HOMEWORK 2
PROOF TECHNIQUES *

10-607 COMPUTATIONAL FOUNDATIONS FOR MACHINE LEARNING

START HERE: Instructions

- **Collaboration Policy**: Please read the collaboration policy in the syllabus.
- **Late Submission Policy:** See the late submission policy in the syllabus.
- **Submitting your work:** You will use Gradescope to submit answers to all questions.
  - **Written:** For written problems such as short answer, multiple choice, derivations, proofs, or plots, please use the provided template. Submissions can be handwritten onto the template, but should be labeled and clearly legible. If your writing is not legible, you will not be awarded marks. Alternatively, submissions can be written in LaTeX. Each derivation/proof should be completed in the boxes provided. To receive full credit, you are responsible for ensuring that your submission contains exactly the same number of pages and the same alignment as our PDF template.
  - **Latex Template:** https://www.overleaf.com/read/tqnqbmjpppwd

Question	Points
Proof by Construction	5
Proof by Cases	12
Proof by Contradiction	5
Proof by Induction	10
Proof by Contraposition	5
Total:	37

*Compiled on Sunday 20th October, 2024 at 14:12

Instructions for Specific Problem Types

For “Select One” questions, please fill in the appropriate bubble completely:

Select One: Who taught this course?

- ● Matt Gormley
- ◯ Marie Curie
- ◯ Noam Chomsky

If you need to change your answer, you may cross out the previous answer and bubble in the new answer:

Select One: Who taught this course?

- ● Henry Chai
- ◯ Marie Curie
- ~~●~~ Noam Chomsky

For “Select all that apply” questions, please fill in all appropriate squares completely:

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- □ I don’t know

Again, if you need to change your answer, you may cross out the previous answer(s) and bubble in the new answer(s):

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- □ I don’t know

For questions where you must fill in a blank, please make sure your final answer is fully included in the given space. You may cross out answers or parts of answers, but the final answer must still be within the given space.

Fill in the blank: What is the course number?

10-606	10-60~~6~~7

Definitions

Please use the following definitions and inference rules in your proofs for the problems below.

Definition 1: The system $\mathbb{Z}$ of integers can be characterized by the following definitions and axioms. The set $\mathbb{Z}$ has two binary operations, addition ($a + b$) and multiplication (ab), and two distinguished integers, 0 and 1, with $0 \neq 1$. $\mathbb{Z}$ is closed under addition and multiplication; that is, whenever $a, b \in \mathbb{Z}$ then the numbers $a + b$ and ab are also in $\mathbb{Z}$.

Assume $a, b, c \in \mathbb{Z}$. Then the following axioms hold:

1. Commutativity: $a + b = b + a,\ ab = ba.$
2. Associativity: $a + (b + c) = (a + b) + c,\ a(bc) = (ab)c.$
3. Distributivity: $a(b + c) = ab + ac.$
4. Existence of Zero: $0 + a = a.$
5. Existence of One: $1a = a.$
6. Negation: For any $a \in \mathbb{Z}$, there exists a unique $b \in \mathbb{Z}$ such that $a + b = 0.$

Each of these axioms means that we can use the inference rule of substitution of equal quantities with the stated equation, where a, b, c are any terms of type $\mathbb{Z}$. Note that you won't need all these facts but we include them for completeness.

Definition 2: Let $a, b \in \mathbb{Z}$. We say $a \mid b$ (a divides b) if there exists $m \in \mathbb{Z}$ such that $am = b$.

1 Proof by Construction (5 points)

1. (5 points) For any two finite sets A and B, prove that $\neg(A \cup B) = \neg A \cap \neg B$.

Solution

$\neg(A \cup B) = x \in \neg(A \cup B) \leftrightarrow x \notin A \cup B.$

so $x \notin A$ and $x \notin B$

$\leftrightarrow x \notin A$ and $x \notin B$

$\neg A \cap \neg B =$ so $\neg A$ and $\neg B$

$x \in \neg A \cap \neg B \leftrightarrow x \in \neg A$ and $x \in \neg B$

Hence, Proved. $\neg(A \cup B) = \neg A \cap \neg B.$

2 Proof by Cases (12 points)

Using proof by cases, show that $5x^2 + x + 1$ is not divisible by 3 for any $x \in \mathbb{Z}$.

1. (4 points) Case 1: Assume $x = 3k,\ k \in \mathbb{Z}$. Show that $\neg(3 \mid 5x^2 + x + 1)$.

Solution

$5m^2+m+1 = 5(3k)^2 + 3k+1$ (By substitution)

$= 45k^2 + 3k + 1$

$= 3(15k^2 + k) + 1$

$3(15k^2+k) \rightarrow$ divisible by 3 but added 1 leaves remainder 1; hence $5m^2+m+1$ is not divisible by 3. Hence, proved.

2. (4 points) Case 2: Assume $x = 3k + 1,\ k \in \mathbb{Z}$. Show that $\neg(3 \mid 5x^2 + x + 1)$.

Solution

$5m^2+m+1 = 5(3k+1)^2 + (3k+1) + 1$ (By substitution)

$= 5(9k^2 + 1 + 6k) + (3k+1) + 1$

$= (45k^2 + 30k + 5) + (3k+1) + 1$

$= 45k^2 + 30k + 5 + 3k + 1 + 1$

$= 45k^2 + 33k + 6 + 1$

$= 3(15k^2 + 11k + 2) + 1$

$3(15k^2 + 11k + 1) \rightarrow$ divisible by 3

$+1 \rightarrow$ leaves a remainder of 1

So not divisible by 3

3. (4 points) Case 3: Assume $x = 3k + 2,\ k \in \mathbb{Z}$. Show that $\neg(3 \mid 5x^2 + x + 1)$.

Solution

$5x^2 + x + 1 = 5(3k+2)^2 + (3k+2) + 1$ (By substitution)

$= 5(9k^2 + 4 + 12k) + 3k + 2 + 1$

$= 45k^2 + 20 + 60k + 3k + 2 + 1$

$= 45k^2 + 63k + 23$

$= 45k^2 + 63k + 21 + 2$

$= 3(15k^2 + 21k + 7) + 2$

$3(15k^2 + 21k + 7) \to$ is divisible by 3

$+2 \to$ is not divisible

So not divisible by 3; Hence, Proved.

3 Proof by Contradiction (5 points)

1. (5 points) Using proof by contradiction, show that there is no least positive real number.

Solution

$v \rightarrow$ least positive real number. (no number smaller than v)
v = positive
dividing v by 2 (a positive number) = another positive number.

$\frac{v}{2} < v$ for example if $v = 1$ & $\frac{v}{2} = 0.5$
$0.5 < 1$

so $\frac{v}{2}$ would be a positive real number smaller than v.

Hence, no least positive real number.

4 Proof by Induction (10 points)

Prove that the following statement holds for any $n \in \mathbb{Z}^+$:

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Use the following induction hypothesis, and fill in the base case and inductive step below:

Induction hypothesis: For some $k \leq n$, $k \in \mathbb{Z}^+$,

$$\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$$

.

1. (3 points) Base Case: Show the statement holds for $k = 0$.

Solution

L.H.S $= \sum_{i=1}^{0} i^3 = 0$

R.H.S $= \left(\frac{0(0+1)}{2}\right)^2$

$= \left(\frac{0}{2}\right)^2 = 0$ $\therefore$ L.H.S = R.H.S

Hence, Proved for $k=0$

2. (7 points) Induction Step: Assume the hypothesis is true for some k; use this assumption to show the hypothesis holds for $k + 1$.

Solution

$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$

$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$ (substituting $\sum_{i=1}^{k} i^3$)

$= \frac{k^2(k+1)^2}{4} + (k+1)^3$

$= \frac{k^2(k+1)^2}{4} + (k+1)^2 \cdot (k+1)$

$= (k+1)^2 \cdot \frac{k^2}{4} + (k+1)^2 \cdot (k+1)$

$= (k+1)^2 \left(\frac{k^2}{4} + (k+1)\right)$

$= (k+1)^2 \left(\frac{k^2+4k+4}{4}\right)$

$= (k+1)^2 \left(\frac{(k+2)^2}{4}\right)$

$= \left(\frac{(k+1)^2 (k+2)^2}{2}\right)^2$ $\therefore$ Hence, Proved

5 Proof by Contraposition (5 points)

1. (5 points) Recall the rule of contraposition: For propositions p and q, if $\neg q \implies \neg p$, $p \implies q$. Use this rule to prove that for $a, b \geq 2,\ a, b \in \mathbb{Z}$, either $\neg(a \mid b)$ or $\neg(a \mid b+1)$.

Solution

Assume $(a|b)$ and $a|(b+1)$ (Negation)
$a|b$ so $b = am$ (int m exists)
$a|(b+1)$ so $b+1 = an$ (int n exists)

$b+1 = an$
$am+1 = an$ (by substitution)
$an - am = 1$
$a(n-m) = 1$
for equation to be true a has to divide 1
but $a \geq 2$.
So if $(a|b)$ and $a|(b+1)$ leads to contradiction

Therefore by contraposition, it is true that $a, b \geq 2$, so either $\neg(a|b)$ or $\neg(a|b+1)$

6 Collaboration Questions

After you have completed all other components of this assignment, report your answers to these questions regarding the collaboration policy. Details of the policy can be found in the syllabus.

1. Did you receive any help whatsoever from anyone in solving this assignment? If so, include full details.
2. Did you give any help whatsoever to anyone in solving this assignment? If so, include full details.
3. Did you find or come across code that implements any part of this assignment? If so, include full details.

Your Answer

1. I have not recived any help whatsoever.
2. I did not give any help whatsoever to anyone.
3. I did not find or across code that implements this assignment.