

# 微分流形初步作业

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# 目录



# 第一章 预备知识

最后一次 1.1 证: 令  $V = \mathbb{R}^n$ ,  $A = \mathbb{R}^n$ , 则  $\forall x, y \in A$ , 令  $\vec{xy} = y - x$  则有:

- ①  $\forall x \in A, \vec{xx} = x - x = 0$ ,
- ②  $\forall x \in A, v \in V$ , 令  $y = x + v \in V$ . 则  $\vec{xy} = y - x = v$ . 若另有  $y'$ , s. t.  $\vec{xy'} = v$ , 则  $y' - x = v \Rightarrow y' = x + v = y$  从而  $y$  唯一,
- ③  $\forall x, y, z \in A. \vec{xy} + \vec{yz} = (y - x) + (z - y) = z - x = \vec{xz}$ .

所以  $\mathbb{R}^n$  是一个  $n$  维仿射空间, 它以  $\mathbb{R}^n$  自身为它的伴随向量空间。

1.2 证:

- ①  $\forall P, Q \in E^n, 0 \leq d(P, Q) = |\vec{PQ}| < +\infty$ , 且  $d(P, Q) = 0 \Leftrightarrow |\vec{PQ}| = 0 \Leftrightarrow P = Q$ ,
- ②  $\forall P, Q \in E^n, d(P, Q) = |\vec{PQ}| = |\vec{QP}| = d(Q, P)$ ,
- ③  $\forall P, Q, R \in E^n, d(P, R) = |\vec{PR}| = |\vec{PQ} + \vec{QR}| \leq |\vec{PQ}| + |\vec{QR}| \leq d(Q, P)$ .

从而  $E^n$  关于距离函数  $d$  成为一个度量。

1.3 证:

(1) 记  $E^n$  的全体开子集为  $\tau$ ,

- ① 显然  $\emptyset, E^n \in \tau$ ,
- ②  $\forall A \in \tau, B \in \tau$ , 若  $A \cap B = \emptyset$ , 则  $A \cap B \in \tau$ ,  
若  $A \cap B \neq \emptyset$ , 则  $\forall P \in A \cap B$ , 即  $P \in A$  且  $P \in B$ ,  
则  $\exists \varepsilon_1, \varepsilon_2 > 0$ , s. t.  $P \in B_{\varepsilon_1}(P) \subset A, P \in B_{\varepsilon_2}(P) \subset B$ , 取  $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ , 则  
 $P \in B_\varepsilon(P) = B_{\varepsilon_1}(P) \cap B_{\varepsilon_2}(P) \subset A \cap B$ , 因而  $A \cap B \in \tau$ .
- ③ 若  $A_\alpha (\alpha \in I) \in \tau$ , 则  $\forall P \in \bigcup_{\alpha \in I} A_\alpha, \exists i \in I$ , s. t.  $P \in A_i \in \tau$ ,  
则  $\exists \varepsilon > 0$ , s. t.  $P \in B_\varepsilon(P) \in A_i$ , 从而  $B_\varepsilon(P) \subset \bigcup_{\alpha \in I} A_\alpha$ , 从而  $\bigcup_{\alpha \in I} A_\alpha \in \tau$ .

所以,  $\tau$  为  $E^n$  的一个拓扑。

- (2)  $\forall P, Q \in E$  且  $P \neq Q$ . 则记  $d = d(P, Q)$ , 取  $\varepsilon_1 = \varepsilon_2 = \frac{d}{3}$ , 则  $P \in B_{\varepsilon_1}(P)$  (开),  $Q \in B_{\varepsilon_2}(Q)$  (开), 且  $B_{\varepsilon_1}(P) \cup B_{\varepsilon_2}(Q) = \emptyset$ ,  
 否则, 若  $\exists R \in B_{\varepsilon_1}(P) \cap B_{\varepsilon_2}(Q)$ , 则  $d(Q, R) < \frac{d}{3}, d(P, R) < \frac{d}{3}$ ,  
 $d = d(P, Q) \leq d(P, R) + d(Q, R) < \frac{2d}{3} < d$  矛盾! 所以  $B_{\varepsilon_1}(P) \cup B_{\varepsilon_2}(Q) = \emptyset$  成立.  
 从而,  $E_n$  满足  $T_2$  分离性公理, 为 Hausdorff 空间.

- (3) 取开集族  $\mathcal{B} = \{B_\varepsilon(P) | P \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}\}$ , 其中  $\mathbb{Q}$  为有理数, 故  $\mathcal{B}$  为可数的, 下证其为拓扑基:

$$\forall P \in E_n, \quad \forall U = B_\varepsilon(P) \in \mathcal{N}(P),$$

$$\exists \varepsilon' > 0, \text{ 且 } \varepsilon' \in \mathbb{Q}, \text{ s. t. } \frac{3\varepsilon}{4} < \varepsilon' < \varepsilon. \quad \exists Q \in \mathbb{Q}^n, \text{ s. t. } |\overrightarrow{PQ}| < \frac{\varepsilon'}{4},$$

$$\text{令 } B = B_{\varepsilon'}(Q), \text{ 则 } B \subset U \text{ 且 } B \in \mathcal{B}.$$

所以,  $\mathcal{B}$  为  $E^n$  中可数拓扑基, 从而  $E^n$  第二可数.

#### 1.4 证:

- (1) 任取  $E_n$  中直线  $l$ , 在  $l$  上依次任取 3 个不同的点  $P, Q, R$ , 则有  $|\overrightarrow{OQ}| = t|\overrightarrow{OP}| + |\overrightarrow{OR}|$ ,  
 其中  $t \in (0, 1)$ . 记  $\sigma(P) = P' \quad (\forall P \in E_n)$   
 则  $|\overrightarrow{O'Q'}| = |\overrightarrow{OQ}| = t|\overrightarrow{OP}| + (1-t)|\overrightarrow{OR}| = t|\overrightarrow{O'P'}| + (1-t)|\overrightarrow{O'R'}| \quad (t \in (0, 1))$   
 $\therefore P', Q', R'$  三点共线且保持分比, 所以  $\sigma$  将直线映为直线.

- (2) 任取  $E_n$  中两平行直线  $l_1, l_2$ , 则由 (1) 知  $l_1, l_2$  在  $\sigma$  下仍为直线, 记为  $l'_1, l'_2$ . 任取不同点  $A, B \in l_1$ , 不同点  $C, D \in l_2$ ,  
 则  $\overrightarrow{AB}, \overrightarrow{CD}$  非零, 且  $\overrightarrow{AB} // \overrightarrow{CD}$  从而  $\exists \lambda \neq 0, \text{ s. t. } \overrightarrow{AB} = \lambda \overrightarrow{CD}$ ,  
 而  $|\overrightarrow{A'B'}| = |\overrightarrow{AB}| = |\lambda| \cdot |\overrightarrow{CD}| = |\lambda| \cdot |\overrightarrow{C'D'}| \quad \therefore \overrightarrow{A'B'} // \overrightarrow{C'D'} \quad \therefore l'_1 // l'_2$   
 $\therefore$  由  $l_1, l_2$  任意性知  $\sigma$  把  $E_n$  中平行直线映为平行直线.

- (3) 记  $\sigma(O) = O', \overrightarrow{OP_i} = \delta_i, \sigma(\delta_i) = \delta'_i = \overrightarrow{O'P'_i}, i = 1, 2, \dots, n$   
 则由  $\{O, \delta_i\}$  为正交标架知

$$|\overrightarrow{OP_i}| \cdot |\overrightarrow{OP_j}| = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \quad i, j = 1, 2, \dots, n$$

$\therefore \{O', \delta'_i\}$  也为正交坐标系.

#### 1.5 证: $\forall t \leq 0. \forall x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$ .

由  $d(x, y) = \sqrt{\sum_{i=1}^n (y_i - x_i)^2}$  知  $d(tx, ty) = td(x, y)$ .

又由  $\sigma$  为等距变换知  $d(\sigma(tx), \sigma(ty)) = d(tx, ty) = td(x, y) = td(\sigma(x), \sigma(y))$ .

取  $y = \theta = (0, \dots, 0)$ , 则  $d(\sigma(tx), \sigma(\theta)) = t \cdot d(\sigma(x), \sigma(\theta))$ ,

则由  $\sigma$  保持共线性而  $\overrightarrow{Ox}$  与  $\overrightarrow{O(tx)}$  共线, 知  $\overrightarrow{\sigma(\theta)\sigma(x)}$  与  $\overrightarrow{\sigma(\theta)\sigma(tx)}$  共线

$\therefore \overrightarrow{\sigma(\theta)\sigma(tx)} = \pm t \overrightarrow{\sigma(\theta)\sigma(x)}$  不妨取  $+t$ , ( $-t$  同理可证), 则有

$$\sigma(tx) - \sigma(\theta) = t(\sigma(x) - \sigma(\theta)) \Rightarrow \sigma(tx) = t\sigma(x) - (1-t)\sigma(\theta) \dots \dots \dots (1),$$

对 (1) 式左右两边关于  $t$  求导得:

$$\begin{aligned}
 \text{左边} &= \frac{\partial}{\partial t} \sigma_1(tx) \\
 &= \left( \frac{\partial}{\partial t} \sigma_1(tx), \dots, \frac{\partial}{\partial t} \sigma_n(tx) \right) \\
 &= (x_1, \dots, x_n) \cdot \begin{pmatrix} \partial_1 \sigma_1(tx) & \cdots & \partial_1 \sigma_n(tx) \\ \vdots & & \vdots \\ \partial_n \sigma_1(tx) & \cdots & \partial_n \sigma_n(tx) \end{pmatrix} \\
 \text{左边} &= \sigma(x) - \sigma(0)
 \end{aligned}$$

则左边 = 右边, 且令  $t=0$  后有:

$$\sigma(x) = \sigma(0) + (x_1, \dots, x_n) \cdot \begin{pmatrix} \partial_1 \sigma_1(tx) & \cdots & \partial_1 \sigma_n(tx) \\ \vdots & & \vdots \\ \partial_n \sigma_1(tx) & \cdots & \partial_n \sigma_n(tx) \end{pmatrix} \Big|_{t=0}$$

记  $a_0^j = \sigma_j(0)$      $a_i^j = (\partial_i \sigma_j)|_{(t=0)}$     其中  $i = 1, 2, \dots, n$ .  $j = 1, 2, \dots, n$ . 有:

$$\sigma(x_1, \dots, x_n) = (a_0^1, \dots, a_0^n) + (x_1, \dots, x_n) \cdot \begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \vdots & & \vdots \\ a_n^1 & \cdots & a_n^n \end{pmatrix}$$

取  $\varepsilon_i = (0, \dots, \overset{\text{第 } i \text{ 个}}{\underset{\downarrow}{1}}, 0, \dots, 0)$ , 则,

$$1 = d(\varepsilon_i, 0) = d(\sigma(\varepsilon_i), \sigma(0)) \Rightarrow (a_i^1)^2 + (a_i^2)^2 + \cdots + (a_i^n)^2 = 1 \quad (\forall i = 1, 2, \dots, n)$$

( $i \neq j$  时)

$$\begin{aligned}
 2 &= 1^2 + 1^2 = d^2(\varepsilon_i, \varepsilon_j) \\
 &= d^2(\sigma(\varepsilon_i), \sigma(\varepsilon_j)) \\
 &= (a_j^1 - a_i^1)^2 + \cdots + (a_j^n - a_i^n)^2 \\
 &= ((a_i^1)^2 + \cdots + (a_i^n)^2) + ((a_j^1)^2 + \cdots + (a_j^n)^2) - 2(a_j^1 a_i^1 + \cdots + a_j^n a_i^n) \\
 &= 1 + 1 - 2(a_j^1 a_i^1 + \cdots + a_j^n a_i^n)
 \end{aligned}$$

$$\Rightarrow a_j^1 a_i^1 + \cdots + a_j^n a_i^n = 0 \quad (i \neq j \text{ 时})$$

从而  $(a_i^j)_{n \times n}$  为单位正交矩阵.

**1.6 证:** 设  $Q$  关于  $\{O; \delta_i\}$  的坐标为:  $x = (x^1, \dots, x^n)$ , 即  $Q - O = x^1 \delta_1 + \cdots + x^n \delta_n$ .

由第 5 题知  $\sigma$  为线性变换, 而  $\sigma(O) = P$ ,  $\sigma(\delta_i) = e_i$ ,

$$\therefore \sigma(Q) - \sigma(O) = \sigma(Q - O) = \sigma(x^1 \delta_1 + \cdots + x^n \delta_n) = x^1 \sigma(\delta_1) + \cdots + x^n \sigma(\delta_n) = x^1 e_1 + \cdots + x^n e_n$$

即点  $Q'$  关于  $\{P; e_i\}$  的坐标等于点  $Q$  关于  $\{O; \delta_i\}$  的坐标.

**1.7 证:** 设  $\{O, \delta_i\}$  为  $E_n$  中某一直角坐标系,  $f$  在  $\{O, \delta_i\}$  之下表示成

$$\begin{aligned}\overrightarrow{Of(t)} &= \sum_{i=1}^n x^i(t) \delta_i, \quad \forall t \in \mathbb{R} \\ &= (\delta_1, \dots, \delta_n) \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix}\end{aligned}$$

任取另一直角坐标系  $\{P, e_i\}$  则有唯一表示:

$$\overrightarrow{OP} = \sum_{i=1}^n a^i \delta_i = (\delta_1, \dots, \delta_n) \begin{pmatrix} a^1(t) \\ \vdots \\ a^n(t) \end{pmatrix} \quad e_i = \sum_{i=1}^n a^i \delta_i = (\delta_1, \dots, \delta_n) \begin{pmatrix} a_i^1 \\ \vdots \\ a_i^n \end{pmatrix} \quad (i = 1, 2, \dots, n)$$

从而

$$(e_1, \dots, e_n) = (\delta_1, \dots, \delta_n) \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_n^n \end{pmatrix}$$

记  $(a_j^i) = A$ . 由  $(e_1, \dots, e_n)$  与  $(\delta_1, \dots, \delta_n)$  均为正交向量组知  $|A| \neq 0$ .

$\therefore A$  可逆. 记  $A^{-1} = (b_j^i)_{n \times n}$   $\therefore (\delta_1, \dots, \delta_n) = (e_1, \dots, e_n) A^{-1}$

$$\overrightarrow{Of(t)} = (\delta_1, \dots, \delta_n) \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix} = (e_1, \dots, e_n) A^{-1} \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix}$$

$$\overrightarrow{OP} = (\delta_1, \dots, \delta_n) \begin{pmatrix} a^1(t) \\ \vdots \\ a^n(t) \end{pmatrix} = (e_1, \dots, e_n) A^{-1} \begin{pmatrix} a^1(t) \\ \vdots \\ a^n(t) \end{pmatrix}$$

而  $\overrightarrow{Of(t)} = \overrightarrow{OP} + \overrightarrow{Pf(t)}$  从而

$$\begin{aligned}\overrightarrow{Pf(t)} &= \overrightarrow{Of(t)} - \overrightarrow{OP} \\ &= (e_1, \dots, e_n) A^{-1} \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix} - (e_1, \dots, e_n) A^{-1} \begin{pmatrix} a^1(t) \\ \vdots \\ a^n(t) \end{pmatrix} \\ &= (e_1, \dots, e_n) A^{-1} \begin{pmatrix} x^1(t) - a^1(t) \\ \vdots \\ x^n(t) - a^n(t) \end{pmatrix} \\ &\triangleq \sum_{k=1}^n e_k y^k(t) \quad \text{其中 } y^k(t) = \sum_{j=1}^n b_j^k (x^j(t) - a^j(t)) \quad (k = 1, 2, \dots, n)\end{aligned}$$

由  $x^j(t)$  连续 (或  $r$  次连续可微) 可得  $y^k(t)$  连续 (或  $r$  次连续可微), 从而映射  $f: \mathbb{R} \rightarrow E^n$  的连续性和  $r$  次连续可微性与  $E^n$  中直角坐标系的选取无关。



**1.8 证:** 记  $f(t)$  在  $\{O; \delta_i\}$  与  $\{P; e_i\}$  下的坐标为  $(x^1(t), \dots, x^n(t)), (y^1(t), \dots, y^n(t))$ , 其中

$$\overrightarrow{OP} = \sum_{i=1}^n a_i \delta_i, \quad e_i = \sum_{j=1}^n a_i^j \delta_j \quad (i = 1, \dots, n)$$

由的 (1.14)

$$x^i(t) = a^i + \sum_{j=1}^n y^j(t) a_j^i$$

则等式两边同时对  $t$  求导有

$$\frac{dx^i}{dt}(t_0) = \sum_{j=1}^n \frac{dy^j(t)}{dt} a_j^i$$

又由 (2.6)

$$\begin{aligned} f'(t_0) &= \sum_{i=1}^n \frac{dx^i}{dt}(t_0) \delta_i \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{dy^j}{dt} a_j^i \delta_i \\ &= \sum_{j=1}^n \left( \frac{y^j(t)}{dt} \sum_{i=1}^n a_j^i \delta_i \right) \\ &= \sum_{j=1}^n \frac{dy^j}{dt} e_j \end{aligned}$$

从而可见其形式不变, 即切向量定义式 (2.6) 与直角坐标系的选取无关。

**1.9 证:**

(1)

$$\begin{aligned} D_v(g + \lambda h) &= \langle \nabla(g + \lambda h)(P), v \rangle \\ &= \langle \nabla g(P) + \lambda \nabla h(P), v \rangle \\ &= \langle \nabla g(P), v \rangle + \lambda \langle \nabla h(P), v \rangle \\ &= D_v(g) + \lambda D_v(h) \end{aligned}$$

(2)

$$\begin{aligned} D_v(g \cdot h) &= \langle \nabla(gh)(P), v \rangle \\ &= \langle ((\nabla g)h)(P), v \rangle + \langle (g(\nabla h))(P), v \rangle \\ &= h(P) \langle \nabla g(P), v \rangle + g(P) \langle \nabla h(P), v \rangle \\ &= h(P) D_v g + g(P) D_v h \end{aligned}$$

**1.10 证:**  $E_n \rightarrow R$  上的函数  $x^i : P = \lambda^1 \delta_1 + \dots + \lambda^n \delta_n \rightarrow \lambda^i \quad (i = 1, 2, \dots, n)$

$\forall P = (\lambda^1, \dots, \lambda^n) \in E^n$ , 在  $P$  的邻域内取  $Q = P + \Delta P = (\lambda^1 + \Delta \lambda^1, \dots, \lambda^n + \Delta \lambda^n)$ , 则

$$\lim_{\Delta \lambda^j \rightarrow 0} \frac{x^i(Q) - x^i(P)}{\Delta \lambda^j} = \lim_{\Delta \lambda^j \rightarrow 0} \frac{\Delta \lambda^i}{\Delta \lambda^j} = \delta_i^j \in C^\infty$$

从而  $x^i \in C^\infty \quad (\forall i = 1, 2, \dots, n)$ .

**1.11 证:**

(1) 在  $E^m$  中, 取两新旧直角坐标分别为  $\{O; \delta_i\}, \{P; e_i\}$ , 且满足

$$\begin{cases} \overrightarrow{OP} = \sum_{i=1}^m a^i \delta_i \\ e_j = \sum_{i=1}^m a_j^i \delta_i \end{cases} \quad j = 1, \dots, m.$$

在  $E^n$  中, 取两新旧直角坐标分别为  $\{O; \xi_i\}, \{P; \eta_i\}$ , 且满足

$$\begin{cases} \overrightarrow{OQ} = \sum_{i=1}^m b^i \xi_i \\ \eta_j = \sum_{i=1}^m b_j^i \xi_i \end{cases} \quad j = 1, \dots, m.$$

记原映射为  $F(\lambda^1, \dots, \lambda^m) = (f^1(\lambda^1, \dots, \lambda^m), \dots, f^n(\lambda^1, \dots, \lambda^m))$ .

设在新坐标下表示为  $G(\mu^1, \dots, \mu^m) = (g^1(\mu^1, \dots, \mu^m), \dots, g^n(\mu^1, \dots, \mu^m))$ .

则有  $f^l = b^l + \sum_{j=1}^n b_j^l g^j \quad \lambda^k = a^k + \sum_{j=1}^m a_j^k \mu^j \quad (k = 1, \dots, m)$ . 再记  $(a_j^k)_{m \times m}$  的逆为  $(C_k^j)_{m \times m}$

$$\begin{aligned} \therefore \frac{\partial f^l}{\partial \lambda^k} &= \sum_{j=1}^n b_j^l \frac{\partial g^j}{\partial \lambda^k} \\ &= \sum_{j=1}^n b_j^l \left( \sum_{i=1}^m \frac{\partial g^j}{\partial \mu^i} C_k^i \right) \\ &= \sum_{j=1}^n \sum_{i=1}^m b_j^l \frac{\partial g^j}{\partial \mu^i} C_k^i \\ \therefore \left( \frac{\partial f^l}{\partial \lambda^k} \right)_{m \times n} &= A^{-1} \left( \frac{\partial g^j}{\partial \mu^i} \right)_{m \times n} B \end{aligned}$$

记  $J_f, J_g$  分别为  $F(\lambda^1, \dots, \lambda^m)$  与  $G(\mu^1, \dots, \mu^m)$  的 Jacobi 矩阵. 则有  $J_f = A^{-1} J_g B$ .

(2) 由于  $A, B$  可逆, 所以  $A^{-1}, B$  均可表为若干初等行、列变换的乘积,

$\therefore r(J_f) = r(J_g)$ , 在任意点  $x_0$ .

(3) 在  $E^m$  中任取点  $P$ , 记  $Q = f(P)$ .

在  $P$  点邻域分别取两新旧曲纹坐标系  $(u^1, \dots, u^m), (v^1, \dots, v^m)$ . 两者之间由同胚映射  $g$  关联:  $v^i = g^i(u^1, \dots, u^m) \quad i = 1, \dots, m$ . 记  $g^{-1} = \bar{g}$  则  $u^i = \bar{g}^i(v^1, \dots, v^m) \quad i = 1, \dots, m$ .

同样在  $Q$  点邻域分别取两新旧曲纹坐标系  $(s^1, \dots, s^n), (t^1, \dots, t^n)$ . 两者之间由同胚映射  $h$  关联:  $t^i = h^i(s^1, \dots, s^n) \quad i = 1, \dots, m$

原函数  $f(u^1, \dots, u^m)$  中分量记为  $f^i(u^1, \dots, u^m) \quad (i = 1, \dots, n)$  在  $E^m$  与  $E^n$  间曲纹坐

标变换下为  $\tilde{f}(v^1, \dots, v^m)$ , 其在新坐标下分量为:  $\tilde{f}^i(v^1, \dots, v^m) = h^i(f^1, \dots, f^n)$  其中  $f^j(u^1, \dots, u^m) = f^j(\bar{g}(v^1, \dots, v^m), \dots, \bar{g}(v^1, \dots, v^m))$  ( $i = 1, \dots, n; j = 1, \dots, n$ ) 从而在变换后 Jacobi 矩阵  $J_{n \times m}$  为

$$\begin{aligned} J_j^i &= \frac{\partial \tilde{f}^i}{\partial v^j} \\ &= \frac{\partial h^i}{\partial f^k} \frac{\partial f^k}{\partial u^r} \frac{\partial \bar{g}^r}{\partial v^j} \quad (i, k = 1, \dots, n, r, j = 1, \dots, m) \end{aligned}$$

在 P 点, 记  $g$  的雅克比矩阵  $(\frac{\partial g^i}{\partial u^j})_{m \times m}$  为  $G$ ,  $h$  的雅克比矩阵  $(\frac{\partial h^i}{\partial v^j})_{n \times n}$  为  $H$ , 变换前  $f$  的雅克比矩阵  $(\frac{\partial f^i}{\partial u^j})_{n \times m}$  为  $J_0$  则有:

$$J = H J_0 G^{-1}$$

**1.12 证:** 记原方程组为  $x^i = f^i(u^1, \dots, u^n)$ ,  $i = 1, \dots, n$ . 则  $f = f^i \delta_i$  的雅克比矩阵的行列式为

$$\begin{aligned} \left| \frac{\partial(f^1, \dots, f^n)}{\partial(u^1, \dots, u^n)} \right| &= \begin{vmatrix} -x^1 \tan u^1 & -x^1 \tan u^2 & \cdots & -x^1 \tan u^{n-2} & -x^1 \tan u^{n-1} & \frac{x^1}{u^n} \\ -x^2 \tan u^1 & -x^2 \tan u^2 & \cdots & -x^2 \tan u^{n-2} & x^2 \cot u^{n-1} & \frac{x^2}{u^n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x^{n-1} \tan u^1 & x^{n-1} \cot u^2 & \cdots & 0 & 0 & \frac{x^{n-1}}{u^n} \\ x^n \cot u^1 & 0 & \cdots & 0 & 0 & \frac{x^n}{u^n} \end{vmatrix} \\ &= x^1 x^2 \cdots x^n \begin{vmatrix} -\tan u^1 & -\tan u^2 & \cdots & -\tan u^{n-2} & -\tan u^{n-1} & 1 \\ -\tan u^1 & -\tan u^2 & \cdots & -\tan u^{n-2} & \cot u^{n-1} & 1 \\ -\tan u^1 & -\tan u^2 & \cdots & \cot u^{n-2} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\tan u^1 & \cot u^2 & \cdots & 0 & 0 & 1 \\ \cot u^1 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix} \\ &= \frac{x^1 x^2 \cdots x^n}{u^n} \cdot \left[ (-1)^{1+n} (-1)^{\frac{(n-1)(n-2)}{2}} \cot u^1 \cdots \cot u^{n-1} \right. \\ &\quad \left. + (-1)^{2+n} (-1)^{\frac{(n-1)(n-2)}{2}} \cot u^1 \cdots \cot u^{n-2} (-\tan u^{n-1}) \right] \\ &= (-1)^{\frac{n^2-n+4}{2}} \left( \prod_{i=1}^{n-2} r^i \prod_{j=1}^i \cos u^j \right) x^1 x^2 (u^n)^{n-3} \frac{1}{\sin u^{n-1} \cos u^{n-1}} \end{aligned}$$

当  $x^1, x^2 \neq 0$  时, 有  $u^n, \cos u^1, \dots, \cos u^{n-1}, \sin u^{n-1}$  非零, 又由  $r^1, \dots, r^n$  为正数可知:  $\left| \frac{\partial(f^1, \dots, f^n)}{\partial(u^1, \dots, u^n)} \right| \neq 0$ , 即  $f$  的秩为  $n$ . 从而  $(u^1, \dots, u^n)$  给出  $E^n$  中除坐标面  $\{(0, 0, x^3, \dots, x^n) : x^3, \dots, x^n \in \mathbb{R}\}$  以外的任意一点的邻域内的曲线坐标系。

**1.13 解:** 设  $(x, y, z)$  是  $E^3$  中的直角坐标系, 令

$$\begin{cases} x = \rho \cos \psi \cos \theta \\ y = \rho \cos \psi \sin \theta \\ z = \rho \sin \psi \end{cases} \quad (0 < \rho < +\infty, -\frac{\pi}{2} < \psi < \frac{\pi}{2}, -\pi < \theta < \pi)$$

则  $(\rho, \psi, \theta)$  给出球坐标系, 记  $u^1 = \rho, u^2 = \psi, u^3 = \theta$  则可求其自然标架场如下:

$$\begin{aligned}\vec{r}_1 &= (\cos\psi\cos\theta, \cos\psi\sin\theta, \sin\psi) \\ \vec{r}_2 &= (-\rho\sin\psi\cos\theta, -\rho\sin\psi\sin\theta, \rho\cos\psi) \\ \vec{r}_3 &= (-\rho\cos\psi\sin\theta, \rho\cos\psi\cos\theta, 0)\end{aligned}$$

则  $\forall Q = r(P), \{Q, r_i\}$  为  $E^3$  中球坐标系诱导的自然标架场。

下求其度量系数:

$$\begin{aligned}g_{11} &= \langle \vec{r}_1, \vec{r}_1 \rangle = 1 & g_{12} &= g_{21} = \langle \vec{r}_1, \vec{r}_2 \rangle = 0 \\ g_{31} &= g_{13} = \langle \vec{r}_1, \vec{r}_3 \rangle = 0 \\ g_{22} &= \langle \vec{r}_2, \vec{r}_2 \rangle = \rho^2 & g_{23} &= g_{32} = \langle \vec{r}_2, \vec{r}_3 \rangle = 0 \\ g_{33} &= \langle \vec{r}_3, \vec{r}_3 \rangle = \rho^2 \cos^2\psi\end{aligned}$$

$\therefore$  度量系数为  $g_{11} = \rho^2, g_{33} = \rho^2 \cos^2\psi, g_{ij} = 0$  ( $i \neq j$  时, 其中  $i, j = 1, 2, 3$ ).

则  $g^{11} = 1, g^{22} = \frac{1}{\rho^2}, g^{33} = \frac{1}{\rho^2 \cos^2\psi}, g^{ij} = 0$  ( $i \neq j$  时, 其中  $i, j = 1, 2, 3$ ).

$$\text{且 } \frac{\partial g_{11}}{\partial u^i} = 0 \ (i = 1, 2, 3) \quad \frac{\partial g_{ij}}{\partial u^k} \ (i, j, k = 1, 2, 3) \quad \frac{\partial g_{22}}{\partial u^1} = 2\rho \quad \frac{\partial g_{22}}{\partial u^2} = 0 \quad \frac{\partial g_{22}}{\partial u^3} = 0 \quad \frac{\partial g_{33}}{\partial u^1} = 2\rho \cos^2\psi \quad \frac{\partial g_{33}}{\partial u^2} = -\rho^2 \sin 2\psi \quad \frac{\partial g_{33}}{\partial u^3} = 0$$

从而由

$$\Gamma_{il}^k = \frac{1}{2} g^{kj} \left( \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{il}}{\partial u^j} \right)$$

得:

$$\begin{aligned}\Gamma_{33}^1 &= \frac{1}{2} g^{11} \left( \frac{\partial g_{31}}{\partial u^3} + \frac{\partial g_{13}}{\partial u^1} - \frac{\partial g_{33}}{\partial u^1} \right) = \rho \cos^2\psi \\ \Gamma_{33}^2 &= \frac{1}{2} g^{22} \left( \frac{\partial g_{32}}{\partial u^3} + \frac{\partial g_{32}}{\partial u^1} - \frac{\partial g_{33}}{\partial u^2} \right) = -\sin\psi \\ \Gamma_{33}^3 &= \frac{1}{2} g^{33} \left( \frac{\partial g_{33}}{\partial u^3} + \frac{\partial g_{33}}{\partial u^3} - \frac{\partial g_{33}}{\partial u^3} \right) = 0\end{aligned}$$

又由

$$\begin{aligned}\frac{\partial \vec{r}_1}{\partial u^1} &= 0 & \frac{\partial \vec{r}_1}{\partial u^2} &= \frac{1}{\rho} \vec{r}_2 & \frac{\partial \vec{r}_1}{\partial u^3} &= \frac{1}{\rho} \vec{r}_3 \\ \frac{\partial \vec{r}_2}{\partial u^2} &= -\rho \vec{r}_1 & \frac{\partial \vec{r}_2}{\partial u^3} &= -\tan\psi \vec{r}_3\end{aligned}$$

从而 Christoffel 系数为:

$$\begin{aligned}\Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{\rho} & \Gamma_{31}^3 &= \Gamma_{13}^3 = \frac{1}{\rho \cos\psi} \\ \Gamma_{22}^1 &= -\rho & \Gamma_{23}^3 &= \Gamma_{32}^3 = -\tan\psi \\ \Gamma_{33}^1 &= -\rho \cos^2\psi & \Gamma_{33}^2 &= -\sin\psi\end{aligned}$$

其余为 0.

**1.14 解:** 设  $(x, y, z)$  为  $E^3$  中直角坐标系, 设

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \\ z = t \end{cases} \quad (\text{其中 } 0 < \rho < +\infty, -\pi < \theta < \pi, -\infty < t < +\infty)$$

则  $(\rho, \theta, t)$  为柱坐标系。记  $u^1 = \rho, u^2 = \theta, u^3 = t$ , 则有:

$$\begin{aligned}\vec{r}_1 &= (\cos\theta, \sin\theta, 0) \\ \vec{r}_2 &= (-\rho\sin\theta, \rho\cos\theta, 0) \\ \vec{r}_3 &= (0, 0, 1)\end{aligned}$$

且  $g_{11} = 1, g_{22} = \rho^2, g_{33} = 1, g_{ij} = 0 (i \neq j \text{ 时})$  由于:

$$\begin{aligned}\frac{\partial \vec{r}_1}{\partial u^1} &= 0 & \frac{\partial \vec{r}_1}{\partial u^2} &= \frac{1}{\rho} \vec{r}_2 & \frac{\partial \vec{r}_1}{\partial u^3} &= 0 \\ \frac{\partial \vec{r}_2}{\partial u^1} &= -\rho \vec{r}_1 & \frac{\partial \vec{r}_2}{\partial u^2} &= 0 & \frac{\partial \vec{r}_2}{\partial u^3} &= 0 \\ \frac{\partial \vec{r}_3}{\partial u^1} &= 0 & \frac{\partial \vec{r}_3}{\partial u^2} &= 0 & \frac{\partial \vec{r}_3}{\partial u^3} &= 0\end{aligned}$$

所以 Christoffel 系数为:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{\rho} \quad \Gamma_{22}^1 = -\rho$$

其余为 0

**1.15 证:**  $\because v = v^i \vec{e}_i$

$$\begin{aligned}\therefore dv &= v^j d\vec{e}_j + \vec{e}_i dv^i \\ &= v^j \left( \frac{\partial \vec{e}_j}{\partial u^k} du^k \right) + \vec{e}_i dv^i \\ &= v^j \Gamma_{jk}^i \vec{e}_i du^k + \vec{e}_i dv^i \\ &= (v^j \Gamma_{jk}^i du^k + dv^i) \vec{e}_i\end{aligned}$$

$$\therefore v \text{ 为平行向量场} \Leftrightarrow v^j \Gamma_{jk}^i + dv^i = 0 (\forall i = 1, 2, \dots, n)$$

**1.16 证:** 定义加法  $'+' : (f+g)(x) = f(x) + g(x)$ , 数乘:  $(\lambda f)(x) = \lambda f(x)$

先证  $(\mathcal{L}(v_1, \dots, v_r; \mathbb{R}), +)$  构成加法群。

$$(1) \quad \forall f, g \in \mathcal{L}(v_1, \dots, v_r; \mathbb{R}) \quad \forall i \in \{1, \dots, r\} \quad \forall x_1 = (x^1, \dots, x_1^i, \dots, x^r), x_2 = (x^1, \dots, x_2^i, \dots, x^r) \in V_1 \times \dots \times V_r \quad \forall k_1, k_2 \in \mathbb{R} \text{ 有}$$

$$\begin{aligned}& (f+g)(x^1, \dots, k_1 x_1^i + k_2 x_2^i, \dots, x^r) \\ &= f(x^1, \dots, k_1 x_1^i + k_2 x_2^i, \dots, x^r) + g(x^1, \dots, k_1 x_1^i + k_2 x_2^i, \dots, x^r) \\ &= (k_1 f(x_1) + k_2 f(x_2)) + (k_1 g(x_1) + k_2 g(x_2)) \\ &= k_1 (f+g)(x_1) + k_2 (f+g)(x_2)\end{aligned}$$

$$\therefore f+g \in \mathcal{L}(V_1, \dots, V_r; \mathbb{R})$$

$$(2) \quad \forall f, g, h \in \mathcal{L}(V_1, \dots, V_r; \mathbb{R}) \quad \forall x \in V_1 \times \dots \times V_r, \text{ 有}$$

$$[(f+g)+h](x) = (f+g)(x) + h(x) = f(x) + g(x) + h(x)$$

$$[f+(g+h)](x) = f(x) + (g+h)(x) = f(x) + g(x) + h(x)$$

$$\therefore (f+g)+h = f+(g+h)$$

(3) 取  $\theta$  为  $\forall x \in V_1 \times \cdots \times V_r, \theta(x) = 0$

则显然  $f + \theta = \theta + f, \forall f$

(4)  $\forall f$ , 取  $-f$  为  $(-f)(x) = -f(x)$  则  $f + (-f) = (-f) + f = \theta$

(5)  $\forall f, g \in \mathcal{L}(V_1 \cdots V_r; \mathbb{R})$  显然  $f + g = g + f$ .

所以  $(\mathcal{L}(V_1, \cdots, V_r; \mathbb{R}), +)$  构成 Abel 群。  $\forall \lambda, \mu \in \mathbb{R} \quad \forall f, g \in \mathcal{L}(V_1 \times \cdots \times V_r; \mathbb{R})$ , 又有

(6)  $(\lambda, \mu)f = \lambda f + \mu f$

(7)  $(\lambda\mu)g = \lambda(\mu g)$

(8)  $\lambda(f + g) = \lambda f + \lambda g$

(9)  $1 \cdot f = f$

所以  $\mathcal{L}(V_1, \cdots, V_r; \mathbb{R})$  构成  $\mathbb{R}$  上的线性空间。

### 1.17 证:

(1)  $\forall k_1, k_2 \in \mathbb{R} \quad u_1, u_2 \in V$ , 则由  $\tilde{f}$  为 2 重线性函数知

$$\begin{aligned} f(k_1 u_1 + k_2 u_2) &= \sum_{i=1}^n \tilde{f}(k_1 u_1 + k_2 u_2, \delta^i) \delta_i \\ &= \sum_{i=1}^n (k_1 \tilde{f}(u_1, \delta^i) + k_2 \tilde{f}(u_2, \delta^i)) \delta_i \\ &= k_1 \sum_{i=1}^n \tilde{f}(u_1, \delta^i) \delta_i + k_2 \sum_{i=1}^n \tilde{f}(u_2, \delta^i) \delta_i \\ &= k_1 f(u_1) + k_2 f(u_2) \end{aligned}$$

所以  $f$  为线性变换。

(2) 若另取基底  $\{e^i\}$ , 且  $e^i = a_j^i \delta_j$  其对偶基底  $\{e_i\}$ , 则有  $\delta_j = a_j^i e_i$ , 其中  $(b_i^j)$  为  $(a_i^j)$  的逆矩阵。且由

$$\delta^i(e_j) = \delta_i(a_j^k \delta_k) = a_j^k \delta_k^i = a_j^i = a_k^i e^k(e_j)$$

知  $\delta^i = a_k^i e^k$ .

$$\therefore f(u) = \tilde{f}(u, \delta^i) \delta_i = \tilde{f}(u, a_k^i e^k) (b_i^j e_j) = a_k^i b_i^j \tilde{f}(u, e^k) e_j = \delta_k^j \tilde{f}(u, e^k) e_j = \tilde{f}(u, e^j) e_j$$

$\therefore f$  的定义与基底的选取无关。

**1.18 证:** 取定  $t, \forall v = (v_1, \cdots, v_r) \in V \times \cdots \times V \quad v^1 \in V^*$ , 取  $F : V^* \times V_1 \cdots \times V_r \rightarrow \mathbb{R}$  为  $F(v^1, v_1, \cdots, v_r) = v^1(t(v))$ , 显然  $F$  是  $V^* \times V \times \cdots \times V$  上的一个  $1+r$  重线性函数, 即  $(1, r)$  型张量。

反之, 若给定一个  $(1, r)$  型张量  $F : V^* \times V_1 \times \cdots \times V_r, \forall v = (v_1, \cdots, v_r) \in V \times \cdots \times V \rightarrow \mathbb{R}$ , 令

$t(v) = F(\delta^i, v_1, \dots, v_r)\delta_i$ , 显然  $t$  为  $r$  重线性映射。

所以  $t$  等同一个  $(1, r)$  型张量。

**1.19 证:**

$$(1) \quad \forall \alpha_1, \dots, \alpha_2 \in \mathcal{L}(V_1, \dots, V_p; \mathbb{R}), \quad \forall \beta \in \mathcal{L}(W_1, \dots, W_q; \mathbb{R})$$

$$\forall v \in V_1 \times \dots \times V_p, \quad w \in W_1 \times \dots \times W_q$$

$$\begin{aligned} & (\alpha_1 + \alpha_2) \otimes \beta(v, w) \\ &= (\alpha_1 + \alpha_2)(v) \cdot \beta(w) \\ &= [\alpha_1(v) + \alpha_2(v)] \cdot \beta(w) \\ &= \alpha_1(v) \cdot \beta(w) + \alpha_2(v) \cdot \beta(w) \\ &= \alpha_1 \otimes \beta(v, w) + \alpha_2 \otimes \beta(v, w) \\ &= (\alpha_1 \otimes \beta + \alpha_2 \otimes \beta)(v, w) \end{aligned}$$

$$\therefore (\alpha_1 + \alpha_2) \otimes \beta = \alpha_1 \otimes \beta + \alpha_2 \otimes \beta \text{ 同理 } \beta \otimes (\alpha_1 + \alpha_2) = \beta \otimes \alpha_1 + \beta \otimes \alpha_2$$

$$(2) \quad \forall \alpha \in \mathcal{L}(V_1, \dots, V_r; \mathbb{R}), \quad \beta \in \mathcal{L}(W_1, \dots, W_s; \mathbb{R}), \quad \gamma \in \mathcal{L}(Z_1, \dots, Z_t; \mathbb{R})$$

$$\forall v_1 \times \dots \times v_r, \quad w \in W_1 \times \dots \times W_s, \quad z \in Z_1 \times \dots \times Z_t, \text{ 则}$$

$$\begin{aligned} & (\alpha \otimes \beta) \otimes \gamma(v, w, z) \\ &= (\alpha \otimes \beta)(v, w) \cdot \gamma(z) \\ &= (\alpha(v) \cdot \beta(w)) \cdot \gamma(z) \\ &= \alpha(v) \cdot [\beta(w)\gamma(z)] \\ &= \alpha(v) \cdot [(\beta \otimes \gamma)(w, z)] \\ &= [\alpha \otimes (\beta \otimes \gamma)](v, w, z) \end{aligned}$$

$$\therefore (\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

故多重线性函数的张量积服从分配律和结合律。

**1.20 证:** 设  $\{\xi_i\}$  为  $V$  的基,  $\{\eta_i\}$  为  $W$  的基, 记  $A = \{V \otimes W | v \in V, w \in W\}$ , 下证  $A$  不构成线性空间:

由于  $\dim(V) \geq 2, \dim(W) \geq 2$ . 所以分别可取  $V, W$  的两组基:  $\xi_1, \xi_2, \eta_1, \eta_2$ , 则

$$\xi_1 \otimes \eta_1 \in A, \quad \xi_2 \otimes \eta_2 \in A,$$

若

$$\xi_1 \otimes \eta_1 + \xi_2 \otimes \eta_2 \in A$$

则  $\exists a^i (i = 1, 2, \dots, n) \text{ s.t.}$

$$\xi_1 \otimes \eta_1 + \xi_2 \otimes \eta_2 = (a^i \xi_i) \otimes (b^j \eta_j) = a^i b^j \xi_i \otimes \eta_j$$

移项并合并后, 由于  $\{\xi_i \eta_j\}$  线性无关, 所以有

$$a^i b^j = \begin{cases} 1 & i = j = 1 \text{ 以及 } i = j = 2 \text{ 时;} \\ 0 & \text{其余情况.} \end{cases}$$

$a^1 b^1 = a^2 b^2 = 1$ , 从而  $a^1, b^1, a^2, b^2 \neq 0$ , 但  $a^1 b^2 = 0$ , 矛盾!

$\therefore \xi_1 \otimes \eta_1 + \xi_2 \otimes \eta_2 \notin A$ ,  $A$  不构成线性空间。

### 1.21 证:

(1) 由题知  $f(\delta_i) = b_i^j \delta_j$ .

若另取基底  $\{e_i\}$  且  $e_i = a_i^j \delta_j$  从而  $\delta_j = c_j^i e_i$  (其中  $(c_j^i)$  为  $(a_i^j)$  的逆).

则  $f(e_i) = f(a_i^j \delta_j) = a_i^j f(\delta_j) = a_i^j b_j^k \delta_k = a_i^j b_j^k c_k^l e_l$ .

即  $f(e_i) = (a_i^j b_j^k c_k^l) e_l$ , 记  $d_i^l = a_i^j b_j^k c_k^l$ .

$\therefore f$  在基  $\{e_i\}$  下矩阵是  $(d_i^l)$ .

此时

$$\begin{aligned} B_3 &= d_i^j d_k^i d_j^k \\ &= a_i^s b_s^t c_t^j a_k^e b_e^f c_f^i a_j^g b_g^h c_h^k \\ &= (a_i^s c_f^i)(a_k^e c_h^k)(a_j^g c_t^j) b_s^t b_e^f b_g^h \\ &= (\delta_f^s b_e^f)(\delta_h^e b_g^h)(\delta_i^g b_s^t) \\ &= b_e^s b_g^e b_s^g \end{aligned}$$

$\therefore B_3$  与基底的选取无关。

(2) 令  $F: V^* \times V^* \rightarrow \mathbb{R}$  为  $F(\delta^i, u) = \delta^i(f(u))$ . ( $\forall u \in V$ ). 则易证  $F$  为 (1,1) 型张量, 且

$$\begin{aligned} B_3 &= b_i^j b_k^i b_j^k \\ &= F(\delta^j, \delta_i) \cdot F(\delta^i, \delta_k) \cdot F(\delta^k, \delta_j) \\ &= F(\delta^j, \delta_i) \cdot [(C_1^2 F \otimes F)(\delta^i, \delta_j)] \\ &= C_2^1 [F \otimes C_1^2 (F \otimes F)](\delta^j, \delta_j) \\ &= C_1^1 \{C_2^1 [F \otimes C_1^2 (F \otimes F)]\}. \end{aligned}$$

**1.22 解:** 令  $F: V^* \times V \rightarrow \mathbb{R}$   $F(v^i, v_j) = v^i(v_j)$ . 易证  $F$  为 (1,1) 型张量. 任取  $V$  的基底  $\{\delta_i\}$ , 其对偶基底  $\{\delta^i\}$ , 则  $\{\delta_i \otimes \delta^j\}$  为  $F$  的基底, 且  $\delta^i(\delta_j) = \delta_j^i$ . 从而  $F(\delta^i, \delta_j) = \delta_j^i$ .

$$F = F(\delta^i, \delta_j) \delta_i \otimes \delta^j = \delta_i^j \delta_i \otimes \delta^j = \sum_i \delta_i \otimes \delta^i$$

$\therefore \delta_i^j$  为  $F$  的分量。



**1.23 解:** 若存在 (0,2) 型张量  $F$  满足题设条件, 则任取  $V$  的一组基底  $\{\delta_i\}$ ,  $F(\delta_i, \delta_j) = \delta_{ij}$ . 另任取一基底  $\{e_i\}$ , 且  $e_i = a_i^j \delta_j$ , 则

$$\begin{aligned} F(e_i, e_j) = \delta_{ij} &\Leftrightarrow F(a_i^k \delta_k, a_j^r \delta_r) = \delta_{ij} \\ &\Leftrightarrow a_i^k a_j^r F(\delta_k, \delta_r) = \delta_{ij} \\ &\Leftrightarrow a_i^k a_j^k = \delta_{ij} \\ &\Leftrightarrow (a_i^j) \text{ 为单位正交阵} \end{aligned}$$

但  $a_i^j$  未必为单位正交阵, 矛盾!  
所以不存在。

**1.24 证:**  $\forall v_1, \dots, v_q \in V$ ,  $(\sigma(f))(v_1, \dots, v_q) = f(v_{\sigma(1)}, \dots, v_{\sigma(q)}) = \alpha^1(v_{\sigma(1)}) \cdots \alpha^q(v_{\sigma(q)})$ .

$$\begin{aligned} \alpha^{\tau(1)}(v_1) \otimes \cdots \otimes \alpha^{\tau(q)}(v_q) &= \alpha^{\tau(1)} \otimes \cdots \otimes \alpha^{\tau(q)}(v_1, \dots, v_q) \\ (\text{由 } \tau = \sigma^{-1} \text{ 知 } i = \sigma(\tau(i))) &= \alpha^{\tau(1)}(v_{\sigma(\tau(1))}) \cdots \alpha^{\tau(q)}(v_{\sigma(\tau(q))}) \\ &= \alpha^1(v_{\sigma(1)}) \cdots \alpha^q(v_{\sigma(q)}) \end{aligned}$$

$$\therefore \sigma(f) = \alpha^{\tau(1)} \otimes \cdots \otimes \alpha^{\tau(q)}$$

**1.25 证:**

(1)

$$\begin{aligned} \sigma \text{ 是对称张量} &\Leftrightarrow \sigma(\xi) = \xi \\ &\Leftrightarrow \frac{1}{q!} \sum_{\sigma \in \varphi(q)} \sigma(\xi) = \xi \\ S_q(\xi) &= \xi. \end{aligned}$$

(2)

$$\begin{aligned} \sigma \text{ 是反对称张量} &\Leftrightarrow \sigma(\xi) = \text{sign}(\sigma)\xi \\ &\Leftrightarrow \text{sign}(\sigma) \cdot \sigma(\xi) = \xi \\ &\Leftrightarrow \sum_{\sigma \in \varphi(q)} \text{sign}(\sigma) \cdot \sigma(\xi) = \xi \cdot q! \\ &\Leftrightarrow A_q(\xi) = \xi \end{aligned}$$

**1.26 证:**  $\forall \xi \in V_2^0, \{\delta^i\}$  为  $V$  中的基, 记  $\xi = \xi_{ij}\delta^i\delta^j$  为二阶协变张量. 对于  $\delta \triangleq \delta^i \otimes \delta^j$ .

$$\begin{aligned} S_2(\delta) &= \frac{1}{2} \cdot (\delta^i \otimes \delta^j + \delta^j \otimes \delta^i) \text{ (由 24 题)} \\ A_2(\delta) &= \frac{1}{2}(\delta^i \otimes \delta^j - \delta^j \otimes \delta^i) \\ \Rightarrow \delta &= \delta^i \otimes \delta^j = S_2(\delta) + A_2(\delta) \\ \therefore \xi &= \xi_{ij}\delta^i \otimes \delta^j \\ &= \delta_{ij}(S_2(\delta^i \otimes \delta^j) + A_2(\delta^i \otimes \delta^j)) \\ &= S_2(\xi) + A_2(\xi) \end{aligned}$$

其中  $S_2(\xi)$  为对称张量,  $A_2(\xi)$  为反对称张量。

**1.27 证:** 取  $V$  中基底  $\{\delta_i\}$ , 由  $\varphi \in V_2^0$ , 依题意知,  $\forall \delta_i\delta_j\delta_k$  有

$$\begin{aligned} \varphi(\delta_i, \delta_j, \delta_k) &= \varphi(\delta_j, \delta_i, \delta_k) \\ &= -\varphi(\delta_j, \delta_k, \delta_i) \\ &= -\varphi(\delta_k, \delta_j, \delta_i) \\ &= \varphi(\delta_k, \delta_i, \delta_j) \\ &= -\varphi(\delta_i, \delta_j, \delta_k) \\ \Rightarrow \varphi(\delta_i, \delta_j, \delta_k) &= 0. \\ \therefore \varphi &= \varphi(\delta_i, \delta_j, \delta_k)\delta^i \otimes \delta^j \otimes \delta^k = 0. \end{aligned}$$

**1.28 证:** 取  $V$  的基  $\{\delta_i\}$ ,  $\because a(x, x) = 0$ ,  $\therefore a(\delta_i, \delta_j) = 0$  ( $i = j$ ) 时.  
 $\forall i \neq j$  取  $x = \delta_i + \delta_j$  则

$$\begin{aligned} 0 &= a(x, x) \\ &= a(\delta_i + \delta_j, \delta_i + \delta_j) \\ &= a_{ii} + a_{ij} + a_{ji} + a_{jj} \\ &= a_{ij} + a_{ji}. \end{aligned}$$

记  $a = a_{ij}\delta^i \otimes \delta^j$  有

$$\begin{aligned} S_2(a) &= \frac{1}{2}a_{ij}(\delta^i \otimes \delta^j + \delta^j \otimes \delta^i) \\ &= \frac{1}{2}(a_{ij}\delta^i \otimes \delta^j + a_{ji}\delta^i \otimes \delta^j) \\ &= \frac{1}{2}(a_{ij} + a_{ji})\delta^i \otimes \delta^j \\ &= 0. \end{aligned}$$

**1.29 证:** 由于  $a$  为任意对称张量, 可取  $a$  使得  $a^{ij} \neq 0 (\forall i, j)$  (例如, 取  $a = \sum_i \sum_j \delta_i \otimes \delta_j$ ), 则

$$0 = a^{ij}b_{ij} = a^{ji}b_{ji} = a^{ij}b_{ji}$$

$\therefore a^{ij}(b^{ij} + b^{ji}) = 0$ , 此时  $b_{ij} + b_{ji} = 0 \Rightarrow b_{ij} = -b_{ji}$ ,  
 $\therefore b$  为反对称张量。

**1.30 证:** 取  $V$  的基  $\{\delta_i\}$ , 由  $x$  的任意性,  $\forall i, j$  取  $x = \delta_i + \delta_j \in V$ . 由  $f(x, x) = \tilde{f}(x, x)$  知

$$\begin{aligned} f(\delta_i + \delta_j, \delta_i + \delta_j) &= \tilde{f}(\delta_i + \delta_j, \delta_i + \delta_j) \\ \Rightarrow f(\delta_i, \delta_i) + f(\delta_i, \delta_j) + f(\delta_j, \delta_i) + f(\delta_j, \delta_j) &= \tilde{f}(\delta_i, \delta_i) + \tilde{f}(\delta_i, \delta_j) + \tilde{f}(\delta_j, \delta_i) + \tilde{f}(\delta_j, \delta_j) \\ \Rightarrow f(\delta_i, \delta_j) + f(\delta_j, \delta_i) &= \tilde{f}(\delta_i, \delta_j) + \tilde{f}(\delta_j, \delta_i) \\ \Rightarrow 2f(\delta_i, \delta_j) &= 2\tilde{f}(\delta_i, \delta_j) \\ \Rightarrow f_{ij} &= \tilde{f}_{ij} \end{aligned}$$

记  $f = f_{ij}\delta^i \otimes \delta^j$   $\tilde{f} = \tilde{f}_{ij}\delta^i \otimes \delta^j$ , 则  $f = \tilde{f}$ .

**1.31 证:** 原行列式  $= \sum_{\sigma} \delta_{i_1, \dots, i_r}^{\sigma(i_1), \dots, \sigma(i_r)} \delta_{j_1}^{\sigma(i_1)} \dots \delta_{j_r}^{\sigma(i_r)} = \delta_{i_1, \dots, i_r}^{j_1, \dots, j_r}$ .

**1.32 证:** (1) 记置换

$$\pi = \begin{pmatrix} i_1, \dots, i_{r+s} \\ j_1, \dots, j_{r+s} \end{pmatrix} \quad \pi_1 = \begin{pmatrix} k_1, \dots, k_r \\ j_1, \dots, j_r \end{pmatrix} \quad \pi_2 = \begin{pmatrix} i_1, \dots, i_r, i_{r+1}, \dots, i_{r+s} \\ k_1, \dots, k_r, j_{r+1}, \dots, j_{r+s} \end{pmatrix}$$

则  $\pi = \pi_1 \pi_2$ , 从而  $\text{sgn}(\pi) = \text{sgn}(\pi_1) \cdot \text{sgn}(\pi_2)$ .

即, 取定某排列  $(l_1, \dots, l_r)$ , 有

$$\delta_{j_1, \dots, j_{r+s}}^{i_1, \dots, i_{r+s}} = \delta_{j_1, \dots, j_r}^{l_1, \dots, l_r} \cdot \delta_{l_1, \dots, l_r, j_{r+1}, \dots, j_{r+s}}^{i_1, \dots, i_{r+s}} \quad (\text{注: 此时不对上下指标求和})$$

而  $(l_1, \dots, l_r)$  的排列共有  $r!$  种取法.

$$\therefore \delta_{j_1, \dots, j_{r+s}}^{i_1, \dots, i_{r+s}} = \frac{1}{r!} \sum_{k_1, \dots, k_r} \delta_{j_1, \dots, j_r}^{k_1, \dots, k_r} \delta_{k_1, \dots, k_r, j_{r+1}, \dots, j_{r+s}}^{i_1, \dots, i_{r+s}}.$$

(2)

$$\delta_{i_1, \dots, i_r}^{i_1, \dots, i_r} = A_n^r = \frac{n!}{(n-r)!}.$$

**1.33 证:** 记  $a = (\alpha_1, \dots, \alpha_n)$ , 其中  $\alpha_i = (a_i^1, \dots, a_i^n)^T$ , 则

$$\begin{aligned} \det a &= \det(\alpha_1, \dots, \alpha_n) \\ &= \frac{1}{n!} \delta_{1, \dots, n}^{i_1, \dots, i_n} \det(\alpha_{i_1}, \dots, \alpha_{i_n}) \\ &= \frac{1}{n!} \delta_{1, \dots, n}^{i_1, \dots, i_n} \delta_{j_1, \dots, j_n}^{1, \dots, n} a_{i_1}^{j_1} \cdots a_{j_n}^{i_n} \\ &= \frac{1}{n!} \delta_{j_1, \dots, j_n}^{i_1, \dots, i_n} a_{i_1}^{j_1} \cdots a_{j_n}^{i_n}. \end{aligned}$$

**1.34 证:**