Random Matrices

Pavel Kurilovich

1. What?

Usual random variable

Usual random variable

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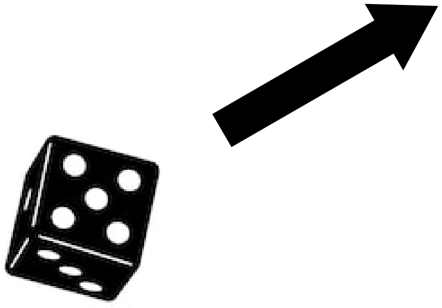
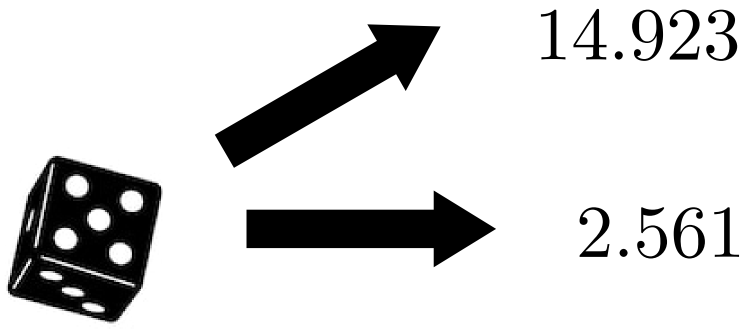


14.923

Usual random variable



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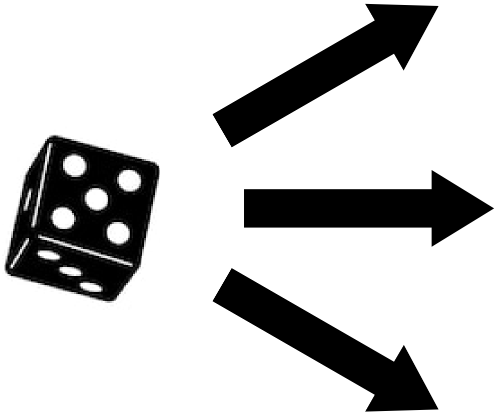
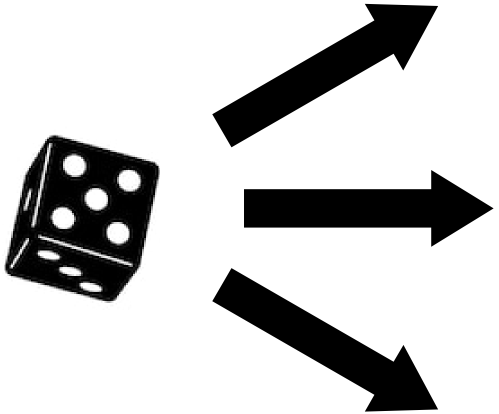


14.923

2.561

31.415

Usual random variable



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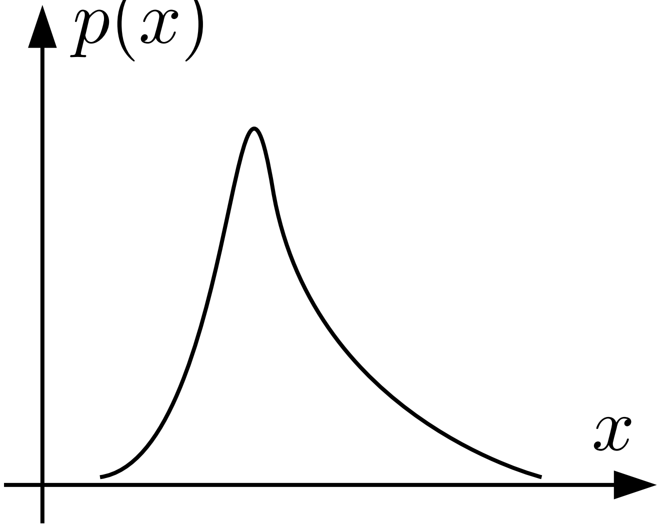

Random matrix

Random matrix
= matrix of random variables

Random matrix

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Random matrix

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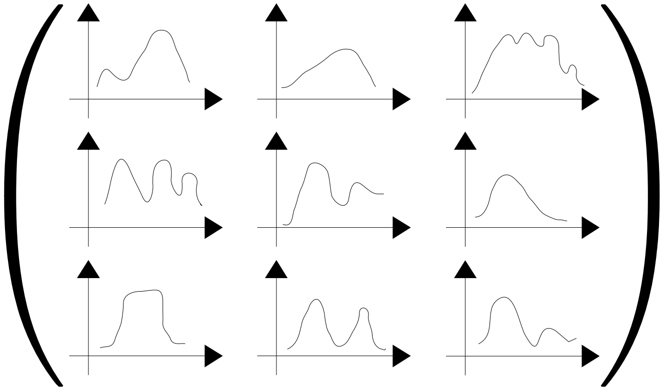
$$\begin{pmatrix} 1.1 & 2.7 & 0.1 \\ 3.5 & 0.5 & 1.2 \\ 9.6 & 8.2 & 3.1 \end{pmatrix}$$

Random matrix
= matrix of random variables

$$\begin{pmatrix} 1.1 & 2.7 & 0.1 \\ 3.5 & 0.5 & 1.2 \\ 9.6 & 8.2 & 3.1 \end{pmatrix}$$

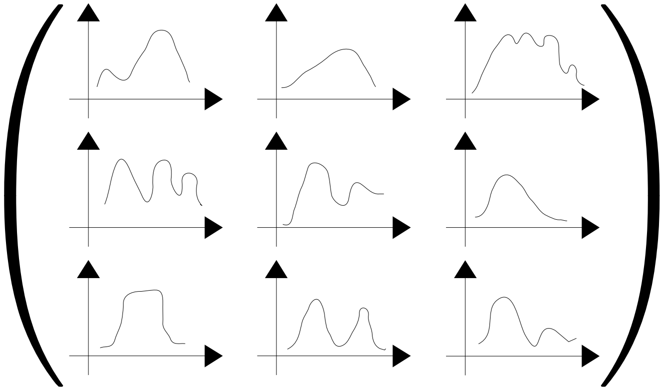
$$\begin{pmatrix} 7.0 & 9.9 & 1.6 \\ 2.1 & 0.9 & 3.2 \\ 5.3 & 1.2 & 6.1 \end{pmatrix}$$

Random matrix
= matrix of random variables



Random matrix

= matrix of random variables



What can we study?

Random matrix: $H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$

Averages?

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NOT INTERESTING

Random matrix: $H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$

Statistics of spectrum (eigenvalues)?

$$\det\left(H - E_i\right) = 0$$

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Statistics of spectrum (eigenvalues)?

$$\det\left(H - E_i\right) = 0$$

HIGHLY NON-TRIVIAL

Not all $N \times N$ matrices possess N eigenvalues!

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Symmetry constrains for random matrices:

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Symmetry constrains for random matrices:

Complex, $H^{\dagger} = H$

Real, $H^{T} = H$

and so on...

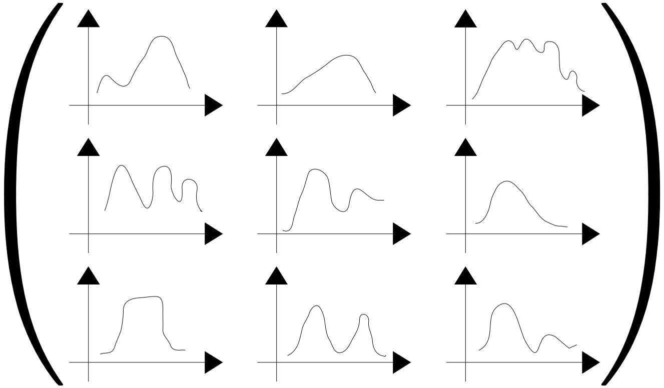
⇒

Nice **real** spectrum!

N eigenvalues

Important class:

Gaussian random matrices



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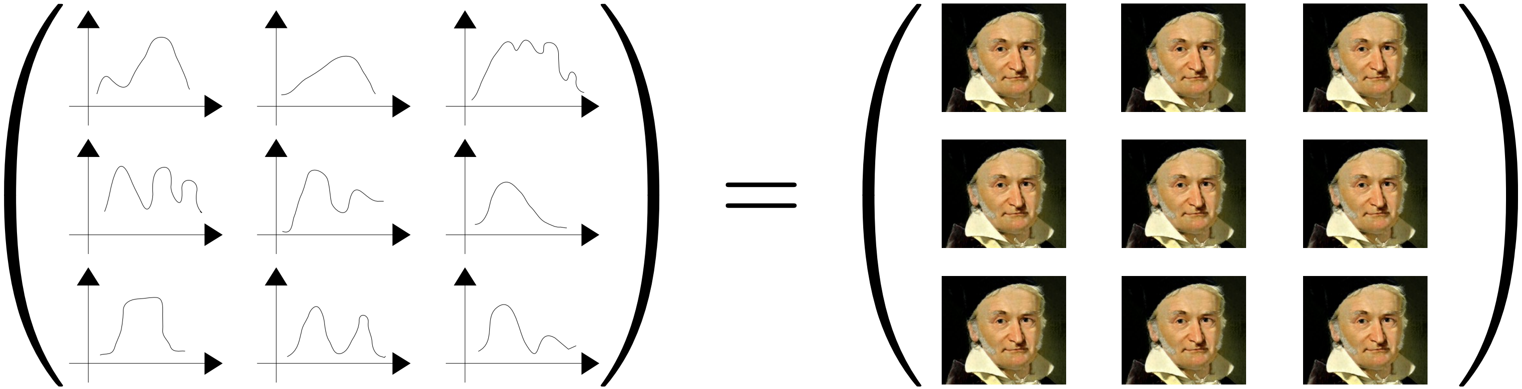
Gaussian random matrices



Focus of the talk:

Complex Hermitian Gaussian random matrices

Particular realization: **G**aussian **U**nitary **E**nsemble (**GUE**)

$$H \in N \times N, \quad H = H^\dagger$$

$$p(H) \propto \exp\left(-\frac{N}{2}\mathrm{tr}H^2\right)$$

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$$p(H) \propto \exp\left(-\frac{N}{2}\left[\sum_i H_{ii}^2 + 2\sum_{i<j}|H_{ij}|^2\right]\right)$$

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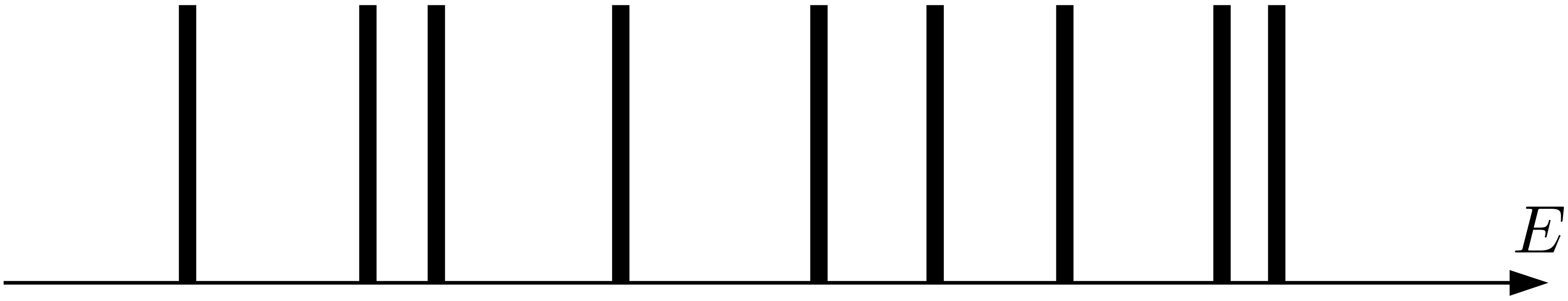

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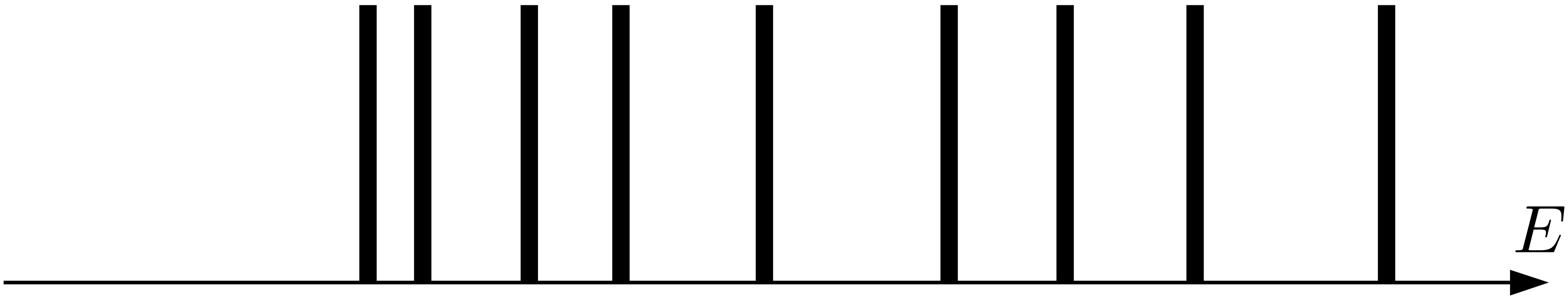

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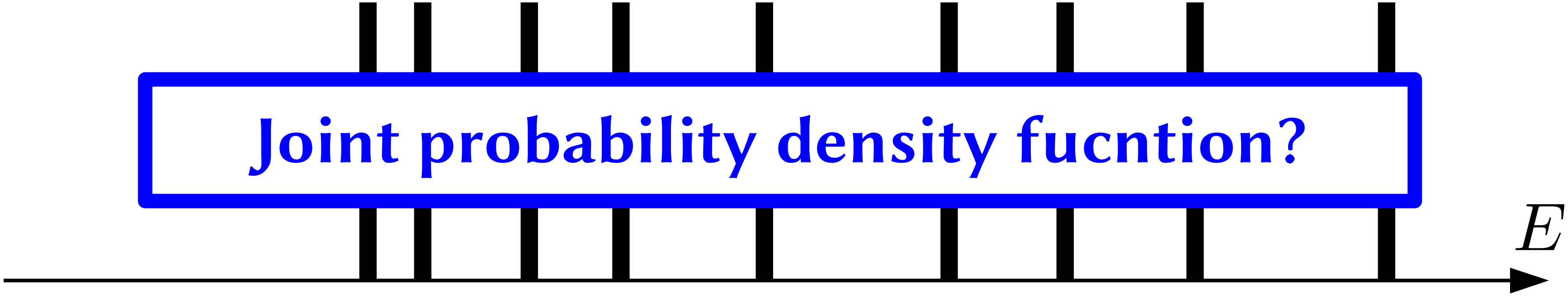


Joint probability density fucntion?

2. Why?

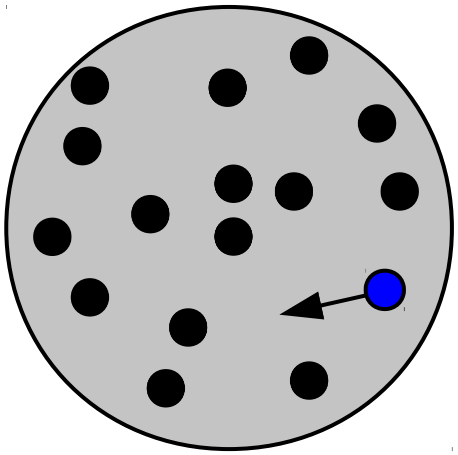
Different physical models!

Originally:

Energy levels of heavy nuclei

Different physical models!

Moreover:



Disordered quantum dots (tiny metal grains...)

Also QCD, quantum optics, neuroscience...

3. How?

Gaussian Unitary Ensemble

$$H \in N \times N, \quad H = H^\dagger, \quad p(H) \propto \exp\left(-\frac{N}{2}\mathrm{tr}H^2\right)$$

Joint probability density fucntion of eigenvalues?

Is this a Gaussian distribution?

Gaussian Unitary Ensemble

$$H \in N \times N, \quad H = H^\dagger, \quad p(H) \propto \exp\left(-\frac{N}{2}\mathrm{tr}H^2\right)$$

Attempt 1:

Gaussian Unitary Ensemble

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Attempt 1:

$$H = U^{-1}\hat{E}U, \quad \hat{E} = \mathrm{diag}\left(E_1, ..., E_N\right)$$

$$p(E_i) \propto \exp\left(-\frac{N}{2}\sum_i E_i^2\right)$$

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WRONG

Gaussian Unitary Ensemble

$$H \in N \times N, \quad H = H^\dagger, \quad p(H) \propto \exp\left(-\frac{N}{2}\mathrm{tr}H^2\right)$$

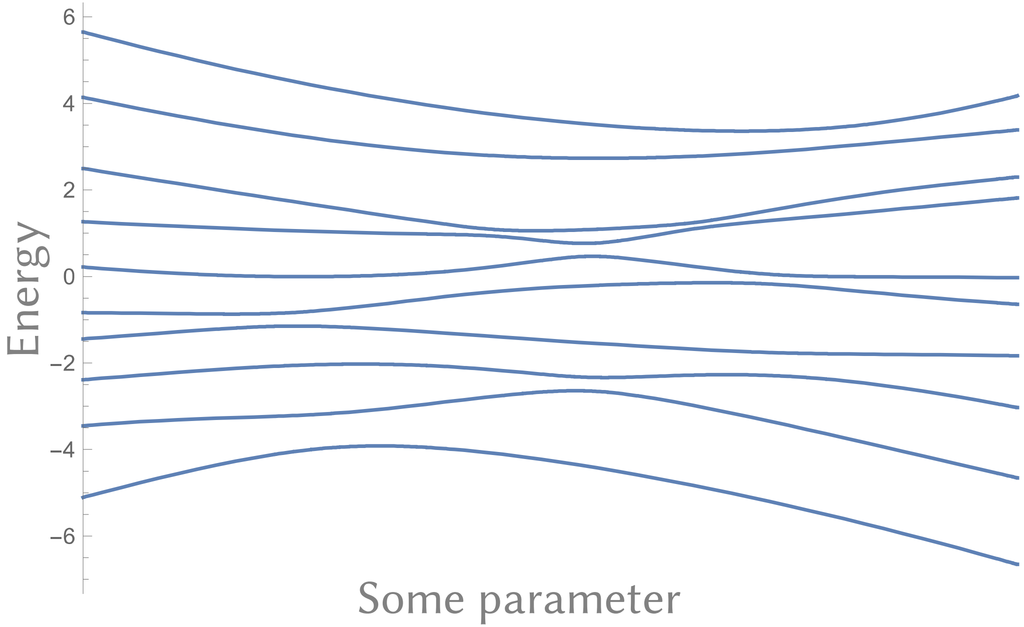
Taking Jacobian into the account:

$$p(E_i) \propto \left(\prod_{i<j} |E_i - E_j|^2\right) \exp\left(-\frac{N}{2}\sum_i E_i^2\right)$$

Repulsion of eigenvalues!

$$p(E_i) \propto \left(\prod_{i<j} |E_i - E_j|^2 \right) \exp\left(-\frac{N}{2} \sum_i E_i^2 \right)$$

Repulsion of eigenvalues – usual for QM



Something simple to look at...

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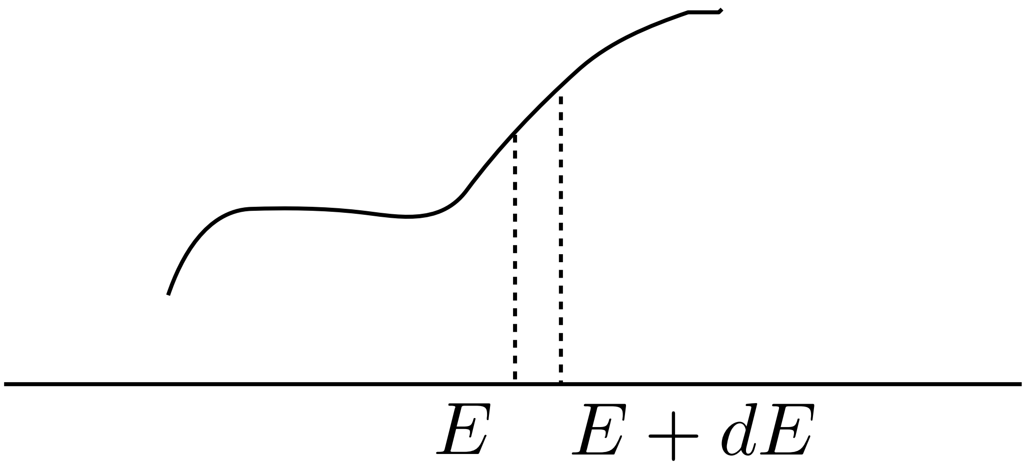
Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E+dE)$



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$$p(E_i) \propto \left(\prod_{i<j} |E_i - E_j|^2 \right) \exp\left(-\frac{N}{2} \sum_i E_i^2 \right)$$

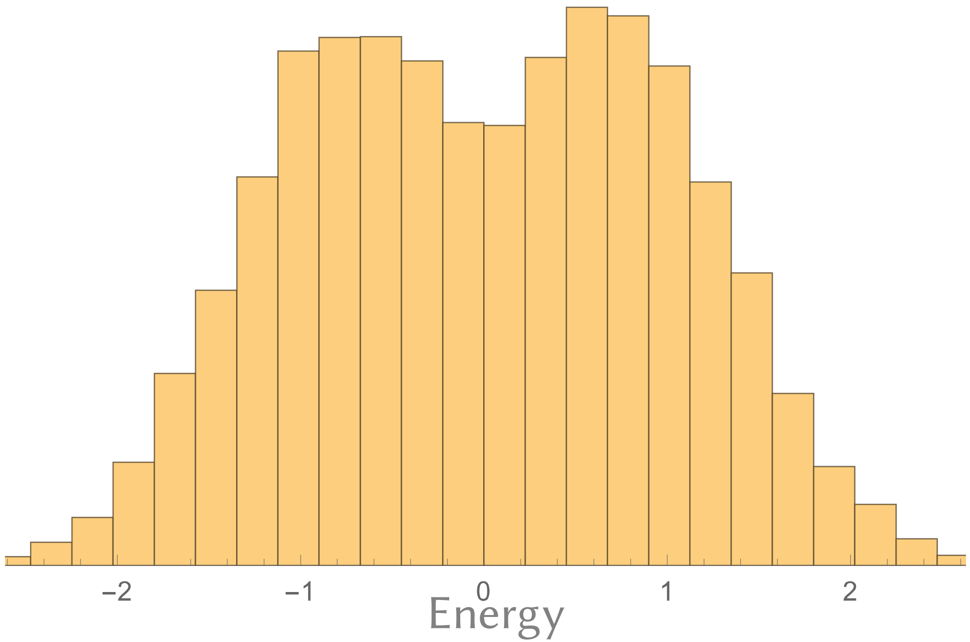
PAIN to work with

Possible to circumvent for $N \gg 1$

Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

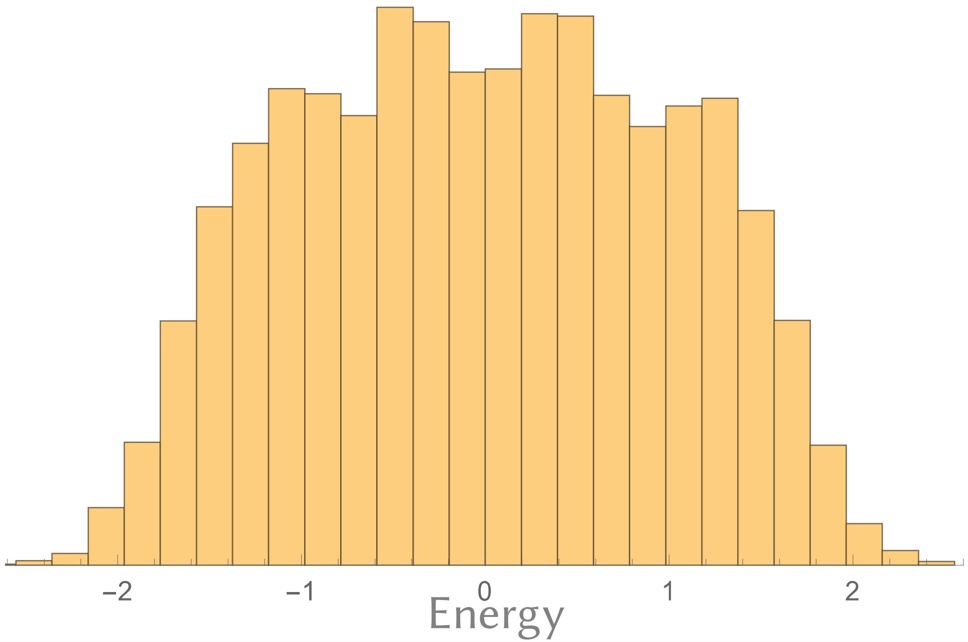
$N = 2$



Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

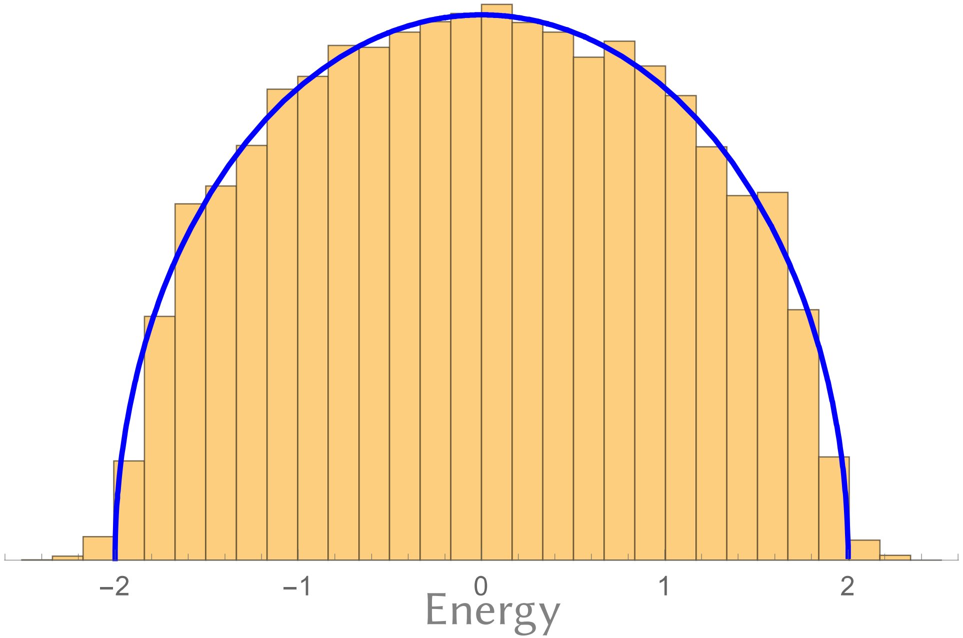
$N = 4$



Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

$N = 10$



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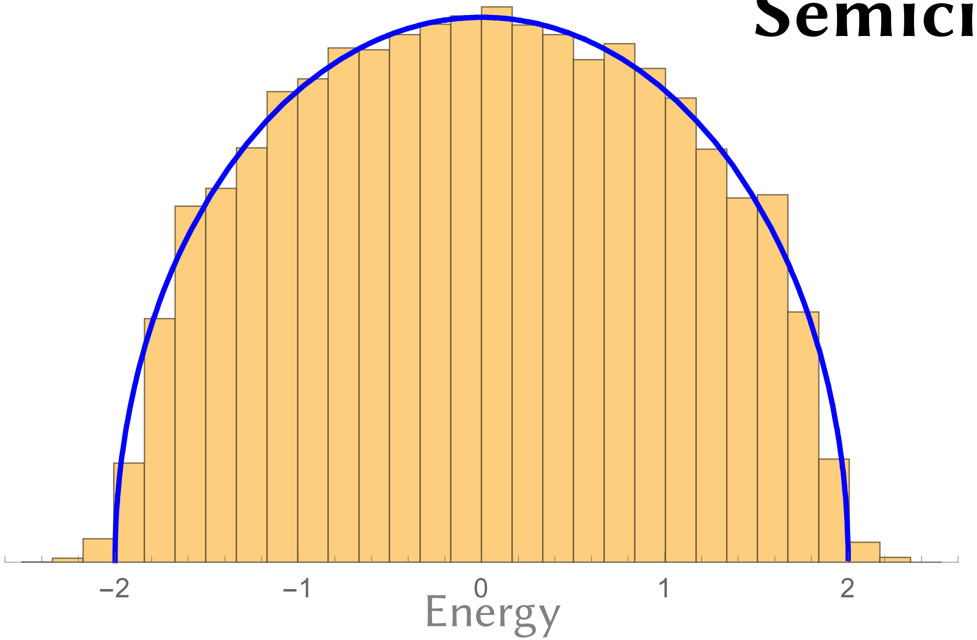
$N = 10$

Semicircle (Wigner)!

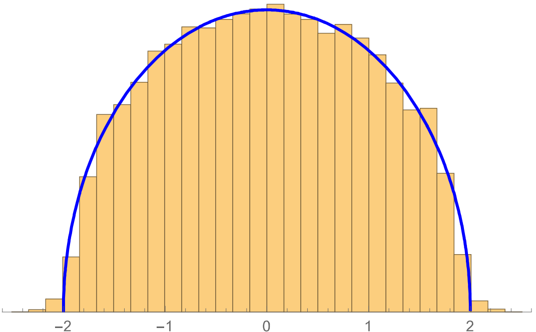


Spectral density: $\nu(E) = dn(E)/dE$

$dn(E)$ - average number of eigenvalues in $(E, E + dE)$

For $N \gg 1$

$$\nu(E) \approx \frac{N}{2\pi}\sqrt{4 - E^2}$$



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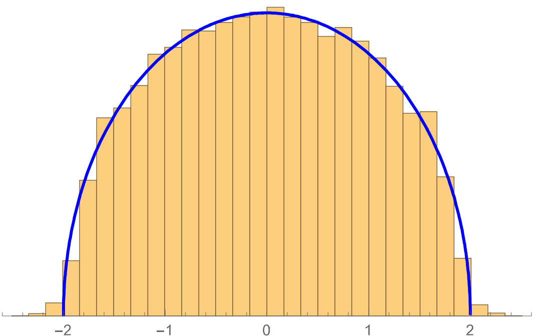
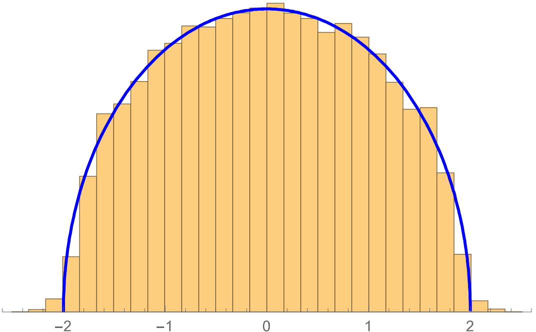


How to derive this result?

Feynman diagrams

Spectral density: $\nu(E) = dn(E)/dE$

$$\nu(E) = \langle \Sigma_n \delta(E - E_n) \rangle = -\frac{1}{\pi} \langle \mathrm{Im}\left(\mathrm{tr}\left[E - H + i0\right]^{-1}\right\rangle$$



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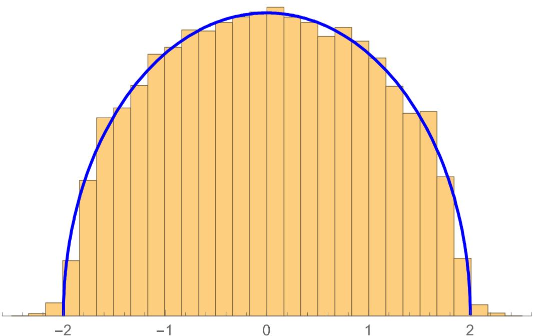
Green's function (resolvent)



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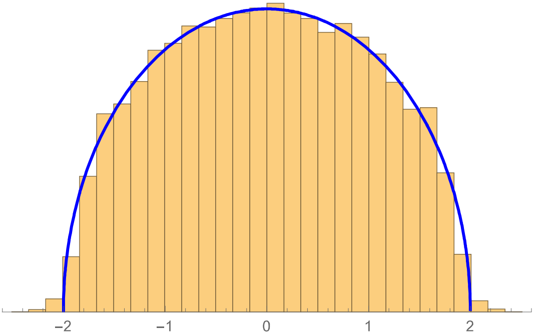
$$\frac{1}{E-H} = \frac{1}{E} + \frac{1}{E^2}H + \frac{1}{E^3}H^2 + \frac{1}{E^4}H^3 + \ldots$$



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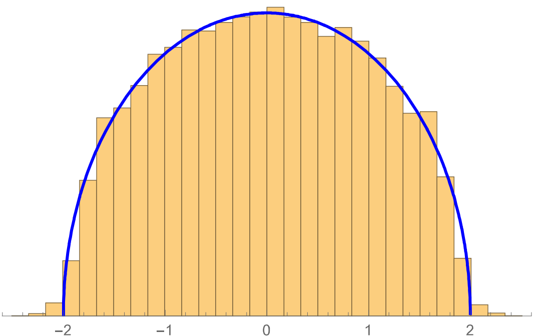
$$\langle H_{mn} H^{\star}_{kp} \rangle \propto \delta_{mk} \delta_{np}$$



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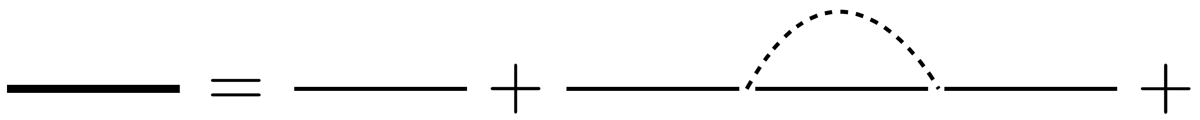

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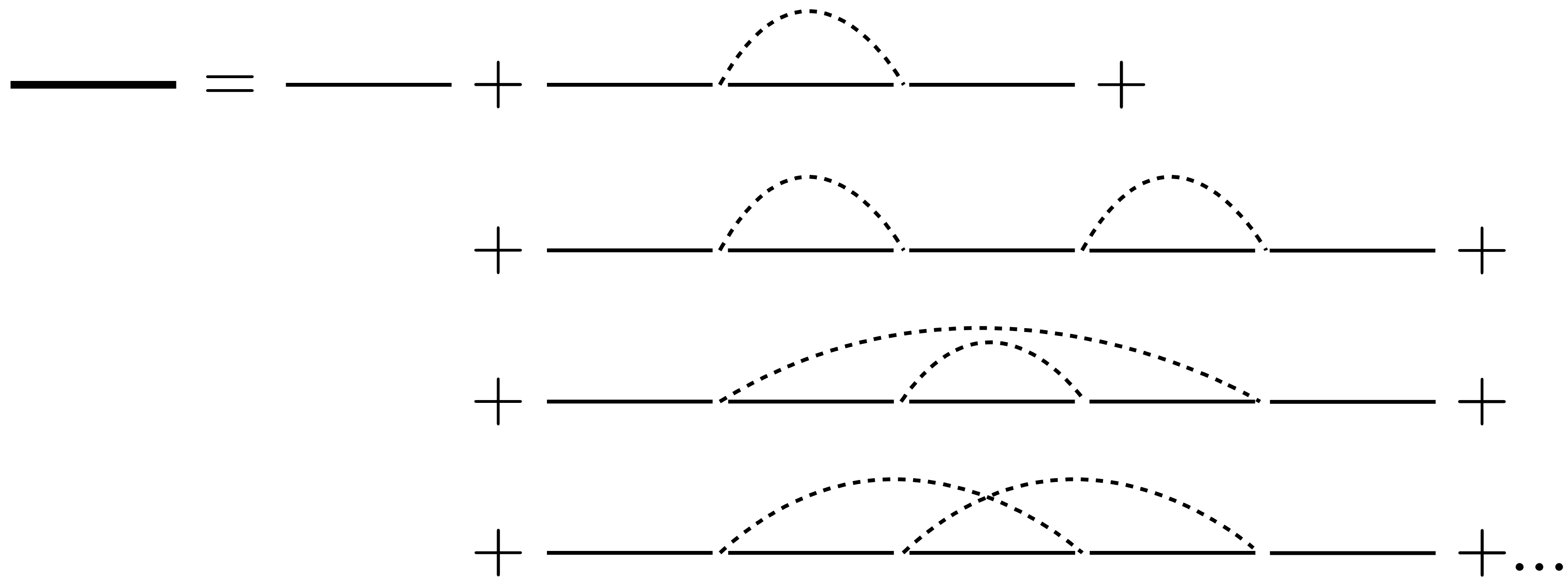

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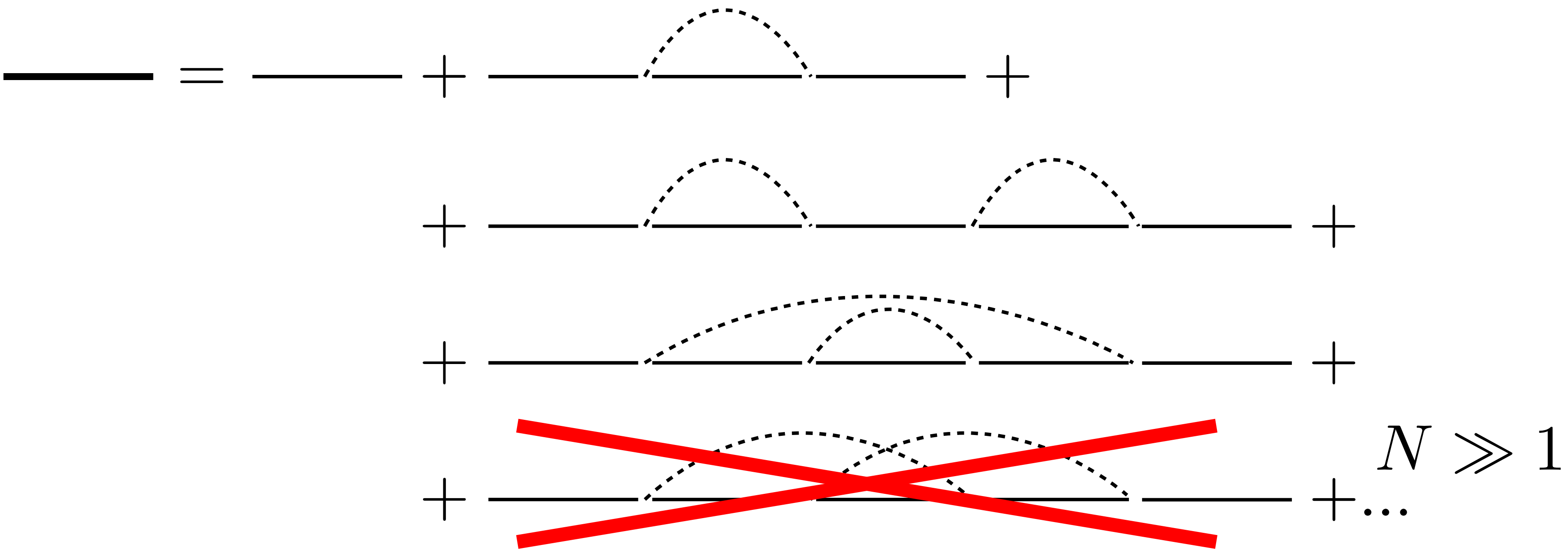

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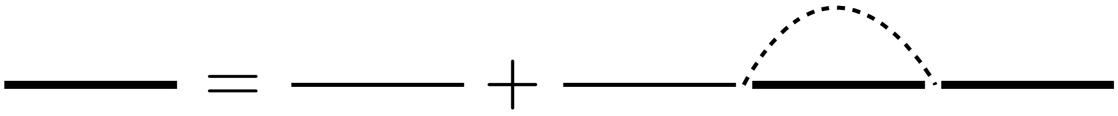

Dyson equation

$$\frac{1}{E-H} = \frac{1}{E} + \frac{1}{E^2}H + \frac{1}{E^3}H^2 + \frac{1}{E^4}H^3 + \ldots$$

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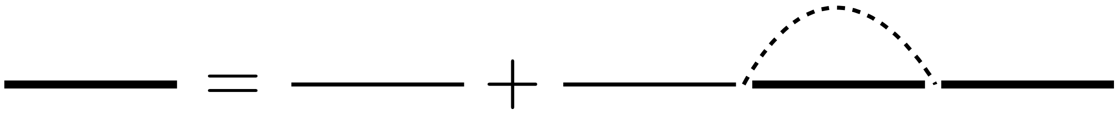

Dyson equation

$$\left(\frac{1}{E-H}\right)_{mn} \approx \delta_{mn}\frac{E - i\sqrt{4-E^2}}{2}$$

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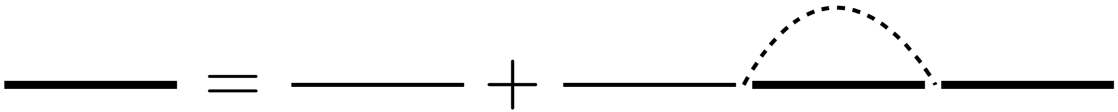

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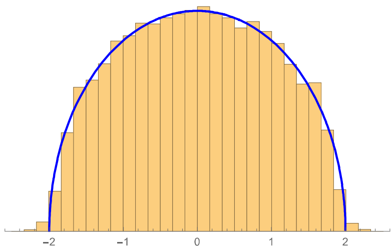

Summary

1. Random matrices = matrices of random variables

2. Eigenvalue problem: level repulsion for GUE

3. Spectral density: Wigner semicircle

Thank you for attention!