\mathbb{R}$ 上的线性空间。

1.17 证:

(1) $\forall\, k_1, k_2 \in \mathbb{R}\ u_1, u_2 \in V$, 则由 $\tilde{f}$ 为2重线性函数知

$$\begin{aligned}
f(k_1u_1 + k_2u_2) =& \sum_{i=1}^{n} \tilde{f}(k_1u_1 + k_2u_2, \delta^i)\delta_i \\
=& \sum_{i=1}^{n} (k_1\tilde{f}(u_1, \delta^i) + k_2\tilde{f}(u_2, \delta^i))\delta_i \\
=& k_1 \sum_{i=1}^{n} \tilde{f}(u_1, \delta^i)\delta_i + k_2 \sum_{i=1}^{n} \tilde{f}(u_2, \delta^i)\delta_i \\
=& k_1 f(u_1) + k_2 f(u_2)
\end{aligned}$$

所以 f 为线性变换。

(2) 若另取基底 $\{e^i\}$, 且 $e^i = a^i_j\delta_j$ 其对偶基底 $\{e_i\}$, 则有 $\delta_j = a^i_j e_i$, 其中 (b^j_i) 为 (a^j_i) 的逆矩阵。且由

$$\delta^i(e_j) = \delta_i(a^k_j\delta_k) = a^k_j\delta^i_k = a^i_j = a^i_k e^k(e_j)$$

知 $\delta^i = a^i_k e^k$.
$\therefore f(u) = \tilde{f}(u, \delta^i)\delta_i = \tilde{f}(u, a^i_k e^k)(b^j_i e_j) = a^i_k b^j_i \tilde{f}(u, e^k)e_j = \delta^j_k \tilde{f}(u, e^k)e_j = \tilde{f}(u, e^j)e_j$
$\therefore$ f 的定义与基底的选取无关。

1.18 证: 取定 $t, \forall\ v = (v_1, \cdots v_r) \in V \times \cdots \times V \quad v^1 \in V^\star$, 取 $F : V^\star \times V_1 \cdots \times V_r \to \mathbb{R}$ 为 $F(v^1, v_1, \cdots\ v_r) = v^1(t(v))$ ，显然 F 是 $V^\star \times V \times \cdots V$ 上的一个 $1 + r$ 重线性函数，即 $(1, r)$ 型张量。
反之，若给定一个 $(1, r)$ 型张量 $F : V^\star \times V_1 \times \cdots V_r$, $\forall\ v = (v1, \cdots, v_r) \in V \times \cdots \times \to \mathbb{R}$, 令

$t(v) = F(\delta^i, v_1, \cdots, v_r)\delta_i$, 显然 t 为 r 重线性映射。
所以 t 等同一个 $(1, r)$ 型张量。

1.19 证:

(1) $\forall\ \alpha_1, \cdots, \alpha_2 \in \mathscr{L}(V_1, \cdots, V_p; \mathbb{R}), \quad \forall\ \beta \in \mathscr{L}(W_1, \cdots, W_q; \mathbb{R})$
$\forall\ v \in V_1 \times \cdots \times V_p, \quad w \in W_1 \times \cdots \times W_q$

$$\begin{aligned}
&(\alpha_1 + \alpha_2) \otimes \beta(v, w) \\
=&(\alpha_1 + \alpha_2)(v) \cdot \beta(w) \\
=&[\alpha(_1(v) + \alpha_2(v)] \cdot \beta(w) \\
=&\alpha_1(v) \cdot \beta(w) + \alpha_2(v) \cdot \beta(w) \\
=&\alpha_1 \otimes \beta(v, w) + \alpha_2 \otimes \beta(v, w) \\
=&(\alpha_1 \otimes \beta + \alpha_2 \otimes \beta)(v, w)
\end{aligned}$$

$\therefore\ (\alpha_1 + \alpha_2) \otimes \beta = \alpha_1 \otimes \beta + \alpha_2 \otimes \beta$ 同理 $\beta \otimes (\alpha_1 + \alpha_2) = \beta \otimes \alpha_1 + \beta \otimes \alpha_2$

(2) $\forall\ \alpha \in \mathscr{L}(V_1, ..., Vr; \mathbb{R}),\ \beta \in \mathscr{L}(W_1, ..., W_s; \mathbb{R}),\ \gamma \in \mathscr{L}(Z_1, ..., Z_t; \mathbb{R})$
$\forall\ V_1 \times \cdots \times V_r,\ w \in W_1 \times \cdots \times W_s,\ z \in Z_1 \times \cdots \times Z_t$, 则

$$\begin{aligned}
&(\alpha \otimes \beta) \otimes \gamma(v, w, z) \\
=&(\alpha \otimes \beta)(v, w) \cdot \gamma(z) \\
=&(\alpha(v) \cdot \beta(w)) \cdot \gamma(z) \\
=&\alpha(v) \cdot [\beta(w)\gamma(z)] \\
=&\alpha(v) \cdot [(\beta \otimes \gamma(w, z))] \\
=&[\alpha \otimes (\beta \otimes \gamma)](v, w, z)
\end{aligned}$$

$\therefore (\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$
故多重线性函数的张量积服从分配率和结合律。

1.20 证: 设 $\{\xi_i\}$ 为 V 的基，$\{\eta_i\}$ 为 W 的基，记 $A = \{V \otimes W | v \in V, w \in W\}$, 下证 A 不构成线性空间：
由于 $dim(V) \geqslant 2, dim(W) \geqslant 2$. 所以分别可取 V, W 的两组基：$\xi_1, \xi_2, \eta_1, \eta_2$,则

$$\xi_1 \otimes \eta_1 \in A,\ \xi_2 \otimes \eta_2 \in A,$$

若

$$\xi_1 \otimes \eta_1 + \xi_2 \otimes \eta_2 \in A$$

则 $\exists\ a^i\ (i = 1, 2, ..., n)\ s.t.$

$$\xi_1 \otimes \eta_1 + \xi_2 \otimes \eta_2 = (a^i \xi_i) \otimes (b^j \eta_j) = a^i b^j \xi_i \otimes \eta_j$$

移项并合并后，由于 $\{\xi_i\eta_j\}$ 线性无关，所以有

$$a^ib^j=\begin{cases}1 & i=j=1 \text{ 以及 } i=j=2 \text{ 时;}\\ 0 & \text{其余情况.}\end{cases}$$

$a^1b^1=a^2b^2=1$, 从而 $a^1,b^1,a^2,b^2\neq 0$, 但 $a^1b^2=0$, 矛盾!
$\therefore \xi_1\otimes\eta_1+\xi_2\otimes\eta_2\notin A$, A 不构成线性空间。

1.21 证:

(1) 由题知 $f(\delta_i)=b_i^j\delta_i$.
若另取基底 $\{e_i\}$ 且 $e_i=a_i^j\delta_j$ 从而 $\delta_j=c_j^ie_i$ (其中 (c_i^j) 为 (a_i^j) 的逆).
则 $f(e_i)=f(a_i^j\delta_j)=a_i^jf(\delta_j)=a_i^jb_j^k\delta_k=a_i^jb_j^kc_k^le_l$.
即 $f(e_i)=(a_i^jb_j^kc_k^l)e_l$, 记 $d_i^l=a_i^jb_j^kc_k^l$.
$\therefore f$ 在基 $\{e_i\}$ 下矩阵是 (d_i^l).
此时

$$\begin{aligned}B_3&=d_i^jd_k^id_j^k\\&=a_i^sb_s^tc_t^ja_k^eb_e^fc_f^ia_j^gb_g^hc_h^k\\&=(a_i^sc_f^i)(a_k^ec_h^k)(a_j^gc_t^j)b_s^tb_e^fb_g^h\\&=(\delta_f^sb_e^f)(\delta_h^eb_g^h)(\delta_t^gb_s^t)\\&=b_e^sb_g^eb_s^g\end{aligned}$$

$\therefore B_3$ 与基底的选取无关。

(2) 令 $F:V^\star\times V^\star\to\mathbb{R}$ 为 $F(\delta^i,u)=\delta^i(f(u))$. $(\forall u\in V)$. 则易证 F 为 (1,1) 型张量，且

$$\begin{aligned}B_3&=b_i^jb_k^ib_j^k\\&=F(\delta^j,\delta_i)\cdot F(\delta^i,\delta_k)\cdot F(\delta^k,\delta_j)\\&=F(\delta^j,\delta_i)\cdot[(C_1^2F\otimes F)(\delta^i,\delta_j)]\\&=C_2^1[F\otimes C_1^2(F\otimes F)](\delta^j,\delta_j)\\&=C_1^1\{C_2^1[F\otimes C_1^2(F\otimes F)]\}.\end{aligned}$$

1.22 解: 令 $F:V^\star\times V\to\mathbb{R}$ $F(v^i,v_j)=v^i(v_j)$. 易证 F 为 (1,1) 型张量. 任取 V 的基底 $\{\delta_i\}$, 其对偶基底 $\{\delta^i\}$, 则 $\{\delta_i\otimes\delta^j\}$ 为 F 的基底,且 $\delta^i(\delta_j)=\delta_j^i$. 从而 $F(\delta^i,\delta_j)=\delta_i^j$.

$$F=F(\delta^i,\delta_j)\delta_i\otimes\delta^j=\delta_i^j\delta_i\otimes\delta^j=\sum_i\delta_i\otimes\delta^i$$

$\therefore \delta_i^j$ 为 F 的分量。

1.23 解: 若存在 (0,2) 型张量 F 满足题设条件，则任取 V 的一组基底 $\{\delta_i\}$, $F(\delta_i,\delta_j)=\delta_{ij}$. 另任取一基底 $\{e_i\}$ ，且 $e_i=a_i^j\delta_j$, 则

$$\begin{aligned}F(e_i,e_j)=\delta_{ij} &\Leftrightarrow F(a_i^k\delta_k,a_j^r\delta_r)=\delta_{ij}\\ &\Leftrightarrow a_i^k a_j^r F(\delta_k,\delta_r)=\delta_{ij}\\ &\Leftrightarrow a_i^k a_j^k=\delta_{ij}\\ &\Leftrightarrow (a_i^j)\text{ 为单位正交阵}\end{aligned}$$

但 a_i^j 未必为单位正交阵，矛盾!
所以不存在。

1.24 证: $\forall\, v_1,\cdots,v_q\in V$, $(\sigma(f))(v_1,\cdots,v_q)=f(v_{\sigma(1)},\cdots,v_{\sigma(q)})=\alpha^1(v_{\sigma(1)})\cdots\alpha^q(v_{\sigma(q)})$.

$$\begin{aligned}\alpha^{\tau(1)}(v_1)\otimes\cdots\otimes\alpha^{\tau(q)}(v_1,\cdots,v_q)&=\alpha^{\tau(1)}\otimes\cdots\otimes\alpha^{\tau(q)}(v_1,\cdots,v_q)\\ (\text{由 }\tau=\sigma^{-1}\text{ 知 }i=\sigma(\tau(i)))\quad&=\alpha^{\tau(1)}(v_{\sigma(\tau(1))})\cdots\alpha^{\tau(q)}(v_{\sigma(\tau(q))})\\ &=\alpha^1(v_{\sigma(1)})\cdots\alpha^q(v_{\sigma(q)})\end{aligned}$$

$\therefore\ \sigma(f)=\alpha^{\tau(1)}\otimes\cdots\otimes\alpha^{\tau(q)}$

1.25 证:

(1)

$$\begin{aligned}\sigma\text{ 是对称张量 }&\Leftrightarrow \sigma(\xi)=\xi\\ &\Leftrightarrow \frac{1}{q!}\sum_{\sigma\in\varphi(q)}\sigma(\xi)=\xi\\ &S_q(\xi)=\xi.\end{aligned}$$

(2)

$$\begin{aligned}\sigma\text{ 是反对称张量 }&\Leftrightarrow \sigma(\xi)=sign(\sigma)\xi\\ &\Leftrightarrow sign(\sigma)\cdot\sigma(\xi)=\xi\\ &\Leftrightarrow \sum_{\sigma\in\varphi(q)}sign(\sigma)\cdot\sigma(\xi)=\xi\cdot q!\\ &\Leftrightarrow A_q(\xi)=\xi\end{aligned}$$

1.26 证: $\forall\, \xi \in V_2^0, \{\delta^i\}$ 为 V 中的基，记 $\xi = \xi_{ij}\delta^i\delta^j$ 为二阶协变张量. 对于$\delta \triangleq \delta^i \otimes \delta^j$.

$$
\begin{aligned}
S_2(\delta) &= \frac{1}{2} \cdot (\delta^i \otimes \delta^j + \delta^j \otimes \delta^i) \ (\text{由24题}) \\
A_2(\delta) &= \frac{1}{2}(\delta^i \otimes \delta^j - \delta^j \otimes \delta^i) \\
\Rightarrow \delta &= \delta^i \otimes \delta^j = S_2(\delta) + A_2(\delta) \\
\therefore \xi =& \xi_{ij}\delta^i \otimes \delta^j \\
&= \delta_{ij}(S_2(\delta^i \otimes \delta^j) + A_2(\delta^i \otimes \delta^j)) \\
&= S_2(\xi) + A_2(\xi)
\end{aligned}
$$

其中 $S_2(\xi)$ 为对称张量，$A_2(\xi)$ 为反对称张量。

1.27 证：取 V 中基底 $\{\delta_i\}$, 由 $\varphi \in V_2^0$, 依题意知, $\forall\ \delta_i\delta_j\delta_k$ 有

$$
\begin{aligned}
\varphi(\delta_i, \delta_j, \delta_k) &= \varphi(\delta_j, \delta_i, \delta_k) \\
&= -\varphi(\delta_j, \delta_k, \delta_i) \\
&= -\varphi(\delta_k, \delta_j, \delta_i) \\
&= \varphi(\delta_k, \delta_i, \delta_j) \\
&= -\varphi(\delta_i, \delta_j, \delta_k) \\
\Rightarrow& \varphi(\delta_i, \delta_j, \delta_k) = 0. \\
\therefore \varphi &= \varphi(\delta_i, \delta_j, \delta_k)\delta^i \otimes \delta^j \otimes \delta^k = 0.
\end{aligned}
$$

1.28 证：取 V 的基 $\{\delta_i\}$, $\because a(x,x) = 0$, $\therefore a(\delta_i, \delta_j) = 0 \quad (i = j)$ 时.
$\forall i \neq j$ 取 $x = \delta_i + \delta_j$ 则

$$
\begin{aligned}
0 &= a(x,x) \\
&= a(\delta_i + \delta_j, \delta_i + \delta_j) \\
&= a_{ii} + a_{ij} + a_{ji} + a_{jj} \\
&= a_{ij} + a_{ji}.
\end{aligned}
$$

记 $a = a_{ij}\delta^i \otimes \delta^j$ 有

$$
\begin{aligned}
S_2(a) &= \frac{1}{2}a_{ij}(\delta^i \otimes \delta^j + \delta^j \otimes \delta^i) \\
&= \frac{1}{2}(a_{ij}\delta^i \otimes \delta^j + a_{ji}\delta^i \otimes \delta^j) \\
&= \frac{1}{2}(a_{ij} + a_{ji})\delta^i \otimes \delta^j \\
&= 0.
\end{aligned}
$$

1.29 证：由于 a 为任意对称张量，可取 a 使得 $a^{ij}\neq 0(\forall\, i,j)$ (例如，取 $a=\sum_i\sum_j\delta_i\otimes\delta_j$)，则

$$0=a^{ij}b_{ij}=a^{ji}b_{ji}=a^{ij}b_{ji}$$

$\therefore a^{ij}(b^{ij}+b^{ji})=0$, 此时$b_{ij}+b_{ji}=0\Rightarrow b_{ij}=-bji$,
$\therefore b$ 为反对称张量。

1.30 证：取 V 的基 $\{\delta_i\}$ ，由 x 的任意性， $\forall\, i,j$ 取 $x=\delta_i+\delta_j\in V$. 由 $f(x,x)=\tilde{f}(x,x)$ 知

$$\begin{aligned}
f(\delta_i+\delta_j,\delta_i+\delta_j)&=\tilde{f}(\delta_i+\delta_j,\delta_i+\delta_j)\\
\Rightarrow f(\delta_i,\delta_i)+f(\delta_i,\delta_j)+f(\delta_j,\delta_i)+f(\delta_j,\delta_j)&=\tilde{f}(\delta_i,\delta_i)+\tilde{f}(\delta_i,\delta_j)+\tilde{f}(\delta_j,\delta_i)+\tilde{f}(\delta_j,\delta_j)\\
\Rightarrow f(\delta_i,\delta_j)+f(\delta_j,\delta_i)&=\tilde{f}(\delta_i,\delta_j)+\tilde{f}(\delta_j,\delta_i)\\
\Rightarrow 2f(\delta_i,\delta_j)&=2\tilde{f}(\delta_i,\delta_j)\\
\Rightarrow f_{ij}&=\tilde{f}_{ij}
\end{aligned}$$

记 $f=f_{ij}\delta^i\otimes\delta^j\quad\tilde{f}=\tilde{f}_{ij}\delta^i\otimes\delta^j$, 则 $f=\tilde{f}$.

1.31 证： 原行列式$=\sum_\sigma\delta^{\sigma(i_1),\ldots,\sigma(i_r)}_{i_1,\ldots,i_r}\delta^{\sigma(i_1)}_{j_1}\cdots\delta^{\sigma(i_r)}_{j_r}=\delta^{j_1,\ldots,j_r}_{i_1,\ldots,i_r}$.

1.32 证： (1) 记置换

$$\pi=\begin{pmatrix} i_1,\cdots,i_{r+s}\\ j_1,\cdots,j_{r+s}\end{pmatrix}\quad \pi_1=\begin{pmatrix} k_1,\cdots,k_r\\ j_1,\cdots,j_r\end{pmatrix}\quad \pi_2=\begin{pmatrix} i_1,\cdots,i_r,i_{r+1},\cdots,i_{r+s}\\ k_1,\cdots,k_r,j_{r+1},\cdots,j_{r+s}\end{pmatrix}$$

则 $\pi=\pi_1\pi_2$, 从而 $sgn(\pi)=sgn(\pi_1)\cdot sgn(\pi_2)$.
即，取定某排列 $(l_1,\cdots,l_r)$, 有

$$\delta^{i_1,\cdots,i_{r+s}}_{j_1,\cdots,j_{r+s}}=\delta^{l_1,\cdots,l_r}_{j_1,\cdots,j_r}\cdot\delta^{i_1,\cdots,i_{r+s}}_{l_1,\cdots,l_r,j_{r+1},\cdots,j_{r+s}}\quad\text{(注：此时不对上下指标求和)}$$

而 $(l_1,\cdots,l_r)$ 的排列共有 $r!$ 种取法.

$$\therefore\delta^{i_1,\cdots,i_{r+s}}_{j_1,\cdots,j_{r+s}}=\frac{1}{r!}\sum_{k_1,\cdots,k_r}\delta^{k_1,\cdots,k_r}_{j_1,\cdots,j_r}\delta^{i_1,\cdots,i_{r+s}}_{k_1,\cdots,k_r,j_{r+1},\cdots,j_{r+s}}.$$

(2)

$$\delta^{i_1,\cdots,i_r}_{i_1,\cdots,i_r}=A^r_n=\frac{n!}{(n-r)!}.$$

1.33 证： 记 $a=(\alpha_1,...,\alpha_n)$, 其中 $\alpha_i=(a_i^1,...,a_i^n)^T$，则

$$\begin{aligned}\det a &= \det(\alpha_1,...,\alpha_n)\\ &= \frac{1}{n!}\delta^{i_1,...,i_n}_{1,...,n}\det(\alpha_{i_1},...,\alpha_{i_n})\\ &= \frac{1}{n!}\delta^{i_1,...,i_n}_{1,...,n}\delta^{1,...,n}_{j_1,...,j_n}a^{j_1}_{i_1}\cdots a^{i_n}_{j_n}\\ &= \frac{1}{n!}\delta^{i_1,...,i_n}_{j_1,...,j_n}a^{j_1}_{i_1}\cdots a^{i_n}_{j_n}.\end{aligned}$$

=======

第二章　预备知识

1.1 证: 令$V=\mathbb{R}^n,\ A=\mathbb{R}^n$, 则 $\forall x,\ y\in A$, 令 $\overrightarrow{xy}=y-x$ 则有:

① w $\forall\ x\in A,\ \overrightarrow{xx}=x-x=0$,

② $\forall\ x\in A,\ v\in V$, 令 $y=x+v\in V$. 则 $\overrightarrow{xy}=y-x=v$. 若另有 y', $s.\ t.\ \overrightarrow{xy'}=v$, 则 $y'-x=v\Rightarrow y'=x+v=y$ 从而 y 唯一,

③ $\forall\ x,y,z\in A.\ \overrightarrow{xy}+\overrightarrow{yz}=(y-x)+(z-y)=z-y=\overrightarrow{yz}$.

所以 $\mathbb{R}^n$ 是一个n维仿射空间，它以 $\mathbb{R}^n$ 自身为它的伴随向量空间。

1.2 证:

① $\forall\ P,Q\in E^n,\ 0\leqslant d(P,Q)=\left|\overrightarrow{PQ}\right|<+\infty$, 且$d(P,Q)=0\Leftrightarrow\left|\overrightarrow{PQ}\right|=0\Leftrightarrow P=Q$,

② $\forall\ P,Q\in E^n,\ d(P,Q)=\left|\overrightarrow{PQ}\right|=\left|\overrightarrow{QP}\right|=d(Q,P)$,

③ $\forall\ P,Q,R\in E^n,\ d(P,R)=\left|\overrightarrow{PR}\right|=\left|\overrightarrow{PQ}+\overrightarrow{QR}\right|\leqslant\left|\overrightarrow{PQ}\right|+\left|\overrightarrow{QR}\right|\leqslant d(Q,P)$.

从而 E^n 关于距离函数 d 成为一个度量。

1.3 证:

(1) 记 E^n 的全体开子集为 τ,

① 显然 $\emptyset,\ E^n\in\tau$,

② $\forall\ A\in\tau,\ B\in\tau$, 若 $A\cap B=\emptyset$, 则 $A\cap B\in\tau$,
若 $A\cap B\neq\emptyset$, 则 $\forall\ P\in A\cap B$, 即 $P\in A$ 且 $P\in B$,
则 $\exists\ \varepsilon_1,\ \varepsilon_2>0,\ s.\ t.\ P\in B_{\varepsilon_1}(P)\subset A,\ P\in B_{\varepsilon_2}(P)\subset B$, 取 $\varepsilon=min\{\varepsilon_1,\varepsilon_2\}$, 则 $P\in B_\varepsilon(P)=B_{\varepsilon_1}(P)\cap B_{\varepsilon_2}(P)\subset A\cap B$, 因而 $A\cap B\in\tau$.

③ 若 $A_\alpha(\alpha\in I)\in\tau$, 则 $\forall\ P\in\bigcup\limits_{\alpha\in I}A_\alpha,\ \exists\ i\in I,\ s.\ t.\ P\in A_i\in\tau$,
则 $\exists\ \varepsilon>0,\ s.\ t.\ P\in B_\varepsilon(P)\in A_i$, 从而 $B_\epsilon(P)\subset\bigcup\limits_{\alpha\in I}A_\alpha$, 从而 $\bigcup\limits_{\alpha\in I}A_\alpha\in\tau$.

所以, τ 为 E^n 的一个拓扑。

(2) $\forall\ P,Q\in E$ 且 $P\neq Q$. 则记 $d=d(P,Q)$, 取 $\varepsilon_1=\varepsilon_2=\frac{d}{3}$, 则 $P\in B_{\varepsilon_1(P)}$（开），$Q\in B_{\varepsilon_2}(Q)$（开），且 $B_{\varepsilon_1}(P)\cup B_{\varepsilon_2}(Q)=\emptyset$,
否则, 若 $\exists\ R\in B_{\varepsilon_1}(P)\cap B_{\varepsilon_2}(Q)$, 则 $d(Q,R)<\frac{d}{3}, d(Q,R)<\frac{d}{3}$,
$d=d(P,Q)\leq d(P,R)+d(Q,R)<\frac{2d}{3}<d$ 矛盾！ 所以 $B_{\varepsilon_1}(P)\cup B_{\varepsilon_2}(Q)=\emptyset$ 成立.
从而, E_n 满足 T_2 分离性公理, 为 Hausdorff 空间.

(3) 取开集族 $\mathscr{B}=\{B_\varepsilon(P)|P\in\mathbb{Q}^n,\varepsilon\in\mathbb{Q}\}$, 其中 $\mathbb{Q}$ 为有理数，故 $\mathscr{B}$ 为可数的，下证其为拓扑基：
$\forall\ P\in E_n,\quad \forall\ U=B_\varepsilon(P)\in N(P)$,
$\exists\ \varepsilon'>0$, 且 $\varepsilon'\in\mathbb{Q},\ s.\ t.\ \frac{3\varepsilon}{4}<\varepsilon'<\varepsilon.\quad \exists\ Q\in\mathbb{Q}^n,\ s.\ t.\ \left|\overrightarrow{PQ}\right|<\frac{\varepsilon'}{4}$,
令 $B=B_{\varepsilon'}(Q)$, 则 $B\subset U\in$ 且 $B\in\mathscr{B}$.
所以, $\mathscr{B}$ 为 E^n 中可数拓扑基, 从而 E^n 第二可数。

1.4 证:

(1) 任取 E_n 中直线 l, 在 l 上依次任取3个不同的点 P,Q,R, 则有 $\left|\overrightarrow{OQ}\right|=t\left|\overrightarrow{OP}\right|+\left|\overrightarrow{OR}\right|$, 其中 $t\in(0,1)$. 记 $\sigma(P)=P'\quad(\forall\ P\in E_n)$
则 $\left|\overrightarrow{O'Q'}\right|=\left|\overrightarrow{OQ}\right|=t\left|\overrightarrow{OP}\right|+(1-t)\left|\overrightarrow{OR}\right|=t\left|\overrightarrow{O'P'}\right|+(1-t)\left|\overrightarrow{O'R'}\right|\quad(t\in(0,1))$
$\therefore\ P',Q',R'$ 三点共线且保持分比，所以 σ 将直线映为直线.

(2) 任取 E_n 中两平行直线 l_1,l_2, 则由（1）知 l_1,l_2 在 σ 下仍为直线，记为 l_1',l_2'. 任取不同点 $A,B\in l_1$, 不同点 $C,D\in l_2$,
则 $\overrightarrow{AB},\overrightarrow{CD}$ 非零，且 $\overrightarrow{AB}//\overrightarrow{CD}$ 从而 $\exists\ \lambda\neq 0,\ s.\ t.\ \overrightarrow{AB}=\lambda\,\overrightarrow{CD}$,
而 $\left|\overrightarrow{A'B'}\right|=\left|\overrightarrow{AB}\right|=|\lambda|\cdot\left|\overrightarrow{CD}\right|=|\lambda|\cdot\left|\overrightarrow{C'D'}\right|\quad\therefore\overrightarrow{A'B'}//\overrightarrow{C'D'}\quad\therefore\ l_1'//l_2'$
$\therefore$ 由 l_1,l_2 任意性知 σ 把 E_n 中平行直线映为平行直线.

(3) 记 $\sigma(O)=O',\overrightarrow{OP_i}=\delta_i,\sigma(\delta_i)=\delta_i'=\overrightarrow{O'P'},i=1,2,...n$
则由 $\{O,\delta_i\}$ 为正交标架知

$$\left|\overrightarrow{OP_i}\right|\cdot\left|\overrightarrow{OP_j}\right|=\begin{cases}0, & i\neq j,\\ 1, & i=j,\end{cases}\qquad i,j=1,2,...,n$$

$\therefore\ \{O',\delta_i\}$ 也为正交坐标系.

1.5 证: $\forall\ t\leq 0.\ \forall\ x=(x_1,...,x_n),\ y=(y_1,...,y_n)\in\mathbb{R}^n$.
由 $d(x,y)=\sqrt{\sum\limits_{i=1}^{n}(y^i-x^i)^2}$ 知 $d(tx,ty)=td(x,y)$.
又由 σ 为等距变换知 $d(\sigma(tx),\sigma(ty)=d(tx,ty)=td(x,y)=td(\sigma(x),\sigma(y))$.
取 $y=\mathit{0}=(0,...,0)$, 则 $d(\sigma(tx),\sigma(\mathit{0})=t\cdot d(\sigma(x),\sigma(\mathit{0}))$,
则由 σ 保持共线性质而 $\overrightarrow{Ox}$ 与 $\overrightarrow{O(tx)}$ 共线, 知 $\overrightarrow{\sigma(\mathit{0})\sigma(x)}$ 与 $\overrightarrow{\sigma(\mathit{0})}\sigma(tx)$ 共线
$\therefore\ \overrightarrow{\sigma(\mathit{0})\sigma(tx)}=\pm t\overrightarrow{\sigma(\mathit{0})\sigma(x)}$ 不妨取 $+t,(-t$ 同理可证), 则有
$\sigma(tx)-\sigma(\mathit{0})=t(\sigma(x)-\theta(\mathit{0}))\Rightarrow\sigma(tx)=t\sigma(x)-(1-t)\sigma(\mathit{0})\cdots\cdots\cdots(1)$,

对（1）式左右两边关于 t 求导得：

$$
\begin{aligned}
\text{左边} &= \frac{\partial}{\partial t}\sigma_1(tx) \\
&= (\frac{\partial}{\partial t}\sigma_1(tx), \dots, \frac{\partial}{\partial t}\sigma_n(tx)) \\
&= (x_1, \dots,\ x_n) \cdot \begin{pmatrix} \partial_1\sigma_1(tx) & \cdots & \partial_1\sigma_n(tx) \\ \vdots & & \vdots \\ \partial_n\sigma_1(tx) & \cdots & \partial_n\sigma_n(tx) \end{pmatrix} \\
\text{左边} &= \sigma(x) - \sigma(0)
\end{aligned}
$$

则左边=右边，且令 t=0 后有：

$$
\sigma(x) = \sigma(0) + (x_1, \dots,\ x_n) \cdot \left. \begin{pmatrix} \partial_1\sigma_1(tx) & \cdots & \partial_1\sigma_n(tx) \\ \vdots & & \vdots \\ \partial_n\sigma_1(tx) & \cdots & \partial_n\sigma_n(tx) \end{pmatrix} \right|_{t=0}
$$

记 $a_0^j = \sigma_i(0) \qquad a_i^j = (\partial_i\sigma_j)|_(t=0) \qquad$ 其中 $i = 1, 2, \dots, n. \quad j = 1, 2, \dots, n.$ 有:

$$
\sigma(x_1, \dots x_n) = (a_0^1, \dots, a_0^n) + (x_1, \dots, x_n) \cdot \begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \vdots & & \vdots \\ a_n^1 & \cdots & a_n^n \end{pmatrix}
$$

取 $\varepsilon_i = (0, \dots, \overset{\text{第i个}}{\overset{\downarrow}{1}}, 0, \dots 0)$, 则,

$1 = d(\varepsilon_i, 0) = d(\sigma(\varepsilon_i), \sigma(0)) \Rightarrow (a_i^1)^2 + (a_i^2)^2 + \cdots + (a_i^n)^2 = 1 \ (\forall\ i = 1, 2, \dots, n)$

($i \neq j$ 时)

$$
\begin{aligned}
2 = 1^2 + 1^2 &= d^2(\varepsilon_i, \varepsilon_j) \\
&= d^2(\sigma(\varepsilon_i), \sigma(\varepsilon_j)) \\
&= (a_j^1 - a_i^1)^2 + \cdots + (a_j^n - a_i^n)^2 \\
&= ((a_i^1)^2 + \cdots + (a_i^n)^2) + ((a_j^1)^2 + \cdots + (a_j^n)^2) - 2(a_j^1 a_i^1 + \cdots + a_j^n a_i^n) \\
&= 1 + 1 - 2(a_j^1 a_i^1 + \cdots + a_j^n a_i^n)
\end{aligned}
$$

$\Rightarrow a_j^1 a_i^1 + \cdots + a_j^n a_i^n = 0$ ($i \neq j$ 时)

从而 $(a_i^j)_{n\times n}$为单位正交矩阵.

1.6 证: 设 Q 关于 $\{O; \delta_i\}$ 的坐标为: $x = (x^1, \dots, x^n)$, 即 $Q - O = x^1\delta_1 + \cdots + x^n\delta_n$.

由第5题知 σ 为线性变换，而 $\sigma(O) = P,\ \sigma(\delta_i) = e_i$,

$\therefore\ \sigma(Q) - \sigma(O) = \sigma(Q - O) = \sigma(x^1\delta_1 + \cdots + x^n\delta_n) = x^1\sigma(\delta_1) + \cdots + x^n\sigma(\delta_n) = x^1e_1 + \cdots + x^ne_n$

即点 Q' 关于 $\{P; e_i\}$ 的坐标等于点 Q 关于 $\{O; \delta_i\}$ 的坐标.

1.7 证:设 $\{O,\delta_i\}$ 为 E_n 中某一直角坐标系， f 在 $\{O,\delta_i\}$ 之下表示成

$$\begin{aligned}\overrightarrow{Of(t)} &= \sum_{i=1}^{n} x^i(t)\delta_i,\ \forall\ t\in\mathbb{R}\\ &= (\delta_1,...,\delta_n)\begin{pmatrix} x^1(t)\\ :\\ x^n(t)\end{pmatrix}\end{aligned}$$

任取另一直角坐标系 $\{P,e_i\}$ 则有唯一表示：

$$\overrightarrow{OP} = \sum_{i=1}^{n} a^i\delta_i = (\delta_1,...,\delta_n)\begin{pmatrix} a^1(t)\\ :\\ a^n(t)\end{pmatrix} \qquad e_i = \sum_{i=1}^{n} a^i\delta_i = (\delta_1,...,\delta_n)\begin{pmatrix} a_i^1\\ :\\ a_i^n\end{pmatrix} \quad (i=1,2,...n)$$

从而

$$(e_1,...e_n) = (\delta_1,...\delta_n)\begin{pmatrix} a_1^1 & \cdots & a_n^1\\ : & & :\\ a_1^n & \cdots & a_n^n\end{pmatrix}$$

记 $(a_j^i) = A$. 由 $(e_1,...e_n)$ 与 $(\delta_1,...,\delta_n)$ 均为正交向量组知 $|A|\neq 0$.
$\therefore A$ 可逆。记 $A^{-1} = (b_j^i)_{n\times n}$ $\therefore (\delta_1,...,\delta_n) = (e_1,...e_n)A^{-1}$

$$\overrightarrow{Of(t)} = (\delta_1,...,\delta_n)\begin{pmatrix} x^1(t)\\ :\\ x^n(t)\end{pmatrix} = (e_1,...e_n)A^{-1}\begin{pmatrix} x^1(t)\\ :\\ x^n(t)\end{pmatrix}$$

$$\overrightarrow{OP} = (\delta_1,...,\delta_n)\begin{pmatrix} a^1(t)\\ :\\ a^n(t)\end{pmatrix} = (e_1,...e_n)A^{-1}\begin{pmatrix} a^1(t)\\ :\\ a^n(t)\end{pmatrix}$$

而 $\overrightarrow{Of(t)} = \overrightarrow{OP} + \overrightarrow{Pf(t)}$ 从而

$$\begin{aligned}\overrightarrow{Pf(t)} &= \overrightarrow{Of(t)} - \overrightarrow{OP}\\ &= (e_1,...e_n)A^{-1}\begin{pmatrix} x^1(t)\\ :\\ x^n(t)\end{pmatrix} - (e_1,...e_n)A^{-1}\begin{pmatrix} a^1(t)\\ :\\ a^n(t)\end{pmatrix}\\ &= (e_1,...e_n)A^{-1}\begin{pmatrix} x^1(t)-a^1(t)\\ :\\ x^n(t)-a^n(t)\end{pmatrix}\\ &\triangleq \sum_{k=1}^{n} e_k y^k(t) \qquad \text{其中 } y^k(t) = \sum_{j=1}^{n} b_j^k(x^j(t)-a^j(t)) \quad (k=1,2,...,n)\end{aligned}$$

由 $x^j(t)$ 连续（或r次连续可微）可得 $y^k(t)$ 连续（或r次连续可微），从而映射 $f:\mathbb{R}\to E^n$ 的连续性和r次连续可微性与 E^n 中直角坐标系的选取无关。

1.8 证: 记 $f(t)$ 在 $\{O;\delta_i\}$ 与 $\{P;e_i\}$ 下的坐标为 $(x^1(t),...,x^n(t)),(y^1(t),...,y^n(t))$, 其中

$$\overrightarrow{OP}=\sum_{i=1}^{n}a_i\delta_i,\qquad e_i=\sum_{j=1}^{n}a_i^j\delta_j\quad(i=1,...n)$$

由的（1.14）

$$x^i(t)=a^i+\sum_{j=1}^{n}y^j(t)a_j^i$$

则等式两边同时对t求导有

$$\frac{dx^i}{dt}(t_0)=\sum_{j=1}^{n}\frac{dy^j(t)}{dt}a_j^i$$

又由（2.6）

$$\begin{aligned}
f'(t_0)&=\sum_{i=1}^{n}\frac{dx^i}{dt}(t_0)\delta_i\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{dy^j}{dt}a_j^i\delta_i\\
&=\sum_{j=1}^{n}\left(\frac{y^j(t)}{dt}\sum_{i=1}^{n}a_j^i\delta_i\right)\\
&=\sum_{j=1}^{n}\frac{dy^j}{dt}e_j
\end{aligned}$$

从而可见其形式不变，即切向量定义式（2.6）与直角坐标系的选取无关。

1.9 证:

(1)

$$\begin{aligned}
D_v(g+\lambda h)&=\langle\nabla(g+\lambda h)(P),v\rangle\\
&=\langle\nabla g(P)+\lambda\nabla h(P),V\rangle\\
&=\langle\nabla g(P),v\rangle+\lambda\langle\nabla h(P),v\rangle\\
&=D_v(g)+\lambda D_v(h)
\end{aligned}$$

(2)

$$\begin{aligned}
D_v(g\cdot h)&=\langle\nabla(gh)(P),v\rangle\\
&=\langle((\nabla g)h)(P),v\rangle+\langle(g(\nabla h))(P),v\rangle\\
&=h(P)\langle\nabla g(P),v\rangle+g(P)\langle\nabla h(P),v\rangle\\
&=h(P)D_vg+g(P)D_vh
\end{aligned}$$

1.10 证: $E_n \to R$ 上的函数 $x^i : P = \lambda^1\delta_1 + ... + \lambda^n\delta_n \to \lambda^i \quad (i = 1, 2, ..., n)$
$\forall\ P = (\lambda^1, ..., \lambda^n) \in E^n$, 在 P 的邻域内取 $Q = P + \triangle P = (\lambda^1 + \triangle\lambda^1, ..., \lambda^n + \triangle\lambda^n)$, 则

$$\lim_{\triangle\lambda^j \to 0} \frac{x^i(Q) - x^i(P)}{\triangle\lambda^j} = \lim_{\triangle\lambda^j \to 0} \frac{\triangle\lambda^i}{\triangle\lambda^j} = \delta_i^j \in C^\infty$$

从而 $x^i \in C^\infty \quad (\forall\ i = 1, 2, ..., n)$.

1.11 证:

(1) 在 E^m 中，取两新旧直角坐标分别为 $\{O; \delta_i\}, \{P; e_i\}$, 且满足

$$\begin{cases} \overrightarrow{OP} = \sum_{i=1}^m a^i\delta_i \\ e_j = \sum_{i=1}^m a_j^i\delta_i \end{cases} \qquad j = 1, .., m.$$

在 E^n 中，取两新旧直角坐标分别为 $\{O; \xi_i\}, \{P; \eta_i\}$, 且满足

$$\begin{cases} \overrightarrow{OQ} = \sum_{i=1}^m b^i\xi_i \\ \eta_j = \sum_{i=1}^m b_j^i\xi_i \end{cases} \qquad j = 1, .., m.$$

记原映射为 $F(\lambda^1, .., \lambda^m) = (f^1(\lambda^1, .., \lambda^m), ..., f^n(\lambda^1, .., \lambda^m))$.
设在新坐标下表示为 $G(\mu^1, ..., \mu^m) = (g^1(\mu^1, .., \mu^m), ..., g^n(\mu^1, .., \mu^m))$.
则有 $f^l = b^l + \sum_{j=1}^n b_j^l g^j \quad \lambda^k = a^k + \sum_{j=1}^m a_j^k\mu^j \ (k = 1, .., m)$. 再记 $(a_j^k)_{m\times m}$ 的逆为 $(C_k^j)_{m\times m}$

$$\begin{aligned} \therefore \frac{\partial f^l}{\partial\lambda^k} &= \sum_{j=1}^n b_j^l \frac{\partial g^j}{\partial\lambda^k} \\ &= \sum_{j=1}^n b_j^l (\sum_{i=1}^m \frac{\partial g^j}{\partial\mu^i} C_k^i) \\ &= \sum_{j=1}^n sum_{i=1}^m b_j^l frac\partial g^j \partial\mu^i C_k^i \end{aligned}$$

$$\therefore (\frac{\partial f^l}{\partial\lambda^k})_{m\times n} = A^{-1} (\frac{\partial g^j}{\partial\mu^i})_{m\times n} B$$

记 J_f, J_g 分别为 $F(\lambda^1, .., \lambda^m)$ 与 $G(\mu^1, .., \mu^m)$ 的Jacobi矩阵。则有 $J_f = A^{-1}J_gB$.

(2) 由于A、B可逆，所以 A^{-1}, B 均可表为若干初等行、列变换的乘积，
$\therefore r(J_f) = r(J_g)$, 在任意点 x_0.

(3) 在 E^m 中任取点P，记 $Q = f(P)$.
在 P 点邻域分别取两新旧曲纹坐标系 $(u^1, ..., u^m), (v^1, ..., v^m)$. 两者之间由同胚映射 g 关联: $\quad v^i = g^i(u^1, ..., u^m) \quad i = 1, .., m.$ 　　记 $g^{-1} = \bar{g}$ 则 $u^i = \bar{g}^i(v^1, .., v^m) \quad i = 1, .., m$.
同样在 Q 点邻域分别取两新旧曲纹坐标系 $(s^1, ..., s^n), (t^1, ..., t^n)$. 两者之间由同胚映射 h 关联: $\quad t^i = h^i(s^1, ..., s^n) \quad i = 1, .., m$
原函数 $f(u^1, .., u^m)$ 中分量记为 $f^i(u^1, .., u^m)$ $(i = 1, .., n)$ 在 E^m 与 E^n 间曲纹坐标变换

下为 $\tilde{f}(v^1,..,v^m)$, 其在新坐标下分量为: $\tilde{f}^i(v^1,..,v^m)=h^i(f^1,..,f^n)$ 其中 $f^j(u^1,..,u^m)=f^j(\bar{g}(v^1,..,v^m),...,\bar{g}(v^1,..,v^m))$ $(i=1,..,n;j=1,..,n)$
从而在变换后Jacobi矩阵 $J_{n\times m}$ 为

$$\begin{aligned}J^i_j&=\frac{\partial\tilde{f}^i}{\partial v^j}\\&=\frac{\partial h^i}{\partial f^k}\frac{\partial f^k}{\partial u^r}\frac{\partial\bar{g}^r}{\partial v^j}\quad(i,k=1,..,n,r,j=1,..,m)\end{aligned}$$

在 P 点，记 g 的雅克比矩阵 $(\frac{\partial g^i}{\partial u^j})_{m\times m}$ 为 G, h 的雅克比矩阵 $(\frac{\partial h^i}{\partial v^j})_{n\times n}$ 为 H, 变换前 f 的雅克比矩阵 $(\frac{\partial f^i}{\partial u^j})_{n\times m}$ 为 J_0 则有：

$$J=HJ_0G^{-1}$$

1.12 证: 记原方程组为$x^i=f^i(u^1,..,u^n)$, $i=1,..,n$. 则 $f=f^i\delta_i$ 的雅克比矩阵的行列式为

$$\begin{aligned}\left|\frac{\partial(f^1,..,f^n)}{\partial(u^1,..,u^n)}\right|&=\begin{vmatrix}-x^1tanu^1&-x^1tanu^2&\cdots&-x^1tanu^{n-2}&-x^1tanu^{n-1}&\frac{x^1}{u^n}\\-x^2tanu^1&-x^2tanu^2&\cdots&-x^2tanu^{n-2}&x^2cotu^{n-1}&\frac{x^2}{u^n}\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\-x^{n-1}tanu^1&x^{n-1}cotu^2&\cdots&0&0&\frac{x^{n-1}}{u^n}\\x^ncotu^1&0&\cdots&0&0&\frac{x^n}{u^n}\end{vmatrix}\\&=x^1x^2\cdots x^n\begin{vmatrix}-tanu^1&-tanu^2&\cdots&-tanu^{n-2}&-tanu^{n-1}&1\\-tanu^1&-tanu^2&\cdots&-tanu^{n-2}&cotu^{n-1}&1\\-tanu^1&-tanu^2&\cdots&cotu^{n-2}&0&1\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\-tanu^1&\cot u^2&\cdots&0&0&1\\\cot u^1&0&\cdots&0&0&1\end{vmatrix}\\&=\frac{x^1x^2\cdots x^n}{u^n}\cdot\Big[(-1)^{1+n}(-1)^{\frac{(n-1)(n-2)}{2}}cotu^1\cdots cotu^{n-1}\\&\qquad+(-1)^{2+n}(-1)^{\frac{(n-1)(n-2)}{2}}cotu^1\cdots cotu^{n-2}(-tanu^{n-1})\Big]\\&=(-1)^{\frac{n^2-n+4}{2}}\left(\prod_{i=1}^{n-2}r^i\prod_{j=1}^{i}cosu^j\right)x^1x^2(u^n)^{n-3}\frac{1}{sinu^{n-1}cosu^{n-1}}\end{aligned}$$

当 $x^1,x^2\neq 0$ 时，有 $u^n,cosu^1,\cdots,cosu^{n-1},sinu^{n-1}$ 非零，又由 $r^1,\cdots r^n$ 为正数可知: $\left|\frac{\partial(f^1,..,f^n)}{\partial(u^1,..,u^n)}\right|\neq 0$, 即 f 的秩为 n . 从而 $(u^1,\cdots u^n)$ 给出 E^n 中除坐标面 $\{(0,0,x^3,\cdots,x^n):x^3,\cdots,x^n\in\mathbb{R}\}$ 以外的任意一点的邻域内的曲纹坐标系。

1.13 解: 设 (x,y,z) 是 E^3 中的直角坐标系,令

$$\begin{cases}x=\rho\cos\psi\cos\theta\\y=\rho\cos\psi\sin\theta\qquad(0<\rho<+\infty,-\frac{\pi}{2}<\psi<\frac{\pi}{2},-\pi<\theta<\pi)\\z=\rho\sin\psi\end{cases}$$

则 (ρ,ψ,θ) 给出球坐标系，记 $u^1=\rho,u^2=\psi,u^3=\theta$ 则可求其自然标架场如下：

$$\overrightarrow{r_1}=(cos\psi cos\theta,cos\psi sin\theta,sin\psi)$$
$$\overrightarrow{r_2}=(-\rho sin\psi cos\theta,-\rho sin\psi sin\theta,\rho cos\psi)$$
$$\overrightarrow{r_3}=(-\rho cos\psi sin\theta,\rho cos\psi cos\theta,0)$$

则 $\forall\ Q=r(P),\{Q,r_i\}$ 为 E^3 中球坐标系诱导的自然标架场。
下求其度量系数：

$$g_{11}=\langle\overrightarrow{r_1},\overrightarrow{r_1}\rangle=1 \qquad g_{12}=g_{21}=\langle\overrightarrow{r_1},\overrightarrow{r_2}\rangle=0$$
$$g_{31}=g_{13}=\langle\overrightarrow{r_1},\overrightarrow{r_3}\rangle=0$$
$$g_{22}=\langle\overrightarrow{r_2},\overrightarrow{r_2}\rangle=\rho^2 \qquad g_{23}=g_{32}=\langle\overrightarrow{r_2},\overrightarrow{r_3}\rangle=0$$
$$g_{33}=\langle\overrightarrow{r_3},\overrightarrow{r_3}\rangle=\rho^2cos^2\psi$$

$\therefore$ 度量系数为 $g_{11}=\rho^2,\ g_{33}=\rho^2cos^2\psi,\ g_{ij}=0\ (i\neq j$ 时,其中 $i,j=1,2,3)$.
则 $g^{11}=1,\ g^{22}=\frac{1}{\rho^2},\ g^{33}=\frac{1}{\rho^2cos^2\psi},\ g^{ij}=0\ (i\neq j$ 时,其中 $i,j=1,2,3)$.
且 $\frac{\partial g_{11}}{\partial u^i}=0\ (i=1,2,3)$ $\frac{\partial g_{ij}}{\partial u^k}\ (i,j,k=1,2.3)$ $\frac{\partial g_{22}}{\partial u^1}=2\rho$ $\frac{\partial g_{22}}{\partial u^2}=0$ $\frac{\partial g_{22}}{\partial u^3}=0$ $\frac{\partial g_{33}}{\partial u^1}=2\rho cos^2\psi$ $\frac{\partial g_{33}}{\partial u^2}=-\rho^2sin2\psi$ $\frac{\partial g_{33}}{\partial u^3}=0$
从而由

$$\Gamma_{il}^k=\frac{1}{2}g^{kj}(\frac{\partial g_{ij}}{\partial u^l}+\frac{\partial g_{jl}}{\partial u^i}-\frac{\partial g_{il}}{\partial u^j})$$

得：

$$\Gamma_{33}^1=\frac{1}{2}g^{11}(\frac{\partial g_{31}}{\partial u^3}+\frac{\partial g_{13}}{\partial u^1}-\frac{\partial g_{33}}{\partial u^1})=\rho cos^2\psi$$
$$\Gamma_{33}^2=\frac{1}{2}g^{22}(\frac{\partial g_{32}}{\partial u^3}+\frac{\partial g_{32}}{\partial u^1}-\frac{\partial g_{33}}{\partial u^2})=-sins\psi$$
$$\Gamma_{33}^3=\frac{1}{2}g^{33}(\frac{\partial g_{33}}{\partial u^3}+\frac{\partial g_{33}}{\partial u^3}-\frac{\partial g_{33}}{\partial u^3})=0$$

又由

$$\frac{\partial\overrightarrow{r_1}}{\partial u^1}=0 \qquad \frac{\partial\overrightarrow{r_1}}{\partial u^2}=\frac{1}{\rho}\overrightarrow{r_2} \qquad \frac{\partial\overrightarrow{r_1}}{\partial u^3}=\frac{1}{\rho}\overrightarrow{r_3}$$
$$\frac{\partial\overrightarrow{r_2}}{\partial u^2}=-\rho\overrightarrow{r_1} \qquad \frac{\partial\overrightarrow{r_2}}{\partial u^3}=-tan\psi\overrightarrow{r_3}$$

从而 Christoffel 系数为：

$$\Gamma_{12}^2=\Gamma_{21}^2=\frac{1}{rho} \qquad \Gamma_{31}^3=\Gamma_{13}^3=\frac{1}{rho}$$
$$\Gamma_{22}^1=-rho \qquad \Gamma_{23}^3=\Gamma_{32}^3=-tan\psi$$
$$\Gamma_{33}^1=-\rho cos^2\psi \qquad \Gamma_{33}^2=-sin^2\psi$$

其余为 0.

1.14 解: 设 (x,y,z) 为 E^3 中直角坐标系，设

$$\begin{cases}x=\rho cos\theta\\ y=\rho sin\theta \qquad (\text{其中 } 0<r<+\infty,\ -\pi<\theta<\pi,\ -\infty<t<+\infty)\\ z=t\end{cases}$$

则 (ρ,θ,t) 为柱坐标系。记 $u^1=\rho,\ u^2=\theta,\ u^3=t$, 则有：

$$\begin{aligned}\overrightarrow{r_1}&=(cos\theta,sin\theta,0)\\ \overrightarrow{r_2}&=(-\rho sin\theta,\rho cos\theta,0)\\ \overrightarrow{r_3}&=(0,0,1)\end{aligned}$$

且 $g_{11}=1,\ g_{22}=\rho^2,\ g_{33}=1,\ g_{ij}=0\,(i\neq j$时$)$ 由于：

$$\begin{array}{rrr}\frac{\partial\overrightarrow{r_1}}{\partial u^1}=0 & \frac{\partial\overrightarrow{r_1}}{\partial u^2}=\frac{1}{\rho}\overrightarrow{r_2} & \frac{\partial\overrightarrow{r_1}}{\partial u^3}=0\\ & \frac{\partial\overrightarrow{r_2}}{\partial u^2}=-\rho\overrightarrow{r_1} & \frac{\partial\overrightarrow{r_2}}{\partial u^3}=0\\ & & \frac{\partial\overrightarrow{r_3}}{\partial u^3}=0\end{array}$$

所以 Christoffel 系数为：

$$\mathit{\Gamma}_{12}^2=\mathit{\Gamma}_{21}^2=\frac{1}{\rho}\quad \mathit{\Gamma}_{22}^1=-\rho$$

其余为0

1.15 证: $\because\ v=v^i\overrightarrow{e_i}$

$$\begin{aligned}\therefore\ dv&=v^j d\overrightarrow{e_j}+\overrightarrow{e_i}dv^i\\ &=v^j(\frac{\partial\overrightarrow{e_j}}{\partial u^k}du^k)+\overrightarrow{e_i}dv^i\\ &=v^j\mathit{\Gamma}_{jk}^i\overrightarrow{e_i}du^k+\overrightarrow{e_i}dv^i\\ &=(v^j\mathit{\Gamma}_{jk}^i du^k+dv^i)\overrightarrow{e_i}\end{aligned}$$

$\therefore\ v$为平行向量场 $\Leftrightarrow v^j\mathit{\Gamma}_{jk}^i+dv^i=0(\forall\ i=1,2,...,n)$

1.16 证: 定义加法 $'+':(f+g)(x)=f(x)+g(x)$, 数乘: $(\lambda f)(x)=\lambda f(x)$
先证 $(\mathscr{L}(v_1,\cdots,v_r;\mathbb{R}),+)$ 构成加法群。

(1) $\forall\, f,g\in\mathscr{L}(v_1,\cdots,v_r;\mathbb{R})\quad\forall\, i\in\{1,\cdots,r\}\quad\forall\, x_1=(x^1,\cdots,x_1^i,\cdots,x^r),x_2=(x^1,\cdots,x_2^i,\cdots,x^r)\in V_1\times\cdots\times V_r\quad\forall\ k_1,k_2\in\mathbb{R}$ 有

$$\begin{aligned}&(f+g)(x^1,\cdots,k_1x_1^i+k_2x_2^i,\cdots x^r)\\ =&f(x^1,\cdots,k_1x_1^i+k_2x_2^i,\cdots x^r)+g(x^1,\cdots,k_1x_1^i+k_2x_2^i,\cdots x^r)\\ =&(k_1f(x_1)+k_2f(x_2))+(k_1g(x_1)+k_2g(x_2))\\ =&k_1(f+g)(x_1)+k_2(f+g)(x_2)\end{aligned}$$

$\therefore f+g\in\mathscr{L}(V_1,\cdots,V_r;\mathbb{R})$

(2) $\forall\ f,g,h\in\mathscr{L}(V_1,\cdots,V_r;\mathbb{R})\quad\forall\ x\in V_1\times,\cdots,V_r$, 有

$$[(f+g)+h](x)=(f+g)(x)+h(x)=f(x)+g(x)+h(x)$$

$$[f+(g+h)](x)=f(x)+(g+h)(x)=f(x)+g(x)+h(x)$$

$\therefore (f+g)+h=f+(g+h)$

(3) 取 θ 为 $\forall\ x \in V_1 \times \cdots \times V_r, \theta(x) = 0$
则显然 $f + \theta = \theta + f, \forall\ f$

(4) $\forall\ f$, 取 $-f$ 为 $(-f)(x) = -f(x)$ 则 $f + (-f) = (-f) + f = \theta$

(5) $\forall\ f, g \in \mathscr{L}(V_1 \cdots V_r; \mathbb{R})$ 显然 $f + g = g + f$.

所以 $(\mathscr{L}(v_1, \cdots, v_r; \mathbb{R}), +)$ 构成Abel群。 $\forall\ \lambda, \mu \in \mathbb{R}\quad \forall f, g \in \mathscr{L}(V_1 \times \cdots \times V_r; \mathbb{R})$, 又有

(6) $(\lambda, \mu)f = \lambda f + \mu f$

(7) $(\lambda\mu)g = \lambda(\mu g)$

(8) $\lambda(f + g) = \lambda f + \lambda g$

(9) $1 \cdot f = f$

所以 $\mathscr{L}(V_1, \cdots, V_r; \mathbb{R})$ 构成 $\mathbb{R}$ 上的线性空间。

1.17 证:

(1) $\forall\ k_1, k_2 \in \mathbb{R}\ u_1, u_2 \in V$, 则由 $\tilde{f}$ 为2重线性函数知

$$\begin{aligned}
f(k_1u_1 + k_2u_2) &= \sum_{i=1}^{n} \tilde{f}(k_1u_1 + k_2u_2, \delta^i)\delta_i \\
&= \sum_{i=1}^{n} (k_1\tilde{f}(u_1, \delta^i) + k_2\tilde{f}(u_2, \delta^i))\delta_i \\
&= k_1 \sum_{i=1}^{n} \tilde{f}(u_1, \delta^i)\delta_i + k_2 \sum_{i=1}^{n} \tilde{f}(u_2, \delta^i)\delta_i \\
&= k_1 f(u_1) + k_2 f(u_2)
\end{aligned}$$

所以 f 为线性变换。

(2) 若另取基底 $\{e^i\}$, 且 $e^i = a^i_j\delta_j$ 其对偶基底 $\{e_i\}$, 则有 $\delta_j = a^i_j e_i$, 其中 (b^j_i) 为 (a^j_i) 的逆矩阵。且由

$$\delta^i(e_j) = \delta_i(a^k_j\delta_k) = a^k_j\delta^i_k = a^i_j = a^i_k e^k(e_j)$$

知 $\delta^i = a^i_k e^k$.
$\therefore f(u) = \tilde{f}(u, \delta^i)\delta_i = \tilde{f}(u, a^i_k e^k)(b^j_i e_j) = a^i_k b^j_i \tilde{f}(u, e^k)e_j = \delta^j_k \tilde{f}(u, e^k)e_j = \tilde{f}(u, e^j)e_j$
$\therefore$ f 的定义与基底的选取无关。

1.18 证: 取定 $t, \forall\ v = (v_1, \cdots v_r) \in V \times \cdots \times V \quad v^1 \in V^\star$, 取 $F : V^\star \times V_1 \cdots \times V_r \to \mathbb{R}$ 为 $F(v^1, v_1, \cdots\ v_r) = v^1(t(v))$ ，显然 F 是 $V^\star \times V \times \cdots V$ 上的一个 $1 + r$ 重线性函数，即 $(1, r)$ 型张量。
反之，若给定一个 $(1, r)$ 型张量 $F : V^\star \times V_1 \times \cdots V_r$, $\forall\ v = (v1, \cdots, v_r) \in V \times \cdots \times \to \mathbb{R}$, 令

$t(v)=F(\delta^i,v_1,\cdots,v_r)\delta_i$, 显然 t 为 r 重线性映射。
所以 t 等同一个 $(1,r)$ 型张量。

1.19 证:

(1) $\forall\ \alpha_1,\cdots,\alpha_2\in\mathscr{L}(V_1,\cdots,V_p;\mathbb{R}),\quad\forall\ \beta\in\mathscr{L}(W_1,\cdots,W_q;\mathbb{R})$
$\forall\ v\in V_1\times\cdots\times V_p,\quad w\in W_1\times\cdots\times W_q$

$$\begin{aligned}
&(\alpha_1+\alpha_2)\otimes\beta(v,w)\\
=&(\alpha_1+\alpha_2)(v)\cdot\beta(w)\\
=&[\alpha(_1(v)+\alpha_2(v)]\cdot\beta(w)\\
=&\alpha_1(v)\cdot\beta(w)+\alpha_2(v)\cdot\beta(w)\\
=&\alpha_1\otimes\beta(v,w)+\alpha_2\otimes\beta(v,w)\\
=&(\alpha_1\otimes\beta+\alpha_2\otimes\beta)(v,w)
\end{aligned}$$

$\therefore\ (\alpha_1+\alpha_2)\otimes\beta=\alpha_1\otimes\beta+\alpha_2\otimes\beta$ 同理 $\beta\otimes(\alpha_1+\alpha_2)=\beta\otimes\alpha_1+\beta\otimes\alpha_2$

(2) $\forall\ \alpha\in\mathscr{L}(V_1,...,Vr;\mathbb{R}),\ \beta\in\mathscr{L}(W_1,...,W_s;\mathbb{R}),\ \gamma\in\mathscr{L}(Z_1,...,Z_t;\mathbb{R})$
$\forall\ V_1\times\cdots\times V_r,\ w\in W_1\times\cdots\times W_s,\ z\in Z_1\times\cdots\times Z_t$, 则

$$\begin{aligned}
&(\alpha\otimes\beta)\otimes\gamma(v,w,z)\\
=&(\alpha\otimes\beta)(v,w)\cdot\gamma(z)\\
=&(\alpha(v)\cdot\beta(w))\cdot\gamma(z)\\
=&\alpha(v)\cdot[\beta(w)\gamma(z)]\\
=&\alpha(v)\cdot[(\beta\otimes\gamma(w,z))]\\
=&[\alpha\otimes(\beta\otimes\gamma)](v,w,z)
\end{aligned}$$

$\therefore(\alpha\otimes\beta)\otimes\gamma=\alpha\otimes(\beta\otimes\gamma)$
故多重线性函数的张量积服从分配率和结合律。

1.20 证: 设 $\{\xi_i\}$ 为 V 的基， $\{\eta_i\}$ 为 W 的基，记 $A=\{V\otimes W|v\in V,w\in W\}$, 下证 A 不构成线性空间：
由于 $dim(V)\geqslant 2,dim(W)\geqslant 2$. 所以分别可取 V,W 的两组基： $\xi_1,\xi_2,\eta_1,\eta_2$,则

$$\xi_1\otimes\eta_1\in A,\ \xi_2\otimes\eta_2\in A,$$

若

$$\xi_1\otimes\eta_1+\xi_2\otimes\eta_2\in A$$

则 $\exists\ a^i\ (i=1,2,...,n)\ s.t.$

$$\xi_1\otimes\eta_1+\xi_2\otimes\eta_2=(a^i\xi_i)\otimes(b^j\eta_j)=a^ib^j\xi_i\otimes\eta_j$$

移项并合并后，由于 $\{\xi_i\eta_j\}$ 线性无关，所以有

$$a^i b^j = \begin{cases} 1 & i=j=1 \text{ 以及 } i=j=2 \text{ 时;} \\ 0 & \text{其余情况.} \end{cases}$$

$a^1b^1 = a^2b^2 = 1$, 从而 $a^1, b^1, a^2, b^2 \neq 0$, 但 $a^1b^2 = 0$, 矛盾!
$\therefore \xi_1 \otimes \eta_1 + \xi_2 \otimes \eta_2 \notin A$, A 不构成线性空间。

1.21 证:

(1) 由题知 $f(\delta_i) = b_i^j \delta_i$.
若另取基底 $\{e_i\}$ 且 $e_i = a_i^j \delta_j$ 从而 $\delta_j = c_j^i e_i$ (其中 (c_i^j) 为 (a_i^j) 的逆).
则 $f(e_i) = f(a_i^j \delta_j) = a_i^j f(\delta_j) = a_i^j b_j^k \delta_k = a_i^j b_j^k c_k^l e_l$.
即 $f(e_i) = (a_i^j b_j^k c_k^l) e_l$, 记 $d_i^l = a_i^j b_j^k c_k^l$.
$\therefore f$ 在基 $\{e_i\}$ 下矩阵是 (d_i^l).
此时

$$\begin{aligned} B_3 &= d_i^j d_k^i d_j^k \\ &= a_i^s b_s^t c_t^j a_k^e b_e^f c_f^i a_j^g b_g^h c_h^k \\ &= (a_i^s c_f^i)(a_k^e c_h^k)(a_j^g c_t^j) b_s^t b_e^f b_g^h \\ &= (\delta_f^s b_e^f)(\delta_h^e b_g^h)(\delta_t^g b_s^t) \\ &= b_e^s b_g^e b_s^g \end{aligned}$$

$\therefore B_3$ 与基底的选取无关。

(2) 令 $F : V^\star \times V^\star \to \mathbb{R}$ 为 $F(\delta^i, u) = \delta^i(f(u))$. $(\forall u \in V)$. 则易证 F 为 (1,1) 型张量，且

$$\begin{aligned} B_3 &= b_i^j b_k^i b_j^k \\ &= F(\delta^j, \delta_i) \cdot F(\delta^i, \delta_k) \cdot F(\delta^k, \delta_j) \\ &= F(\delta^j, \delta_i) \cdot [(C_1^2 F \otimes F)(\delta^i, \delta_j)] \\ &= C_2^1 [F \otimes C_1^2 (F \otimes F)](\delta^j, \delta_j) \\ &= C_1^1 \{C_2^1 [F \otimes C_1^2 (F \otimes F)]\}. \end{aligned}$$

1.22 解: 令 $F : V^\star \times V \to \mathbb{R}$ $\quad F(v^i, v_j) = v^i(v_j)$. 易证 F 为 (1,1) 型张量. 任取 V 的基底 $\{\delta_i\}$, 其对偶基底 $\{\delta^i\}$, 则 $\{\delta_i \otimes \delta^j\}$ 为 F 的基底,且 $\delta^i(\delta_j) = \delta_j^i$. 从而 $F(\delta^i, \delta_j) = \delta_i^j$.

$$F = F(\delta^i, \delta_j) \delta_i \otimes \delta^j = \delta_i^j \delta_i \otimes \delta^j = \sum_i \delta_i \otimes \delta^i$$

$\therefore \delta_i^j$ 为 F 的分量。

1.23 解: 若存在 (0,2) 型张量 F 满足题设条件，则任取 V 的一组基底 $\{\delta_i\}$, $F(\delta_i,\delta_j)=\delta_{ij}$. 另任取一基底 $\{e_i\}$ ，且 $e_i=a_i^j\delta_j$, 则

$$\begin{aligned}
F(e_i,e_j)=\delta_{ij} \Leftrightarrow & F(a_i^k\delta_k,a_j^r\delta_r)=\delta_{ij}\\
\Leftrightarrow & a_i^k a_j^r F(\delta_k,\delta_r)=\delta_{ij}\\
\Leftrightarrow & a_i^k a_j^k=\delta_{ij}\\
\Leftrightarrow & (a_i^j) \text{ 为单位正交阵}
\end{aligned}$$

但 a_i^j 未必为单位正交阵，矛盾!
所以不存在。

1.24 证: $\forall\, v_1,\cdots,v_q\in V$, $(\sigma(f))(v_1,\cdots,v_q)=f(v_{\sigma(1)},\cdots,v_{\sigma(q)})=\alpha^1(v_{\sigma(1)})\cdots\alpha^q(v_{\sigma(q)})$.

$$\begin{aligned}
\alpha^{\tau(1)}(v_1)\otimes\cdots\otimes\alpha^{\tau(q)}(v_1,\cdots,v_q) &= \alpha^{\tau(1)}\otimes\cdots\otimes\alpha^{\tau(q)}(v_1,\cdots,v_q)\\
(\text{由 } \tau=\sigma^{-1} \text{ 知 } i=\sigma(\tau(i))) \quad &= \alpha^{\tau(1)}(v_{\sigma(\tau(1))})\cdots\alpha^{\tau(q)}(v_{\sigma(\tau(q))})\\
&= \alpha^1(v_{\sigma(1)})\cdots\alpha^q(v_{\sigma(q)})
\end{aligned}$$

$\therefore\ \sigma(f)=\alpha^{\tau(1)}\otimes\cdots\otimes\alpha^{\tau(q)}$

1.25 证:

(1)

$$\begin{aligned}
\sigma \text{ 是对称张量 } &\Leftrightarrow \sigma(\xi)=\xi\\
&\Leftrightarrow \frac{1}{q!}\sum_{\sigma\in\varphi(q)}\sigma(\xi)=\xi\\
&S_q(\xi)=\xi.
\end{aligned}$$

(2)

$$\begin{aligned}
\sigma \text{ 是反对称张量 } &\Leftrightarrow \sigma(\xi)=sign(\sigma)\xi\\
&\Leftrightarrow sign(\sigma)\cdot\sigma(\xi)=\xi\\
&\Leftrightarrow \sum_{\sigma\in\varphi(q)} sign(\sigma)\cdot\sigma(\xi)=\xi\cdot q!\\
&\Leftrightarrow A_q(\xi)=\xi
\end{aligned}$$

1.26 证: $\forall\, \xi \in V_2^0, \{\delta^i\}$ 为 V 中的基, 记 $\xi = \xi_{ij}\delta^i\delta^j$ 为二阶协变张量. 对于$\delta \triangleq \delta^i \otimes \delta^j$.

$$
\begin{aligned}
S_2(\delta) &= \frac{1}{2} \cdot (\delta^i \otimes \delta^j + \delta^j \otimes \delta^i) \text{ (由24题)} \\
A_2(\delta) &= \frac{1}{2}(\delta^i \otimes \delta^j - \delta^j \otimes \delta^i) \\
\Rightarrow \delta &= \delta^i \otimes \delta^j = S_2(\delta) + A_2(\delta) \\
\therefore \xi =& \xi_{ij}\delta^i \otimes \delta^j \\
&= \delta_{ij}(S_2(\delta^i \otimes \delta^j) + A_2(\delta^i \otimes \delta^j)) \\
&= S_2(\xi) + A_2(\xi)
\end{aligned}
$$

其中 $S_2(\xi)$ 为对称张量，$A_2(\xi)$ 为反对称张量。

1.27 证：取 V 中基底 $\{\delta_i\}$, 由 $\varphi \in V_2^0$, 依题意知, $\forall\ \delta_i\delta_j\delta_k$ 有

$$
\begin{aligned}
\varphi(\delta_i, \delta_j, \delta_k) &= \varphi(\delta_j, \delta_i, \delta_k) \\
&= -\varphi(\delta_j, \delta_k, \delta_i) \\
&= -\varphi(\delta_k, \delta_j, \delta_i) \\
&= \varphi(\delta_k, \delta_i, \delta_j) \\
&= -\varphi(\delta_i, \delta_j, \delta_k) \\
\Rightarrow & \varphi(\delta_i, \delta_j, \delta_k) = 0. \\
\therefore \varphi &= \varphi(\delta_i, \delta_j, \delta_k)\delta^i \otimes \delta^j \otimes \delta^k = 0.
\end{aligned}
$$

1.28 证：取 V 的基 $\{\delta_i\}$, $\because a(x,x) = 0$, $\therefore a(\delta_i, \delta_j) = 0 \quad (i = j)$ 时.
$\forall i \neq j$ 取 $x = \delta_i + \delta_j$ 则

$$
\begin{aligned}
0 &= a(x,x) \\
&= a(\delta_i + \delta_j, \delta_i + \delta_j) \\
&= a_{ii} + a_{ij} + a_{ji} + a_{jj} \\
&= a_{ij} + a_{ji}.
\end{aligned}
$$

记 $a = a_{ij}\delta^i \otimes \delta^j$ 有

$$
\begin{aligned}
S_2(a) &= \frac{1}{2}a_{ij}(\delta^i \otimes \delta^j + \delta^j \otimes \delta^i) \\
&= \frac{1}{2}(a_{ij}\delta^i \otimes \delta^j + a_{ji}\delta^i \otimes \delta^j) \\
&= \frac{1}{2}(a_{ij} + a_{ji})\delta^i \otimes \delta^j \\
&= 0.
\end{aligned}
$$

1.29 证：由于 a 为任意对称张量，可取 a 使得 $a^{ij} \neq 0(\forall\, i,j)$ (例如，取 $a = \sum_i \sum_j \delta_i \otimes \delta_j$)，则

$$0 = a^{ij} b_{ij} = a^{ji} b_{ji} = a^{ij} b_{ji}$$

$\therefore a^{ij}(b^{ij} + b^{ji}) = 0$, 此时$b_{ij} + b_{ji} = 0 \Rightarrow b_{ij} = -bji$,
$\therefore b$ 为反对称张量。

1.30 证：取 V 的基 $\{\delta_i\}$ ，由 x 的任意性， $\forall\, i,j$ 取 $x = \delta_i + \delta_j \in V$. 由 $f(x,x) = \tilde{f}(x,x)$ 知

$$\begin{gathered}
f(\delta_i + \delta_j, \delta_i + \delta_j) = \tilde{f}(\delta_i + \delta_j, \delta_i + \delta_j) \\
\Rightarrow f(\delta_i, \delta_i) + f(\delta_i, \delta_j) + f(\delta_j, \delta_i) + f(\delta_j, \delta_j) = \tilde{f}(\delta_i, \delta_i) + \tilde{f}(\delta_i, \delta_j) + \tilde{f}(\delta_j, \delta_i) + \tilde{f}(\delta_j, \delta_j) \\
\Rightarrow f(\delta_i, \delta_j) + f(\delta_j, \delta_i) = \tilde{f}(\delta_i, \delta_j) + \tilde{f}(\delta_j, \delta_i) \\
\Rightarrow 2f(\delta_i, \delta_j) = 2\tilde{f}(\delta_i, \delta_j) \\
\Rightarrow f_{ij} = \tilde{f}_{ij}
\end{gathered}$$

记 $f = f_{ij}\delta^i \otimes \delta^j \quad \tilde{f} = \tilde{f}_{ij}\delta^i \otimes \delta^j$, 则 $f = \tilde{f}$.

1.31 证： 原行列式$= \sum_\sigma \delta^{\sigma(i_1),\ldots,\sigma(i_r)}_{i_1,\ldots,i_r} \delta^{\sigma(i_1)}_{j_1} \cdots \delta^{\sigma(i_r)}_{j_r} = \delta^{j_1,\ldots,j_r}_{i_1,\ldots,i_r}$.

1.32 证： (1) 记置换

$$\pi = \begin{pmatrix} i_1, \cdots, i_{r+s} \\ j_1, \cdots, j_{r+s} \end{pmatrix} \quad \pi_1 = \begin{pmatrix} k_1, \cdots, k_r \\ j_1, \cdots, j_r \end{pmatrix} \quad \pi_2 = \begin{pmatrix} i_1, \cdots, i_r, i_{r+1}, \cdots, i_{r+s} \\ k_1, \cdots, k_r, j_{r+1}, \cdots, j_{r+s} \end{pmatrix}$$

则 $\pi = \pi_1\pi_2$, 从而 $sgn(\pi) = sgn(\pi_1) \cdot sgn(\pi_2)$.
即，取定某排列 $(l_1, \cdots, l_r)$, 有

$$\delta^{i_1,\cdots,i_{r+s}}_{j_1,\cdots,j_{r+s}} = \delta^{l_1,\cdots,l_r}_{j_1,\cdots,j_r} \cdot \delta^{i_1,\cdots,i_{r+s}}_{l_1,\cdots,l_r,j_{r+1},\cdots,j_{r+s}} \quad \text{(注：此时不对上下指标求和)}$$

而 $(l_1, \cdots, l_r)$ 的排列共有 $r!$ 种取法.

$$\therefore \delta^{i_1,\cdots,i_{r+s}}_{j_1,\cdots,j_{r+s}} = \frac{1}{r!} \sum_{k_1,\cdots,k_r} \delta^{k_1,\cdots,k_r}_{j_1,\cdots,j_r} \delta^{i_1,\cdots,i_{r+s}}_{k_1,\cdots,k_r,j_{r+1},\cdots,j_{r+s}}.$$

(2)

$$\delta^{i_1,\cdots,i_r}_{i_1,\cdots,i_r} = A^r_n = \frac{n!}{(n-r)!}.$$

1.33 证： 记 $a=(\alpha_1,...,\alpha_n)$, 其中 $\alpha_i=(a_i^1,...,a_i^n)^T$，则

$$
\begin{aligned}
\det a &= \det(\alpha_1,...,\alpha_n)\\
&= \frac{1}{n!}\delta^{i_1,...,i_n}_{1,...,n}\det(\alpha_{i_1},...,\alpha_{i_n})\\
&= \frac{1}{n!}\delta^{i_1,...,i_n}_{1,...,n}\delta^{1,...,n}_{j_1,...,j_n}a^{j_1}_{i_1}\cdots a^{i_n}_{j_n}\\
&= \frac{1}{n!}\delta^{i_1,...,i_n}_{j_1,...,j_n}a^{j_1}_{i_1}\cdots a^{i_n}_{j_n}.
\end{aligned}
$$

¿¿¿¿¿¿¿ 06aadd420f9273e09b8e8b858953f1a93049c2ec **1.34 证：**