

Random Matrices

Pavel Kurilovich



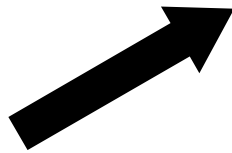
1. What?

Usual random variable

Usual random variable

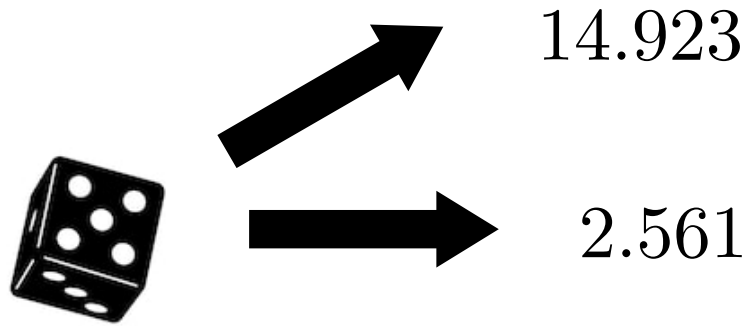


Usual random variable

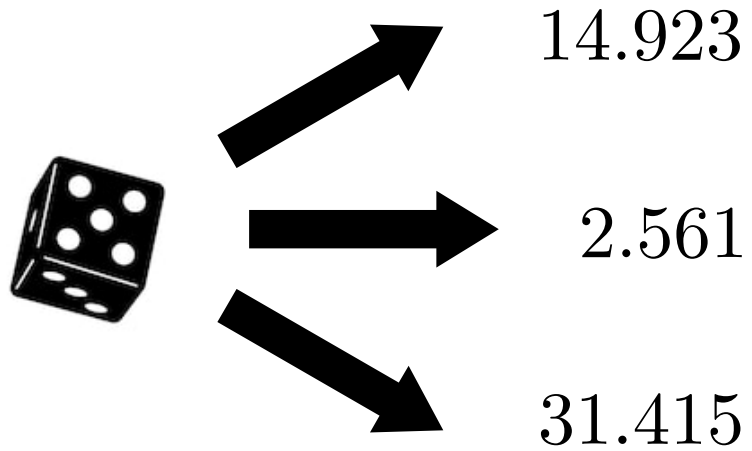


14.923

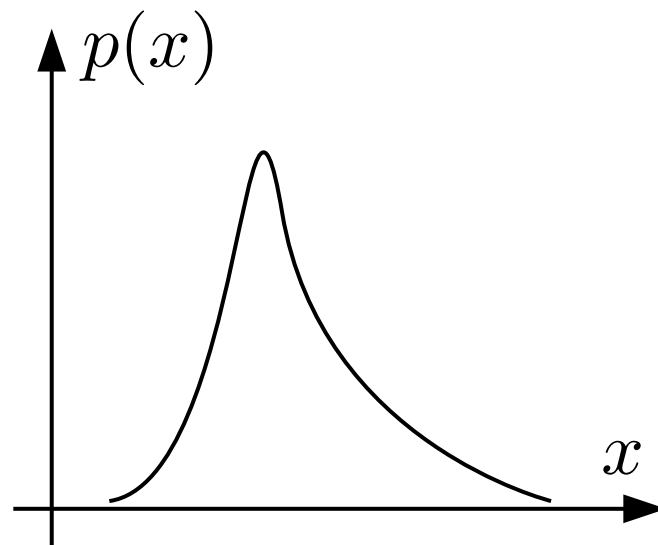
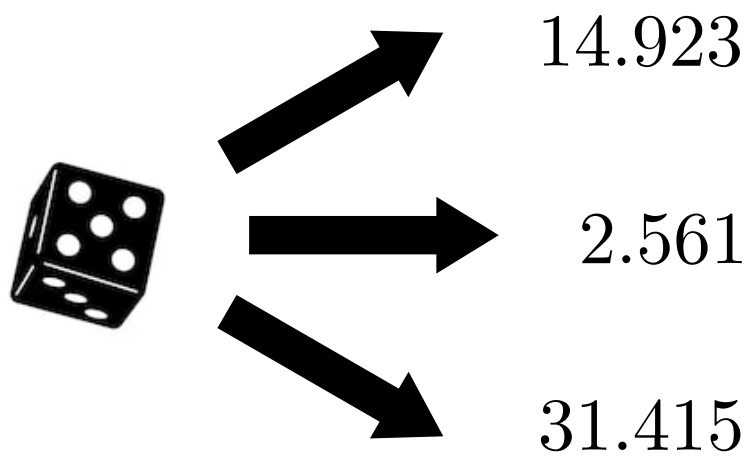
Usual random variable



Usual random variable



Usual random variable



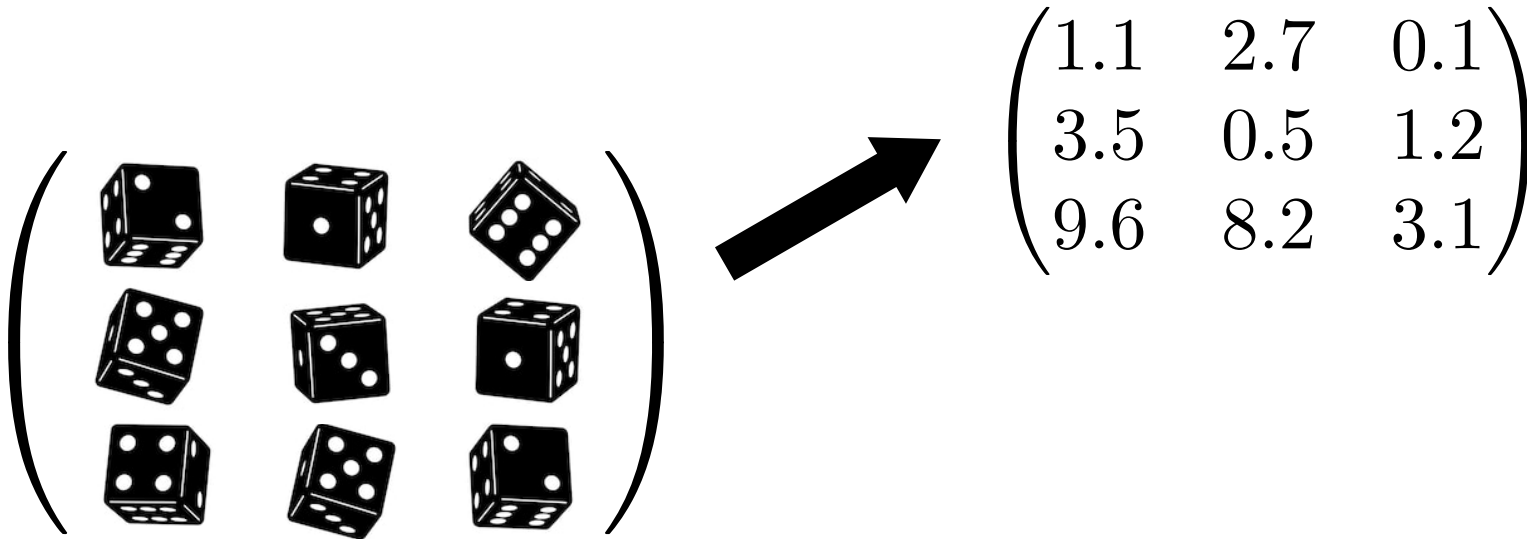
Random matrix

Random matrix
= **matrix** of **random variables**

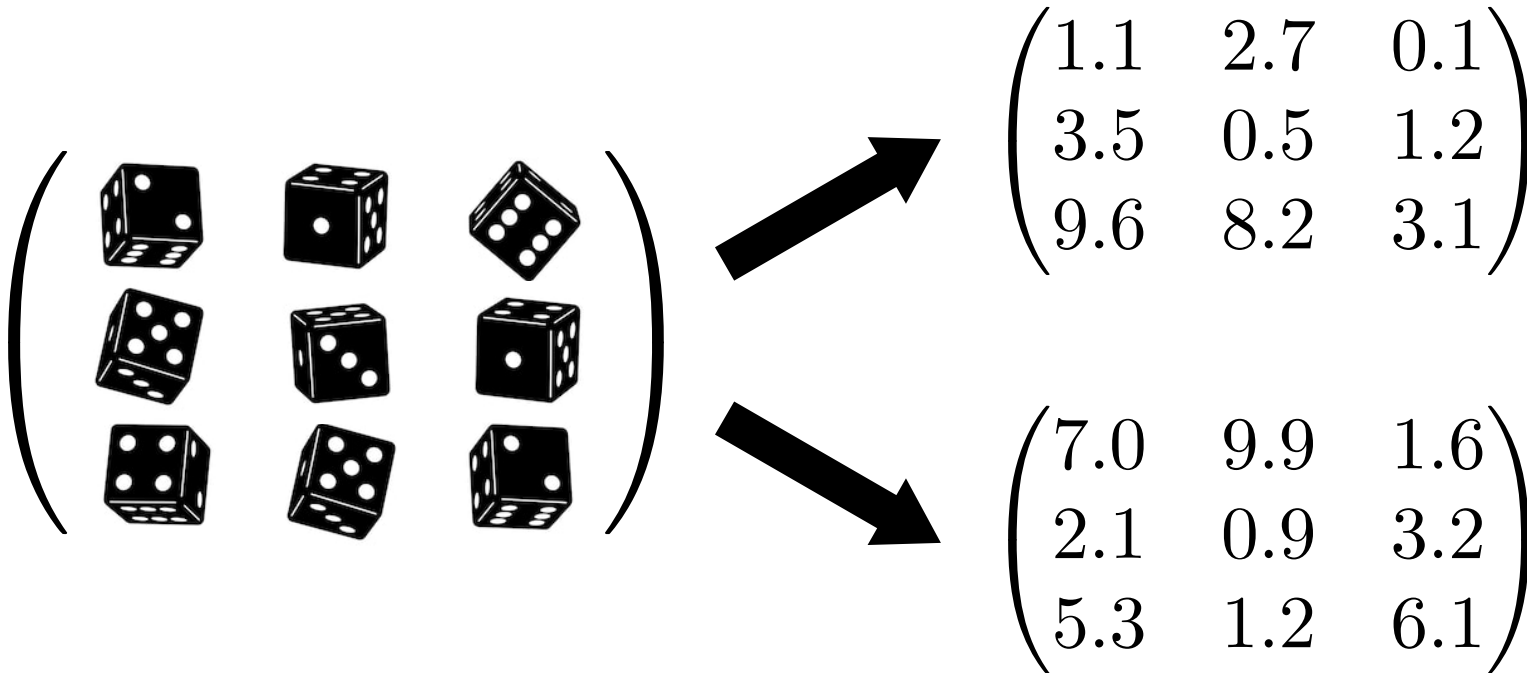
Random matrix
= **matrix** of **random variables**



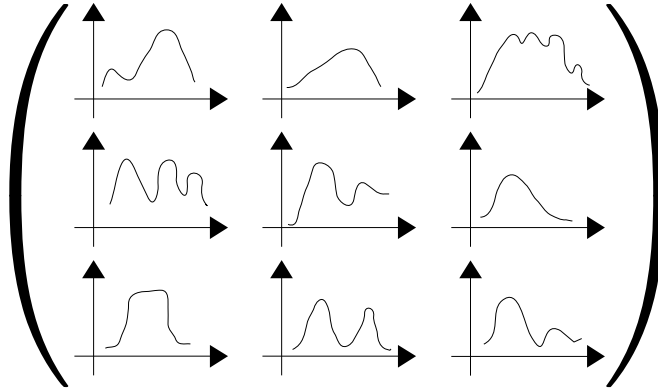
Random matrix
= **matrix** of **random variables**



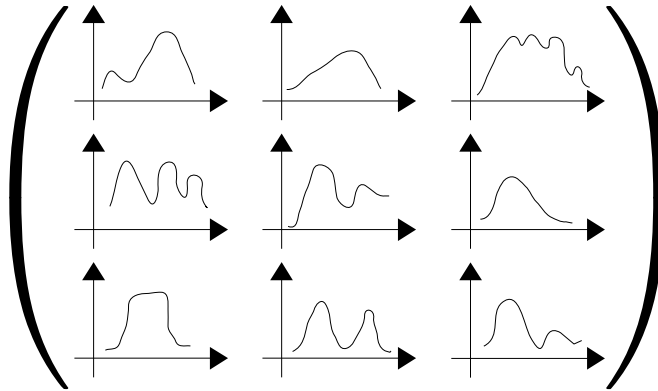
Random matrix
= **matrix** of **random variables**



Random matrix
= **matrix** of **random variables**



Random matrix
= **matrix** of **random variables**



What can we study?

Random matrix: $H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$

Averages?

Random matrix: $H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$

Averages?

$$\langle H \rangle = \begin{pmatrix} \langle H_{11} \rangle & \langle H_{12} \rangle & \langle H_{13} \rangle \\ \langle H_{21} \rangle & \langle H_{22} \rangle & \langle H_{23} \rangle \\ \langle H_{31} \rangle & \langle H_{32} \rangle & \langle H_{33} \rangle \end{pmatrix}$$

Random matrix: $H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$

Averages?

$$\langle H \rangle = \begin{pmatrix} \langle H_{11} \rangle & \langle H_{12} \rangle & \langle H_{13} \rangle \\ \langle H_{21} \rangle & \langle H_{22} \rangle & \langle H_{23} \rangle \\ \langle H_{31} \rangle & \langle H_{32} \rangle & \langle H_{33} \rangle \end{pmatrix}$$

NOT INTERESTING

Random matrix: $H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$

Statistics of spectrum (eigenvalues)?

$$\det (H - E_i) = 0$$

Random matrix: $H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$

Statistics of spectrum (eigenvalues)?

$$\det (H - E_i) = 0$$

HIGHLY NON-TRIVIAL

Not all $N \times N$ matrices possess N eigenvalues!

Not all $N \times N$ matrices possess N eigenvalues!

Symmetry constrains for random matrices:

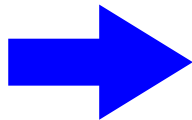
Not all $N \times N$ matrices possess N eigenvalues!

Symmetry constrains for random matrices:

Complex, $H^\dagger = H$

Real, $H^T = H$

and so on...

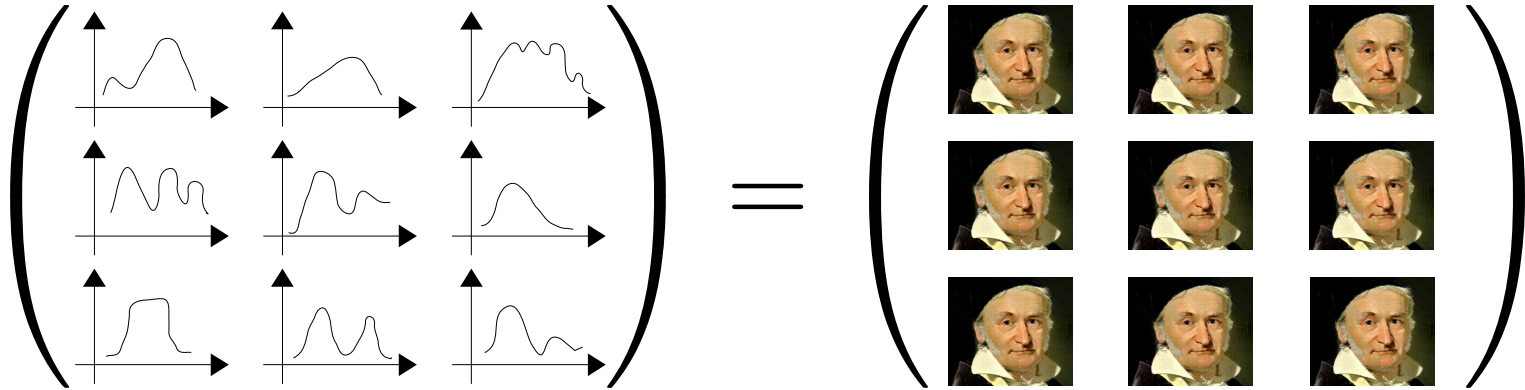


Nice **real** spectrum!

N eigenvalues

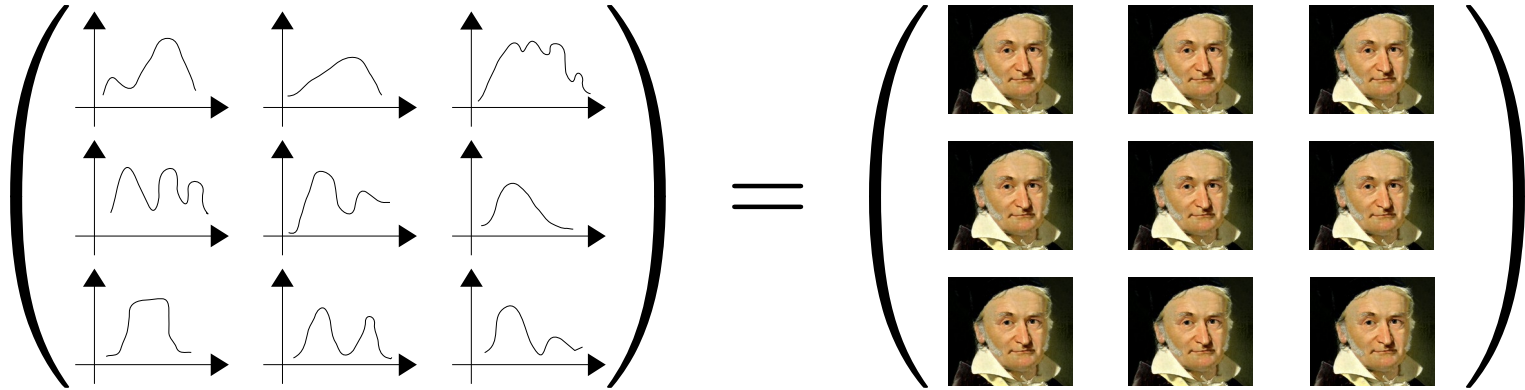
Important class:

Gaussian random matrices



Important class:

Gaussian random matrices



Focus of the talk:

Complex Hermitian Gaussian random matrices

Particular realization: **G**aussian **U**nitary **E**nsemble (**GUE**)

$$H \in N \times N, \quad H = H^\dagger$$

$$p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$

Particular realization: **G**aussian **U**nitary **E**nsemble (**GUE**)

$$H \in N \times N, \quad H = H^\dagger$$

$$p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$

$$p(H) \propto \exp \left(-\frac{N}{2} \left[\sum_i H_{ii}^2 + 2 \sum_{i < j} |H_{ij}|^2 \right] \right)$$

Particular realization: **G**aussian **U**nitary **E**nsemble (**GUE**)

$$H \in N \times N, \quad H = H^\dagger$$

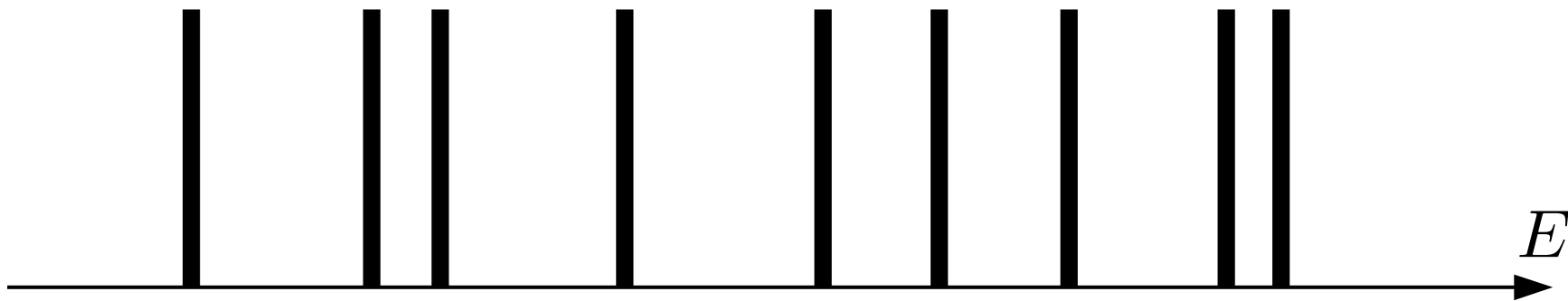
$$p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$



Particular realization: **G**aussian **U**nitary **E**nsemble (**GUE**)

$$H \in N \times N, \quad H = H^\dagger$$

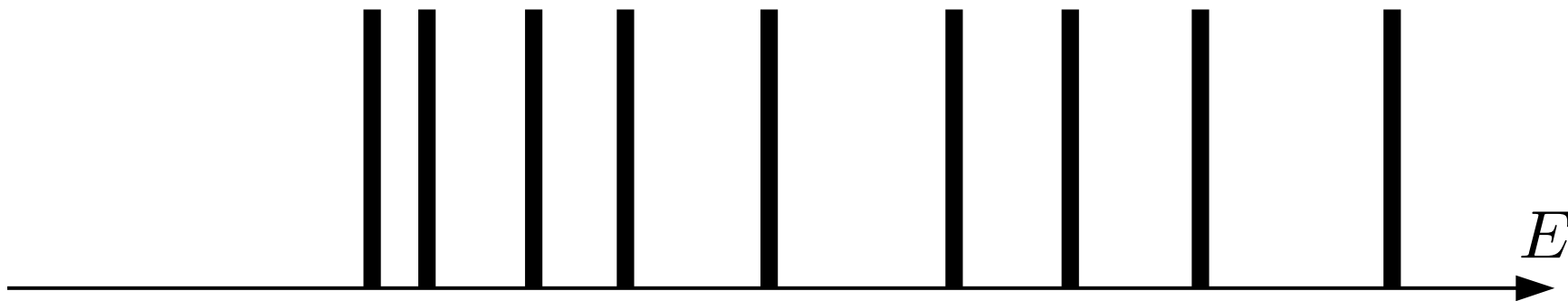
$$p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$



Particular realization: **G**aussian **U**nitary **E**nsemble (**GUE**)

$$H \in N \times N, \quad H = H^\dagger$$

$$p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$



Particular realization: **G**aussian **U**nitary **E**nsemble (**GUE**)

$$H \in N \times N, \quad H = H^\dagger$$

$$p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$



Joint probability density fucntion?

The diagram shows a horizontal axis labeled E with ten vertical tick marks representing energy levels. A blue rectangular box with a thick border is superimposed over the middle of these tick marks, containing the text 'Joint probability density fucntion?'.

2. Why?

Different physical models!

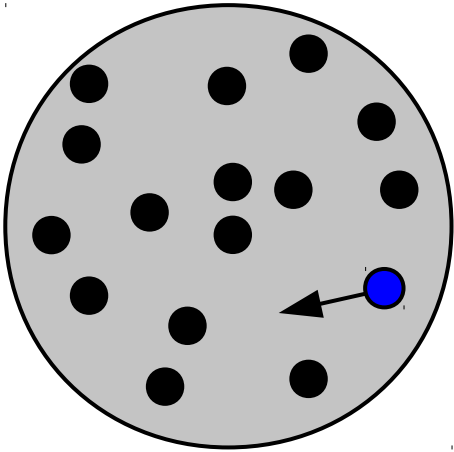
Originally:



Energy levels of heavy nuclei

Different physical models!

Moreover:



Disordered quantum dots (tiny metal grains...)

Also QCD, quantum optics, neuroscience...

3. How?

Gaussian Unitary Ensemble

$$H \in N \times N, \quad H = H^\dagger, \quad p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$

Joint probability density function of eigenvalues?

Is this a Gaussian distribution?



Gaussian Unitary Ensemble

$$H \in N \times N, \quad H = H^\dagger, \quad p(H) \propto \exp\left(-\frac{N}{2}\text{tr}H^2\right)$$

Attempt 1:

Gaussian Unitary Ensemble

$$H \in N \times N, \quad H = H^\dagger, \quad p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$

Attempt 1:

$$H = U^{-1} \hat{E} U, \quad \hat{E} = \text{diag} (E_1, \dots, E_N)$$

$$p(E_i) \propto \exp \left(-\frac{N}{2} \sum_i E_i^2 \right)$$

Gaussian Unitary Ensemble

$$H \in N \times N, \quad H = H^\dagger, \quad p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$

Attempt 1:

$$H = U^{-1} \hat{E} U, \quad \hat{E} = \text{diag} (E_1, \dots, E_N)$$

$$p(E_i) \propto \exp \left(-\frac{N}{2} \sum_i E_i^2 \right)$$

WRONG

Gaussian Unitary Ensemble

$$H \in N \times N, \quad H = H^\dagger, \quad p(H) \propto \exp \left(-\frac{N}{2} \text{tr} H^2 \right)$$

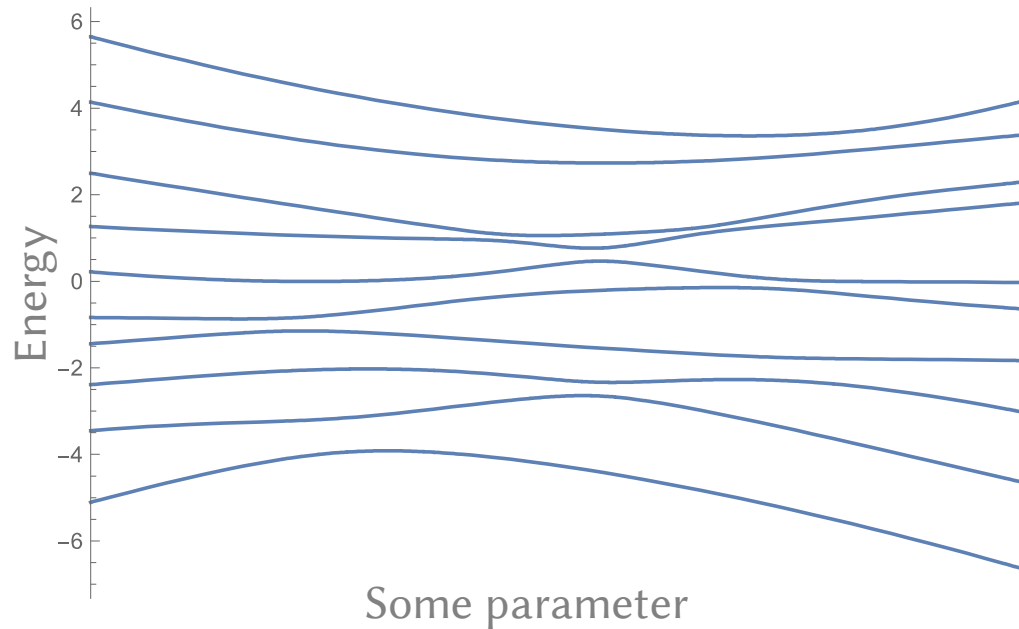
Taking Jacobian into the account:

$$p(E_i) \propto \left(\prod_{i < j} |E_i - E_j|^2 \right) \exp \left(-\frac{N}{2} \sum_i E_i^2 \right)$$

Repulsion of eigenvalues!

$$p(E_i) \propto \left(\prod_{i < j} |E_i - E_j|^2 \right) \exp \left(-\frac{N}{2} \sum_i E_i^2 \right)$$

Repulsion of eigenvalues – usual for QM

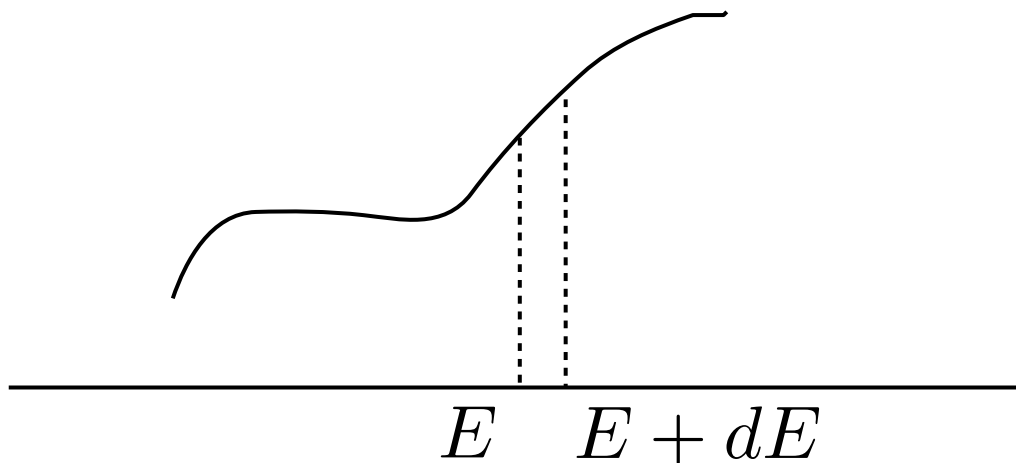


Something simple to look at...

Something simple to look at...

Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$



Something simple to look at...

Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

$$p(E_i) \propto \left(\prod_{i < j} |E_i - E_j|^2 \right) \exp \left(-\frac{N}{2} \sum_i E_i^2 \right)$$

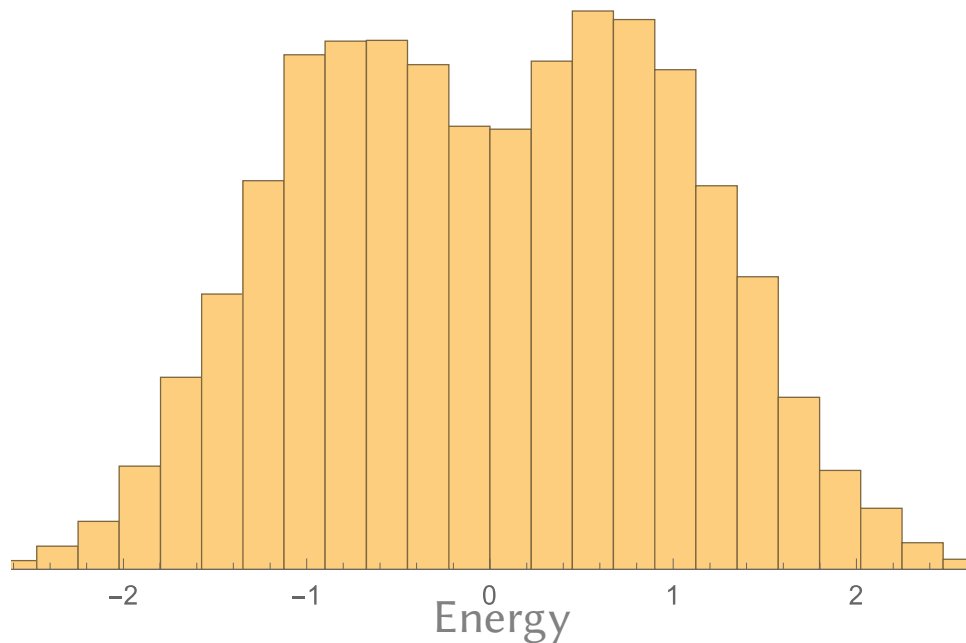
PAIN to work with

Possible to circumvent for $N \gg 1$

Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

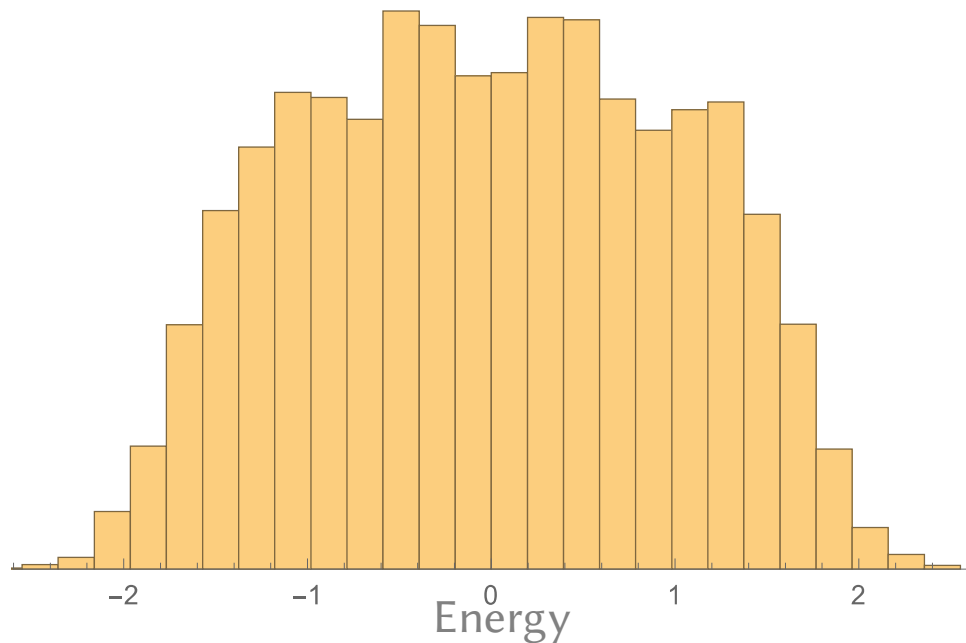
$$N = 2$$



Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

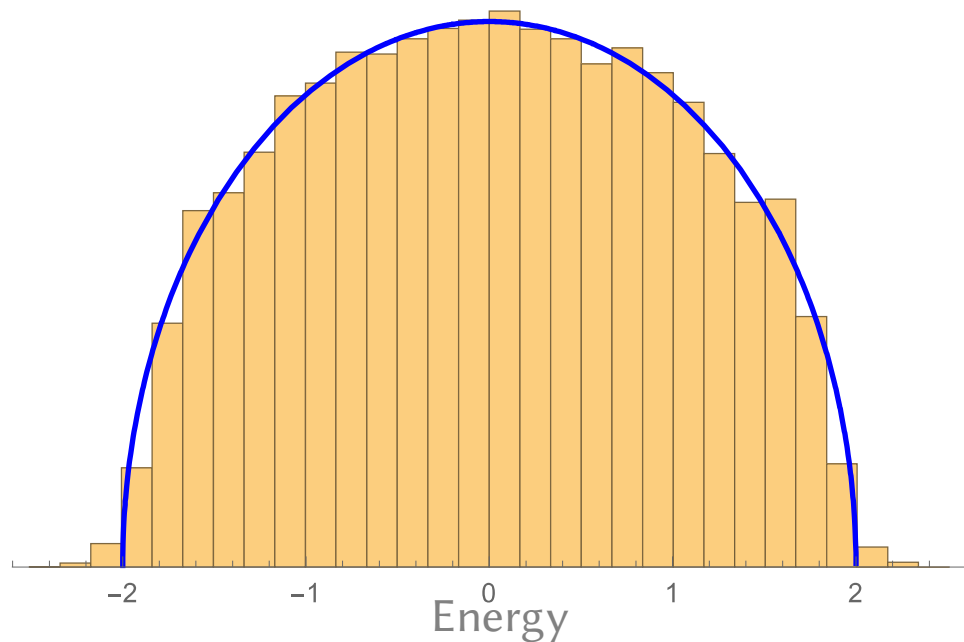
$$N = 4$$



Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

$$N = 10$$

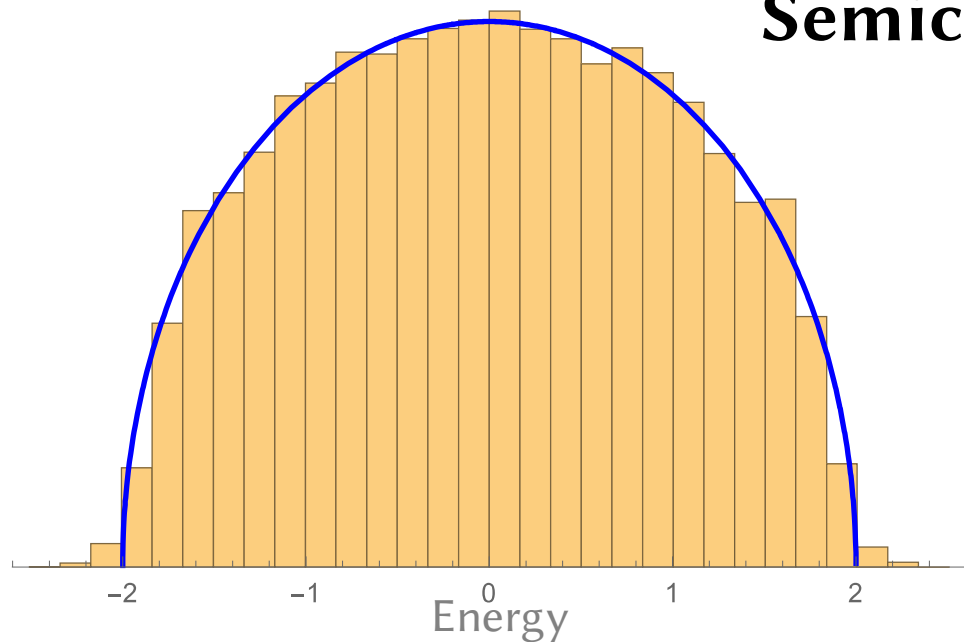


Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

$N = 10$

Semicircle (Wigner)!

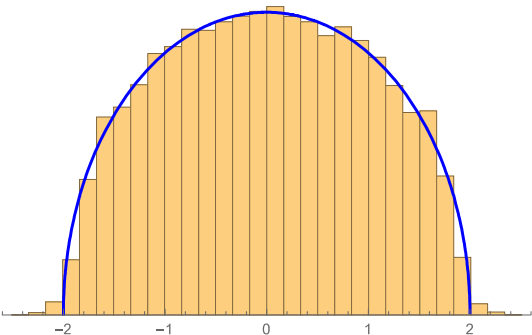


Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

For $N \gg 1$

$$\nu(E) \approx \frac{N}{2\pi} \sqrt{4 - E^2}$$

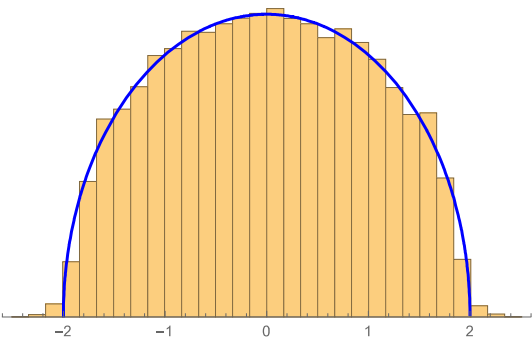


Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

For $N \gg 1$

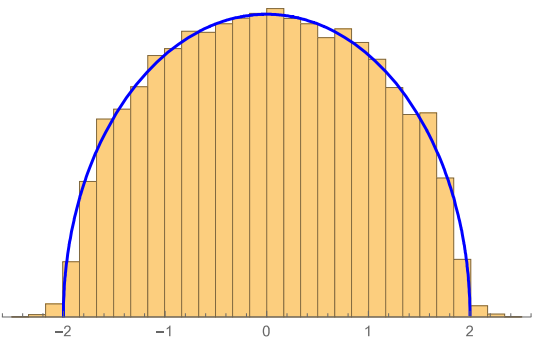
$$\nu(E) \approx \frac{N}{2\pi} \sqrt{4 - E^2}$$



How to derive this result?
Feynman diagrams

Spectral density: $\nu(E) = dn(E)/dE$

$$\nu(E) = \langle \sum_n \delta(E - E_n) \rangle = -\frac{1}{\pi} \langle \text{Im} (\text{tr} [E - H + i0]^{-1}) \rangle$$

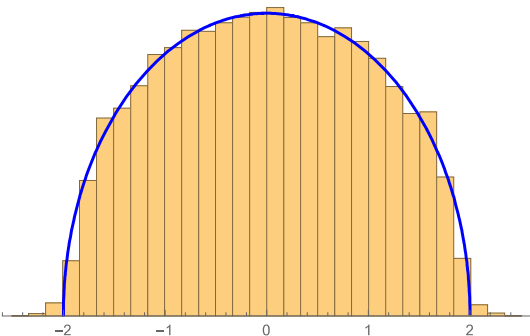


Spectral density: $\nu(E) = dn(E)/dE$

$$\nu(E) = \langle \sum_n \delta(E - E_n) \rangle = -\frac{1}{\pi} \langle \text{Im} (\text{tr} [E - H + i0]^{-1}) \rangle$$



Green's function (resolvent)



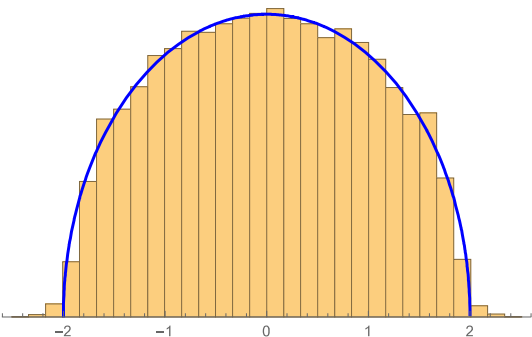
Spectral density: $\nu(E) = dn(E)/dE$

$$\nu(E) = \langle \sum_n \delta(E - E_n) \rangle = -\frac{1}{\pi} \langle \text{Im} (\text{tr} [E - H + i0]^{-1}) \rangle$$



Green's function (resolvent)

$$\frac{1}{E - H} = \frac{1}{E} + \frac{1}{E^2} H + \frac{1}{E^3} H^2 + \frac{1}{E^4} H^3 + \dots$$



Spectral density: $\nu(E) = dn(E)/dE$

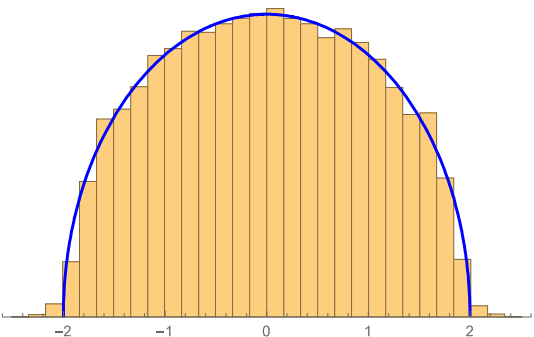
$$\nu(E) = \langle \sum_n \delta(E - E_n) \rangle = -\frac{1}{\pi} \langle \text{Im} (\text{tr} [E - H + i0]^{-1}) \rangle$$



Green's function (resolvent)

$$\frac{1}{E - H} = \frac{1}{E} + \frac{1}{E^2} H + \frac{1}{E^3} H^2 + \frac{1}{E^4} H^3 + \dots$$

$$\langle H_{mn} H_{kp}^* \rangle \propto \delta_{mk} \delta_{np}$$



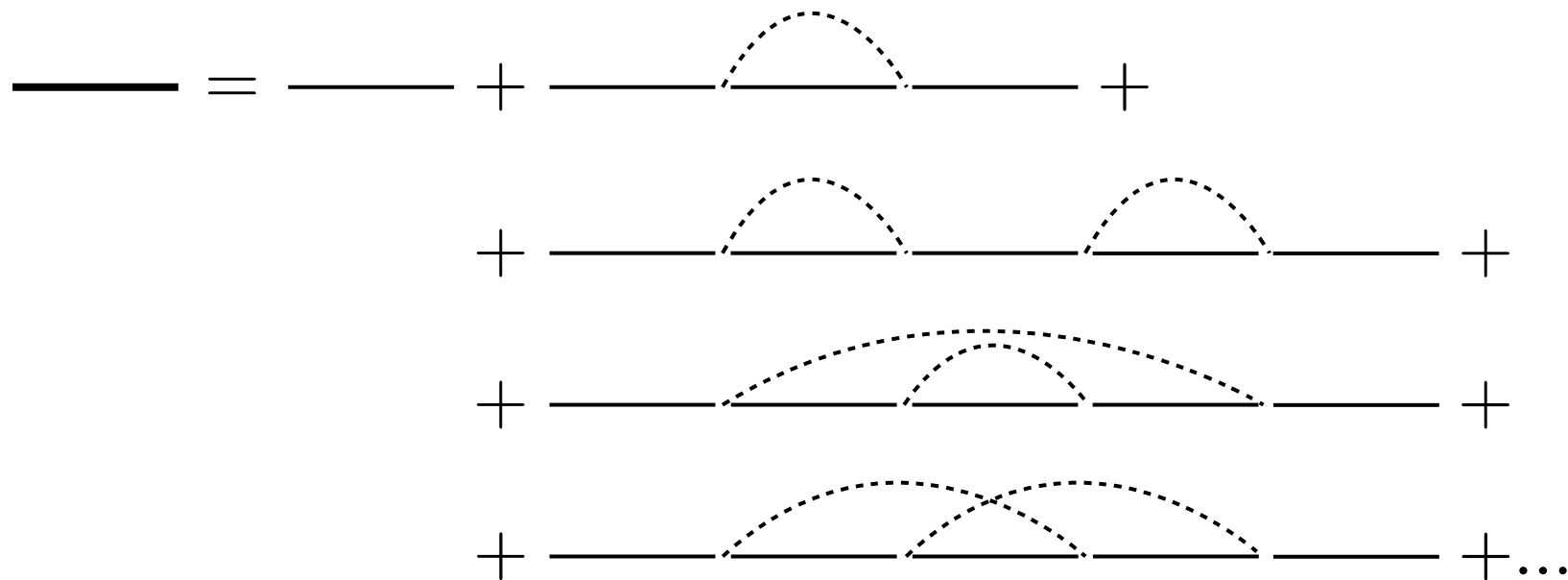
$$\frac{1}{E-H} = \frac{1}{E} + \frac{1}{E^2}H + \frac{1}{E^3}H^2 + \frac{1}{E^4}H^3 + \dots$$

$$\langle H_{mn}H_{kp}^{\star} \rangle \propto \delta_{mk}\delta_{np}$$

$$\text{thick line} = \text{thin line} + \text{thin line} \overset{\text{dashed arc}}{\text{---}} \text{thin line} + \dots$$

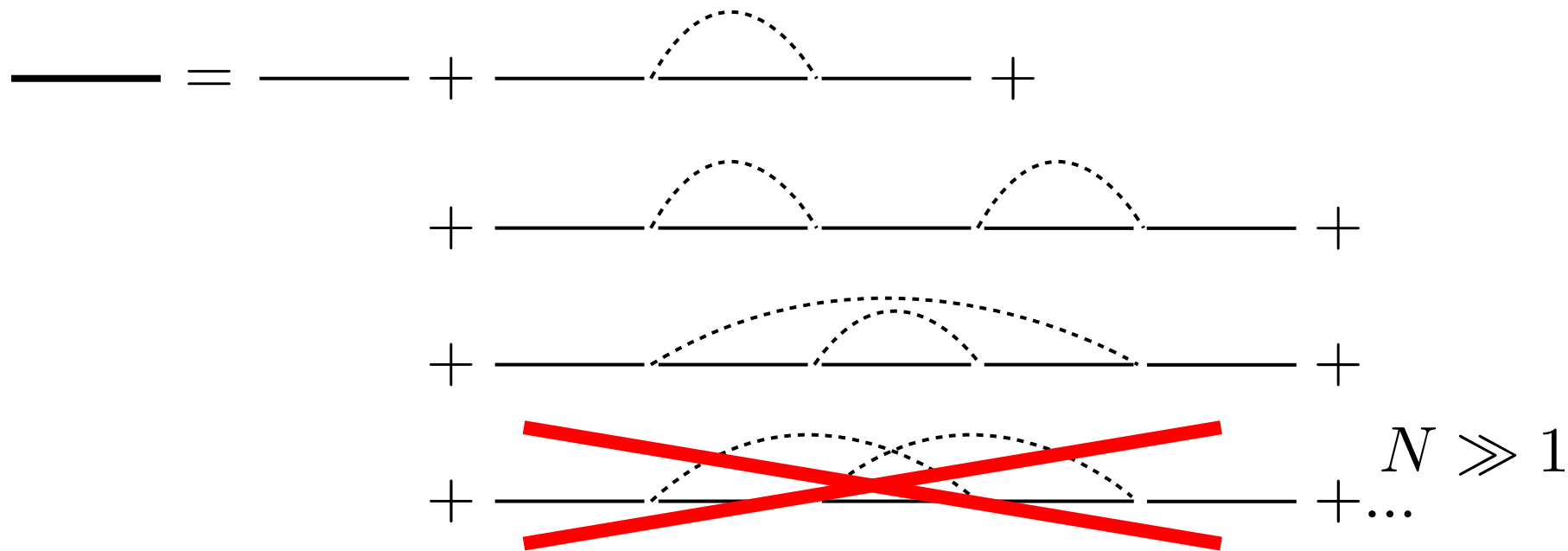
$$\frac{1}{E-H} = \frac{1}{E} + \frac{1}{E^2}H + \frac{1}{E^3}H^2 + \frac{1}{E^4}H^3 + \dots$$

$$\langle H_{mn} H_{kp}^{\star} \rangle \propto \delta_{mk} \delta_{np}$$



$$\frac{1}{E-H} = \frac{1}{E} + \frac{1}{E^2}H + \frac{1}{E^3}H^2 + \frac{1}{E^4}H^3 + \dots$$

$$\langle H_{mn} H_{kp}^* \rangle \propto \delta_{mk} \delta_{np}$$



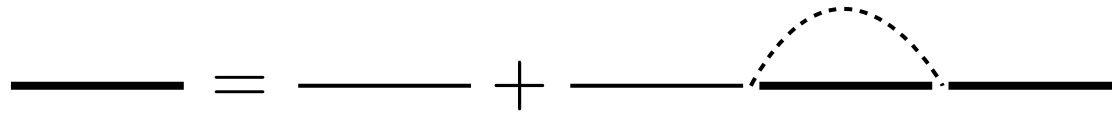
The diagram shows a series of Feynman diagrams representing the expansion of a propagator. The first row shows a thick solid line equal to a thin solid line plus a thin solid line with a single dashed loop, plus a thin solid line with two dashed loops, plus a thin solid line with three dashed loops, and so on. The last row is crossed out with a large red 'X' and followed by an ellipsis and the text $N \gg 1$.

$$\text{Thick solid line} = \text{Thin solid line} + \text{Thin solid line with 1 loop} + \text{Thin solid line with 2 loops} + \text{Thin solid line with 3 loops} + \dots$$

$N \gg 1$

$$\frac{1}{E-H} = \frac{1}{E} + \frac{1}{E^2}H + \frac{1}{E^3}H^2 + \frac{1}{E^4}H^3 + \dots$$

$$\langle H_{mn}H_{kp}^{\star} \rangle \propto \delta_{mk}\delta_{np}$$

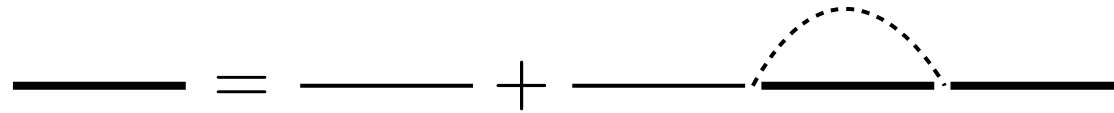


$$\text{thick line} = \text{thin line} + \text{thin line} \text{---} \text{thick line} \text{---} \text{thin line}$$

Dyson equation

$$\frac{1}{E-H} = \frac{1}{E} + \frac{1}{E^2}H + \frac{1}{E^3}H^2 + \frac{1}{E^4}H^3 + \dots$$

$$\langle H_{mn}H_{kp}^{\star} \rangle \propto \delta_{mk}\delta_{np}$$



$$\text{thick line} = \text{thin line} + \text{thin line} \text{---} \text{dashed arc} \text{---} \text{thick line}$$

Dyson equation

$$\left(\frac{1}{E-H}\right)_{mn} \approx \delta_{mn} \frac{E-i\sqrt{4-E^2}}{2}$$

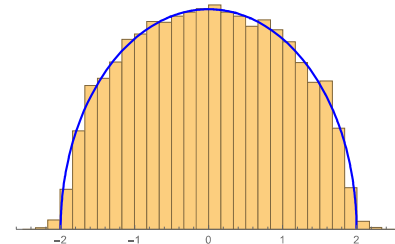
$$\frac{1}{E-H} = \frac{1}{E} + \frac{1}{E^2}H + \frac{1}{E^3}H^2 + \frac{1}{E^4}H^3 + \dots$$

$$\langle H_{mn} H_{kp}^* \rangle \propto \delta_{mk} \delta_{np}$$

$$\text{---} = \text{---} + \text{---} \text{---} \text{---}$$

Dyson equation

$$\left(\frac{1}{E-H} \right)_{mn} \approx \delta_{mn} \frac{E - i\sqrt{4-E^2}}{2}$$



Summary

1. Random matrices = matrices of random variables
2. Eigenvalue problem: level repulsion for GUE
3. Spectral density: Wigner semicircle

Thank you for attention!