EXERCISE 3.1

Let

$$A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 3 \end{bmatrix}$$

1. Find (i) $A + B$ (ii) $A - B$
   (iii) $2A + 3B$ (iv) $3A - 5B$
   (v) AB (vi) BA
   Is $AB = BA$?

2. Evaluate

   (i) $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & 5 \\ -2 & 2 \end{bmatrix}$ (ii) $[x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

   (iii) $\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^2$ (iv) $\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^3$

   (v) $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2$ (vi) $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3$

3. Prove that the product of matrices

   $$A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

   is the zero matrix when θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$

4. The direction cosines of two lines are $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2, μ_3. Prove that the product

   $$\begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{bmatrix}\begin{bmatrix} \mu_1^2 & \mu_1\mu_2 & \mu_1\mu_3 \\ \mu_1\mu_2 & \mu_2^2 & \mu_2\mu_3 \\ \mu_1\mu_3 & \mu_2\mu_3 & \mu_3^2 \end{bmatrix}$$

   is zero if and only if the lines are perpendicular to each other.

5. Show that, in general, for any two matrices A and B which are conformable for addition and multiplication,

   $$(A + B)^2 \neq A^2 + 2AB + B^2$$

   and $$A^2 - B^2 \neq (A - B)(A + B)$$

   Under what conditions equality holds in each case?

6. If

   $$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \text{ show that}$$

   $$A^2 - 4A - 5I = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix} \text{ and}$$

   $$A^3 - 3A^2 - 7A - 3I = 0$$

7. Show that the matrix

   $$A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \text{ is periodic having period 2}$$

8. Show that

   $$\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix} \text{ is nilpotent. What is its nilpotency index?}$$

9. Show that

   $$\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \text{ is involutory.}$$

10. Show that every square matrix A with entries from R, can be written as a symmetric matrix $B = \frac{1}{2}(A + A^T)$ and a skew symmetric matrix $D = \frac{1}{2}(A - A^T)$

11. Show that every square matrix A with entries from C, can be written as a Hermitian matrix $B = \frac{1}{2}\,(A + (\bar{A})^T)$, and as a skew Hermitian matrix $D = \frac{1}{2}\,(A + (-\bar{A})^T)$

12. If A and B are symmetric matrices, then prove that AB is symmetric if and only if A and B commute.

If A is an $m \times m$ symmetric (skew symmetric) matrix and P is an $m \times n$ matrix, then prove that $B = P^T AP$ is symmetric (skew symmetric)

14. Show that AA^T and A^TA are symmetric for any square matrix A.

If A is a square matrix over C, then show that $A + (\bar{A})^T$, $A(\bar{A})^T$ and $(\bar{A})^T A$ are all Hermitian

16. Show that every square matrix over C can be expressed in a unique way as $P + iQ$, where P and Q are Hermitian

17. Show that every Hermitian matrix can be written as $A + iB$, where A is real and symmetric and B is real and skew symmetric.

18. Show that, for any matrices A and B over C and $k \in C$,

(i) $\bar{\bar{A}} = A$ (ii) $\overline{kA} = \bar{k}\bar{A}$

(iii) $\overline{A+B} = \bar{A} + \bar{B}$ (iv) $\overline{AB} = \bar{A}\,\bar{B}$

(v) $(\bar{A})^T = \overline{(A)^T}$

19. If A is a matrix over R and $AA^T = 0$, show that $A = 0$

20. If A is a matrix over C and $A(\bar{A})^T = 0$, then show that $\bar{A} = 0 = A$.

21. Show that for any matrix A, $(A^p)^T = (A^T)^p$, where p is a positive integer.

22. Let $x = [x_1\ \ x_2\ \ x_3]$, $y = [y_1\ \ y_2\ \ y_3]$

be two 1×3 matrices. Define a product $x \times y$ (called a **vector product** of x and y) as follows

$$x \times y = [x_2y_3 - y_2x_3\quad x_3y_1 - y_3x_1\quad x_1y_2 - y_1x_2]$$

Show that

(i) $y \times x = -(x \times y)$ (ii) $x \times x = 0$ for all x.

(iii) $(x \times y) \times z \neq x \times (y \times z)$, where $z = [z_1\ \ z_2\ \ z_3]$

23. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Define **inner product** of A and B by

$$A \cdot B = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

Show that: (i) $A \cdot B = 0$ but neither $A = 0$ nor $B = 0$

(ii) $A \cdot B = B \cdot A$

(iii) $(\alpha A + \beta B)\cdot C = \alpha A \cdot C + \beta B \cdot C$, $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

(iv) $A \cdot A \geq 0$ and $A \cdot A = 0 \Leftrightarrow A = 0$.

24. Let

$$A = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

and

$$B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 3 & 6 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Compute AB using the indicated partitionings

25. Let

$$A = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$B = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find AB using the indicated partitionings

5. Show that in general for any two matrices A and B which are comformable for addition and multiplication.

$(A+B)^2 \neq A^2-2AB+B^2$

and $A^2-B^2 \neq (A-B)(A+B)$

Under what condition equality holds in each case?

Since-

$(A+B)^2 = (A+B)(A+B)$

∴ Matrices holds distributive Law

$= A(A+B)+B(A+B)$

$= A^2+AB+BA+B^2$

Since $(AB \neq BA)$

In matrices commutative law with multiplication does not hold.

Hence,

$(A+B)^2 \neq A^2-2AB+B^2$

Similarly

$(A-B)(A+B) = A^2+AB-BA-B^2$

Since $(AB \neq BA)$

When $AB = BA$ then equality holds.

6.9f

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$ Show that

$$A^2 - 4A - 5I = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$ and

$$A^3 - 3A^2 - 7A - 3I = 0$$

$$A \times A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+2 & 2+2+2 & 1+4+1 \\ 2+2+4 & 4+1+4 & 2+2+2 \\ 2+4+2 & 4+2+2 & 2+4+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & 6 & 6 \\ 8 & 9 & 6 \\ 8 & 8 & 7 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$

L.H.S

$$= \begin{bmatrix} 7 & 6 & 6 \\ 8 & 9 & 6 \\ 8 & 8 & 7 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 & 6 \\ 8 & 9 & 6 \\ 8 & 8 & 7 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 4 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 7-4 & 6-8 & 6-4 \\ 8-8 & 9-4 & 6-8 \\ 8-8 & 8-8 & 7-4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3-5 & -2-0 & 2-0 \\ 0-0 & 5-5 & -2-0 \\ 0-0 & 0-0 & 3-5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$

L.H.S = R.H.S

Hence proved

$$A^2 \times A = \begin{bmatrix} 7 & 6 & 6 \\ 8 & 9 & 6 \\ 8 & 8 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7+12+12 & 14+6+12 & 7+12+6 \\ 8+18+12 & 16+9+12 & 8+18+6 \\ 8+16+14 & 16+8+14 & 8+16+7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 31 & 32 & 25 \\ 38 & 37 & 32 \\ 38 & 38 & 31 \end{bmatrix}$$

$$A^3 - 3A^2 - 7A - 3I = 0$$

L.H.S

$$= \begin{bmatrix} 31 & 32 & 25 \\ 38 & 37 & 32 \\ 38 & 38 & 31 \end{bmatrix} - 3\begin{bmatrix} 7 & 6 & 6 \\ 8 & 9 & 6 \\ 8 & 8 & 7 \end{bmatrix} - 7\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 32 & 25 \\ 38 & 37 & 32 \\ 38 & 38 & 31 \end{bmatrix} - \begin{bmatrix} 21 & 18 & 18 \\ 24 & 27 & 18 \\ 24 & 24 & 21 \end{bmatrix} - \begin{bmatrix} 7 & 14 & 7 \\ 14 & 7 & 14 \\ 14 & 14 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 31-21 & 32-18 & 25-18 \\ 38-24 & 37-27 & 32-18 \\ 38-24 & 38-24 & 31-21 \end{bmatrix} - \begin{bmatrix} 7 & 14 & 7 \\ 14 & 7 & 14 \\ 14 & 14 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 & 7 \\ 14 & 10 & 14 \\ 14 & 14 & 10 \end{bmatrix} - \begin{bmatrix} 7 & 14 & 7 \\ 14 & 7 & 14 \\ 14 & 14 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10-7 & 14-14 & 7-7 \\ 14-14 & 10-7 & 14-14 \\ 14-14 & 14-14 & 10-7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-3 & 0-0 & 0-0 \\ 0-0 & 3-3 & 0-0 \\ 0-0 & 0-0 & 3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

L.H.S = R.H.S

Hence proved.

7. Show that the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

is periodic having period 2.

$$A \times A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6-12 & -2-4-0 & -6-18+18 \\ -3-6+18 & 6+4+0 & 18+18-27 \\ 2-0-6 & 4+0+0 & -12+0+9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}$$

$$A^2 \times A = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+18-12 & 10-12-0 & 30-54+18 \\ 9-30+18 & -18+20+0 & -54+90-27 \\ -4+12-6 & 8-8-0 & 24-36+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

8. Show that $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is nilpotent.

What is its nilpotency index?

Let

$$A \times A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+3-4 & -3-9+12 & -4-12+16 \\ -1-3+4 & 3+9-12 & 4+12-16 \\ 1+3-4 & -3-9+12 & -4-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence prove that it is nilpotent and its nilpotency index is 2.

9. Show that

$$\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$ **is involutary**

Let that it is A

$$A \times A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence prove that it is involutary.

V.imp

12. If A and B are symmetric matrices, then prove that AB is symmetric if and only if A and B commute.

It is given that A and B are symmetric then

$A^t = A \quad , \quad B^t = B$

Now we suppose that AB is symmetric

$\therefore (AB)^t = AB \qquad\qquad (AB)^t = B^t A^t$

$B^t A^t = AB$

$BA = AB$

Therefore A, B commute

Now suppose A and B commute then

$AB = BA$

We shall show that (AB) is symmetric.

$(AB)^t = B^t A^t$

But $B^t = B$.

$A^t = A$

$(AB)^t = BA$

But $BA = AB$

So, $(AB)^t = AB$

Hence if A and B are symmetric Matrices then AB is symmetric.

14. **Show that AA^t and A^tA are symmetric for any square Matrix A.**

First we prove

$B = AA^t$ ① is symmetric

For that we shall show $B^t = B$

∴ taking transpose of ①

$(B)^t = (AA^t)^t$ But $(AB)^t = B^tA^t$

$(B)^t = (A^t)^t (A)^t$

∴ $(A^t)^t = A$

So,

$(B)^t = AA^t$

$B^t = B$

∴ $B = AA^t$ is symmetric

Now we prove $D = A^t A$ (2) is symmetric

We shall show that $D^t = D$

Taking transpose of (2)

$D^t = (A^t A)^t$ But $(AB)^t = B^t A^t$

$D^t = A^t \cdot (A^t)^t$

$D^t = A^t A$ $(A^t)^t = A$

$D^t = D$

$\therefore D$ is symmetric.

Example 18: Reduce the matrix $A = \begin{bmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$ **into the echelon form**

$$A = \begin{bmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{bmatrix} \begin{matrix} R_2+4R_1 \\ R_3-6R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3+R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{bmatrix} \quad \frac{R_2}{9}$$