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Solutions Manual For

CALCULUS WITH ANALYTIC GEOMETRY

By

A BOARD OF EXPERIENCED PROFESSORS

ILMI KITAB KHANA

Kabir Street, Urdu Bazar, Lahore. 54000

Solutions Manual For

CALCULUS With ANALYTIC GEOMETRY

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CONTENTS

Chapter 1 REAL NUMBERS, LIMITS AND CONTINUITY

Exercise Set 1.1 (Page 17)

1. If $a, b \in \boldsymbol{R}$ and $a + b = 0$, prove that $a = -b$

Sol. Since $b \in \boldsymbol{R}$, there is an element $-b \in \boldsymbol{R}$ such that

$$b + (-b) = 0 \tag{1}$$

By hypothesis, $a + b = 0$ (2)

Adding $-b$ to both sides of (2) and applying the above property of additive inverse, we get

$$a + b + (-b) = -b \quad \text{or} \quad a + (b + (-b)) = -b$$

(Associative property of addition)

or $\quad a + 0 = -b \qquad$ by (1)

i.e., $\quad a = -b \qquad$ as required.

2. Prove that $(-a)(-b) = ab$ for all $a, b \in \boldsymbol{R}$.

Sol. We have, $ab + a(-b) + (-a)(-b) = ab + [a(-b) + (-a)(-b)]$,

(Associative property of addition)

or $\quad a[b + (-b)] + (-a)(-b) = ab + [a + (-a)](-b)$

(Distributive property)

or $\quad a.0 + (-a)(-b) = ab + 0.(-b)$

i.e., $\quad (-a)(-b) = ab$, since $a.0 = 0 = 0.(-b)$.

3. Prove that $\big||a| - |b|\big| \le |a - b|$ for every $a, b \in \boldsymbol{R}$.

Sol. By Theorem 1.5 (v), we have

$$|a + b| \le |a| + |b| \tag{1}$$

Replacing b by $-b$, (1) becomes

$$|a - b| \le |a| + |-b| = |a| + |b|, \tag{2}$$

since $|-b| = |b|$

Replace a by $b - a$ in (2) to get

$$|-a| \le |b - a| + |b|$$

or $\quad |a| - |b| \le |b - a| = |a - b| \qquad$ (3)

Again, in $|b - a| \le |a| + |b|$, replace b by $a - b$ to have

$$|-b| \le |a| + |a - b| \quad \text{or} \quad |b| - |a| \le |a - b|$$

Multiplying both sides of the inequality by -1, we get

$$|a| - |b| \ge -|a - b|$$

or $\quad -|a - b| \le |a| - |b| \qquad$ (4)

Combining (3) and (4), we have $-|a - b| \le |a| - |b| \le |a - b|$

or $\left||a|-|b|\right| \le |a-b|$, by Theorem 1.5 (*iv*).

4. Express $3 < x < 7$ in modulus notation

Sol. We know that $|x-a| < l$ implies $a-l < x < a+l$

Now $3 < x < 7$

Therefore, by comparison,

$$a - l = 3 \quad (1)$$
$$a + l = 7 \quad (2)$$

Adding (1) and (2), we get $2a = 10$ or $a = 5$

Subtracting (1) from (2), we have $2l = 4$ or $l = 2$

Hence the given inequality can be expressed in the modulus notation as $|x-5| < 2$

5. Let $\delta > 0$ and $a \in \boldsymbol{R}$. Show that $a - \delta < x < a + \delta$ if and only if $|x-a| < \delta$.

Sol. Suppose $a - \delta < x < a + \delta$. These inequalities can be written as

$$a - \delta < x \quad (1)$$
and $$x < a + \delta \quad (2)$$

From (1) and (2), we have respectively

$$-\delta < x - a \quad (3)$$
and $$x - a < \delta \quad (4)$$

Combining (3) and (4), we get

$-\delta < x - a < \delta$ or $|x-a| < \delta$ by Theorem 1.5 (*iv*)

Conversely, let $|x-a| < \delta$. By Theorem 1.5 (*iv*), we have

$-\delta < x - a < \delta$ or $a - \delta < x < a + \delta$ as desired

6. Give an example of a set of rational numbers which is bounded above but does not have a rational supremum.

Sol. Consider the set S of rational numbers defined by

$$S = \{x \in Q : x^2 < 2\}$$

The supremum of S is $\sqrt{2}$ which is not a rational number.

Solve each of the following inequalities (Problems 7 – 15)

7. $|2x+5| > |2-5x|$

Sol. Associated equation is $|2x+5| = |2-5x|$

This is equivalent to

$$2x + 5 = 2 - 5x \quad (1)$$
or $$2x + 5 = -2 + 5x \quad (2)$$

From (1), we get $x = -\dfrac{3}{7}$ and from (2), we have $x = \dfrac{7}{3}$

These are the boundary numbers for the given inequality. The number line is divided by the boundary numbers into regions as shown:

Region A, test $x = -1$: $|-2+5| > |2+5|$ False
Region B, test $x = 0$: $|5| > |2|$ True
Region C, test $x = 3$: $|6+5| > |2-15|$ False

Thus the solution set is

$$\left\{x : \frac{-3}{7} < x < \frac{7}{3}\right\} = \left]-\frac{3}{7}, \frac{7}{3}\right[$$

8. $$\left|\frac{x+8}{12}\right| < \frac{x-1}{10} \quad (1)$$

Sol. (1) is equivalent to the compound inequality

$$-\frac{x-1}{10} < \frac{x+8}{12} < \frac{x-1}{10} \quad \text{or} \quad -6x + 6 < 5x + 40 < 6x - 6$$

This is equivalent to $-11x < 34$ and $46 < x$

i.e., $-\dfrac{34}{11} < x$ and $46 < x$

The solution set is

$$\left\{x : -\frac{34}{11} < x\right\} \cap \{x : 46 < x\} = \{x : 46 < x\} =]46, \infty[$$

Alternative Method:

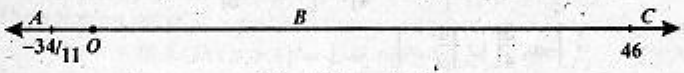
Associated equation is $\dfrac{x+8}{12} = \pm\dfrac{x-1}{10}$

i.e., $5x + 40 = \pm 6(x-1)$

i.e., $5x + 40 = 6x - 6$ and $5x + 40 = -6x + 6$

or $x = 46$ and $x = -\dfrac{34}{11}$

These boundary numbers divide the number line as shown:



Region A, test $x = -4$: $\left|\dfrac{-4+8}{12}\right| < \dfrac{-4-1}{10}$ False

Region B, test $x = 45$: $\left|\dfrac{45+8}{12}\right| < \dfrac{45-1}{10}$ False

Region C, test $x = 47$: $\left|\dfrac{47-8}{12}\right| < \dfrac{47-1}{10}$ True

The solution set is $\{x : x > 46\} =]46, \infty[$.

9. $|x| + |x-1| > 1$

Sol. The associated equation is

$$|x| + |x-1| = 1 \quad \text{or} \quad \pm x \pm (x-1) = 1$$

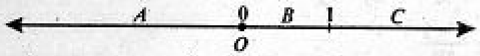
This is equivalent to

$$x + x - 1 = 1 \quad (1)$$

$$x + x - 1 = -1 \qquad (2)$$
$$x - (x - 1) = 1 \qquad (3)$$
$$-x + x - 1 = 1 \qquad (4)$$

From (1) and (2), we find $x = 1, 0$.

These boundary numbers divide the number line as shown:



Region A, test $x = -1$: $|-1| + |-1-1| > 1$ True

Region B, test $x = \frac{1}{2}$: $\left|-\frac{1}{2}\right| + \left|\frac{1}{2} - 1\right| > 1$ False

Region C, test $x = 2$: $|2| + |2-1| > 1$ True

The solution set is $\{x : x < 0\} \cup \{x : x > 1\} =]-\infty, 0[\cup]1, \infty[$.

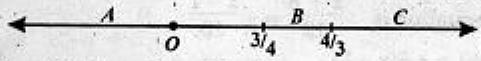
10. $2x^2 - 25x + 12 > 0$

Sol. The associated equation is

$$12x^2 - 25x + 12 = 0$$

$$x = \frac{25 \pm \sqrt{625 - 576}}{24} = \frac{25 \pm 7}{24} = \frac{4}{3}, \frac{3}{4}$$

These boundary numbers divide the number line as shown.



Region A, test $x = 0$: $12 > 0$ True

Region B, test $x = 1$: $12 - 25 + 12 > 0$ False

Region C, test $x = 2$: $48 - 50 + 12 > 0$ True

The solution set is $\left\{x : x < \frac{3}{4}\right\} \cup \left\{x : x > \frac{4}{3}\right\}$

$$= \left]-\infty, \frac{3}{4}\right[\cup \left]\frac{4}{3}, \infty\right[$$

11. $\frac{x-1}{2} - \frac{1}{x} > \frac{4}{x} + 5$

Sol. The associated equation is $\frac{x-1}{2} - \frac{1}{x} = \frac{4}{x} + 5$

or $x^2 - x - 2 = 8 + 10x$ or $x^2 - 11x - 10 = 0$

$$x = \frac{11 \pm \sqrt{121 + 40}}{2} = \frac{11 \pm \sqrt{161}}{2} = \frac{11 + \sqrt{161}}{2}, \frac{11 - \sqrt{161}}{2}$$

The point $x = 0$ is a free boundary number. These boundary numbers divide the number line as shown:

Region A, test $x = -1$: $-1 + 1 > -4 + 5$ False

Region B, text $x = -5.0$: $-\frac{1.5}{2} - \frac{1}{-0.5} > \frac{4}{-0.5} + 5$

i.e., $-0.75 + 2 > -8 + 5$ True

Region C, test $x = 5$: $\frac{5-1}{2} - \frac{1}{5} > \frac{4}{5} + 5$ False

Region D, test $x = 13$: $\frac{13-1}{2} - \frac{1}{13} > \frac{4}{13} + 5$ True

The solution set is $\left]\frac{11 - \sqrt{161}}{2}, 0\right[\cup \left]\frac{11 + \sqrt{161}}{2}, \infty\right[$

12. $|x^2 - x + 1| > 1$ (1)

Sol. $|x^2 - x + 1| > 1$ is equivalent to

$$x^2 - x + 1 > 1 \text{ or } x^2 - x + 1 < 1$$

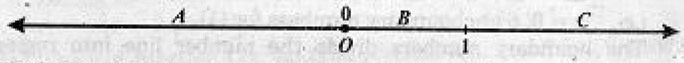
For the inequality $x^2 - x + 1 > 1$, the associated equation

$$x^2 - x + 1 = 1$$

gives $x(x - 1) = 0$

or $x = 0, 1$ are the boundary numbers for $x^2 - x + 1 > 0$

The number line is divided by 0 and 1 as shown:



Region A, test $x = -1$: $1 + 1 + 1 > 1$ True

Region B, text $x = \frac{1}{2}$: $\frac{1}{4} - \frac{1}{2} + 1 > 1$ False

Region C, test $x = 2$: $4 - 2 + 1 > 1$ True

The solution set of $x^2 - x + 1 > 1$ is

$$\{x : x < 0\} \cup \{x : x > 1\} =]-\infty, 0[\cup]1, \infty[.$$

For $x^2 - x + 1 < -1$, one must have $x^2 - x + 2 < 0$

i.e., $\left(x - \frac{1}{2}\right)^2 + \frac{7}{4} < 0$ which is impossible for real x.

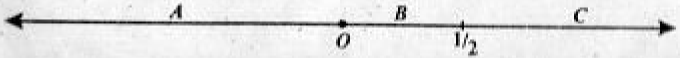
Thus the solution set of (1) is $]-\infty, 0[\cup]1, \infty[$.

13. $x^{-2} - 4x^{-1} + 4 > 0$

Sol. The given inequality is equivalent to

$$\frac{1}{x^2} - \frac{4}{x} + 4 > 0 \text{ or } \left(\frac{1 - 2x}{x}\right)^2 > 0$$

or $x = \frac{1}{2}$ is a boundary number. $x = 0$ is a free boundary number since x occurs in the denominator of the inequality. The boundary numbers divide the number line into regions as shown:



Region A, test $x = -1$: $\left(\frac{1+2}{-1}\right)^2 > 0$ True

Region B, test $x = \frac{1}{4}$: $\left(\frac{1-\frac{1}{2}}{\frac{1}{4}}\right)^2 > 0$ True

Region C, test $x = 1$: $\left(\frac{1-2}{1}\right)^2 > 0$ True

The solution set is $\{x : x < 0\} \cup \left\{x : 0 < x < \frac{1}{2}\right\} \cup \left\{x : x > \frac{1}{2}\right\}$

$$=]-\infty, 0[\;\cup\;]0, \tfrac{1}{2}[\;\cup\;]\tfrac{1}{2}, \infty[$$

14. $\frac{2x}{x+2} \geq \frac{x}{x-2}$ (1)

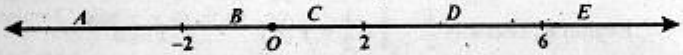
Sol. $x = -2, 2$ are free boundary numbers for (1). The associated equation

$$\frac{2x}{x+2} = \frac{x}{x-2} \text{ is equivalent to}$$

$$2x^2 - 4x = x^2 + 2x \quad \text{or} \quad x^2 - 6x = 0$$

i.e., $x = 0, 6$ are boundary numbers for (1).

The boundary numbers divide the number line into region as shown:



Region A, test $x = -3$: $\frac{-6}{-3+2} \geq \frac{-3}{-3-2}$ True

Region B, test $x = -1$: $\frac{-2}{-1+2} \geq \frac{-1}{-1-2}$ False

Region C, test $x = 1$: $\frac{2}{1+2} \geq \frac{2}{1-2}$ True

Region D, test $x = 3$: $\frac{6}{3+2} \geq \frac{3}{3-2}$ False

Region E, test $x = 7$: $\frac{14}{7+2} \geq \frac{7}{7-2}$ True

Since equality sign occurs in the inequality, the boundary numbers 0, 6 are in the solution set.

The solution set is

$$\{x : x < -2\} \cup \{x : 0 \leq x < 2\} \cup \{x : x \geq 6\}$$

$$=]-\infty, -2[\;\cup\; [0, 2[\;\cup\; [6, \infty[$$

...lems 23 – 30):

15. $x^4 - 5x^3 - 4x^2 + 20x \leq 0$

Sol. The associated equation is

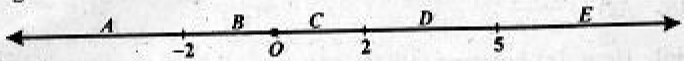
$$x^4 - 5x^3 - 4x^2 + 20x = 0 \quad (1)$$

i.e., $x(x^3 - 5x^2 - 4x + 20) = 0$

or $x[x^2(x-5) - 4(x-5)] = 0$ or $x(x-2)(x+2)(x-5) = 0$

$x = 0, -2, 2, 5$ are the boundary numbers for (1).

Locate the boundary numbers on a number line and check each region whether it belongs to the solution set or not.



Region A, test $x = -3$: $-3(-3-2)(-3+2)(-3-5) \leq 0$ False

Region B, test $x = -1$ $-1(-1-2)(-1+2)(-1-5) \leq 0$ True

Region C, test $x = 1$: $1(1-2)(1+2)(1-5) \leq 0$ False

Region D, test $x = 3$: $3(3-2)(3+2)(3-5) \leq 0$ True

Region E, test $x = 6$: $6(6-2)(6+2)(6-5) \leq 0$ False

The solution set consists of regions B and D. The boundary numbers are in these regions and since equality occurs in (1), they belong to the solution sets. The solution set is

$$\{x : -2 \leq x \leq 0\} \cup \{x : 2 \leq x \leq 5\} = [-2, 0] \cup [2, 5]$$

16. The cost function $C(x)$ and the revenue function $R(x)$ for producing x units of certain product are given by

$$C(x) = 5x + 350\,; \qquad R(x) = 50 - x^2$$

Find the values of x that yield a profit.

Sol. A profit is produced if revenue exceeds cost. Therefore for a profit

$R(x) > C(x)$

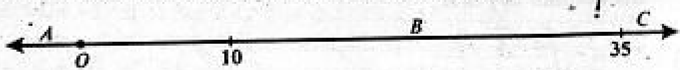
i.e., $50x - x^2 > 5x + 350$

or $x^2 - 45x + 350 < 0$ (1)

The associated equation of (1) is

$$x^2 - 45x + 350 = 0 \quad \text{or} \quad (x-10)(x-35) = 0$$

which gives $x = 10, 35$ as the boundary points. Locate these points on a number line and check which regions belong to the solution.



Region A, test $x = 0$: $(-10)(-35) < 0$ False

Region B, test $x = 15$: $(15-10)(15-35) < 0$ True

Region C, test $x = 40$: $(40-10)(40-35) < 0$ False

Thus the solution set for a profit is $\{x : 10 < x < 35\}$

Function f from R to R is defined by the given formula. Determine the domain of the function (Problems 17 – 22)

7. $f(x) = \sqrt{1 - x^2}$

ol. As soon as the numerical value of x exceeds 1, $f(x)$ becomes imaginary.

Hence the domain of definition of this function is $|x| \le 1$.

. $f(x) = \dfrac{a + x}{a - x}$

ol. Here $f(x)$ becomes infinite when $x = a$ and for every other real value of x, we get the corresponding real value of $f(x)$. Hence the domain of this function is the set of all real numbers except $x = a$.

. $f(x) = \dfrac{1}{\sqrt{(1 - x)(2 - x)}}$

l. Here $f(x)$ becomes infinite when $x = 1$ or $x = 2$. Also when $x \in \,]1, 2[$, the value of $f(x)$ becomes imaginary *i.e.*, $f(x)$ is not defined for any value of x where $1 < x < 2$. Therefore, the domain of definition of this function is the set of all real numbers x except when $x \in [1, 2]$.

$f(x) = \sqrt{3 + x} + \sqrt{7 - x}$

. Here when x exceeds 7, the value of $f(x)$ does not remain real. Similarly, when $x < -3$, $f(x)$ does not remain real. For every other real value of x, $f(x)$ is defined in the set of real numbers. Hence the required domain is the closed interval $[-3, 7]$.

$$f(x) \begin{cases} x^2 - 1 & \text{if } x \le 2 \\ \sqrt{x - 1} & \text{if } x > 2 \end{cases}$$

. The function is defined by two rules for all real numbers. Hence the domain of f is R.

$f(x) = \sqrt{\dfrac{x - 4}{x + 1}}$

The function is not defined when $x = -1$. For $-1 < x < 4$, the numerator is negative while the denominator is positive and so the value of the function is imaginary. Hence Dom $f = R - [-1, 4[$

Draw the graph of the following functions (Problems 23 – 30):

23. $f(x) = |x| + |x - 1|$ for all $x \in R$

Sol. This can be rewritten as

$$y = f(x) = \begin{cases} -x + 1 - x = 1 - 2x, & \text{when } x < 0 \\ x + 1 - x = 1, & \text{when } 0 \le x \le 1 \\ x + x - 1 = 2x - 1, & \text{when } x > 1 \end{cases}$$

So, the graph of the function will consist of three parts.

$y = 1 - 2x$, when $x < 0$ (1)

$y = 1$, when $0 \le x \le 1$ (2)

$y = 2x - 1$, when $x > 1$ (3)

For (1), we have the following table of values

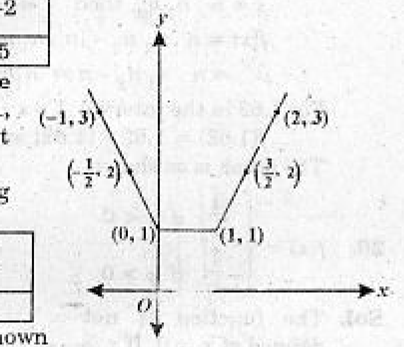
x	$-\frac{1}{2}$	-1	-2
y	2	3	5

For (2), its graph is a line segment in the interval $0 \le x \le 1$, parallel to the x-axis at a unit distance.

For (3), we have the following table of values to be plotted

x	$\frac{3}{2}$	2	3
y	2	3	5

Thus we have the graph as shown



24. $f(x) = [x] + [x + 1]$ for all $x \in R$

Sol. This can be rewritten as (if $y = f(x)$)

$y = 1$, if $0 \le x < 1$

$y = 3$, if $1 \le x < 2$

$y = 5$, if $2 \le x < 3$

$y = 7$, if $3 \le x < 4$

... ...

and $y = -1$, if $-1 \le x < 0$

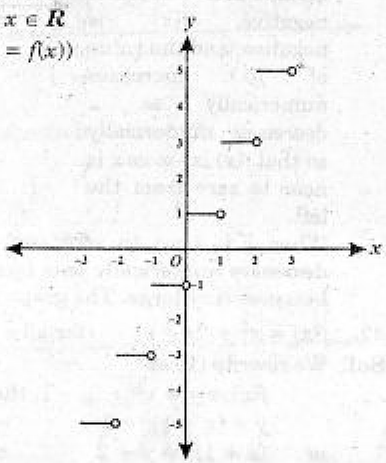
$y = -3$, if $-2 \le x < -1$

$y = -5$, if $-3 \le x < -2$

$y = -7$, if $-4 \le x < -3$

... ...

The graph is as shown



25. $f(x) = x - [x]$ for all $x \in [-3, 3]$ ***(Saw-tooth function)***

Sol. When x is an integer, (whether positive or negative), then $f(x) = 0$

If x is a negative fraction, say $x = -n \,.\, n_1 n_2$, where n, n_1, n_2 are positive integers, then

$$f(x) = -n \,.\, n_1 n_2 - [-n \,.\, n_1 n_2]$$
$$= -n \,.\, n_1 n_2 + n + 1 = 1 - .n_1 n_2$$

For -1.91 in the interval $-2 \le x < -1$

$$f(-1.91) = -1.91 - [-1.91]$$
$$= -1.91 - (-2)$$
$$= -1.91 + 2 = .09$$

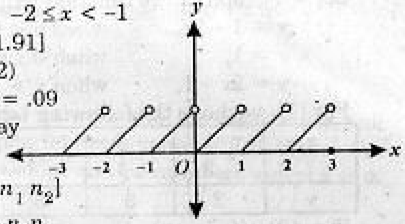
If x is a positive fraction, say $x = n \,.\, n_1 n_2$, then

$$f(x) = n \,.\, n_1 n_2 - [n \,.\, n_1 n_2]$$
$$= n \,.\, n_1 n_2 - n = .n_1 n_2$$

For 1.62 in the interval $1 \le x < 2$

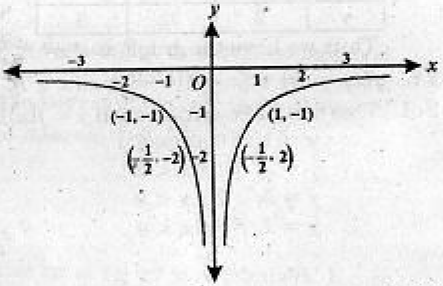
$$f(1.62) = 1.62 - [1.62] = 1.62 - 1 = .62$$

The graph is as shown



26. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < 0 \\ -\dfrac{1}{x} & \text{if } x > 0 \end{cases}$

Sol. The function is not defined at $x = 0$. If x is negative, $f(x)$ is negative, and the value of $f(x)$ increases numerically as x decreases numerically so that $f(x)$ is $-\infty$ as x is near to zero from the left.



When x is near to zero and positive, then $f(x)$ is $-\infty$ and $f(x)$ decreases numerically as x increases so that $f(x)$ is near to 0 as x becomes very large. The graph is as shown.

27. $f(x) = x^2 + 2x - 1$ for all $x \in \boldsymbol{R}$ (1)

Sol. We rewrite (1) as

$$f(x) = y = x^2 + 2x - 1, \text{ that is,}$$
$$y = (x + 1)^2 - 2$$

or $(x + 1)^2 = y + 2$ or $X^2 = Y$ (2)

where $X = x + 1$ and $Y = y + 2$

Now (2) is a parabola which is symmetric about the Y-axis and has its vertex at

$$X = 0, Y = 0$$

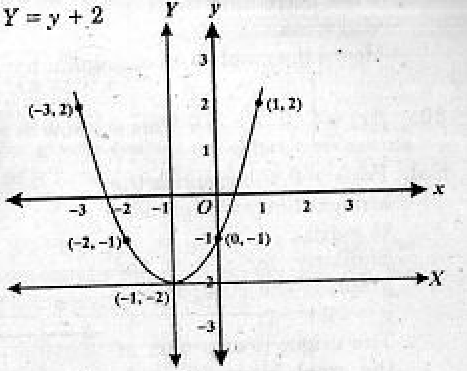
i.e., $x + 1 = 0$

and $y + 2 = 0$

or $x = -1$

and $y = -2$

It has the graph as shown.



28. $f(x) = \dfrac{1}{x^2}$, $x \ne 0$

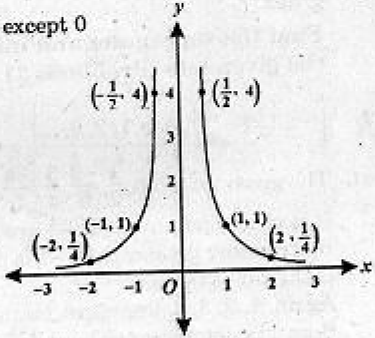
Sol. $f(x) = y = \dfrac{1}{x^2}$

y is defined for all values of x except 0

y is always positive, therefore the graph lies entirely above the x-axis.

$f(x_2) > f(x_1)$ if $x_2 > x_1$ for negative values of x that is, f is increasing in the interval $(-\infty, 0)$

$f(x_2) < f(x_1)$ if $x_2 > x_1$ for positive values of x that is, f is decreasing in the interval $(0, \infty)$.



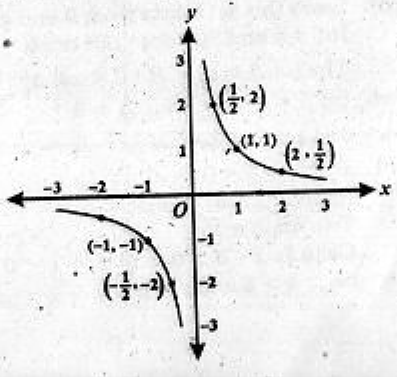
Hence we have the graph as shown in the figure.

29. $f(x) = \dfrac{1}{x}$, $x \ne 0$

Sol. $f(x) = y = \dfrac{1}{x}$

Here y is defined for all values of x except $x = 0$

When x is +ve, y is also +ve and when x is –ve, y is also –ve, therefore the graph lies in the first and third quadrants.



y is a decreasing function of x *i.e.*, as x increases, y decreases and vice versa.

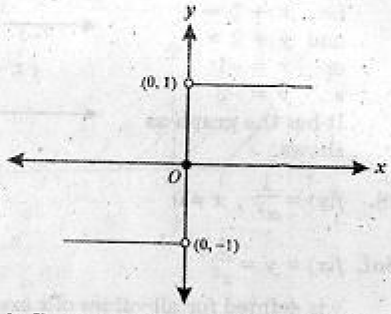
Hence the graph is a rectangular hyperbola as shown in the figure.

30. $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ This is know as **singum (sgn)** ***function***

Sol. For $x > 0$, the graph is the straight line $y = 1$ (parallel to x-axis).

Similarly, for $x < 0$, the graph is the straight line $y = -1$.

The origin is also part of the graph since $f(0) = 0$. The points $(0, 1)$ and $(0, -1)$ are not on the graph.



Find the supremum and infimum (if they exist) of each of the given sets (Problems 31 – 34)

31. $\left\{(-1)^n\left(1-\frac{1}{n}\right), n = 1, 2, 3, \ldots\right\}$

Sol. The given set is $\left\{0, \frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, \frac{-6}{7}, \ldots\right\}$

It is clear that ..., -3, -2, -1 are lower bounds of the set. Since any real number greater than -1 is not a lower bound, we infer that -1 is the Inf of the set.

Again, 1, 2, 3, ... are upper bounds of the set. But any real smaller than 1 is not an upper bound. Thus 1 is the Sup.

32. The set of all nonnegative integers

Sol. Since this set starts from 0 and extends to $+\infty$, Inf $= 0$ and Sup does not exist.

33. The set $A = \{x \in \mathbf{R} : 0 < x \leq 3\}$

Sol. Inf $A = 0$ and Sup $A = 3$

34. The set $B = \{x \in \mathbf{R} : x^2 - 2x - 3 < 0\}$

Sol. $x^2 - 2x - 3 < 0$

Implies $(x - 3)(x + 1) < 0$

Two cases arise:

Case I: $x - 3 > 0$ and $x + 1 < 0$

i.e., $x > 3$ and $x < -1$.

Since there is no real number which is greater than 3 and less than -1, so this is not possible.

Case II: $x - 3 < 0$ and $x + 1 > 0$

$\Rightarrow \quad x < 3$ and $x > -1$. Thus $-1 < x < 3$

$\Rightarrow \quad$ Inf $= -1$ and Sup $= 3$

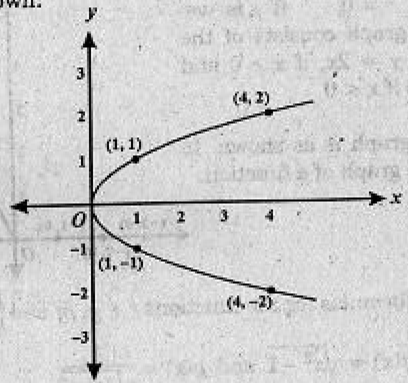
Sketch the graph of the given equation. Also determine which one is the graph of a function (Problems 35 – 38)

35. $y^2 = x$

Sol. It is clear that x is always positive. The graph passes through the origin. As y increases (numerically) x increases and is positive. We have the following table of some particular values:

x	0	1	4	9	16
y	0	± 1	± 2	± 3	$+4$

The graph is as shown:



By the vertical line test, the graph is not of a function.

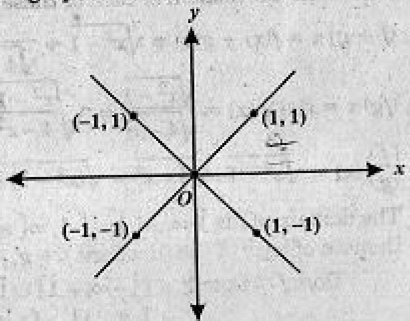
36. $|x| = |y|$

Sol. Here, we have

$x = \pm y$

which is a pair of straight lines through the origin.

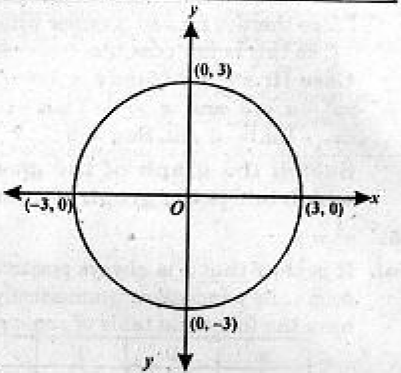
It is not the graph of a function.



37. $x^2 + y^2 = 9$

Sol. It is a circle with centre at the origin and having radius 3.

The graph is not a function.



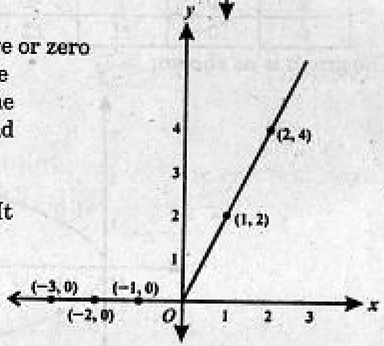
38. $y = |x| + x$

Sol. We have

$$y = x + x, \text{ if } x \text{ is +ve or zero}$$
$$= 0, \quad \text{if } x \text{ is −ve}$$

The graph consists of the lines $y = 2x$, if $x \geq 0$ and $y = 0$ if $x < 0$

The graph is as shown. It is the graph of a function.



39. Find formulas for the functions $f + g$, fg and $\dfrac{f}{g}$, where

$$f(x) = \sqrt{x^2 - 1} \text{ and } g(x) = \frac{1}{\sqrt{4 - x^2}}$$

Also write the domain of each of these functions.

Sol. $(f + g)\,x = f(x) + g(x) = \sqrt{x^2 - 1} + \dfrac{1}{\sqrt{4 - x^2}}$

$$(fg)\,x = f(x)\,g(x) = \frac{\sqrt{x^2 - 1}}{\sqrt{4 - x^2}} = \sqrt{\frac{x^2 - 1}{4 - x^2}}$$

$$\left(\frac{f}{g}\right)(x) = \sqrt{x^2 - 1}\,.\,\sqrt{4 - x^2} = \sqrt{(x^2 - 1)(4 - x^2)}$$

The domain of f is $]-\infty, -1] \cup [1, \infty[$ and the domain of g is $]-2, 2[$

Domain of each of the functions $f + g$, fg and f/g is

$$\text{Dom } f \cap \text{Dom } g = (]-\infty, -1] \cup [1, \infty[) \cap]-2, 2[$$
$$=]-2, -1] \cup [1, 2[$$

40. Find formulas for fog and gof, where

$$f(x) = \sqrt{x^3 - 3} \text{ and } g(x) = x^2 + 3$$

Sol. We know that $(fog)(x) = f(g(x))$

$= f(x^2 + 3)$, by defining rule of g

$= \sqrt{(x^2 + 3)^2 - 3}$ by defining rule of f

$= \sqrt{x^4 + 6x^2 + 6}$

Again $(gof) = g(f(x))$

$= g(\sqrt{x^2 - 3})$ by defining rule of f

$= x^2 - 3 + 3 = x^2$

Exercise Set 1.2 (Page 35)

Evaluate the indicated limits (Problems 1 – 30):

1. $\displaystyle\lim_{x \to 2} \frac{x - 2}{\sqrt{2 + x}}$

Sol. $\displaystyle\lim_{x \to 2} \frac{x - 2}{\sqrt{2 + x}} = \frac{\lim_{x \to 2}(x - 2)}{\lim_{x \to 2}\sqrt{2 + x}} = \frac{0}{2} = 0$

2. $\displaystyle\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Sol. $\displaystyle\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1}$

$$= \lim_{x \to 1} (x^2 + x + 1) = 3$$

3. $\displaystyle\lim_{x \to 1} \left(\frac{1}{1 - x} - \frac{3}{1 - x^3}\right)$

Sol. The given limit

$$= \lim_{x \to 1} \frac{1 + x + x^2 - 3}{1 - x^3} = \lim_{x \to 1} \frac{x^2 + x - 2}{1 - x^3}$$

$$= \lim_{x \to 1} \frac{(x + 2)(x - 1)}{-(x - 1)(x^2 + x + 1)} = \lim_{x \to 1} \frac{x + 2}{-(x^2 + x + 1)}$$

$$= \frac{3}{-3} = -1$$

4. If $P_n(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$, prove that

$$\lim_{x \to a} P_n(x) = P_n(a)$$

Sol. $\lim\limits_{x \to a} P_n(x)$

$$= \lim_{x \to a} \{a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n\}$$

$$= \lim_{x \to a} a_0 x^n + \lim_{x \to a} a_1 x^{n-1} + \dots + \lim_{x \to a} a_{n-1} x + \lim_{x \to a} a_n, \quad \text{by Theorem 1.26 } (i)$$

$$= a_0 a^n + a_1 a^{n-1} + \dots + a_{n-1} a + a_n \quad \begin{cases} \text{by Theorem 1.26 } (ii) \\ \text{and Theorem 1.24} \end{cases}$$

$$= P_n(a)$$

5. $\lim\limits_{x \to 0} \dfrac{\csc x - \cot x}{x}$ ... (1)

Sol. (1) may be written as

$$\lim_{x \to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x \sin x}$$

$$= \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x \sin x (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x \sin x (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x \,.\, \sin x (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{1 + \cos x}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \cdot \lim_{x \to 0} \left(\frac{1}{1 + \cos x}\right)$$

$$= 1 \,.\, \frac{1}{2} = \frac{1}{2}$$

6. $\lim\limits_{x \to 0} \dfrac{\sin ax}{\sin bx}$

Sol. The given limit is

$$\lim_{x \to 0} \left[\left(\frac{\sin ax}{ax}\right)\left(\frac{bx}{\sin bx}\right)\left(\frac{ax}{bx}\right)\right]$$

$$= \lim_{x \to 0} \left(\frac{\sin ax}{ax}\right) \cdot \lim_{x \to 0} \frac{bx}{\sin bx} \cdot \lim_{x \to 0} \frac{ax}{bx}$$

$$= 1 . 1 . \frac{a}{b} = \frac{a}{b}$$

7. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ (1)

Sol. (1) may be written as

$$\lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2 (1 + \cos x)} = \lim_{x \to 0} \left(\frac{1 - \cos^2 x}{x^2} \cdot \frac{1}{1 + \cos x}\right)$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x \to 0} \frac{1}{1 + \cos x}$$

$$= (1)^2 \cdot \frac{1}{2} = \frac{1}{2}$$

8. $\lim\limits_{y \to x} \dfrac{y^{\frac{2}{3}} - x^{\frac{2}{3}}}{y - x}$

Sol. $$\lim_{y \to x} \frac{y^{\frac{2}{3}} - x^{\frac{2}{3}}}{y - x} = \lim_{y \to x} \frac{\left(y^{\frac{1}{3}} - x^{\frac{1}{3}}\right)\left(y^{\frac{1}{3}} + x^{\frac{1}{3}}\right)}{\left(y^{\frac{1}{3}} - x^{\frac{1}{3}}\right)\left(y^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}$$

$$= \lim_{y \to x} \frac{y^{\frac{1}{3}} + x^{\frac{1}{3}}}{y^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + x^{\frac{2}{3}}} = \frac{2x^{\frac{1}{3}}}{3x^{\frac{2}{3}}} = \frac{2}{3} x^{\frac{1}{3}}$$

9. $\lim\limits_{x \to \pi} \dfrac{\tan(\sin x)}{\sin x}$

Sol. Let $\sin x = \theta$

When $x \longrightarrow \pi$, $\theta \longrightarrow 0$. Therefore,

$$\lim_{x \to \pi} \frac{\tan(\sin x)}{\sin x} = \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \times \frac{1}{\cos \theta}$$

$$= \lim_{\theta \to \theta} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{1}{\cos \theta} = 1 \times 1 = 1$$

10. $\lim\limits_{x \to 0} x \sin\left(\dfrac{1}{x}\right)$

Sol. Here $|f(x) - 0|$

$$= \left|x \sin \frac{1}{x} - 0\right| = \left|x \sin \frac{1}{x}\right|$$

$$= |x| \left|\sin \frac{1}{x}\right| \le |x|, \left(\text{since } \left|\sin \frac{1}{x}\right| \le 1\right)$$

If we take $\varepsilon = \delta$, then $|f(x) - 0| < \varepsilon$

whenever $|x| < \delta$. Hence $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0$

Alternative Method:

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = \lim_{x \to 0} x \, . \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$
$$= 0. \text{ some number in } [-1, 1] = 0$$

11. $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

Sol. Here we find $\lim_{x \to 0^-} \sin\left(\frac{1}{x}\right)$

Since $\left|\sin\left(\frac{1}{x}\right)\right| \le 1$, then limit repeatedly takes the values 1 and -1 or any value between -1 and 1. Hence this limit cannot exist.

Similarly, $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ does not exist.

Thus $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

12. $\lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x+1}$

Sol. The given limit is $\lim_{x \to \infty} \frac{x\sqrt{1+\frac{1}{x^2}}}{x(1+\frac{1}{x})} = \lim_{x \to \infty} \frac{\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \frac{1}{1} = 1$

13. $\lim_{x \to \infty} \frac{4x^3 - 2x^2 + 1}{3x^3 - 5}$

Sol. Dividing both the numerator and denominator by x^3, we get the given limit

$$= \lim_{x \to \infty} \frac{4 - \frac{2}{x^2} + \frac{1}{x^3}}{3 - \frac{5}{x^3}} = \frac{4}{3}$$

14. $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$

Sol. $\lim_{x \to \infty} \left[\left(1 + \frac{2}{x}\right)^{x/2}\right]^2 = e^2, \left(\text{since } \lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{x/2} = e\right)$

15. $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x$

Sol. $\lim_{x \to \infty} \left[\left(1 - \frac{1}{x}\right)^{-x}\right]^{-1} = e^{-1} = \frac{1}{e}$

16. $\lim_{x \to \infty} \left(\frac{x}{1+x}\right)^x$

Sol. The given limit is

$$\lim_{x \to \infty} \left(\frac{1+x}{x}\right)^{-x} = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{-x} = \lim_{x \to \infty} \left[\left(1 + \frac{1}{x}\right)^x\right]^{-1} = e^{-1} = \frac{1}{e}$$

17. $\lim_{x \to \infty} \frac{a^x - 1}{x}, (a > 1)$

Sol. Let $a^x - 1 = z$

or $a^x = 1 + z$ or $x = \log_a (1 + z)$

If $x \longrightarrow \infty$, then $z \longrightarrow \infty$. Therefore,

$$\lim_{x \to \infty} \frac{a^x - 1}{x} = \lim_{z \to \infty} \frac{z}{\log_a (1+z)} = \lim_{z \to \infty} \frac{1}{\log_a (1+z)^{1/z}}$$

$$= \frac{1}{0} = \infty. \text{ (We assume } \infty^0 = 1)$$

18. $\lim_{x \to \infty} \frac{x^4 - 2x^2 + 6}{x^2 + 7}$ (1)

Sol. Dividing both the numerator and denominator by x^4, (1) becomes

$$= \lim_{x \to \infty} \frac{1 - \frac{2}{x^2} + \frac{6}{x^4}}{\frac{1}{x^2} + \frac{7}{x^4}} = \frac{\lim_{x \to \infty}\left(1 - \frac{2}{x^2} + \frac{6}{x^2}\right)}{\lim_{x \to \infty}\left(\frac{1}{x^2} + \frac{7}{x^4}\right)} = \frac{1}{0} = \infty$$

19. $\lim_{x \to \pm\infty} \left[\frac{x^2}{x+1} - \frac{x^2}{x+3}\right]$

Sol. The given limit is

$$\lim_{x \to \pm\infty} \left[\frac{x^3 + 3x^2 - x^3 - x^2}{(x+1)(x+3)}\right] = \lim_{x \to \pm\infty} \left[\frac{2x^2}{x^2 + 4x + 3}\right]$$
$$= \lim_{x \to \pm\infty} \frac{2}{1 + \frac{4}{x} + \frac{3}{x^2}} = 2$$

20. $\lim\limits_{x \to \infty} (x - \sqrt{x^2 - a^2})$

Sol. The given limit is

$$\lim_{x \to \infty} \frac{(x - \sqrt{x^2 - a^2})\,(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} = \lim_{x \to \infty} \frac{x^2 - x^2 + a^2}{x + \sqrt{x^2 - a^2}} = 0$$

21. $\lim\limits_{x \to \infty} \dfrac{x^2 + 1}{x^{3/2}}$

Sol. $$\lim_{x \to \infty} \frac{x^2 + 1}{x^{3/2}} = \lim_{x \to \infty} \frac{1 + \dfrac{1}{x^2}}{\dfrac{1}{x^{1/2}}} = \infty$$

22. $\lim\limits_{x \to \infty} \dfrac{5x^3 + 3x^2 - 1}{x - 4x^4}$

Sol. Since the limit of a quotient of polynomials as $x \longrightarrow \pm\infty$ is the same as the limit of the quotient of the highest power terms in the numerator and denominator, we have

$$\lim_{x \to \pm\infty} \frac{5x^3 + 2x^2 - 1}{x - 4x^4} = \lim_{x \to \pm\infty} \frac{5}{4x} = \lim_{x \to \pm\infty} \frac{5}{4x} = 0$$

23. $\lim\limits_{x \to \infty} \dfrac{3 - 2x^4}{1 + x}$

Sol. The given limit equals

$$\lim_{x \to \infty} \frac{-2x^4}{x} = \lim_{x \to \infty} -2x^3 = -\infty$$

24. $\lim\limits_{x \to -1} \dfrac{x^{1/3} + 1}{x + 1}$

Sol. $$\lim_{x \to -1} \frac{x^{1/3} + 1}{x + 1} = \lim_{h \to 0} \frac{(-1 + h)^{1/3} + 1}{h} = \lim_{h \to 0} \frac{-(1 - h)^{1/3} + 1}{h}$$

$$= \lim_{h \to 0} \frac{-\left[1 + \frac{1}{3}(-h) + \dfrac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}(-h)^2 + \ldots\right] + 1}{h}$$

$$= \lim_{h \to 0} \frac{\left(-1 + \frac{1}{3}h + \frac{1}{9}h^2 + \ldots\right) + 1}{h}$$

$$= \lim_{h \to 0} \left(\frac{1}{3} + \frac{1}{9}h + \ldots\right) = \frac{1}{3}$$

25. $\lim\limits_{x \to 3^-} \left(\dfrac{1}{x - 3} - \dfrac{1}{|x - 3|}\right)$

Sol. $$\lim_{x \to 3^-} \left[\frac{1}{x - 3} - \frac{1}{|x - 3|}\right] = \lim_{x \to 3^-} \left[\frac{1}{x - 3} - \frac{1}{3 - x}\right]$$

$$= \lim_{x \to 3^-} \left[\frac{1}{x - 3} + \frac{1}{x - 3}\right]$$

$$= \lim_{x \to 3^-} \left[\frac{2}{x - 3}\right] = -\infty$$

26. $\lim\limits_{x \to -2^-} \dfrac{x^2 + 2x - 8}{x^2 - 4}$

Sol. $$\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = \lim_{x \to -2^-} (x^2 + 2x - 8) \times \lim_{x \to -2^-} \frac{1}{x^2 - 4}$$

$$= (4 - 4 - 8) \cdot \frac{1}{(-2)^2 - 4} = -\infty$$

27. $\lim\limits_{x \to 1^-} \dfrac{\sqrt{1 - x^2}}{1 - x}$ (1)

Sol. We write (1) as

$$\lim_{x \to 1^-} \frac{\sqrt{(1 - x)\,(1 + x)}}{1 - x} = \lim_{x \to 1^-} \sqrt{\frac{1 + x}{1 - x}} = \frac{\sqrt{2}}{0} = \infty$$

28. $\lim\limits_{x \to 1^+} \dfrac{x - 1}{\sqrt{x^2 - 1}}$

Sol. $$\lim_{x \to 1^+} \frac{x - 1}{\sqrt{x^2 - 1}} = \lim_{x \to 1^+} \sqrt{\frac{x - 1}{x + 1}} = \sqrt{\frac{0}{2}} = 0$$

29. $\lim\limits_{x \to 2^-} \dfrac{\sqrt{4 - x^2}}{\sqrt{6 - 5x + x^2}}$ (1)

Sol. (1) may be written as

$$\lim_{x \to 2^-} \sqrt{\frac{(2 - x)\,(2 + x)}{(2 - x)\,(3 - x)}} = \lim_{x \to 2^-} \sqrt{\frac{2 + x}{3 - x}} = \frac{\sqrt{4}}{1} = 2$$

30. $\lim\limits_{x \to \infty} \dfrac{x + \sin x}{x}$

Sol. $$\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \frac{x}{x} + \lim_{x \to \infty} \frac{\sin x}{x}$$

$$= \lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{\sin x}{x} = 1 + 0,$$

$$= 1, \begin{cases}\text{since } \sin x \text{ remains} \\ \text{bounded for all values of } x\end{cases}$$

31. Let $f(x) = \begin{cases} x^2 + 3 & \text{if } x \le 1 \\ x + 1 & \text{if } x > 1 \end{cases}$.

Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$

Sol. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x + 1) = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 + 3) = 4$

32. $f(x) = \begin{cases} 3 & \text{if } x \le -2 \\ -\frac{1}{2}x^2 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$.

Find $\lim_{x\to \pm 2^+} f(x)$ and $\lim_{x\to \pm 2^-} f(x)$

Sol. $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (3) = 3$

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \left(-\frac{1}{2}x^2\right) = -2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \left(-\frac{1}{2}x^2\right) = -2$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3) = 3$

33. Let $f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 2 \\ \sqrt{x + 7} & \text{if } x > 2 \end{cases}$

Find $\lim f(x)$ as $x \to 2$.

Sol. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 1) = 3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sqrt{x + 7} = \sqrt{2 + 7} = 3$

Thus $\lim_{x\to 2^-} = \lim_{x\to 2^+} f(x) = 3 = \lim_{x\to 2} f(x)$

34. Let $f(x) = \begin{cases} \cos x & \text{if } x \le 0 \\ 1 - x & \text{if } x > 0 \end{cases}$.

Find $\lim_{x\to 0} f(x)$.

Sol. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos x = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1 - x) = 1$

Thus, $\lim_{x\to 0} f(x) = 1$

35. Let $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x^3 & \text{if } x > 1 \end{cases}$.

Show that $\lim_{x\to 1} f(x) = 1$

Sol. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2) = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^3) = 1$

Hence, $\lim_{x\to 1} f(x) = 1$

36. Let $f(x) = \begin{cases} x + 2 & \text{if } x \le -1 \\ ax^2 & \text{if } x > -1 \end{cases}$.

Find a so that $\lim_{x\to -1} f(x)$ exists.

Sol. $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x + 2) = 1$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} ax^2 = a$

If $\lim_{x\to -1} f(x)$ exists, we must have

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$$

Therefore, $1 = a$.

37. Evaluate $\lim_{x\to 3^-} \frac{3 - x}{|x - 3|}$

Sol. As $x \to 3^-$, so $|x - 3| = -(x - 3) = 3 - x$,

Therefore, $\lim_{x\to 3^-} \frac{3 - x}{|x - 3|} = \lim_{x\to 3^-} \frac{3 - x}{3 - x} = 1$

38. Evaluate $\lim_{x\to 0^-} \frac{x}{x - |x|}$

Sol. $\lim_{x\to 0^-} \frac{x}{x - |x|} = \lim_{x\to 0^-} \frac{x}{x + x}$ $\quad (|x| = -x \text{ as } x \to 0^-)$

$$= \lim_{x\to 0^-} \frac{x}{2x} = \frac{1}{2}$$

39. Find $\lim\limits_{h\to 0} \dfrac{|-1+h|-1}{h}$

Sol. $\lim\limits_{h\to 0} \dfrac{|-1+h|-1}{h} = \lim\limits_{h\to 0} \dfrac{|-(1-h)|-1}{h} = \lim\limits_{h\to 0} \dfrac{1-h-1}{h} = -1$

40. Evaluate, [...] being the bracket function:

(*i*) $\lim\limits_{x\to 1} [2x](x-1)$ (*ii*) $\lim\limits_{x\to 0} [x][x+1]$

(*iii*) $\lim\limits_{x\to 0} x\left[\dfrac{1}{x}\right]$ (*iv*) $\lim\limits_{x\to 0} x^3\left[\dfrac{1}{x}\right]$

Sol.

(*i*) $\lim\limits_{x\to 1^-} [2x](x-1) = \lim\limits_{x\to 1^-} [2x] \cdot \lim\limits_{x\to 1^-} (x-1) = (1)(0) = 0$

$\lim\limits_{x\to 1^+} [2x](x-1) = \lim\limits_{x\to 1^+} [2x] \cdot \lim\limits_{x\to 1^+} (x-1) = 2\cdot 0 = 0$

Hence $\lim\limits_{x\to 1} [2x](x-1) = 0$

(*ii*) $\lim\limits_{x\to 0^-} [x][x+1] = \lim\limits_{x\to 0^-} [x] \cdot \lim\limits_{x\to 0^-} [x+1] = (-1)(0) = 0$

$\lim\limits_{x\to 0^-} [x][x+1] = \lim\limits_{x\to 0^+} [x] \cdot \lim\limits_{x\to 0^+} [x+1] = (0)(1) = 0$

Thus $\lim\limits_{x\to 0} [x][x+1] = 0$

(*iii*) $\lim\limits_{x\to 0} x\left[\dfrac{1}{x}\right]$

In general, $\dfrac{1}{x} - 1 \le \left[\dfrac{1}{x}\right] \le \dfrac{1}{x}$, by definition of the bracket function.

For $x < 0$, $1 = x\left(\dfrac{1}{x}\right) \le x\left[\dfrac{1}{x}\right] \le x\left(\dfrac{1}{x} - 1\right) = 1 - x$

So $\lim\limits_{x\to 0^-} x\left[\dfrac{1}{x}\right] = 1$ by Theorem 1.32 (v)

For $x > 0$, $1 - x = x\left(\dfrac{1}{x} - 1\right) \le x\left[\dfrac{1}{x}\right] \le x\left(\dfrac{1}{x}\right) = 1$

$\lim\limits_{x\to 0^+} x\left[\dfrac{1}{x}\right] = 1$

It follows that $\lim\limits_{x\to 0} x\left[\dfrac{1}{x}\right] = 1$

(*iv*) $\lim\limits_{x\to 0} x^3\left[\dfrac{1}{x}\right]$

In general, $\dfrac{1}{x} - 1 \le \left[\dfrac{1}{x}\right] \le \dfrac{1}{x}$

For $x < 0$, $x^2 = x^3\left(\dfrac{1}{x}\right) \le x^3\left[\dfrac{1}{x}\right] \le x^3\left(\dfrac{1}{x} - 1\right) = x^2 - x^3$

and so $\lim\limits_{x\to 0^-} x^3\left[\dfrac{1}{x}\right] = 0$

For $x > 0$

$x^2 - x^3 = x^3\left(\dfrac{1}{x} - 1\right) \le x3\left[\dfrac{1}{x}\right] \le x^3\left(\dfrac{1}{x}\right) = x^2$

and thus $\lim\limits_{x\to 0^+} x^3\left[\dfrac{1}{x}\right] = 0$

It follows that $\lim\limits_{x\to 0} x^3\left[\dfrac{1}{x}\right] = 0$

Exercise Set 1.3 (Page 42)

Discuss the continuity of the following functions at the indicated points/set (Problems 1 – 7):

1. $f(x) = |x-3|$ at $x = 3$

Sol. $f(3) = |3-3| = 0$

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} |x-3| = \lim\limits_{h\to 0} |3-h-3|$, putting $x = 3-h$

$= \lim\limits_{h\to 0} |-h| = 0$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} |x-3| = \lim\limits_{h\to 0} |3+h-3|$, putting $x = 3+h$

$= \lim\limits_{h\to 0} |h| = 0$

Thus $\lim\limits_{x\to 3^+} f(x)(= \lim\limits_{x\to 3^-} f(x)(= f(3)$

Hence f is continuous at $x = 3$

2. $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \ne 3 \\ 0 & \text{if } x = 3 \end{cases}$ at $x = 3$

Sol. $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = \lim\limits_{x\to 3} (x+3) = 6$

But $f(3) = 0$ (given)

Therefore, $\lim\limits_{x\to 3} f(x) \ne f(3)$

Hence f is discontinuous at $x = 3$

3. $f(x) = \begin{cases} x-4 & \text{if } -1 < x \le 2 \\ x^2 - 6 & \text{if } 2 < x < 5 \end{cases}$ at $x = 2$

Sol. $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x-4) = -2$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x^2 - 6) = -2$

When $x = 2, f(x) = x - 4$

Thus $f(2) = 2 - 4 = -2$

Therefore, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$

Hence the function is continuous at $x = 2$.

4. $f(x) = \begin{cases} \dfrac{x^3 - 27}{x^2 - 9} & \text{if } x \ne 3 \\ 6 & \text{if } x = 3 \end{cases}$ at $x = 3$

Sol. $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \dfrac{x^3 - 27}{x^2 - 9} = \lim\limits_{x \to 3} \dfrac{(x-3)(x^2 + 3x + 9)}{(x-3)(x+3)}$

$= \lim\limits_{x \to 3} \dfrac{x^2 + 3x + 9}{x + 3} = \dfrac{27}{6} = \dfrac{9}{2}$

But $f(3) = 6$ (given)

Thus $\lim\limits_{x \to 3} f(x) \ne f(3)$

Hence the function is discontinuous at $x = 3$.

5. $f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$

Sol. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \sin \dfrac{1}{x} = \sin(-\infty)$ which is not a definite number.

It can have any value in $[-1, 1]$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \sin \dfrac{1}{x} = \sin \infty$, which is also not a definite number.

Hence the limit of $f(x)$ does not exist as $x \to 0$, so the function is discontinuous at $x = 0$.

6. $f(x) = \sin x$ for all $x \in \mathbf{R}$.

Sol. Let θ be any arbitrary real number.

$\lim\limits_{x \to \theta^-} f(x) = \lim\limits_{x \to \theta^-} \sin x = \sin \theta$

$\lim\limits_{x \to \theta^+} f(x) = \lim\limits_{x \to \theta^+} \sin x = \sin \theta$

Also $f(\theta) = \sin \theta$

Thus $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} = f(\theta)$

and so $f(x)$ is continuous at $x = \theta$

Since θ is any real number, $\sin x$ is continuous at every $x \in \mathbf{R}$.

7. $f(x) = \begin{cases} \dfrac{x^2}{a} - a & \text{if } 0 < x < a \\ 0 & \text{if } x = a \\ a - \dfrac{a^2}{x} & \text{if } x > a \end{cases}$ at $x = a$

Sol. $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^-} \left(\dfrac{x^2}{a} - a\right) = 0$

$\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^+} \left(a - \dfrac{a^2}{x}\right) = 0$

Therefore $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} = f(a) = 0$

Hence $f(x)$ is continuous at $x = a$.

8. Determine the points of continuity of the function $f(x) = x - [x]$ for all $x \in \mathbf{R}$.

Sol. Here $[x]$ is the bracket function and stands for the greatest integer not greater than x.

So, when $0 \le x < 1$, $[x] = 0$; $[x] = -1$, if $-1 \le x < 0$

when $1 \le x < 2$, $[x] = 1$; $[x] = -2$, if $-2 \le x - 1$

when $2 \le x < 3$, $[x] = 2$; $[x] = -3$, if $-3 \le x - 2$ etc.

Hence the given function may be defined as

$$f(x) = x \quad \text{in } 0 \le x < 1 \qquad (1)$$
$$= x - 1 \quad \text{in } 1 \le x < 2 \qquad (2)$$
$$= x - 2 \quad \text{in } 2 \le x < 3 \qquad (3)$$

etc.

Also $f(x) = x + 1$ in $-1 \le x < 0$ (4)

$= x + 2$ in $-2 \le x < -1$ (5)

At $x = 1$; $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x - 1) = 0$

Therefore, the limit of $f(x)$ at $x = 1$ does not exist.

Hence, f is not continuous at $x = 1$.

At $x = 0$, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x + 1) = 1$, from (4)

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0$, from (1)

Thus $\lim_{x \to 0} f(x)$ does not exist.

So, the function is discontinuous at $x = 0$
Similarly, it is discontinuous for integral values of x, both positive and negative but it is continuous at every other real value of x.

9. Discuss the continuity of $x - |x|$ at $x = 1$.

Sol. Let $f(x) = x - |x|$

$$\lim_{x \to 1^-} x - |x| = 0$$
$$\lim_{x \to 1^+} x - |x| = 0$$
$$f(1) = 1 - 1 = 0$$

Thus $\lim_{x \to 1} f(x) = f(1)$ and so the function is continuous at $x = 1$.

10. Show that the function $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ 1 - x & \text{if } x \text{ is rational} \end{cases}$$

is continuous at $x = \frac{1}{2}$

Sol. $f\left(\frac{1}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

$$\lim_{x \to 1/2} f(x) = \lim_{x \to 1/2} (1 - x) = \frac{1}{2}$$

Hence the limit of $f(x)$ exists at $x = \frac{1}{2}$ and is equal to the value of

$f(x)$ at $x = \frac{1}{2}$

Thus $f(x)$ is continuous at $x = \frac{1}{2}$

11. Show that the function $f:]\,0, 1\,] \to \boldsymbol{R}$ defined by $f(x) = \frac{1}{x}$ is continuous on $]\,0, 1]$. Is $f(x)$ bounded on this interval? Explain.

Sol. $f(x)$ is defined for all real values of x such that $0 < x \le 1$ and its limit exists at each such x and equals to its value there, so it is continuous on $]0, 1]$.
When $x = 1$, $f(x) = 1$ which is its lower bound.
So it is bounded below. $f(x)$ increases indefinitely as x becomes small. Thus $f(x)$ is not bounded above. Hence $f(x)$ is not bounded on $]0, 1]$.

12. Let $f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

Is f continuous at $x = 0$?

Sol. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos\left(\frac{1}{x}\right)$,

which may be any value in $[-1, 1]$. Thus the limit does not exist.
Similarly, $\lim_{x \to 0^+} f(x)$ does not exist.

Hence $\lim_{x \to 0} f(x)$ does not exist and the function cannot be continuous at $x = 0$.

13. Let $f(x) = \begin{cases} (x - a) \sin\left(\frac{1}{x - a}\right) & \text{if } x \ne a \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = a$.

Sol. Here $|f(x) - f(a)| = \left|(x - a) \sin \frac{1}{x - a} - 0\right|$

$$= \left|(x - a) \sin \frac{1}{x - a}\right| = |x - a| \left|\sin \frac{1}{x - a}\right|$$
$$\le |x - a|, \left[\text{since } \left|\sin \frac{1}{x - a}\right| \le 1\right]$$
$$< \varepsilon \quad \text{if } |x - a| < \delta = \varepsilon$$

Thus by definition, $f(x)$ is continuous at $x = a$.

14. Let $f(x) = \begin{cases} x \cos\left(\frac{1}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

Show that f is continuous at $x = 0$

Sol. Here $|f(x) - f(0)| = \left|x \cos \frac{1}{x} - 0\right| = \left|x \cos \frac{1}{x}\right|$

$$= |x| \left|\cos \frac{1}{x}\right| \le |x|, \left(\text{since } \left|\cos \frac{1}{x}\right| \le 1\right)$$
$$< \varepsilon \text{ if } |x| < \delta = \varepsilon$$

Hence $f(x)$ is continuous at $x = 0$ by definition.

15. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = 0$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

$= 0$ (some number between -1 and 1) $= 0$

$f(0) = 0$

Thus $f(x)$ is continuous at $x = 0$.

16. Let $f(x) = \begin{cases} x \sin\left(\dfrac{|x|}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = 0$

Sol. Now $\lim\limits_{x \to 0^-} x \sin \dfrac{|x|}{x} = \lim\limits_{x \to 0^-} x \lim\limits_{x \to 0^-} \sin \dfrac{-x}{x}$

$$= 0 \,.\, \sin(-1) = 0$$

$$\lim_{x \to 0^+} x \sin \frac{|x|}{x} = \lim_{x \to 0^+} x \lim_{x \to 0^+} \sin \frac{x}{x} = 0 \,.\, \sin 1 = 0$$

Thus $\lim\limits_{x \to 0} f(x) = 0$

Also $f(0) = 0$

Hence f is continuous at $x = 0$

17. Find c such that the function $f(x) = \begin{cases} \dfrac{1-\sqrt{x}}{x-1} & \text{if } 0 \leq x < 1 \\ c & \text{if } x = 1 \end{cases}$

is continuous for all $x \in [0, 1]$.

Sol. Let a be any point of $[0, 1[$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{1-\sqrt{x}}{x-1} = \frac{1-\sqrt{a}}{a-1} = f(a)$$

Thus f is continuous at a and since a is an arbitrary point of $[0, 1[$, f is continuous on $[0, 1[$. In order that f be continuous at the point $x = 1$, we must have $\lim\limits_{x \to 1} f(x) = f(1) = c$

Now $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \dfrac{1-\sqrt{x}}{x-1} = \lim\limits_{x \to 1^-} \dfrac{-(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = -\dfrac{1}{2}$

Thus f is continuous at $x = 1$ if $c = -\dfrac{1}{2}$.

Hence f is continuous on $[0, 1]$ for $c = -\dfrac{1}{2}$.

In Problems 18 – 20, find the points of discontinuity of the given functions.

18. $f(x) = \begin{cases} x + 4 & \text{if } -6 \leq x < -2 \\ x & \text{if } -2 \leq x < 2 \\ x - 4 & \text{if } 2 \leq x < 6 \end{cases}$

Sol. $\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} (x + 4) = 2$

$\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} (x) = -2$

Thus $\lim\limits_{x \to -2^-} f(x) \neq f(x) \lim\limits_{x \to -2^+}$ and so the function is discontinuous at $= -2$.

Again $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} x = 2$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x - 4) = -2$

Thus $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$ and therefore the function is discontinuous at $x = 2$.

The points of discontinuity of f are $x = -2, 2$.

19. $g(x) = \begin{cases} x^3 & \text{if } x < 1 \\ -4 - x^2 & \text{if } 1 \leq x \leq 10 \\ 6x^2 + 46 & \text{if } x > 10 \end{cases}$

Sol. $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^-} x^3 = 1$

$\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^+} (-4 - x^2) = -5$

Thus $\lim\limits_{x \to 1} g(x)$ does not exist and so g is discontinuous at $x = 1$

Next, we find $\lim g(x)$ as $x \to 10$

$$\lim_{x \to 10^-} g(x) = \lim_{x \to 10^-} (-4 - x^2) = -104$$

$$\lim_{x \to 10^+} g(x) = \lim_{x \to 10^+} (6x^2 + 46) = 646$$

Hence $\lim\limits_{x \to 10} g(x)$ does not exist and so the function is discontinuous at $x = 10$

20. $f(x) = \begin{cases} x + 2 & \text{if } 0 \leq x < 1 \\ x & \text{if } 1 \leq x < 2 \\ x + 5 & \text{if } 2 \leq x < 3 \end{cases}$

Sol. We check the continuity of f at $x = 1$ and $x = 2$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 2) = 3$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x = 1$$

Thus $\lim\limits_{x \to 1} f(x)$ does not exist and therefore the function is discontinuous at $x = 1$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x = 2$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x + 5) = 7$$

Therefore, $\lim_{x \to 2} f(x)$ does not exist.

The function is also discontinuous at $x = 2$.

21. Find constants a and b such that the function f defined by

$$f(x) = \begin{cases} x^3 & \text{if } 0 < -1 \\ ax + b & \text{if } -1 \le x < 1 \\ x^2 + 2 & \text{if } x \ge 1 \end{cases} \quad \text{is continuous for all } x.$$

Sol. It is easy to see that the given function is continuous for all x possibly except at $x = -1$ and 1.

Now $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^3 = -1$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (ax + b) = -a + b$

If the function is continuous at $x = -1$, we must have

$$-a + b = -1 \quad (1)$$

Again $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (ax + b) = a + b$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 2) = 3$

For continuity of f at $x = 1$, we must have

$$a + b = 3 \quad (2)$$

Solving (1) and (2) simultaneously, we obtain $a = 2, b = 1$.

Find the interval on which the given function is continuous. Also find points where it is discontinuous. (Problems 22 – 26).

22. $f(x) = \dfrac{x^2 - 5}{x - 1}$

Sol. The function $f(x) = \dfrac{x^2 - 5}{x - 1}$ is not defined at $x = 1$. Thus $f(x)$ is discontinuous at $x = 1$.

The numerator $x^2 - 5$ is continuous at every point of $\boldsymbol{R}$ and so is the denominator $x - 1$. Hence $f(x)$ is continuous at every point of $\boldsymbol{R} - \{1\}$.

23. $f(x) = \dfrac{x}{|x|}$

Sol. $f(x)$ is not defined at $x = 0$ and so it is discontinuous at $x = 0$. The function is continuous at every other point of $\boldsymbol{R}$.

24. $f(x) = \dfrac{\sin x}{x}$

Sol. The function $\dfrac{\sin x}{x}$ is not defined at $x = 0$. Hence it is discontinuous at $x = 0$. The function is continuous at every other point of $\boldsymbol{R}$ since $\sin x$ and x are continuous on $\boldsymbol{R}$.

25. $f(x) = \tan x$

Sol. $f(x) = \dfrac{\sin x}{\cos x}$

The function is not defined at $x = (2n + 1)\dfrac{\pi}{2}$, where n is an integer. Thus, $f(x)$ is discontinuous at these points, $f(x)$ is continuous on all other points of $\boldsymbol{R}$.

26. $f(x) = \begin{cases} \sin x & \text{if } x \le \pi/4 \\ \cos x & \text{if } x > \pi/4 \end{cases}$

Sol. $\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} \sin x = \dfrac{1}{\sqrt{2}}$

$\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^+} \cos x = \dfrac{1}{\sqrt{2}}$

More-over, $f\left(\dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

The function is continuous at $x = \dfrac{\pi}{4}$. We also know that $\sin x$ and $\cos x$ are continuous at every point of $\boldsymbol{R}$. Hence $f(x)$ is continuous at every point of $\boldsymbol{R}$.

In Problems 27 – 34, examine whether the given function is continuous at $x = 0$

27. $f(x) = \begin{cases} (1 + 3x)^{1/x} & \text{if } x \ne 0 \\ e^2 & \text{if } x = 0 \end{cases}$

Sol. $f(x) = (1 + 3x)^{1/x}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} [(1 + 3x)^{1/3x}]^3 = e^3$

$f(0) = e^2$

Since $e^3 \ne e^2$, $f(x)$ is discontinuous at $x = 0$

28. $f(x) = \begin{cases} (1 + x)^{1/x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} (1 + x)^{1/x} = e$ but $f(0) = 1$

Since $e \neq 1$, $f(x)$ is discontinuous $x = 0$

29. $f(x) = \begin{cases} (1+2x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$

Sol. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} (1+2x)^{1/x} = [\lim\limits_{x \to 0} (1+2x)^{1/2x}]^2 = e^2$

And $f(0) = e^2$

Thus $f(x)$ is continuous at $x = 0$

30. $f(x) = \begin{cases} (e^{-1/x}) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} e^{-1/x^2} = \lim\limits_{x \to 0} \dfrac{1}{e^{1/x^2}} = 0$

But $f(0) = 1$, so $f(x)$ is discontinuous at $x = 0$

31. $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. When $x \to 0^-, \dfrac{1}{x} \to -\infty$ and so $e^{1/x} \to e^{-\infty} = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{e^{1/x}}{1+e^{1/x}} = \frac{0}{1+0} = 0$$

As $x \to 0^+, \dfrac{1}{x} \to \infty$ and so $e^{1/x} \to e^{\infty} = \infty$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{1/x}}{1+e^{1/x}} = \frac{\infty}{\infty}$$

Thus $\lim\limits_{x \to 0} f(x)$ does not exist.

Hence $f(x)$ is discontinuous at $x = 0$.

32. $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} = \lim\limits_{x \to 0} \dfrac{1}{1-\dfrac{1}{e^{1/x^2}}} = 1$

$f(0) = 1$

Thus $\lim\limits_{x \to 0} f(x) = f(0)$ and so $f(x)$ is continuous at $x = 0$

33. $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{\sin 2x}{2x} . 2 = 1.2 = 2$

$f(0) = 1$

Since $\lim\limits_{x \to 0} f(x) \neq f(0)$, $f(x)$ is discontinuous at $x = 0$

34. $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin 2x} & \text{if } x \neq 0 \\ \dfrac{2}{3} & \text{if } x = 0 \end{cases}$

Sol. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{\sin 3x}{3x} \cdot \dfrac{2x}{\sin 2x} \times \dfrac{3}{2}$

$= 1 \times \dfrac{3}{2} = \dfrac{3}{2}$

$f(0) = \dfrac{2}{3}$

Thus $\lim\limits_{x \to 0} f(x) \neq f(0)$ and so $f(x)$ is discontinuous at $x = 0$.

35. Let $f(x) = x^2$ and $g(x) = \begin{cases} -4 & \text{if } x \leq 0 \\ |x-4| & \text{if } x > 0 \end{cases}$

Determine whether *fog* and *gof* are continuous at $x = 0$

Sol. $(fog)(x) = f(g(x))$

$= f(-4)$, if $x \leq 0$

$= f(|x-4|)$, if $x > 0$

Thus $(fog)(x) = 16$, if $x \leq 0$

$= (x-4)^2$, if $x > 0$

Now $\lim\limits_{x \to 0^-} (fog)(x) = 16$

$\lim\limits_{x \to 0^+} (fog)(x) = \lim\limits_{x \to 0^+} (x-4)^2 = 16$

$(fog)(0) = 16$

Thus *fog* is continuous at $x = 0$

Again, $(gof)\,x = g(f(x)) = g(x^2)$

$= -4$, if $x^2 \leq 0$

$= |x^2 - 4|$, if $x^2 > 0$

$\lim\limits_{x \to 0^-} (gof)(x) = -4$

$\lim\limits_{x \to 0^+} (gof)(x) = \lim\limits_{x \to 0^+} |x^2 - 4| = 4$

Thus $\lim\limits_{x \to 0} (gof)(x)$ does not exist and so *gof* is discontinuous at $x=0$.

Chapter 2 THE DERIVATIVE

Exercise Set 2.1 (Page 51)

1. Show that the function $f(x) = |x| + |x-1|$ is continuous for every value of x but is not differentiable at $x = 0$ and $x = 1$.

Sol. Let a be an arbitrary real number. We check the continuity of $f(x)$ at $x = a$.

$$\lim_{x \to a} f(x) = \lim_{x \to a} (|x| + |x-1|)$$
$$= \lim_{x \to a} |x| + \lim_{x \to a} |x-1|$$
$$= |a| + |a-1|.$$
$$f(a) = |a| + |a-1|.$$

Thus $f(a) = \lim_{x \to a} f(x)$ and so $f(x)$ is continuous. But a is any real number. Therefore, the given function is continuous for every real value of x.

Now $$Lf'(0) = \lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \frac{|x| + |x-1| - |-1|}{x}$$
$$= \lim_{x \to 0^-} \left(\frac{-x}{x} + \frac{-x+1}{x} - \frac{1}{x}\right) = \lim_{x \to 0^-} \left(-1 - 1 + \frac{1}{x} - \frac{1}{x}\right)$$
$$= -2$$
$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^+} \frac{|x| + |x-1| - |-1|}{x}$$
$$= \lim_{x \to 0^+} \left(\frac{x}{x} + \frac{1-x}{x} - \frac{1}{x}\right) = 0$$

Thus $Lf'(0) \neq Rf'(0)$ and so the function is not differentiable at $x = 0$

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^-} \frac{|x| + |x-1| - 1}{x-1}$$
$$= \lim_{x \to 1^-} \frac{x+1-x-1}{x-1} = 0$$
$$Rf'(1) = \lim_{x \to 1^+} \frac{|x| + |x-1| - 1}{x-1} = \lim_{x \to 1^+} \frac{x+x-1-1}{x-1}$$
$$= \lim_{x \to 1^+} \frac{2(x-1)}{x-1} = 2$$

Therefore, $Lf'(1) \neq Rf'(1)$ and so the function is not differentiable at $x = 1$.

2. Let $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2x-1 & \text{if } 1 < x \le 2. \end{cases}$

Discuss the continuity and differentiability of f at $x = 1$.

Sol. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x-1) = 1$

Also $f(1) = 1$

Thus Left hand limit = Right hand limit

= Value of $f(x)$ at $x = 1$

Hence $f(x)$ is continuous at $x = 1$

For its differentiability at $x = 1$, we have

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^-} \frac{x-1}{x-1} = 1$$

and $$Rf'(1) = \lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^+} \frac{2x-1-1}{x-1} = 2$$

Thus $Lf'(1) \neq Rf'(1)$

Hence f is not differentiable at $x = 1$.

3. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Show that f is continuous and differentiable at $x = 0$.

Sol. $$|f(x) - f(0)| = \left|x^2 \sin\frac{1}{x} - 0\right| = \left|x^2 \sin\frac{1}{x}\right|$$
$$\le x^2 < \delta^2 = \varepsilon, \left(\text{since } \left|\sin\frac{1}{x}\right| \le 1\right)$$

whenever $|x| < \delta$

Thus $f(x)$ is continuous at $x = 0$

Now, $$Rf'(0) = \lim_{h \to 0} \frac{(0+h)^2 \sin\left|\frac{1}{0+h}\right| - 0}{h}$$
$$= \lim_{h \to 0} h \sin\frac{1}{h} = 0$$
$$Lf'(0) = \lim_{h \to 0} \frac{(0-h)^2 \sin\frac{1}{0-h} - 0}{-h} = \lim_{h \to 0} \frac{h^2 \sin\left(-\frac{1}{h}\right)}{-h}$$

$$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$

Thus $Rf'(0) = Lf'(0) = 0$

Hence f is differentiable at $x = 0$

4. Is the function f defined by $f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & \text{if } x \neq a \\ 0 & \text{if } x = a. \end{cases}$

continuous and differentiable at $x = a$?

Sol. $|f(x) - f(a)| = \left|(x-a)\sin\frac{1}{x-a} - 0\right|$

$$= \left|(x-a)\sin\frac{1}{x-a}\right| = |x-a|\left|\sin\frac{1}{x-a}\right|$$

$$\leq |x-a|, \left\{\text{since } \left|\sin\frac{1}{x-a}\right| \leq 1\right\}$$

$$< \delta = \varepsilon, \text{ whenever } |x-a| < \delta$$

Hence the function is continuous at $x = a$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{h\sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} \sin\frac{1}{h}, \text{ which does not exist.}$$

Hence the function is not differentiable at $x = a$.

5. Let $f(x) = \begin{cases} x \text{ actan}\left(\frac{1}{x}\right) & \text{if } x \neq a \\ 0 & \text{if } x = 0. \end{cases}$

Discuss the continuity and differentiability of f at $x = 0$.

Sol. $|f(x) - f(0)| = \left|x \arctan\frac{1}{x} - 0\right| = \left|x\arctan\frac{1}{x}\right|$

$$= |x|\left|\arctan\frac{1}{x}\right| < \varepsilon\frac{\pi}{2}, \text{ wherever } |x| < \delta$$

Hence $f(x)$ is continuous at $x = 0$,

$$f'(0) = \lim_{h \to 0} \frac{(h)\arctan\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h \to 0} \arctan\left(\frac{1}{h}\right), \text{ which does not exist.}$$

Hence $f(x)$ is not differentiable at $x = 0$.

6. Find $Lf'(2)$ and $Rf'(2)$ for the function $f(x) = |x^2 - 4|$.

Sol. $Lf'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{|x^2 - 4|}{x - 2}$

$$= \lim_{x \to 2^-} \frac{|x-2|\,(x+2)}{x-2} = \lim_{x \to 2^-} \frac{(2-x)(x+2)}{x-2} = -4$$

Similarly,

$$Rf'(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2^+} \frac{|x^2-4|}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(x-2)(x+2)}{x-2} = 4$$

Thus $Lf'(2) \neq Rf'(2)$ and so f is not differentiate at $x = 2$

7. Find the values of a and b so that the function f is continuous and differentiable at $x = 1$, where

$$f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax + b & \text{if } x \geq 1. \end{cases}$$

Sol. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3 = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (ax + b) = a + b$

For continuity at $x = 1$, we must have

$$f(1) = a + b = 1 \qquad (1)$$

Now, $Lf'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^3 - 1}{x - 1}$

$$= \lim_{x \to 1^-} \frac{(x-1)(x^2+x+1)}{x-1} = 3$$

$$Rf'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^+} \frac{ax + b - (a+b)}{x-1} = a$$

For differentiability at $x = 1$, we must have

$Lf'(1) = Rf'(1)$, i.e., $a = 3$

Now we have

$a + b = 1$ and $a = 3$

Therefore, $b = 1 - 3 = -2$

8. Let $f(x) = \begin{cases} \sin 2x & \text{if } 0 < x \leq \frac{\pi}{6} \\ ax + b & \text{if } \frac{\pi}{6} < x \leq 1. \end{cases}$

Derive the values of a and b so that f is continuous and differentiable at $x = \frac{\pi}{6}$.

Sol. $\lim_{x \to \frac{\pi}{6}^-} f(x) = \lim_{x \to \frac{\pi}{6}^-} \sin 2x = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\lim_{x \to \frac{\pi}{6}^+} f(x) = \lim_{x \to \frac{\pi}{6}^+} ax + b = \frac{\pi}{6}a + b$$

For continuity at $x = \frac{\pi}{6}$, we must have

$$\frac{\pi}{6}a + b = \frac{\sqrt{3}}{2} = f\left(\frac{\pi}{6}\right)$$

For differentiability at $x = \frac{\pi}{6}$, we have

$$Lf'\left(\frac{\pi}{6}\right) = \lim_{x \to \frac{\pi}{6}^-} \frac{f(x) - f\left(\frac{\pi}{6}\right)}{x - \frac{\pi}{6}} = \lim_{x \to \frac{\pi}{6}^-} \frac{\sin 2x - \sin\left(2 \cdot \frac{\pi}{6}\right)}{x - \frac{\pi}{6}}$$

$$= \lim_{x \to \frac{\pi}{6}^-} \frac{2\cos\left(x + \frac{\pi}{6}\right)\sin\left(x - \frac{\pi}{6}\right)}{x - \frac{\pi}{6}} = 2\cos\frac{\pi}{3} = 1$$

$$Rf'\left(\frac{\pi}{6}\right) = \lim_{x \to \frac{\pi}{6}^+} \frac{f(x) - f\left(\frac{\pi}{6}\right)}{x - \frac{\pi}{6}} = \lim_{x \to \frac{\pi}{6}^+} \frac{ax + b - a\frac{\pi}{6} - b}{x - \frac{\pi}{6}} = a$$

Thus $f(a)$ is differentiable at $x = \frac{\pi}{6}$ if $a = 1$

Now we have, $\frac{\pi}{6}a + b = \frac{\sqrt{3}}{2}$ and $a = 1$

Therefore, $a = 1$ and $b = \frac{\sqrt{3}}{2} - \frac{\pi}{6} = \frac{3\sqrt{3} - \pi}{6}$

9. Let $f(x) = x \tanh\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Discuss the continuity and differentiability of f at $x = 0$.

Sol. Here $f(x) = x \tanh\frac{1}{x} = x\,\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = x\,\frac{e^{2/x} - 1}{e^{2/x} + 1}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x \cdot \frac{e^{2/x} - 1}{e^{2/x} + 1}\right) = 0 \cdot (-1) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x\,\frac{e^{2/x} - 1}{e^{2/x} + 1} = \lim_{x \to 0^+} x\,\frac{1 - e^{-2/x}}{1 + e^{-2/x}} = 0.1 = 0$$

Hence f is continuous at $x = 0$

For differentiability, we have

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{e^{2/x} - 1}{e^{2/x} + 1} = -1$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{1 - e^{-2/x}}{1 + e^{-2/x}} = 1$$

$f'(0)$ does not exist, so f is not differentiable at $x = 0$.

Find the slope of the tangent line to the given curve at the indicated point (Problems 10 – 12):

10. $y = x^2$ at $(2, 4)$

Sol. $y = x^2$

$$\frac{dy}{dx} = 2x$$

$\left[\frac{dy}{dx}\right]_{(2,4)} = 4$, which is the required slope.

11. $y = \frac{1}{x}$ at $(1, 1)$

Sol. $\frac{dy}{dx} = -\frac{1}{x^2}$

$\left[\frac{dy}{dx}\right]_{(1,1)} = -\frac{1}{(1)^2} = -1$, which is the required slope.

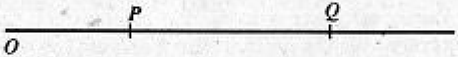
12. $y = x^2 - 7x + 3$ at $(7, 3)$

Sol. $\frac{dy}{dx} = 2x - 7$

$\left[\frac{dy}{dx}\right]_{(7,3)} = 14 - 7 = 7$, which is the required slope.

13. Let v be the velocity of a particle at any given time t. Deduce that the acceleration of the particle at this instant is $\frac{dv}{dt}$.

Sol. Let v be the velocity at P after time t and $v + \delta v$ be the velocity at Q after a time $t + \delta t$.



The change in velocity $= \delta v$; Change in time $= \delta t$

$$\text{Acceleration} = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}$$

Find the velocity and acceleration at $t = 0, 1, 2$ in each of the following (Problems 14 – 16):

14. $s = \dfrac{1}{t+1}$

Sol. $\dfrac{ds}{dt} = -\dfrac{1}{(t+1)^2} = v$ (1)

$v_0 = -\dfrac{1}{(0+1)^2} = -1$; $v_1 = -\dfrac{1}{(1+1)^2} = -\dfrac{1}{4}$; $v_2 = \dfrac{1}{(2+1)^2} = -\dfrac{1}{9}$

From (1), $a = \dfrac{d^2s}{dt^2} = \dfrac{2}{(t+1)^3}$

Let a_0, a_1, a_2 be the accelerations at $t = 0, 1, 2$ respectively. Then

$$a_0 = \frac{2}{1} = 2 \; ; \; a_1 = \frac{2}{(1+1)^3} = \frac{2}{8} = \frac{1}{4} \; ; \; a_2 = \frac{2}{(2+1)^3} = \frac{2}{27}$$

15. $s = t^2 + 2t + 5$

Sol. $v = \dfrac{ds}{dt} = 2t + 2$; $v_0 = 2$

$v_1 = 2(1) + 2 = 4$; $v_2 = 2(2) + 2 = 6$

$a = \dfrac{d^2s}{dt^2} = 2$; $a_0 = 2, a_1 = 2, a_2 = 2$

16. $s = t^2(t-1)$

Sol. $v = \dfrac{ds}{dt} = 3t^2 - 2t$; $v_0 = 0$

$v_1 = 3 - 2 = 1$; $v_2 = 3(2)^2 - 2(2) = 12 - 4 = 8$

$a = \dfrac{d^2s}{dt^2} = 6t - 2$; $a_0 = -2$

$a_1 = 6(1) - 2 = 4$; $a_2 = 6(2) - 2 - 10$

17. A point moves in a straight line so that its distance s (in metres) after time t (in seconds) is $s = 4t^2 - 16t + 12$. Find

(*i*) the average velocity in the interval $[1, 1 + \Delta t]$

(*ii*) the velocity at $t = 1$.

Sol. The average velocity $\Delta s/\Delta t$ in the interval $[1, 1 + \Delta t]$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(1+\Delta t) - s(1)}{1 + \Delta t - 1}$$

$$= \frac{4(1+\Delta t)^2 - 16(1+\Delta t) + 12 - 4 + 16 - 12}{\Delta t}$$

$$= \frac{4\Delta t(\Delta t - 2)}{\Delta t} = 4\Delta t - 8$$

The velocity at $t = 1$ is

$$\left.\frac{ds}{dt}\right]_{t=1} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} (4\Delta t - 8) = -8$$

i.e., -8 m/sec.

18. The position of a body (in feet) at time t seconds is $s = t^3 - 6t^2 + 9t$. Find the body's acceleration each time its velocity is zero.

Sol. $s = t^3 - 6t^2 + 9t$

$\dfrac{ds}{dt} = 3t^2 - 12t + 9$ (1)

$\dfrac{d^2s}{dt^2} = 6t - 12$ (2)

Velocity of the body is zero if

$3t^2 - 12t + 9 = 0$ or $3(t-1)(t-3) = 0$

i.e., $t = 1, 3$

Acceleration of the body

(*i*) when $t = 1$ is $\left.\dfrac{d^2s}{dt^2}\right]_{t=1} = 6 - 12 = -6$

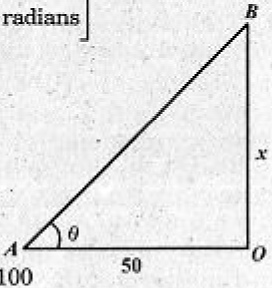
i.e., -6 ft/sec^2

(*ii*) when $t = 3$ is $\left.\dfrac{d^2s}{dt^2}\right]_{t=3} = 18 - 12 = 6$

i.e., 6 ft/sec^2

19. A ladder is placed 50 metres from a wall at an angle θ with the horizontal. Top of the ladder is x metres above the ground. If the bottom of the ladder is pushed toward the wall, find the rate of change of x with respect to θ when $\theta = 45°$.

$\left[\textbf{Hint: } \dfrac{d}{d\theta}(\tan\theta) = \sec^2\theta \quad \text{if } \theta \text{ is in radians}\right]$



Sol. Let θ be in radians.

From the figure, we have

$\tan\theta = \dfrac{x}{50}$ or $x = 50\tan\theta$

Rate of change of x w.r.t. θ is

$$\frac{dx}{d\theta} = 50\sec^2\theta$$

$$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{4}} = 50\left(\sec\frac{\pi}{4}\right)^2 = 50 \times 2 = 100$$

i.e., 100 m/radian $= \left(\dfrac{100}{\frac{180}{\pi}}\right)$ m/deg $= \dfrac{100 \times \pi}{180}$ m/deg ≈ 1.75 m/deg

20. The number of litres of water in a tank, t minutes after the water starts draining out of the tank, is given by $f(t) = 200\,(300 - t)^2$

(*i*) What is the average rate at which the water flows out during the first 5 minutes?

(*ii*) How fast is the water running out at the end of 5 minutes?

Sol. Average rate at which the water flows out during the first five minutes

$$= \frac{f(5) - f(1)}{5 - 1} = \frac{200(30 - 5)^2 - 200(30 - 1)^2}{4}$$

$$= \frac{200(25^2 - 29^2)}{4} = -50 \times 216 = -10800$$

i.e., 10800 lit/min.

$$f'(t) = -400(30 - t)$$

The rate at which water runs out after 5 min

$$= f'(5) = -400(30 - 5) = -400 \times 25 = -10000$$

i.e., 10000 lit./min.

21. The heights (in feet) of a rocket t seconds after its launching, is given by $s = -t^3 + 96t^2 + 195t + 10,\ (t \geq 0)$.

Find

(*i*) the velocity of the rocket at any time t.

(*ii*) the velocity of the rocket when $t = 0, 30, 70$ seconds. Interpret the results

(*iii*) the maximum height attained by the rocket.

Sol.

(*i*) The velocity v of the rocket at any time t is

$$v = \frac{ds}{dt} = -3t^2 + 192t + 195$$

(*ii*) The velocity when $t = 0$ is 195 ft/sec.

When $t = 30$, $v = -3(30)^2 + 192.30 + 195 = 3255$

When $t = 50$, $v = -3(50)^2 + 192.50 + 195 = 2295$

When $t = 70$, $v = -3(70)^2 + 192.70 + 195 = -1065$

Interpretations:

The velocity with which rocket is launched = 195 ft/sec.

At $t = 30$ sec, it accelerates to 3255 ft/sec.

After 50 sec, the velocity is 2295 ft/sec. which is less than the velocity at $t = 30$. Thus the velocity is decreasing after some time.

The velocity is 0 if

$$-3t^2 + 192t + 195 = 0$$

i.e., $t^2 - 64t - 65 = 0$

or $t = 65, -1$

After 65 sec. of its fight, the velocity of the rocket is 0 and after 70 sec. it is -1065, which means it is coming back to the earth with a velocity of 1065 ft/sec. at this instant.

(*iii*) The maximum height is attained when the rocket has velocity 0, *i.e.,* at $t = 65$.

Maximum height attained

$$= -(65)^3 + 96(65)^2 + 195(65) + 10$$

$$= 143660 \quad \text{i.e.,} \quad 143660 \text{ ft.}$$

22. The rupee cost $C(x)$ of producing x washing machines is given by

$$C(x) = 2000 + 100x - 0.1x^2$$

(*i*) Find the marginal cost at $x = 100$.

(*ii*) Show that the marginal cost at $x = 100$ is approximately the cost of producing the 101st washing machine.

Sol.

(*i*) $C(x) = 2000 + 100x - 0.1x^2$

$C'(x) = 100 - 0.2x$

Marginal cost at $x = 100$ is

$$C'(100) = 100 - 0.2(100) = 100 - 20 = 80$$

i.e., Rs. 80

(*ii*) Cost of producing the 101st washing machine is

$$C(101) - C(100)$$

$$= [2000 + 100 \times 101 - 0.1\,(101)^2] - [2000 + 100 \times 100 - 0.1\,(100)^2]$$

$$= (2000 + 10100 - 1020.10) - (2000 + 10000 - 1000.00)$$

$$= 1079.9 - 1000 = 79.90$$

i.e., Rs. 79.90 $\approx$ Rs. 80

Thus marginal cost at $x = 100$ is approximately the cost of producing the 101st washing machine.

23. The revenue $R(x)$ (in rupees) from selling x units of desks is given by

$$R(x) = 2000\left(1 - \frac{1}{x + 2}\right).$$

(*i*) Find the marginal revenue when x number of desks are sold.

(*ii*) Use $R'(x)$ to estimate the increase in revenue that will result by selling the 9th desk.

Sol.

(*i*) Marginal revenue when x desks are sold

$$R'(x) = 2000\left(\frac{1}{(x + 2)^2}\right) = \frac{2000}{(x + 2)^2}$$

(*ii*) Approximate increase in revenue that will result by selling the 9th desk

$$= R'(8) = \frac{2000}{10^2} = 20 \quad i.e., \quad \text{Rs. } 20.$$

24. The cost $C(x)$ (in rupees) of producing x units of fans is

$$C(x) = 100x + 200000$$

and the revenue $R(x)$ (in rupees) of selling these x number of fans is $\quad R(x) = -0.02x^2 + 400x.$

Find the profit function $P(x)$ and the marginal profit at $x = 2000$. Calculate the actual profit realized from the sale of 2001 st fan.

Sol. Profit function $P(x)$ is given by

$$P(x) = R(x) - C(x)$$
$$= -0.02x^2 + 400x - 100x - 200000$$
$$= -0.02x^2 + 300x - 200000$$

Marginal profit is

$$P'(x) = -0.04x + 300$$

Marginal profit at $x = 2000$ is

$$P'(2000) = -0.04\,(2000) + 300$$
$$= -80.00 + 300 = 220 \quad \Rightarrow \quad i.e., \text{ Rs. } 220.$$

Actual profit realized from the sale of 2001st fan

$$= P(2001) - P(2000)$$
$$= [-0.02(2001)^2 + 300(2001) - 200000]$$
$$- [-0.02(2000)^2 + 300(2000) - 200000]$$
$$= -0.02[(2001)^2 - (2000)^2] + 300$$
$$= -0.02(4001) + 300$$
$$= -80.02 + 300 = 219.98 \quad i.e., \text{ Rs. } 219.98$$

Thus the marginal profit at $x = 2000$ is approximately equal to the profit realized from the sale of 2001 st fan.

Exercise Set 2.2 (Page 66)

Differentiate with respect to x, (Problems 1 – 1[illegible]):

1. $\sqrt{a^2 + x^2}$

Sol. Let $\quad y = \sqrt{a^2 + x^2} = (a^2 + x^2)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}(a^2 + x^2)^{-1/2}\frac{d}{dx}(a^2 + x^2)$$
$$= \frac{1}{2}(a^2 + x^2)^{-1/2}(2x) = \frac{x}{\sqrt{a^2 + x^2}}$$

2. $\sqrt[3]{x^2 + x + 1}$

Sol. Let $\quad y = (x^2 + x + 1)^{1/3}$

$$\frac{dy}{dx} = \frac{1}{3}(x^2 + x + 1)^{-2/3}\frac{d}{dx}(x^2 + x + 1)$$
$$= \frac{1}{3}(x^2 + x + 1)^{-2/3}(2x + 1) = \frac{2x + 1}{3(x^2 + x + 1)^{2/3}}$$

3. $\dfrac{\sqrt{a + x} - \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}$

Sol. Let $\quad y = \dfrac{\sqrt{a + x} - \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}$

$$= \frac{(\sqrt{a + x} - \sqrt{a - x})\,(\sqrt{a + x} - \sqrt{a - x})}{(\sqrt{a + x} - \sqrt{a - x})\,(\sqrt{a + x} - \sqrt{a - x})}, \text{ (rationalizing)}$$
$$= \frac{(a + x) + (a - x) - 2(\sqrt{a^2 - x^2})}{(a + x) - (a - x)}$$
$$= \frac{2a - 2\sqrt{a^2 - x^2}}{2x} = \frac{a - \sqrt{a^2 - x^2}}{x}$$

$$\frac{dy}{dx} = \frac{x\left[0 - \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)\right] - [a - \sqrt{a^2 - x^2}]\,.\,1}{x^2}$$

$$= \frac{x\,.\,\dfrac{x}{\sqrt{a^2 - x^2}} - a + \sqrt{a^2 - x^2}}{x^2}$$

$$= \frac{\dfrac{x^2}{\sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2} - a}{x^2} = \frac{\dfrac{x^2 + a^2 - x^2}{\sqrt{a^2 - x^2}} - a}{x^2}$$

$$= \frac{\dfrac{a^2}{\sqrt{a^2 - x^2}} - a}{x^2} = \frac{a^2 - a\sqrt{a^2 - x^2}}{x^2\sqrt{a^2 - x^2}} = \frac{a(a - \sqrt{a^2 - x^2})}{x^2\sqrt{a^2 - x^2}}$$

4. $\dfrac{\sqrt{\sin x}}{\sin\sqrt{x}}$

Sol. $\dfrac{dy}{dx} = \dfrac{\sin\sqrt{x}\left\{\frac{1}{2}(\sin x)^{-1/2}\cos x\right\} - \sqrt{\sin x}\left\{\cos\sqrt{x}\left(\frac{1}{2}\right)x^{-1/2}\right\}}{(\sin\sqrt{x})^2}$

$$= \frac{\sin\sqrt{x}\,.\,\dfrac{\cos x}{2\sqrt{\sin x}} - \dfrac{\sqrt{\sin x}\,.\,\cos\sqrt{x}}{2\sqrt{x}}}{(\sin\sqrt{x})^2}$$

$$= \frac{\dfrac{\sqrt{x}\sin\sqrt{x}\cos x - \sin x\cos\sqrt{x}}{2\sqrt{x}\sqrt{\sin x}}}{(\sin\sqrt{x})^2}$$

$$= \frac{\sqrt{x}\sin\sqrt{x}\cos x - \sin x\cos\sqrt{x}}{2\sqrt{x}\sqrt{\sin x}\sin^2\sqrt{x}}$$

5. $\sqrt{\log_{10}(x^2+1)}$

Sol. Let $y = \sqrt{\log_{10}(x^2+1)} = \sqrt{\dfrac{\ln(x^2+1)}{\ln 10}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\ln 10}} \cdot \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{(2x)}{(x^2+1)}$$

$$= \frac{x}{\sqrt{\ln 10}\,(x^2+1)\sqrt{\ln(x^2+1)}}$$

6. $\tan(\sin x)$

Sol. $\dfrac{dy}{dx} = \sec^2(\sin x)\,.\,\dfrac{d}{dx}(\sin x) = \sec^2(\sin x)\,.\,\cos x$

$$= \cos x\,.\,\sec^2(\sin x)$$

7. $\arctan\left(\dfrac{x\sin\alpha}{1 - x\cos\alpha}\right)$

Sol. $$\frac{dy}{dx} = \frac{\dfrac{d}{dx}\left(\dfrac{x\sin a}{1-x\cos a}\right)}{1 + \left(\dfrac{x\sin a}{1-x\cos a}\right)^2}$$

$$= \frac{\dfrac{(1-x\cos a)\sin a - x\sin a(-\cos a)}{(1-x\cos a)^2}}{\dfrac{(1-x\cos a)^2 + (x\sin a)^2}{(1-x\cos a)^2}}$$

$$= \frac{\sin a - x\sin a\cos a + x\sin a\cos a}{(1-x\cos a)^2 + (x\sin a)^2}$$

$$= \frac{\sin a}{1 - 2x\cos a + x^2\cos^2 a + x^2\sin^2 a} = \frac{\sin a}{1 - 2x\cos a + x^2}$$

8. $\ln\left(\dfrac{x^2+x+1}{x^2-x+1}\right)$

Sol. $y = \ln(x^2+x+1) - \ln(x^2-x+1)$

$$\frac{dy}{dx} = \frac{1}{x^2+x+1}\frac{d}{dx}(x^2+x+1) - \frac{1}{x^2-x+1}\frac{d}{dx}(x^2-x+1)$$

$$= \frac{2x+1}{x^2+x+1} - \frac{2x-1}{x^2-x+1}$$

$$= \frac{(2x+1)(x^2-x+1) - (2x-1)(x^2+x+1)}{(x^2+x+1)(x^2-x+1)} = \frac{2-2x^2}{1+x^2+x^4}$$

9. x^{x^2}

Sol. $\ln y = x^2\ln x$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = x^2\,.\,\frac{1}{x} + 2x\ln x$$

$$= x + 2x\ln x = x(1 + 2\ln x)$$

$$\frac{dy}{dx} = y\,.\,x(2\ln x + 1)$$

$$= x^{x^2}\,.\,x(2\ln x + 1) = x^{x^2+1}(2\ln x + 1)$$

10. $\ln(x^2+x)$

Sol. Let $y = \ln(x^2+x)$

Differentiating w.r.t, we have

$$\frac{dy}{dx} = \frac{1}{x^2+x}\frac{d}{dx}(x^2+x) = \frac{2x+1}{x^2+x}$$

11. $(\arcsin x)^{x^{1/x}}$

Sol. Taking logarighm of both sides, we have

$$\ln y = x^{1/x}\ln(\arcsin x)$$

Differentiating both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = x^{1/x}\,.\,\frac{\dfrac{1}{\sqrt{1-x^2}}}{\arcsin x} + \ln(\arcsin x)\frac{d}{dx}(x^{1/x}) \qquad (1)$$

Now, let $u = x^{1/x}$ or $\ln u = \dfrac{1}{x}\ln x$

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{x}\,.\,\frac{1}{x} + \ln x\left(-\frac{1}{x^2}\right) = \frac{1}{x^2} + \ln x\left(-\frac{1}{x^2}\right)$$

$$\frac{du}{dx} = u\left(\frac{1}{x^2} - \frac{1}{x^2}\ln x\right) = x^{1/x}\frac{1}{x^2}(1 - \ln x)$$

$$= x^{\frac{1}{x}-2}(\ln e - \ln x) = x^{\frac{1}{x}-2}\ln\frac{e}{x}$$

Putting in (1)

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^{1/x}}{\sqrt{1-x^2}\arcsin x} + x^{\frac{1}{x}-2}\ln\frac{e}{x}\ln(\arcsin x)$$

$$\frac{dy}{dx} = (\arcsin x)^{x^{1/x}} \times \left[x^{\frac{1}{x}-2}\ln\frac{e}{x}\ln(\arcsin x) + \frac{x^{1/x}}{\sqrt{1-x^2}(\arcsin x)}\right]$$

12. $|x^2-9|$

Sol.
$$y = x^2-9, \quad \text{if } |x| \ge 3$$
$$= -x^2+9, \quad \text{if } |x| < 3$$

Therefore, $\dfrac{dy}{dx} = 2x, \quad \text{if } |x| \ge 3$
$$= -2x, \quad \text{if } |x| < 3$$

13. $\sqrt{x+\sqrt{x+\sqrt{x}}}$

Sol.
$$\frac{dy}{dx} = \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2} \times \frac{d}{dx}\left(x+\sqrt{x+\sqrt{x}}\right)$$
$$= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \times \left[1+\frac{1}{2}(x+\sqrt{x})^{-1/2}\frac{d}{dx}(x+\sqrt{x})\right]$$
$$= \frac{1}{2y}\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)\right]$$

14. $(x+|x|)^{1/2}$

Sol. Here
$$y = (x+x)^{1/2} = (2x)^{1/2} \quad \text{if } x > 0$$
$$= (0+0) = 0 \quad \text{if } x = 0$$
$$= (x-x)^{1/2} = 0 \quad \text{if } x < 0$$

Thus $\dfrac{dy}{dx} = \dfrac{1}{2}(2x)^{-1/2}\cdot 2 = \dfrac{1}{\sqrt{2x}}$ if $x > 0$
$$= 0 \quad \text{if } x \le 0$$

15. Differentiate $\arctan\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$ with respect to $\arccos x^2$.

Sol. Let $u = \arccos x^2$

Then $x^2 = \cos u$

and
$$y = \arctan\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$$
$$= \arctan\left(\frac{\sqrt{1+\cos u}-\sqrt{1-\cos u}}{\sqrt{1+\cos u}+\sqrt{1-\cos u}}\right)$$
$$= \arctan\left(\frac{\sqrt{2\cos^2\frac{u}{2}}-\sqrt{2\sin^2\frac{u}{2}}}{\sqrt{2\cos^2\frac{u}{2}}+\sqrt{2\sin^2\frac{u}{2}}}\right)$$
$$= \arctan\left(\frac{\cos\frac{u}{2}-\sin\frac{u}{2}}{\cos\frac{u}{2}+\sin\frac{u}{2}}\right) = \arctan\left(\frac{1-\tan\frac{u}{2}}{1+\tan\frac{u}{2}}\right)$$

$$= \arctan\left(\tan\left(\frac{\pi}{4}-\frac{u}{2}\right)\right) = \frac{\pi}{4}-\frac{u}{2}$$

Thus $\dfrac{dy}{du} = -\dfrac{1}{2}$

Find $\dfrac{dy}{dx}$ (Problems 16 – 20):

16. $y = x^{\sin y}$

Sol. Taking logarithm of both sides, we have
$$\ln y = \sin y\,(\ln x)$$
Now differentiating w.r.t. x, we get
$$\frac{1}{y}\frac{dy}{dx} = \frac{\sin y}{x} + (\ln x)\cos y\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx}(x - xy\cos y\ln x) = y\sin y$$
i.e., $\dfrac{dy}{dx} = \dfrac{y\sin y}{x - xy(\cos y\ln x)}$

17. $x^y = e^{x-y}$

Sol. Taking logarithm of both sides, we have
$$y\ln x = (x-y)\ln e$$
$$= x - y \qquad (1)$$
Differentiating both sides w.r.t. x, we have
$$y\cdot\frac{1}{x} + \ln x\cdot\frac{dy}{dx} = 1 - \frac{dy}{dx} \quad \text{or} \quad (\ln x + 1)\frac{dy}{dx} = 1 - \frac{y}{x} = \frac{x-y}{x}$$
or $\dfrac{dy}{dx} = \dfrac{x-y}{x(\ln x+1)} \qquad (2)$

From (1), we have
$$(1+\ln x)y = x \quad \text{or} \quad y = \frac{x}{1+\ln x}$$
Putting in (2), we get
$$\frac{dy}{dx} = \frac{x-\frac{x}{1+\ln x}}{x(\ln x+1)} = \frac{x + x\ln x - x}{x(\ln x+1)^2} = \frac{\ln x}{(\ln x+1)^2}$$

Alternative Method:
$$y\ln x = x - y \Rightarrow y(1+\ln x) = x \quad \text{or} \quad y = \frac{x}{1+\ln x}$$
Differentiating both sides w.r.t. x, we have
$$\frac{dy}{dx} = \frac{(1+\ln x)\cdot 1 - x\cdot\frac{1}{x}}{(1+\ln x)^2} = \frac{1+\ln x-1}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2}$$

18. $y^x + x^y = c$

Sol. Let $u = y^x$ and $v = x^y$

Taking logarithm of both sides of the first equation

$$\ln u = x \ln y$$

Differentiating w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{y} + \ln y$$

$$\frac{du}{dx} = u\left(\frac{x}{y}\frac{dy}{dx} + \ln y\right) = y^x\left(\frac{x}{y}\frac{dy}{dx} + \ln y\right) \quad (1)$$

Now from $v = x^y$, taking logarithm, we get

$$\log v = y \ln x$$

Differentiating w.r.t. x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{y}{x} + \ln x \frac{dy}{dx}$$

$$\frac{dv}{dx} = v\left[\frac{y}{x} + \ln x \,.\, \frac{dy}{dx}\right] = x^y\left[\frac{y}{x} + \ln x \,.\, \frac{dy}{dx}\right] \quad (2)$$

The given equation is $u + v = c$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad (3)$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (1) and (2) into (3), we have

$$y^x\left[\frac{x}{y}\frac{dy}{dx} + \ln y\right] + x^y\left[\frac{y}{x} + \ln x \,.\, \frac{dy}{dx}\right] = 0$$

or $\quad (xy^{x-1} + x^y \ln x)\dfrac{dy}{dx} = -\left[y^x \ln y + x^y \,.\, \dfrac{y}{x}\right]$

or $\quad \dfrac{dy}{dx} = -\dfrac{y^2 \ln y + yx^{y-1}}{xy^{x-1} + x^y \ln x}$

19. $\dfrac{x+y}{x-y} = x^2 + y^2$

Sol. Differentiating w.r.t. x, we have

$$\frac{(x-y)(1+y') - (x+y)(1-y')}{(x-y)^2} = 2x + 2yy',$$

where $\quad y' = \dfrac{dy}{dx}$

or $\quad \dfrac{y'(x-y+x+y) + x - y - x - y}{(x-y)^2} = 2x + 2yy'$

or $\quad xy' - y = x(x-y)^2 + yy'(x-y)^2$

or $\quad y'[x - y(x-y)^2] = y + x(x-y)^2$

$$y' = \frac{dy}{dx} = \frac{y + x(x-y)^2}{x - y(x-y)^2}$$

20. $x + \arcsin y = xy$

Sol. $x + \arcsin y = xy \quad (1)$

Differentiating (1) w.r.t. x, we have $\quad 1 + \dfrac{1}{\sqrt{1-y^2}}\dfrac{dy}{dx} = x\dfrac{dy}{dx} + y$

or $\quad \dfrac{dy}{dx}\left(\dfrac{1}{\sqrt{1-y^2}} - x\right) = y - 1 \quad$ or $\quad \dfrac{dy}{dx}\left(\dfrac{1 - x\sqrt{1-y^2}}{\sqrt{1-y^2}}\right) = y - 1$

Therefore, $\quad \dfrac{dy}{dx} = \dfrac{(y-1)\sqrt{1-y^2}}{1 - x\sqrt{1-y^2}}$

In Problems 21 – 30, find $f'(x)$ where:

21. $f(x) = x^2\sqrt{2ax - x^2}$

Sol. $f'(x) = x^2 \cdot \frac{1}{2}(2ax - x^2)^{-1/2}\frac{d}{dx}(2ax - x^2) + \sqrt{2ax - x^2} \,.\, 2x$

$$= \frac{x^2}{2\sqrt{2ax - x^2}} \,.\, (2a - 2x) + 2x\sqrt{2ax - x^2}$$

$$= \frac{x^2(a-x)}{\sqrt{2ax - x^2}} + 2x\sqrt{2ax - x^2}$$

$$= \frac{x^2(a-x) + 2x(2ax - x^2)}{\sqrt{2ax - x^2}} = \frac{5ax^2 - 3x^3}{\sqrt{2ax - x^2}}$$

22. $f(x) = \ln\left(\dfrac{e^x}{1+e^x}\right)$

Sol. $f(x) = \ln\dfrac{e^x}{1+e^x} = \ln e^x - \ln(1+e^x) = x - \ln(1+e^x)$

$$f'(x) = 1 - \frac{e^x}{1+e^x} = \frac{1 + e^x - e^x}{1+e^x} = \frac{1}{1+e^x}$$

23. $f(x) = x^{\ln x}$

Sol. Taking ln of both sides, we get

$$\ln(f(x)) = \ln x \,.\, \ln x = (\ln x)^2$$

Differentiating both sides, we have

$$\frac{f'(x)}{f(x)} = 2(\ln x) \,.\, \frac{1}{x} = \frac{2\ln x}{x}$$

or $\quad f'(x) = f(x) \,.\left(\dfrac{2\ln x}{x}\right) = x^{\ln x}\left(\dfrac{2\ln x}{x}\right) = \dfrac{2}{x}\ln x \,.\, x^{\ln x}$

24. $f(x) = \ln\left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right)$

Sol. $f(x) = \ln(1+\sqrt{x}) - \ln(1-\sqrt{x})$

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}}{1+\sqrt{x}} - \frac{-\frac{1}{2\sqrt{x}}}{1-\sqrt{x}} = \frac{1}{2\sqrt{x}\,(1+\sqrt{x})} + \frac{1}{2\sqrt{x}\,(1-\sqrt{x})}$$

$$= \frac{1-\sqrt{x}+1+\sqrt{x}}{2\sqrt{x}\,(1+\sqrt{x})\,(1-\sqrt{x})} = \frac{2}{2\sqrt{x}\,(1-x)} = \frac{1}{\sqrt{x}\,(1-x)}$$

25. $f(x) = e^{ax}\cos(b\arctan x)$

Sol. $f(x) = e^{ax}\cos(b\arctan x)$

$$f'(x) = e^{ax}\cdot a\cos(b\arctan x) + e^{ax}\left[-\frac{b}{1+x^2}\sin(b\arctan x)\right]$$

$$= e^{ax}\left[a\cos(b\arctan x) - \frac{b\sin(b\arctan x)}{1+x^2}\right]$$

$$= \frac{e^{ax}}{1+x^2}\,[a(1+x^2)\cos(b\arctan x) - b\sin(b\arctan x]$$

26. $f(x) = \dfrac{1}{\sqrt{b^2-a^2}}\ln\dfrac{\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}\right)}{\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}\right)}$

Sol. $f(x) = \dfrac{1}{\sqrt{b^2-a^2}}\left[\ln\left(\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}\right)\right) - \ln\left(\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}\right)\right)\right]$

$$f'(x) = \frac{1}{\sqrt{b^2-a^2}}\left(\frac{\frac{1}{2}\sqrt{b-a}\sec^2\left(\frac{x}{2}\right)}{\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}\right)} - \frac{-\frac{1}{2}\sqrt{b-a}\sec^2\left(\frac{x}{2}\right)}{\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}\right)}\right)$$

$$= \frac{\sqrt{b-a}\sec^2\left(\frac{x}{2}\right)}{2\sqrt{b^2-a^2}}\left[\frac{2\sqrt{b+a}}{(b+a)-(b-a)\tan^2\left(\frac{x}{2}\right)}\right]$$

$$= \frac{\cos^2\frac{x}{2}}{2\sqrt{b+a}\cos^2\frac{x}{2}}\left[\frac{2\sqrt{b+a}}{(b+a)\cos^2\left(\frac{x}{2}\right)-(b-a)\sin^2\frac{x}{2}}\right]$$

$$= \frac{1}{2\sqrt{b+a}}\,\frac{2\sqrt{b+a}}{b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)+a} = \frac{1}{a+b\cos x}$$

27. $f(x) = x\,a^x\sinh x$

Sol. $f'(x) = 1\,.\,a^x\sinh x + x\,.\,a^x\ \ln a\sinh x + xa^x\cosh x$

$$= a^x\sinh x + xa^x\sinh x\ln a + xa^x\cosh x.$$

28. $f(x) = \dfrac{-\cos x}{2\sin^2 x} + \dfrac{1}{2}\ln\tan\left(\dfrac{x}{2}\right)$

Sol. $f'(x) = -\dfrac{1}{2}\left[\dfrac{\sin^2 x\,(-\sin x) - \cos x\,(2\sin x\cos x)}{\sin^4 x}\right] + \dfrac{1}{2}\,\dfrac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}}$

$$= -\frac{1}{2}\left[\frac{-\sin^2 x - 2\cos^2 x}{\sin^3 x}\right] + \frac{1}{4}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

$$= \frac{1}{2}\left[\frac{\sin^2 x + 2\cos^2 x}{\sin^3 x}\right] + \frac{1}{4\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \frac{\sin^2 x + 2\cos^2 x}{2\sin^3 x} + \frac{1}{2\sin x}$$

$$= \frac{\sin^2 x + 2\cos^2 x + \sin^2 x}{2\sin^3 x} = \frac{2}{2\sin^3 x} = \csc^3 x$$

29. $f(x) = \operatorname{arcsec}(\csc x + \sqrt{x})$

Sol. $f'(x) = \dfrac{1}{(\csc x+\sqrt{x})\sqrt{(\csc x+\sqrt{x})^2-1}} \times \dfrac{d}{dx}(\csc x + \sqrt{x})$

$$= \frac{1}{(\csc x+\sqrt{x})\sqrt{(\csc x+\sqrt{x})^2-1}} \times \left(-\csc x\cot x + \frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1-2\sqrt{x}\csc x\cot x}{2\sqrt{x}\,(\csc x+\sqrt{x})\sqrt{\cot^2 x + x + 2\sqrt{x}\csc x}}$$

30. $f(x) = \left(1+\dfrac{1}{x}\right)^{x^2}$

Sol. Taking ln of both sides, we have $\ln f(x) = x^2\ln\left(1+\dfrac{1}{x}\right)$

Differentiating w.r.t. x, we obtain

$$\frac{f'(x)}{f(x)} = 2x\,.\,\ln\left(1+\frac{1}{x}\right) + x^2\cdot\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)$$

$$= 2x\cdot\ln\left(1+\frac{1}{x}\right) - \frac{1}{1+\frac{1}{x}}$$

$$f'(x) = f(x)\left[2x\ln\left(1+\frac{1}{x}\right) - \frac{x}{x+1}\right]$$

$$=\left(1+\frac{1}{x}\right)^{x^2}\left[2x\ln\left(1+\frac{1}{x}\right)-\frac{x}{x+1}\right]$$

Differentiate with respect to x each of the following (Problems 31 – 42).

31. $\arctan\left(\dfrac{1+2x}{2-x}\right)$

Sol. We have

$$\frac{d}{dx}\left[\arctan\left(\frac{1+2x}{2-x}\right)\right]=\frac{1}{1+\left(\dfrac{1+2x}{2-x}\right)^2}\,\frac{d}{dx}\left(\frac{1+2x}{2-x}\right)$$

$$=\frac{(2-x)^2}{(2-x)^2+(1+2x)^2}\cdot\frac{2(2-x)-(1+2x)(-1)}{(2-x)^2}=\frac{5}{5+5x^2}=\frac{1}{1+x^2}$$

32. $\ln(\arcsin e^x)$

Sol. Let $y=\ln(\arcsin e^x)$. Then

$$\frac{dy}{dx}=\frac{1}{\arcsin e^x}\cdot\frac{d}{dx}(\arcsin e^x)$$

$$=\frac{1}{\arcsin e^x}\cdot\frac{1}{\sqrt{1-(e^x)^2}}\cdot e^x=\frac{e^x}{\sqrt{1-(e^x)^2}\arcsin e^x}$$

33. $(\arcsin x^2)^\pi$

Sol. Let $u=\arcsin x^2$. Then

$$y=u^\pi$$

$$\frac{dy}{du}=\pi u^{\pi-1}=\pi(\arcsin x^2)^{\pi-1} \qquad (1)$$

$$\frac{du}{dx}=\frac{1}{\sqrt{1-x^4}}\cdot 2x \qquad (2)$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{2x\,\pi(\arcsin x^2)^{\pi-1}}{\sqrt{1-x^4}}$$

34. $f\left(\dfrac{x^2+1}{x^2-1}\right)$

Sol. Let $y=f\left(\dfrac{x^2+1}{x^2-1}\right)$

Set $u=\dfrac{x^2+1}{x^2-1}$ so that $y=f(u)$

$$\frac{dy}{du}=f'(u)$$

$$\frac{du}{dx}=\frac{(x^2-1)\,2x-2x\,(x^2+1)}{(x^2-1)^2}=\frac{-4x}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=f'(u)\cdot\frac{-4x}{(x^2-1)^2}=f'\left(\frac{x^2+1}{x^2-1}\right)\cdot\frac{-4x}{(x^2-1)^2}$$

35. $\dfrac{1-\cosh x}{1+\cosh x}$

Sol. Using the identities $\cosh 2x=2\sinh^2 x+1$ and $\cosh 2x=2\cosh^2 x-1$, we have

$$y=\frac{-2\sinh^2\left(\frac{x}{2}\right)}{2\cosh^2\left(\frac{x}{2}\right)}=-\tanh^2\left(\frac{x}{2}\right)$$

Therefore, $\dfrac{dy}{dx}=-2\tanh\left(\dfrac{x}{2}\right)\operatorname{sech}^2\left(\dfrac{x}{2}\right)\cdot\dfrac{1}{2}$

$$=-\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)$$

36. $\ln(\tanh 2x)$

Sol. $\dfrac{dy}{dx}=\dfrac{1}{\tanh 2x}\cdot\dfrac{d}{dx}(\tanh 2x)$

$$=\frac{2\operatorname{sech}^2 2x}{\tanh 2x}=\frac{2}{\cosh^2 2x\cdot\dfrac{\sinh 2x}{\cosh 2x}}=\frac{4}{2\cosh 2x\sinh 2x}$$

$$=\frac{4}{\sinh 4x}=4\operatorname{csch} 4x$$

37. $\log_{10}\left(\dfrac{x+1}{x}\right)$

Sol. $y=\log_{10}\left(\dfrac{x+1}{x}\right)=\dfrac{\ln\left(\dfrac{x+1}{x}\right)}{\ln 10}=\dfrac{1}{\ln 10}[\ln(x+1)-\ln x]$

$$y'=\frac{1}{\ln 10}\left(\frac{1}{x+1}-\frac{1}{x}\right)=\frac{1}{\ln 10}\cdot\frac{-1}{x(x+1)}=-\frac{1}{(\ln 10)\,x\,(x+1)}$$

38. $\arccos\left(\sqrt{1-x^2}\right)$

Sol. Let $y=\arccos\sqrt{1-x^2}$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-(1-x^2)}}\,\frac{d}{dx}\left(\sqrt{1-x^2}\right)=\frac{-1}{\sqrt{x^2}}\cdot\frac{1}{2}\,\frac{-2x}{\sqrt{1-x^2}}=\frac{x}{|x|\sqrt{1-x^2}}$$

39. $\operatorname{arcsec}(\sinh x)$

Sol. Let $y=\operatorname{arcsec}(\sinh x)$

$$\frac{dy}{dx}=\frac{1}{\sinh x\sqrt{\sinh^2 x-1}}\,\frac{d}{dx}(\sinh x)=\frac{\cosh x}{\sinh x\sqrt{\sinh^2 x-1}}$$

40. $\arcsin(\text{arccot} \ln x)$

Sol. Let $y = \arcsin(\text{arccot} \ln x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\text{arccot}\ln x)^2}}\frac{d}{dx}(\text{arccot}\ln x)$$
$$= \frac{1}{\sqrt{1-(\text{arccot}\ln x)^2}}\cdot\frac{-1}{1+(\ln x)^2}\cdot\frac{1}{x}$$
$$= \frac{-1}{x(1+\ln^2 x)\sqrt{1-(\text{arccot}\ln x)^2}}$$

41. $\cosh^{-1}(1+x^2)$

Sol. Let $y = \cosh^{-1}(1+x^2)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1+x^2)^2-1}}\frac{d}{dx}(1+x^2) = \frac{2x}{\sqrt{2x^2+x^4}} = \frac{2x}{|x|\sqrt{2+x^2}}$$

42. $\sinh^{-1}(\tanh x)$

Sol. $$\frac{dy}{dx} = \frac{1}{\sqrt{\tanh^2 x+1}}\cdot\frac{d}{dx}(\tanh x)$$
$$= \frac{\text{sech}^2 x}{\sqrt{\tanh^2 x + 1}}$$

In Problems 43 – 54 find $\frac{dy}{dx}$:

43. $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Sol. Differentiating, both sides w.r.t. x, we get

$$\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\frac{dy}{dx}=0$$

or $\quad x^{-1/2}+y^{-1/2}\frac{dy}{dx}=0 \quad$ or $\quad y^{-1/2}\frac{dy}{dx} = -x^{-1/2}$

or $\quad \frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}} \quad$ or $\quad \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

44. $xy^2-2xy+x=1$

Sol. Differentiating both sides w.r.t x, we get

$$1\,.\,y^2+2xy\frac{dy}{dx}-2x\frac{dy}{dx}-2y+1=0$$

or $\quad \frac{dy}{dx}(2xy-2x) = -y^2+2y-1$

or $\quad \frac{dy}{dx} = \frac{-y^2+2y-1}{2x(y-1)}$, provided $2x(y-1)\neq 0$

45. $x^3+y^3-3axy=0$

Sol. Differentiating w.r.t. x, we get

$$3x^2+3y^2\frac{dy}{dx}-3a\left[x\frac{dy}{dx}+y\,.\,1\right]=0$$

or $\quad x^2+y^2\frac{dy}{dx}-a\left[x\frac{dy}{dx}+y\right]=0$

or $\quad (y^2-ax)\frac{dy}{dx} = ay-x^2 \quad$ or $\quad \frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$

46. $(x^2+y^2)^3 = y$

Sol. Differentiating both sides w.r.t. x, we have

$$3(x^2+y^2)^2\frac{d}{dx}(x^2+y^2) = \frac{dy}{dx}$$

or $\quad 3(x^2+y^2)^2\left(2x+2y\frac{dy}{dx}\right)=\frac{dy}{dx}$

or $\quad \frac{dy}{dx}(1-6y(x^2+y^2)^2) = 6x(x^2+y^2)^2$

$$\frac{dy}{dx} = \frac{6x(x^2+y^2)^2}{1-6y(x^2+y^2)^2} \quad \text{provided } 1-6y(x^2+y^2)^2\neq 0$$

47. $\arctan\left(\frac{y}{x}\right)+yx^2=1$

Sol. Differentiating both sides w.r.t. x, we have

$$\frac{1}{1+\frac{y^2}{x^2}}\frac{d}{dx}\left(\frac{y}{x}\right)+x^2\frac{dy}{dx}+2xy=0$$

or $\quad \frac{x^2}{x^2+y^2}\times\frac{x\frac{dy}{dx}-y\,.\,1}{x^2}+x^2\frac{dy}{dx}+2xy=0$

or $\quad \frac{dy}{dx}\left[\frac{x}{x^2+y^2}+x^2\right]=\frac{y}{x^2+y^2}-2xy$

or $\quad \frac{dy}{dx}\left[\frac{x+x^2(x^2+y^2)}{x^2+y^2}\right]=\frac{y-2xy(x^2+y^2)}{x^2+y^2}$

or $\quad \frac{dy}{dx}=\frac{y-2xy(x^2+y^2)}{x+x^2(x^2+y^2)}=\frac{y(1-2x^3-2xy^2)}{x(1+x^3+xy^2)}$

48. $\arctan(x+y) = \arcsin(e^y+x)$.

Sol. Differentiating both the sides w.r.t. x implicitly, we have

$$\frac{1}{1+(x+y)^2}\left(1+\frac{dy}{dx}\right)=\frac{1}{\sqrt{1-(e^y+x)^2}}\left(e^y\frac{dy}{dx}+1\right)$$

or $\frac{dy}{dx}\left[\frac{1}{1+(x+y)^2}-\frac{e^y}{\sqrt{1-(e^y+x)^2}}\right]$

$$=\frac{1}{\sqrt{1-(e^y+x)^2}}-\frac{1}{1+(x+y)^2}$$

or $\left[\sqrt{1-(e^y+x)^2}-e^y(1+(x+y)^2)\right]\frac{dy}{dx}$

$$=1+(x+y)^2-\sqrt{1-(e^y+x)^2}$$

or $\frac{dy}{dx}=\frac{1+(x+y)^2-\sqrt{1-(e^y+x)^2}}{\sqrt{1-(e^y+x)^2}-e^y(1+(x+y)^2)}$

49. $y=\arcsin(\ln x)-\ln(\arctan x)$

Sol. Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(\ln x)^2}}\frac{d}{dx}(\ln x)-\frac{1}{\arctan x}\frac{d}{dx}(\arctan x)$$

$$=\frac{1}{\sqrt{1-(\ln x)^2}}\cdot\frac{1}{x}-\frac{1}{\arctan x}\cdot\frac{1}{1+x^2}$$

$$=\frac{1}{x\sqrt{1-(\ln x)^2}}-\frac{1}{(1+x^2)\arctan x}$$

50. $y\arcsin x-x\arctan y=1$

Sol. Differentiating both sides of the above equation, w.r.t. x, we get

$$\frac{dy}{dx}\cdot\arcsin x+y\cdot\frac{1}{\sqrt{1-x^2}}-\left(1.\arctan y+x\cdot\frac{1}{1+y^2}\frac{dy}{dx}\right)=0$$

or $\left(\arcsin x-\frac{x}{1+y^2}\right)\frac{dy}{dx}=\arctan y-\frac{y}{1-x^2}$

or $\left(\frac{(1+y^2)\arcsin x-x}{1+y^2}\right)\frac{dy}{dx}=\frac{\sqrt{1-x^2}\arctan y-y}{\sqrt{1-x^2}}$

or $\frac{dy}{dx}=\frac{(1+y^2)(\sqrt{1-x^2}\arctan y-y)}{\sqrt{1-x^2}((1+y^2)\arcsin x-x)}$

51. $\arcsin(\ln xy)=x+y^2$

Sol. Differentiating both sides of the above equation, w.r.t. x, we get

$$\frac{1}{\sqrt{1-(\ln xy)^2}}\cdot\frac{d}{dx}(\ln xy)=1+2y\frac{dy}{dx}$$

or $\frac{1}{\sqrt{1-(\ln xy)^2}}\cdot\frac{1}{xy}\left(1.y+x\frac{dy}{dx}\right)=1+2y\frac{dy}{dx}$

or $\frac{1}{\sqrt{1-(\ln xy)^2}}\cdot\frac{1}{y}\cdot\frac{dy}{dx}-2y\frac{dy}{dx}=1-\frac{1}{\sqrt{1-(\ln xy)^2}}\cdot\frac{1}{x}$

or $\left[\frac{1}{y\sqrt{1-(\ln xy)^2}}-2y\right]\frac{dy}{dx}=\frac{x\sqrt{1-(\ln xy)^2}-1}{x\sqrt{1-(\ln xy)^2}}$

or $\frac{1-2y^2\sqrt{1-(\ln xy)^2}}{y\cdot\sqrt{1-(\ln xy)^2}}\cdot\frac{dy}{dx}=\frac{x\sqrt{1-(\ln xy)^2}-1}{x\sqrt{1-(\ln xy)^2}}$

or $\frac{dy}{dx}=\frac{y(x\sqrt{1-(\ln xy)^2}-1)}{x(1-2y^2\sqrt{1-(\ln xy)^2})}$

52. $\operatorname{arcsec}(x^2+y)-e^x=\frac{1}{x+y}$

Sol. $\operatorname{arcsec}(x^2+y)-e^x=(x+y)^{-1}$ (1)

Differentiating both sides of (1), w.r.t. x, we get

$$\frac{1}{(x^2+y)\sqrt{(x^2+y)^2-1}}\left(2x+\frac{dy}{dx}\right)-e^x=-(x+y)^{-2}\times\left(1+\frac{dy}{dx}\right)$$

or $\left[\frac{1}{(x^2+y)\sqrt{(x^2+y)^2-1}}+\frac{1}{(x+y)^2}\right]\frac{dy}{dx}$

$$=e^x-\frac{1}{(x+y)^2}-\frac{2x}{(x^2+y)\sqrt{(x^2+y)^2-1}}$$

or $\left(\frac{(x+y)^2+(x^2+y)\sqrt{(x^2+y)^2-1}}{(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1}}\right)\frac{dy}{dx}$

$$=\frac{e^x.(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1}-(x^2+y)\sqrt{(x^2+y)^2-1}-2x(x+y)^2}{(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1}}$$

or $\frac{dy}{dx}=\frac{e^x(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1}-(x^2+y)\sqrt{(x^2+y)^2-1}-2x(x+y)^2}{(x+y)^2+(x^2+y)\sqrt{(x^2+y)^2-1}}$

53. $x=a(t-\sin t),\ y=a(1-\cos t)$

Sol. Differentiating, above equations w.r.t. t, we get

$$\frac{dx}{dt}=a(1-\cos t)$$

and $\frac{dy}{dt}=a(0-(-\sin t))=a\sin t$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{a\sin t}{a(1-\cos t)}=\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}}=\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}=\cot\frac{t}{2}$$

54. $x = \dfrac{3at}{1+t^2}, y = \dfrac{3at^2}{1+t^2}$

Sol. $\dfrac{dx}{dt} = \dfrac{(1+t^2)\,3a - 3at\,(2t)}{(1+t^2)^2} = \dfrac{3a[1+t^2-2t^2]}{(1+t^2)^2} = \dfrac{3a(1-t^2)}{(1+t^2)^2}$

From $y = \dfrac{3at^2}{1+t^2}$, we have

$$\frac{dy}{dt} = 3a\,\frac{(1+t^2)\,2t - t^2\,(2t)}{(1+t^2)^2} = 3a\,\frac{2t}{(1+t^2)^2} = \frac{6at}{(1+t^2)^2}$$

Thus $\dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx} = \dfrac{6at}{(1+t^2)^2} \times \dfrac{(1+t^2)^2}{3a(1-t^2)} = \dfrac{2t}{1-t^2}$

Alternative Method:

$$y = \frac{3at^2}{1+t^2} = t\cdot\frac{3at}{1+t^2} = tx \quad i.e.,\ y = tx \tag{1}$$

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = \frac{dt}{dx}\cdot x + t\,.\,1 = \frac{dt}{dx}\cdot x + t \tag{2}$$

But from $x = \dfrac{3at}{1+t^2}$

$$\frac{dx}{dt} = 3a\,.\,\frac{(1+t^2)\,.\,1 - t\,.\,2t}{(1+t^2)^2} = \frac{3a(1-t^2)}{(1+t^2)^2}$$

Now (2) becomes $\dfrac{dy}{dx} = \dfrac{(1+t^2)^2}{3a(1-t^2)}\cdot\dfrac{3at}{1+t^2} + t$

$$= \frac{t(1+t^2)}{1-t^2} + t = \frac{t+t^3+t-t^3}{1-t^2} = \frac{2t}{1-t^2}$$

Differentiate with respect to x (problems 55–60)

55. $y = \sqrt[3]{\dfrac{x(x^2+1)}{(x-1)^2}}$

Sol. $y = \left[\dfrac{x(x^2+1)}{(x-1)^2}\right]^{1/3}$ (1)

Taking natural logrithms of both sides of (1), we get

$$\ln y = \frac{1}{3}\ln\left[\frac{x(x^2+1)}{(x-1)^2}\right]$$

$$= \frac{1}{3}[\ln x + \ln(x^2+1) - 2\ln(x-1)] \tag{2}$$

Differentiating both sides of (2), w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x^2+1}\cdot 2x - 2\cdot\frac{1}{x-1}\right]$$

$$= \frac{1}{3}\cdot\frac{(x^2+1)(x-1) + 2x\,.\,x(x-1) - 2x(x^2+1)}{x(x^2+1)(x-1)}$$

$$= \frac{x^3 - x^2 + x - 1 + 2x^3 - 2x^2 - 2x^3 - 2x}{3x(x^2+1)(x-1)}$$

or $\dfrac{dy}{dx} = y\cdot\dfrac{x^3-3x^2-x-1}{3x(x^2+1)(x-1)}$

$$= \frac{x^{1/3}(x^2+1)^{1/3}}{(x-1)^{2/3}}\cdot\frac{x^3-3x^2-x-1}{3x(x^2+1)(x-1)} = \frac{x^3-3x^2-x-1}{3x^{2/3}(x^2+1)^{2/3}(x-1)^{5/3}}$$

56. $y = \dfrac{\sqrt{x}\,(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/3}}$

Sol. Taking natural logarithms of both sides of the above equation, we get

$$\ln y = \ln\frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/3}}$$

or $\ln y = \dfrac{1}{2}\ln x + \dfrac{2}{3}\ln(1-2x) - \dfrac{3}{4}\ln(2-3x) - \dfrac{4}{3}\ln(3-4x)$ (1)

Differentiating both sides of (1), w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{x} + \frac{2}{3}\frac{1}{1-2x}(-2) - \frac{3}{4}\frac{1}{2-3x}(-3) - \frac{4}{3}\frac{1}{3-4x}(-4)$$

$$= \frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{3(3-4x)}$$

$$= \left[\frac{1}{2x} + \frac{9}{4(2-3x)}\right] + \left[\frac{16}{3(3-4x)} - \frac{4}{3(1-2x)}\right]$$

$$= \frac{2(2-3x)+9x}{4x(2-3x)} + \frac{16(1-2x) - 4(3-4x)}{3(3-4x)(1-2x)}$$

$$= \frac{4-6x+9x}{4x(2-3x)} + \frac{16-32x-12+16x}{3(3-4x)(1-2x)}$$

$$= \frac{4+3x}{4x(2-3x)} + \frac{4-16x}{3(3-4x)(1-2x)}$$

or $\dfrac{dy}{dx} = y\left[\dfrac{4+3x}{4x(2-3x)} + \dfrac{4(1-4x)}{3(1-2x)(3-4x)}\right]$

$$= \frac{\sqrt{x}\,(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/3}}\left[\frac{4+3x}{4x(2-3x)} + \frac{4(1-4x)}{3(1-2x)(3-4x)}\right]$$

57. $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$

Sol. Let $u = (\tan x)^{\cot x}$ (1)

and $v = (\cot x)^{\tan x}$ (2)

Then $y = u + v$ (3)

Taking natural logarithms of both sides of (1), we get

$$\ln u = \cot x \ln \tan x \quad (4)$$

Differentiating (4) w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = (-\operatorname{cosec}^2 x) \ln \tan x + \cot x \cdot \frac{1}{\tan x} \cdot \sec^2 x$$

$$= -\operatorname{cosec}^2 x \ln \tan x + \frac{1}{\tan^2 x} \cdot \frac{1}{\cos^2 x}$$

$$= -\operatorname{cosec}^2 x \ln \tan x + \operatorname{cosec}^2 x \left(\because \frac{1}{\tan x \cos x} = \frac{1}{\sin x}\right)$$

$$= \operatorname{cosec}^2 x (1 - \ln \tan x)$$

$$= \operatorname{cosec}^2 x (\ln e - \ln \tan x)$$

$$= \operatorname{cosec}^2 x \ln \frac{e}{\tan x} = \operatorname{cosec}^2 x \,.\, \ln (e \cot x)$$

or $$\frac{du}{dx} = u \,.\, \operatorname{cosec}^2 x \,.\, \ln (e \cot x)$$

$$= (\tan x)^{\cot x} \,.\, \operatorname{cosec}^2 x \,.\, \ln (e \cot x)$$

Taking natural logarithms of both sides of (2), we have

$$\ln v = \tan x \ln \cot x \quad (5)$$

Differentiating both sides of (5), w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \sec^2 x \,.\, \ln \cot x + \tan x \,.\, \frac{1}{\cot x} \cdot (-\operatorname{cosec}^2 x)$$

$$= \sec^2 x \ln \cot x - \frac{1}{\cot^2 x} \cdot \frac{1}{\sin^2 x}$$

$$= \sec^2 x \ln \cot x - \sec^2 x \quad \left(\because \frac{1}{\cot x} \cdot \frac{1}{\sin x} = \frac{1}{\cos x}\right)$$

$$= -\sec^2 x (1 - \ln \cot x)$$

$$= -\sec^2 x \ln \frac{e}{\cot x} \quad (\because 1 = \ln e)$$

$$= -\sec^2 x \ln (e \tan x)$$

or $$\frac{dv}{dx} = v\,(-\sec^2 x \ln (e \tan x))$$

$$= -(\cot x)^{\tan x} \sec^2 x \ln (e \tan x)$$

Differentiating bothsides of (3) w.r.t. x, we have

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\tan x)^{\cot x} \operatorname{cosec}^2 x \ln (e \cot x) - (\cot x)^{\tan x} \sec^2 x \ln (e \tan x)$$

58. $y = x^x e^x \sin (\ln x)$

Sol. Taking natural logarithms of both sides, we get

$$\ln y = \ln x^x + \ln e^x + \ln (\sin (\ln x))$$

or $$\ln y = x \ln x + x + \ln (\sin (\ln x))$$

Differentiating both sides of the above equation w.r.t x, we have

$$\frac{1}{y}\frac{dy}{dx} = \left(1 \,.\, \ln x + x \,.\, \frac{1}{x}\right) + 1 + \frac{1}{\sin (\ln x)} \cdot \cos (\ln x) \,.\, \frac{1}{x}$$

$$= \ln x + 1 + 1 + \frac{1}{x} \cdot \frac{\cos (\ln x)}{\sin (\ln x)}$$

or $$\frac{dy}{dx} = y\left(\ln x + 2 + \frac{1}{x} \cdot \frac{\cos (\ln x)}{\sin (\ln x)}\right)$$

$$= x^x e^x \sin (\ln x) \left(2 + \ln x + \frac{1}{x} \cot (\ln x)\right)$$

59. $y = \dfrac{(x+2)^2}{(x+1)(x^2+3)^3}$

Sol. Taking natural logarithms of both sides of the given equation, we get

$$\ln y = \ln (x+2)^2 - [\ln (x+1) + \ln (x^2+3)^3]$$

or $$\ln y = 2 \ln (x+2) - \ln (x+1) - 3 \ln (x^2+3)$$

Differentiating both sides of the above equation, we have

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \,.\, \frac{1}{x+2} - \frac{1}{x+1} - \frac{3}{x^2+3} \cdot 2x$$

$$= \frac{2(x+1)(x^2+3) - (x+2)(x^2+3) - 6x(x+2)(x+1)}{(x+2)(x+1)(x^2+3)}$$

$$= \frac{2(x^3+x^2+3x+3) - (x^3+2x^2+3x+6) - 6x(x^2+3x+2)}{(x+1)(x+2)(x^2+3)}$$

$$= \frac{2x^3 + 2x^2 + 6x + 6 - x^3 - 2x^2 - 3x - 6 - 6x^3 - 18x^2 - 12x}{(x+1)(x+2)(x^2+3)}$$

or $$\frac{dy}{dx} = y\left(\frac{-5x^3 - 18x^2 - 9x}{(x+1)(x+2)(x^2+3)}\right)$$

$$= -y \cdot \frac{5x^3 + 18x^2 + 9x}{(x+1)(x+2)(x^2+3)}$$

$$= -\frac{(x+2)^2}{(x+1)(x^2+3)^3}\cdot\frac{5x^3+18x^2+9x}{(x+1)(x+2)(x^2+3)}$$

$$= -\frac{(x+2)(5x^3+18x+9x)}{(x+1)^2(x^2+3)^4}$$

60. $y = \exp.\left(\text{arccsc}\left(\frac{1}{x}\right)\right)$

Sol. Taking ln of both sides, we have

$$\ln y = \text{arc csc}\left(\frac{1}{x}\right)$$

Therefore, $\frac{1}{y}\frac{dy}{dx} = \frac{-1}{\frac{1}{x}\sqrt{\frac{1}{x^2}-1}}\frac{d}{dx}\left(\frac{1}{x}\right),\ \frac{1}{|x|} > 1$

$$= \frac{1}{x^2}\cdot\frac{x^2}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}},\quad |x| < 1$$

$$= \frac{\exp.\left(\text{arccsc}\left(\frac{1}{x}\right)\right)}{\sqrt{1-x^2}},\quad |x| < 1$$

Exercise Set 2.3 (Page 75)

1. Find Δy, dy, $\Delta y - dy$ if

(*i*) $y = x^3 - 1, x = 1, \Delta x = -0.5$

(*ii*) $y = \sqrt{3x-2}, x = 2, \Delta x = 0.3$

Sol.

(*i*) Here $y = x^3 - 1$ (1)

$$\Delta y = (x+\Delta x)^3 - 1 - x^3 + 1$$

Setting $x = 1, \Delta x = -0.5$, we get

$$\Delta y = (1-0.5)^3 - 1^3$$
$$= 0.125 - 1 = -0.875$$

From (1), we have

$\frac{dy}{dx} = 3x^2$. Therefore, $dy = 3x^2 dx$

When $x = 1, dx = \Delta x = -0.5$,

$$dy = 3(-0.5) = -1.5$$
$$\Delta y - dy = -0.875 + 1.5 = 0.625$$

(*ii*) $y = \sqrt{3x-2}$ (1)

From (1), we have $\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$.

Therefore, $dy = \frac{3}{2\sqrt{3x-2}}dx$

When $x = 2,\ dx = \Delta x = 0.3$,

$$dy = \frac{3}{2\times 2}(0.3) = 0.2250$$
$$\Delta y = \sqrt{3(x+\Delta x)-2} - \sqrt{3x-2}$$

Setting $x = 2,\ \Delta x = 0.3$, we get

$$\Delta y = \sqrt{3(2.3)-2} - \sqrt{3\times 2-2} = \sqrt{4.9} - 2 \approx 2.2136 - 2$$
$$\approx 0.2136$$
$$\Delta y - dy \approx 0.2136 - 0.2250 = -0.0114$$

2. Use differentials to approximate

(*i*) $\sqrt{26.2}$	(*ii*) $\sqrt{80.9}$	(*iii*) $\sqrt[3]{123}$
(*iv*) cos 61°	(*v*) $(3.02)^4$	(*vi*) tan 29°

(*i*) $\sqrt{26.2}$

Sol. We consider $y = f(x) = \sqrt{x}$ (1)

with $x = 25$ and $\Delta x = 1.2$

From (1), we have $dy = \frac{1}{2\sqrt{x}}dx$ (2)

Substituting $x = 25, dx = \Delta x = 1.2$ in (2), we get

$$dy = \frac{1}{10}(1.2) = 0.12$$

Now, $dy \approx \Delta y = y + \Delta y - y = \sqrt{x+\Delta x} - \sqrt{x}$

Therefore, $0.12 \approx \sqrt{26.2} - \sqrt{25} = \sqrt{26.2} - 5$

or $\sqrt{26.2} \approx 5 + 0.12 = 5.1200$

Using calculator, we find $\sqrt{26.2} \approx 5.1186$

Error in the approximation = 0.0014

(*ii*) $\sqrt{80.9}$

Sol. Let $y = f(x) = \sqrt{x}$ with $x = 81$

and $\Delta x = -0.1 = dx$

$$dy = \frac{1}{2\sqrt{x}}dx = \frac{1}{2\sqrt{81}}\times(-0.1)$$
$$= \frac{-0.1}{18} \approx -0.005556$$

Now $dy \approx \Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{80.9} - 9$.

Therefore, $-0.005556 \approx \sqrt{80.9} - 9$

or $\sqrt{80.9} \approx 9 - 0.005556 = 8.994444$

Using calculator, $\sqrt{80.9} \approx 8.994443$
Error in approximation = 0.000001

(iii) $\sqrt[3]{123}$

Sol. Let $y = f(x) = x^{1/3}$
with $x = 125$ and $\Delta x = -2$

$$dy = \frac{1}{3x^{2/3}}\,dx = \frac{1}{3(125)^{2/3}}\cdot(-2) = \frac{-2}{75} \approx -0.0267$$

$$dy \approx \Delta y = f(x+\Delta x) - f(x).$$

Therefore, $-0.0267 \approx \sqrt[3]{123} - \sqrt[3]{125}$

or $\sqrt[3]{123} \approx 5 - 0.0267 = 4.9733$

But $\sqrt[3]{123} = 4.9732$ by using calculator
Error in the approximation = 0.0001

(iv) $\cos 61^\circ$

Sol. Let $y = f(x) = \cos x$

with $x = 60^\circ = \frac{\pi}{3}$ and $\Delta x = 1^\circ = \frac{\pi}{180}$

$$dy = -\sin x\,dx \qquad (1)$$

Putting $x = \frac{\pi}{3}$, $dx = \frac{\pi}{180}$ in (1), we have

$$dy = \left(-\sin\frac{\pi}{3}\right)\frac{\pi}{180} = -\frac{\sqrt{3}}{2}\cdot\frac{\pi}{180}$$

$$dy \approx \Delta y = f(x+\Delta x) - f(x)$$

i.e., $\frac{-\sqrt{3}\,\pi}{360} \approx \cos 61^\circ - \cos 60^\circ$

or $\cos 61^\circ \approx \frac{1}{2} - \frac{\sqrt{3}\,\pi}{360} \approx 0.5 - 0.01512 = .48488$

From tables, $\cos 61^\circ \approx 0.48481$
Error in approximation = .00007

(v) $(3.02)^4$

Sol. Let $y = f(x) = x^4$ with $x = 3$ and $\Delta x = 0.02$

$$dy = 4x^3\,dx$$

$$dy\big]_{x=3} = 4 \times 3^3\,(0.02) = 2.16$$

Since $dy \approx \Delta y = (x+\Delta x)^4 - x^4$, therefore,

$$2.16 \approx (3.02)^4 - 3^4$$

or $(3.02)^4 \approx 81 + 2.16 = 83.16$

But $(3.02)^4 = 83.1817$

Error in approximation = −0.0217.

(vi) $\tan 29^\circ$

Sol. We let $f(x) = \tan x$, with $x = \frac{\pi}{6}$ and $\Delta x = \frac{-\pi}{180}$

$$dy = \sec^2 x\,dx$$

$$dy\big]_{x=\frac{\pi}{6}} = \sec^2\frac{\pi}{6}\left(-\frac{\pi}{180}\right) = \frac{4}{3}\times\frac{-\pi}{180} \approx -0.0233$$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$
Therefore, $-0.0233 \approx \tan 29^\circ - \tan 30^\circ$

or $\tan 29^\circ \approx \frac{1}{\sqrt{3}} - 0.0233 \approx 0.5774 - 0.0233$
≈ 0.5541

But $\tan 29^\circ \approx 0.5543$
Error in approximation = −0.0002

3. The side of a cube is measured with a possible error of ±2%. Find the percentage error in the surface area of one face of the cube.

Sol. Let x be edge of the cube.
Area A of a face is

$$A = x^2$$

$$dA = 2x\,dx$$

$$\frac{dA}{A} = \frac{2x}{x^2}\cdot dx = 2\frac{dx}{x}$$

But $\frac{dx}{x} = \pm 0.02$

Therefore, $\frac{dA}{A} = 2(\pm 0.02) = \pm 0.04$

The percentage error in the surface area is ±4%

4. A box with a square base has its height twice its width. If the width of the box is 8.5 inches with a possible error of ±0.3 inches, find the possible error in the volume of the box.

Sol. Let x in be the width of the box. Then its volume V is

$$V = 2x^3$$

$dV = 6x^2\,dx$. But $dx = \pm\frac{3}{10}$ (in)

Therefore change in volume

$$dV = 6(8.5)^2\left(\pm\frac{3}{10}\right) = \pm 130.05\ (\text{in}^3)$$

Thus the error in the volume of box is ± 130.05 cubic inches.

5. The radius x of a circle increases from $x = 10$ centimetres (cm) to $x + \Delta x = 10.1$ cm. Find the corresponding change in the area of the circle. Also find the percentage change in the area.

Sol. Let A be area of the circle of radius x. Then $A = \pi x^2$

$$dA = 2\pi x\, dx$$

Now, $x = 10$ cm and $\Delta x = 0.1$ cm

Change in the area of the circle is

$$\Delta A \approx dA = 2\pi(10)(0.1) = 2\pi\ (\text{cm}^2)$$

Relative change in the area $= \dfrac{2\pi}{\pi(10)^2} = \dfrac{2}{100} = 0.02$

Percentage change $= \dfrac{2}{100} \times 100 = 2\%$

6. The diameter of a plant was 8 inches. After one year the circumference of the plant increased by 2 inches. How much did

(*i*) the diameter of the plant increase?

(*ii*) the cross-sectional area of the plant change?

Sol. If x is the radius of the plant, then its circumference $C = 2\pi x$

Therefore, $dC = 2\pi dx$

Change in circumference is $dC = 2$

and so the change Δx in radius is given by

$$2 = 2\pi dx \quad \text{or} \quad dx = \frac{1}{\pi}$$

Thus the diameter increased by $\dfrac{2}{\pi}$ inches.

Area A of the cross-section is

$$A = \pi x^2$$
$$dA = 2\pi x\, dx$$

When $x = 4$, $dx = \dfrac{1}{\pi}$ and change in area is

$$\Delta A \approx dA = 2\pi(4)\frac{1}{\pi} = 8 \quad i.e., \quad 8 \text{ in}^2$$

7. Sand pouring from a chute forms a conical pile whose altitude is always equal to the radius. If the radius of the pile is 10cm, find the approximate change in radius when volume increases by 2 cm^3.

Sol. The volume of the conical pile of radius r and height r is

$$V = \frac{1}{3}\pi r^3$$
$$dV = \pi r^2\, dr$$

Now $\Delta V \approx dV = 2$ cm^3, when $r = 10$

Change in radius $= \Delta r = dr$

Therefore, $2 \approx \pi(10)^2\, dr$ or $dr \approx \dfrac{1}{50\pi}$

i.e., change in radius $\approx \dfrac{1}{50\pi}$ cm.

8. A dome is in the shape of a hemisphere with radius 60 feet. The dome is to be painted with a layer of 0.01 inch thickness. Use differentials to estimate the amount of the paint required.

Sol. If V is volume of hemisphere with radius r, then

$$V = \frac{2\pi r^3}{3}$$
$$dV = 2\pi r^2\, dr \qquad (1)$$

We need change in the volume when change Δr in r is $\dfrac{1}{1200}$ ft.

Letting $r = 60$ and $dr = \dfrac{1}{1200}$ in (1), we have

$$dV = 2\pi \times 60 \times 60 \times \frac{1}{1200} = 6\pi\ (\text{ft}^3), \text{ so}$$

$\Delta V \approx 6\pi$ (ft.^{3}), that is,

the amount of paint $\approx 6\pi$ cubic ft.

9. The side of a building is in the shape of a square surmounted by an equilateral triangle. If the length of the base is 15 metres with an error of 1%, find the percentage error in the area of the side.

Sol. Let x m be the length of the base. Then area A of the side is given by

$$A = x^2 + \frac{\sqrt{3}}{4}x^2 = \left(1 + \frac{\sqrt{3}}{4}\right)x^2$$
$$dA = 2\left(1 + \frac{\sqrt{3}}{4}\right)x\,dx \qquad (1)$$

It is given that $\dfrac{dx}{x} = 0.01 \Rightarrow dx = x\left(\dfrac{1}{100}\right)$ and $x = 15$(m).

Now we need to find $\dfrac{dA}{A} \times 100$

From (1), we have

$$dA = 2\left(1 + \frac{\sqrt{3}}{4}\right)15 \times 15\left(\frac{1}{100}\right), \text{ so}$$

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2 \times 15^2\left(\frac{\sqrt{3}}{4} + 1\right) \times \frac{1}{100}}{\left(1 + \frac{\sqrt{3}}{4}\right)15^2} = \frac{2}{100}$$

Thus the percentage error in the area of the side is approximately equal to 2%.

10. A boy makes a paper cup in the shape of a right circular cone with height four times its radius. If the radius is changed from 2 cm to 1.5 cm but the height remains four times the radius, find the approximate decrease in the capacity of the cup.

Sol. If r is the radius of the base and h is height of the cup, then its volume V is given by

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (4r) = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\, dr \qquad (1)$$

Now it is given that

$$r = 2 \text{ cm and } \Delta r = dr = -0.5$$

Then from (1), change in the capacity of the cup

$$\Delta V \approx dV = 4\pi(2)^2\left(-\frac{1}{2}\right) = -8\pi$$

The –ve sign shows that there is decrease in the capacity of the cup, which is approximately equal to 8π cm^3.

11. To estimate the height of Minar-i-Pakistan, the shadow of a 3 metre pole placed 24 metres from the Minar is measured. If the length of the shadow is 1 metre with a percentage error of 1%, find the height of the Minar. Also find the percentage error in the height so found.

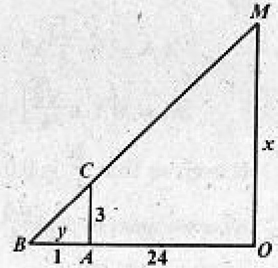
Sol. If x m is height of the Minar, then from the figure

$$\frac{x}{25} = \frac{3}{1}$$

Therefore, $x = 25 \times 3 = 75$.

Height of the Minar = 75 m.

If y is the actual length of the shadow of the pole, then



$$\frac{y+24}{x} = \frac{y}{3}$$

or $\quad 3y + 72 = xy$

or $\quad 3dy = x\,dy + y\,dx$

or $\quad (3-x)\,dy = y\,dx$

or $\quad (3-x)\dfrac{dy}{y} = dx \qquad (1)$

Now $\dfrac{dy}{y} = \pm 0.01$. When $x = 75$, relative error in the height $= \dfrac{dx}{x}$.

From (1), we get

$$\frac{-72}{75}(\pm 0.01) = \frac{dx}{75} \quad \text{or} \quad \frac{dx}{75} = \pm\frac{24}{25}\times\frac{1}{100}$$

Percentage error

$$= \pm\frac{24}{25}\times\frac{1}{100}\times 100 = \pm 0.96\%.$$

12. Oil spilled from a tanker spreads in a circle whose radius increases at the rate of 2 ft/sec. How fast is the area increasing when the radius of the circle is 40 feet?

Sol. Let r be the radius of the circle at any instant t. Then area A of the circle is

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \qquad (1)$$

We have to find $\dfrac{dA}{dt}$ when $\dfrac{dr}{dt} = 2$ and $r = 40$. Substituting into (1), we have

$$\frac{dA}{dt} = 2\pi \times 40 \times 2 = 160\pi$$

Thus area of the circle changes at the rate of $160\,\pi$ ft^2/sec.

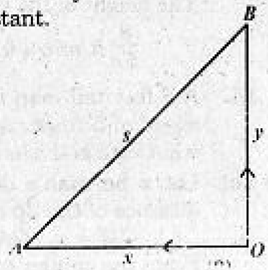
13. From a point O, two cars leave at the same time. One car travels west and after t seconds its position is $x = t^2 + t$ feet. The other car travels north and it covers $y = t^2 + 3t$ feet in t seconds. At what rate is the distance between the two cars changing after 5 seconds.

Sol. Let A, B be the positions of the two cars at any instant t and let s be the distance between them at this instant.

$$s^2 = x^2 + y^2 \qquad (1)$$

$$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \qquad (2)$$

We have to find $\dfrac{ds}{dt}$ at the instant when $t = 5$.



We have

$$x = t^2 + t \qquad (3)$$

$$y = t^2 + 3t \qquad (4)$$

Differentiating (3) and (4) w.r.t. t, we have

$$\frac{dx}{dt} = 2t + 1, \qquad \left.\frac{dx}{dt}\right|_{t=5} = 11$$

$$\frac{dy}{dt} = 2t + 3, \qquad \left.\frac{dy}{dt}\right|_{t=5} = 13$$

After 5 sec. the distances of the two cars from O are

$$x = 5^2 + 5 = 30$$
$$y = 5^2 + 15 = 40$$

and $s^2 = 30^2 + 40^2$ from (1)

i.e., $s = 50$

Substituting into (2), we get

$$50\frac{ds}{dt} = 30 \times 11 + 40 \times 13 = 330 + 520 = 850$$

or $$\frac{ds}{dt} = \frac{850}{50} = 17$$

Therefore, the distance between the two cars is changing at the rate of 17 ft./sec.

14. Sand falls from a container at the rate of 10 ft^3/min and forms a conical pile whose height is always double the radius of the base. How fast is the height increasing when the pile is 5 feet high?

Sol. Let h be the height of the pile at any instant t. Radius of the pile = $\frac{h}{2}$. Volume V of the pile is $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

It is given that $\frac{dV}{dt} = 10$ and we need to find $\frac{dh}{dt}$ at the instant when $h = 5$. Therefore, from (1), we have

$$10 = \frac{\pi}{4}\cdot 5^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{10 \times 4}{25\,\pi} = \frac{8}{5\,\pi}$$

The height of the pile is changing at the rate of

$\frac{8}{5\pi}$ ft/min ≈ 0.51 ft./min.

15. A 6 feet tall man is walking toward a lamp post 16 feet high at a speed of 5 ft/sec. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing?

Sol. Let x be man's distance from the lamp post OP and z be the distance of the tip of his shadow from O.

i.e., $OM = x$, $OA = z$

From the similar triangles, we have

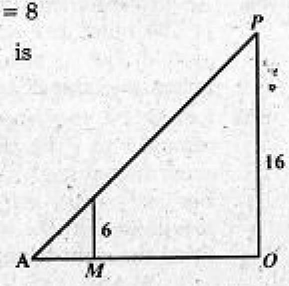
$$\frac{16}{z} = \frac{6}{z - x} \quad \text{or} \quad 16z - 6z = 16x$$

i.e., $5z = 8x$ (1)

Differentiating (1), w.r.t. t, we get

$$5\frac{dz}{dt} = 8\frac{dx}{dt}$$

We substitute $\frac{dx}{dt} = 5$ and find that $\frac{dz}{dt} = 8$



Therefore the tip of man's shadow is moving at the rate of 8 ft./sec.

If y is the length of the shadow, then $MA = y$. From the similar triangles we have

$$\frac{16}{x + y} = \frac{6}{y} \quad \text{or} \quad 16y - 6y = 6x$$

i.e., $5y = 3x$

Therefore, $5\frac{dy}{dt} = 3\frac{dx}{dt}$

Substituting $\frac{dx}{dt} = 5$, we have $5\frac{dy}{dt} = 3 \times 5 \Rightarrow \frac{dy}{dt} = 3$

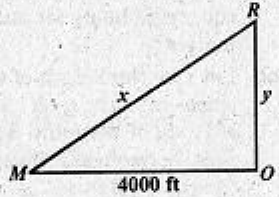
Thus the shadow is changing at the rate of 3 ft./sec.

16. At a distance of 4000 feet from a launching site, a man is observing a rocket being launched. If the rocket lifts off vertically and is rising at a speed of 600 ft/sec. when it is at an altitude of 3000 feet, how fast is the distance between the rocket and the man changing at this instant?

Sol. Let y be altitude of the rocket and x be the distance between the man and the rocket at any instant t.

We have

$$x^2 = y^2 + 4000^2 \qquad (1)$$



When $y = 3000$ ft, we have from (1),

$$x^2 = 3000^2 + 4000^2$$
$$= 9000000 + 16000000$$
$$= 25000000$$
$$x = 5000$$

Differentiating (1) w.r.t. t, we get

$$x\frac{dx}{dt} = y\frac{dy}{dt} \qquad (2)$$

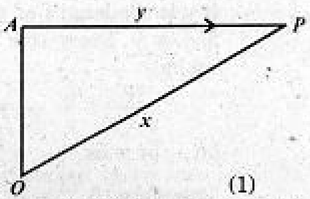
When $y = 3000$, $\frac{dy}{dt} = 600$ (given) and we have to find $\frac{dx}{dt}$ at this instant. From (2), we have

$$\frac{dx}{dt} = \frac{3000}{5000} \times 600 = 360$$

Thus the distance between the rocket and man is changing at the rate of 360 ft/sec.

17. An airplane flying horizontally at an altitude of 3 miles and a speed of 480 miles per hour passes directly above an observer on the ground. How fast is the distance of the observer to the airplane increasing after 30 seconds?

Sol. Let O be the observer on the ground and P be the airplane at some instant t.



Let $OP = x$, $AP = y$

It is given that $OA = 3$.

From the right triangle, we have

$$3^2 + y^2 = x^2 \qquad (1)$$

The distance travelled by the plane 30 sec. after it has passed above the observer = 4 miles. Substituting $y = 4$ in (1) we get $x = 5$.

We have to find $\frac{dx}{dt}$ at the instant when $t = 30$ sec. and $\frac{dy}{dt} = 480$.

Implicit differentiation of (1) gives

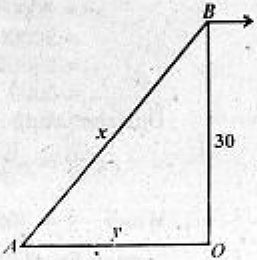
$$y\frac{dy}{dt} = x\frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$$

$$\left.\frac{dx}{dt}\right|_{t = 30 \text{ sec.}} = \frac{4}{5} \times 480 = 384$$

The rate of change of the distance of the plane from the observer = 384 miles/hr.

18. A boy flies a kite at an altitude of 30 metres. If the kite flies horizontally away from the boy at the rate of 2 m/sec, how fast is the string being let out when the length of the string released is 70 metres?

Sol. Let x be the length of the string let out at some instant t, K be the kite at an altitude of 30m and let $OA = y$. The kite flies horizontally away from the boy at the rate of 2 m/sec.



From ΔAOK, we have

$$x^2 = 30^2 + y^2 \qquad (1)$$

Therefore, $$x\frac{dx}{dt} = y\frac{dy}{dt} \qquad (2)$$

When $x = 70$, we have from (1),

$$y^2 = 70^2 - 30^2 = 4900 - 900 = 4000 = 20\sqrt{10}$$

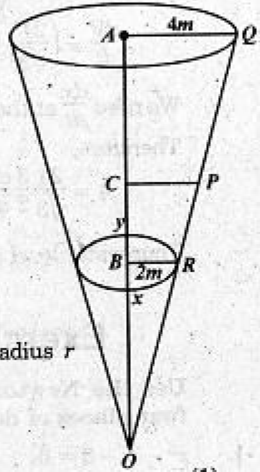
and $\frac{dy}{dt} = 2$ at this instant.

Substituting in (2), we get $\frac{dx}{dt} = \frac{20\sqrt{10} \times 2}{70} = \frac{4\sqrt{10}}{7}$

Thus the string is being let out at the rate of $\frac{4\sqrt{10}}{7}$ m/sec.

19. A water tank is in the shape of frustrum of a cone with height 6 metres and upper and lower radii 4 metres and 2 metres respectively. If water pours into the tank at the rate of 20 m³/min, how fast is the water level rising when the water is half way up?

Sol. Extend the tank downward so as to form a cone. Let $BO = x$ m so that the height of the cone is $x + 6$.



Suppose that at some instant water level is at C where $BC = y$ and let $CP = r$.

From similar Δ's AOQ and BOR, we get

$$\frac{6 + x}{4} = \frac{x}{2}$$

i.e., $x = 6\ (= BO)$.

From Δ's COP and BOR, we have

$$\frac{y + 6}{r} = \frac{6}{2} \quad i.e., \quad r = \frac{y + 6}{3}$$

Volume V of the frustum with upper radius r and lower radius 2 is

$$V = \frac{1}{3}\pi r^2 (y + 6) - \frac{1}{3}\pi(2)^2 \times 6$$

$$= \frac{\pi}{9 \times 3}(y + 6)^3 - \frac{1}{3}\pi(2)^2 \times 6 \qquad (1)$$

Differentiating (1) w.r.t. t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{9} \cdot (y + 6)^2 \frac{dy}{dt} \qquad (2)$$

It is given that $\frac{dV}{dt} = 20$ and we need find $\frac{dy}{dt}$ when water is half way up *i.e.*, when $y = 3$. Therefore, from (2), we have

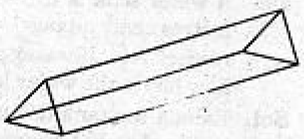
$$20 = \frac{\pi}{9} \cdot 9^2 \frac{dy}{dt} \quad \text{or} \quad \frac{dy}{dt} = \frac{20}{9\pi}$$

Thus water level is rising at the rate of $\frac{20}{9\pi}$ m/min.

20. A 12 metre long water trough, with vertical cross-sections in the shape of equilateral triangles (one vertex down), is being filled at the rate of 4 m³/min. How fast is the water level rising at the instant when the depth of the water is $1\frac{1}{2}$ metres?

Sol. When the water is x ft deep, a vertical cross section of water has area $= \frac{1}{2}x.\, x \csc 60^\circ = \frac{x^2}{\sqrt{3}}$.

Volume of water at this instant is



$$V = 12 \times \frac{x^2}{\sqrt{3}}$$

$$\frac{dV}{dt} = \left(\frac{24}{\sqrt{3}}\right) x \frac{dx}{dt}$$

We need $\frac{dx}{dt}$ at the instant when $x = \frac{3}{2}$ m and $\frac{dV}{dt} = 4$.

Therefore,

$$4 = \frac{24}{\sqrt{3}}\frac{3}{2}\frac{dx}{dt} \quad \text{or} \quad 4 = 12\sqrt{3}\,\frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{3\sqrt{3}}$$

Thus water level is rising at the rate of $\frac{1}{3\sqrt{3}}$ m/min.

Exercise Set 2.4 (Page 80)

Use the Newton-Raphson method to approximate, up to four places of decimal, a root of each of the following:

1. $x^3 - 3x - 3 = 0$ with $x_0 = 2$

Sol. Let $f(x) = x^3 - 3x - 3$.

Then $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

Using Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$,

we have, $x_{n+1} = x_n - \dfrac{x_n^3 - 3x_n - 3}{3x_n^2 - 3} = \dfrac{3x_n^3 - 3x_n - x_n^3 + 3x_n + 3}{3x_n^2 - 3}$

or $$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 - 3} \qquad (1)$$

For $n = 0$, the equation (1) becomes $x_1 = \dfrac{2x_0^3 + 3}{2x_0^2 - 3}$

Now putting $x_0 = 2$ in the above equation, we get

$$x_1 = \frac{2(2)^3 + 3}{3(2)^2 - 3} = \frac{2 \times 8 + 3}{3 \times 4 - 3} = \frac{16 + 3}{12 - 3} = \frac{19}{9}$$

For $n = 1$, the equation (1) becomes

$$x_2 = \frac{2x_1^3 + 3}{3x_1^2 - 3} = \frac{2\left(\frac{19}{9}\right)^3 + 3}{3\left(\frac{19}{9}\right)^2 - 3} \qquad \left(\because x_1 = \frac{19}{9}\right)$$

$$\approx \frac{2(9.4088) + 3}{3(4.4568) - 3} = \frac{18.8176 + 3}{13.3704 - 3} = \frac{21.8176}{10.3704}$$

$$\approx 2.1038$$

For n_2, the equation (1) becomes

$$x_3 = \frac{2x_2^3 + 3}{3(x_2)^2 - 3}$$

$$\approx \frac{2(2.1038)^3 + 3}{3(2.1038)^2 - 3} \qquad [\because x_2 \approx 2.1038]$$

$$\approx \frac{2(9.3114) + 3}{3(4.4260) - 3} = \frac{18.6228 + 3}{13.2780 - 3} = \frac{21.6228}{10.2780}$$

$$\approx 2.1038$$

Thus the required root is approximately equal to 2.1038.

2. $x^3 - 5x + 3 = 0$ with $x_0 = 0$

Sol. If $f(x) = x^3 - 5x + 3$, then $f'(x) = 3x^2 - 5$

Using Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$,

we have $x_{n+1} = x_n - \dfrac{x_n^3 - 5x_n + 3}{3x_n^2 - 5} = \dfrac{3x_n^3 - 5x_n - x_n^3 + 5x_n - 3}{3x_n^2 - 5}$

or $$x_{n+1} = \frac{2x_n^3 - 3}{3x_n^2 - 5} \qquad (1)$$

For $n = 0$, the equation (1) becomes $x_1 = \dfrac{2x_0^3 - 3}{3x_0^2 - 5}$

Now putting $x_0 = 0$ in the above equation, we get

$$x_1 = \frac{2(0)^3 - 3}{3(0)^2 - 5} = \frac{-3}{-5} = \frac{3}{5} = 0.6$$

For $n = 1$, the equation (1) becomes

$$x_2 = \frac{2x_1^3 - 3}{3x_1^2 - 5} = \frac{2(.6)^3 - 3}{3(.6) - 5} \qquad [\because x_1 = 0.6]$$

$$= \frac{2(0.216) - 3}{3(0.36) - 5} = \frac{0.432 - 3}{1.08 - 5} = \frac{-2.568}{-3.92} = \frac{2.568}{3.92} \approx 0.6551$$

For $n = 2$, the equation (1) becomes

$$x_3 = \frac{2x_2^3 - 3}{3x_2^2 - 5} \approx \frac{2(0.6551)^3 - 3}{3(0.6551)^2 - 5} \qquad (\because x_2 \approx 0.6551)$$

$$\approx \frac{2(0.28114) - 3}{3(0.42916) - 5} = \frac{0.56228 - 3}{1.28748 - 5} = \frac{-2.43772}{-3.71252} \approx 0.6566$$

For $n = 3$, the equation (1) becomes

$$x_4 = \frac{2x_3^3 - 3}{3x_3^2 - 5} \approx \frac{2(0.6566)^3 - 3}{3(0.6566)^2 - 5} \qquad (\because x_3 \approx 0.6566)$$

$$\approx \frac{2(0.28308) - 3}{3(0.43112) - 5} = \frac{0.56616 - 3}{1.29336 - 5} = \frac{-2.43384}{-3.70664} \approx 0.6566$$

Thus the required root is approximately equal to 0.6566.

3. $e^{-x} - \sin x = 0$ with $x_0 = 0.5$

Sol. If $f(x) = e^{-x} - \sin x$, then

$f'(x) = -e^{-x} - \cos x = -(e^{-x} + \cos x)$

Using Newton-Raphson formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,

we have $x_{n+1} = x_n - \dfrac{e^{-x_n} - \sin x_n}{-(e^{-x_n} + \cos x_n)}$

or $$x_{n+1} = x_n + \frac{e^{-x_n} - \sin x_n}{e^{-x_n} + \cos x_n} \qquad (1)$$

For $n = 0$, the equation (1) becomes

$$x_1 = x_0 + \frac{e^{-x_0} - \sin x_0}{e^{-x_0} + \cos x_0}$$

Putting $x_0 = 0.5$, in the above equation, we get

$$x_1 = 0.5 + \frac{e^{-.5} - \sin(0.5)}{e^{-.5} + \cos(0.5)} \qquad (e \approx 2.71828 \text{ and } 0.5 \text{ rad} \approx 28.6478^\circ)$$

$$\approx 0.5 + \frac{0.6065 - 0.4794}{0.6065 + 0.8776} = 0.5 + \frac{0.1271}{1.4841}$$

$$\approx 0.5 + 0.0856 = 0.5856$$

For $n = 1$, the equation (1) becomes

$$x_2 = x_1 + \frac{e^{-x_1} - \sin(x_1)}{e^{-x_1} + \cos x_1}$$

$$\approx 0.5856 + \frac{e^{-0.5856} - \sin(0.5856)}{e^{-0.5856} + \cos(0.5856)} \qquad (\because x_1 \approx 0.5856)$$

$$\approx 0.5856 + \frac{0.5568 - 0.5527}{0.5568 + 0.8334} = 0.5856 + \frac{0.0041}{1.3902}$$

$$\approx 0.5856 + 0.0029 = 0.5885$$

For $n = 2$, the equation (1) becomes

$$x_3 = x_2 + \frac{e^{-x_2} - \sin x_2}{e^{-x_2} + \cos x_2}$$

$$\approx 0.5885 + \frac{e^{-0.5885} - \sin(0.5885)}{e^{-0.5885} + \cos(0.5885)} \qquad (\because x_2 \approx 0.5885)$$

$$\approx 0.5885 + \frac{0.55515 - 0.55511}{0.55515 + 0.83177} = 0.5885 + \frac{0.00004}{1.38692}$$

$$\approx 0.5885 + 0.000029 = 0.588529 \approx 0.5885$$

Thus the required root is approximately equal to 0.5885.

4. $e^x - 3x = 0$ with $x_0 = 0$

Sol. If $f(x) = e^x - 3x$, then $f'(x) = e^x - 3$

Using Newton-Raphson formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,

we have

$$x_{n+1} = x_n - \frac{e^{x_n} - 3x_n}{e^{x_n} - 3} = \frac{x_n e^{x_n} - 3x_n - e^{x_n} + 3x_n}{e^{x_n} - 3} = \frac{x_n e^{x_n} - e^{x_n}}{e^{x_n} - 3}$$

or $$x_{n+1} = \frac{e^{x_n}(x_n - 1)}{e^{x_n} - 3} \qquad (1)$$

For $n = 0$, the equation (1) becomes $x_1 = \dfrac{e^{x_0}(x_0 - 1)}{e^{x_0} - 3}$

Putting $x_0 = 0$, we get

$$x_1 = \frac{e^0(0 - 1)}{e^0 - 3} = \frac{1 \times (-1)}{1 - 3} = \frac{-1}{-2} = \frac{1}{2} = 0.5$$

For $n = 1$, the equation (1) becomes

$$x_2 = \frac{e^{x_1}(x_1 - 1)}{e^{x_1} - 3} = \frac{e^{05}(0.5 - 1)}{e^{0.5} - 3} \qquad (\because x_1 = 0.5)$$

$$\approx \frac{1.6487(0.5 - 1)}{1.6487 - 3} = \frac{1.6487 \times (-0.5)}{-1.3513} \approx \frac{-0.8244}{-1.3513}$$

$$\approx 0.6101$$

For $n = 2$, the equation (1) becomes

$$x_3 = \frac{e^{x_2}(x_2 - 1)}{e^{x_2} - 3} \approx \frac{e^{0.6101}(0.6101 - 1)}{e^{0.6101} - 3} \approx \frac{1.8406(-0.3899)}{1.8406 - 3}$$

$$= \frac{-0.7176}{-1.1594} \approx 0.6189$$

For $n = 3$, the equation (1) becomes

$$x_4 = \frac{e^{x_3}(x_3 - 1)}{e^{x_3} - 3} \approx \frac{e^{0.6189}(0.6189 - 1)}{e^{0.6189} - 3} \approx \frac{1.8569 \times (-0.3811)}{1.8569 - 3}$$

$$\approx \frac{-0.7077}{-1.1431} \approx 0.6191$$

For $n = 4$, the equation (1) becomes

$$x_5 = \frac{e^{x_4}(x_4 - 1)}{e^{x_4} - 3} \approx \frac{e^{0.6191}(0.6191 - 1)}{e^{0.6191} - 3} \qquad (\because\ x_4 \approx 0.6191)$$

$$\approx \frac{1.8573 \times (0.3809)}{1.8573 - 3} \approx \frac{-0.7074}{-1.1427} \approx 0.6191$$

Thus the required root is approximately equal to 0.6191.

5. $4 \sin x = e^x$ in the interval $]0, 0.5[$

Sol. If $f(x) = 4 \sin x - e^x$, then $f'(x) = 4 \cos x - e^x$

$$f(0) = 4 \sin 0 - e^0 = 4 \times 0 - 1 = -1$$

$$f(0.5) = 4 \sin (0.5) - e^{0.5} \quad (0.5 \text{ rad} \approx 28.6478^\circ)$$

$$\approx 4(0.4794) - 1.6487 = 1.9176 - 1.6487$$

$$\approx 0.2689$$

As $f(0)$ and $f(0.5)$ are of opposite signs, so there is a root between 0 and 0.5.

Let us take $x_0 = \frac{\pi}{12} \approx 0.2618$.

Using Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$,

we get, $$x_{n+1} = x_n - \frac{4 \sin x_n - e^{x_n}}{4 \cos x_n - e^{x_n}} \qquad (1)$$

For $n = 0$, the equation (1) becomes

$$x_1 = x_0 - \frac{4 \sin x_0 - e^{x_0}}{4 \cos x_0 - e^{x_0}} = \frac{\pi}{12} - \frac{4 \sin \frac{\pi}{12} - e^{\pi/12}}{4 \cos \frac{\pi}{12} - e^{\pi/12}}$$

$$\approx 0.2618 - \frac{4(0.2588) - 1.2993}{4(0.9659) - 1.2993}$$

$$\approx 0.2618 - \frac{1.0352 - 1.2993}{3.8636 - 1.2993} = 0.2618 - \frac{-0.2641}{2.5643}$$

$$\approx 0.2618 + 0.10299 = 0.36479 \approx 0.3648$$

For $n = 1$, the equation (1) becomes

$$x_2 = x_1 - \frac{4 \sin x_1 - e^{x_1}}{4 \cos x_1 - e^{x_1}}$$

$$\approx 0.3648 - \frac{4\sin (0.3648) - e^{0.3648}}{4\cos (0.3648) - e^{0.3648}} \quad (0.3648 \text{ rad} \approx 20.9015^\circ)$$

$$\approx 0.3648 - \frac{4(0.3568) - 1.4402}{4(0.9342) - 1.4402}$$

$$\approx 0.3648 - \frac{1.4272 - 1.4402}{3.7368 - 1.4402} = 0.3648 - \frac{-0.0130}{2.2966}$$

$$\approx 0.3648 + 0.00566 = 0.37046 \approx 0.3705$$

For $n = 2$, the equation (1) becomes

$$x_3 = x_2 - \frac{4 \sin x_2 - e^{x_2}}{4 \cos x_2 - e^{x_2}}$$

$$\approx 0.3705 - \frac{4 \sin (0.3705) - e^{0.3705}}{4 \cos (0.3705) - e^{0.3705}} \quad (\because\ x_2 \approx 0.3705)$$

$$\approx 0.3705 - \frac{4(0.3621) - 1.4485}{4(0.9321) - 1.4485} \quad (0.3705 \text{ rad.} \approx 21.2280^\circ)$$

$$\approx 0.3705 - \frac{1.4484 - 1.4485}{3.7286 - 1.4485} = 0.3705 - \frac{-0.0001}{2.2801}$$

$$\approx 0.3705 + 0.00004386 = 0.37054386$$

$$\approx 0.3705$$

Thus the required root is approximately equal to .3705.

6. $\sin x = 1 - x$ with $x_0 = 0$

Sol. If $f(x) = \sin x - 1 + x$, then

$$f'(x) = \cos x + 1$$

Using Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$,

we get, $$x_{n+1} = x_n - \frac{\sin x_n - 1 + x_n}{\cos x_n + 1} \qquad (1)$$

For $n = 0$, the equation (1) becomes

$$x_1 = x_0 - \frac{\sin x_0 - 1 + x_0}{\cos x_0 + 1}$$

Putting $x = 0$ in the above equation, we have

$$x_1 = 0 - \frac{\sin 0 - 1 + 0}{\cos 0 + 1} = -\frac{0-1}{1+1} = \frac{1}{2} = 0.5$$

For $n = 1$, the equation (1) becomes

$$x_2 = x_1 - \frac{\sin x_1 - 1 + x_1}{\cos x_1 + 1}$$

$$= 0.5 - \frac{\sin (0.5) - 1 + 0.5}{\cos (0.5) + 1} \quad (\because x_1 = 0.5)$$

$$\approx 0.5 - \frac{0.4794 - 1 + 0.5}{.8776 + 1} = 0.5 - \frac{-0.0206}{1.8776} = 0.5 + \frac{0.0206}{1.8776}$$

$$\approx 0.5 + 0.01097 = 0.51097 \approx 0.5110$$

For $n = 2$, the equation (1) becomes

$$x_3 = x_2 - \frac{\sin x_2 - 1 + x_2}{\cos x_2 + 1}$$

$$\approx 0.5110 - \frac{\sin (0.5110) - 1 + 0.5110}{\cos (0.5110) + 1}$$

$$\approx 0.5110 - \frac{0.4890 - 1 + 0.5110}{0.8723 + 1} = 0.5110 - \frac{1-1}{1.8723} \approx 0.5110$$

Thus the required root is approximately equal to 0.5110.

Alternative Method:

Let $f(x) = 1 - x - \sin x$, then

$f'(x) = -1 - \cos x = -(1 + \cos x)$

Using Newton-Raphson formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,

$$\text{we get } x_{n+1} = x_n - \frac{1 - x_n - \sin x_n}{-(1 + \cos x_n)}$$

$$= x_n + \frac{1 - x_n - \sin x_n}{1 + \cos x_n} \qquad (1)$$

For $n = 0$, the equation (1) becomes

$$x_1 = x_0 + \frac{1 - x_0 - \sin x_0}{1 + \cos x_0}$$

Putting $x_0 = 0$ in the above equation, we have

$$x_1 = 0 + \frac{1-0-0}{1+1} = \frac{1}{2} = 0.5$$

For $n = 1$, the equation (1) becomes

$$x_2 = x_1 + \frac{1 - x_1 - \sin x_1}{1 + \cos x_1}$$

$$= 0.5 + \frac{1 - 0.5 - \sin (0.5)}{1 + \cos (0.5)} \quad (0.5 \text{ rad.} \approx 28.6478^\circ)$$

$$\approx 0.5 + \frac{1 - 0.5 - 0.4794}{1 + 0.8776} = 0.5 + \frac{0.0206}{1.8776}$$

$$\approx 0.5 + 0.01097 = 51097 \approx 0.5110$$

For $n = 2$, the equation (1) becomes

$$x_3 = x_2 + \frac{1 - x_2 - \sin x_2}{1 + \cos x_2}$$

$$\approx 0.5110 + \frac{1 - 0.5110 - \sin (0.5110)}{1 + \cos (0.5110)} \quad (0.511 \text{ rad.} \approx 29.2781^\circ)$$

$$\approx 0.5110 + \frac{1 - 0.5110 - 0.4890}{1 + 0.8723} = 0.5110 + \frac{1-1}{1.8723}$$

$$\approx 0.5110$$

Thus the required root is approximately equal to 0.5110.

Exercise Set 2.5 (Page 86)

In each of Problems 1 – 4, find the nth order derivative:

1. $\dfrac{x}{x^2 - a^2}$

Sol. $\dfrac{x}{x^2 - a^2} = \dfrac{x}{(x-a)(x+a)}$

Let $\dfrac{x}{(x-a)(x+a)} = \dfrac{A}{x-a} + \dfrac{B}{x+a}$

or $x = A(x + a) + B(x - a)$

Putting $x = a$ and $x = -a$ in the above equation successively, we get

$$a = 2Aa \qquad \text{or} \qquad A = \frac{1}{2}$$

$$\text{and} \quad -a = -2Ba \qquad \text{or} \qquad B = \frac{1}{2}$$

Thus $\dfrac{x}{x^2 - a^2} = \dfrac{1}{2}\left[\dfrac{1}{x-a} + \dfrac{1}{x+a}\right]$

Therefore,

$$\frac{d^n}{dx^n}\left(\frac{x}{x^2 - a^2}\right) = \frac{1}{2}\left[\frac{(-1)^n n!}{(x-a)^{n+1}} + \frac{(-1)^n n!}{(x+a)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2}\left[\frac{1}{(x-a)^{n+1}} + \frac{1}{(x+a)^{n+1}}\right]$$

2. $\dfrac{x^4}{(x-1)(x-2)}$

Sol. $\dfrac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \dfrac{15x-14}{(x-1)(x-2)}$

Now, $\dfrac{15x-14}{(x-1)(x-2)} = \dfrac{-1}{x-1} + \dfrac{16}{x-2}$, on resolving into partial fractions

Hence $y = \dfrac{x^4}{(x-1)(x-2)}$

$= x^2 + 3x + 7 + \dfrac{-1}{x-1} + \dfrac{16}{x-2}$

For $n > 2$, we have

$$y^{(n)} = 16\frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}} = (-1)^n n!\left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$$

3. $e^{ax} \sin(bx + c)$

Sol. Differentiating w.r.t. x, we get

$y' = e^{ax} . a\sin(bx+c) + e^{ax} b\cos(bx+c)$

$= e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$

Now, putting $a = r\cos\theta$, $b = r\sin\theta$, we get

$$r = \sqrt{a^2+b^2}, \quad \text{and} \quad \theta = \arctan\frac{b}{a} \tag{1}$$

Then $y' = e^{ax}[r\sin(bx+c)\cos\theta + r\cos(bx+c)\sin\theta]$

$= re^{ax}[\sin(bx+c)\cos\theta + \cos(bx+c)\sin\theta]$

$= re^{ax}\sin(bx+c+\theta)$

Similarly, we can get

$y'' = r^2e^{ax}\sin(bx+c+2\theta)$ and generalizing

$y^{(n)} = r^ne^{ax}\sin(bx+c+n\theta)$

$= (a^2+b^2)^{n/2} e^{ax}\sin\left[bx + c + n\arctan\dfrac{b}{a}\right]$

on putting the values of r and θ from (1).

4. $e^{ax}\cos^2 x \sin x$

Sol. $y = e^{ax}\cos^2 x\sin x$

$= \dfrac{1}{2}e^{ax}[(1+\cos 2x)\sin x]$

$= \dfrac{1}{2}e^{ax} e^{ax}\sin x + \dfrac{1}{2}e^{ax}(\cos 2x\sin x)$

$= \dfrac{1}{2}e^{ax}\sin x + \dfrac{1}{4}e^{ax}[\sin 3x - \sin x]$

$= \dfrac{1}{4}e^{ax}\sin x + \dfrac{1}{4}e^{ax}\sin 3x$

Hence $y^{(n)} = \dfrac{1}{4}(a^2+1)^{n/2} e^{ax}\sin\left(x + n\arctan\dfrac{1}{a}\right) +$

$\dfrac{1}{4}(a^2+9)^{n/2}\sin\left(3x + n\arctan\dfrac{3}{a}\right)$

5. If $x^y = e^{x-y}$, find $\dfrac{d^ny}{dx^n}$

Sol. $x^y = e^{x-y}$

or $y\ln x = x - y$ or $y(1+\ln x) = x$ (1)

Let $u = y$ and $v = 1 + \ln x$

Then $v^{(n)} = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

Differentiating (1) n times by Leibniz' Theorem, we have

$$y^{(n)}(1+\ln x) + n\,y^{(n-1)} \times \frac{1}{x} + \frac{n(n-1)}{2}y^{(n-2)}\left(\frac{-1}{x^2}\right)$$

$$+ \ldots + ny'\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + y\frac{(-1)^{n-1}(n-1)!}{x^n} = 0,$$

which gives $y^{(n)}$ in terms of $y^{(n-1)}, y^{(n-2)}, \ldots, y'$.

6. If $f(x) = \ln(1+\sqrt{1-x})$, prove that

$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$

Sol. $f(x) = \ln(1+\sqrt{1-x})$

$f'(x) = \dfrac{1}{1+\sqrt{1-x}}\left(\dfrac{-1}{2\sqrt{1-x}}\right)$

or $2\sqrt{1-x}\,f'(x) = \dfrac{-1}{1+\sqrt{1-x}} = \dfrac{-(1-\sqrt{1-x})}{(1+\sqrt{1-x})(1-\sqrt{1-x})}$

$= \dfrac{-(1-\sqrt{1-x})}{x}$

or $2x\sqrt{1-x}\,f'(x) = -1 + \sqrt{1-x}$

Differentiating the above equation, we get

$2x\sqrt{1-x}\,f''(x) + 2f'(x)\left[1\sqrt{1-x} - \dfrac{x}{2\sqrt{1-x}}\right] = -\dfrac{1}{2\sqrt{1-x}}$

or $4x(1-x)f''(x) + 2f'(x)[2(1-x) - x] = -1$

or $4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$

Differentiating n times by Leibniz' Theorem, we get

$2x(1-x)f^{(n+2)}(x) + \{(2n+2) - (4n+3)x\}f^{(n+1)}(x)$

$-n(2n+1)f^{(n)}(x) = 0$. (Verify!)

7. If $y = \arctan x$, show that

$$(1 + x^2)\, y^2 + 2xy = 0$$

Hence find the value of $y^{(n)}$ when $x = 0$

Sol. Differentiating $y = \arctan x$, w.r.t. x, we get

$$y' = \frac{1}{1+x^2} \qquad y'(0) = 1$$

or $\quad (1 + x^2)\, y' = 1$

Differentiating the above equation, we have

$$(1 + x^2)\, y'' + 2xy' = 0, y''(0) = 0$$

Differentiating n times by Leibniz' Theorem, we get

$$(1+x^2)y^{(n+2)}+n(2x)y^{(n+1)}+\frac{n(n-1)}{2!}(2)y^{(n)}+2xy^{(n+1)}+n(2)y^{(n)}=0$$

or $\quad (1+x^2)\, y^{(n+2)} + 2(n + 1)\, y^{(n+1)}\, x + (n^2+n)\, y^{(n)} = 0$

or $\quad (1+x^2)\, y^{(n+2)} + 2(n + 1)\, xy^{(n+1)} + n(n+1)\, y^{(n)} = 0$

Putting $x = 0, y^{(n+2)}(0) = -n(n + 1)\, y^{(n)}(0)$ (1)

Putting $n = 2, y^{(4)}(0) = -2.3\, y''(0) = (0)$

Putting $n = 4\,, y^{(6)}(0) = -6.7\,.\, y^{(4)}(0) = (0)$ and so on.

Generalizing, we have $y^{(2n)}(0) = (0)$

Putting $n = 1$ in (1), we get $y'''(0) = -1.2\,.\, y'(0)$

$$= -2.1 = (-1)^1\, 2!$$

Putting $n = 3, y^{(5)}(0) = -3.4\, y'''(0) = -3.4\,(-2.1)$

$$= (-1)^2\, 4!$$

Putting $n = 5, y^{(7)}(0) = -5.6\,.\, y^{(5)}(0)$

$$= -6.5\,.\,(-1)^2\, 4!$$

$$= (-1)^3\,.\,6\,.\,5\,.\,4! = (-1)^3\, 6!$$

and so on, generalizing, we get

$$y^{(2n+1)}(0) = (-1)^n\, (2n)!$$

8. If $y = \sin(a \arcsin x)$, prove that

$$(1 - x^2)\, y^{(n+2)} = (2n + 1)\, xy^{(n+1)} + (n^2 - a^2)\, y^{(n)}$$

Sol. $y = \sin(a \arcsin x)$

$$y' = \cos(a \arcsin x) \times \frac{a}{\sqrt{1-x^2}}$$

$$(1 - x^2)^{1/2}\, y' = a\cos(a \arcsin x)$$

Squaring both sides, we have

$$(1 - x^2)\, y'^2 = a^2\cos^2(a \arcsin x) = a^2\,[1 - \sin^2(a \arcsin x)]$$

or $\quad (1 - x^2)\, y'^2 = a^2(1 - y^2)$

Differentiating again, we have

$$(1 - x^2)\, 2y'\, y'' - 2x\, y'^2 = a^2(-2yy')$$

$$(1 - x^2)\, y'' - xy' = -a^2 y$$

$$(1 - x^2)\, y'' - xy' + a^2 y = 0$$

Differentiating n times by Leibniz' Theorem, we get

$$(1 - x^2)\, y^{(n+2)} + n\,.\,(-2x)\, y^{(n+1)} + \frac{n(n-1)}{2}(-2)\, y^{(n)}$$

$$- xy^{(n+1)} - n\,.\, y^{(n)} + a^2\, y^{(n)} = 0$$

or $\quad (1-x^2)\, y^{(n+2)} - (2n+1)\, xy^{(n+1)} - (n^2-a^2)\, y(n) = 0$

or $\quad (1 - x^2)\, y^{(n+2)} = (2n + 1)\, xy^{(n+1)} + (n^2 - a^2)\, y^{(n)}$

9. If $y = e^{m \arcsin x}$ show that

$$(1 - x^2)\, y^{(n+2)} - (2n + 1)\, xy^{(n+1)} - (n^2 + m^2)\, y^{(n)} = 0$$

Find the value of $y^{(n)}$ at $x = 0$

Sol. $$y' = e^{m \arcsin x}\frac{m}{\sqrt{1-x^2}} = \frac{my}{\sqrt{1-x^2}}$$

or $\quad y'^2\,(1 - x^2) = m^2 y^2$ (1)

Differentiating (1), we have

$$2y'\, y''(1 - x^2) + y'^2\,(-2x) = m^2\, 2y\, y'$$

or $\quad (1 - x^2)\, y'' - xy' = m^2 y.$

Using Leibniz' Theorem, we get from the above equation

$$(1-x^2)\, y^{(n+2)}+n(-2x)\, y^{(n+1)} + \frac{n(n-1)}{2!}(-2)y^{(n)} - xy^{(n+1)} - ny^{(n)} = m^2\, y^{(n)}$$

or $\quad (1 - x^2)\, y^{(n+2)} - (2n + 1)\, xy^{(n+1)} - (n^2 + m^2)\, y^{(n)} = 0$

when $x = 0,\ y(0) = 1,\ y'(0) = m, y''(0) = m^2$

$$y^{(n+2)}(0) = (n^2 + m^2)\, y^{(n)}(0)$$

$$y'''(0) = (1 + m^2)\, y'(0) = (1^2 + m^2)\, m$$

$$y^{(4)}(0) = (2^2 + m^2)\, y''(0) = (2^2 + m^2)m^2$$

$$y^{(5)}(0) = (3^2 + m^2)\, y'''(0) = (3^2 + m^2)\,(1^2 + m^2)\, m$$

$$y^{(6)}(0) = (4^2 + m^2)\, y^{(4)}(0) = (4^2 + m^2)\,(2^2 + m^2)\, m^2$$

Thus it follows that when n is even

$$y^{(n)}(0) = [(n-2)^2+m^2]\,[(n-4)^2+m] \ldots [4^2+m^2]\,[2^2+m^2]\, m^2$$

When n is odd

$$y^{(n)}(0) = [(n-2)^2+m^2]\,[(n-4)^2+m^2] \ldots [1^2+m^2]\, m$$

10. Find $y^{(n)}(0)$ if

(*i*) $y = \ln[x + \sqrt{1+x^2}]$

(*ii*) $y = (x + \sqrt{1+x^2})^m$

Sol.

(*i*) $y = \ln(x + \sqrt{1+x^2})$

Differentiating, we have

$$y' = \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{x}{\sqrt{1+x^2}}\right] = \frac{1}{(x+\sqrt{1+x^2})} \cdot \frac{(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}} \qquad (1)$$

or $(1+x^2)\,y'^2 = 1$

Differentiating, we get

$$(1+x^2)\,.\,2y'\,y'' + 2xy'2 = 0$$

or $(1+x^2)\,y'' + xy' = 0$ (2)

Differentiating n times by Leibniz' Theorem, we obtain

$$(1+x^2)\,y^{(n+2)} + n.2xy^{(n+1)} + \frac{n(n-1)}{2}(2)\,y^{(n)} + xy^{(n+1)} + n.1.\,y^{(n)} = 0$$

or $(1+x^2)\,y^{(n+2)} + (2n+1)\,xy^{(n+1)} + n^2 y^{(n)} = 0$ (3)

Putting $x = 0$ in (1), (2), (3), we get

$$y'(0) = 1\,,\, y''(0) = 0$$

$$y^{(n+2)}(0) = -n^2\,y^{(n)}(0)$$

Putting $n = 1\,,\; y'''(0) = -(1)^2\,y'(0) = -1 = -(1)^2$

$n = 2\,,\; y^{(4)}(0) = -(2)^2\,y''(0) = -(2)^2\,(0) = 0$

$n = 3\,,\; y^{(5)}(0) = -(3)^2\,y'''(0) = -(3)^2\,(-(1)^2)$

$n = 4\,,\; y^{(6)}(0) = -(4)^2\,y'(0) = 0$ and so and so on

Thus $y^{(2m)}(0) = 0.$

Putting $n = 5$,

$$y^{(7)}(0) = -5^2\,y^{(5)}(0)$$

$$= -5^2\,.\,-3^2\,.\,-(1)^2$$

$$= (-1)^3\,.\,1^2\,.\,3^2\,.\,5^2$$

Similarly, $y^{(9)}(0) = (-1)^4\,.\,1^2\,.\,3^2\,.\,5^2\,.\,7^2$ and generalizing

$$y^{(2n+1)}(0) = (-1)^n\,.\,1^2\,.\,3^2\,.\,5^2 \ldots (2n-1)^2$$

(*ii*) $y = (x + \sqrt{1+x^2})^m$ (1)

$$y' = m(x+\sqrt{1+x^2})^{m-1}\left[1 + \frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{m(x+\sqrt{1+x^2})^m}{\sqrt{1+x^2}} = \frac{m\,y}{\sqrt{1+x^2}}$$

(2)

or $y'^2 = \dfrac{m\,y^2}{1+x^2}$ or $(1+x^2)\,y'^2 = m^2\,y^2$

Differentiating it again, we have

$$(1+x^2)\,2y'\,y'' + 2xy'^2 = m^2\,.\,2yy'$$

$$(1+x^2)\,y'' + xy' - m^2 y = 0 \qquad (3)$$

Differentiating it n times by Leibniz' Theorem, we get

$$(1+x^2)y^{(n+2)} + n.2xy^{(n+1)} + \frac{n(n-1)}{2!}2y^{(n)} + xy^{(n+1)} + ny^{(n)} - m^2y^{(n)} = 0$$

or $(1+x^2)y^{(n+2)} + (2^n+1)xy^{(n+1)} + (n^2-m^2)y^{(n)} = 0$ (4)

Putting $x = 0$ successively in (1), (2), (3), (4), we obtain

$$y(0) = 1,\, y'(0) = m,\, y''(0) = m^2$$

and $y^{(n+2)}(0) = (m^2 - n^2)\,y^{(n)}(0)$ (5)

Putting $n = 1, 3, 5, \ldots$ in (5), we get

$$y'''(0) = (m^2 - 1^2)\,y'(0) = (m^2 - 1^2)\,m$$

$$y^{(5)}(0) = (m^2 - 3^2)\,y'''(0) = (m^2 - 3^2)\,(m^2 - 1^2).m$$

$$y^{(7)}(0) = (m^2 - 5^2)\,(m^2 - 3^2)\,(m^2 - 1^2)\,m$$

and generalizing, we find

$$y^{(2n+1)}(0) = \{m^2-(2n-1)^2\}\,\{m^2-(2n-3)^2\} \ldots (m^2-5^2)\,(m^2-3^2)(m^2-1^2)m$$

Similarly, putting $n = 2, 4, 6, \ldots$ in (5) and generalizing, we have

$$y^{(2n)}(0) = \{m^2-(2n-2)^2\}\,\{m^2-(2n-4)^2\} \ldots \{m^2-4^2\}\,\{m^2-2^2\}\,m^2$$

11. If $y = a\cos(\ln x) + b\sin(\ln x)$, prove that

$$x^2 y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2+1)\,y^{(n)} = 0$$

Sol. $y = a\cos(\ln x) + b\sin(\ln x)$

$$y' = -a\cos(\ln x)\,\frac{1}{x} + b\cos(\ln x)\,.\,\frac{1}{x}$$

or $xy' = b\cos(\ln x) - a\sin(\ln x)$

Differentiating it again, we get

$$xy'' + y' = -\frac{b\sin(\ln x)}{x} - \frac{a\cos(\ln x)}{x}$$

or $\quad x^2y'' + xy' = -[a\cos(\ln x) + b\sin(\ln x)] = -y$

or $\quad x^2y'' + xy' + y = 0 \qquad (1)$

Differentiating (1) n times by Leibniz' Theorem, we have

$$x^2y^{(n+2)} + n.2xy^{(n+1)} + \frac{n(n-1)}{2!}2y^{(n)} + xy^{(n+1)} + ny^{(n)} + y^{(n)} = 0$$

or $\quad x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$

12. Show that

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{(-1)^n\,n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - \frac{1}{3}\cdots - \frac{1}{n}\right]$$

Sol. Let $\quad u = \frac{1}{x}, v = \ln x$

$$u^{(n)} = \frac{(-1)^n\,n!}{x^{n+1}},\ v' = \frac{1}{x}\ ,\ v'' = \frac{-1}{x^2}$$

$$u^{(n-1)} = \frac{(-1)^{n-1}\,(n-1)!}{x^n}\ ,\ v^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Now, by Leibniz' Theorem, we have

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{d^n}{dx^n}(u\ v)$$

$$= {}^nC_o u^{(n)}v + {}^nC_1 u^{(n-1)}v' + \ldots + {}^nC_{n-1}u'\,v^{(n-1)} + {}^nC_n\,uv^{(n)}$$

$$= \frac{(-1)^n\,n!}{x^{n+1}}\ln x + n\frac{(-1)^{n-1}\,(n-1)!}{x^n}\left(\frac{1}{x}\right) + \frac{n(n-1)}{2}\cdot$$

$$\frac{(-1)^{n-2}\,(n-2)!}{x^{n-1}}\left(-\frac{1}{x^2}\right) + \ldots + \frac{1}{x}\cdot\frac{(-1)^{n-1}\,(n-1)!}{x^n}$$

$$= \frac{(-1)^n\,n!}{x^{n+1}}\ln x - \frac{(-1)^n n!}{x^{n+1}} - \frac{1}{2}\frac{(-1)^n\,n!}{x^{n+1}} - \ldots + \frac{-1}{n}\,\frac{(-1)^n\,n!}{x^{n+1}}$$

$$= \frac{(-1)^n\,n!}{x^{n+1}}\left(\ln x - 1 - \frac{1}{2} - \ldots - \frac{1}{n}\right)$$

Exercise Set 2.6 (Page 97)

Evaluate the given limits (Problems 1 – 5):

1. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{5-x^2}{4+x+y}$

Sol. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{5-x^2}{4+x+y} = \frac{\lim_{(x,y)\to(0,0)}(5-x^2)}{\lim_{(x,y)\to(0,0)}(4+x+y)} = \frac{5-0}{4+0+0} = \frac{5}{4}$

2. $\displaystyle \lim_{(x,y)\to(1,-1)} e^{-xy}$

Sol. $\displaystyle \lim_{(x,y)\to(1,-1)} e^{-xy} = e^{-(1)(-1)} = e$

3. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{e^{xy}\sin xy}{xy}$

Sol. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{e^{xy}\sin xy}{xy} = \lim_{(x,y)\to(0,0)} e^{xy}\cdot \lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} = 1$

4. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

Sol. Setting $x = r\cos\theta, y = r\sin\theta$, we have the given limit

$$= \lim_{r\to 0}\frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}, \text{ since } r^2 = x^2+y^2$$

and $\quad (x,y)\to(0,0) \quad$ implies $r \to 0$

$$= \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

5. $\displaystyle \lim_{(x,y)\to(2,2)} \frac{x^3-2xy+3x^2-2y}{x^2y+4y^2-6x^2+24y}$

Sol. $\displaystyle \lim_{(x,y)\to(2,2)} \frac{x^3-2xy+3x^2-2y}{x^2y+4y^2-6x^2+24y}$

$$= \frac{\lim_{(x,y)\to(2,2)}(x^3-2xy+3x^2-2y)}{\lim_{(x,y)\to(2,2)}(x^2y+4y^2-6x^2+24y)}$$

$$= \frac{8-8+12-4}{8+16-24+48} = \frac{8}{48} = \frac{1}{6}$$

In each of Problems 6 – 10, show that the given limit does not exist:

6. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{xy}{x^2 + y^2}$

Sol. The limit does not exist if we show that $f(x, y) = \dfrac{xy}{x^2 + y^2}$ approaches to different values as $(x, y) \to (0, 0)$ along different paths. Let (x, y) approach $(0, 0)$ along the line $y = mx$.

Then $f(x, y) = \dfrac{mx^2}{x^2(1 + m^2)} = \dfrac{m}{1 + m^2}$, since $x \neq 0$

Thus $f(x, y)$ will have different values for different values of m. Hence the given limit does not exist.

7. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

Sol. Let $x = r\cos\theta$, $y = r\sin\theta$. Then $r^2 = x^2 + y^2$.

As $(x, y) \to (0, 0)$, $\quad r \to 0$. Therefore,

$$\lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{(x, y) \to (0, 0)} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}$$

$$= \lim_{r \to 0} (\cos^2\theta - \sin^2\theta) = \cos 2\theta$$

which is independent of r and can have any value. Thus the given limit does not exist.

8. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{ax^2 + by}{cy^2 + dx}$

Sol. Let $(x, y) \to (0, 0)$ along the line $y = mx$

$$\lim_{(x, y) \to (0, 0)} \frac{ax^2 + by}{cy^2 + dx} = \lim_{(x, y) \to (0, 0)} \frac{ax^2 + bmx}{cm^2x^2 + dx}$$

$$= \lim_{(x, y) \to (0, 0)} \frac{ax + bm}{cm^2x + d}$$

$$= \frac{bm}{d}, \text{ which has different values for}$$

different m. Hence the given limit does not exist.

9. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{(x^2 + y^2)^2}{x^4 + y^4}$

Sol. Let $(x, y) \to (0, 0)$ along the line $ax + by = 0$. Then

$$\lim_{(x, y) \to (0, 0)} \frac{(x^2 + y^2)^2}{x^4 + y^4} = \lim_{(x, y) \to (0, 0)} \frac{\left(x^2 + \frac{a^2}{b^2}x^2\right)^2}{x^4 + \frac{a^4}{b^4}x^4}$$

$$= \lim_{(x, y) \to (0, 0)} \frac{(a^2 + b^2)^2}{a^4 + b^4} = \frac{(a^2 + b^2)^2}{a^4 + b^4}$$

which depends on the values of a and b.
Hence the limit does not exist.

10. $\lim\limits_{(x, y) \to (0, 0)} \left(\dfrac{xy^2}{x^2 + y^4}\right)$

Sol. Let $(x, y) \to (0, 0)$ along $y = mx$. Then

$$f(x, y) = \frac{xy^2}{x^2 + y^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

$$\frac{m^2x}{1 + m^4x^2} \to 0 \text{ as } x \to 0$$

Thus along every straight line through the origin,

$f(x, y) \to 0$ as $(x, y) \to (0, 0)$

Next, let $(x, y) \to (0, 0)$ along $x = y^2$. Then

$$f(x, y) = \frac{y^4}{y^4 + y^4} = \frac{1}{2}, \text{ so that } f(x, y) \to \frac{1}{2} \text{ as } (x, y) \to (0, 0)$$

along the parabola $x = y^2$. Hence the limit does not exist.

11. Let $f(x, y) = \begin{cases} \dfrac{xy^2}{x^3 + y^3} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Show that f is not continuous at the origin.

Sol. We evaluate $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$.

Let $(0, 0)$ be approached along the line $y = mx$

$$\lim_{(x, y) \to (0, 0)} \frac{x^2 y^2}{x^3 + y^3} = \lim_{(x, y) \to (0, 0)} \frac{m^2x^3}{x^3 + m^3 x^3}$$

$$= \frac{m^2}{1 + m^3}, \text{ since } x \neq 0,$$

which will have different values for different m. Therefore, $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist. Hence the function is not continuous at $(0, 0)$.

12. Find a such that the function

$$f(x, y) = \begin{cases} \dfrac{3xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0) \\ a & \text{if } (x, y) = (0, 0) \end{cases}$$

is continuous at $(0, 0)$.

Sol. Let $x = r\cos\theta$, $y = r\sin\theta$. Then $r \to 0$ as $(x, y) \to (0, 0)$

$$f(x, y) = \frac{3r^2 \sin\theta\cos\theta}{r} = 3r\sin\theta\cos\theta$$

$3r\sin\theta\cos\theta \to 0$ as $r \to 0$.

Thus $\lim\limits_{(x, y) \to (0, 0)} f(x, y) = 0$

The function will be continuous at the origin if $f(0, 0) = 0$ which requires $a = 0$.

13. Let $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Examine the continuity at (0, 0). Do $f_x(0, 0)$ and $f_y(0, 0)$ exist?

Sol. Let $x = r = \cos\theta, y = r\sin\theta$. Then $r \to 0$ as $(x, y) \to (0, 0)$

$$f(x, y) = \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2} = r(\cos^3\theta + \sin^3\theta)$$

$$r(\cos^3\theta + \sin^3\theta) \to 0 \text{ as } r \to 0.$$

Therefore, f is continuous at (0, 0).

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = 1$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k}$$

$$= \lim_{k \to 0} \frac{k - 0}{k} = 1$$

Thus $f_x(0, 0)$ and $f_y(0, 0)$ exist.

14. Let $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Prove that f is not continuous at (0, 0). Do $f_x(0, 0)$ and $f(0, 0)$ exist?

Sol. Let (0, 0) be approached along the line $y = mx$

$$f(x, y) = \frac{mx^3}{x^4 + m^2x^2} = \frac{mx}{x^2 + m^2} \to 0 \text{ as } x \to 0.$$

But if we approach (0, 0) along $y = x^2$, then

$$f(x, y) = \frac{x^4}{x^4 + x^4} = \frac{1}{2} \text{ and so } \lim_{(x, y) \to (0, 0)} f(x, y) = \frac{1}{2}$$

Thus $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

Hence f is not continuous at (0, 0).

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$f_y(0, 0) = \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0$$

Thus $f_x(0, 0) = f_y(0, 0) = 0$.

Find the first order derivatives of the given functions (Problems 15 – 22):

15. $f(x, y) = x^{y^2}$

Sol. $f(x, y) = x^{y^2}$

$$f_x = y^2 . x^{y^2 - 1} \ ; \ f_y = 2y . x^{y^2} \ln x, \left[\frac{d}{dx}(a^x) = a^x \ln a\right]$$

16. $f(x, y) = e^{x^2 + y^2}$

Sol. $f(x, y) = e^{x^2 + y^2}$

$$f_x = (e^{x^2 + y^2})\, 2x = 2xe^{x^2 + y^2} \ ; \ f_y = (e^{x^2 + y^2})\, 2y = 2y\, e^{x^2 + y^2}$$

17. $f(x, y) = \arctan\left(\dfrac{y}{x}\right)$

Sol. $f(x, y) = \arctan\left(\dfrac{y}{x}\right)$

$$f_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1 + \left(1 + \frac{y}{x}\right)} \cdot \left(-\frac{y}{x^2}\right)$$

$$= -\frac{1}{x^2\left\{1 + \left(\frac{y}{x}\right)^2\right\}} = -\frac{y}{x^2\left\{\left(\frac{x^2 + y^2}{x^2}\right)\right\}} = -\frac{y}{x^2 + y^2}$$

$$f_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{\frac{x^2 + y^2}{x^2}} \cdot \frac{1}{x} = \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

18. $f(x, y) = \arctan(x + y)$

Sol. Let $z = \arctan(x + y)$

Then $\dfrac{\partial z}{\partial x} = \dfrac{1}{1 + (x + y)^2} . 1 = \dfrac{1}{1 + (x + y)^2}$

and $\dfrac{\partial z}{\partial y} = \dfrac{1}{1 + (x + y)^2} . 1 = \dfrac{1}{1 + (x + y)^2}$

19. $f(x, y) = e^{ax} \sin by$

Sol. Let $z = e^{ax} \sin by$

Then $\dfrac{\partial z}{\partial x} = (e^{ax} . a) \sin by = a\, e^{ax} \sin by$

and $\dfrac{\partial z}{\partial y} = e^{ax} (\cos by)\, b = b\, e^{ax} \cos by$

20. $f(x, y) = \ln(x^2 + y^2)$

Sol. Let $z = \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$

21. $f(x,y) = \ln\left[\dfrac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right]$

Sol. $f(x,y) = \ln\left\{\dfrac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right\}$

$$= \ln\{\sqrt{x^2+y^2}-x\} - \ln\{\sqrt{x^2+y^2}+x\}$$

$$f_x = \frac{1}{\sqrt{x^2+y^2}-x} \cdot \frac{\partial}{\partial x}(\sqrt{x^2+y^2}-x) - \frac{1}{\sqrt{x^2+y^2}+x} \cdot \frac{\partial}{\partial x}(\sqrt{x^2+y^2}+x)$$

$$= \frac{1}{\sqrt{x^2+y^2}-x} \cdot \left\{\frac{1}{2}(x^2+y^2)^{-1/2}(2x) - 1\right\} - \frac{1}{\sqrt{x^2+y^2}+x} \cdot \left\{\frac{1}{2}(x^2+y^2)^{-1/2}(2x) + 1\right\}$$

$$= \frac{1}{\sqrt{x^2+y^2}-x} \cdot \left\{\frac{x}{(x^2+y^2)^{1/2}} - 1\right\} - \frac{1}{\sqrt{x^2+y^2}+x} \cdot \left\{\frac{x}{(x^2+y^2)^{1/2}(2x)} + 1\right\}$$

$$= \frac{1}{\sqrt{x^2+y^2}-x} \cdot \left\{\frac{x-(x^2+y^2)^{1/2}}{(x^2+y^2)^{1/2}}\right\} - \frac{1}{\sqrt{x^2+y^2}+x} \cdot \left\{\frac{x+(x^2+y^2)^{1/2}}{(x^2+y^2)^{1/2}}\right\}$$

$$= \frac{-1}{(x^2+y^2)^{1/2}} - \frac{1}{(x^2+y^2)^{1/2}} = \frac{-2}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{1}{\sqrt{x^2+y^2}-x} \cdot \frac{\partial}{\partial y}(\sqrt{x^2+y^2}-x) - \frac{1}{\sqrt{x^2+y^2}+x} \cdot \frac{\partial}{\partial y}(\sqrt{x^2+y^2}+x)$$

$$= \frac{1}{\sqrt{x^2+y^2}-x} \cdot \left\{\frac{1}{2}(x^2+y^2)^{-1/2} \cdot (2y)\right\} - \frac{1}{\sqrt{x^2+y^2}} \cdot \left\{\frac{1}{2}(x^2+y^2)^{-1/2} \cdot (2y)\right\}$$

$$= \frac{1}{\sqrt{x^2+y^2}-x} \cdot \left(\frac{y}{\sqrt{x^2+y^2}}\right) - \frac{1}{\sqrt{x^2+y^2}+x} \cdot \left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{y}{\sqrt{x^2+y^2}} \cdot \left\{\frac{\sqrt{x^2+y^2}+x-\sqrt{x^2+y^2}+x}{(\sqrt{x^2+y^2}-x)(\sqrt{x^2+y^2}+x)}\right\}$$

$$= \frac{y}{\sqrt{x^2+y^2}} \cdot \left(\frac{2x}{x^2+y^2-x^2}\right)$$

$$= \frac{2xy}{y^2\sqrt{x^2+y^2}} = \frac{2x}{y\sqrt{x^2+y^2}}$$

22. $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

Sol. $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$

Now, $f_x = -\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2}\dfrac{\partial}{\partial x}(x^2+y^2+z^2)$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot (2x) = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$f_y = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y) = \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$f_z = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z) = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

Find all the four second order partial derivatives (Problems 23 – 26):

23. e^{x-y}

Sol. Let $z = e^{x-y}$ Then

$$\frac{\partial z}{\partial x} = e^{x-y} \cdot 1 = e^{x-y}$$

$$\frac{\partial^2 z}{\partial x^2} = e^{x-y} \cdot 1 = e^{x-y}$$

and $\dfrac{\partial z}{\partial y} = e^{x-1} \cdot (-1) = -e^{x-y}$

$$\frac{\partial^2 z}{\partial y^2} = (-e^{x-y})(-1) = e^{x-y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-e^{x-y}) = -e^{x-y} \cdot 1 = -e^{x-y}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right)(e^{x-y}) = e^{x-y} \cdot -1 = -e^{x-y}$$

24. $\dfrac{x+y}{x-y}$

Sol. Let $z = \dfrac{x+y}{x-y}$

$$\frac{\partial z}{\partial x} = \frac{(x-y) \cdot 1 - (x+y) \cdot 1}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = (-2y)(-2) \cdot \frac{1}{(x-y)^3} \cdot 1 = \frac{4y}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-2y}{(x-y)^2}\right)$$

$$= \frac{(x-y)^2(-2) - (-2y)(2)(x-y)(-1)}{(x-y)^4}$$

$$= \frac{-2(x-y)[x-y+2y]}{(x-y)^4} = \frac{-2(x+y)}{(x-y)^3}$$

Also $\dfrac{\partial z}{\partial y} = \dfrac{(x-y)(1) - (x+y)(-1)}{(x-y)^2} = \dfrac{2x}{(x-y)^2}$

$$\frac{\partial^2 z}{\partial y^2} = (2x)(x-y)^{-3}(-2)(-1) = \frac{4x}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{(x-y)^2}\right)$$

$$= \frac{(x-y)^2 \cdot 2 - 2x \cdot 2(x-y) \cdot 1}{(x-y)^4}$$

$$= \frac{2(x-y)[x-y-2x]}{(x-y)^4}$$

$$= \frac{2(-x-y)}{(x-y)^3} = \frac{-2(x+y)}{(x-y)^3}$$

25. e^{xy}

Sol. Let $u = e^{xy}$. Then

$$\frac{\partial u}{\partial x} = e^{xy} \times yx^{y-1} = yx^{y-1} e^{xy}$$

$$\frac{\partial u}{\partial y} = e^{xy} \frac{\partial}{\partial y}(x^y) = e^{xy} x^y \ln x$$

$$\frac{\partial^2 u}{\partial x^2} = y(y-1)x^{y-2} e^{xy} + yx^{y-1} \cdot e^{xy} \cdot (yx^{y-1})$$

$$= [y(y-1)x^{y-2} + y^2 x^{2y-2}] e^{xy}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}(y x^{y-1} e^{xy})$$

$$= x^{y-1} e^{xy} \frac{\partial}{\partial x}(y) + yx^{y-1} \frac{\partial}{\partial x}(e^{xy}) + ye^{xy} \frac{\partial}{\partial x}(x^{y-1})$$

$$= x^{y-1} e^{xy} + yx^{y-1} e^{xy}(x^y \ln x) + ye^{xy} x^{y-1} \ln x$$

$$= e^{xy}(x^{y-1} + yx^{2y-1} \ln x + yx^{y-1} \ln x)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(e^{xy} x^y \ln x) = e^{xy} \frac{\partial}{\partial x}(x^y) \ln x + \frac{\partial}{\partial x}(e^{xy}) x^y \ln x$$

$$= e^{xy} x^y (\ln x)^2 + e^{xy} x^y \ln x (x^y \ln x)$$

$$= e^{xy}[x^y (\ln x)^2 + x^{2y} (\ln x)^2]$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(e^{xy} x^y \ln x)$$

$$= e^{xy} x^y \frac{\partial}{\partial x}(\ln x) + e^{xy} \ln x \frac{\partial}{\partial x}(x^y) + x^y \ln x \frac{\partial}{\partial x}(e^{xy})$$

$$= e^{xy} x^y \times \frac{1}{x} + e^{xy} \ln x \, yx^{y-1} + x^y \ln x \, e^{xy} yx^{y-1}$$

$$= e^{xy} x^{y-1} + e^{xy} y \ln x \, x^{y-1} + e^{xy}(x^{2y-1} y \ln x)$$

$$= e^{xy}[x^{y-1} + yx^{y-1} \ln x + x^{2y-1} y \ln x]$$

26. $\tan(\arctan x + \arctan y)$

Sol. Let $z = \tan(\arctan x + \arctan y)$

$$= \frac{\tan \arctan x + \tan \arctan y}{1 - \tan(\text{arc}\tan x)\tan(\arctan y)} = \frac{x+y}{1-xy}$$

$$\frac{\partial z}{\partial x} = \frac{(1-xy) \cdot 1 - (x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(1-xy) \cdot 1 - (x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = (1+y^2)[-2(1-xy)^{-3}(-y) = \frac{2y(1+y^2)}{(1-xy)^3}$$

$$\frac{\partial^2 z}{\partial y^2} = (1+x^2).[-2(1-2y)^{-3} \times (-x)] = \frac{2x(1+x^2)}{(1-xy)^3}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{1+y^2}{(1-xy)^2}\right]$$

$$= \frac{(1-xy)^2 \cdot 2y - (1+y^2) \cdot 2(1-xy)(-x)}{(1-xy)^4}$$

$$= \frac{2y(1-xy)^2 + 2x(1+y^2)(1-xy)}{(1-xy)^4}$$

$$= \frac{2y(1-xy) + 2x(1+y^2)}{(1-xy)^3} = \frac{2(x+y)}{(1-xy)^3}$$

Similarly, or by symmetry (students are advised to workout themselves)

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{2(y+x)}{(1-xy)^3}.$$

In Problems 27 – 32 verify that $f_{xy} = f_{yx}$:

27. $f(x, y) = e^{xy}\cos(bx+c)$

Sol. $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{xy})\cos(bx+c) + e^{xy}\frac{\partial}{\partial x}\cos(bx+c)$

$$= ye^{xy}\cos(bx+c) - be^{xy}\sin(bx+c)$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \cos(bx+c)\left\{y\frac{\partial}{\partial y}(e^{xy}) + e^{xy}\frac{\partial}{\partial y}(y)\right\}$$

$$- b\sin(bx+c)\frac{\partial}{\partial y}(e^{xy})$$

$$= \cos(bx+c)\{xy\,e^{xy} + e^{xy}\} - b\sin(bx+c)\,xe^{xy}$$

or $f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = xy\,e^{xy}\cos(bx+c) + e^{xy}\cos(bx+c)$

$$-bx\,e^{xy}\sin(bx+c) \qquad (1)$$

$$f_y = \frac{\partial f}{\partial y} = x\,e^{xy}\cos(bx+c)$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = x\,e^{xy}\frac{\partial}{\partial x}\cos(bx+c) + x\cos(bx+c)\frac{\partial}{\partial x}(e^{xy})$$

$$+ e^{xy}\cos(bx+c)\frac{\partial}{\partial x}(x)$$

$$= -bx\,e^{xy}\sin(bx+c) + xy\cos(bx+c)\,e^{xy} + e^{xy}\cos(bx+c)$$

$$= xy\,e^{xy}\cos(bx+c) + e^{xy}\cos(bx+c) - bx\,e^{xy}\sin(bx+c) \qquad (2)$$

From (1) and (2), we see that $f_{xy} = f_{yx}$

28. $f(x, y) = \ln(e^x + e^y)$

Sol. $f(x, y) = \ln(e^x + e^y)$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{e^x+e^y}\frac{\partial}{\partial x}(e^x+e^y)$$

$$= \frac{1}{e^x+e^y}\cdot e^y = \frac{e^y}{e^x+e^y}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{(e^x+e^y)\,.\,0 - e^y(e^x)}{(e^x+e^y)^2} = \frac{e^x e^y}{(e^x+e^y)^2}$$

or $f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = -\frac{e^{x+y}}{(e^x+e^y)^2} \qquad (1)$

Now $f_x = \frac{\partial f}{\partial x} = \frac{1}{(e^x+e^y)}\cdot\frac{\partial}{\partial x}(e^x+e^y) = \frac{1}{e^x+e^y} = \frac{e^x}{e^x+e^y}$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{(e^x+e^y)\,.\,0 - e^x\,.\,e^y}{(e^x+e^y)^2} = -\frac{e^x e^y}{(e^x+e^y)^2}$$

or $f_{xy} = \frac{\partial^2 f}{\partial x\,\partial y} = -\frac{e^{x+y}}{(e^x+e^x)^2} \qquad (2)$

Hence (1) and (2), we get $f_{xy} = f_{yx}$

29. $f(x, y) = \ln\left(\frac{x^2+y^2}{xy}\right)$

Sol. $f(x, y) = \ln\left(\frac{x^2+y^2}{xy}\right) = \ln(x^2+y^2) - \ln(xy)$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial x}(x^2+y^2) - \frac{1}{xy}\cdot\frac{\partial}{\partial x}(xy)$$

$$= \frac{1}{x^2+y^2}(2y) - \frac{1}{xy}\,.\,(x) = \frac{2y}{x^2+y^2} - \frac{x}{xy} = \frac{2y}{x^2+y^2} - \frac{1}{y}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{(x^2+y^2)\frac{\partial}{\partial x}(2y) - (2y)\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} - \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$

$$= \frac{(x^2+y^2)\,(0) - 2y\,(2x)}{(x^2+y^2)^2} - 0 = -\frac{4xy}{(x^2+y^2)^2}$$

or $f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = -\frac{4xy}{(x^2+y^2)^2} \qquad (1)$

$$f_x = \frac{1}{(x^2+y^2)}\frac{\partial}{\partial x}(x^2+y^2) - \frac{1}{(xy)}\frac{\partial}{\partial x}(xy)$$

$$= \frac{1}{x^2+y^2}\cdot(2x) - \frac{1}{(xy)}\cdot y = \frac{2x}{x^2+y^2} - \frac{1}{x}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{(x^2+y^2)\frac{\partial}{\partial y}(2x) - (2x)\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} - \frac{\partial}{\partial y}\left(\frac{1}{x}\right)$$

$$= \frac{(x^2+y^2)0 - (2x)\,(0+2y)}{(x^2+y^2)^2} - 0 = -\frac{4xy}{(x^2+y^2)^2}$$

or $f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = -\frac{4xy}{(x^2+y^2)^2} \qquad (2)$

From (1) and (2), we get $f_{xy} = f_{yx}$

30. $f(x, y) = x^y + y^x$

Sol. $f(x, y) = x^y + y^x$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^y) + \frac{\partial}{\partial y}(y^x) = x^y\ln x + xy^{x-1}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = x^y\frac{\partial}{\partial x}(\ln x) + \ln x\frac{\partial}{\partial x}(x^y) + x\frac{\partial}{\partial x}(y^{x-1}) + y^{x-1}\frac{\partial}{\partial x}(x)$$

$$= x^{y-1} + yx^{y-1}\ln x + xy^{x-1}\ln y + y^{x-1}$$

or $\quad f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y} = x^{y-1}(1 + y\ln x) + y^{x-1}(1 + x\ln y) \qquad (1)$

Now $f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(x^y) + \dfrac{\partial}{\partial x}(y^x) = yx^{y-1} + y^x \ln y \,.\, 1$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = y \,.\, \frac{y}{\partial y}(x^{y-1}) + (x^{y-1})\frac{\partial}{\partial y}(y) + y^x \frac{\partial}{\partial y}(\ln y) + \text{In}\, y \frac{\partial}{\partial y}(y^x)$$

$$= y \,.\, x^{y-1} \,.\, \ln x + x^{y-1} + y^x \,.\, \frac{1}{y} + \ln y(x \,.\, y^{x-1})$$

$$= yx^{y-1}\ln x + x^{y-1} + y^{x-1} + xy^{x-1}\ln y$$

$$= x^{y-1}(y\ln x + 1) + y^{x-1}(1 + x\ln y) \qquad (2)$$

From (1) and (2), we get $f_{xy} = f_{yx}$

31. $f(x, y) = x\sin xy + y\cos xy$

Sol. $f_x = \sin xy + xy\cos xy - y^2 \sin xy$

$$f_{xy} = x\cos xy + x\cos xy - x^2y\sin xy - 2y\sin xy - xy^2\cos xy$$

$$= (2x - xy^2)\cos xy - (x^2y + 2y)\sin xy \qquad (1)$$

$$f_y = x^2\cos xy + \cos xy - xy\sin xy$$

$$f_{yx} = 2x\cos xy - x^2y\sin xy - y\sin xy - y\sin xy - xy^2\cos xy$$

$$= (2x - xy^2)\cos xy - (x^2y + 2y)\sin xy \qquad (2)$$

It is clear from (1) and (2) that $f_{xy} = f_{yx}$

32. $f(x, y) = \dfrac{xy}{\sqrt{1 + x^2 + y^2}}$

Sol. $f(x, y) = \dfrac{xy}{\sqrt{1 + x^2 + y^2}}$

$$f_y = \frac{\partial f}{\partial y} = x \cdot \frac{\sqrt{1 + x^2 + y^2} \,.\, 1 - y \,.\, \frac{1}{2}\frac{2y}{\sqrt{1 + x^2 + y^2}}}{(\sqrt{1 + x^2 + y^2})^2}$$

$$= \frac{x}{1 + x^2 + y^2}\left[\frac{1 + x^2 + y^2 - y^2}{\sqrt{1 + x^2 + y^2}}\right] = \frac{x(1 + x^2)}{(1 + x^2 + y^2)^{3/2}}$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{x(1 + x^2)}{(1 + x^2 + y^2)^{3/2}}\right]$$

$$= \frac{(1 + x^2 + y^2)^{3/2}(1 + 3x^2) - x(1 + x^2) \,.\, {}^3/_2 (1 + x^2 + y^2)^{1/2} \,.\, 2x}{(1 + x^2 + y^2)^3}$$

$$= (1 + x^2 + y^2)^{1/2}\left[\frac{(1 + x^2 + y^2)(1 + 3x^2) - (x + x^3) \,.\, 3x}{(1 + x^2 + y^2)^3}\right]$$

$$= \frac{1}{(1 + x^2 + y^2)^{5/2}}[1 + x^2 + y^2 + 3x^2 + 3x^4 + 3x^2y^2 - 3x^2 - 3x^4]$$

$$= \frac{1 + x^2 + y^2 + 3x^2y^2}{(1 + x^2 + y^2)^{5/2}} \qquad (1)$$

Also

$$f_x = \frac{\partial f}{\partial x} = y \cdot \frac{\sqrt{1 + x^2 + y^2} \,.\, 1 - x \,.\, \frac{1}{2}\frac{2x}{\sqrt{1 + x^2 + y^2}}}{(1 + x^2 + y^2)}$$

$$= \frac{y}{1 + x^2 + y^2} \cdot \frac{1 + y^2}{\sqrt{1 + x^2 + y^2}} = \frac{y(1 + y^2)}{(1 + x^2 + y^2)^{3/2}}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left[\frac{y(1 + y^2)}{(1 + x^2 + y^2)^{3/2}}\right]$$

$$= \frac{(1 + x^2 + y^2)^{3/2}(1 + 3y^2) - y(1 + y^2)\frac{3}{2}(1 + x^2 + y^2)^{1/2} \,.\, 2y}{(1 + x^2 + y^2)^3}$$

$$= (1 + x^2 + y^2)^{1/2}\frac{[(1 + x^2 + y^2)(1 + 3y^2) - 3y^2(1 + y^2)]}{(1 + x^2 + y^2)^3}$$

$$= \frac{[1 + x^2 + y^2 + 3x^2y^2]}{(1 + x^2 + y^2)^{5/2}} \qquad [1 + x^2 + y^2 + 3x^2y^2] \qquad (2)$$

From (1) and (2), we see that

$$f_{xy} = f_{yx}$$

Show that each of the following functions satisfies Laplace's equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. (Problems 33 – 36):

33. $f(x, y) = \sin x \sinh y$

Sol. $\dfrac{\partial f}{\partial x} = \cos x \sinh y \quad$ and $\quad \dfrac{\partial^2 f}{\partial x^2} = -\sin x \sinh y \qquad (1)$

$\dfrac{\partial f}{\partial y} = \sin x \cosh y \quad$ and $\quad \dfrac{\partial^2 f}{\partial y^2} = \sin x \sinh y \qquad (2)$

Adding (1) and (2), we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\sin x \sinh y + \sin x \sinh y = 0$$

34. $f(x, y) = e^{-x}\cos y$

Sol. $\dfrac{\partial f}{\partial x} = -e^{-x}\cos y \quad$ and $\quad \dfrac{\partial^2 f}{\partial x^2} = e^{-x}\cos y \qquad (1)$

$\dfrac{\partial f}{\partial y} = e^{-x}(-\sin y) = -e^{-x}\sin y \quad$ and $\quad \dfrac{\partial^2 f}{\partial y^2} = -e^{-x}\cos y \qquad (2)$

Adding (1) and (2), we have

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=e^{-x}\cos y-e^{-x}\cos y=0$$

35. $f(x, y)=\ln\sqrt{x^2+y^2}$

Sol. $f(x, y)=\ln\sqrt{x^2+y^2}=\frac{1}{2}\ln(x^2+y^2)$

$$\frac{\partial f}{\partial x}=\frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x=\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2} \qquad (1)$$

$$\frac{\partial f}{\partial y}=\frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y=\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2} \qquad (2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0$$

36. $f(x, y)=\arctan\left(\frac{2xy}{x^2-y^2}\right)$

Sol. $f(x, y)=\arctan\left(\frac{2xy}{x^2-y^2}\right)$

$$\frac{\partial f}{\partial x}=\frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}}\times\frac{(x^2-y^2)\,2y-2xy\,.\,2x}{(x^2-y^2)^2}$$

$$=\frac{(x^2-y^2)^2}{(x^2+y^2)^2}\times\frac{-2y(x^2+y^2)}{(x^2-y^2)^2}=\frac{-2y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2}=(-2y)\,(-1)\,(x^2+y^2)^{-2}\cdot 2x=\frac{4xy}{(x^2+y^2)^2} \qquad (1)$$

$$\frac{\partial f}{\partial y}=\frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}}\times\frac{(x^2-y^2)\,2x-2xy\,.\,(-2y)}{(x^2-y^2)^2}$$

$$=\frac{(x^2-y^2)^2}{(x^2+y^2)^2}\times\frac{2x\,(x^2+y^2)}{(x^2-y^2)^2}=\frac{2x}{(x^2+y^2)}$$

$$\frac{\partial^2 f}{\partial y^2}=2x\,(-1)\,(x^2+y^2)^{-2}\times 2y\,\frac{-4xy}{(x^2+y^2)^2} \qquad (2)$$

Adding (1) and (2), we see that

$$\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial y^2}=0$$

37. $f(x, y)=x^2\arctan\left(\frac{y}{x}\right)-y^2\arctan\left(\frac{x}{y}\right)$, show that $\frac{\partial^2 f}{\partial x\,\partial y}=\frac{x^2-y^2}{x^2+y^2}$.

Sol. $f(x, y)=x^2\arctan\frac{y}{x}-y^2\operatorname{arc\,tan}\frac{x}{y}$

$$\frac{\partial f}{\partial y}=x^2\cdot\frac{\partial}{\partial y}\left(\arctan\frac{y}{x}\right)+\arctan\frac{y}{x}\cdot\frac{\partial}{\partial y}\cdot(x^2)$$

$$-y^2\cdot\frac{\partial}{\partial y}\left(\arctan\frac{x}{y}\right)+\operatorname{arc\,tan}\frac{x}{y}\cdot\frac{\partial}{\partial y}(y^2)$$

$$=\frac{x^2\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2}+\left(\arctan\frac{y}{x}\right)\cdot 0-y^2\frac{1\left(-\frac{x}{y^2}\right)}{1+\left(\frac{x}{y}\right)^2}-2y\arctan\frac{x}{y}$$

$$=\frac{x}{\frac{x^2+y^2}{x^2}}+\frac{x}{\frac{x^2+y^2}{y^2}}-2y\arctan\frac{x}{y}$$

$$=\frac{x^3}{x^2+y^2}+\frac{xy^2}{x^2+y^2}-2y\arctan\frac{x}{y}$$

$$=\frac{x(x^2+y^2)}{x^2+y^2}-2y\arctan\frac{x}{y}=x-2y\arctan\frac{x}{y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(x-2y\arctan\frac{x}{y}\right)=1-2y\frac{1}{1+\left(\frac{x}{y}\right)^2}\times\frac{1}{y}$$

or $$\frac{\partial^2 f}{\partial x\partial y}=1-\frac{2}{1+\frac{x^2}{y^2}}=1-\frac{2y^2}{x^2+y^2}=\frac{x^2+y^2-2y^2}{x^2+y^2}=\frac{x^2-y^2}{x^2+y^2}$$

38. If $f(x, y)=\frac{x^2+y^2}{x+y}$, that $(f_x-f_y)^2=4(1-f_x-f_y)$

Sol. $f(x, y)=\frac{x^2+y^2}{x+y}$

$$f_x=\frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2)-(x^2+y^2)\left(\frac{\partial}{\partial x}\right)(x+y)}{(x+y)^2}$$

$$=\frac{(x+y)\,(2x)-(x^2+y^2)\,.\,1}{(x+y)^2}=\frac{x^2+2xy-y^2}{(x+y)^2}$$

$$f_y=\frac{(x+y)\,2y-(x^2+y^2)\,.\,1}{(x+y)^2}=\frac{2xy+y^2-x^2}{(x+y)^2}$$

$$f_x-f_y=\frac{x^2+2xy-y^2-(2xy+y^2-x^2)}{(x+y)^2}$$

$$= \frac{2x^2 - 2y^2}{(x+y)^2} = \frac{2(x^2 - y^2)}{(x+y)^2} = \frac{2(x-y)}{x+y}$$

$$(f_x - f_y)^2 = \frac{4(x-y)^2}{(x+y)^2}$$

Again, $1 - f_x - f_y = 1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{2xy + y^2 - x^2}{(x+y)^2}$

$$= 1 - \left[\frac{2xy + 2xy}{(x+y)^2}\right] = 1 - \frac{4xy}{(x+y)^2}$$

$$= \frac{(x+y)^2 - 4xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2}$$

Therefore, $4(1 - f_x - f_y) = \frac{4(x-y)^2}{(x+y)^2}$

Hence $(f_x - f_y)^2 = 4(1 - f_x - f_y)$

39. Show that the function $f(x, y) = \sin xy$ satisfies the differential equation $x^2 f_{xx} - y^2 f_{yy} = 0$.

Sol. $f(x, y) = \sin xy$

$f_x = y\cos(xy)$ and $f_{xx} = -y^2 \sin(xy)$

$f_y = x\cos(xy)$ and $f_{yy} = -x^2 \sin(xy)$

$x^2 f_{xx} - y^2 f_{yy} = -x^2 y^2 \sin(xy) + y^2 x^2 \sin(xy) = 0$ as required

40. Let $f(x, y) = \begin{cases} x^2 \arctan\left(\frac{y}{x}\right) - y^2 \arctan\left(\frac{x}{y}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

Sol. $f_x(x, y) = x^2 \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(\frac{-y}{x^2}\right) + 2x \arctan\frac{y}{x} - y^2 \cdot \frac{1}{\left(1 + \frac{x^2}{y^2}\right)} \cdot \frac{1}{y}$

$$= \frac{-x^2 y}{x^2 + y^2} + 2x \arctan\frac{y}{x} - \frac{y^3}{x^2 + y^2}$$

$$= \frac{-y(x^2 + y^2)}{x^2 + y^2} + 2x \tan\frac{y}{x}$$

$$= 2x \arctan\frac{y}{x} - y$$

Hence $f_x(0, k) = -k$.

Similarly, $f_y(x, y) = x - 2y \arctan\frac{x}{y}$

and so $f_y(h, 0) = h$

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

$$f_{xy}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = \lim_{k \to 0} \frac{-k}{k} = -1$$

And $f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$

Thus $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

41. (*i*) Let $f(x, y, z) = x^3 + 3yz + \sin xyz$. Prove that $f_{xyz} = f_{zxy}$

(*ii*) If $f(x, y, z, w) = \frac{xy}{z + w}$, show that $f_{xyzw} = \frac{2}{(x + w)^3}$

Sol.

(*i*) $f(x, y, z) = x^3 + 3yz + \sin xyz$

$f_x = 3x^2 + yz\cos xyz$

$f_{xy} = z\cos xyz + yz(xz)(-\sin xyz) = z\cos xyz - xyz^2 \sin xyz$

$f_{xyz} = \cos xyz - xyz\sin xyz - 2xyz(\sin xyz) - x^2y^2z^2(\cos xyz)$ (1)

$f_z = 3y + xy\cos xyz$

$f_{zx} = y\cos xyz - xy^2 z\sin xyz$

$f_{zxy} = \cos xyz - xyz\sin xyz - 2xyz\sin xyz - x^2y^2z^2\cos xyz$ (2)

The result follows from (1) and (2).

(*ii*) If $f(x, y, z, w) = \frac{xy}{z + w}$, show that $f_{xyzw} = \frac{2}{(x + w)^3}$.

Sol. $f(x, y, z, w) = \frac{xy}{x + w}$

$f_x = \frac{y}{z + w}$ and $f_{xy} = \frac{1}{z + w}$

$$f_{xyz} = \frac{-1}{(z + w)^2}$$

$$f_{xyzw} = \frac{0 - (-1) \cdot 2(z + w)}{(z + w)^4} = \frac{2}{(z + w)^3} \text{ as required.}$$

In Problems 42 – 45, find $\frac{dy}{dx}$ by using partial derivatives:

42. $y^2 + x^2 y + ax^4 = 0$

Sol. Let $f(x, y) = y^2 + x^2 y + ax^4 = 0$

Now $f_x = 2xy + 4ax^3$

and $f_y = 2y + x^2$

Therefore, $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{2xy + 4ax^3}{2y + x^2}$

43. $3x^2 - y^2 + x^3 = 0$

Sol. Let $f(x, y) = 3x^2 - y^2 + x^3 = 0$

Now $f_x = 6x + 3x^2$ and $f_y = -2y$

Therefore, $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{6x + 3x^2}{-2y} = \dfrac{6x + 3x^2}{2y}$

44. $x^2 + xy + y^2 + ax + by = 0$

Sol. Let $f(x, y) = x^2 + xy + y^2 + ax + by = 0$

Now $f_x = 2x + y + a$

and $f_y = x + 2y + b$

Thus $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{2x + y + a}{x + 2y + b}$

45. $x^3 + x^2 + xy^2 + \sin y = 0.$

Sol. Let $f(x, y) = x^3 + x^2 + xy^2 + \sin y = 0$

Now, $f_x = 3x^2 + 2x + y^2$

$f_y = 2xy + \cos y$

Therefore, $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{3x^2 + 2x + y^2}{2xy + \cos y}$

Chapter 3 GENERAL THEOREMS, INDETERMINATE FORMS

Exercise Set 3.1 (Page 107)

Discuss the validity of Rolle's Theorem. Find c (wherever possible) such that $f'(c) = 0$ (Problems 1 – 6):

1. $f(x) = x^2 - 3x + 2$ on $[1, 2]$

Sol. $f(1) = (1)^2 - 3(1) + 2 = 1 - 3 + 2 = 0$

and $f(2) = (2)^2 - 3(2) + 2 = 4 - 6 + 2 = 0$

Thus $f(1) = f(2)$

Moreover, $f(x)$ is continuous on $[1, 2]$ and differentiable on $]1, 2[$

Hence all the conditions of Rolle's Theorem are satisfied.

Therefore, there must exist a point c in $]1, 2[$ such that $f'(c) = 0$

Now $f'(x) = 2x - 3$ and so $f'(c) = 2c - 3$

$$f'(c) = 0 \Rightarrow 2c - 3 = 0 \quad \text{or} \quad c = \frac{3}{2}$$

Hence Rolle's Theorem is valid and $c = \dfrac{3}{2}$

2. $f(x) = \sin^2 x$ on $[0, \pi]$

Sol. $f(0) = \sin^2(0) = 0$ and $f(\pi) = \sin^2(\pi) = 0$

Thus $f(0) = 0 = f(\pi)$.

Moreover, $f(x)$ is continuous on $[0, \pi]$ and differentiable on $]0, \pi[$.

Hence all the conditions of Rolle's Theorem are satisfied.

There must exist a point c in the interval $]0, \pi[$ such that $f'(c) = 0$

Now $f'(c) = 0$

Now $f'(x) = 2 \sin x \cos x$

$f'(c) = 2 \sin c \cos c = \sin 2c$

$$f'(c) = 0 \Rightarrow \sin 2c = 0 \quad \text{or} \quad 2c = 0, \pi \Rightarrow c = 0, \frac{\pi}{2}$$

Thus $c = \dfrac{\pi}{2}$, since $0 \notin]0, \pi[$

Hence Rolle's Theorem is valid and $c = \dfrac{\pi}{2}$

3. $f(x) = 1 - x^{3/4}$ on $[-1, 1]$

Sol. $f(-1) = 1 - (-1)^{3/4} = 1 - \left(\dfrac{\pm 1 \pm i}{\sqrt{2}}\right) \neq 0$

$f(1) = 1 - (1)^{3/4} = 1 - 1 = 0$

Thus $f(-1) \neq f(1)$ and one of the conditions of Rolle's Theorem is not satisfied. The Rolle's Theorem is not valid and we cannot calculate the value of c.

4. $f(x) = \dfrac{1 - x^2}{1 + x^2}$ on $[-1, 1]$

Sol. $f(-1) = 0 = f(1)$. $f(x)$ is continuous on $[-1, 1]$ and differentiable on $]-1, 1[$.

$$f'(x) = \frac{(1 + x^2)(-2x) - (1 - x^2) \cdot 2x}{(1 + x^2)^2} = \frac{-4x}{(1 + x^2)^2}$$

and $f'(0) = \dfrac{-4(0)}{(1 + 0)^2} = 0$

Thus Rolle's Theorem holds for the given function and $c = 0$.

5. $f(x) = x(x + 3)e^{-\frac{1}{2}x}$ on $[-3, 0]$

Sol. $f(-3) = -3(-3 + 3)e^{-\frac{1}{2}(-3)} = -3\,(0)\,e^{\frac{3}{2}} = 0$

$f(0) = 0\,(0 + 3)e^{-\frac{1}{2}(0)} = 0$

Thus $f(-3) = 0 = f(0)$

Moreover, $f(x)$ is continuous on $[-3, 0]$ and differentiable on $]-3, 0[$. By Rolle's Theorem, there must exist a point c in the interval $]-3, 0[$ such that $f'(c) = 0$

Now, $f'(x) = (x^2 + 3x)\left(-\frac{1}{2} \cdot e^{-\frac{1}{2}x}\right) + e^{-\frac{1}{2}x} \cdot (2x + 3)$

$$= e^{-\frac{x}{2}}\left[\frac{-x^2 - 3x}{2} + 2x + 3\right] = e^{-\frac{x}{2}}\left[\frac{-x^2 - 3x + 4x + 6}{2}\right]$$

$$= -\frac{1}{2} \cdot e^{-\frac{x}{2}}\,(x^2 - x - 6)$$

$f'(c) = 0$ gives $c^2 - c - 6 = 0$

$\Rightarrow \quad (c - 3)(c + 2) = 0 \quad \Rightarrow \quad c = 3, -2$

But only $c = -2$ lies in the interval $]-3, 0[$.

Hence the Rolle's Theorem is valid and $c = -2$.

6. $f(x) = 2 + (x - 1)^{3/2}$ on $[0, 2]$

Sol. $f(0) = 2 + (0 - 1)^{3/2} = 2 + (-1)^{3/2} = 2 - i$ or $2 + i$ $\quad (i = \sqrt{-1})$

$f(2) = 2 + (2 - 1)^{3/2} = 2 + (1)^{3/2} = 3$

Thus $f(0) \neq f(2)$

One of the conditions of Rolle's Theorems is not satisfied and so Rolle's Theorem is not valid.

Find c (wherever possible) of the Mean Value Theorem (Problems 7 – 10):

7. $f(x) = x^3 - 3x - 1$ on $\left[\frac{-11}{7}, \frac{13}{7}\right]$

Sol. $f\left(-\frac{11}{7}\right) = \left(-\frac{11}{7}\right)^3 - 3\left(-\frac{11}{7}\right) - 1 = \frac{-1331}{343} + \frac{33}{7} - 1 = \frac{-57}{343}$

$f\left(\frac{13}{7}\right) = \left(\frac{13}{7}\right)^3 - 3\left(\frac{13}{7}\right) - 1 = \frac{2197}{343} - \frac{39}{7} - 1 = \frac{-57}{343}$

Now, $f'(x) = 3x^2 - 3$; $f'(c) = 3c^2 - 3$

By the Mean Value Theorem,

$f(b) - f(a) = (b - a)f'(c)$ implies

$$\frac{-57}{343} - \left(\frac{-57}{343}\right) = \left[\left(\frac{13}{7}\right) - \left(\frac{-11}{7}\right)\right](3c^2 - 3)$$

or $\quad \dfrac{-57}{343} + \dfrac{57}{343} = \left(\dfrac{13}{7} + \dfrac{11}{7}\right)(3c^2 - 3)$ or $0 = \left(\dfrac{13}{7} + \dfrac{11}{7}\right)(3c^2 - 3)$

$\Rightarrow \quad 3c^2 - 3 = 0 \quad$ or $\quad c = \pm 1.$

8. $f(x) = \sqrt{x - 2}$ on $[2, 4]$

Sol. $f(2) = 0, \qquad f(4) = \sqrt{4 - 2} = \sqrt{2}$

$$f'(x) = \frac{1}{2\sqrt{x - 2}}$$

By the Mean Value Theorem, we have

$\dfrac{f(4) - f(2)}{4 - 2} = f'(c)$ i.e., $\dfrac{\sqrt{2}}{2} = \dfrac{1}{2\sqrt{c - 2}}$

or $\quad \sqrt{2}\sqrt{c - 2} = 1 \quad$ or $2c - 4 = 1$ or $c = \dfrac{5}{2}$

9. $f(x) = x^3 - 5x^2 + 4x - 2$ on $[1, 3]$

Sol. $f(1) = 1 - 5 + 4 - 2 = -2$

$f(3) = 27 - 45 + 12 - 2 = -8$

$f'(x) = 3x^2 - 10x + 4$

By the Mean Value Theorem, we have

$$\frac{f(3) - f(1)}{3 - 1} = f'(c)$$

$\dfrac{-8 + 2}{2} = 3c^2 - 10c + 4$ or $3c^2 - 10c + 7 = 0$

$c = \dfrac{10 \pm \sqrt{100 - 84}}{6} = \dfrac{10 \pm 4}{6} = \dfrac{7}{3}, 1.$ Thus $c = \dfrac{7}{3}$

10. $f(x) = x^{2/3}$ on $[-1, 1]$.

Sol. Here $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

$f'(0)$, is not defined at $x = 0$.

Thus $f(x)$ is not differentiable on $]-1, 1[$ and the M.V.T. fails.

In Problems 11–15, use the Mean Value Theorem to show that:

11. $|\sin x - \sin y| \le |x - y|$ for any real numbers x, y.

Sol. Let $f(t) = \sin t$. Then, $f(t)$ is continuous and differentiable for every real t, we apply the M.V.T. to $f(t) = \sin t$ in the interval $[x, y]$ where x, y are any real numbers. Therefore,

$$\frac{\sin y - \sin x}{y - x} = \cos z,\ z \in]x, y[\qquad (f'(t) = \cos t)$$

Taking modulus of both sides, we get

$$\left|\frac{\sin y - \sin x}{y - x}\right| = |\cos z|$$

or $\quad |\sin x - \sin y| = |x - y|\ |\cos z| \le |x - y|,$

since $\quad |\cos z| \le 1$.

12. $\left|\frac{\cos ax - \cos bx}{x}\right| \le |b - a|$, if $x \ne 0$.

Sol. Let $f(t) = \cos t$. $f(t)$ is continuous and differentiable for all real t. We apply the M.V.T. to $f(t)$ in the interval $[ax, bx]$ where $x \ne 0$. Therefore,

$$\frac{\cos bx - \cos ax}{bx - ax} = -\sin z,\ z \in]ax, bx[\qquad (f'(t) = -\sin t)$$

or $\quad \left|\frac{\cos bx - \cos ax}{x}\right| = |b - a|\ |\sin(-z)| \le |b - a|,$

since $|\sin(-z)| \le 1$.

13. $|\tan x + \tan y| \ge |x + y|$ for all real numbers x and y in the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

Sol. The function $f(t) = \tan t$ is continuous on $[-x, y] \subset \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and differentiable in $]-x, y[$. Thus M.V.T. applies and we have

$$\frac{\tan y - \tan(-x)}{y + x} = \sec^2 z,\ z \in]-x, y[\qquad (f'(t) = \sec^2 t)$$

or $\quad \left|\frac{\tan x + \tan y}{y + x}\right| = |\sec^2 z|$

i.e., $\quad |\tan x + \tan y| = |x+y|\ |\sec^2 z| \ge |x+y|$, since $|\sec^2 z| \ge 1$.

14. $(1 + x)^a > 1 + ax$, where $a > 1$ and $x > 0$. [Bernoulli's Inequality]

Sol. Let $f(x) = (1 + x)^a - (1 + ax)$. Then $f(0) = 0$ and f satisfies the conditions of the M.V.T. on $[0, x]$

By applying the M.V.T. to f on $[0, x]$, we have

$$\frac{f(x) - f(0)}{x - 0} = f'(c), \qquad c \in]0, x[$$

i.e., $\quad (1 + x)^a - (1 + ax) = xf'(c) = x[a(1 + c)^{a-1} - a]$

$= ax[(1 + c)^{a-1} - 1] > 0,$

since $x > 0, a > 1, (1 + c)^{a-1} > 1$

Therefore, $(1 + x)^a > 1 + ax$.

15. $\frac{1}{6} < \sqrt{27} - 5 < \frac{1}{5}$. Also approximate $\sqrt{168}$ by using the Mean Value Theorem.

Sol. Consider the function f defined by $f(x) = \sqrt{x}$ on the interval $[25, 27]$. f satisfies the hypothesis of the M.V.T. on $[25, 27]$. Therefore,

$$\frac{\sqrt{27} - \sqrt{25}}{27 - 25} = f'(c) = \frac{1}{2\sqrt{c}}, \qquad (1)$$

$c \in]25, 27[\quad$ *i.e.,* $\quad 25 < c < 27$, or $\quad \sqrt{25} < \sqrt{c} < \sqrt{27}$

i.e., $\quad \frac{1}{\sqrt{27}} < \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{25}} = \frac{1}{5}$

Now, $\frac{1}{6} = \frac{1}{\sqrt{36}} < \frac{1}{\sqrt{27}}$ and so $\frac{1}{6} < \frac{1}{\sqrt{c}} < \frac{1}{5} \qquad (2)$

From (1), we have $\sqrt{27} - 5 = \frac{1}{\sqrt{c}}$. Substitution in (2) yields

$\frac{1}{6} < \sqrt{27} - 5 < \frac{1}{5}$ as desired.

Consider $f(x) = \sqrt{x}$ on $[168, 169]$

By the MVT, $f(169) - f(168) = f'(c)(169 - 168)$,

where $\quad 168 < c < 169$.

Therefore, $\sqrt{169} - \sqrt{168} = \frac{1}{2\sqrt{c}}$.

The exact value of $\sqrt{c}$ is not known but it is near 13 and so

$$13 - \sqrt{168} \approx \frac{1}{26}$$

or $\quad \sqrt{168} \approx 13 - \frac{1}{26} = 12\frac{25}{26} \approx 12.9615.$

16. Let a function f be continuous on $[a, b]$ and $f'(x) = 0$ for all $x \in]a, b[$. Prove that f is constant on $[a, b]$. Use this to show that $\sin^2 x + \cos^2 x = 1$ for all real numbers x.

Sol. Let $f(x)$ be the differentiable in the interval $]a, b[$ and let x_1, x_2, be any two points belonging to this interval such that $x_2 > x_1$.

Applying the Mean Value Theorem on the interval $[x_1, x_2]$, we see that there exists a number c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$$

But $f'(c) = 0$

Hence $f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1)$

Thus f assumes the same value at any two points in $[a, b]$ and so it is constant on $[a, b]$.

Set $f(x) = \sin^2 x + \cos^2 x$

$$f'(x) = 2 \sin x \cos x - 2 \cos x \sin x$$
$$= 0, \text{ for all real } x.$$

Hence $f(x)$ is a constant.

Let $f(x) = \sin^2 x + \cos^2 x = c$. (1)

where c is an arbitrary constant. Since (1) holds for all real x, it holds, in particular, for $x = 0$,

i.e., $\sin^2 0 + \cos^2 0 = c$ or $c = 1$

Thus, $\sin^2 x + \cos^2 x = 1$.

17. Let $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x & \text{if } x > 1. \end{cases}$

Does the Mean Value Theorem hold for f on $\left[\frac{1}{2}, 2\right]$?

Sol. It is easy to see that f is continuous on $\left[\frac{1}{2}, 2\right]$.

We check whether f is differentiable on $\left[\frac{1}{2}, 2\right]$. To this end, we check whether $f'(1)$ exists.

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1}$$
$$= \lim_{x \to 1^-} (x + 1) = 2$$

$$Rf'(1) = \lim_{x \to 1^+} \frac{f(x) - (1)}{x - 1} = \lim_{x \to 1^+} \frac{x - 1}{x - 1} = 1$$

Thus, $Lf'(1) \neq Rf'(1)$.

Therefore, $f'(1)$ does not exist and the M.V.T. does not hold on $\left[\frac{1}{2}, 2\right]$

18. Let n be a positive integer. Apply Rolle's Theorem to the function

$$F(x) = \begin{vmatrix} f(x) & f(a) & f(b) \\ x^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix}$$

to obtain a result that generalizes the Mean Value Theorem. Does the result hold if $n < 0$?.

Sol. $F(a) = \begin{vmatrix} f(a) & f(a) & f(b) \\ a^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix} = 0$ and $F(b) = \begin{vmatrix} f(b) & f(a) & f(b) \\ b^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix} = 0$

Thus $F(a) = F(b)$

By Rolle's Theorem, there exists $c \in]a, b[$ such that $F'(c) = 0$.

Now $F'(x) = \begin{vmatrix} f'(x) & f(a) & f(b) \\ n \cdot x^{n-1} & a^n & b^n \\ 0 & 1 & 1 \end{vmatrix}$

$$= f'(x)(a^n - b^n) - nx^{n-1}(f(a) - f(b))$$

$$F'(c) = 0$$

$\Rightarrow$ $f'(c)(a^n - b^n) = nc^{n-1}(f(a) - f(b))$

or $\dfrac{f(b) - f(a)}{b^n - a^n} = \dfrac{f'(c)}{nc^{n-1}}$

which is the required generalization of the Mean Value Theorem.

If $n < 0$, the theorem holds if $0 \notin [a, b]$.

19. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on the graph of the parabola $y = f(x) = ax^2 + bx + c$. By the Mean Value Theorem, there is a point (x_3, y_3) on the curve where tangent line is parallel to the chord AB. Show that $x_3 = \dfrac{x_1 + x_2}{2}$.

Sol. $f(x) = ax^2 + bx + c$; $f'(x) = 2ax + b$

By the M.V.T., we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_3)$$

or $\dfrac{ax_2^2 + bx_2 + c - ax_1^2 - bx_1 - c}{x_2 - x_1} = 2ax_3 + b$

i.e., $\dfrac{a(x_2 - x_1)(x_2 + x_1) + b(x_2 - x_1)}{x_2 - x_1} = 2ax_3 + b$

or $a(x_2 + x_1) + b = 2ax_3 + b$

i.e., $x_3 = \dfrac{x_1 + x_2}{2}$ as required.

20. Show that $f(x) = x^3 - 3x^2 + 2$ is monotonically increasing on every interval.

Sol. $f(x) = x^3 - 3x^2 + 3x + 2$ and

$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$

It is obvious that $3(x-1)^2 = 0$ if $x = 1$. But $3(x-1)^2 > 0$ if $x \in]-\infty, 1[$ and $x \in]1, \infty[$. That is $f'(x) > 0$ for $x \in]-\infty, 1[$ and $x \in]1, \infty[$

So we conclude that $f(x)$ is increasing on $]-\infty, 1[$ and $]1, \infty[$.

For $x = 1$, $f'(x) = 0$

We can verify that $f(x)$ is increasing on every interval containing 1. Thus $f(x)$ is monotonically increasing on every interval.

21. Prove that $f(x) = 2x - \arctan x - \ln(x + \sqrt{x^2+1})$ is increasing on $[0, \infty[$.

Sol. $f(x) = 2x - \text{arc}\tan x - \ln(x + \sqrt{x^2+1})$ and

$$f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{1}{2}\cdot\frac{1}{\sqrt{x^2+1}}\cdot 2x\right)$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{x^2+1}}\cdot\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{x^2+1}}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}} = 2 - \frac{1}{\sqrt{1+x^2}} - \frac{1}{1+x^2}$$

$$= \frac{2(1+x^2) - \sqrt{1+x^2} - 1}{1+x^2}$$

$$= \frac{(2\sqrt{1+x^2}+1)(\sqrt{1+x^2}-1)}{1+x^2}$$ [If $\sqrt{1+x^2} = t$, then the numerator $= 2t^2 - t - 1 = (2t+1)(t-1)$]

Now $2\sqrt{1+x^2} + 1 > 0$ for all $x \in \boldsymbol{R}$ and $\sqrt{1+x^2} - 1 > 0$ for $x \in R - \{0\}$. That is $f'(x) > 0$ if $x \in \boldsymbol{R} - \{0\}$

For $x = 0$, $\sqrt{1+x^2} - 1 = 0$, that is, $f'(x) = 0$ for $x = 0$

It is obvious that $f'(x)$ is positive for $x \in]0, \infty[$ and $f'(x) = 0$ for $x = 0$, therefore, $f(x)$ is increasing on $[0, \infty[$.

22. Show that $\dfrac{\tan x}{x}$ is an increasing function for $0 < x < \dfrac{\pi}{2}$.

Sol. Let $f(x) = \dfrac{\tan x}{x}$; $f'(x) = \dfrac{x\sec^2 x - \tan x}{x^2}$ (1)

Let $\phi(x) = x\sec^2 x - \tan x$

$\phi'(x) = \sec^2 x + x(2\sec^2 x \tan x) - \sec^2 x$

$= 2x\sec^2 x \tan x$

When $0 < x < \dfrac{\pi}{2}$, $\phi'(x) > 0$. Thus $\phi(x)$ is an increasing function for $0 < x < \dfrac{\pi}{2}$.

Since $\phi(0) = 0$, $\phi(x) > \phi(0)$ for $x > 0$.

i.e., $x\sec^2 x - \tan x > 0$

or $x\sec^2 x > \tan x$ when $0 < x < \dfrac{\pi}{2}$

From (1), we have $f'(x) > 0$ when $0 < x < \dfrac{\pi}{2}$

i.e., $f(x)$ is an increasing function for $0 < x < \dfrac{\pi}{2}$

23. Determine the interval on which

$$f(x) = 2x^3 - 15x + 36x + 1$$

is increasing or decreasing.

Sol. $f(x) = 2x^3 - 15x^2 + 36x + 1$; $f'(x) = 6x^2 - 30x + 36$

For $f(x)$ to be an increasing function

$f'(x) > 0$

i.e., $6x^2 - 30x + 36 > 0$ or $x^2 - 5x + 6 > 0$

or $(x-2)(x-3) > 0$

Now two cases arise.

Case I. $(x-2) > 0$ and $(x-3) > 0$

$\Rightarrow$ $x > 2$ and $x > 3$

Combining the two statements, we can say that $f(x)$ is an increasing function for all $x > 3$ *i.e.,* for all $x \in]3, \infty[$.

Case II. $x - 2 < 0$ and $x - 3 < 0$ $\Rightarrow$ $x < 2$ and $x < 3$

Combining the two statements it is evident that $f(x)$ is an increasing function for all $x < 2$ *i.e.,* for all $x \in]-\infty, 2[$.

Now for $f(x)$ to be a decreasing function

$f'(x) < 0$

i.e., $6x^2 - 30x + 36 < 0$ $\Rightarrow$ $(x-2)(x-3) < 0$

Two cases arise:

Case I. $(x-2) > 0$ and $(x-3) < 0$

$\Rightarrow$ $x > 2$ and $x < 3$

$\Rightarrow$ $x \in]2, 3[$. Thus f is decreasing in $]2, 3[$.

Case II. $(x-2) < 0$ and $(x-3) > 0$

$\Rightarrow$ $x < 2$ and $x > 3$

which is an absurdity because there is no such number which is less than 2 and greater than 3.

24. If $x > 0$, prove that:

$$x - \ln(1+x) > \frac{x^2}{2(1+x)}$$

Sol. Let $f(x) = x - \ln(1+x) - \dfrac{x^2}{2(1+x)}$

$$= x - \ln(1+x) - \frac{1}{2}\left[x - 1 + \frac{1}{1+x}\right]$$

$$= x - \ln(1+x) - \frac{1}{2}x + \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{1+x}$$

$$f'(x) = 1 - \frac{1}{1+x} - \frac{1}{2} + \frac{1}{2(1+x)^2}$$

$$= \frac{1}{2} + \frac{1}{2(1+x)^2} - \frac{1}{1+x}$$

$$= \frac{(1+x)^2 + 1 - 2(1+x)}{2(1+x)^2}$$

$$= \frac{1 + x^2 + 1 - 2}{2(1+x)^2} = \frac{x^2}{2(1+x)^2}$$

which is positive for every $x > 0$.

Thus $f(x)$ is an increasing function for $x > 0$.

But $f(0) = 0$

Hence $f(x) > 0$ for $x > 0$

i.e., $x - \ln(1+x) - \dfrac{x^2}{2(1+x)} > 0$ *i.e.*, $x - \ln(1+x) > \dfrac{x^2}{2(1+x)}$.

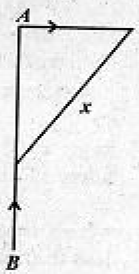
5. A ship sails east from port A at 10 nautical miles pet hour. At the same time, another ship leaves port B, which is 100 nautical miles due south of port A, and sails north at 25 nautical miles per hour. For how long is the distance between the ships decreasing?

ol. Let x be the distance between the ships after t hours. Then

$$x^2 = (10t)^2 + (100 - 25t)^2$$

$$= 100t^2 + 10000 + 625t^2 - 5000t$$

$$= 725t^2 - 5000t + 10000$$



Differentiating w.r.t. t, we have

$$2x\frac{dx}{dt} = 1450t - 5000$$

or $$x\frac{dx}{dt} = 725\,t - 2500$$

or $$\frac{dx}{dt} < 0$$

whenever $725t - 2500 < 0$ (since x is positive)

or $$t < \frac{2500}{725} = \frac{100}{29}$$

Thus $\dfrac{dx}{dt} < 0$ for $0 < t < \dfrac{100}{29}$ and the distance x between the ships decreases for $\dfrac{100}{29}$ hours.

Exercise Set 3.2 (Page 114)

1. Write the Maclaurin's Formula for the function $f(x) = \sqrt{1+x}$ with remainder after two terms.

Sol. Here $f(x) = (1+x)^{1/2}, f(0) = 1$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} = \frac{1}{2\sqrt{1+x}}, f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2} = -\frac{1}{4}\cdot\frac{1}{(1+x)^{3/2}},$$

$$f''(\theta x) = -\frac{1}{4(1+\theta x)^{3/2}}$$

By Maclaurin's Theorem with remainder after two terms, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x), \quad 0 < \theta < 1$$

or $$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{2!}x^2\left\{-\frac{1}{4(1+\theta x)^{3/2}}\right\}$$

$$= 1 + \frac{1}{2}\cdot x - \frac{1}{8}\cdot\frac{x^2}{(1+\theta x)^{3/2}}$$

2. Find, by Maclaurin's Formula, the first four terms of the expansion of $f(x) = e^{ax}\cos bx$ and write the remainder after n terms.

Sol. Here $f(x) = e^{ax}\cos bx$

$$f^{(n)}(x) = (a^2 + b^2)^{n/2} e^{ax}\cos(bx + n\phi) \qquad \phi = \arctan\frac{b}{a}$$

Taking $n = 1, 2, 3$

$$f'(x) = (a^2 + b^2)^{1/2} e^{ax}\cos(bx + \phi)$$

$$f''(x) = (a^2 + b^2)\, e^{ax}\cos(bx + 2\phi)$$

$$f'''(x) = (a^2 + b^2)^{3/2}\cos(bx + 3\phi)$$

Now $f(0) = 1$

$$f'(0) = \sqrt{a^2 + b^2}\cos(\phi) = \sqrt{a^2 + b^2}\cdot\frac{a}{\sqrt{a^2 + b^2}} = a$$

$$f''(0) = (a^2 + b^2)\cos 2\phi = (a^2 + b^2)(\cos^2\phi - \sin^2\phi)$$

$$= (a^2 + b^2)\left(\frac{a^2 - b^2}{a^2 + b^2}\right) = a^2 - b^2$$

$$f'''(0) = (a^2 + b^2)^{3/2} \cos 3\phi$$
$$= (a^2 + b^2)^{3/2} [4 \cos^3 \phi - 3 \cos \phi]$$
$$= (a^2 + b^2)^{3/2} \left[4 \frac{a^3}{(a^2 + b^2)^{3/2}} - \frac{3a}{(a^2 + b^2)^{1/2}}\right]$$
$$= (a^2 + b^2)^{3/2} \left[\frac{4a^3 - 3a(a^2 + b^2)}{(a^2 + b^2)^{3/2}}\right]$$
$$= a^3 - 3ab^2 = a(a^2 - 3b^2)$$
$$f^{(n)}(\theta x) = (a^2 + b^2)^{n/2} e^{a\theta x} \cos\left(b\theta x + n \arctan\frac{b}{a}\right), \quad 0 < \theta < 1$$

By Maclaurin's Theorem with remainder after n terms, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^{(n)}(\theta x)$$

Therefore, $e^{ax} \cos bx = 1 + ax + \dfrac{a^2 - b^2}{2!}x^2 + \dfrac{a(a^2 - 3b^2)}{3!}x^3$

$$+ \dots + \frac{x^n}{n!}(a^2 + b^2)^{n/2} e^{a\theta x} \cos\left(b\theta x + n \arctan\frac{b}{a}\right).$$

Find the Maclaurin's series of the given functions (Problems 3 – 9):

3. $f(x) = \sin x$

Sol.

$f(x) = \sin x, \quad f(0) = 0$

$f'(x) = \cos x, \quad f'(0) = 1$

$f''(x) = -\sin x, \quad f''(0) = 0$

$f'''(x) = -\cos x, \quad f'''(0) = -1$

$f^{(4)}(x) = \sin x, \quad f^{(4)}(0) = 0$

Generalizing, we get

$$f^{(r)}_{(x)} = \sin\left(x + r \cdot \frac{\pi}{2}\right), \quad f^{(r)}_{(0)} = \sin r \cdot \frac{\pi}{2}$$

$$f^{(2n-1)}_{(x)} = \sin\left[x + (2n-1)\frac{\pi}{2}\right], f^{(2n-1)}_{(0)} = (-1)^{n-1}$$

$R_n = \dfrac{x^n}{n!} f^{(n)}(\theta x)$, where R_n is the remainder after n terms and $0 < \theta < 1$.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right) \longrightarrow 0$$

Thus $f(x) = \sin x$ can be expanded into an infinite series. Hence by Maclaurin's Theorem, for all values of x, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^{r-1}}{(r-1)!}f^{(r-1)}_{(0)} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1} \cdot \frac{x^{2n-1}}{(2n-1)!} + \dots$$

4. $f(x) = \cos x$

Sol.

$f(x) = \cos x, \quad f(0) = 1$

$f'(x) = -\sin x, \quad f'(0) = 0$

$f''(x) = -\cos x, \quad f''(0) = -1$

$f'''(x) = \sin x, \quad f'''(0) = 0$

$f^{(4)}(x) = \cos x, \quad f^{(4)}(0) = 1$

$$f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right), \; f^{(n)}(\theta x) = \cos\left(\theta x + \frac{n\pi}{2}\right)$$

$$f^{(n)}(0) = \cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}, & \text{if } n \text{ is even} \end{cases}$$

Now $R_n = \dfrac{x^n}{n!} f^{(n)}(\theta x), \quad 0 < \theta < 1.$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{x^n}{n!} \cos\left(\theta x + \frac{n\pi}{2}\right)$$
$$= \lim_{n\to\infty} \frac{x^n}{n!} \lim_{n\to\infty} \cos\left(\theta x + \frac{n\pi}{2}\right) = 0$$

Hence by Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{(-1)^n}{(2n)!}x^{2n} + \dots$$

5. $f(x) = \tan x$

Sol. $f(x) = \tan x, \quad f(0) = 0$

$f'(x) = \sec^2 x = 1 + \tan^2 x, \quad f'(0) = 1$

$$f''(x) = 2 \tan x \sec^2 x$$
$$= 2 \tan x (1 + \tan^2 x)$$
$$= 2 \tan x + 2 \tan^3 x, \quad f''(0) = 0$$
$$f'''(x) = 2 \sec^2 x + 6 \tan^2 x \sec^2 x$$
$$= 2(1 + \tan^2 x) + 6 \tan^2 x (1 + \tan^2 x)$$
$$= 2 + 2 \tan^2 x + 6 \tan^2 x + 6 \tan^4 x$$
$$= 2 + 8 \tan^2 x + 6 \tan^4 x, \quad f'''(0) = 2$$
$$f^{(4)}(x) = 16 \tan x \sec^2 x + 24 \tan^3 x \sec^2 x$$
$$= 16 \tan x (1 + \tan^2 x) + 24 \tan^3 x (1 + \tan^2 x)$$
$$= 16 \tan x + 16 \tan^3 x + 24 \tan^3 x + 24 \tan^5 x$$
$$= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x, \quad f^{(4)}(0) = 0$$

$f^{(5)}(x) = 16\sec^2 x + 120\tan^2 x\sec^2 x + 120\tan^4 x\sec^2 x,\ f^{(5)}(0) = 16$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

By Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \frac{x^5}{5!}f^{(5)}(0) + \dots$$

Substituting the values, we have

$$\tan x = 0 + x\times 1 + \frac{x^2}{2!}(0) + \frac{x^3}{3\times 2\times 1}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5\times 4\times 3\times 2}\times 16 + \dots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$$

6. $f(x) = \sec x$

Sol. $y = \sec x,\ y(0) = \sec 0 = 1$

$y_1 = \sec x\tan x,\quad y_1(0) = 0$

$y_2 = 2\sec^3 x - \sec x = 2y^3 - y,\quad y_2(0) = 1$

$y_3 = 6y^2 y_1 - y_1,\ y_3(0) = 0$

$y_4 = 6y^2 y_2 + 12yy_1^2 - y_2,\ y_4(0) = 5$

$y_5 = 6y^2 y_3 + 12yy_1y_2 + 12y_1^3 + 24yy_1y_2 - y_3$

$= 6y^2 y_3 + 36yy_1y_2 + 12y_1^3 - y_3,\ y_5(0) = 0$

$y_6 = 6y^2 y_4 + 12yy_1y_3 + 36(y_1^2 y_2 + yy_2^2 + yy_1y_3) + 36y_1^2 y_2 - y_4$

$y_6(0) = 6.1.5 + 36(0 + 1 + 0) + 36\cdot 0\cdot 1 - 5 = 61$

Now by Maclaurin's Theorem,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \dots$$

or $$\sec x = 1 + x.0 + \frac{x^2}{2!}.1 + \frac{x^3}{3!}.0 + \frac{x^4}{4!}.5 + \frac{x^5}{5!}.0 + \frac{x^6}{6!}.61 + \dots$$

$$= 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \dots$$

7. $f(x) = \ln(1 - x)$

Sol. $$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1-x)^n}\cdot(-1)^n = \frac{(-1)^{2n-1}(n-1)!}{(1-x)^n} = -\frac{(n-1)!}{(1-x)^n}$$

$$f^{(n)}(\theta x) = -\frac{(n-1)!}{(1-\theta x)^n},\ 0 < \theta < 1$$

$f'(0) = -1,\ f''(0) = -1,\ f'''(0) = -2, \dots f^{(4)}(0) = -(3!)$

$$f(x) = f(x) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}.f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \dots$$

$$\ln(1-x) = 0 + x(-1) + \frac{x^2}{2\times 1}(-1) + \frac{x^3}{3\times 2\times 1}\times(-2) + \frac{x^4}{4!}(-(3!)) + \dots$$

Therefore, $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$

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8. $f(x) = e^{\sin x}$

Sol. $f(x) = e^{\sin x},\ f(0) = e^0 = 1$

$f'(x) = e^{\sin x}\cos x,\ f'(0) = 1$

$f''(x) = e^{\sin x}\cos^2 x - e^{\sin x}\sin x,\ f''(0) = 1$

$f'''(x) = e^{\sin x}\cos^3 x - 2e^{\sin x}\cos x\sin x - e^{\sin}x\sin x\cos x - e^{\sin x}\cos x$

$= e^{\sin x}\cos^3 x - 3e^{\sin x}\sin x\cos x - e^{\sin x}\cos x$

$= e^{\sin x}\cos^3 x - \frac{3}{2}e^{\sin x}\sin 2x - e^{\sin x}\cos x,\ f'''(0) = 0$

$f^{(4)}(x) = e^{\sin x}\cos^4 x - 3e^{\sin x}\cos^2 x\sin x - \frac{3}{2}e^{\sin x}\sin 2x\cos x$

$-\frac{3}{2}e^{\sin x}2\cos 2x - e^{\sin x}\cos^2 x + e^{\sin x}\sin x,$

$f^{(4)}(0) = 1 - 3 - 1 = -3$

Hence $$e^{\sin x} = 1 + x.1 + \frac{x^2}{2!}.1 + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3) + \dots$$

$$= 1 + x + \frac{x^2}{2!} - \frac{x^4}{8} - \dots$$

9. $f(x) = a^x$

Sol. $f(x) = a^x,\qquad f(0) = 1$

$f'(x) = a^x\ln a,\qquad f'(0) = \ln a$

$f''(x) = a^x(\ln a)^2,\qquad f''(0) = (\ln a)^2$

$f'''(x) = a^x(\ln a)^3,\qquad f'''(0) = \ln(a)^3$

$\vdots \qquad\qquad \vdots$

$f^{(n)}(x) = a^x(\ln a)^n,\qquad f^{(n)}(\theta x) = a^{\theta x}(\ln a)^n$

Therefore,

$$a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \dots + \frac{x^{n-1}}{(n-1)!}(\ln a)^{n-1} + R_n,$$

where $R_n = \frac{x^n}{n!}a^{\theta x}(\ln a)^n = \frac{(x\ln a)^n}{n!}a^{\theta x},\ 0 < \theta < 1.$

To show that $\lim_{n\to\infty} R_n = 0$, we use the Ratio Test

$$\lim_{n\to\infty}\frac{R_{n+1}}{R_n} = \lim_{n\to\infty}\frac{(x\ln a)^{n+1}}{(n+1)!}\times\frac{n!}{(x\ln a)^n}\frac{a^{\theta x}}{a^{\theta x}}$$

$$= \lim_{n\to\infty}\frac{x\ln a}{n+1} = 0 < 1,$$

if x and a are fixed numbers. The series is convergent.

Hence $\lim_{n \to \infty} R_n = 0$

$$a^x = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{2!} + \dots + \frac{(x \ln a)^n}{n!} + \dots \infty$$

10. Apply Taylor's Theorem to prove that

$$(a+b)^m = a^m + \frac{m}{1!} a^{m-1} b + a^{m-2} b^2 + \dots$$

for all real m, $a > 0$, $-a < b < a$.

Sol. Consider $f(x+b) = (x+b)^m$

$$f(x) = x^m \ ; \ f'(x) = m x^{m-1}$$

$$f''(x) = m(m-1)x^{m-2} \ ; \ f'''(x) = m(m-1)(m-2)x^{m-3}$$

By Taylor's Theorem,

$$f(x+b) = f(x) + bf'(x) + \frac{b^2}{2!} f''(x) + \frac{b^3}{3!} f'''(x) + \dots$$

$$= x^m + bm x^{m-1} + \frac{b^2}{2!} m(m-1) x^{m-2} + \frac{b^3}{3!} m(m-1)(m-2) x^{m-3} + \dots$$

Therefore,

$$f(a+b) = (a+b)^m = a^m + \frac{m}{1!} a^{m-1} b + \frac{m(m-1)}{2!} a^{m-2} b^2 + \dots$$

Here $R_n = \frac{b^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(a + \theta b), \quad 0 < \theta < 1.$

$$f^{(n)}(x) = m(m-1)(m-2)(m-n+1) x^{m-n}$$

$$f^{(n)}(a+\theta b) = m(m-1)(m-2)\dots(m-n+1)(a+\theta b)^{m-}$$

$$R_n = \frac{b^n}{(n-1)!}(1-\theta)^{n-1} m(m-1)(m-1)(m-2)(m-n+1)(a+\theta b)^{m-n}$$

$$= \frac{b^n (1-\theta)^{n-1} n!}{(m-n)!\,(n-1)!} \cdot (a+\theta b)^{m-n}$$

$R_n \to 0$ as $n \to \infty$ for all real m, $a > 0$, $-a < b < a$

Hence $(a+b)^m = a^m + \frac{m}{1!} a^{m-1} b + \frac{m(m-1)}{2!} a^{m-2} b^2 + \dots$

11. Prove that:

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)}{2!} f''(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(a + (x-a)\theta)$$

stating the conditions under which it holds.

Sol. $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$

$$+ \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(a + \overline{x-a}\,\theta)$$

The conditions for which the above theorem holds are as follows:

The function f is such that

(i) $f, f'', f''', \dots, f^{(n-1)}$ are continuous on $[a, x]$

(ii) $f^{(n)}$ exists in $]a, x[$.

then there exists a real number θ, $0 < \theta < 1$, such that

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$+ \frac{(x-a)^{n-1}}{(x-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(a + \overline{x-a}\,\theta)$$

Here $f(x) = f(a + \overline{x-a})$

Expanding by Taylor's Theorem with Lagrange's form of remainder, we have

$$f(x) = f(a + \overline{x-a})$$

$$= f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$+ \frac{(x-a)^{n-1}}{(x-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n(a + \overline{x-a}\,\theta)$$

12. Use Taylor's Theorem to prove that

$$\ln \sin(x+h) = \ln \sin x + h \cot x - \frac{1}{2} h^2 \csc^2 x + \frac{1}{3} h^3 \cot x \csc^2 x + \dots$$

Sol. Let $f(x) = \ln \sin(x+h)$

Then $f(x) = \ln \sin x \ ; \ f'(x) = \frac{1}{\sin x} . \cos x = \cot x$

$$f''(x) = -\csc^2 x \ ; \ f'''(x) = -2 \csc x (-\csc x \cot x)$$

$$= 2 \csc^2 x \cot x$$

By Taylor's Theorem, we get

$$\ln \sin(x+h) = \ln \sin x + h \cot x + \frac{h^2}{2!}(-\csc^2 x) + \frac{h^3}{3!}(2\csc^2 x \cot x) + \dots$$

$$= \ln \sin x + h \cot x - \frac{h^2}{2} \csc^2 x + \frac{h^3}{3} \csc^2 x \cot x + \dots$$

13. Show that, under certain conditions to be stated,
$f(a+h) = f(a) + hf'(a+\theta h)$ where $0 < \theta < 1$. Prove also that the limiting value of θ, when h decreases indefinitely, is $\frac{1}{2}$.

Sol. $f(a+h) = f(a) + hf'(a+\theta h)$, holds by Taylor's Theorem with remainder after one term.

Also by Taylor's theorem with Lagrange's form of remainder after two terms,

$$f(a+h) = f(a)+hf'(a) + \frac{h^2}{2!}f''(a+\theta' h), \text{ where } 0 < \theta' < 1$$

The two equations give

$$f(a) + hf'(a+\theta h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta' h)$$

or $$hf'(a+\theta h) = hf'(a) + \frac{h^2}{2!}f''(a+\theta' h)$$

or $$f'(a+\theta h) = f'(a) + \frac{h}{2}f''(a+\theta' h)$$

or $$f'(a+\theta h) - f'(a) = \frac{h}{2}f''(a+\theta' h)$$

or $$\theta h f''(a+\theta\theta'' h) = \frac{h}{2}f''(a+\theta' h), \qquad 0 < \theta'' < 1$$

or $$\theta f''(a+\theta\theta'' h) = \frac{1}{2}f''(a+\theta' h)$$

Letting $h \longrightarrow 0$, we have

$$\theta f''(a) = \frac{1}{2}f''(a) \qquad \text{or} \qquad \theta = \frac{1}{2}.$$

14. If the functions f, ϕ and Ψ are continuous on $[a, b]$ and differentiable in $]a, b[$, show that there exists a point $\xi \in]a, b[$ such that

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0$$

Hence deduce Lagrange's and Cauchy's mean value theorems.

Sol. We form a new function $F(x) = f(x) + A\,\phi(x) + B\,\psi(x)$
where A and B are constants to be chosen such that

$$f(a) + A\,\phi(a) + B\,\psi(a) = 0 \qquad (1)$$
$$f(b) + A\,\phi(b) + B\,\psi(b) = 0 \qquad (2)$$

The function, $F(x)$ satisfies the conditions of Rolle's Theorem. Hence there exists a point $\xi \in [a, b]$ such that

$$f'(\xi) + A\,\phi'(\xi) + B\,\psi'(\xi) = 0 \qquad (3)$$

Eliminating A and B from (1), (2) and (3), we obtain

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0 \qquad (4)$$

Deduction. Take $\psi(x) = k$, where k is some constant. Then

$$\psi(a) = \psi(b) = k$$

and $\psi'(\xi) = 0$.

Hence from (4), we have

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a) & \phi(a) & k \\ f(b) & \phi(b) & k \end{vmatrix} = 0$$

or $$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a)-f(b) & \phi(a)-\phi(b) & 0 \\ f(b) & \phi(b) & k \end{vmatrix} = 0 \quad \text{by } R_2 - R_3$$

or $$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a)-f(b) & \phi(a)-\phi(b) & 0 \\ f(b) & \phi(b) & 1 \end{vmatrix} = 0$$

or $f'(\xi)[\phi(a) - \phi(b)] - \phi'(\xi)[f(a) - f(b)] = 0$

or $[\phi(b) - \phi(a)]f'(\xi) = [f(b) - f(a)]\,\phi'(\xi)$

$$\Rightarrow \quad \frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(\xi)}{\phi'(\xi)} \qquad (5)$$

which is Cauchy's Mean Value Theorem.

Now take $\phi(x) = x$ so that $\phi(a) = a$, $\phi(b) = b$ and $\phi'(\xi) = 1$

Putting these values in (5), we get

$$\frac{f(b)-(a)}{b-a} = \frac{f'(\xi)}{1}$$

which is Lagrang's Mean Value Theorem.

15. Assuming f'' to be continuous on $[a, b]$ and differentiable on $]a, b[$, show that

$$f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} = \frac{1}{2}(c-a)(c-b)f''(\xi)$$

where both c and ξ belong to $]a, b[$.

Sol. Consider the function

$$\phi(x) = f(x) + Ax + Bx^2 \qquad (1)$$

where A and B are constants to be determined such that

$$\phi(a) = \phi(b) = \phi(c)$$

Thus we have

$$f(a) + Aa + Ba^2 = f(b) + Ab + Bb^2 = f(c) + Ac + Bc^2$$

or $(a-b)A + (a^2-b^2)B + [f(a) - f(b)] = 0$

and $(b-c)A + (b^2-c^2)B + [f(b) - f(c)] = 0$

13. Show that, under certain conditions to be stated, $f(a+h) = f(a) + hf'(a+\theta h)$ where $0 < \theta < 1$. Prove also that the limiting value of θ, when h decreases indefinitely, is $\frac{1}{2}$.

Sol. $f(a+h) = f(a) + hf'(a+\theta h)$, holds by Taylor's Theorem with remainder after one term.

Also by Taylor's theorem with Lagrange's form of remainder after two terms,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta' h), \text{ where } 0 < \theta' < 1$$

The two equations give

$$f(a) + hf'(a+\theta h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta' h)$$

or $\quad hf'(a+\theta h) = hf'(a) + \frac{h^2}{2!}f''(a+\theta' h)$

or $\quad f'(a+\theta h) = f'(a) + \frac{h}{2}f''(a+\theta' h)$

or $\quad f'(a+\theta h) - f'(a) = \frac{h}{2}f''(a+\theta' h)$

or $\quad \theta h f''(a+\theta\theta'' h) = \frac{h}{2}f''(a+\theta' h), \qquad 0 < \theta'' < 1$

or $\quad \theta f''(a+\theta\theta'' h) = \frac{1}{2}f''(a+\theta' h)$

Letting $h \longrightarrow 0$, we have

$$\theta f''(a) = \frac{1}{2}f''(a) \qquad \text{or} \qquad \theta = \frac{1}{2}$$

14. If the functions f, ϕ and Ψ are continuous on $[a, b]$ and differentiable in $]a, b[$, show that there exists a point $\xi \in]a, b[$ such that

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0$$

Hence deduce Lagrange's and Cauchy's mean value theorems.

Sol. We form a new function $F(x) = f(x) + A\phi(x) + B\psi(x)$ where A and B are constants to be chosen such that

$$f(a) + A\phi(a) + B\psi(a) = 0 \tag{1}$$
$$f(b) + A\phi(b) + B\psi(b) = 0 \tag{2}$$

The function, $F(x)$ satisfies the conditions of Rolle's Theorem. Hence there exists a point $\xi \in [a, b]$ such that

$$f'(\xi) + A\phi'(\xi) + B\psi'(\xi) = 0 \tag{3}$$

Eliminating A and B from (1), (2) and (3), we obtain

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0 \tag{4}$$

Deduction. Take $\psi(x) = k$, where k is some constant. Then

$$\psi(a) = \psi(b) = k$$

and $\quad \psi'(\xi) = 0.$

Hence from (4), we have

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a) & \phi(a) & k \\ f(b) & \phi(b) & k \end{vmatrix} = 0$$

or $\quad \begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a)-f(b) & \phi(a)-\phi(b) & 0 \\ f(b) & \phi(b) & k \end{vmatrix} = 0 \text{ by } R_2 - R_3$

or $\quad \begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a)-f(b) & \phi(a)-\phi(b) & 0 \\ f(b) & \phi(b) & 1 \end{vmatrix} = 0$

or $\quad f'(\xi)[\phi(a)-\phi(b)] - \phi'(\xi)[f(a)-f(b)] = 0$

or $\quad [\phi(b)-\phi(a)]f'(\xi) = [f(b)-f(a)]\phi'(\xi)$

$$\Rightarrow \quad \frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(\xi)}{\phi'(\xi)} \tag{5}$$

which is Cauchy's Mean Value Theorem.

Now take $\phi(x) = x$ so that $\phi(a) = a$, $\phi(b) = b$ and $\phi'(\xi) = 1$

Putting these values in (5), we get

$$\frac{f(b)-(a)}{b-a} = \frac{f'(\xi)}{1}$$

which is Lagrang's Mean Value Theorem.

15. Assuming f'' to be continuous on $[a, b]$ and differentiable on $]a, b[$, show that

$$f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} = \frac{1}{2}(c-a)(c-b)f''(\xi)$$

where both c and ξ belong to $]a, b[$.

Sol. Consider the function

$$\phi(x) = f(x) + Ax + Bx^2 \tag{1}$$

where A and B are constants to be determined such that

$$\phi(a) = \phi(b) = \phi(c)$$

Thus we have

$$f(a) + Aa + Ba^2 = f(b) + Ab + Bb^2 = f(c) + Ac + Bc^2$$

or $\quad (a-b)A + (a^2-b^2)B + [f(a)-f(b)] = 0$

and $\quad (b-c)A + (b^2-c^2)B + [f(b)-f(c)] = 0$

Solving these equation simultaneously by cross-multiplication, we have

$$\frac{A}{(a^2-b^2)[f(b)-f(c)]-(b^2-c^2)]f(a)-f(b)]}$$
$$=\frac{B}{(b-c)[f(a)-f(b)]-(a-b)[f(b)-(c)]}$$
$$=\frac{1}{(a-b)(b^2-c^2)-(a^2-b^2)(b-c)}$$

i.e., $$A=\frac{(a^2-b^2)[f(b)-f(c)]-(b^2-c^2)[f(a)-f(b)]}{(a-b)(b-c)(c-a)} \quad (2)$$

$$B=\frac{(b-c)[f(a)-f(b)]-(a-b)[f(b)-f(c)]}{(a-b)(b-c)(c-a)} \quad (3)$$

Now $f''(x)$ exists in $[a, b]$. Therefore, $f(x)$, $f'(x)$ exist and are differentiable in $[a, b]$.

Also, $Ax+Bx^2$ is differentiable in $[a, b]$ and so

$\phi(x)$ is differentiable in $[a, b]$

Now, consider the interval $[a, c]$

Since this interval is included in $[a, b]$, we see that

$\phi(x)$ is differentiable in $[a, c]$.

Also $\phi(a)=\phi(c)$ (4)

Hence by Rolle's Theorem, there is a value α in $[a, c]$, such that

$\phi'(\alpha)=0$ (5)

Similarly applying Rolle's Theorem to the interval $[c, b]$ we conclude that there is a value β in $[c, b]$, such that

$\phi'(\beta)=0$ (6)

Now, consider the function

$F(x)=\phi'(x)=f'(x)+A+2Bx$

We have $F'(x)=f''(x)+2B$ (7)

which exists in $[a, b]$. Therefore,

$F(x)$ is differentiable in $[a, b[$ and hence also in $]\alpha, \beta[$ which is contained in $[a, b]$.

Also $F(\alpha)=\phi'(\alpha)=0$, by (5)

and $F(\beta)=\phi'(\beta)=0$, by (6)

Thus $F(\alpha)=F(\beta)$.

Hence by Rolle's Theorem there is a number ξ in $]\alpha, \beta[$ and so also in $]a, b[$ such that

$F'(\xi)=0$

i.e., $f''(\xi)+2B=0$, by (7)

or $$f''(\xi)+2\frac{(b-c)[f(a)-f(b)]-(a-b)[f(b)-f(c)]}{(a-b)(b-c)(c-a)}=0$$

$$\Rightarrow \quad \frac{1}{2}(c-a)(c-b)f''(\xi)=\frac{(b-c)[f(a)-f(b)]-(a-b)[f(b)-f(c)]}{(a-b)}$$

or $$\frac{1}{2}(c-a)(c-b)f''(\xi)=\frac{b-c}{a-b}f(a)-\left(\frac{b-c+a-b}{a-b}\right)f(b)+f(c)$$
$$=-\frac{b-c}{b-a}f(a)-\frac{c-a}{b-a}f(b)+f(c)$$
$$=f(c)-f(a)\frac{b-c}{b-a}-f(b)\frac{c-a}{b-a}$$

Hence $$f(c)-f(a)\frac{b-c}{b-a}-f(b)\frac{c-a}{b-a}=\frac{1}{2}(c-a)(c-b)f''(\xi)$$

16. Show that the number θ which occurs in the Taylor's Theorem with Lagrange's form of remainder after n terms approaches the limit $\frac{1}{1+n}$ as $h\to 0$ provided that $f^{(n+1)}(x)$ is continuous and different from zero at $x=a$.

Sol. Applying Taylor's Theorem with remainder after n terms and $(n+1)$ terms successively, we obtain

$$f(a+h)=f(a)+hf'(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h)$$
$$f(a+h)=f(a)+hf'(a)+\ldots+\frac{h^n}{n!}f^n(a)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta' h)$$

These gives $$\frac{h^n}{n!}f^{(n)}(a+\theta h)=\frac{h^n}{n!}f^{(n)}(a)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta' h)$$

or $$f^{(n)}(a+\theta h)-f^{(n)}(a)=\frac{h}{n+1}f^{(n+1)}(a+\theta' h)$$

Applying Lagrange's Mean Value Theorem to the left-hand side, we have

$$\theta hf^{(n+1)}(a+\theta''\theta h)=\frac{h}{n+1}f^{(n+1)}(a+\theta' h),\ 0<\theta''<1$$

or $$\theta=\frac{1}{n+1}\frac{f^{(n+1)}(a+\theta' h)}{f^{(n+1)}(a+\theta''\theta h)} \quad \text{or} \quad \lim_{h\to 0}\theta=\frac{1}{n+1}$$

Exercise Set 3.3 (Page 126)

Evaluate the given limits (Problems 1 – 48):

1. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\sin x}$ $\left(\dfrac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\sin x} = \lim\limits_{x \to 0} \dfrac{e^x + e^{-x}}{\cos x} = 2$ $\left(\dfrac{0}{0}\right)$

2. $\lim\limits_{x \to 0} \dfrac{e^{x^2} - 1}{\cos x - 1}$ $\left(\dfrac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{e^{x^2} - 1}{\cos x - 1} = \lim\limits_{x \to 0} \dfrac{2xe^{x^2}}{-\sin x}$ $\left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{2[x(2x)e^{x^2} + e^{x^2}]}{-\cos x} = \lim_{x \to 0} \frac{2e^{x^2}(2x^2 + 1)}{-\cos x}$$

$$= \frac{2e^0(0+1)}{-1} = -2$$

3. $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x - \sin x}$ $\left(\dfrac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x - \sin x} = \lim\limits_{x \to 0} \dfrac{1 - \sec^2 x}{1 - \cos x}$ $\left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{-2\sec^2 x \tan x}{\sin x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-2\sec^4 x - 4\sec^2 x \tan^2 x}{\cos x} = -2$$

4. $\lim\limits_{x \to \pi} \dfrac{\sin^2 x}{\cos 3x + 1}$ $\left(\dfrac{0}{0}\right)$

Sol. $\lim\limits_{x \to \pi} \dfrac{\sin^2 x}{\cos 3x + 1} = \lim\limits_{x \to \pi} \dfrac{2\sin x \cos x}{-3\sin 3x} = \lim\limits_{x \to \pi} \dfrac{\sin 2x}{-3\sin 3x}$ $\left(\dfrac{0}{0}\right)$

$$= \lim_{x \to \pi} \frac{2\cos 2x}{-9\cos 3x} = \frac{2}{9}$$

5. $\lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{1 - \cos x}}$ $\left(\dfrac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{1 - \cos x}} = \lim\limits_{x \to 0} \dfrac{\sin x}{\left(2\sin^2 \frac{x}{2}\right)^{1/2}} = \lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{2}\sin \frac{x}{2}}$

$$= \lim_{x \to 0} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sqrt{2}\sin\frac{x}{2}} = \lim_{x \to 0} \sqrt{2}\cos\frac{x}{2} = \sqrt{2}$$

6. $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^2 \tan x}$

Sol. We have $\dfrac{\tan x - \sin x}{x^2 \tan x} = \dfrac{\frac{\sin x}{\cos x} - \sin x}{x^2 \cdot \frac{\sin x}{\cos x}} = \dfrac{\sin x(1 - \cos x)}{x^2 \sin x} = \dfrac{1 - \cos x}{x^2}$

Now, $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^2 \tan x} = \lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ $\left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}\lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

7. $\lim\limits_{x \to 1} \dfrac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$

Sol. $\lim\limits_{x \to 1} \dfrac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$ $\left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 1} \frac{n(n+1)x^n - n(n+1)x^{n-1}}{2(x-1)} \quad \left(\frac{0}{0}\right)$$

$$= \frac{n(n+1)}{2} \lim_{x \to 1} \frac{x^n - x^{n-1}}{x - 1} = \frac{n(n+1)}{2} \lim_{x \to 1} \frac{x^{n-1}(x-1)}{x-1}$$

$$= \frac{n(n+1)}{2} \lim_{x \to 1} (x^{n-1}) = \frac{n(n+1)}{2} \cdot 1 = \frac{n(n+1)}{2}$$

8. $\lim\limits_{x \to 0} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x}$ $\left(\dfrac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x} = \lim\limits_{x \to 0} \dfrac{e^x + 2\sin x - e^{-x}}{x\cos x + \sin x}$ $\left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{e^x + 2\cos x + e^{-x}}{-x\sin x + \cos x + \cos x}$$

$$= \lim_{x \to 0} \frac{e^x + 2\cos x + e^{-x}}{-x\sin x + 2\cos x} = \frac{4}{2} = 2$$

9. $\lim\limits_{x \to 0} \dfrac{\ln(1 - x^2)}{\ln \cos x}$ $\left(\dfrac{0}{0}\right)$

Sol. $\lim\limits_{x\to 0}\dfrac{\ln(1-x^2)}{\ln\cos x} = \lim\limits_{x\to 0}\dfrac{\dfrac{1}{(1-x^2)}(-2x)}{\dfrac{1}{\cos x}(-\sin x)}$

$$= \lim_{x\to 0}\frac{\dfrac{-2x}{(1-x^2)}}{\dfrac{-\sin x}{\cos x}} = \lim_{x\to 0}\frac{2x\cos x}{(1-x^2)\sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{2[-x\sin x+\cos x]}{(1-x^2)\cos x+\sin x\,(-2x)}$$

$$= \lim_{x\to 0}\frac{-2x\sin x+2\cos x}{(1-x^2)\cos x-2x\sin x} = \frac{2}{1} = 2$$

10. $\lim\limits_{x\to 0}\dfrac{\cosh x-\cos x}{x\sin x} \qquad \left(\dfrac{0}{0}\right)$

Sol. $\lim\limits_{x\to 0}\dfrac{\cosh x-\cos x}{x\sin x} = \lim\limits_{x\to 0}\dfrac{\sinh x+\sin x}{x\cos x+\sin x} \qquad \left(\dfrac{0}{0}\right)$

$$= \frac{\cosh x+\cos x}{-x\sin x+\cos x+\cos x} = \frac{2}{2} = 1$$

11. $\lim\limits_{x\to 1}\dfrac{1-x+\ln x}{1-\sqrt{2x-x^2}} \qquad \left(\dfrac{0}{0}\right)$

Sol. $\lim\limits_{x\to 1}\dfrac{1-x+\ln x}{1-\sqrt{2x-x^2}} = \lim\limits_{x\to 1}\dfrac{-1+\dfrac{1}{x}}{-\dfrac{1}{2}(2x-x^2)^{-1/2}(2-2x)}$

$$= \lim_{x\to 1} -\frac{\dfrac{(1-x)}{x}}{(2x-x^2)^{-1/2}(1-x)} = \lim_{x\to 1} -\frac{(1-x)(2x-x^2)^{1/2}}{x(1-x)}$$

$$= \lim_{x\to 1} -\frac{(2x-x^2)^{1/2}}{x} = -\frac{(2-1)^{1/2}}{1} = -1$$

12. $\lim\limits_{x\to 0}\dfrac{x\cos x-\ln(1+x)}{x^2}$

Sol. $\lim\limits_{x\to 0}\dfrac{x\cos x-\ln(1+x)}{x^2} \qquad \left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{-x\sin x+\cos x-\dfrac{1}{1+x}}{2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-\{x\cos x+\sin x\}-\sin x+(1+x)^{-2}}{2} = \frac{1}{2}$$

13. $\lim\limits_{x\to 0}\dfrac{\sin x-\ln(e^x\cos x)}{x\sin x}$

Sol. $\lim\limits_{x\to 0}\dfrac{\sin x-\ln(e^x\cos x)}{x\sin x} \qquad \left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{\cos x-\dfrac{1}{e^x\cos x}(-e^x\sin x+\cos x\,e^x)}{x\cos x+\sin x}$$

$$= \lim_{x\to 0}\frac{\cos x+\dfrac{e^x(\sin x-\cos x)}{e^x\cos x}}{x\cos x+\sin x}$$

$$= \lim_{x\to 0}\frac{\cos x+\tan x-1}{x\cos x+\sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-\sin x+\sec^2 x}{-x\sin x+\cos x+\cos x} = \frac{1}{2}$$

14. $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\ln(1-x)}{x\tan^2 x}$

Sol. $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\ln(1-x)}{x\tan^2 x} \qquad \left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{\cos x+\sin x-\dfrac{1}{1-x}}{x\,.\,2\tan x\sec^2 x+\tan^2 x}$$

$$= \lim_{x\to 0}\frac{\cos x+\sin x-\dfrac{1}{(1-x)}}{2x\tan x(1+\tan^2 x)+\tan^2 x}$$

$$= \lim_{x\to 0}\frac{\cos x+\sin x-\dfrac{1}{1-x}}{2x\tan x+2x\tan^3 x+\tan^2 x}$$

$$= \lim_{x\to 0}\frac{-\sin x+\cos x-\dfrac{1}{(1-x)^2}}{2x\sec^2 x+2\tan x+6x\tan^2 x\sec^2 x+2\tan^3 x+2\tan x\sec^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-\cos x-\sin x-\dfrac{2}{(1-x)^3}}{2\sec^2 x+4x\sec^2 x\tan x+2\sec^2 x+6\tan^2 x\sec^2 x}$$

$$+ 12x \sec^2 x \tan^3 x + 12x \tan x \sec^4 x$$
$$+ 6 \tan^2 x \sec^2 x + 4 \sec^2 x \tan^2 x + 2 \sec^4 x$$
$$= \frac{-1-2}{2+2+2} = \frac{-3}{6} = \frac{-1}{2}$$

15. If $\lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$ exists, find the values of a and the limit.

Sol. $\lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$ $\quad \left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{2 \cos 2x + a \cos x}{3x^2} \qquad (1)$$

The denominator of (1) $\to 0$ as $x \to 0$
and since the given limit exists and is finite, the numerator
$2 \cos 2x + a \cos x$ of (1) must $\to 0$ as $x \to 0$
i.e., $2 + a = 0$ or $a = -2$
With this value of a, (1) becomes

$$\lim_{x \to 0} \frac{2 \cos 2x - 2 \cos x}{3x^2} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to 0} \frac{-4 \sin 2x + 2 \sin x}{6x} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to 0} \frac{-8 \cos 2x + 2 \cos x}{6} = \frac{-8+2}{6} = -1$$

16. $\lim\limits_{x \to 0} \dfrac{\ln (\sin 3x)}{\ln (\sin x)}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

Sol.
$$\lim_{x \to 0} \frac{\ln (\sin 3x)}{\ln (\sin x)} = \lim_{x \to 0} \frac{\frac{1}{\sin 3x} 3 \cos 3x}{\frac{1}{\sin x} \cos x}$$
$$= \lim_{x \to 0} \frac{3 \sin x \cos 3x}{\sin 3x \cos x} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to 0} \frac{3[\sin x (-3 \sin 3x) + \cos 3x \cos x]}{-\sin 3x \sin x + \cos x (3 \cos 3x)}$$
$$= \lim_{x \to 0} \frac{-9 \sin x \sin 3x + 3 \cos x \cos 3x}{-\sin x \sin 3x + 3 \cos x \cos 3x} = \frac{3}{3} = 1$$

17. $\lim\limits_{x \to 0} \left(\dfrac{1}{x \arcsin x} - \dfrac{1}{x^2}\right)$

Sol. The given limit $= \lim\limits_{x \to 0} \left(\dfrac{x - \arcsin x}{x^2 \arcsin x}\right)$ $\quad \left(\dfrac{0}{0}\right)$

Put $\arcsin x = y$ so that $x = \sin y$ and $y \to 0$ as $x \to 0$.

The limit becomes $\lim\limits_{y \to 0} \left(\dfrac{\sin y - y}{y \sin^2 y}\right)$ $\quad \left(\dfrac{0}{0}\right)$

$$= \lim_{y \to 0} \left(\frac{\cos y - 1}{2y \sin y \cos y + \sin^2 y}\right) = \lim_{y \to 0} \frac{\cos y - 1}{y \sin 2y + \sin^2 y} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{y \to 0} \left(\frac{-\sin y}{2y \cos 2y + \sin 2y + \sin 2y}\right)$$
$$= \lim_{y \to 0} \left(\frac{-\sin y}{2y \cos 2y + 2 \sin 2y}\right) \quad \left(\frac{0}{0}\right)$$
$$= \lim_{y \to 0} \left(\frac{-\cos y}{2 \cos 2y - 4y \sin 2y + 4 \cos 2y}\right) = \frac{-1}{2+4} = -\frac{1}{6}$$

18. $\lim\limits_{x \to a} \dfrac{\ln (x - a)}{\ln (e^x - e^a)}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

Sol.
$$\lim_{x \to a} \frac{\ln (x-a)}{\ln (e^x - e^a)} = \lim_{x \to a} \frac{\frac{1}{(x-a)}}{\frac{1}{e^x - e^a} \cdot e^x} \quad \left(\frac{\infty}{\infty}\right)$$
$$= \lim_{x \to a} \frac{e^x - e^a}{e^x (x-a)} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to a} \frac{e^x}{e^x + (x-a) e^x} = \frac{e^a}{e^a} = 1$$

19. $\lim\limits_{x \to 0} \dfrac{\ln (\tan x)}{\ln x}$

Sol. $\lim\limits_{x \to 0} \dfrac{\ln (\tan x)}{\ln x}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{\tan x}\right) \sec^2 x}{\frac{1}{x}} = \lim_{x \to 0} \frac{x}{\sin x \cos x} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to 0} \frac{2x}{2 \sin x \cos x} = \lim_{x \to 0} \frac{2x}{\sin 2x} = 1, \quad \text{since } \lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1$$

20. $\lim\limits_{x \to 0} \log_{\tan x} (\tan 2x)$

Sol. $\lim\limits_{x \to 0} \log_{\tan x} (\tan 2x)$

$$= \lim_{x \to 0} \frac{\ln \tan 2x}{\ln \tan x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{\tan 2x} \cdot 2\sec^2 2x}{\frac{1}{\tan x} \cdot \sec^2 x} = \lim_{x \to 0} \frac{\frac{\cos 2x}{\sin 2x} \cdot \frac{2}{\cos^2 2x}}{\frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}}$$

$$= \lim_{x \to 0} \frac{\frac{2}{\sin 2x \cos 2x}}{\frac{1}{\sin x \cos x}} = \lim_{x \to 0} \frac{2 \sin x \cos x}{\sin 2x \cos 2x}$$

$$= \lim_{x \to 0} \frac{\sin 2x}{\sin 2x \cos 2x} = \lim_{x \to 0} \frac{1}{\cos 2x} = \frac{1}{1} = 1$$

21. $\lim_{x \to a} (x - a) \csc\left(\frac{\pi x}{a}\right)$

Sol. $\lim_{x \to a} (x - a) \csc\left(\frac{\pi x}{a}\right)$ $(0 \times \infty)$

$$= \lim_{x \to a} \frac{x - a}{\sin \frac{\pi x}{a}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{1}{\frac{\pi}{a} \cos \frac{\pi x}{a}} = \frac{a}{\pi} \lim_{x \to a} \frac{1}{\cos \frac{\pi x}{a}} = \frac{a}{\pi} \cdot \frac{1}{-1} = -\frac{a}{\pi}$$

22. $\lim_{x \to 1} (1 - x) \tan\left(\frac{\pi x}{2}\right)$

Sol. $\lim_{x \to 1} (1 - x) \tan\left(\frac{\pi x}{2}\right)$ $(0 \times \infty)$

$$= \lim_{x \to 1} \frac{(1 - x)}{\cot \frac{\pi x}{2}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{-1}{\left(-\csc^2 \frac{\pi x}{2}\right) \cdot \frac{\pi}{2}} = \lim_{x \to 1} \frac{\sin^2 \left(\frac{\pi x}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}$$

23. $\lim_{x \to 0} x \ln (\tan x)$

Sol. $\lim_{x \to 0} x \ln (\tan x) = \lim_{x \to 0} \frac{\ln \tan x}{\frac{1}{x}}$ $\left(\frac{\infty}{\infty}\right)$

$$= \lim_{x \to 0} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{-\frac{1}{x^2}} = \lim_{x \to 0} \frac{\frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} -\frac{x^2}{\sin x \cos x} = \lim_{x \to 0} -\frac{2x^2}{\sin 2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} -\frac{4x}{2 \cos 2x} = 0$$

24. $\lim_{x \to 0} x \tan\left(\frac{\pi}{2} - x\right)$ $(0 \times \infty)$

Sol. $\lim_{x \to 0} x \tan\left(\frac{\pi}{2} - x\right) = \lim_{x \to 0} \frac{\tan\left(\frac{\pi}{2} - x\right)}{\frac{1}{x}}$ $\left(\frac{\infty}{\infty}\right)$

$$= \lim_{x \to 0} \frac{-\sec^2\left(\frac{\pi}{2} - x\right)}{-x^{-2}} = \lim_{x \to 0} \frac{x^2}{\cos^2\left(\frac{\pi}{2} - x\right)} = \lim_{x \to 0} \frac{x^2}{\sin^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2x}{2 \sin x \cos x} = \lim_{x \to 0} \frac{2x}{\sin 2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2}{2 \cos 2x} = \frac{2}{2} = 1$$

25. $\lim_{x \to \pi/2} \tan x \ln (\sin x)$ $(\infty \times 0)$

Sol. $\lim_{x \to \pi/2} \tan x \ln (\sin x) = \lim_{x \to \pi/2} \frac{\ln \sin x}{\cot x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\sin x} \cdot \cos x}{-\csc^2 x} = \lim_{x \to \pi/2} -\frac{\cos x \sin^2 x}{\sin x}$$

$$= \lim_{x \to \pi/2} -\cos x \sin x = 0$$

26. $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$ $(\infty - \infty)$

Sol. $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x \to 0} \frac{e^x - 1 - x}{x (e^x - 1)} = \lim_{x \to 0} \frac{e^x - 1 - x}{x e^x - x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{e^x - 1}{x e^x + e^x - 1} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x}{xe^x + e^x + e^x} = \lim_{x \to 0} \frac{e^x}{e^x(x+2)} = \lim_{x \to 0} \frac{1}{x+2} = \frac{1}{2}$$

27. $\lim_{x \to 0} \left[\frac{a}{x} - \cot\left(\frac{x}{a}\right)\right]$

Sol. $\lim_{x \to 0} \left[\frac{a}{x} - \cot\left(\frac{x}{a}\right)\right]$ $(\infty - \infty)$

$$= \lim_{x \to 0} \left(\frac{a}{x} - \frac{\cos\frac{x}{a}}{\sin x\frac{x}{a}}\right) = \lim_{x \to 0} \frac{a\sin\frac{x}{a} - x\cos\frac{x}{a}}{x\sin\frac{x}{a}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos\frac{x}{a} - \left\{-\frac{x}{a}\sin\frac{x}{a} + \cos\frac{x}{a}\right\}}{\frac{x}{a}\cos\frac{x}{a} + \sin\frac{x}{a}} = \lim_{x \to 0} \frac{\frac{x}{a}\cdot\sin\frac{x}{a}}{\frac{x}{a}\cdot\cos\frac{x}{a} + \sin\frac{x}{a}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{a}\cdot\frac{x}{a}\cos\frac{x}{a} + \frac{1}{a}\cdot\sin\frac{x}{a}}{-\frac{x}{a^2}\sin\frac{x}{a} + \frac{1}{a}\cdot\cos\frac{x}{a} + \frac{1}{a}\cdot\cos\frac{x}{a}} = \frac{0+0}{-0 + \frac{1}{a} + \frac{1}{a}} = 0$$

28. $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

Sol. $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$ $(\infty - \infty)$

$$= \lim_{x \to 1} \frac{x\ln x - x + 1}{(x-1)\ln x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{x\cdot\frac{1}{x} + \ln x - 1}{(x-1)\frac{1}{x} + \ln x} = \lim_{x \to 1} \frac{\ln x}{1 - \frac{1}{x} + \ln x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{1+1} = \frac{1}{2}$$

29. $\lim_{x \to \pi/2} (\sec x - \tan x)$

Sol. $\lim_{x \to \pi/2} (\sec x - \tan x)$ $(\infty - \infty)$

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x \to \pi/2} \left(\frac{1 - \sin x}{\cos x}\right) \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = 0$$

30. $\lim_{x \to 1} \left(\frac{2}{x^2-1} - \frac{1}{x-1}\right)$

Sol. $\lim_{x \to 1} \left(\frac{2}{x^2-1} - \frac{1}{x-1}\right) = \lim_{x \to 1} \left\{\frac{2}{(x-1)(x+1)} - \frac{1}{(x-1)}\right\}$ $(\infty - \infty)$

$$= \lim_{x \to 1} \frac{2 - (x+1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1-x}{x^2-1} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{-1}{2x} = -\frac{1}{2}$$

31. $\lim_{x \to \infty} (\sqrt{x^2 + 5x} - x)$

Sol. This is of the form $\infty - \infty$. But application of L. Hospital's rule to this limit results in complicated derivatives. We proceed as under:

$$\lim_{x \to \infty} (\sqrt{x^2 + 5x} - x)$$

$$= \lim_{x \to \infty} \frac{(\sqrt{x^2 + 5x} - x)(\sqrt{x^2 + 5x} + x)}{\sqrt{x^2 + 5x} + x}$$

$$= \lim_{x \to \infty} \frac{x^2 + 5x - x^2}{\sqrt{x^2 + 5x} + x} = \lim_{x \to \infty} \frac{5x}{\sqrt{x^2 + 5x} + x}$$

$$= \lim_{x \to \infty} \frac{5}{\sqrt{1 + \frac{5}{x}} + 1} \quad \text{(dividing the numerator and the denominator by } x\text{)}$$

$$= \frac{5}{2}$$

Alternative Method:

As $\sqrt{x^2 + 5x} - x = \sqrt{x^2\left(1 + \frac{5}{x}\right)} - x = x\left(\sqrt{1 + \frac{5}{x}} - 1\right)$, so

$$\lim_{x \to \infty} \sqrt{x^2 + 5x} - x = \lim_{x \to \infty} \frac{\sqrt{1 + \frac{5}{x}} - 1}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{1}{2}\cdot\frac{-\frac{5}{x^2}}{\sqrt{1 + \frac{5}{x}}}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{5}{2}\cdot\frac{1}{\sqrt{1 + \frac{5}{x}}} = \frac{5}{2}\cdot\frac{1}{1} = \frac{5}{2}$$

32. $\lim\limits_{x \to \infty} (e^x + e^{-x})^{2/x}$

Sol. Let $y = (e^x + e^{-x})^{2/x}$. Then $\ln y = \frac{2}{x} \ln (e^x + e^{-x})$ and

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{2 \ln (e^x + e^{-x})}{x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{2 \cdot \frac{1}{e^x + e^{-x}} \cdot (e^x - e^{-x})}{1} = \lim_{x \to \infty} \frac{2e^x (1 - e^{-2x})}{e^x (1 + e^{-2x})}$$

$$= \lim_{x \to \infty} \frac{2(1 - e^{-2x})}{1 + e^{-2x}} = \frac{2(1-0)}{1+0} = 2$$

$$\lim_{x \to \infty} \ln y = 2 \Rightarrow \lim_{x \to \infty} y = e^2$$

Thus $\lim\limits_{x \to \infty} (e^x + e^{-x})^{2/x} = e^2$

33. $\lim\limits_{x \to 0} \left(\frac{1}{x}\right)^{\tan x}$ $\qquad \infty^0$

Sol. Let $y = \left(\frac{1}{x}\right)^{\tan x}$ or $\ln y = \tan x \ln \left(\frac{1}{x}\right)$

$$= -\tan x \ln x = -\frac{\ln x}{\cot x}. \text{ Then}$$

$$\lim_{x \to 0} (\ln y) = -\lim_{x \to 0} \frac{\ln x}{\cot x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= -\lim_{x \to 0} \frac{\frac{1}{x}}{-\csc^2 x} = \lim_{x \to 0} \frac{\sin^2 x}{x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 \sin x \cos x}{1} = \frac{0}{1} = 0$$

Hence $\lim\limits_{x \to 0} y = e^0 = 1$

34. $\lim\limits_{x \to \pi/2} (\cos x)^{-x + \pi/2}$

Sol. Let $y = (\cos x)^{-x + \frac{\pi}{2}}$ $\qquad$ or $\qquad \ln y = \left(\frac{\pi}{2} - x\right) \ln \cos x$

and $\lim\limits_{x \to \pi/2} (\ln y) = \lim\limits_{x \to \pi/2} \dfrac{\ln \cos x}{\left(\dfrac{1}{\frac{\pi}{2} - x}\right)}$ $\qquad \left(\frac{\infty}{\infty}\right)$

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{\left(\frac{\pi}{2} - x\right)^{-2}} = \lim_{x \to \pi/2} -\frac{\tan x}{\left(\frac{\pi}{2} - x\right)^{-2}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \pi/2} -\frac{\left(\frac{\pi}{2} - x\right)^2}{\cot x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \pi/2} -\frac{2 \cdot \left(\frac{\pi}{2} - x\right)(-1)}{-\csc^2 x} = \lim_{x \to \pi/2} \frac{2\left(\frac{\pi}{2} - x\right)}{-\csc^2 x} = \frac{0}{-1} = 0$$

Hence $\lim\limits_{x \to \pi/2} y = e^0 = 1$ or $\lim\limits_{x \to \pi/2} (\cos x)^{\pi/2 - x} = 1$

35. $\lim\limits_{x \to \infty} \left(\frac{x+a}{x-a}\right)^x$

Sol. Let $y = \left(\frac{x+a}{x-a}\right)^x$. Then $\ln y = x \ln \left(\frac{x+a}{x-a}\right)$ and $\lim\limits_{x \to \infty} x \ln \left(\frac{x+a}{x-a}\right)$

$$= \lim_{z \to 0} \frac{1}{z} \ln \frac{1 + az}{1 - az}, \text{ on setting } z = \frac{1}{x}$$

$$= \lim_{z \to 0} \frac{\ln \frac{1 + az}{1 - az}}{z} = \lim_{z \to 0} \frac{\ln (1 + az) - \ln (1 - az)}{z} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{z \to 0} \frac{\frac{a}{1 + az} + \frac{a}{1 - az}}{1} = \lim_{z \to 0} \frac{2a}{1 - a^2 z^2} = 2a$$

Thus $\lim\limits_{x \to \infty} \ln y = 2a$ or $\lim\limits_{x \to \infty} y = e^{2a}$

36. $\lim\limits_{x \to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$

Sol. Let $y = \left(\frac{\sinh x}{x}\right)^{1/x^2}$ or $\ln y = \frac{1}{x^2} \ln \left(\frac{\sinh x}{x}\right)$

Now $\dfrac{\ln \frac{\sinh x}{x}}{x^2}$ is of the form $\frac{0}{0}$ as $x \to 0$, since $\frac{\sinh x}{x} \to 1$ as $x \to 0$.

Therefore, $\lim\limits_{x \to 0} \ln y = \lim\limits_{x \to 0} \dfrac{\ln \left(\frac{\sinh x}{x}\right)}{x^2} = \lim\limits_{x \to 0} \dfrac{\frac{\cosh x}{\sinh x} - \frac{1}{x}}{2x}$

$$= \lim_{x \to 0} \frac{x \cosh x - \sinh x}{2x^2 \sinh x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cosh x + x \sinh x - \cosh x}{4x \sinh x + 2x^2 \cosh x}$$

$$= \lim_{x \to 0} \frac{\sinh x}{4 \sinh x + 2x \cosh x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cosh x}{4 \cosh x + 2 \cosh x + 2x \sinh x} = \frac{1}{6}$$

Thus $\lim_{x \to 0} y = e^{1/6}$.

37. $\lim_{x \to 0} (\tan x)^{\sin 2x}$

Sol. Let $y = (\tan x)^{\sin 2x}$ or $\ln y = \sin 2x \ln \tan x = \dfrac{\ln \tan x}{\csc 2x}$

$$\lim_{x \to 0} (\ln y) = \lim_{x \to 0} \frac{\ln \tan x}{\csc 2x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{\sec^2 x}{\tan x}}{-2 \csc 2x \cot 2x}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x \cos x}}{\frac{-2 \cos 2x}{\sin^2 2x}} = \lim_{x \to 0} -\frac{\sin^2 2x}{\cos 2x\,(2 \sin x \cos x)}$$

$$= \lim_{x \to 0} -\frac{\sin 2x}{\cos 2x} = -\frac{0}{1} = 0$$

Hence $\lim_{x \to 0} y = e^0 = 1$

38. $\lim_{x \to 0} (1 + \sin x)^{\cot x}$

Sol. Let $y = (1 + \sin x)^{\cot x}$

or $\ln y = \cot x \ln (1 + \sin x) = \dfrac{\ln (1 + \sin x)}{\tan x} \qquad \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\frac{1}{1 + \sin x} \cos x}{\sec^2 x} = \lim_{x \to 0} \frac{\cos^3 x}{1 + \sin x} = 1$$

Thus $\lim_{x \to 0} y = e^1 = e$

39. $\lim_{x \to \pi/2} (\sec x)^{\cot x}$

Sol. Let $y = (\sec x)^{\cot x}$ or $\ln y = \cot x \ln \sec x = \dfrac{\ln \sec x}{\tan x}$

$$\lim_{x \to \pi/2} (\ln y) = \lim_{x \to \pi/2} \frac{\ln \sec x}{\tan x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\sec x} \sec x \tan x}{\sec^2 x}$$

$$= \lim_{x \to \pi/2} \frac{\tan x}{\sec^2 x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \pi/2} \frac{\sec^2 x}{2 \tan x \sec^2 x}$$

$$= \lim_{x \to \pi/2} \frac{1}{2 \tan x} = \lim_{x \to \pi/2} \frac{1}{2} \cot x = 0$$

Therefore, $\lim_{x \to \pi/2} y = e^0 = 1$

40. $\lim_{x \to 1} (1 - x^2)^{1/\ln (1 - x)}$

Sol. Let $y = (1 - x^2)^{1/\ln (1 - x)}$

or $\ln y = \dfrac{1}{\ln (1 - x)} \ln (1 - x^2) = \dfrac{\ln (1 - x^2)}{\ln (1 - x)}$

$$\lim_{x \to 1} (\ln y) = \lim_{x \to 1} \frac{\ln (1 - x^2)}{\ln (1 - x)} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 1} \frac{\frac{1}{(1 - x^2)}(-2x)}{\frac{1}{(1 - x)}(-1)} = \lim_{x \to 1} \frac{2x\,(1 - x)}{(1 - x^2)}$$

$$= \lim_{x \to 1} \frac{2x}{(1 + x)} = 1$$

Thus $\lim_{x \to 1} y = e^1 = e$

41. $\lim_{x \to 1} \left[\tan\left(\frac{x\pi}{4}\right)\right]^{\tan\left(\frac{\pi x}{2}\right)}$

Sol. Let $y = \left[\tan\left(\frac{x\pi}{4}\right)\right]^{\tan\left(\frac{\pi x}{2}\right)}$

$$\ln y = \tan\frac{\pi x}{2}\ln\tan\frac{\pi x}{4} = \frac{\ln\tan\frac{\pi x}{4}}{\cot\frac{\pi x}{2}}$$

$$\lim_{x\to 1}\ln y = \lim_{x\to 1}\frac{\ln\tan\frac{\pi x}{4}}{\cot\frac{\pi x}{2}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1}\frac{\frac{1}{\tan\frac{\pi x}{4}}\cdot\left(\sec^2\frac{\pi x}{4}\right)\cdot\frac{\pi}{4}}{\frac{\pi}{2}\cdot\left(-\csc^2\frac{\pi x}{2}\right)}$$

$$= \lim_{x\to 1} -\frac{\sin^2\frac{\pi x}{2}}{2\sin\frac{\pi x}{4}\cos\frac{\pi x}{4}} = \lim_{x\to 1} -\frac{\sin^2\frac{\pi x}{2}}{\sin\frac{\pi x}{2}}$$

$$= \lim_{x\to 1}\left(-\sin\frac{\pi x}{2}\right) = -1$$

Thus $\lim_{x\to 1} y = e^{-1} = \frac{1}{e}$

42. $\lim_{x\to \pi/2}(1-\sin x)^{\cos x}$

Sol. Let $y = (1-\sin x)^{\cos x}$

or $\ln y = \cos x\ln(1-\sin x) = \frac{\ln(1-\sin x)}{\sec x}$

$$\lim_{x\to \pi/2}\ln y = \lim_{x\to \pi/2}\frac{\ln(1-\sin x)}{\sec x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to \pi/2}\frac{\frac{-\cos x}{1-\sin x}}{\sec x\tan x}$$

$$= \lim_{x\to \pi/2}\left[\frac{-\cos x}{1-\sin x}\times\frac{\cos x\times\cos x}{\sin x}\right]$$

$$= \lim_{x\to \pi/2}\frac{-\cos^3 x}{\sin x-\sin^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \frac{-3\cos^2 x(-\sin x)}{\cos x - 2\sin x\cos x} = \lim_{x\to \pi/2}\frac{3\sin x\cos x}{1-2\sin x}$$

$$= \lim_{x\to \pi/2}\frac{3}{2}\cdot\frac{\sin 2x}{1-2\sin x} = 0$$

Therefore, $\lim_{x\to \pi/2} y = e^0 = 1$

Alternative Method:

$$\lim_{x\to \pi/2}\frac{\frac{-\cos x}{1-\sin x}}{\sec x\tan x} = \lim_{x\to \pi/2}\frac{\frac{-\cos^2 x}{1-\sin \pi}}{\tan x} = \lim_{x\to \pi/2}\frac{-\cos x(1-\sin^2 x)}{\sin x(1-\sin x)}$$

$$= \lim_{x\to \pi/2}\frac{\cos x(1+\sin x)}{\sin x} = -\frac{0(2)}{1} = 0$$

Thus $\lim_{x\to \pi/2} y = e^0 = 1$

43. $\lim_{x\to 0}(\cot x)^{\sin 2x}$

Sol. Let $y = (\cot x)^{\sin 2x}$ or $\ln y = \sin 2x\ln(\cot x)$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln\cot x}{\csc 2x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{\frac{-\csc^2 x}{\cot x}}{-2\csc 2x\cot 2x} = \lim_{x\to 0}\frac{\frac{1}{\sin x\cos x}}{2\frac{\cos 2x}{\sin^2 2x}}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{\cos 2x} = 0$$

$\ln(\lim_{x\to 0} y) = 0$ or $\lim_{x\to 0} y = e^0 = 1$

44. $\lim_{x\to 0}\frac{\tanh x-\sinh x}{x^2}$

Sol.
$$\lim_{x\to 0}\frac{\tanh x-\sinh x}{x^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\operatorname{sech}^2 x-\cosh x}{2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-2\operatorname{sech}^2 x\tanh x-\sinh x}{2} = 0$$

45. $\lim_{x\to 0}\frac{\sqrt{x}-\sqrt{\sin x}}{x^{5/2}}$

Sol.
$$\lim_{x\to 0}\frac{\sqrt{x}-\sqrt{\sin x}}{x^{5/2}} = \lim_{x\to 0}\frac{\sqrt{x}-\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right\}^{1/2}}{x^{5/2}}$$

$$= \lim_{x \to 0} \frac{\sqrt{x} - \sqrt{x}\left\{1 - \frac{x^2}{6} + \frac{x^4}{120} - ...\right\}^{1/2}}{x^{5/2}}$$

$$= \lim_{x \to 0} \frac{\sqrt{x} - \sqrt{x}\left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + ...\right)\right]^{1/2}}{x^{5/2}}$$

$$= \lim_{x \to 0} \frac{\sqrt{x} - \sqrt{x}\left[1 - \frac{1}{2}\left(\frac{x^2}{6} - \frac{x^4}{120} + ...\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\cdot\left(-\left(\frac{x^2}{6} - \frac{x^4}{1200} + ...\right)^2\right) + ...\right]}{x^{5/2}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}\cdot\frac{x^2}{6} + \text{higher powers of } x}{x^2} = \frac{1}{12}$$

46. $\lim_{x \to 0} \frac{\sinh x - \sin x}{\sin^3 x}$

Sol. $\lim_{x \to 0} \frac{\sinh x - \sin x}{\sin^3 x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\cosh x - \cos x}{3 \sin^2 x \cos x}, \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\sinh x + \sin x}{6 \sin^2 x \cos x - 3 \sin^3 x}, \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cosh x + \cos x}{6 \cos^3 x - 12 \sin^2 x \cos x - 9 \sin^2 x \cos x} = \frac{2}{6} = \frac{1}{3}$$

47. $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2}$

Sol. To expand $(1 + x)^{1/x}$ into an infinite series, we let

$$y = (1+x)^{1/x} \quad \text{or} \quad \ln y = \frac{1}{x}\ln(1+x)$$

$$= \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - ...\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - ...$$

Therefore, $y = \exp\left(1 - \frac{x}{2} + \frac{x^2}{3} - ...\right) = e \cdot e^{-\frac{x}{2} + \frac{x^2}{3} - ...}$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - ...\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - ...\right)^2 + ...\right]$$

$$= e\left(1 - \frac{x}{2} + \frac{11x^2}{24} - ...\right)$$

Hence, $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2}$

$$= \lim_{x \to 0} \frac{e\left(1 - \frac{x}{2} + \frac{11x^2}{24} - ...\right) - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x \to 0} \left(\frac{11}{24}e + \text{powers of } x\right) = \frac{11}{24}e$$

48. $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

Sol. $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right) - \left\{1 + x + \frac{x^2}{2} - \frac{x^4}{8} + ...\right\}}{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right)}$$

$$= \lim_{x \to 0} \frac{\frac{x^3}{3!} + \text{higher powers in } x}{\frac{x^3}{6} + \text{higher powers in } x} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1.$$

49. Use L' Hospital's Rule to prove that

$$\lim_{x \to \infty} \left[\frac{a^{1/x} + b^{1/x}}{2}\right]^x = \sqrt{ab}, \; a > 0, b > 0.$$

Sol. The given limit is of the from 1^∞.

Let $y = \left[\frac{a^{1/x} + b^{1/x}}{2}\right]^x$ or $\ln y = x \ln\left(\frac{a^{1/x} + b^{1/x}}{2}\right)$

Now, limit $x \ln\left(\frac{a^{1/x} + b^{1/x}}{2}\right)$ is of the form $\infty \times 0$ as $x \to \infty$

$$\lim_{x \to \infty} x \ln\left(\frac{a^{1/x} + b^{1/x}}{2}\right) = \lim_{x \to \infty} \frac{\ln(a^{1/x} + b^{1/x}) - \ln 2}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{1}{a^{1/x} + b^{1/x}} \cdot \left[-\frac{1}{x^2}(a^{1/x}\ln a + b^{1/x}\ln b)\right]}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{a^{1/x}\ln a + b^{1/x}\ln b}{a^{1/x} + b^{1/x}} = \frac{\ln a + \ln b}{2} = \ln(ab)^{1/2}$$

Thus $\lim_{x \to \infty} \ln y = \ln \sqrt{ab}$ or $\lim_{x \to \infty} y = \sqrt{ab}$ as required.

50. If f is a thrice differentiable function, prove, by using L'Hospital's Rule, that

(*i*) $\lim_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h} = f'(x)$

(*ii*) $\lim_{h \to 0} \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$

(*iii*) $\lim_{h \to 0} \dfrac{f(x+h) - f(x) - hf'(x) - \dfrac{h^2}{2} f''(x)}{h^3} = \dfrac{f'''(x)}{6}$

Sol.

(*i*) $\lim_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{h \to 0} \dfrac{f'(x+h) + f'(x-h)}{2}$

$= \dfrac{2f'(x)}{2} = f'(x)$

(*ii*) $\lim_{h \to 0} \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{h \to 0} \dfrac{f'(x+h) - f'(x-h)}{2h}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{h \to 0} \dfrac{f''(x+h) + f''(x-h)}{2} = \dfrac{2f''(x)}{2} = f''(x)$

(*iii*) $\lim_{h \to 0} \dfrac{f(x+h) - f(x) - hf'(x) - \dfrac{h^2}{2} f''(x)}{h^3}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{h \to 0} \dfrac{f'(x+h) - f'(x) - hf''(x)}{3h^2}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{h \to 0} \dfrac{f''(x+h) - f''(x)}{6h}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{h \to 0} \dfrac{f'''(x+h)}{6} = \dfrac{f'''(x)}{6}$

51. Determine a, b, c, d and e such that

$$\lim_{x \to 0} \frac{\cos ax + bx^3 + cx^2 + dx + e}{x^4} = \frac{2}{3}.$$

Sol. If the limit is to be of the indeterminate form $\dfrac{0}{0}$, then

$\cos 0 + e = 0$ *i.e.*, $e = -1$.

Now $\lim_{x \to 0} \dfrac{\cos ax + bx^3 + cx^2 + dx - 1}{x^4}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{x \to 0} \dfrac{-a \sin ax + 3bx^2 + 2cx + d}{4x^3}$

It will be of the form $\dfrac{0}{0}$ if $d = 0$

$$\lim_{x \to 0} \frac{-a \sin ax + 3bx^2 + 2cx}{4x^3} = \lim_{x \to 0} \frac{-a^2 \cos ax + 6bx + 2c}{12x^2}$$

If this limit is of the form $\dfrac{0}{0}$, then $-a^2 + 2c = 0$

$\lim_{x \to 0} \dfrac{-a^2 \cos ax + 6bx + 2c}{12x^2}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{x \to 0} \dfrac{a^3 \sin ax + 6b}{24x}$

It is of the form $\dfrac{0}{0}$ if $b = 0$.

Therefore, $\lim_{x \to 0} \dfrac{a^3 \sin ax}{24x} = \dfrac{2}{3}$ or $\lim_{x \to 0} \dfrac{a^4 \cos ax}{24} = \dfrac{2}{3}$

i.e., $\dfrac{a^4}{24} = \dfrac{2}{3}$ or $a^4 = 16 \Rightarrow a^2 = \pm 4 \Rightarrow a = \pm 2, \pm 2i$

We take real values of a

From above $a^2 = 2c$ yields $c = 2$.

Thus $a = \pm 2$, $b = 0$, $c = 2$, $d = 0$, $e = -1$.

Chapter 4

TECHNIQUES OF INTEGRATION

Exercise Set 4.1 (Page 132)

Write down the indefinite integral of each of the following:

1. 0

Sol. A constant.

2. $\sqrt{x}$

Sol. $\int \sqrt{x}\, dx = \frac{x^{3/2}}{3/2} = \frac{2}{3} x^{3/2}$

3. $\frac{1+x}{x}$

Sol. $\int \frac{1+x}{x}\, dx = \int \left(\frac{1}{x} + 1\right) dx = \ln|x| + x$

4. $\frac{x^2-1}{x^2+1}$

Sol. $\int \frac{x^2-1}{x^2+1}\, dx = \int \left\{1 - \frac{2}{x^2+1}\right\} dx$

$$= \int 1 \,.\, dx - 2\int \frac{dx}{x^2+1} = x - 2\arctan x$$

5. $\tan^2 x$

Sol. $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x$

6. $\cot^2 x$

Sol. $\int \cot^2 x\, dx = \int (\operatorname{cossec}^2 x - 1)\, dx = -\cot x - x$

7. $\cos^2 x$

Sol. $\int \cos^2 x\, dx = \frac{1}{2}\int 2\cos^2 x\, dx = \frac{1}{2}\int (1 + \cos 2x)\, dx$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]$$

8. $\sin^2 x$

Sol. $\int \sin^2 x\, dx = \frac{1}{2}\int 2\sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\, dx$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]$$

9. $\sqrt{1-\cos x}$

Sol. $\int \sqrt{1-\cos x}\, dx = \int \sqrt{2\sin^2\frac{x}{2}}\, dx = \sqrt{2}\int \sin\frac{x}{2}\, dx = \sqrt{2}\left(\dfrac{-\cos\frac{x}{2}}{\frac{1}{2}}\right)$

$$= \sqrt{2}\left(-2\cos\frac{x}{2}\right) = -2\sqrt{2}\cos\frac{x}{2}$$

10. $\sqrt{4-x^2}$

Sol. $\int \sqrt{4-x^2}\, dx = \frac{x}{2}\sqrt{4-x^2} + \frac{(2)^2}{2}\arcsin\frac{x}{2}$

$$= \frac{x\sqrt{4-x^2}}{2} + 2\arcsin\frac{x}{2}$$

11. $\sqrt{4+x^2}$

Sol. $\int \sqrt{4+x^2}\, dx = \frac{x\sqrt{4+x^2}}{2} + \frac{(2)^2}{2}\ln\left|\frac{x+\sqrt{4+x^2}}{2}\right|$

$$= \frac{x\sqrt{4+x^2}}{2} + 2\ln\left|\frac{x+\sqrt{4+x^2}}{2}\right|$$

12. $\sqrt{x^2-4}$

Sol. $\int \sqrt{x^2-4}\, dx = \frac{x\sqrt{x^2-4}}{2} - \frac{(2)^2}{2}\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right|$

$$= \frac{x}{2}\sqrt{x^2-4} - 2\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right|$$

Exercise Set 4.2 (Page 135)

Evaluate (Problems 1 – 22):

1. $\int \dfrac{dx}{\sqrt{a^2+x^2}}$

Sol. Put $x = a\sinh\theta$ or $dx = a\cosh\theta\, dx$

Now $\int \dfrac{dx}{\sqrt{a^2+x^2}} = \int \dfrac{a\cosh\theta\, d\theta}{\sqrt{a^2+a^2\sinh^2\theta}} = \int \dfrac{a\cosh\theta}{a\cosh\theta}d\theta$

$$= \int 1\,.\,d\theta = \theta = \sinh^{-1}\frac{x}{a}$$

2. $\int \dfrac{dx}{\sqrt{x^2-a^2}}$

Sol. Put $x = a\cosh\theta$, $dx = a\sinh\theta\, d\theta$. Then

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh\theta\, d\theta}{\sqrt{a^2\cosh^2\theta - a^2}} = \int \frac{a\sinh\theta}{a\sinh\theta}d\theta$$

$$= \int d\theta = \theta = \cosh^{-1}\frac{x}{a}$$

3. $\int \tan x\, dx$

Sol. $\int \tan x\, dx = \int \dfrac{\sin x}{\cos x}dx = -\int \dfrac{-\sin x}{\cos x}dx$

$$= -\ln|\cos x| = \ln|\cos x|^{-1} = \ln|\sec x|$$

4. $\int \cot x\, dx$

Sol. $\int \cot x\, dx = \int \dfrac{\cos x}{\sin x}dx = \ln|\sin x|$

5. $\int \sec x\, dx$

Sol. $\int \sec x\, dx = \int \dfrac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}dx = \int \dfrac{\sec^2 x + \sec x\tan x}{(\sec x + \tan x)}dx$

$$= \ln|\sec x + \tan x| = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right|$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| = \ln\left|\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right|$$

$$= \ln\left|\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right| = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$$

6. $\int \csc x\, dx$

Sol. $\int \csc x\, dx = \int \frac{\operatorname{cosec} x\,(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}\, dx$

$$= \ln|\operatorname{cosec} x - \cot x| = \ln\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| = \ln\left|\frac{1-\cos x}{\sin x}\right|$$

$$= \ln\left|\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right| = \ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right| = \ln\left|\tan\frac{x}{2}\right|$$

7. $\int (ax^2 + 2bx + c)^n\,(ax+b)dx$

Sol. Let $t = ax^2 + 2bx + c$ so that

$$dt = 2\,(ax+b)\,dx \qquad \text{or} \qquad (ax+b)\,dx = \frac{1}{2}\,dt$$

Now $\int (ax^2+2bx+c)^n\,(ax+b)dx = \frac{1}{2}\int t^n\,.\,dt$

$$= \frac{1}{2}\frac{t^{n+1}}{n+1} = \frac{1}{2}\cdot\frac{1}{n+1}\cdot(ax^2+2bx+c)^{n+1}$$

8. $\int \sqrt{\frac{1+x}{1-x}}\,dx$

Sol. $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}\sqrt{1+x}}{\sqrt{1-x^2}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx$

$$= \int \frac{dx}{\sqrt{1-x^2}}\,dx + \int \frac{x\,dx}{\sqrt{1-x^2}}\,dx$$

$$= \arcsin x - \frac{1}{2}\int (1-x^2)^{-1/2}\,(-2x)\,dx$$

$$= \arcsin x - \frac{1}{2}\frac{(1-x)^{1/2}}{\frac{1}{2}} = \arcsin x - \sqrt{1-x^2}$$

9. $\int \frac{dx}{a+\sqrt{bx+c}}$ $\qquad (a > 0)$

Sol. Let $bx + c = z^2$ so that $b\,dx = 2z\,dz$ or $dx = \frac{2}{b}z\,dz$

Now, $\int \frac{dx}{a+\sqrt{bx+c}} = \frac{2}{b}\int\frac{z\,dz}{a+z} = \frac{2}{b}\int\left(1-\frac{a}{a+z}\right)dz$

$$= \frac{2}{b}\left[z - a\ln(a+z)\right] = \frac{2}{b}\left[\sqrt{bx+c} - a\ln\left(a+\sqrt{bx+c}\right)\right]$$

10. $\int \frac{dx}{(1+x^2)\arctan x}$

Sol. $I = \int \frac{dx}{(1+x^2)\arctan x} = \int \frac{1}{\arctan x}\cdot\frac{1}{1+x^2}\,dx$

Put $\arctan x = t$ so that $\frac{1}{1+x^2}dx = dt$. Then

$$I = \int \frac{dt}{t} = \ln|t| = \ln|\arctan x|$$

11. $\int \frac{\sin x + \cos x}{\sin x - \cos x}\,dx$

Sol. Put $\sin x - \cos x = t$ so that $(\cos x + \sin x)\,dx = dt$. Then

$$\int \frac{\sin x+\cos x}{\sin x-\cos x}\,dx = \int\frac{dt}{t} = \ln|t| = \ln|\sin x - \cos x|$$

12. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Sol. $I = \int \sin\sqrt{x}\,.\,\frac{1}{\sqrt{x}}\,dx$

Put $\sqrt{x} = t$ so that $\frac{1}{2}x^{1/2}\,dx = dt$ *i.e.*, $\frac{dx}{\sqrt{x}} = 2\,dt$. Then

$$I = 2\int \sin t\,dt = -2\cos t = -2\cos\sqrt{x}$$

13. $\int \sqrt{e^{2x}+e^{3x}}\,dx$

Sol. $\int \sqrt{e^{2x}+e^{3x}}\,dx = \int (\sqrt{1+e^x})\cdot e^x\,dx$

Let $1 + e^x = z$ so that $e^x\,dx = dz$

and $\int (\sqrt{1+e^x})\,e^x\,dx = \int \sqrt{z}\,dz = \frac{2}{3}z^{3/2} = \frac{2}{3}(1+e^x)^{3/2}$

14. $\displaystyle\int \frac{dx}{e^x + e^{-x}}$

Sol. Put $e^x = t$ so that $e^x\,dx = dt$. Then

$$I = \int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{e^{2x} + 1} = \int \frac{dt}{t^2 + 1} = \arctan t = \arctan e^x.$$

15. $\displaystyle\int \frac{e^{2x}\,dx}{\sqrt{e^x - 1}}$

Sol. Let $e^x - 1 = z^2$ so that $e^x\,dx = 2z\,dz$ and

$$\int \frac{e^{2x}\,dx}{\sqrt{e^x - 1}} = \int \frac{e^x}{\sqrt{e^x - 1}} \cdot e^x dx = \int \frac{(1 + z^2)\,2z\,dz}{z} = 2\left(z + \frac{z^3}{3}\right)$$

$$= 2z\left(1 + \frac{z^2}{3}\right) = 2\sqrt{e^x - 1}\left(1 + \frac{e^x - 1}{3}\right) = \frac{2}{3}\sqrt{e^x - 1}\,(2 + e^x)$$

16. $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

Sol. Let $\ln x = t$ so that $\frac{1}{x}\,dx = dt$. Then

$$\int \frac{\cos(\ln x)}{x}\,dx = \int \cos t\,dt = \sin t = \sin(\ln x)$$

17. $\displaystyle\int \frac{2x + 5}{\sqrt{x^2 + 5x + 7}}\,dx$

Sol. Let $x^2 + 5x + 7 = t$ so that $(2x + 5)\,dx = dt$ and

$$\int \frac{2x + 5}{\sqrt{x^2 + 5x + 7}}\,dx = \int (x^2 + 5x + 7)^{-1/2}\,(2x + 5)\,dx$$

$$= \int t^{-1/2}\,dt = 2\,t^{1/2} = 2\sqrt{x^2 + 5x + 7}$$

18. $\displaystyle\int \frac{(x + 2)dx}{\sqrt{2x^2 + 8x + 5}}$

Sol. Let $2x^2 + 8x + 5 = t$ so that $(4x + 8)\,dx = dt$

or $(x + 2)\,dx = \frac{1}{4}dt$. Then

$$\int \frac{(x + 2)dx}{\sqrt{2x^2 + 8x + 5}} = \int (2x^2 + 8x + 5)^{-1/2}\,(x + 2)\,dx = \int t^{-1/2} \cdot \frac{1}{4}\,dt$$

$$= \frac{1}{4} \cdot \frac{t^{1/2}}{\frac{1}{2}} = \frac{1}{2}t^{1/2} = \frac{1}{2}\sqrt{2x^2 + 8x + 5} = \frac{\sqrt{2x^2 + 8x + 5}}{2}$$

19. $\displaystyle\int \frac{\sqrt{x^2 - a^2}}{x^4}\,dx$

Sol. Put $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\,d\theta$. Then

$$\int \frac{\sqrt{x^2 - a^2}}{x^4}\,dx = \int \frac{\sqrt{a^2\sec^2\theta - a^2}}{a^4\sec^4\theta} \,.\, a\sec\theta\tan\theta \,.\, d\theta$$

$$= \int \frac{a\tan\theta \,.\, \tan\theta}{a^3\sec^3\theta}\,d\theta = \frac{1}{a^2}\int \frac{\sin^2\theta}{\cos^2\theta} \times \cos^3\theta\,d\theta$$

$$= \frac{1}{a^2}\int \sin^2\theta \times \cos\theta\,d\theta$$

$$= \frac{1}{a^2} \cdot \frac{\sin^3\theta}{3} \qquad \left(\text{As } \cos\theta = \frac{a}{x}, \text{ so } \sin^2\theta = 1 - \cos^2\theta\right.$$

$$= \frac{1}{3a^2} \cdot \frac{(x - a^2)^{3/2}}{x^3} \qquad = 1 - \frac{a^2}{x^2}$$

$$= \frac{(x^2 - a^2)^{3/2}}{3a^2\,x^3} \qquad \left. = \frac{x^2 - a^2}{x^2}\right)$$

20. $\displaystyle\int \cos^6 x \sin^3 x\,dx$

Sol. $I = \int -\cos^6 x \,.\, \sin^2 x \,.\, (-\sin x\,dx) = -\int \cos^6 x\,(1 - \cos^2 x)\,(-\sin x\,dx)$

Putting $\cos x = t$ and $-\sin x\,dx = dt$ in I, we have

$$I = -\int t^6\,(1 - t^2)\,dt = \int t^8\,dt - \int t^6\,dt$$

$$= \frac{t^9}{9} - \frac{t^7}{7} = \frac{1}{9}\cos^9 x - \frac{1}{7}\cos^7 x$$

21. $\displaystyle\int \tan^3\theta\sec^3\theta\,d\theta$

Sol. We have

$$I = \int \tan^3\theta\sec^3\theta\,d\theta = \int \tan^2\theta\sec^2\theta\,(\tan\theta\sec\theta)\,d\theta$$

$$= \int (\sec^2\theta - 1)\sec^2\theta\tan\theta\sec\theta\,d\theta$$

Now let $\sec\theta = z$ so that $\sec\theta\tan\theta\,d\theta = dz$

and $I = \int (z^2 - 1)z^2\,dz = \int (z^4 - z^2)\,dz = \frac{z^5}{5} - \frac{z^3}{3} = \frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3}$

22. $\int \cot^3 x \csc^4 x\, dx$

Sol. Let $\cot x = z$ so that $-\csc^2 x\, dx = dz$ and

$\int \cot^3 x \csc^4 x\, dx = -\int \cot^3 x \csc^2 x (-\csc^2 x)\, dx$

$-\int z^3(1+z^2)\, dz = -\int (z^3 + z^5)\, dz = -\left(\frac{z^4}{4} + \frac{z^6}{6}\right) = -\left(\frac{\cot^4 x}{4} + \frac{\cot^6 x}{6}\right)$

Find an antiderivative of each of the following (Problems 23 – 40):

23. $\dfrac{1}{\sqrt{2x^2 + 3x + 4}}$

Sol. $\displaystyle\int \frac{dx}{\sqrt{2}\sqrt{x^2 + \frac{3}{2}x + 2}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 + 2 - \left(\frac{3}{4}\right)^2}} \quad \text{adding and subtracting } \left(\frac{3}{4}\right)^2$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}} = \frac{1}{\sqrt{2}} \sinh^{-1} \frac{x + \frac{3}{4}}{\frac{\sqrt{23}}{4}}$$

$$= \frac{1}{\sqrt{2}} \sinh^{-1} \frac{4x + 3}{\sqrt{23}}$$

24. $\sqrt{a^2 - x^2}$

Sol. Put $x = a \sin\theta$ so that $dx = a\cos\theta\, d\theta$ and

$$\int \sqrt{a^2 - x^2}\, dx = \int a\cos\theta\, a\cos\theta\, d\theta = \frac{a^2}{2} \int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\frac{\sin 2\theta}{2} = \frac{a^2}{2}\theta + \frac{a^2}{2} \cdot \frac{2\sin\theta\cos\theta}{2}$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sin\theta\sqrt{1 - \sin^2\theta}$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}$$

25. $(2x + 3)\sqrt{2x + 1}$

Sol. Let $2x + 1 = z^2$. Then $2\, dx = 2z\, dz$

or $dx = z\, dz$

Substituting, we get

$$\int (2x + 3)\sqrt{2x + 1}\, dx = \int z(z^2 + 2)\, z\, dz = \int (z^4 + 2z^2)\, dz$$

$$= \frac{z^5}{5} + \frac{2z^3}{3} = \frac{(2x + 1)^{5/2}}{5} + \frac{2(2x + 1)^{3/2}}{3}$$

26. $(1 + x^2)^{-3/2}$

Sol. Let $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$

Substituting, we have

$$\int \frac{dx}{(1 + x^2)^{3/2}} = \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int \cos\theta\, d\theta = \sin\theta$$

Since $\tan\theta = x$, $\sin\theta = \dfrac{x}{\sqrt{1 + x^2}}$

Therefore, $\displaystyle\int \frac{dx}{(1 + x^2)^{3/2}} = \frac{x}{\sqrt{1 + x^2}}$

27. $\dfrac{x^2}{\sqrt{x^2 + 1}}$

Sol. $\displaystyle\int \frac{x^2}{\sqrt{x^2 + 1}}\, dx = \int \frac{(x^2 + 1) - 1}{\sqrt{x^2 + 1}}\, dx = \int \sqrt{x^2 + 1}\, dx - \int \frac{dx}{\sqrt{x^2 + 1}}$

But $\displaystyle\int \sqrt{x^2 + 1}\, dx = \frac{x\sqrt{x^2 + 1}}{2} + \frac{1}{2}\sinh^{-1} x$

Hence $\displaystyle\int \frac{x^2}{\sqrt{x^2 + 1}}\, dx = \frac{x\sqrt{x^2 + 1}}{2} + \frac{1}{2}\sinh^{-1} x - \sinh^{-1} x$

$$= \frac{x\sqrt{x^2 + 1}}{2} - \frac{1}{2}\sinh^{-1} x.$$

28. $(2x + 4)\sqrt{2x^2 + 3x + 1}$

Sol. $\int (2x + 4)(2x^2 + 3x + 1)^{1/2}\, dx$

$$= \frac{1}{2}\int (4x + 3 + 5)(2x^2 + 3x + 1)^{1/2}\, dx$$

$$= \frac{1}{2}\int (4x + 3)(2x^2 + 3x + 1)^{1/2}\, dx + \frac{5}{2}\int (2x^2 + 3x + 1)^{1/2}\, dx$$

$$= \frac{(2x^2 + 3x + 1)^{3/2}}{3} + \frac{5}{\sqrt{2}}\int \left\{x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 + \frac{1}{2} - \left(\frac{3}{4}\right)^2\right\}^{1/2} dx$$

$$= \frac{(2x^2 + 3x + 1)^{3/2}}{3} + \frac{5}{\sqrt{2}}\int \left\{\left(x + \frac{3}{4}\right)^2 + \frac{1}{2} - \frac{9}{16}\right\}^{1/2} dx$$

$$= \frac{(2x^2+3x+1)^{3/2}}{3} + \frac{5}{\sqrt{2}}\int\left\{\left(x+\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right\}^{1/2} dx \qquad (1)$$

By the formula

$\int\sqrt{x^2-a^2} = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$, we have

$$\int\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}\,dx = \frac{\left(x+\frac{3}{4}\right)\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}}{2} - \frac{\left(\frac{1}{4}\right)^2}{2}\cosh^{-1}\frac{x+\frac{3}{4}}{\frac{1}{4}}$$

$$= \frac{\left(\frac{4x+3}{4}\right)\sqrt{x^2+\frac{3}{2}x+\frac{9}{16}-\frac{1}{16}}}{2} - \frac{1}{32}\cosh^{-1}\left(\frac{4x+3}{1}\right)$$

$$= \frac{4x+3}{8}\sqrt{x^2+\frac{3}{2}x+\frac{1}{2}} - \frac{1}{32}\cosh^{-1}(4x+3)$$

$$= \frac{4x+3}{8\sqrt{2}}\sqrt{2x^2+3x+1} - \frac{1}{32}\cosh^{-1}(4x+3) \qquad (2)$$

Putting from (2) into (1), we get

$$\int(2x+4)\sqrt{2x^2+3x+1}\,dx = \frac{(2x^2+3x+1)}{3}$$

$$+\frac{5}{\sqrt{2}}\left[\frac{4x+3}{8\sqrt{2}}\sqrt{2x^2+3x+1} - \frac{1}{32}\cosh^{-1}(4x+4)\right]$$

$$= \frac{(2x^2+3x+1)^{3/2}}{3} + \frac{5(4x+3)}{16}\sqrt{2x^2+3x+1} - \frac{5}{32\sqrt{2}}\cosh^{-1}(4x+3)$$

29. $\dfrac{1}{3\sin x + 4\cos x}$

Sol. $\displaystyle\int\frac{dx}{3\sin x + 4\cos x}$

Put $3 = r\cos\theta$, $4 = r\sin\theta$, we get $r = 5$ and $\tan\theta = \frac{4}{3}$

Now $\displaystyle\int\frac{dx}{3\sin x+4\cos x} = \int\frac{dx}{5(\sin x\cos\theta + \cos x\sin\theta)}$

$$= \int\frac{dx}{5\sin(x+\theta)} = \frac{1}{5}\int\operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{5}\ln\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right| = \frac{1}{5}\ln\left|\tan\left(\frac{x}{2}+\frac{1}{2}\arctan\frac{4}{3}\right)\right|$$

30. $\dfrac{\tan x}{\cos x + \sec x}$

Sol. $\displaystyle I = \int\frac{\tan x}{\cos x+\sec x}dx = \int\frac{\frac{\sin x}{\cos x}}{\cos x+\frac{1}{\cos x}}dx = \int\frac{\sin x\,dx}{\cos^2 x+1}$

Put $\cos x = \theta$. Then $-\sin x\,dx = d\theta$ or $\sin x\,dx = -d\theta$ and

$$I = -\int\frac{d\theta}{\theta^2+1} = -\arctan\theta = -\arctan(\cos x).$$

31. $\dfrac{1}{\sin(x-a)\sin(x-b)}$

Sol. $\displaystyle\int\frac{dx}{\sin(x-a)\sin(x-b)}$

$a - b = (x-b) - (x-a)$

or $\sin(a-b) = \sin\{(x-b)-(x-a)\}$ (1)

Thus $\displaystyle\int\frac{dx}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(a-b)}\int\frac{\sin(a-b)}{(\sin(x-a)\sin(x-b))}dx$

$$= \frac{1}{\sin(a-b)}\int\frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)}dx - \frac{1}{\sin(a-b)}\int\frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int\frac{\cos(x-a)}{\sin(x-a)}dx - \frac{1}{\sin(a-b)}\int\frac{\cos(x-b)}{\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\ln|\sin(x-a)| - \frac{1}{\sin(a-b)}\ln|\sin(x-b)|$$

$$= \frac{1}{\sin(a-b)}\ln\left|\frac{\sin(x-a)}{\sin(x-b)}\right|$$

32. $\tan x\ln(\sec x)$

Sol. Put $\ln\sec x = t$. Then $\dfrac{(\sec x\tan x)}{\sec x}dx = dt$ or $\tan x\,dx = dt$

$$\int(\ln(\sec x))\tan x\,dx = \int t\,dt = \frac{t^2}{2} = \frac{1}{2}(\ln\sec x)^2$$

33. $\dfrac{1}{(3\tan x+1)\cos^2 x}$

Sol. Put $3\tan x + 1 = t$ Then $3\sec^2 x\,dx = dt$ or $\sec^2 x\,dx = \frac{1}{3}dt$

$$\int \frac{\sec^2 x\,dx}{3\tan x+1} = \int \frac{\frac{1}{3}dt}{t} = \frac{1}{3}\ln|t| = \frac{1}{3}\ln|3\tan x + 1|$$

34. $e^{\sin x}\cos x$

Sol. Put $\sin x = t$. Then $\cos x\,dx = dt$ and

$$\int e^{\sin x}\cos x\,dx = \int e^t\,dt = e^t = e^{\sin x}.$$

35. $\sqrt{1+3\cos^2 x}\,\sin 2x$

Sol. Put $1 + 3\cos^2 x = t$. Then $-6\cos x\sin x\,dx = dt$

or $-3\sin 2x\,dx = dt$ or $\sin 2x\,dx = -\frac{1}{3}dt$

$$\int\sqrt{1+3\cos^2 x}\,.\,\sin 2x\,dx = -\frac{1}{3}\int t^{1/2}\,dt = -\frac{2}{9}t^{3/2}$$

$$= -\frac{2}{9}(1+3\cos^2 x)^{3/2}$$

36. $\dfrac{\sin 2x}{\sqrt{1+\cos^2 x}}$

Sol. Put $1+\cos^2 x = t$ so that $-2\cos x\sin x\,dx = dt$

or $-\sin 2x\,dx = dt$ or $\sin 2x\,dx = -dt$

$$\int\frac{\sin 2x\,dx}{\sqrt{1+\cos^2 x}} = -\int\frac{dt}{\sqrt{t}} = -\int t^{-1/2}\,dt$$

$$= -2t^{1/2} = -2\sqrt{1+\cos^2 x}$$

37. $\dfrac{1}{2\sin^2 x + 3\cos^2 x}$

Sol. $I = \displaystyle\int\frac{dx}{2\sin^2 x + 3\cos^2 x}$

Dividing Num. and Denom. by $\cos^2 x$, we have

$I = \displaystyle\int\frac{\sec^2 x\,dx}{2\tan^2 x + 3}$ Put $\tan x = t$ so that $\sec^2 x\,dx = dt$

and $I = \displaystyle\int\frac{dt}{2t^2+3} = \frac{1}{2}\int\frac{dt}{t^2+\frac{3}{2}} = \frac{1}{2}\cdot\frac{\sqrt{2}}{\sqrt{3}}\arctan\frac{t}{\sqrt{\frac{3}{2}}}$

$$= \frac{1}{\sqrt{6}}\arctan\sqrt{\frac{2}{3}}\,t = \frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}\tan x\right)$$

38. $\dfrac{1}{\sqrt{x}}\sec\sqrt{x}\tan\sqrt{x}$

Sol. $I = \displaystyle\int\frac{1}{\sqrt{x}}\sec\sqrt{x}\tan\sqrt{x}\,dx = \int(\sec\sqrt{x}\tan\sqrt{x})\frac{1}{\sqrt{x}}\,dx$

Put $\sqrt{x} = u$ so that $\dfrac{1}{2\sqrt{x}}dx = du$ or $\dfrac{1}{\sqrt{x}}dx = 2du$

Then $I = 2\int\sec u\tan u\,du = 2\sec u = 2\sec\sqrt{x}$

39. $[\pi^{\sin x} + (\sin x)^{\pi}]\cos x$

Sol. $I = \int[\pi^{\sin x} + (\sin x)^{\pi}]\cos x\,dx$

Put $\sin x = u$ so that $\cos x\,dx = du$ and

$$I = \int(\pi^u + u^{\pi})\,du = \frac{\pi^u}{\ln\pi} + \frac{u^{\pi+1}}{\pi+1}$$

$$= \frac{\pi^{\sin x}}{\ln\pi} + \frac{(\sin x)^{\pi+1}}{\pi+1}$$

40. $\dfrac{\cos x}{3\sin x + 4\sqrt{\sin x}}$

Sol. $I = \displaystyle\int\frac{\cos x}{3\sin x + 4\sqrt{\sin x}}\,dx$

Put $\sqrt{\sin x} = z$ or $\sin x = z^2$. Therefore, $\cos x\,dx = 2z\,dz$

and $I = \displaystyle\int\frac{2z\,dz}{3z^2+4z} = \int\frac{2\,dz}{4+3z} = \frac{2}{3}\int\frac{1}{4+3z}\cdot 3dz$

$$= \frac{2}{3}\ln|4+3z| = \frac{2}{3}\ln(4+3\sqrt{\sin x})$$

Exercise Set 4.3 (Page 142)

Evaluate (Problems 1 – 20):

1. $\int x\sec^2 x\,dx$

Sol. $I = \int x\sec^2 x\,dx$

Integrating by parts, regarding x as first function, we have

$$I = x\tan x - \int\tan x\,dx = x\tan x + \int\frac{1}{\cos x}\,.\,(-\sin x)\,dx$$

$$= x\tan x + \ln|\cos x|$$

2. $\int x\csc^2 x\,dx$

Sol. $\int x\csc^2 x\,dx = -\int x(-\csc^2 x)\,dx = -[x\cot x - \int(\cot x) - 1\,dx]$

$$= -x\cot x + \int\frac{\cos x}{\sin x}dx = -x\cot x + \ln|\sin x|$$

3. $\int x^n \ln x\,dx$

Sol. $\int x^n \ln x\,dx = \int(\ln x)\,x^n\,dx$

$= (\ln x)\cdot\dfrac{x^{n+1}}{n+1} - \displaystyle\int\frac{x^{n+1}}{n+1}\cdot\frac{1}{x}dx$, (Integrating by parts)

$= \ln x\cdot\dfrac{x^{n+1}}{n+1} - \dfrac{1}{n+1}\displaystyle\int x^n\,dx = \ln x\cdot\frac{x^{n+1}}{n+1} - \frac{1}{n+1}\cdot\frac{x^{n+1}}{n+1}$

$= \dfrac{x^{n+1}}{n+1}\cdot\ln x - \dfrac{x^{n+1}}{(n+1)^2}$

4. $\int x^2\arctan x\,dx$

Sol. $\int x^2\arctan x\,dx = \int(\arctan x)\,x^2\,dx = (\arctan x)\dfrac{x^3}{3} - \displaystyle\int\frac{x^3}{3}\cdot\frac{1}{1+x^2}dx$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx \qquad (1)$$

Now $\displaystyle\int\frac{x^3}{1+x^2}dx = \int\left(x - \frac{x}{x^2+1}\right)dx = \frac{x^2}{2} - \frac{1}{2}\int\frac{2x}{x^2+1}dx$

$$= \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)$$

Putting this value into (1), we have

$\displaystyle\int x^2\arctan x\,dx = \frac{x^3}{3}\arctan x - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)\right]$

$$= \frac{x^3}{3}\arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1)$$

5. $\int\sec^3 x\,dx$

Sol. $\int\sec^3 x\,dx = \int\sec x\sec^2 x\,dx = \sec x\tan x - \int\tan x\sec x\tan x\,dx$

$$= \sec x\tan x - \int\sec x(\sec^2 x - 1)\,dx$$

$$= \sec x\tan x - \int\sec^3 x\,dx + \int\sec x\,dx$$

or $\quad 2\int\sec^3 x\,dx = \sec x\tan x + \int\sec x\,dx$

$$= \sec x\tan x + \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$$

or $\quad\displaystyle\int\sec^3 x\,dx = \frac{\sec x\tan x}{2} + \frac{1}{2}\ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$

6. $\int\csc^3 x\,dx$

Sol. $\int\csc^3 x\,dx = \int\csc x\,.\,\csc^2 x\,dx$

$$= \csc x(-\cot x) - \int(-\cot x)(-\csc x\cot x)\,dx$$

$$= -\csc x\cot x - \int(\csc^2 x - 1)\csc x\,dx$$

$$= -\csc x\cot x - \int\csc^3 x\,dx + \int\csc x\,dx$$

or $\quad 2\int\csc^3 x\,dx = -\csc x\cot x + \int\csc x\,dx$

$$= -\csc x\cot x + \ln\left|\tan\frac{x}{2}\right|$$

$$\int\csc^3 x\,dx = -\frac{1}{2}\csc x\cot x + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|$$

7. $\displaystyle\int\frac{x-\sin x}{1-\cos x}dx$

Sol. $\displaystyle\int\frac{x-\sin x}{1-\cos x}dx = \int\frac{x - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}dx$

$$= \int x\left(\csc^2\frac{x}{2}\right)\frac{1}{2}dx - \int\cot\frac{x}{2}dx$$

$$= x\left(-\cot\frac{x}{2}\right) - \int\left(-\cot\frac{x}{2}\right).\,1\,dx - \int\cot\frac{x}{2}dx$$

$$= -x\cot\frac{x}{2} + \int\cot\frac{x}{2}dx - \int\cot\frac{x}{2}dx = -x\cot\frac{x}{2}$$

8. $\int x\arcsin x\,dx$

Sol. $I = \int x\arcsin x\,dx = \int(\arcsin x)\,x\,dx$

Integrating by parts taking arcsin x as first function, we have

$$I = (\arcsin x)\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx \qquad (1)$$

Now $\int \frac{-x^2}{\sqrt{1-x^2}}\,dx = \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx = \int\sqrt{1-x^2}\,dx - \int\frac{1}{\sqrt{1-x^2}}\,dx$

$$= x\cdot\frac{\sqrt{1-x^2}}{2} + \frac{1}{2}\arcsin x - \arcsin x$$

$$= x\cdot\frac{\sqrt{1-x^2}}{2} - \frac{1}{2}\arcsin x.$$

Putting this value into (1), we have

$$I = \frac{x^2}{2}\arcsin x + \frac{1}{4}x\cdot\sqrt{1-x^2} - \frac{1}{4}\arcsin x$$

$$= \frac{2x^2-1}{4}\arcsin x + \frac{1}{4}x\cdot\sqrt{1-x^2}$$

9. $\int x^3\sqrt{x^2+1}\,dx$

Sol. Let $u = x^2$ and $v = \sqrt{x^2+1}\,(2x)$.

Integrating by parts, we get

$$\int x^3\sqrt{x^2+1}\,dx = \frac{1}{2}\int x^2\sqrt{x^2+1}\,(2x)\,dx$$

$$= \frac{1}{2}\left[x^2\,\frac{2}{3}(x^2+1)^{3/2} - \int\frac{2}{3}(x^2+1)^{3/2}\,2x\,dx\right]$$

$$= \frac{1}{3}x^2(x^2+1)^{3/2} - \frac{1}{3}\cdot\frac{2}{5}(x^2+1)^{5/2}$$

$$= \frac{1}{3}x^2(x^2+)^{3/2} - \frac{2}{15}(x^2+1)^{5/2}$$

10. $\int e^x\,\frac{1+x\ln x}{x}\,dx$

Sol. $\int e^x\left(\frac{1+x\ln x}{x}\right)dx = \int e^x\cdot\frac{1}{x}\,dx + \int e^x\ln x\,dx$

$$= e^x\ln x - \int \ln x\,.\,e^x\,dx + \int e^x\ln x\,dx = e^x\ln x$$

11. $\int e^x\cdot\frac{1-\sin x}{1-\cos x}\,dx$

Sol. $\int e^x\cdot\frac{1-\sin x}{1-\cos x}\,dx = \int e^x\cdot\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$

$$= \int e^x\left(\frac{1}{2}\csc^2\frac{x}{2}\right)dx - \int e^x\cot\frac{x}{2}\,dx$$

$$= e^x\left(-\cot\frac{x}{2}\right) + \int\left(\cot\frac{x}{2}\right)e^x\,dx - \int e^x\cot\frac{x}{2}\,dx$$

$$= -e^x\cot\frac{x}{2}$$

12. $\int \arctan\left(\sqrt{\frac{1-x}{1+x}}\right)dx$

Sol. Put $x = \cos\theta$ or $dx = -\sin\theta\,d\theta$

$$I = \int\arctan\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)(-\sin\theta)\,d\theta$$

$$= \int\arctan\left(\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\right)(-\sin\theta)\,d\theta$$

$$= \int\arctan\left(\tan\frac{\theta}{2}\right)(-\sin\theta)\,d\theta$$

$$= \int\frac{\theta}{2}(-\sin\theta)\,d\theta = \frac{1}{2}\left[\theta\cos\theta - \int\cos\theta\,d\theta\right]$$

$$= \frac{1}{2}\left[\theta\cos\theta - \sin\theta\right] = \frac{1}{2}\left[x\arccos x - \sqrt{1-x^2}\,\right]$$

13. $\int\arcsin\left(\sqrt{\frac{x}{x+a}}\right)dx$

Sol. Put $x = a\tan^2\theta$ or $dx = 2a\tan\theta\sec^2\theta\,d\theta$

$$I = \int\arcsin\sqrt{\frac{a\tan^2\theta}{a\sec^2\theta}}\,.\,2a\tan\theta\sec^2\theta\,d\theta$$

$$= \int\arcsin\left(\frac{\tan\theta}{\sec\theta}\right).\,2a\tan\theta\sec^2\theta\,d\theta$$

$$= \int\arcsin(\sin\theta)\,.\,2a\tan\theta\,.\,\sec^2\theta\,d\theta$$

$$= 2a\int\theta\,.\,(\tan\theta\sec^2\theta)\,d\theta$$

$$= 2a\left[\theta\,\frac{\tan^2\theta}{2} - \int\frac{\tan^2\theta}{2}\,d\theta\right]$$

$$= a\,\theta\tan^2\theta - a\int\tan^2\theta\,d\theta = a\theta\tan^2\theta - a\int(\sec^2\theta - 1)\,d\theta$$
$$= a\,\theta\tan^2\theta - a\,(\tan\theta - \theta)$$
$$= (a\tan^2\theta)\,\theta - a\tan\theta + a\theta$$
$$= x\arctan\sqrt{\frac{x}{a}} - a\sqrt{\frac{x}{a}} + a\arctan\sqrt{\frac{x}{a}}$$
$$= (x + a)\arctan\sqrt{\frac{x}{a}} - \sqrt{ax}$$

14. $\int e^{ax}\sin(bx + c)\,dx$

Sol. Taking e^{ax} as first function and $\sin(bx + c)$ as the second function and then integrating by parts, we have

$$\int e^{ax}\sin(bx + c)\,dx = e^{ax}\frac{-\cos(bx + c)}{b} - \int -\frac{\cos(bx + c)}{b}e^{ax}.\,a\,dx$$
$$= -\frac{e^{ax}}{b}\cos(bx + c) + \frac{a}{b}\int e^{ax}\cos(bx + c)\,dx$$
$$= -\frac{1}{b}e^{ax}\cos(bx + c) + \frac{a}{b}\left[a^{ax}\frac{\sin(bx + c)}{b} - \int\frac{\sin(bx + c)}{b}a\,.\,e^{ax}\,dx\right]$$
$$= -\frac{1}{b}e^{ax}\cos(bx + c) + \frac{a}{b^2}e^{ax}\sin(bx + c) - \frac{a^2}{b^2}\int e^{ax}\sin(bx + c)\,dx$$

or $\left(1 + \dfrac{a^2}{b^2}\right)\int e^{ax}\sin(bx + c)\,dx = e^{ax}\left[-\dfrac{\cos(bx + c)}{b} + \dfrac{a\sin(bx + c)}{b^2}\right]$

or $\left(\dfrac{a^2 + b^2}{b^2}\right)\int e^{ax}\sin(bx + c)\,dx = \dfrac{e^{ax}}{b^2}[a\sin(bx + c) - b\cos(bx+c)]$

i.e., $\int e^{ax}\sin(bx+c)\,dx = \dfrac{e^{ax}}{a^2 + b^2}[a\sin(bx+c) - b\cos(bx+c)]$ (1)

Put $a = r\cos\theta$ and $b = r\sin\theta$

then $r = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}$

Making these substitutions in the R.H.S. of (1), we get

$$\int e^{ax}\sin(bx + x) = \frac{e^{ax}}{\sqrt{a^2 + b^2}}[\sin(bx + c)\cos\theta - \cos(bx + c)\sin\theta]$$
$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}}\sin(bx + c - \theta)$$
$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}}\sin\left(bx + c - \arctan\frac{b}{a}\right)$$

15. $\int\ln(x + \sqrt{1 + x^2})\,dx$

Sol. $\int\ln(x + \sqrt{1 + x^2})\,dx = \int(\ln(x + \sqrt{1 + x^2}))\,.\,1\,dx$

$$= \ln(x + \sqrt{1 + x^2})\,.\,x - \int x\cdot\frac{1 + \frac{x}{\sqrt{1 + x^2}}}{x + \sqrt{1 + x^2}}\,dx$$
$$= x\ln(x + \sqrt{1 + x^2}) - \int\frac{x}{\sqrt{1 + x^2}}\,dx$$
$$= x\ln(x + \sqrt{1 + x^2}) - \int(1 + x^2)^{-1/2}\,.\,x\,dx$$
$$= x\ln(x + \sqrt{1 + x^2}) - \frac{1}{2}\frac{(1 + x^2)^{1/2}}{\frac{1}{2}}$$
$$= x\ln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2}$$

16. $\int\dfrac{x^2 + 1}{(x + 1)^2}e^x\,dx$

Sol. $\int\dfrac{x^2 + 1}{(x + 1)^2}e^x\,dx = \int\dfrac{(x + 1)^2 - 2x}{(x + 1)^2}e^x\,dx = \int\left[1 - \dfrac{2x}{(x + 1)^2}\right]e^x\,dx$

$$= \int\left[1 - 2\frac{(x + 1 - 1)}{(x + 1)^2}\right]e^x\,dx$$
$$= \int\left[1 - 2\left\{\frac{1}{x + 1} - \frac{1}{(x + 1)^2}\right\}\right]e^x\,dx$$
$$= \int e^x\,dx - 2\int\frac{e^x\,dx}{x + 1}dx + 2\int\frac{e^x\,dx}{(x + 1)^2}$$
$$= e^x - 2\left[\frac{1}{x + 1}e^x - \int e^x\left(\frac{-1}{(x + 1)^2}\right)dx\right] + 2\int\frac{e^x\,dx}{(x + 1)^2}$$

(Integrating the first and second terms and leaving the third term as it is)

$$= e^x - \frac{2}{x + 1}e^x - 2\int\frac{e^x\,dx}{(x + 1)^2} + 2\int\frac{e^x}{(x + 1)^2}dx$$
$$= e^x - \frac{2}{x + 1}e^x = \left(\frac{x - 1}{x + 1}\right)e^x$$

17. $\int\cos(\ln x)\,dx$

Sol. Take $\cos(\ln x)$ as first function, 1 as second function and integrate by parts to get

$$I = x\,(\cos(\ln x)) - \int x\left(-\frac{\sin(\ln x)}{x}\right)dx$$

$$= x\cos(\ln x) + \int \sin(\ln x)\;dx$$

$$= x\cos(\ln x) + x\sin(\ln x) - \int \frac{\cos(\ln x)}{x}\,x\,dx$$

$$= x\cos(\ln x) + x\sin(\ln x) - I$$

or $\quad 2I = x\cos(\ln x) + x\sin(\ln x)$

or $\quad I = \frac{1}{2}\,[x\cos(\ln x) + x\sin(\ln x)]$

18. $\int \sqrt{x}\, e^{-\sqrt{x}}\, dx$

Sol. $I = \int \sqrt{x}\, e^{-\sqrt{x}}\, dx$

Put $\sqrt{x} = z$ or $\frac{1}{2\sqrt{x}}\,dx = dz$ or $dx = 2z\,dz$

$$I = \int z\,.\,e^{-z}\,.\,2z\,dz = 2\int z^2 e^{-z}\,dz = 2\left[z^2\,.\,\frac{e^{-z}}{-1} - \int \frac{e^{-z}}{-1}\,.\,2z\,dz\right]$$

$$= 2\left[z^2\frac{e^{-z}}{-1} + 2\int z e^{-z}\,dz\right] = 2\left[-z^2 e^{-z} + 2\left\{z\,\frac{e^{-z}}{-1} - \int \frac{e^{-z}}{-1}\,dz\right\}\right]$$

$$= 2\left[-z^2 e^{-z} - 2ze^{-z} + 2\int e^{-z}\,dz\right] = 2\,[-z^2 e^{-z} - 2\,ze^{-z} - 2e^{-z}]$$

$$= -2\,(x\,e^{-\sqrt{x}} + 2\sqrt{x}\;e^{-\sqrt{x}} + 2e^{-\sqrt{x}})$$

19. $\int x^3 e^{2x}\,dx$

Sol. $\int x^3 e^{2x}\,dx = x^3\,.\,\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,.\,3x^2\,dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\int x^2 e^{2x}\,dx$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\left[x^2\,.\,\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,.\,2x\,dx\right]$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\int x e^{2x}\,dx$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\left[x\cdot\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\cdot 1\,dx\right]$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x\,e^{2x} - \frac{3}{4}\cdot\frac{e^{2x}}{2}$$

$$= \frac{e^{2x}}{8}\,(4x^3 - 6x^2 + 6x - 3)$$

20. $\int x^5 e^{x^3}\,dx$

Sol. We take the first function as $u = x^3$ and the second function as $v = x^2 e^{x^3}$. Then $\int v\,dx = \int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3}$

Hence using the by-parts formula, we have

$$\int x^5 e^{x^3}\,dx = \int x^3\,.\,e^{x^3}\,.\,x^2\,dx = x^3\,.\,\frac{e^{x^3}}{3} - \int 3x^2\,.\,\frac{e^{x^3}}{3}\,dx$$

$$= \frac{1}{3}x^3 e^{x^3} - \int e^{x^3}\,.\,x^2\,dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3}$$

21. Show that

$$\int x^n \arctan x\,dx = \frac{x^{n+1}}{n+1}\arctan x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}\,dx$$

Hence evaluate $\int x^3 \arctan x\,dx$.

Sol. Integrate by parts with

$u = \arctan x$ and $v = x^n$ so that

$$\int (\arctan x)\,x^n\,dx = (\arctan x)\,\frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}\cdot\frac{dx}{1+x^2}$$

$$= \frac{x^{n+1}}{n+1}\arctan x - \frac{1}{x+1}\int \frac{x^{n+1}}{1+x^2}\,dx \quad \text{as required.}$$

Setting $n = 3$ in the above equation, we get

$$\int x^3 \arctan x\,dx = \frac{x^4}{4}\arctan x - \frac{1}{4}\int \frac{x^4}{1+x^2}\,dx$$

Now $\int \frac{x^4}{1+x^2}\,dx = \int\left(x^2 - 1 + \frac{1}{1+x^2}\right) = \frac{x^3}{3} - x + \arctan x$

Therefore,

$$\int x^3 \arctan x\,dx = \frac{x^4}{4}\arctan x - \frac{x^3}{12} + \frac{x}{4} - \frac{\arctan x}{4}$$

$$= \frac{1}{4}\,(x^4 - 1)\arctan x - \frac{x^3}{12} + \frac{x}{4}$$

22. Find a reduction formula for $\int x^n e^{ax}\,dx$ and apply it to evaluate $\int x^3 e^{ax}\,dx$.

Sol. $\int x^n e^{ax}\,dx = x^n\,.\,\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}\,.\,nx^{n-1}\,dx$

$$= x^n \cdot \frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx \qquad (1)$$

is the required reduction formula.

Taking $n = 3$

$$\int x^3 e^{ax}\,dx = \frac{e^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax}\,dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left[x^2 \cdot \frac{e^{ax}}{a} - \frac{2}{a}\int x e^{ax}\,dx\right] \quad \left(\begin{array}{l}\text{Putting } n = 2\\ \text{in the formula (1)}\end{array}\right)$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^2}\int x e^{ax}\,dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^2}\left[x \cdot \frac{e^{ax}}{a} - \frac{1}{a}\int e^{ax}\,dx\right]$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^3}x\, e^{ax} - \frac{6}{a^3}\int e^{ax}\,dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^3}x\, e^{ax} - \frac{6}{a^4}e^{ax}$$

$$= e^{ax}\left[\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right] = \frac{e^{ax}}{a^4}\,[a^3x^3 - 3a^2x^2 + 6ax - 6]$$

23. Find reduction formulas for $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$, where n is a positive integer.

Sol. $I_n = \int \sin^n x\,dx$

We write $I_n = \int \sin^{n-1}x \sin x\,dx$ and integrate by parts taking $\sin^{n-1}x$ as the first function.

$$I_n = \sin^{n-1}x\,(-\cos x) - \int (n-1)\sin^{n-2}x \cos x\,(-\cos x)\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \cos^2 x \sin^{n-2}x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int (1 - \sin^2 x)\sin^{n-2}x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)I_n$$

Hence $\quad I_n\,(n - 1 + 1) = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$

or $\quad I_n = -\dfrac{\cos x \sin^{n-1}x}{n} + \dfrac{n-1}{n}I_{n-2}$

which is the required reduction formula.

Next, let us write

$$I_n = \int \cos^n x\,dx = \int \cos^{n-1}x \cos x\,dx$$

Integrating by parts, we have

$$I_n = \cos^{n-1}x \sin x - \int (n-1)\cos^{n-2}x\,(-\sin x)\sin x\,dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,(1 - \cos^2 x)\,dx$$

$$= \sin x \cos^{n-1}x - (n-1)I_n + (n-1)I_{n-2}$$

$$nI_n = \sin x \cos^{n-1}x + (n-1)I_{n-2}$$

or $\quad I_n = \dfrac{\sin x \cos^{n-1}x}{n} + \dfrac{n-1}{n}I_{n-2}$

24. Find a reduction formula for $\int x^n \sin ax\,dx$, where $n > 1$ is an integer. Hence evaluate $\int x^4 \sin 4x\,dx$.

Sol. Let $u = x^n$ and $v = \sin ax$ so that the by-parts formula gives

$$\int x^n \sin ax\,dx = x^n\left(-\frac{\cos ax}{a}\right) - \int nx^{n-1}\left(-\frac{\cos ax}{a}\right)dx$$

$$= -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx \qquad (1)$$

Again, $\int x^{n-1}\cos ax\,dx = x^{n-1}\left(\dfrac{\sin ax}{a}\right) - \int (n-1)\,x^{n-2}\left(\dfrac{\sin ax}{a}\right)dx$

$$= \frac{1}{a}x^{n-1}\sin ax - \frac{n-1}{a}\int x^{n-2}\sin ax\,dx \qquad (2)$$

From (1) and (2), we obtain

$$\int x^n \sin ax\,dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a^2}x^{n-1}\sin ax - \frac{n(n-1)}{a^2}\int x^{n-2}\sin ax\,dx$$

which is the required reduction formula.

Now $\int x^4 \sin 4x = -\dfrac{1}{4}x^4 \cos 4x + \dfrac{4}{16}x^3 \sin 4x - \dfrac{4 \times 3}{16}\int x^2 \sin 4x\,dx \qquad (3)$

Again,

$$\int x^2 \sin 4x\,dx = -\frac{x^2}{4}\cos 4x + \frac{2}{16}x \sin 4x - \frac{2}{16}\int \sin 4x\,dx \qquad (4)$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \frac{1}{32}x^4.$$

From (3) and (4), we have

$$\int x^4 \sin 4x\, dx = -\frac{1}{4}x^4 \cos 4x + \frac{1}{4}x^3 \sin 4x - \frac{3}{4}\left[-\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x\right]$$

$$= -\frac{1}{4}x^4 \cos 4x + \frac{1}{4}x^3 \sin 4x + \frac{3}{16}x^2 \cos 4x - \frac{3}{32}x \sin 4x - \frac{3}{128} \times \cos 4x$$

25. Find a reduction formula for $\int x^m (\ln x)^n\, dx$, $m \neq -1$ and n is an integer greater than 1. Hence evaluate $\int x^3 (\ln x)^2\, dx$.

Sol. Let $u = (\ln x)^n$ and $v = x^m$. Then the by-part rule gives

$$\int (\ln x)^n \cdot x^m\, dx = (\ln x)^n \cdot \frac{x^{m+1}}{m+1} - \int \frac{x^{m+1}}{m+1} \cdot \frac{n(\ln x)^{n-1}}{x}\, dx$$

$$= \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\, dx$$

as the required reduction formula.

In the above reduction formula putting $m = 3$ and $n = 2$, we have

$$\int x^3 (\ln x)^2\, dx = \frac{x^4}{4}(\ln x)^2 - \frac{2}{4}\int x^3 \ln x\, dx$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{1}{2}\left[(\ln x)\frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x}\, dx\right]$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \frac{1}{32}x^4$$

Exercise Set 4.4 (Page 149)

Integrate each of the following with respect to x:

1. $\dfrac{x}{(x-1)(x-2)}$

Sol. Here $\dfrac{x}{(x-1)(x-2)} = \dfrac{-1}{x-1} + \dfrac{2}{x-2}$

$$\int \frac{x\, dx}{(x-1)(x-2)} = -\int \frac{dx}{x-1} + \int \frac{dx}{x-2} = -\ln|x-1| + 2\ln|x-2|$$

$$= \ln \frac{(x-2)^2}{|x-1|}$$

2. $\dfrac{2x-3}{(x^2-1)(2x+3)}$

Sol. Here $\dfrac{2x-3}{(x^2-1)(2x+3)} = \dfrac{2x-3}{(x-1)(x+1)(2x+3)}$

$$= \frac{-1}{10(x-1)} + \frac{-5}{-2(x+1)} + \frac{-6}{\left(\frac{-5}{2}\right)\left(\frac{-1}{2}\right)(2x+3)}$$

Integrating, $\displaystyle\int \frac{2x-3}{(x2-1)(2x+3)}\, dx$

$$= -\frac{1}{10}\int \frac{dx}{x-1} + \frac{5}{2}\int \frac{dx}{x+1} - \frac{24}{5}\int \frac{dx}{2x+3}$$

$$= -\frac{1}{10}\ln|x-1| + \frac{5}{2}\ln|x+1| - \frac{24}{5}\cdot\frac{1}{2}\ln|2x+3|$$

$$= \frac{5}{2}\ln|x+1| - \frac{1}{10}\ln|x-1| - \frac{12}{5}\ln|2x+3|$$

3. $\dfrac{x+1}{x^2+4x+5}$

Sol. $\displaystyle\int \frac{x+1}{x^2+4x+5}\, dx = \frac{1}{2}\int \frac{2x+2}{x^2+4x+5}\, dx = \frac{1}{2}\int \frac{(2x+4)-2}{x^2+4x+5}\, dx$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\, dx - \frac{1}{2}\int \frac{2}{(x^2+4x+5)}\, dx$$

$$= \frac{1}{2}\ln(x^2+4x+5) - \int \frac{dx}{x^2+4x+5}$$

$$= \frac{1}{2}\ln(x^2+4x+5) - \int \frac{dx}{(x+2)^2+1}$$

$$= \frac{1}{2}\ln(x^2+4x+5) - \arctan(x+2)$$

4. $\dfrac{2x^2+3x+1}{x^2+2x+2}$

Sol. $\dfrac{2x^2+3x+1}{x^2+2x+2} = 2 - \dfrac{x+3}{x^2+2x+2}$

Therefore,

$$\int \frac{2x^2+3x+1}{x^2+2x+2}\, dx = 2\int 1\, .\, dx - \int \frac{x+3}{x^2+2x+2}\, dx$$

$$= 2x - \frac{1}{2}\int \frac{(2x+2)+4}{x^2+2x+2} = 2x - \frac{1}{2}\int \frac{(2x+2)+4}{x^2+2x+2}\, dx$$

$$= 2x - \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\, dx - 2\int \frac{dx}{x^2+2x+2}$$

$$= 2x - \frac{1}{2}\ln(x^2 + 2x + 2) - 2\int \frac{dx}{(x+1)^2 + 1}$$

$$= 2x - \frac{1}{2}\ln(x^2 + 2x + 2) - 2\arctan(x + 1).$$

5. $\dfrac{x^2}{(x-1)^3(x+1)}$

Sol. $\dfrac{x^2}{(x-1)^3(x+1)} = \dfrac{1}{2(x-1)^3} + \dfrac{3}{4(x-1)^2} + \dfrac{1}{8(x+1)} - \dfrac{1}{8(x+1)}$

Now, integrating

$$\int \frac{x^2\,dx}{(x-1)^3(x+1)} = \frac{1}{2}\int\frac{dx}{(x-1)^3} + \frac{3}{4}\int\frac{dx}{(x-1)^2} + \frac{1}{8}\int\frac{dx}{x-1} - \frac{1}{8}\int\frac{dx}{x+1}$$

$$= \frac{1}{2}\cdot\frac{(x-1)^{-2}}{-2} + \frac{3}{4}\frac{(x-1)^{-1}}{-1} + \frac{1}{8}\ln|x-1| - \frac{1}{8}|x+1|$$

$$= -\frac{1}{4}\cdot\frac{1}{(x-1)^2} - \frac{3}{4}\frac{1}{x-1} + \frac{1}{8}\ln|x-1| - \frac{1}{8}\ln|x+1|$$

$$= -\frac{1}{4}\cdot\frac{1}{(x-1)^2} - \frac{3}{4}\frac{1}{x-1} + \frac{1}{8}\ln\left|\frac{x-1}{x+1}\right|$$

6. $\dfrac{1}{x(x+1)^3}$

Sol. $\dfrac{1}{x(x+1)^3} = -\dfrac{1}{(x+1)^3} - \dfrac{1}{(x+1)^2} - \dfrac{1}{x+1} + \dfrac{1}{x}$

Integrating, we have

$$\int\frac{dx}{x(x+1)^3} = -\int\frac{dx}{(x+1)^3} - \int\frac{dx}{(x+1)^2} - \int\frac{dx}{x+1} + \int\frac{dx}{x}$$

$$= -\frac{(x+1)^{-2}}{-2} - \frac{(x+1)^{-1}}{-1} - \ln|x+1| + \ln x$$

$$= \frac{1}{2(x+1)^2} + \frac{1}{x+1} - \ln|x+1| + \ln|x|$$

$$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + \frac{1}{2(x+1)^2}$$

7. $\dfrac{x+1}{(x-1)^2(x+2)^2}$

Sol. Resolving $\dfrac{x+1}{(x-1)^2(x+2)^2}$ into partial fractions, we have

$$\frac{x+1}{(x-1)^2(x+2)^2} = \frac{-\frac{1}{27}}{x-1} + \frac{\frac{2}{9}}{(x-1)^2} + \frac{\frac{1}{27}}{x+2} - \frac{\frac{1}{9}}{(x+2)^2}$$ Therefore,

$$\int\frac{x+1}{(x-1)^2(x+2)^2}\,dx = -\frac{1}{27}\int\frac{dx}{x-1} + \frac{2}{9}\int\frac{dx}{(x-1)^2} + \frac{1}{27}\int\frac{dx}{x+2} - \frac{1}{9}\int\frac{dx}{(x+2)^2}$$

$$= -\frac{1}{27}\ln|x-1| - \frac{2}{9}\cdot\frac{1}{x-1} + \frac{1}{27}\ln|x+2| + \frac{1}{9}\cdot\frac{1}{(x+2)}$$

$$= \frac{1}{27}\ln\left|\frac{x+2}{x-1}\right| - \frac{2}{9}\cdot\frac{1}{x-1} + \frac{1}{9}\cdot\frac{1}{x+2}$$

8. $\dfrac{1}{1-x^3}$

Sol. Resolving $\dfrac{1}{1-x^3}$ into partial fractions, we have

$$\frac{1}{1-x^3} = \frac{1}{3}\cdot\frac{1}{1-x} + \frac{1}{3}\cdot\frac{x+2}{x^2+x+1}.$$ Hence

$$\int\frac{1}{1-x^3}\,dx = \frac{1}{3}\int\frac{dx}{1-x} + \frac{1}{3}\int\frac{(x+2)\,dx}{x^2+x+1}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{3}\cdot\frac{1}{2}\int\frac{2x+4}{x^2+x+1}\,dx$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\int\frac{(2x+1)+3}{x^2+x+1}\,dx$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\int\frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int\frac{dx}{x^2+x+1}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \frac{1}{2}\int\frac{dx}{x^2+x+1}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

9. $\dfrac{x^2+1}{x^3+1}$

Sol. Resolving $\dfrac{x^2+1}{x^3+1}$ into partial fractions, we find that

$$\frac{x^2+1}{x^3+1} = \frac{\frac{2}{3}}{x+1} + \frac{\frac{1}{3}x + \frac{1}{3}}{x^2-x+1}.$$ Therefore,

$$\int\frac{x^2+1}{x^3+1}\,dx = \frac{2}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{(x+1)\,dx}{x^2-x+1}$$

$$= \frac{2}{3}\ln|x+1| + \frac{1}{6}\int\frac{2x+2}{x^2-x+1}\,dx$$

$$= \frac{2}{3}\ln|x+1| + \frac{1}{6}\int\frac{(2x-1)+3}{x^2-x+1}\,dx$$

$$= \frac{2}{3}\ln|x+1| + \frac{1}{6}\int\frac{(2x-1)\,dx}{x^2-x+1} + \frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$= \frac{2}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$$

$$= \frac{2}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$$

10. $\dfrac{1}{(x-1)(x^2+4)}$

Sol. Resolving $\dfrac{1}{(x-1)(x^2+4)}$ into partial fractions, we have

$$\frac{1}{(x-1)(x^2+4)} = \frac{\frac{1}{5}}{x-1} - \frac{\frac{1}{5}x+\frac{1}{5}}{x^2+4}. \text{ Thus,}$$

$$\int\frac{dx}{(x-1)(x^2+4)} = \frac{1}{5}\int\frac{dx}{x-1} - \frac{1}{5}\int\frac{x+1}{x^2+4}\,dx$$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\int\frac{2x}{x^2+4}\,dx - \frac{1}{5}\int\frac{dx}{x^2+4}$$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln(x^2+4) - \frac{1}{10}\left(\arctan\frac{x}{2}\right)$$

11. $\dfrac{2x^2-3x-3}{(x-1)(x^2-2x+5)}$

Sol. Let $\dfrac{2x^2-3x-3}{(x-1)(x^2-2x+5)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2-2x+5} = -\dfrac{1}{x-1}$ $+\dfrac{3x-2}{x^2-2x+5}$, after finding the values of A, B, C. Then

$$\int\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}\,dx = -\int\frac{dx}{x-1} + \int\frac{3x-2}{x^2-2x+5}\,dx$$

$$= -\ln|x-1| + 3\int\frac{x-\frac{2}{3}}{x^2-2x+5}\,dx$$

$$= -\ln|x-1| + \frac{3}{2}\int\frac{2x-\frac{4}{3}}{x^2-2x+5}\,dx$$

$$= -\ln|x-1| + \frac{3}{2}\int\frac{(2x-2)+\left(2-\frac{4}{3}\right)}{x^2-2x+5}\,dx$$

$$= -\ln|x-1| + \frac{3}{2}\int\frac{2x-2}{x^2-2x+5}\,dx + \frac{3}{2}\cdot\frac{2}{3}\int\frac{dx}{x^2-2x+5}$$

$$= -\ln|x-1| + \frac{3}{2}\ln(x^2-2x+5) + \int\frac{dx}{(x-1)^2+4}$$

$$= -\ln|x-1| + \frac{3}{2}\ln(x^2-2x+5) + \frac{1}{2}\arctan\left(\frac{x-1}{2}\right)$$

12. $\dfrac{1}{x^4+1}$

Sol. $$\int\frac{dx}{x^4+1} = \int\frac{1}{x^4+1}\,dx = \frac{1}{2}\int\frac{2}{x^4+1}\,dx$$

$$= \frac{1}{2}\int\frac{(x^2+1)-(x^2-1)}{x^4+1}\,dx = \frac{1}{2}\int\frac{x^2+1}{x^4+1}\,dx - \frac{1}{2}\int\frac{x^2-1}{x^4+1}\,dx$$

$$= \frac{1}{2}(I_1 - I_2), \qquad (1)$$

where, $I_1 = \displaystyle\int\frac{x^2+1}{x^4+1}\,dx$

$= \displaystyle\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx.$ Put $x-\dfrac{1}{x}=t$, or $\left(1+\dfrac{1}{x^2}\right)dx = dt$

and $x^2+\dfrac{1}{x^2}-2=t^2$ *i.e.*, $x^2+\dfrac{1}{x^2}=t^2+2$

$$I_1 = \int\frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}} = \frac{1}{\sqrt{2}}\arctan\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}\,x} \qquad (2)$$

and $I_2 = \int \frac{x^2-1}{x^4+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$

Put $x + \frac{1}{x} = t$, or $\left(1-\frac{1}{x^2}\right)dx = dt$

and $x^2 + \frac{1}{x^2} + 2 = t^2$, *i.e.*, $x^2 + \frac{1}{x^2} = t^2 - 2$

$$I_2 = \int \frac{dt}{t^2-2} = \frac{1}{2\sqrt{2}} \ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| = \frac{1}{2\sqrt{2}} \ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|$$

$$= \frac{1}{2\sqrt{2}} \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| \qquad (3)$$

Hence from (1), (2) and (3), we have

$$\int \frac{dx}{x^4+1} = \frac{1}{2}\left[\frac{1}{\sqrt{2}} \arctan\frac{x^2-1}{\sqrt{2}x} - \frac{1}{2\sqrt{2}} \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\right]$$

$$= \frac{1}{2\sqrt{2}} \arctan\frac{x^2-1}{\sqrt{2}x} - \frac{1}{4\sqrt{2}} \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|$$

13. $\frac{x^4}{x^4+2x^2+1}$

Sol. $\frac{x^4}{x^4+2x^2+1} = 1 + \frac{-2x^2-1}{(x^2+1)^2} = 1 - \frac{2}{1+x^2} + \frac{1}{(1+x^2)^2}$

Therefore,

$$\int \frac{x^4}{x^4+2x^2+1}\,dx = \int\left(1 - \frac{2}{1+x^2} + \frac{1}{(1+x^2)^2}\right)dx$$

$$= x - 2\arctan x + \int \frac{dx}{(x^2+1)^2} \qquad (1)$$

In $\int \frac{dx}{(x^2+1)^2}$, put $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$

Then $\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta\, d\theta}{(\sec\theta)^2} = \int \cos^2\theta\, d\theta = \frac{1}{2}\int(1+\cos 2\theta)\, d\theta$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}$$

Putting this value in (1), we get

$$\int \frac{x^4}{x^4+2x^2+1}\,dx = x - \frac{3}{2}\arctan x + \frac{1}{2}\cdot\frac{x}{1+x^2}$$

14. $\frac{x^2+1}{x^4-x^2+1}$

Sol. $\int \frac{x^2+1}{x^4-x^2+1}\,dx = \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}\,dx$

$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}\,dx$ Put $x - \frac{1}{x} = t$ so that $\left(1+\frac{1}{x^2}\right)dx = dt$

and $x^2 + \frac{1}{x^2} - 2 = t^2$ or $x^2 + \frac{1}{x^2} = t^2 + 2$

$$= \int \frac{dt}{t^2+2-1}$$

$$= \int \frac{dt}{t^2+1} = \arctan t = \arctan\left(x - \frac{1}{x}\right) = \arctan\left(\frac{x^2-1}{x}\right)$$

15. $\frac{1}{(e^x-1)^2}$

Sol. Put $e^x = z$ so that $e^x\,dx = dz$ or $dx = \frac{dz}{z}$. Then

$$\int \frac{dx}{(e^x-1)^2} = \int \frac{dz}{z(z-1)^2}$$

Now $\frac{1}{z(z-1)^2} = \frac{1}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{z}$

Therefore, $\int \frac{dz}{z(z-1)^2} = \int \frac{dz}{(z-1)^2} - \int \frac{dz}{z-1} + \int \frac{dz}{z}$

$$= -\frac{1}{z-1} - \ln(z-1) + \ln z$$

i.e., $\int \frac{dx}{(e^x-1)^2} = -\frac{1}{e^x-1} - \ln(e^x-1) + \ln e^x$

$$= \frac{-1}{e^x-1} - \ln(e^x-1) + x = x - \frac{1}{e^x-1} - \ln(e^x-1)$$

16. $\frac{1}{(1+e^x)(1+e^{-x})}$

Sol. $\int \frac{dx}{(1+e^x)(1+e^{-x})} = \int \frac{dx}{(1+e^x)\left(1+\frac{1}{e^x}\right)} = \int \frac{e^x\,dx}{(1+e^x)(e^x+1)}$

$= \int \frac{e^x}{(e^x+1)^2}dx$. Put $e^x+1=t$ or $e^x\,dx = dt = \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{e^x+1}$

17. $\frac{\cos x}{(1+\sin x)(2+\sin x)(3+\sin x)}$

Sol. Put $\sin x = t$ or $\cos x\,dx = dt$. Then

$$\int \frac{\cos x}{(1+\sin x)(2+\sin x)(3+\sin x)}dx \qquad (\sin x \neq -1)$$

$$= \int \frac{1}{(1+t)(2+t)(3+t)}dt = \int\left[\frac{\frac{1}{2}}{1+t} + \frac{-1}{t+2} + \frac{\frac{1}{2}}{t+3}\right]dt$$

$$= \frac{1}{2}\int\frac{dt}{t+1} - \int\frac{dt}{t+2} + \frac{1}{2}\int\frac{dt}{t+3}$$

$$= \frac{1}{2}\ln(t+1) - \ln(t+2) + \frac{1}{2}\ln(t+3)$$

$$= \frac{1}{2}\ln(1+\sin x) - \ln(2+\sin x) + \frac{1}{2}\ln(3+\sin x)$$

18. $\frac{\sec x}{1+\csc x}$

Sol. $\int \frac{\sec x\,dx}{1+\csc x} = \int \frac{\frac{1}{\cos x}}{1+\frac{1}{\sin x}}dx = \int \frac{\sin x\,dx}{\cos x\,(1+\sin x)} = \int \frac{\cos x \sin x\,dx}{\cos^2 x\,(1+\sin x)}$

$$= \int \frac{\cos x \sin x\,dx}{(1-\sin^2 x)(1+\sin x)} \qquad (-1 < \sin x < 1)$$

$$= \int \frac{\cos x \sin x\,dx}{(1-\sin x)(1+\sin x)^2}, \qquad \text{Put } \sin x = t \text{ or } \cos x\,dx = dt$$

$$= \int \frac{t\,dt}{(1-t)(1+t)^2}$$

Now, $\frac{t}{(1-t)(1+t)^2} = \frac{1}{4(1-t)} + \frac{1}{4(1+t)} - \frac{1}{2(1+t)^2}$

Therefore,

$$\int \frac{t}{(1-t)(1+t)^2}dt = \frac{1}{4}\int\frac{dt}{1-t} + \frac{1}{4}\int\frac{dt}{1+t} - \frac{1}{2}\int\frac{dt}{(1+t)^2}$$

$$= \frac{1}{4}\ln(1-t) + \frac{1}{4}\ln(1+t) + \frac{1}{2}\cdot\frac{1}{1+t}$$

Hence $\int \frac{\sec x\,dx}{1+\csc x} = -\frac{1}{4}\ln(1-\sin x) + \frac{1}{4}\ln(1+\sin x)\frac{1}{2}\cdot\frac{1}{1+\sin x}$

$$= \frac{1}{4}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + \frac{1}{2}\cdot\frac{1}{1+\sin x}$$

19. $\frac{\sin x}{\sin 3x}$

Sol. $I = \int \frac{\sin x\,dx}{\sin 3x} = \int \frac{\sin x\,dx}{3\sin x - 4\sin^3 x} = \int \frac{dx}{3 - 4\sin^2 x}$

$$= \int \frac{dx}{3(\sin^2 x + \cos^2 x) - 4\sin^2 x}$$

$$= \frac{dx}{3\cos^2 x - \sin^2 x} = \int \frac{\sec^2 x\,dx}{3 - \tan^2 x}$$

Put $\tan x = t$ or $\sec^2 x\,dx = dt$. Then

$$I = \int \frac{dt}{3-t^2} = \frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| = \frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|$$

20. $\frac{\cot x - 3\cot 3x}{3\tan 3x - \tan x}$

Sol. We know that

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \quad \text{and} \quad \cot 3x = \frac{1-3\tan^2 x}{3\tan x - \tan^3 x}$$

Therefore, $\frac{\cot x - 3\cot 3x}{3\tan 3x - \tan x} = \frac{\frac{1}{\tan x} - 3\cdot\frac{1-3\tan^2 x}{3\tan x - \tan^3 x}}{3\,\frac{3\tan x - \tan^3 x}{1-3\tan^2 x} - \tan x}$

$$= \frac{\frac{3-\tan^2 x - 3 + 9\tan^2 x}{3\tan x - \tan^3 x}}{\frac{9\tan x - 3\tan^3 x - \tan x + 3\tan^3 x}{1-3\tan^2 x}}$$

$$= \frac{8\tan^2 x}{3\tan x - \tan^3 x}\cdot\frac{1-3\tan^2 x}{8\tan x}$$

$$= \frac{1-3\tan^2 x}{3-\tan^2 x} = 3 - \frac{8}{3-\tan^2 x}$$

Therefore, $\int \frac{\cot x - 3\cot 3x}{3\tan 3x - \tan x}dx = 3x - 8\int\frac{dx}{3-\tan^2 x} \qquad (1)$

Put $\tan x = t$, so that $\sec^2 x\,dx = dt \Rightarrow dx = \dfrac{dt}{\sec^2 x} = \dfrac{dt}{1+t^2}$

Now $\displaystyle\int \frac{dx}{3-\tan^2 x} = \int \frac{dt}{(3-t^2)(1+t^2)}$

Let $\dfrac{1}{(3-t^2)(1+t^2)} = \dfrac{A}{\sqrt{3}+t} + \dfrac{B}{\sqrt{3}-t} + \dfrac{Ct+D}{t^2+1}$

$$= \frac{1}{8\sqrt{3}(\sqrt{3}+t)} + \frac{1}{8\sqrt{3}(\sqrt{3}-t)} + \frac{1}{4(t^2+1)},$$

after finding the values of A, B, C and D. Therefore,

$$\int \frac{dt}{(3-t^2)(1+t^2)} = \frac{1}{8\sqrt{3}}\int \frac{dt}{\sqrt{3}+t} + \frac{1}{8\sqrt{3}}\int \frac{dt}{\sqrt{3}-t} + \frac{1}{4}\int \frac{dt}{t^2+1}$$

$$= \frac{1}{8\sqrt{3}}\ln|\sqrt{3}+t| - \frac{1}{8\sqrt{3}}\ln|\sqrt{3}-t| + \frac{1}{4}\arctan t$$

$$= \frac{1}{8\sqrt{3}}\ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + \frac{1}{4}\arctan(\tan x)$$

$$= \frac{1}{8\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + \frac{1}{4}x$$

Putting it in (1), we get

$$\int \frac{\cot x - 3\cot 3x}{3\tan 3x - \tan x}\,dx = 3x - 8\left[\frac{1}{8\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + \frac{1}{4}x\right]$$

$$= 3x - \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| - 2x = x - \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|$$

21. $\dfrac{\cos x}{\sin^2 x + 4\sin x - 5}$

Sol. We make the substitution $\sin x = u$ so that $\cos x\,dx = du$ and

$$\int \frac{\cos x\,dx}{\sin^2 x + 4\sin x - 5} = \int \frac{du}{u^2+4u-5} = \int \frac{du}{(u+5)(u-1)}$$

$$= -\frac{1}{6}\int \frac{du}{u+5} + \frac{1}{6}\int \frac{du}{u-1}$$

$$= -\frac{1}{6}\ln|u+5| + \frac{1}{6}\ln|u-1|$$

$$= \frac{1}{6}\ln\left|\frac{u-1}{u+5}\right| = \frac{1}{6}\ln\left|\frac{\sin x - 1}{\sin x + 5}\right|$$

22. $\dfrac{\sec^2 x}{\tan^3 x - \tan^2 x}$

Sol. Here we put $\tan x = u$. Therefore, $\sec^2 x\,dx = du$ and

$$\int \frac{\sec^2 x}{\tan^3 x - \tan^2 x} = \int \frac{du}{u^2(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u} - \int \frac{du}{u^2}$$

$$= \ln|u-1| - \ln|u| + \frac{1}{u} = \ln\left|\frac{4-1}{u}\right| + \frac{1}{u} = \ln\left|\frac{\tan x - 1}{\tan x}\right| + \cot x.$$

23. $\dfrac{x^2+1}{(x^2+2x+3)^2}$

Sol. It is easy to see that

$$\frac{x^2+1}{(x^2+2x+3)^2} = \frac{1}{x^2+2x+3} - \frac{2x+2}{(x^2+2x+3)^2}$$

Hence $\displaystyle\int \frac{x^2+1}{(x^2+2x+3)^2}\,dx = \int \frac{dx}{x^2+2x+3} - \int \frac{2x+2}{(x^2+2x+3)^2}\,dx$

$$= \int \frac{dx}{(x+1)^2+(\sqrt{2})^2} + \frac{1}{x^2+2x+3}$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{x^2+2x+3}$$

24. $\dfrac{x^3+2x^2-3}{(x^2+9)^2}$

Sol. We have $\dfrac{x^3+2x^2-3}{(x^2+9)^2} = \dfrac{Ax+B}{x^2+9} + \dfrac{Cx+D}{(x^2+9)^2}$

Therefore, $x^3 + 2x^2 - 3 = (Ax+B)(x^2+9) + Cx + D$ (1)

is an identity.

Equating coefficients of like terms in (1), we get

Coefficient of x^3: $1 = A$

Coefficient of x^2: $2 = B$

Coefficient of x: $0 = 9A + C = 9 + C \Rightarrow C = -9$

Constant terms: $-3 = 9B + D = 18 + D \Rightarrow D = -21$

Now, $\displaystyle\int \frac{x^3+2x^2-3}{(x^2+9)^2}\,dx = \int \frac{(x+2)\,dx}{x^2+9} + \int \frac{-9x-21}{(x^2+9)^2}\,dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+9}\,dx + \int \frac{2\,dx}{x^2+9} - \frac{9}{2}\int \frac{2x\,dx}{(x^2+9)^2} - 21\int \frac{dx}{(x^2+9)^2}$$

$$= \frac{1}{2}\ln(x^2+9) + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{1}{x^2+9} - 21\int \frac{dx}{(x^2+9)^2} \quad (2)$$

In $\displaystyle\int \frac{dx}{(x^2+9)^2}$, put $x = 3\tan\theta$ so that $dx = 3\sec^2\theta\,d\theta$ and

$$\int \frac{dx}{(x^2+9)^2} = \int \frac{3\sec^2\theta\, d\theta}{9^2 \sec^4\theta} = \frac{1}{27}\int \cos^2\theta\, d\theta = \frac{1}{27}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{27}\left(\frac{1}{2}\theta + \frac{\sin\theta\cos\theta}{2}\right) = \frac{1}{54}\left[\arctan\left(\frac{x}{3}\right) + \frac{3x}{x^2+9}\right] \quad (3)$$

From (2) and (3), we get $\displaystyle\int \frac{x^3+2x^2-3}{(x^2+9)}\, dx$

$$= \ln\sqrt{x^2+9} + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{1}{x^2+9} - \frac{21}{54}\left[\arctan\frac{x}{3} + \frac{3x}{x^2+9}\right]$$

$$= \ln\sqrt{x^2+9} + \frac{5}{18}\arctan\left(\frac{x}{3}\right) + \frac{27-7x}{6(x^2+9)}$$

25. $\dfrac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}$

Sol. $$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

$$= \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

after finding the values of the constant A, B, C, D and E.

Therefore, $\displaystyle\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\, dx$

$$= \int \frac{dx}{x+2} + \int \frac{2x\, dx}{x^2+3} + \frac{4x\, dx}{(x^2+3)^2} = \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3}$$

Exercise Set 4.5 (Page 156)

Integrate each of the following with respect to x:

1. $x^2\sqrt{25-x^2}$

Sol. We make the substitution

$x = 5\sin\theta$ or $dx = 5\cos\theta\, d\theta$. Then

$$\int x^2 (25-x^2)\, dx = \int 25\sin^2\theta\sqrt{25-25\sin^2\theta}\,.\,5\cos\theta\, d\theta$$

$$= 625\int \sin^2\theta\cos^2\theta\, d\theta = 625\int \frac{\sin^2 2\theta}{4}\, d\theta$$

$$= \frac{625}{4}\int \frac{1-\cos 4\theta}{2}\, d\theta = \frac{625}{8}\left[\theta - \frac{\sin 4\theta}{4}\right] d\theta$$

$$= \frac{625}{8}\left[\theta - \frac{2\sin 2\theta\cos 2\theta}{4}\right]$$

$$= \frac{625}{8}[\theta - \sin\theta\cos\theta(\cos^2 - \sin^2\theta]$$

$$= \frac{625}{8}[\theta - \sin\theta\cos^3\theta + \sin^3\theta\cos\theta]$$

$$= \frac{625}{8}\left[\arcsin\left(\frac{x}{5}\right) - \frac{x}{5}\left(1-\frac{x^2}{25}\right)^{3/2} + \frac{x^3}{125}\sqrt{1-\frac{x^2}{25}}\right]$$

$$= \frac{625}{8}\arcsin\left(\frac{x}{5}\right) - \frac{x}{8}(25-x^2)^{3/2} + \frac{x^3}{8}\sqrt{25-x^2}$$

2. $x(x+4)^{1/3}$

Sol. Put $(x+4)^{1/3} = z$ or $x+4 = z^3$ or $dx = 3z^2\, dz$

$$\int x(x+4)^{1/3}\, dx = \int (z^3-4)\, z\,.\,3z^2\, dz = 3\int (z^6-4z^3)\, dz$$

$$= \frac{3}{7}z^7 - 3z^4 = \frac{3}{7}(x+4)^{7/3} - 3(x+4)^{4/3}$$

3. $e^x\sqrt{1-e^{2x}}$

Sol. Put $e^x = \sin\theta$, so that $e^x\, dx = \cos\theta\, d\theta$

$$\int e^x\sqrt{(1-e^{2x})}\, dx = \int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4} = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$$

$$= \frac{1}{2}\arcsin e^x + \frac{1}{2}e^x\,.\sqrt{1-e^{2x}}$$

4. $\dfrac{x}{(1-x^2)^{3/2}}$

Sol. $$\int \frac{x}{(1-x^2)^{3/2}}\, dx = -\frac{1}{2}\int (1-x^2)^{-3/2}\,.\,(-2x)\, dx$$

$$= -\frac{1}{2}\cdot\frac{(1-x^2)^{-3/2+1}}{-\frac{3}{2}+1} = (1-x^2)^{-1/2} = \frac{1}{\sqrt{1-x^2}}$$

5. $\dfrac{x^2-3}{x\sqrt{x^2+4}}$

Sol. $$I = \int \frac{x^2-3}{x\sqrt{x^2+4}}\, dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2+4}}\, dx - \int \frac{3}{x\sqrt{x^2+4}}\, dx$$

$$= \sqrt{x^2+4} - \int \frac{3}{x\sqrt{x^2+4}}\, dx$$

Put $x = 2\tan\theta$ or $dx = 2\sec^2\theta\, d\theta$

Then $\int \frac{3}{x\sqrt{x^2+4}}\,dx = \int \frac{6\sec^2\theta\, d\theta}{2\tan\theta \cdot 2\sec\theta}$

$= \frac{3}{2}\int \frac{\sec\theta}{\tan\theta}d\theta = \frac{3}{2}\int \csc\theta\, d\theta$

$= \frac{3}{2}\ln|\csc\theta - \cot\theta|$

$= \frac{3}{2}\ln\left|\frac{\sqrt{4+x^2}}{x} - \frac{2}{x}\right|$

Hence $I = \sqrt{x^2+4} - \frac{3}{2}\ln\left|\frac{\sqrt{4+x^2}-2}{x}\right|$

6. $\sqrt{3x^2 - 4x + 1}$

Sol. $\int \sqrt{3x^2-4x+1}\,dx = \sqrt{3}\int \sqrt{x^2 - \frac{4}{3}x + \frac{1}{3}}\,dx$

$= \sqrt{3}\sqrt{\left(x-\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2}\,dx$

$= \sqrt{3}\left[\frac{\left(x-\frac{2}{3}\right)\sqrt{\left(x-\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2}}{2} - \frac{\left(\frac{1}{3}\right)^2}{2}\cosh^{-1}\left(\frac{x-\frac{2}{3}}{\frac{1}{3}}\right)\right]$

$= \sqrt{3}\left[\frac{(3x-2)\sqrt{x^2-\frac{4}{3}x+\frac{1}{3}}}{6} - \frac{1}{18}\cosh^{-1}\left(\frac{3x-2}{1}\right)\right]$

$= \frac{3x-2}{6}\cdot\sqrt{3x^2-4x+1} - \frac{\sqrt{3}}{18}\cosh^{-1}(3x-2)$

7. $\sqrt{x^2+2x+3}$

Sol. $\int\sqrt{x^2+2x+3}\,dx = \int\sqrt{(x+1)^2+(\sqrt{2})^2}\,dx$

$= \frac{(x+1)\sqrt{(x+1)^2+(\sqrt{2})^2}}{2} + \frac{(\sqrt{2})^2}{2}\sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$

$= \frac{(x+1)\sqrt{x^2+2x+3}}{2} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$

8. $\frac{x}{\sqrt{4+3x-2x^2}}$

Sol. $\int\frac{x\,dx}{\sqrt{4+3x-2x^2}} = -\frac{1}{4}\int\frac{-4x\,dx}{\sqrt{4+3x-2x^2}} = -\frac{1}{4}\int\frac{\{(3-4x)-3\}\,dx}{\sqrt{4+3x-2x^2}}$

$= -\frac{1}{4}\int(3-4x)(4+3x-2x^2)^{-1/2}\,dx + \frac{3}{4}\int\frac{dx}{\sqrt{4+3x-x^2}}$

$= -\frac{1}{4}\frac{(4+3x-2x^2)^{1/2}}{\frac{1}{2}} + \frac{3}{4\times\sqrt{2}}\int\frac{dx}{\sqrt{2+\frac{3}{2}x-x^2}}$

$= -\frac{1}{2}\sqrt{4+3x-2x^2} + \frac{3}{4\sqrt{2}}\int\frac{dx}{\sqrt{\frac{9}{16}+2-\left(x-\frac{3}{4}\right)^2}}$

$= -\frac{1}{2}\sqrt{4+3x-2x^2} + \frac{3}{4\sqrt{2}}\int\frac{dx}{\sqrt{\left(\frac{41}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}}$

$= -\frac{1}{2}\sqrt{4+3x-2x^2} + \frac{3}{4\sqrt{2}}\arcsin\left(\frac{x-\frac{3}{4}}{\frac{\sqrt{41}}{4}}\right)$

$= -\frac{1}{2}\sqrt{4+3x-2x^2} + \frac{3}{4\sqrt{2}}\arcsin\left(\frac{4x-3}{\sqrt{41}}\right)$

9. $\frac{1}{\sqrt{3x^2-4x+1}}$

Sol. $\frac{dx}{\sqrt{3x^2-4x+1}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{x^2-\frac{4}{3}x+\frac{1}{3}}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left(x-\frac{2}{3}\right)^2+\frac{1}{3}-\frac{4}{9}}}$

$= \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left(x-\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2}} = \frac{1}{\sqrt{3}}\cosh^{-1}\frac{x-\frac{2}{3}}{\frac{1}{3}} = \frac{1}{\sqrt{3}}\cosh^{-1}(3x-2)$

10. $\frac{x+1}{\sqrt{x^2+2x+4}}$

Sol. $\int\frac{x+1}{\sqrt{x^2+2x+4}}\,dx = \frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x+3}}\,dx$

$= \frac{1}{2}\int(x^2+2x+3)^{-1/2}(2x+2)\,dx$

$= \frac{1}{2}\frac{(x^2+2x+3)^{-1/2}}{\frac{1}{2}} = \sqrt{x^2+2x+3}$

11. $\dfrac{x^2+2x+3}{\sqrt{x^2+x+1}}$

Sol. $I = \displaystyle\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx = \int \frac{(x^2+x+1)+(x+2)}{\sqrt{x^2+x+1}}\,dx$

$$= \int \sqrt{x^2+x+1}\,dx + \int \frac{x+2}{\sqrt{x^2+x+1}}\,dx$$

$$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+x+1}}\,dx = I_1 + I_2$$

$$I_1 = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}{2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\sinh^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{(2x+1)\sqrt{x^2+x+1}}{4} + \frac{3}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \qquad (1)$$

and $I_2 = \displaystyle\frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+x+1}}\,dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}} + \frac{3}{2}\int \frac{dx}{\sqrt{x^2+x+1}}\,dx$

$$= \sqrt{x^2+x+1} + \frac{3}{2}\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \sqrt{x^2+x+1} + \frac{3}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \qquad (2)$$

Adding (1) and (2), we get

$$I = \left(\frac{(2x+1)}{4}+1\right)\sqrt{x^2+x+1} + \left(\frac{3}{8}+\frac{3}{2}\right)\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$= \left(\frac{2x+5}{4}\right)\sqrt{x^2+x+1} + \frac{15}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

12. $\dfrac{1}{(2x+3)\sqrt{x+5}}$

Sol. Put $\sqrt{x+5} = t$ or $x = t^2 - 5$ or $dx = 2t\,dt$

$$\int \frac{dx}{(2x+3)\sqrt{x+5}} = \int \frac{2t\,dt}{(2t^2-7)\,t} \qquad (2x+3 = 2(t^2-5)+3))$$

$$= 2\int \frac{dt}{2t^2-7} = \int \frac{dt}{t^2-\frac{7}{2}}$$

$$= \frac{1}{2}\sqrt{\frac{2}{7}}\int\left(\frac{1}{t-\sqrt{\frac{7}{2}}} - \frac{1}{t+\sqrt{\frac{7}{2}}}\right)dt = \frac{1}{2}\cdot\sqrt{\frac{2}{7}}\ln\left|\frac{t-\sqrt{\frac{7}{2}}}{t+\sqrt{\frac{7}{2}}}\right|$$

$$= \frac{1}{\sqrt{14}}\ln\left|\frac{\sqrt{x+5}-\sqrt{\frac{7}{2}}}{\sqrt{x+5}+\sqrt{\frac{7}{2}}}\right|$$

13. $\dfrac{1}{(1-2x)\sqrt{1+4x}}$

Sol. Put $1+4x = t^2$ or $x = \dfrac{t^2-1}{4} \Rightarrow dx = \dfrac{t\,dt}{2}$

$$\int \frac{dx}{(1-2x)\sqrt{1+4x}} = \frac{1}{2}\int \frac{t\,dt}{\left(1-\frac{t^2-1}{2}\right).\,t} = \frac{1}{2}\int \frac{2\,dt}{(2-t^2+1)} = \int \frac{dt}{3-t^2}$$

$$= \frac{1}{2\sqrt{3}}\int\left(\frac{1}{\sqrt{3}+t} + \frac{1}{\sqrt{3}-t}\right)dt = \frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right|$$

$$= \frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{1+4x}}{\sqrt{3}-\sqrt{1+4x}}\right|$$

14. $\dfrac{x\sqrt{1+x}}{\sqrt{1-x}}$

Sol. $\displaystyle\int \frac{x\sqrt{1+x}}{\sqrt{1-x}}\,dx = \int \frac{x(1+x)}{\sqrt{1-x}\sqrt{1+x}}\,dx = \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{x^2}{\sqrt{1-x^2}}\,dx$

$$= -\frac{1}{2}\int(-2x)(1-x^2)^{-1/2}\,dx - \int \frac{(1-x^2)-1}{\sqrt{1-x^2}}\,dx$$

$$= -\frac{1}{2}\frac{(1-x^2)^{1/2}}{\frac{1}{2}} - \int\sqrt{1-x^2}\,dx + \int \frac{dx}{\sqrt{1-x^2}}$$

$$= -\sqrt{1-x^2} - \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\arcsin x\right] + \arcsin x$$

$$= -\sqrt{1-x^2} - \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\arcsin x$$

15. $\dfrac{x^4}{(x-1)\sqrt{x+2}}$

Sol. Put $x + 2 = t^2$ or $x = t^2 - 2$ or $dx = 2t\,dt$

$$\int \frac{x^4}{(x-1)\sqrt{x+2}}\,dx = \int \frac{(t^2-2)^4 \cdot 2t\,dt}{(t^2-3)\,t} = 2\int \frac{(t^2-2)^4}{t^2-3}\,dt$$

$$= 2\int \frac{t^8 - 8t^6 + 24t^4 - 32t^2 + 16}{t^2-3}\,dx$$

$$= 2\int \left(t^6 - 5t^4 + 9t^2 - 5 + \frac{1}{t^2-3}\right) dx$$

$$= 2\left[\frac{t^7}{7} - t^5 + 3t^3 - 5t + \frac{1}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|\right]$$

$$= \frac{2}{7}t^7 - 2t^5 + 6t^3 - 10t + \frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|$$

$$= \frac{2}{7}(x+2)^{7/2} - 2(x+2)^{5/2} + 6(x+2)^{3/2} - 10\sqrt{x+2}$$

$$+ \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|$$

16. $\dfrac{1}{(x^2-2x+2)\sqrt{x-1}}$

Sol. Put $x - 1 = t^2$ so that $dx = 2t\,dt$. Then

$$\int \frac{dx}{(x^2-2x+2)\sqrt{x-1}} = \int \frac{2t\,dt}{[(t^2+1)^2 - 2(t^2+1) + 2]\,.\,t}$$

$$= 2\int \frac{dt}{t^4 + 2t^2 + 1 - 2t^2 - 2 + 2} = 2\int \frac{dt}{t^4+1}$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{t^2-1}{\sqrt{2}t}\right) - \frac{1}{2\sqrt{2}}\ln\left|\frac{t^2-\sqrt{2}\,t+1}{t^2+\sqrt{2}\,t+1}\right| \quad \text{(By Q.12 Ex. 4.4)}$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x-2}{\sqrt{2(x-1)}}\right) - \frac{1}{2\sqrt{2}}\ln\left|\frac{x-\sqrt{2(x-1)}}{x+\sqrt{2(x-1)}}\right|$$

17. $\dfrac{1}{(x^2+4x+5)\sqrt{x+2}}$

Sol. Put $x + 2 = t^2$ so that $x = t^2 - 2 \Rightarrow dx = 2t\,dt$. Then

$$\int \frac{dx}{(x^2+4x+5)\sqrt{x+2}} = \int \frac{2t\,dt}{[(t^2-2)^2 + 4(t^2-2) + 5]\,t}$$

$$= 2\int \frac{dt}{t^4 - 4t^2 + 4 + 4t^2 - 8 + 5} = 2\int \frac{dt}{t^4+1}$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{t^2-1}{\sqrt{2}\,t} - \frac{1}{2\sqrt{2}}\ln\left|\frac{t^2-\sqrt{2}\,t+1}{t^2+\sqrt{2}\,t+1}\right| \quad \text{(By Q. 12 Ex. 4.4)}$$

$$= \frac{1}{2}\arctan\left(\frac{x+1}{\sqrt{2(x+2)}}\right) - \frac{1}{2\sqrt{2}}\ln\left|\frac{x+3-\sqrt{2(x+2)}}{x+3+\sqrt{2(x+2)}}\right|$$

18. $\dfrac{1}{(x-1)\sqrt{x^2+1}}$

Sol. Put $x - 1 = \dfrac{1}{t}$ *i.e.,* $dx = -\dfrac{1}{t^2}$ and $x^2 = \left(\dfrac{1}{t}+1\right)^2 = \dfrac{1}{t^2} + \dfrac{2}{t} + 1$

$$\text{Now } \int \frac{dx}{(x-1)\sqrt{x^2+1}} = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+\frac{2}{t}+2}} = -\int \frac{dt}{\sqrt{2t^2+2t+1}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2+t+\frac{1}{2}}} = -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}}$$

$$= -\frac{1}{\sqrt{2}}\sinh^{-1}\frac{t+\frac{1}{2}}{\frac{1}{2}} = -\frac{1}{\sqrt{2}}\sinh^{-1}(2t+1)$$

$$= -\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{2}{x-1}+1\right) = -\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{x+1}{x-1}\right)$$

19. $\dfrac{1}{(x+1)\sqrt{x^2-1}}$

Sol. Put $x + 1 = \dfrac{1}{t}$ *i.e.,* $x = \dfrac{1}{t} - 1$, $dx = -\dfrac{1}{t^2}dt$ and $x^2 = \dfrac{1}{t^2} - \dfrac{2}{t} + 1$

$$\frac{dx}{(x+1)\sqrt{x^2-1}} = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}-\frac{2}{t}}} = -\int \frac{dt}{\sqrt{1-2t}}$$

$$= -\int (1-2t)^{-1/2}\,dt = -\frac{(1-2t)^{1/2}}{-2\left(\frac{1}{2}\right)} = \sqrt{1-2t}$$

$$= \sqrt{1-\frac{2}{x+1}} = \sqrt{\frac{x+1-2}{x+1}} = \sqrt{\frac{x-1}{x+1}}$$

20. $\dfrac{1}{ax^n + bx}$

Sol. $I = \displaystyle\int \frac{dx}{ax^n + bx} = \int \frac{dx}{x^n(a + bx^{-n+1})} = \int \frac{1}{a + bx^{n-1}} \cdot x^{-n}\, dx$

Put $a + bx^{-n+1} = z$. Then $b(-n+1)x^{-n}\, dx = dz$

or $x^{-n} \cdot dx = \dfrac{dz}{b(1-n)}$

$$I = \int \frac{1}{z} \cdot \frac{dz}{b(1-n)} = \frac{1}{b(1-n)} \int \frac{dz}{z} \cdot \frac{1}{(x_1 - n)} \cdot \ln |z|$$

$$= \frac{1}{b(1-n)} \ln |a + bx^{-n+1}|$$

21. $\dfrac{x^2 + 2x + 3}{(x+2)\sqrt{x^2+1}}$

Sol. We have $\dfrac{x^2 + 2x + 3}{x + 2} = x + \dfrac{3}{x+2}$

and $\dfrac{x^2 + 2x + 3}{(x+2)\sqrt{x^2+1}} = \dfrac{x}{\sqrt{x^2+1}} + \dfrac{3}{(x+2)\sqrt{x^2+1}}$. Therefore,

$$\int \frac{x^2 + 2x + 3}{(x+2)\sqrt{x^2+1}}\, dx = \int \frac{x}{\sqrt{x^2+2}}\, dx + 3\int \frac{dx}{(x+2)\sqrt{x^2+1}}$$

$$= \frac{1}{2}\int (2x)(x^2+1)^{-1/2}\, dx + 3\int \frac{dx}{(x+2)\sqrt{x^2+1}}$$

$$= \frac{1}{2} \frac{(x^2+1)^{1/2}}{\frac{1}{2}} + 3\int \frac{dx}{(x+2)\sqrt{x^2+1}}$$

$$= \sqrt{x^2+1} + 3\int \frac{dx}{(x+2)\sqrt{x^2+1}} \qquad (1)$$

To evaluate, $\displaystyle\int \frac{dx}{(x+2)\sqrt{x^2+1}}$, put $x + 2 = \dfrac{1}{t}$

or $x = \dfrac{1}{t} - 2$ *i.e.*, $dx = -\dfrac{1}{t^2}\, dx$ and $x^2 = \dfrac{1}{t^2} - \dfrac{4}{t} + 4$

or $x^2 + 1 = \dfrac{1}{t^2} - \dfrac{4}{t} + 5$ or $\sqrt{x^2+1} = \dfrac{\sqrt{1 - 4t + 5t^2}}{t}$. Thus

$$\int \frac{dx}{(x+2)\sqrt{x^2+1}} = \int \frac{-\frac{1}{t^2}\, dt}{\frac{1}{t} \cdot \frac{\sqrt{1 - 4t + 5t^2}}{t}} = -\int \frac{dt}{\sqrt{1 - 4t + 5t^2}}$$

$$= -\frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2 - \frac{4}{5}t + \frac{1}{5}}} = \frac{1}{-\sqrt{5}} \int \frac{dt}{\sqrt{\left(t - \frac{2}{5}\right)^2 + \frac{1}{5} - \frac{4}{25}}}$$

$$= -\frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{\left(t - \frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2}} = -\frac{1}{\sqrt{5}} \sinh^{-1} \frac{t - \frac{2}{5}}{\frac{1}{5}}$$

$$= -\frac{1}{\sqrt{5}} \sinh^{-1}(5t - 2) = -\frac{1}{\sqrt{5}} \sinh^{-1}\left\{5 \cdot \frac{1}{x+2} - 2\right\}$$

$$= -\frac{1}{\sqrt{5}} \sinh^{-1}\left(\frac{5 - 2x - 4}{x+2}\right) = -\frac{1}{\sqrt{5}} \sinh^{-1}\left(\frac{1 - 2x}{x+2}\right)$$

Putting in (1), we get

$$\int \frac{(x^2 + 2x + 3)\, dx}{(x+2)\sqrt{x^2+1}} = \sqrt{x^2+1} - \frac{3}{\sqrt{5}} \sinh^{-1} \frac{1 - 2x}{x+2}$$

22. $\dfrac{1}{x^2\sqrt{x^2+1}}$

Sol. Putting $x = \dfrac{1}{t}$, we get $dx = -\dfrac{1}{t^2}\, dt$

Therefore, $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{-\frac{1}{t^2}\, dt}{\frac{1}{t^2}\sqrt{\frac{1}{t^2} + 1}} = -\int \frac{t}{\sqrt{1+t^2}}\, dx$

$$= -\frac{1}{2}\int 2t(1+t^2)^{-1/2}\, dt = -\frac{1}{2} \frac{(1+t^2)^{1/2}}{\frac{1}{2}}$$

$$= -\sqrt{1+t^2} = -\sqrt{1 + \frac{1}{x^2}} = -\frac{\sqrt{1+x^2}}{x}$$

23. $\dfrac{1}{(1+x^2)\sqrt{1-x^2}}$

Sol. Putting $x = \dfrac{1}{t}$, we get $dx = -\dfrac{1}{t^2}\, dt$. Therefore,

$$\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int \frac{-\frac{1}{t^2}\,dt}{\left(1+\frac{1}{t^2}\right)\sqrt{1-\frac{1}{t^2}}} = -\int \frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$$

Again set $t^2-1=u^2$ *i.e.,* $2t\,dt = 2u\,du$ or $t\,dt = u\,du$

Also $t^2 = u^2+1$ or $t^2+1 = u^2+2$

Hence $\displaystyle\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = -\int \frac{u\,du}{(u^2+2)\,u} = -\int \frac{du}{u^2+2}$

$$= -\frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{\frac{1}{x^2}-1}}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\cdot\sqrt{\frac{1-x^2}{x^2}}\right)$$

$$= -\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{1-x^2}}{\sqrt{2}\,x}\right)$$

24. $\dfrac{1}{(1-2x^2)\sqrt{1-x^2}}$

Sol. Let $I = \displaystyle\int \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$

Put $x = \dfrac{1}{t}$ or $dx = -\dfrac{1}{t^2}dt$ and $1-x^2 = 1-\dfrac{1}{t^2} = \dfrac{t^2-1}{t^2}$

Therefore, $I = \displaystyle\int \frac{-\frac{1}{t^2}\,dt}{\left(1-\frac{2}{t^2}\right)\frac{\sqrt{t^2-1}}{t}} = -\int \frac{t\,dt}{(t^2-2)\sqrt{t^2-1}}$

New set $t^2-1=u^2$ or $t\,dt = u\,du$ and $t^2 = u^2+1$

$$I = \int \frac{u\,du}{(u^2-1)\,u} = -\int \frac{du}{u^2-1} = \int \frac{du}{1-u^2} = \frac{1}{2}\cdot\int\left(\frac{1}{1+u}+\frac{1}{1-u}\right)dx$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| = \frac{1}{2}\ln\left|\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right|$$ Put $x = \cos\theta$. Then

Alternative Answer:

$$\frac{1}{2}\ln\left|\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right| = \frac{1}{2}\ln\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right| = \frac{1}{2}\ln\left|\frac{1+\tan\theta}{1-\tan\theta}\right|$$

$$= \frac{1}{2}\ln\left|\tan\left(\frac{\pi}{4}+\arccos x\right)\right|$$

25. $\dfrac{1}{(2x^2-3x+1)\sqrt{3x^2-2x+1}}$

Sol. We have $\dfrac{1}{2x^2-3x+1} = \dfrac{1}{(x-1)(2x-1)} = \dfrac{-2}{2x-1}+\dfrac{1}{x-1}$

So, $\displaystyle\int \frac{dx}{(2x^2-3x+1)\sqrt{3x^2-2x+1}}$

$$= -\int \frac{2\,dx}{(2x-1)\sqrt{3x^2-2x+1}} + \int \frac{dx}{(x-1)\sqrt{3x^2-2x+1}}$$

$= I_1 + I_2$ (say)

Now $I_1 = \displaystyle\int \frac{-2\,dx}{(2x-1)\sqrt{3x^2-2x+1}}$

Put $2x-1 = \dfrac{1}{t}$ or $2x = 1+\dfrac{1}{t}$

or $2dx = -\dfrac{1}{t^2}dt \Rightarrow -2dx = \dfrac{1}{t^2}dt$

and $2x = 1+\dfrac{1}{t} = \dfrac{t+1}{t}$ or $x = \dfrac{t+1}{2t}$

$$I_1 = \int \frac{\frac{1}{t^2}\,dx}{\frac{1}{t}\sqrt{3\left(\frac{t+1}{2t}\right)^2 - 2\left(\frac{t+1}{2t}\right)+1}}$$

$$= \int \frac{\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{3}{4t^2}(t^2+2t+1) - \frac{t+1}{t}+1}}$$

$$= \int \frac{2\,dt}{\sqrt{3\,(t^2+2t+1) - 4t\,(t+1) + 4t^2}}$$

$$= 2\int \frac{dt}{\sqrt{3t^2+6t+3-4t}} = 2\int \frac{dt}{\sqrt{3t^2+2t+3}}$$

$$= \frac{2}{\sqrt{3}}\int \frac{dt}{\sqrt{t^2+\frac{2}{3}t+1}} = \frac{2}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{3}\right)^2+\left(1-\frac{1}{9}\right)}}$$

$$= \frac{2}{\sqrt{3}} \int \frac{dt}{\sqrt{\left(t+\frac{1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2}} = \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{t+\frac{1}{3}}{\frac{2\sqrt{2}}{3}}\right)$$

$$= \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{3t+1}{2\sqrt{2}}\right) = \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{3\left(\frac{1}{2x-1}\right)+1}{2\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{3+2x-1}{2\sqrt{2}\,(2x-1)}\right) = \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{2x+2}{2\sqrt{2}\,(2x-1)}\right)$$

$$= \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{x+1}{\sqrt{2}\,(2x-1)}\right)$$

Again $I_2 = \int \frac{dx}{(x-1)\sqrt{3x^2-2x+1}}$

Putting $x - 1 = \frac{1}{t}$ or $x = \frac{1}{t} + 1$ or $dx = -\frac{1}{t^2}dt$, we have

$$I_2 = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{3\left(\frac{1}{t}+1\right)^2 - 2\left(\frac{1}{t}+1\right)+1}}$$

$$= \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{3\left(\frac{1}{t^2}+\frac{2}{t}+1\right)-\frac{2}{t}-2+1}}$$

$$= \int \frac{-dx}{\sqrt{3(1+2t+t^2)-2t-t^2}} = -\int \frac{dt}{\sqrt{2t^2+4t+3}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2+2t+\frac{3}{2}}} = -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{(t+1)^2+\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -\frac{1}{\sqrt{2}} \sinh^{-1} \frac{t+1}{\frac{1}{\sqrt{2}}} = -\frac{1}{\sqrt{2}} \sinh^{-1} \sqrt{2}\,(t+1)$$

$$= -\frac{1}{\sqrt{2}} \sinh^{-1} \sqrt{2}\left[\frac{1}{x-1}+1\right] = -\frac{1}{\sqrt{2}} \sinh^{-1} \frac{\sqrt{2}\,x}{x-1}$$

Hence the required integral

$$= \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{x+1}{\sqrt{2}\,(2x-1)}\right) - \frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{\sqrt{2}\,x}{x-1}\right)$$

26. $\dfrac{\sqrt{x}}{1+\sqrt[3]{x}}$

Sol. $I = \int \frac{\sqrt{x}}{1+x^{1/3}}\,dx$

Here $x^{1/2}$ and $x^{1/3}$ are involved

We put $x^{1/6} = z$ *i.e.*, $x = z^6$ or $dx = 6z^5\,dz$

$$I = \int \frac{z^3}{1+z^2} \cdot 6z^5\,dz = 6\int \frac{z^8\,dz}{1+z^2}$$

$$= 6\int \left(z^6 - z^4 + z^2 - 1 + \frac{1}{1+z^2}\right)dz$$

$$= 6\left[\frac{z^7}{7} - \frac{z^5}{5} + \frac{z^3}{3} - z + \arctan z\right]$$

$$= 6\left(\frac{x^{7/6}}{7} - \frac{x^{5/6}}{5} + \frac{x^{1/2}}{3} - x^{1/6} + \arctan x^{1/6}\right)$$

27. $\dfrac{x^3}{\sqrt{1+x^2}}$

Sol. Put $x = \tan\theta$ or $dx = \sec^2\theta\,d\theta$

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{\tan^3\theta}{\sec\theta}\sec^2 d\theta = \int \tan^2\theta(\sec\theta\tan\theta)\,d\theta$$

$= \tan^2\theta\sec\theta \int(2\tan\theta\sec^2\theta) \,.\, \sec\theta\,d\theta$ (integrating by parts).

$$= \tan^2\theta\sec\theta - 2\int(\tan\theta\sec\theta)\sec^2\theta\,d\theta$$

$$= \tan^2\theta\sec\theta - 2\frac{\sec^3\theta}{3} = x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2}$$

28. $\dfrac{1}{\sqrt{x}+2x^{1/3}}$

Sol. Put $x = z^6$ or $dx = 6z^5\,dz$

$$\int \frac{dx}{\sqrt{x}+2x^{1/3}} = \int \frac{6z^5\,dz}{z^3+2z^2} = 6\int \frac{z^3}{z+2}\,dz$$

$$= 6\int \left(z^2 - 2z + 4 - \frac{8}{z+2}\right)dz = 6\left(\frac{z^3}{3} - z^2 + 4z - 8\ln|z+2|\right)$$

$$= 2\sqrt{x} - 6x^{1/3} + 24x^{1/6} - 48\ln|x^{1/6}+2|.$$

Exercise Set 4.6 (Page 163)

Integrate with respect to x (Problems 1 – 4):

1. $\sin^5 x$

Sol. $\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx$, Put $\cos x = t$ or $-\sin x\,dx = dt$

$$= \int (1-\cos^2 x)^2 \sin x\,dx$$

$$= -\int (1-t^2)^2\,dt = -\int (1-2t^2+t^4)\,dt$$

$$= -\left[t - 2\frac{t^3}{3} + \frac{t^5}{5}\right] = -t + \frac{2}{3}t^3 - \frac{t^5}{5}$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$$

2. $\cos^7 x$

Sol. Putting $\sin x = t$ or $\cos x\,dx = dt$, we have

$$\int \cos^7 x\,dx = \int (1-\sin^2 x)^3 \cos x\,dx = \int (1-t^2)^3\,dt$$

$$= \int (1-3t^2+3t^4-t^6)\,dt = t - t^3 + 3\cdot\frac{t^5}{5} - \frac{t^7}{7}$$

$$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x$$

3. $\sin^8 x$

Sol. By the reduction formula

$$\int \sin^8 x\,dx = -\frac{\cos x \sin^7 x}{8} + \frac{7}{8}\int \sin^6 x\,dx$$

$$= -\frac{\cos x \sin^7 x}{8} + \frac{7}{8}\left[-\frac{\cos x \sin^5 x}{6} + \frac{5}{6}\int \sin^4 x\,dx\right]$$

(Repeating the formula)

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x + \frac{35}{48}\int \sin^4 x\,dx$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x + \frac{35}{48}\left[-\frac{\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x\,dx\right]$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x - \frac{35}{192}\cos x \sin^5 x + \frac{35}{64}\int \sin^2 x\,dx$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x - \frac{35}{192}\cos x \sin^3 x + \frac{35}{128}\int (1-\cos 2x)\,dx$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x - \frac{35}{192}\cos x \sin^3 x + \frac{35}{128}x - \frac{35}{128}\sin x \cos x$$

4. $\cos^6 x$

Sol. By the reduction formula

$$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} dx, \text{ we get}$$

$$\int \cos^6 x\,dx = \frac{\sin x \cos^5 x}{6} + \frac{5}{6}\int \cos^4 x\,dx$$

$$= \frac{\sin x \cos^5 x}{6} + \frac{5}{6}\left[\frac{\sin x \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x\,dx\right]$$

$$= \frac{\sin x \cos^5 x}{6} + \frac{5}{24}\sin x \cos^3 x + \frac{5}{8\times 2}\int (1+\cos 2x)\,dx$$

$$= \frac{\sin x \cos^5 x}{6} + \frac{5}{24}\sin x \cos^3 x + \frac{5}{16}\left[x + \frac{\sin 2x}{2}\right]$$

$$= \frac{\sin x \cos^5 x}{6} + \frac{5}{24}\sin x \cos^3 x + \frac{5}{16}x + \frac{5}{16}\sin x \cos x.$$

Find a reduction formula for each of the following (Problems 5 – 8): ($n > 1$ is an integer).

5. $\int \tan^n x\,dx$

Sol. $\int \tan^n x\,dx = \int \tan^{n-2} x\,.\,(\sec^2 x - 1)\,dx$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$$

which is the required reduction formula.

6. $\int \sec^n x\,dx$

Sol. $\int \sec^n x\,dx = \int \sec^{n-2} x\,.\,\sec^2 x\,dx$

$$= \sec^{n-2} x.\tan x - \int \tan x\,.\,(n-2)\sec^{n-3} x\,.\,\sec x \tan x\,dx$$

(Integrating by parts).

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

or $\quad (1 + n - 2) \int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$

or $\quad \int \sec^n x \, dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} \int \sec^{n-2} x \, dx$

7. $\int \cot^n x \, dx$

Sol. $\int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx$

$$= \int \cot^{n-2} x \, . \, (\csc^2 x - 1) \, dx = \int \cot^{n-2} \csc^2 x \, dx - \int \cot^{n-2} x \, dx$$

$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx$$

which is the required reduction formula.

8. $\int \csc^n x \, dx$

Sol. $\int \csc^n x \, dx = \int \csc^{n-2} x \csc^2 x$

$$= -\csc^{n-2} x \cot x - \int (-\cot x)(n-2) \csc^{n-3} x (-\csc x \cot x) \, dx$$

(Integrating by parts)

$$= -\csc^{n-2} x \cot x - (n-2) \int \csc^{n-2} x \cot^2 x \, dx$$

$$= -\csc^{n-2} x \cot x - (n-2) \int \csc^{n-2} x (\csc^2 x - 1) \, dx$$

$$= -\csc^{n-2} x \cot x - (n-2) \int \csc^n x \, dx + (n-2) \int \csc^{n-2} x \, dx$$

or $(1 + n - 2) \int \csc^n x \, dx = -\csc^{n-2} x \cot x + (n-2) \int \csc^{n-2} x \, dx$

or $\quad \int \csc^n x \, dx = -\dfrac{\csc^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1} \int \csc^{n-2} x \, dx$

which is the required reduction formula.

Evaluate (Problems 9 – 12):

9. $\int \tan^6 x \, dx$

Sol. By the reduction formula, we have

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \qquad (1)$$

Putting $n = 6$ in (1), we get

$$\int \tan^6 x \, dx = \frac{\tan^5 x}{5} - \int \tan^4 x \, dx \qquad (2)$$

Again putting $n = 4$ in (1), we get

$$\int \tan^4 x \, dx = \frac{\tan^3 x}{3} - \int \tan^2 x \, dx = \frac{\tan^3 x}{3} - \int (\sec^2 x - 1) \, dx$$

$$= \frac{\tan^3 x}{3} - |\tan x - x| = \frac{\tan^3 x}{3} - \tan x + x \qquad (3)$$

Putting the value of $\int \tan^4 x \, dx$ from (3) into (2), we have

$$\int \tan^6 x \, dx = \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - \tan x + x\right]$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x$$

10. $\int \cot^5 x \, dx$

Sol. By the reduction formula, we have

$$\int \cot^n x \, dx = -\int \frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx \qquad (1)$$

Putting $n = 5$ in (1), we get

$$\int \cot^5 x \, dx = -\frac{\cot^4 x}{4} - \int \cot^3 x \, dx \qquad (2)$$

Again putting $n = 3$, in (1), we get

$$\int \cot^3 x \, dx = -\frac{\cot^2 x}{2} - \int \cot x \, dx$$

$$= -\frac{\cot^2 x}{2} - \ln |\sin x| \qquad (3)$$

Now putting the value of $\int \cot^3 x \, dx$ from (3) into (2), we get

$$\int \cot^5 x \, dx = -\frac{\cot^4 x}{4} - \left[-\frac{\cot^2 x}{2} - \ln |\sin x|\right]$$

$$= -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \ln |\sin x|$$

11. $\int \sec^6 x \, dx$

Sol. By the reduction formula, we have

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \qquad (1)$$

Putting $n = 6$, in (1),we get

$$\int \sec^6 x\, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\int \sec^4 x\, dx \qquad (2)$$

Again putting $n = 4$, in (1), we have

$$\int \sec^4 x\, dx = \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\int \sec^2 x\, dx$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x$$

Putting the value of $\int \sec^4 x\, dx$ in (2), we have

$$\int \sec^6 x\, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x\right]$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x$$

12. $\int \csc^5 x\, dx$

Sol. By the reduction formula, we have

$$\int \csc^n x\, dx = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\, dx \qquad (1)$$

Putting $n = 5$, in (1), we get

$$\int \csc^5 x\, dx = -\frac{\cot x \csc^3 x}{4} + \frac{3}{4}\int \csc^3 x\, dx \qquad (2)$$

Again putting $n = 3$ in (1), we get

$$\int \csc^3 x\, dx = -\frac{\cot x \csc x}{2} + \frac{1}{2}\int \csc x\, dx$$

$$= -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|$$

Now putting the value of $\int \csc^3 x\, dx$ in (2), we get

$$\int \csc^5 x\, dx = -\frac{1}{4}\cot x \csc^3 x + \frac{3}{4}\left[-\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|\right]$$

$$= -\frac{1}{4}\cot x \csc^3 x - \frac{3}{8}\cot x \csc x + \frac{3}{8}\ln\left|\tan\frac{x}{2}\right|$$

Integrate with respect to x (Problems 13 – 24):

13. $\dfrac{1}{a + b\sin x}$

Sol. $I = \displaystyle\int \frac{1}{a + b\sin x}\, dx$

$$= \int \frac{1}{a\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + 2b\sin\frac{x}{2}\cos\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{a + a\tan^2\frac{x}{2} + 2b\tan\frac{x}{2}}\, dx,$$ dividing numerater and denominator by $\cos^2 x/2$

Now put $\tan\frac{x}{2} = z$, so that (1)

$$\left(\sec^2\frac{x}{2}\right)\times\frac{1}{2}\, dx = dz \qquad \text{or} \qquad \sec^2\frac{x}{2}\, dx = 2\, dz$$

$$I = \int \frac{2\, dz}{a + az^2 + 2bz} = 2\int \frac{1}{a\left[\left(z^2 + 2\frac{b}{a}z\right) + 1\right]}\, dx$$

$$= \frac{2}{a}\int \frac{1}{\left(z^2 + 2\frac{b}{a}z + \frac{b^2}{a^2}\right) - \frac{b^2}{a^2} + 1}\, dz$$

$$= \frac{2}{a}\int \frac{1}{\left(z + \frac{b}{a}\right)^2 + \frac{a^2 - b^2}{a^2}}\, dz \qquad (2)$$

<u>**Case I:**</u> If $a^2 > b^2$, then from (2), we get

$$I = \frac{2}{a}\int \frac{1}{\left(z + \frac{b}{a}\right)^2 + \left(\frac{\sqrt{a^2 - b^2}}{a}\right)^2}\, dz = \frac{2}{a}\frac{1}{\frac{\sqrt{a^2 - b^2}}{a}}\arctan\frac{z + \frac{b}{a}}{\frac{\sqrt{a^2 - b^2}}{a}}$$

$$= \frac{2}{\sqrt{a^2 - b^2}}\arctan\frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}},\ \text{[from (1)]}.$$

<u>**Case II:**</u> If $a^2 < b^2$, then from (2), we have

$$I = \frac{2}{a}\int \frac{1}{\left(z + \frac{b}{a}\right)^2 - \frac{b^2 - a^2}{a^2}}\, dz = \frac{2}{a}\int \frac{1}{\left(z + \frac{b}{a}\right)^2 - \left(\frac{\sqrt{b^2 - a^2}}{a}\right)^2}\, dz$$

$$= \frac{2}{a}\frac{1}{2\frac{\sqrt{b^2 - a^2}}{a}}\ln\frac{z + \frac{b}{a} - \frac{\sqrt{b^2 - a^2}}{a}}{z + \frac{b}{a} + \frac{\sqrt{b^2 - a^2}}{a}}$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{a \tan\frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \tan\frac{x}{2} + b + \sqrt{b^2 - a^2}}$$

Alternative Method:

$$I = \int \frac{1}{a + b \sin x}$$

Put $x = \frac{\pi}{2} + z$ then $dx = dz$ and $z = x - \frac{\pi}{2} = -\left(\frac{\pi}{2} - x\right)$

$$I = \int \frac{1}{a + b \sin\left(\frac{\pi}{2} + z\right)} dz = \int \frac{1}{a + b \cos z} dz$$

Case I: If $a^2 > b^2$, then from Example 40, we have

$$I = \frac{1}{\sqrt{a^2 - b^2}} \arccos \frac{b + a \cos z}{a + b \cos z}$$

$$= \frac{1}{\sqrt{a^2 - b^2}} \arccos \frac{b + a \cos\left[-\left(\frac{\pi}{2} - x\right)\right]}{a + b \cos\left[-\left(\frac{\pi}{2} - x\right)\right]} \qquad \left[\begin{array}{l} \text{since } \cos\left[-\left(\frac{\pi}{2} - x\right)\right] \\ = \cos\left(\frac{\pi}{2} - x\right) \\ = \sin x \end{array}\right]$$

$$= \frac{1}{\sqrt{a^2 - b^2}} \arccos \frac{b + a \sin x}{a + b \sin x}$$

Case II: If $a^2 < b^2$, then from Example 40, we have

$$I = \frac{1}{\sqrt{b^2 - a^2}} \cosh^{-1} \frac{b + a \cos z}{a + b \cos z}$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \cosh^{-1} \frac{b + a \cos\left[-\left(\frac{\pi}{2} - x\right)\right]}{a + b \cos\left[-\left(\frac{\pi}{2} - x\right)\right]} \qquad \left[\begin{array}{l} \text{since } \cos\left[-\left(\frac{\pi}{2} - x\right)\right] \\ = \cos\left(\frac{\pi}{2} - x\right) \\ = \sin x \end{array}\right]$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \cosh^{-1} \frac{b + a \sin x}{a + b \sin x}$$

14. $\dfrac{1}{a + b \cosh x}$

Sol. $I = \displaystyle\int \frac{1}{a + b \cosh x} dx$

$$= \int \frac{1}{a\left(\cosh^2\frac{x}{2} - \sinh^2\frac{x}{2}\right) + b\left(\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}\right)} dx$$

$$= \int \frac{1}{(a + b) \cosh^2\frac{x}{2} + (b - a) \sinh^2\frac{x}{2}} dx$$

Divide the numerator and denominator by $\cosh^2\frac{x}{2}$, then

$$I = \int \frac{\operatorname{sech}^2\frac{x}{2}}{(a + b) + (b - a) \tanh^2\frac{x}{2}} dx$$

Put $\tanh\frac{x}{2} = z$, so that (1)

$$\left(\operatorname{sech}^2\frac{x}{2}\right) \times \frac{1}{2} dx = dz \quad \text{or} \quad \operatorname{sech}^2\frac{x}{2} dx = 2\, dz$$

$$I = \int \frac{2\, dz}{(a + b) + (b - a) z^2} \qquad (2)$$

(Assume that $a + b$ is positive)

Case I: If $a^2 > b^2$ *i.e.*, $a^2 - b^2 > 0$ or $(a + b)(a - b) > 0$ as $a + b > 0$ so $a - b > 0$, then from (2), we have

$$I = 2 \int \frac{1}{(a - b)\left(\frac{a + b}{a - b} - z^2\right)} dz = \frac{2}{a - b} \int \frac{1}{\left(\sqrt{\frac{a + b}{a - b}}\right)^2 - z^2} dz$$

$$= \frac{2}{a - b} \frac{1}{2\sqrt{\frac{a + b}{a - b}}} \ln \frac{\sqrt{\frac{a + b}{a - b}} + z}{\sqrt{\frac{a + b}{a - b}} - z}$$

$$= \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{\sqrt{a + b} + \sqrt{a - b} \tanh\frac{x}{2}}{\sqrt{a + b} - \sqrt{a - b} \tanh\frac{x}{2}}$$

Case II: If $a^2 < b^2$

$a^2 - b^2$ is –ve and $a + b$ is +ve, *i.e.*,
$a - b$ is –ve or $b - a$ is +ve

then from (2), we get

$$I = 2 \int \frac{1}{(b - a)\left[\frac{b + a}{b - a} + z^2\right]} dz = \frac{2}{(b - a)} \int \frac{1}{\left(\sqrt{\frac{b + a}{b - a}}\right)^2 + z^2} dz$$

$$= \frac{2}{b-a}\,\frac{1}{\sqrt{\dfrac{b+a}{b-a}}}\arctan\frac{z}{\sqrt{\dfrac{b+a}{b-a}}}$$

$$= \frac{2}{\sqrt{b^2-a^2}}\arctan\left[\sqrt{\frac{b-a}{b+a}}\tanh\frac{x}{2}\right], \qquad \text{from (1)}$$

15. $\dfrac{\cot x}{1+\sin x}$

Sol. $I = \displaystyle\int\frac{\cot x\,dx}{1+\sin x} = \int\frac{\cos x\,dx}{\sin x\,(1+\sin x)}$

Put $\sin x = z$ so that $\cos x\,dx = dz$ and on substitution,

$$I = \int\frac{dz}{z\,(1+z)} = \int\left(\frac{1}{z}-\frac{1}{1+z}\right)dz = \ln[z] - \ln[1+z]$$

$$= \ln[\sin x] - \ln[1+\sin x]$$

$$= \ln\left[\frac{\sin x}{1+\sin x}\right] = \ln\left[\frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}\right] = \ln\left[\frac{2\tan\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2}\right]$$

$$= \ln\left[2\tan\frac{x}{2}\right] - 2\ln\left[1+\tan\frac{x}{2}\right]$$

16. $\dfrac{2-\cos x}{2+\cos x}$

Sol. $I = \displaystyle\int\frac{2-\cos x}{2+\cos x}\,dx$

Let $\tan\frac{x}{2} = z$ so that $\left(\sec^2\frac{x}{2}\right)\frac{1}{2}\,dx = dz$ or $dx = \dfrac{z}{1+z^3}\,dz$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-z^2}{1+z^2}$$

$$I = \int\frac{2-\dfrac{1-z^2}{1+z^2}}{2+\dfrac{1-z^2}{1+z^2}}\cdot\frac{2\,dz}{1+z^2} = z\int\frac{1+3z^2}{(3+z^2)\,(1+z^2)}\,dz$$

$$= \int\left(\frac{8}{3+z^2}-\frac{2}{1+z^2}\right)dz = \frac{8}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}} - 2\arctan z$$

$$= \frac{8}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) - 2\,\text{aretan}\left(\tan\frac{x}{2}\right)$$

$$= \frac{8}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) - x$$

17. $\dfrac{1}{1+\sin x+\cos x}$

Sol. $I = \displaystyle\int\frac{dx}{1+\sin x+\cos x} = \int\frac{dx}{2\cos^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$

$$= \int\frac{\frac{1}{2}\sec^2\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\,dx$$

Now put $\tan\left(\frac{x}{2}\right) = z$ so that $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$. Then

$$I = \int\frac{dz}{1+z} = \ln|1+z| = \ln\left|1+\tan\left(\frac{x}{2}\right)\right|$$

18. $\dfrac{\cos x}{2-\cos x}$

Sol. $I = \displaystyle\int\frac{\cos x}{2-\cos x}\,dx$

Put $\tan\left(\frac{x}{2}\right) = z$ so that $\frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dz$ or $dx = \dfrac{2\,dz}{1+z^2}$

and $\cos x = \dfrac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \dfrac{1-z^2}{1+z^2}$

Then $I = \displaystyle\int\frac{\dfrac{1-z^2}{1+z^2}}{2-\dfrac{1-z^2}{1+z^2}}\cdot\frac{2}{1+z^2}\,dz = \int\frac{1-z^2}{1+3z^2}\cdot\frac{2}{1+z^2}\,dz$

$$= 2\int\left(\frac{2}{1+3z^2}-\frac{1}{1+z^2}\right)dz$$

$$= \frac{4}{3}\int\frac{dz}{\left(\frac{1}{\sqrt{3}}\right)^2+z^2} - \int\frac{2}{1+z^2}\,dz$$

$$= \frac{4}{3}\cdot\frac{1}{\frac{1}{\sqrt{3}}}\arctan\left(\frac{z}{\frac{1}{\sqrt{3}}}\right) - 2\arctan z$$

$$= \frac{4}{\sqrt{3}} \arctan\left(\sqrt{3}\tan\frac{x}{2}\right) - 2\arctan\left(\tan\frac{x}{2}\right)$$

$$= \frac{4}{\sqrt{3}} \arctan\left[\sqrt{3}\tan\left(\frac{x}{2}\right)\right] - x$$

19. $\dfrac{1}{4\sin x - 3\cos x}$

Sol. $I = \displaystyle\int \frac{dx}{4\sin x - 3\cos x}$

Put $\tan\left(\frac{x}{2}\right) = z$ so that $dx = \dfrac{2}{1+z^2}dz$

$$\sin x = \frac{2z}{1+z^2}, \cos x = \frac{1-z^2}{1+z^2}$$

$$I = \int \frac{1}{\frac{8z}{1+z^2} - \frac{3-3z^2}{1+z^2}} \cdot \frac{2}{1+z^2}dz = \int \frac{2}{3z^2+8z-3}dz$$

$$= \frac{2}{3}\int \frac{dz}{z^2 + \frac{8}{3}z - 1} = \frac{2}{3}\int \frac{dz}{\left(z+\frac{4}{3}\right)^2 - \left(\frac{5}{3}\right)^2} = \frac{2}{3}\cdot\frac{3}{10}\ln\left|\frac{z+\frac{4}{3}-\frac{5}{3}}{z+\frac{4}{3}+\frac{5}{3}}\right|$$

$$= \frac{1}{5}\ln\left|\frac{3z-1}{3z+9}\right| = \frac{1}{5}\ln\left|\frac{3\tan\left(\frac{x}{2}\right)-1}{3\tan\left(\frac{x}{2}\right)+9}\right|$$

Alternative Method:

$$\int \frac{2}{3z^2+8z-3}dz = \int\left(\frac{\frac{3}{5}}{3z-1} + \frac{-\frac{1}{5}}{z+3}\right)dz$$

$$= \frac{1}{5}\left[\int \frac{1}{3z-1}3dz - \int \frac{1}{z+3}dz\right] = \frac{1}{5}[\ln|3z-1| - \ln|z+3|]$$

$$= \frac{1}{5}\ln\left|\frac{3z-1}{z+3}\right| = \frac{1}{5}\ln\left|\frac{3\tan\frac{x}{2}-1}{\tan\frac{x}{2}+3}\right|$$

20. $\dfrac{1}{\tan x - \sin x}$

Sol. $I = \displaystyle\int \frac{dx}{\tan x - \sin x} = \int \frac{\frac{2}{1+z^2}dz}{\frac{2z}{1-z^2} - \frac{2z}{1+z^2}}$, where $z = \tan\left(\frac{x}{2}\right)$ and $dx = \dfrac{2}{1+z^2}dz$

$$= \int \frac{1-z^2}{2z^3}dz = \int\left(\frac{1}{2}\cdot z^{-3} - \frac{1}{2}\cdot\frac{1}{z}\right)dz = \frac{1}{2}\cdot\frac{z^{-2}}{-2} - \frac{1}{2}\ln|z|$$

$$= -\frac{1}{4}\left[\tan\left(\frac{x}{2}\right)\right]^{-2} - \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}\right)\right|$$

21. $\dfrac{1}{2\cosh x + \sinh x}$

Sol. $I = \displaystyle\int \frac{dx}{2\cosh x + \sinh x}$

Here we make the substitution $\tanh\left(\frac{x}{2}\right) = z$

Therefore, $\frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)dx = dz$

and $dx = \dfrac{2\,dz}{\operatorname{sech}^2\left(\frac{x}{2}\right)} = \dfrac{2\,dz}{1-\tanh^2\left(\frac{x}{2}\right)} = \dfrac{2\,dz}{1-z^2}$

$$\sinh x = \frac{2\tanh\left(\frac{x}{2}\right)}{1-\tanh^2\left(\frac{x}{2}\right)} = \frac{2z}{1-z^2}$$

$$\cosh x = \frac{\cosh^2\left(\frac{x}{2}\right) + \sinh^2\left(\frac{x}{2}\right)}{\cosh^2\left(\frac{x}{2}\right) - \sinh^2\left(\frac{x}{2}\right)} = \frac{1+z^2}{1-z^2}$$

$$I = \int \frac{2\,dz}{(1-z^2)\left[2\frac{1+z^2}{1-z^2} + \frac{2z}{1-z^2}\right]} = \int \frac{dz}{z^2+z+1}$$

$$= \int \frac{dz}{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(z+\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{3}\arctan\left(\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{\sqrt{2}}{3}\arctan\left(\frac{2z+1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\arctan\left[\frac{2\tanh\left(\frac{x}{2}\right)+1}{\sqrt{3}}\right]$$

22. $\dfrac{\sin x + \cos x}{\tan x}$

Sol. $\displaystyle\int \frac{\sin x + \cos x}{\tan x}\,dx = \left(\cos x + \frac{\cos^2 x}{\sin x}\right)dx$

$\ln \displaystyle\int \frac{\cos^2 x}{\sin x}\,dx$, put $\cos x = z$ so that $-\sin x\,dx = dz$

$$\int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{\cos^2 x}{\sin^2 x}.\sin x\,dx = \int \frac{-z^2}{-z^2+1}\,dz$$

$$= \int\left(1 - \frac{1}{-z^2+1}\right)dz = z - \frac{1}{2}\ln\left|\frac{z+1}{-z+1}\right|$$

$$= z + \frac{1}{2}\ln\left|\frac{-z+1}{z+1}\right|$$

$$I = \sin x + \cos x + \frac{1}{2}\ln\left|\frac{-\cos x+1}{\cos x+1}\right| \qquad (\cos x \neq 1)$$

$$= \sin x + \cos x + \ln\left|\sqrt{\frac{1-\cos x}{1+\cos x}}\right|$$

$$= \sin x + \cos x + \ln\left|\tan\frac{x}{2}\right|$$

23. $\cos x\sqrt{1-\cos x}$

Sol. Put $\cos x = z$ so that $-\sin x\,dx = dz$

$$dx = \frac{-dz}{\sqrt{1-z^2}}$$

$$I = -\int \frac{z\sqrt{1-z}}{\sqrt{1-z^2}}\,dz = -\int \frac{z\,dz}{\sqrt{1+z}}$$

Now put $1 + z = t^2$ so that $dz = 2t\,dt$

$$I = -2\int \frac{t^2-1}{t}\,t\,dt = -2\int (t^2-1)\,dt = -2\frac{t^3}{3} + 2t$$

$$= -\frac{2}{3}(1+z)^{3/2} + 2\sqrt{1+z} = 2\sqrt{1+z}\left[1 - \frac{1+z}{3}\right]$$

$$= 2\sqrt{1+z}\left(\frac{2-z}{3}\right) = \frac{2}{3}(2-\cos x)\sqrt{1+\cos x}$$

24. $\sqrt{a + \sec^2 x}$

Sol. Put $z = \sec x$ so that $dz = \sec x \tan x\,dx$ or $dx = \dfrac{dz}{z\sqrt{z^2-1}}$

$$I = \int \sqrt{a+z^2}.\frac{dz}{z\sqrt{z^2-1}} = \int \sqrt{a+z^2}.\frac{1}{\sqrt{z^2-1}}.\frac{1}{z}\,dz$$

Now put $z^2 = t$, so that $2z\,dz = dt \Rightarrow \dfrac{1}{z}dz = \dfrac{1}{2t}dt$

$$I = \int \sqrt{a+t}\,\frac{1}{\sqrt{t-1}}.\frac{1}{2t}\,dt = \frac{1}{2}\int \sqrt{\frac{a+t}{t-1}}.\frac{1}{t}\,dt$$

In this integral, set

$$\frac{a+t}{t-1} = \omega^2,\ a + t = \omega^2 t - w^2, \quad \text{or} \quad t = \frac{a+\omega^2}{\omega^2-1}$$

$$\frac{t-1-(a+t)}{(t-1)^2}\,dt = 2\omega\,d\omega \quad \text{or} \quad \frac{-1-a}{(t-1)^2}\,dt = 2\omega\,d\omega$$

$$dt = \frac{-2\omega}{1+a}\left[\frac{a+\omega^2}{\omega^2-a} - 1\right]^2 d\omega = \frac{-2\omega}{1+a}\left(\frac{a+1}{\omega^2-1}\right)^2 d\omega$$

$$= \frac{-2w(a+1)}{(w^2-1)^2}\,dw$$

$$I = \frac{1}{2}\int \omega\,\frac{\omega^2-1}{\omega^2+a}.\frac{-2w(a+1)}{(w^2-1)^2}\,dw$$

$$= -\int \frac{(a+1)\,\omega^2}{(\omega^2+a)\,(\omega^2-1)}\,d\omega = -\int\left(\frac{a}{\omega^2+a} + \frac{1}{\omega^2-1}\right)d\omega$$

$$= -a.\frac{1}{\sqrt{a}}\arctan\left(\frac{\omega}{\sqrt{a}}\right) - \frac{1}{2}\ln\left|\frac{\omega-1}{\omega+1}\right|$$

$$= -\sqrt{a}\arctan\sqrt{\frac{a+t}{a(t-1)}} - \frac{1}{2}\ln\left|\frac{\sqrt{\frac{a+t}{t-1}}-1}{\sqrt{\frac{a+t}{t-1}}+1}\right|$$

$$= -\sqrt{a}\arctan\sqrt{\frac{a+\sec^2 x}{a\tan^2 x}} - \frac{1}{2}\ln\left|\frac{\sqrt{\frac{a+\sec^2 x}{\tan^2 x}}-1}{\sqrt{\frac{a+\sec^2 x}{\tan^2 x}}+1}\right|$$

$$= -\sqrt{a}\arctan\left(\frac{\sqrt{a+\sec^2 x}}{\sqrt{a}\tan x}\right) - \frac{1}{2}\ln\left|\frac{\sqrt{a+\sec^2 x} - \tan x}{\sqrt{a+\sec^2 x} + \tan x}\right|$$

$$= \ln\left|\frac{\sqrt{a+\sec^2 x}+\tan x}{\sqrt{a+\sec^2 x}-\tan x}\right|^{1/2} - \sqrt{a}\arctan\left(\frac{\sqrt{a+\sec^2 x}}{\sqrt{a}\tan x}\right)$$

$$= \ln\left|\sqrt{a\sec^2 x}+\tan x\right| - \frac{1}{2}\ln|a+1|$$

$$-\sqrt{a}\arctan\left(\frac{\sqrt{a+\sec^2 x}}{\sqrt{a}\tan x}\right)$$

$$= \ln\left|\sqrt{a+\sec^2 x}+\tan x\right| - \sqrt{a}\arctan\left(\frac{\sqrt{a+\sec^2 x}}{\sqrt{a}\tan x}\right)$$

25. Evaluate $I = \int\sqrt{\dfrac{1-\cos\theta}{\cos\alpha-\cos\theta}}\,d\theta$, α constant and $0<\alpha<\theta<\pi$.

Sol. $$I = \int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\alpha}{2}-2\cos^2\frac{\theta}{2}}}\,d\theta, \quad \left(\begin{array}{l}\text{since } (1+\cos\alpha)-(1+\cos\theta)\\ = 2\cos^2\frac{\alpha}{2}-2\cos^2\frac{\theta}{2}\end{array}\right)$$

$$= \int\frac{\tan\frac{\theta}{2}}{\sqrt{\cos^2\frac{\alpha}{2}\sec^2\frac{\theta}{2}-1}}\,d\theta = \frac{1}{\cos\frac{\alpha}{2}}\int\frac{\tan\frac{\theta}{2}}{\sqrt{\sec^2\frac{\theta}{2}-\sec^2\frac{\alpha}{2}}}\,d\theta$$

Now put $\sec^2\frac{\theta}{2}-\sec^2\frac{\alpha}{2} = u^2$

or $\quad \sec^2\frac{\theta}{2}\tan\frac{\theta}{2}\,d\theta = 2u\,du$

or $\quad d\theta = \dfrac{2u\,du}{\left(u^2+\sec^2\frac{\alpha}{2}\right)\tan\frac{\theta}{2}}$

$$I = \sec\frac{\alpha}{2}\int\frac{\tan\frac{\theta}{2}\cdot 2u\,du}{u\left(u^2+\sec^2\frac{\alpha}{2}\right)\tan\frac{\theta}{2}} = 2\sec\frac{\alpha}{2}\int\frac{du}{u^2+\sec^2\frac{\alpha}{2}}$$

$$= 2\sec\frac{\alpha}{2}\cdot\frac{1}{\sec\frac{\alpha}{2}}\arctan\left(\frac{u}{\sec\frac{\alpha}{2}}\right)$$

$$= 2\arctan\left[\frac{\sqrt{\sec^2\frac{\theta}{2}-\sec^2\frac{\alpha}{2}}}{\sec\frac{\alpha}{2}}\right]$$

$$= 2\arctan\left[\frac{\sqrt{\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta}{2}}}{\cos\frac{\alpha}{2}\cos\frac{\theta}{2}\cdot\sec\frac{\alpha}{2}}\right] = 2\arctan\left[\frac{\sqrt{\cos\alpha-\cos\theta}}{\sqrt{2}\cos\frac{\theta}{2}}\right]$$

26. By using the substitution $z = \tan\left(\frac{x}{2}\right)$, show that

(i) $\displaystyle\int\sec x\,dx = \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$

(ii) $\displaystyle\int\csc\,dx = \frac{1}{2}\left|\frac{1-\cos x}{1+\cos x}\right|$

Sol.

(i) Let $\displaystyle I = \int\sec x\,dx = \int\frac{dx}{\cos x} = \int\frac{dx}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \int\frac{\sec^2\frac{x}{2}\,dx}{1-\tan^2\frac{x}{2}}$

Put $\tan\frac{x}{2} = z$ so that $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dz \Rightarrow \sec^2\frac{x}{2}\,dx = 2dz$

$$I = \int\frac{2\,dz}{1-z^2} = \int\left(\frac{1}{1+z}+\frac{1}{1-z}\right)dz = \ln\left|\frac{1+z}{1-z}\right|$$

$$= \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|$$

$$\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|^2 = \left[\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right]^2 = \frac{1+\sin x}{1-\sin x}$$

$$\Rightarrow \quad \left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| = \left|\frac{1+\sin x}{1-\sin x}\right|^{1/2}$$

Thus $\displaystyle I = \ln\left|\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)\right| = \ln\left|\frac{1+\sin x}{1-\sin x}\right|^{1/2} = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$

(ii) $\displaystyle\int\csc x\,dx = \int\frac{dx}{\sin x}$

Put $\tan\frac{x}{2} = z$ so that $dx = \frac{2\,dz}{1+z^2}$ and $\sin x = \frac{2z}{1+z^2}$

$$\int \csc x\,dx = \int \frac{1}{\sin x}\,dx = \int \frac{1+z^2}{2z}\cdot\frac{2\,dz}{1+z^2} = \int \frac{1}{z}\,dz = \ln|z|$$

$$= \ln\left|\tan\frac{x}{2}\right| = \ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right| = \frac{1}{2}\cdot 2\ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right|$$

$$= \frac{1}{2}\ln\left|\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\right| = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|.$$

Chapter 5 THE DEFINITE INTEGRAL

Exercise Set 5.1 (Page 172)

Evaluate the following by definition (Problems 1 – 8):

1. $\int_{-1}^{1} x\,dx$

Sol. Consider a partition P of $[-1, 1]$ into n subintervals of equal length *i.e.,* $\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}\cdot$ The subintervals are

$$\left[-1, -1+\frac{2}{n}\right], \left[-1+\frac{2}{n}, -1+\frac{4}{n}\right], \ldots, \left[-1+\frac{2(n-1)}{n}, -1+\frac{2n}{n}\right]$$

Take c_r as the right endpoint of each subinterval. Then

$$S(P, f) = S(P, x) = \sum_{r=1}^{n} \Delta x\, f(c_r)$$
$$= \frac{2}{n}\left[f\left(-1+\frac{2}{n}\right) + f\left(-1+\frac{4}{n}\right) + \ldots + f\left(-1+\frac{2n}{n}\right)\right]$$
$$= \frac{2}{n}\left[\left(-1+\frac{2}{n}\right) + \left(-1+\frac{4}{n}\right) + \ldots + \left(-1+\frac{2n}{n}\right)\right]$$
$$= \frac{2}{n}\left[-n + \frac{2}{n}(1+2+\ldots+n)\right]$$
$$= \frac{2}{n}\left[-n + \frac{2}{n}\frac{(n+1)\cdot n}{2}\right] = \frac{2}{n}(-n+n+1) = \frac{2}{n}$$

$$\int_{-1}^{1} x\,dx = \lim_{n\to\infty}\sum_{r=1}^{n} \Delta x\cdot f(c_r) = \lim_{n\to\infty}\frac{2}{n} = 0.$$

2. $\int_{a}^{b} \frac{1}{x}\,dx$

Sol. This problem could be solved by the method of **Examples**. However, we give another method which is more elegant.

Let $P = \{a, a(\Delta x), a(\Delta x)^2, \dots, a(\Delta x)^{n-1}, a(\Delta x)^n = b\}$, be a partition of $[a, b]$, where $\Delta x = \left(\frac{b}{a}\right)^{1/n}$. The subintervals, into which $[a, b]$ is subdivided are

$$[a, a\,\Delta x], [a\Delta x, a(\Delta x)^2], \dots, [a(\Delta x)^{n-1}, a(\Delta x)^n]$$

Taking c_r as the left endpoint of each subinterval, we have

$$S(P, f) = S\left(P, \frac{1}{x}\right)$$

$$= a(\Delta x - 1)\cdot\frac{1}{a} + a\,\Delta x\,(\Delta x - 1)\cdot\frac{1}{a\,\Delta x} + a\,(\Delta x)^2\,(\Delta x - 1)\cdot\frac{1}{a(\Delta x)^2} + \dots + a(\Delta x)^{n-1}(\Delta x - 1)\cdot\frac{1}{a(\Delta x)^{n-1}}$$

$$= (\Delta x - 1)\,[1 + 1 + \dots + 1, n \text{ terms}] = n\,(\Delta x - 1) \qquad (1)$$

As $n \to \infty$, $\Delta x \to 1$, $\left[\text{since } \Delta x = \left(\frac{b}{a}\right)^{1/n}\right]$ so that the length of each subinterval approaches zero.

Now, $\Delta x = \left(\frac{b}{a}\right)^{1/n}$ or $\ln \Delta x = \frac{1}{n}\ln\left(\frac{b}{a}\right)$ *i.e.*, $n = \frac{1}{\ln \Delta x}\cdot\ln\left(\frac{b}{a}\right)$

Hence (1) becomes

$$S(P, f) = \left(\ln\frac{b}{a}\right)\cdot\left(\frac{\Delta x - 1}{\ln \Delta x}\right)$$

Taking limits as $\Delta x \to 1$, we get

$$\int_a^b \frac{1}{x}\,dx = \left(\ln\frac{b}{a}\right)\lim_{\Delta x \to 1}\frac{\Delta x - 1}{\ln \Delta x} \qquad \left(\frac{0}{0}\right)$$

$$= \left(\ln\frac{b}{a}\right)\cdot 1 = \ln b - \ln a$$

3. $\int_a^b x^2\,dx$

Sol. Let $P = \{a, a + \Delta x, a + 2\Delta x, \dots, a + (n-1)\,\Delta x, a + n\,\Delta x = b\}$ be a partition of $[a, b]$, where $\Delta x = \frac{b-a}{n}$.

Taking left endpoint of each subinterval as c_r, we have

$$S(P, f) = S(P, x^2) = \Delta x a^2 + \Delta x(a+\Delta x)^2 + \Delta x(a+2\Delta x)^2 + \dots + \Delta x(a+(n-1)\,\Delta x)^2$$

$$= \Delta x[na^2 + 2a\Delta x(1 + 2 + \dots + (n-1)) + \Delta x^2(1^2 + 2^2 + \dots + (n-))^2]$$

$$= \Delta x\left[na^2 + 2a\Delta x\,\frac{(n-1)\,n}{2} + \Delta x^2\,\frac{(n-1)\cdot n\cdot(2n-1)}{6}\right]$$

$$= \frac{b-a}{n}\cdot na^2 + a\,\frac{(b-a)^2}{n^2}\,(n-1)\cdot n + \frac{(b-a)^3}{n^3}\cdot\frac{(n-1)\,n.(2n-1)}{6}$$

since $\Delta x = \frac{b-a}{n}$

$$= a^2(b-a) + a(b-a)^2\cdot\left(1 - \frac{1}{n}\right) + \frac{(b-a)^3}{6}\cdot\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

Taking limits as $\Delta x \to 0$ and $n \to \infty$, we have

$$\lim_{\substack{n \to \infty \\ \Delta x \to 0}} S(P, f) = \int_a^b x^2\,dx$$

$$= a^2(b-a) + a(b-a)^2 \lim_{n \to \infty}\left(1 - \frac{1}{n}\right) + \frac{(b-a)^3}{6}\cdot\lim_{n \to \infty}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

$$= (b-a)\,a^2 + a(b-a)^2\cdot 1 + \frac{(b-a)^3}{6}\cdot 2$$

$$= (b-a)\left[a^2 + a\,(b-a) + \frac{(b-a)^2}{3}\right] = (b-a)\left[ab + \frac{b^2 - 2ab + a^2}{3}\right]$$

$$= \frac{b-a}{3}\,[3ab + b^2 - 2ab + a^2]$$

$$= \frac{b-a}{3}\,(b^2 + ab + a^2) = \frac{b^3 - a^3}{3}$$

4. $\int_a^b \frac{dx}{\sqrt{x}}$

Sol. Let $\Delta x = \frac{b-a}{n}$. Then taking left end of each subinterval as c_r, we have

$$S(P, f) = S\left(P, \frac{1}{\sqrt{x}}\right) = \Delta x\left[\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a + \Delta x}} + \dots + \frac{1}{\sqrt{a + (n-1)\,\Delta x}}\right]$$

Now $\lim_{\Delta x \to 0}\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \frac{1}{2}\cdot x^{\frac{1}{2}-1} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$$\text{Therefore, } \frac{1}{2} = \lim_{\Delta x \to 0}\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\frac{\Delta x}{\sqrt{x}}} \qquad (1)$$

Putting $x = a, a + \Delta x, a + 2\Delta x, \dots, a + (n-1)\Delta x$ in (1), we obtain

$$\frac{1}{2} = \lim_{\Delta x \to 0}\frac{\sqrt{a + \Delta x} - \sqrt{a}}{\frac{\Delta x}{\sqrt{a}}}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{a + 2\Delta x} - \sqrt{a + \Delta x}}{\dfrac{\Delta x}{\sqrt{a + \Delta x}}}$$

$$= \ldots \qquad \ldots \qquad \ldots$$

$$= \ldots \qquad \ldots \qquad \ldots$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{a + n\Delta x} - \sqrt{a + (n-1)\Delta x}}{\dfrac{\Delta x}{\sqrt{a + (n-1)\,\Delta x}}}$$

$$= \lim_{\Delta x \to 0} \frac{\text{Sum of } n \text{ numerators}}{\text{Sum of } n \text{ denominators}}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{a + n\,\Delta x} - \sqrt{a}}{\Delta x\left(\dfrac{1}{\sqrt{a}} + \dfrac{1}{\sqrt{a + \Delta x}} + \ldots + \dfrac{1}{\sqrt{a + (n-1)\,\Delta x}}\right)}$$

or $$\lim_{\Delta x \to 0} \Delta x\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a + \Delta x}} + \ldots + \frac{1}{\sqrt{a + (n-1)\,\Delta x}}\right)$$

$$= \lim_{\Delta x \to 0} S\left(P, \frac{1}{\sqrt{x}}\right) = \lim_{\Delta x \to 0} 2\left[\sqrt{a + n\,\Delta x} - \sqrt{a}\right]$$

Hence $$\int_a^b \frac{dx}{\sqrt{x}} = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} 2\left[\sqrt{a + \frac{n\,(b-a)}{n}} - \sqrt{a}\right]$$

$$= 2\,(\sqrt{a + b - a} - \sqrt{a}\,) = 2\,(\sqrt{b} - \sqrt{a}\,)$$

5. $\int_a^b \sin x\, dx$

Sol. Let $\Delta x = \dfrac{b-a}{n}$. Then taking left endpoint of each subinterval as c_r, we have

$S\,(P, f) = S\,(P, \sin x) = \Delta x\,[\sin a + \sin\,(a + \Delta x) + \sin\,(a + 2\Delta x) + \ldots + \sin\,(a + (n-1)\,\Delta x)]$

$$= \Delta x \frac{\sin\left(a + \dfrac{n-1}{2}\,\Delta x\right)\sin\dfrac{n\,\Delta x}{2}}{\sin\dfrac{\Delta x}{2}} \qquad \text{(From Trignometry)}$$

$$= \frac{\dfrac{\Delta x}{2}\left[\cos\left(a - \dfrac{\Delta x}{2}\right) - \cos\left(a + \dfrac{2n-1}{2}\,\Delta x\right)\right]}{\sin\dfrac{\Delta x}{2}}$$

Taking limits of both sides as $n \to \infty$ and $\Delta x \to 0$, we get

$$\int_a^b \sin x\, dx = \lim_{\Delta x \to 0} \frac{\dfrac{\Delta x}{2}}{\sin\dfrac{\Delta x}{2}} \times \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \left[\cos\left(a - \frac{\Delta x}{2}\right) - \cos\left(a + \frac{2n-1}{2}\cdot\frac{b-a}{n}\right)\right]$$

$$= 1 \times [\cos a - \cos\,[(a + (b - a))] = \cos a - \cos b$$

6. $\int_a^b \sin^2 x\, dx$

Sol. Let $\Delta x = \dfrac{b-a}{n}$

Then taking left endpoint of each subinterval as c_r, we have

$S\,(P, f) = S\,(P, \sin^2 x)$

$= \Delta x\,[\sin^2 a + \sin^2(a + \Delta x) + \sin^2(a + 2\Delta x) + \ldots + \sin^2(a + (n-1)\,\Delta x)]$

$$= \Delta x\left[\frac{1 - \cos 2a}{2} + \frac{1 - \cos\,(2a + 2\Delta x)}{2} + \frac{1 - \cos\,(2a + 4\Delta x)}{2} + \ldots + \frac{1 - \cos\,(2a + (2n-2)\,\Delta x)}{2}\right]$$

$$= \Delta x\left[\frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}, n \text{ terms}\right] - \frac{\Delta x}{2}\,[\cos 2a + \cos\,(2a + 2\Delta x) + \cos\,(2a + 4\Delta x) + \ldots + (2a + (2n-2)\,\Delta x)]$$

$$= \frac{n\,\Delta x}{2} - \frac{\dfrac{\Delta x}{2}\,[\cos\,(2a + (n-1)\,\Delta x)\sin n\,\Delta x}{\sin \Delta x}$$

$$= \frac{n\,\Delta x}{2} - \frac{\Delta x}{\sin \Delta x}\cdot\frac{1}{4}\,[\sin\,(2a + (2n-1)\,\Delta x) - \sin\,(2a - \Delta x)]$$

$$= \frac{n}{2}\cdot\frac{b-a}{n} - \frac{\Delta x}{\sin \Delta x}\cdot\frac{1}{4}\left[\sin\left(2a + (2n-1)\,\frac{b-a}{n}\right) - \sin\,(2a - \Delta x)\right]$$

Taking limit as $n \to \infty$ and $\Delta x \to 0$, we have

$$\int_a^b \sin^2 x\, dx = \frac{b-a}{2} - 1\cdot\frac{1}{4}\,[\sin\,(2a + 2\,(b-a)) - \sin 2a]$$

$$= \frac{b-a}{2} - \frac{1}{4}\,[\sin 2b - \sin 2a]$$

7. $\int_a^b \cosh x\, dx$

Sol. We have, $S\,(P, f) = S\,(P, \cosh x)$

$= \Delta x[\cosh a + \cosh\,(a + \Delta x) + \cosh\,(a + 2\Delta x) + \ldots + \cosh\,(a + (n-1)\,\Delta x)]$

$$= \Delta x\left[\frac{e^a+e^{-a}}{2}+\frac{e^{a+\Delta x}+e^{-(a+\Delta x)}}{2}+\ldots+\frac{e^{a+(n-1)\Delta x}+e^{-(a+(n-1)\Delta x)}}{2}\right]$$

$$= \frac{\Delta x}{2}[e^a+e^{a+\Delta x}+\ldots+e^{a+(n-1)\Delta x}]+\frac{\Delta x}{2}[e^{-a}+e^{-(a+\Delta x)}+\ldots+e^{-(a+(n-1)\Delta x)}]$$

$$= \frac{\Delta x}{2}e^a[1+e^{\Delta x}+e^{2\Delta x}+\ldots+e^{(n-1)\Delta x}]+\frac{\Delta x}{2}e^{-a}[1+e^{-\Delta x}+e^{-2\Delta x}+\ldots+e^{-(n-1)\Delta x}]$$

$$= \frac{\Delta x}{2}\cdot e^a\left[\frac{1-e^{n\Delta x}}{1-e^{\Delta x}}\right]+\frac{\Delta x}{2}\cdot e^{-a}\left[\frac{1-e^{-n\Delta x}}{1-e^{-\Delta x}}\right]$$

$$= \frac{e^a}{2}(1-e^{n\Delta x})\cdot\frac{\Delta x}{1-e^{\Delta x}}+\frac{e^{-a}}{2}(1-e^{-n\Delta x})\cdot\frac{\Delta x}{1-e^{-\Delta x}}$$

$$= \frac{e^a}{2}(1-e^{b-a})\cdot(-1)\cdot\frac{\Delta x}{e^{\Delta x}-1}+\frac{e^{-a}}{2}(1-e^{-b+a})\cdot\frac{-\Delta x}{e^{-\Delta x}-1}$$

Taking limits as $\Delta x \to 0$, we get

$$\int_a^b \cosh x\,dx = \frac{e^a-e^b}{2}\times(-1)\cdot 1+\frac{e^{-a}-e^{-b}}{2}\times 1$$

$$= \frac{1}{2}[-e^a+e^b+e^{-a}-e^{-b}] = \frac{e^b-e^{-b}}{2}-\frac{e^a-e^{-a}}{2}$$

$$= \sinh b - \sinh a$$

8. $\int_0^{\pi/2} \cos x\,dx$

Sol. We subdivide $\left[0, \frac{\pi}{2}\right]$ into n subintervals each of length $\Delta x = \frac{\pi}{2n}$

Then $S(P, f) = S(P, \cos x)$

$$= \Delta x\,[\cos 0 + \cos\Delta x + \cos 2\Delta x + \ldots + \cos(n-1)\Delta x]$$

where c_r has been taken as left endpoint of each subinterval.

$$= \Delta x\,\frac{\cos\left(0+\frac{n-1}{2}\Delta x\right)\sin\frac{n\Delta x}{2}}{\sin\frac{\Delta x}{2}} = \frac{\frac{\Delta x}{2}\left[\sin\frac{2n-1}{2}\cdot\Delta x+\sin\frac{\Delta x}{2}\right]}{\sin\frac{\Delta x}{2}}$$

Taking limits of both sides as $n \to \infty$ and $\Delta x \to 0$, we have

$$\int_0^{\pi/2}\cos x\,dx = \lim_{\Delta x\to 0}\frac{\frac{\Delta x}{2}}{\sin\frac{\Delta x}{2}}\times\lim_{\substack{\Delta x\to 0\\ n\to\infty}}\left(\sin\frac{2n-1}{2}\cdot\Delta x+\sin\frac{\Delta x}{2}\right)$$

$$= 1\times\lim_{n\to\infty}\sin\left(\frac{2n-1}{2}\cdot\frac{\pi}{2n}\right)$$

$$= \lim_{n\to\infty}\sin\left(1-\frac{1}{2n}\right)\frac{\pi}{2} = \sin\frac{\pi}{2} = 1$$

Determine the limit of each of the following as $n \to \infty$ (Problems 9 – 15):

9. $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{n+n}$

Sol. The given limit is

$$\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\frac{1}{1+\frac{3}{n}}+\ldots+\frac{1}{1+\frac{n}{n}}\right]$$

$$= \int_0^1\frac{dx}{1+x} = [\ln(1+x]_0^1 = \ln 2 - \ln 1 = \ln 2$$

10. $\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+(n-1)^2}$

Sol. The given limit is

$$\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\ldots+\frac{1}{1+\left(\frac{n-1}{n}\right)^2}\right]$$

$$= \int_0^1\frac{dx}{1+x^2} = [\arctan x]_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}$$

11. $\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+\ldots+\frac{n}{(n+n)^2}$

Sol. The given limit is

$$\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\left(1+\frac{1}{n}\right)^2}+\frac{1}{\left(1+\frac{2}{n}\right)^2}+\ldots+\frac{1}{\left(1+\frac{n}{n}\right)^2}\right]$$

$$\int_0^1\frac{dx}{(1+x)^2} = \left[-\frac{1}{1+x}\right]_0^1 = -\frac{1}{2}+1 = \frac{1}{2}$$

12. $\frac{1}{n\sqrt{n}}[\sqrt{n+1}+\sqrt{n+2}+\ldots+\sqrt{n+n}]$

Sol. The given limit is

$$\lim_{n\to\infty}\frac{1}{n}\left[\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right]$$

$$= \int_0^1 \sqrt{1+x}\,dx = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}(2^{3/2}-1) = \frac{2}{3}(2\sqrt{2}-1)$$

13. $\dfrac{1}{n} + \dfrac{1}{\sqrt{n^2-1}} + \dfrac{1}{\sqrt{n^2-2^2}} + \ldots + \dfrac{1}{\sqrt{n^2-(n-1)^2}}$

Sol. The given limit is

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\sqrt{1-\left(\frac{0}{n}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{1}{n}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{2}{n}\right)^2}} + \ldots + \frac{1}{\sqrt{1-\left(\frac{n-1}{n}\right)^2}}\right]$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} = [\arcsin x]_0^1 = \arcsin 1 - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

14. $\left\{\left(1+\dfrac{1^2}{n^2}\right)\left(1+\dfrac{2^2}{n^2}\right)\left(1+\dfrac{3^2}{n^2}\right)\ldots\left(1+\dfrac{n^2}{n^2}\right)\right\}^{1/n}$

Sol. Let $y = \left\{\left(1+\dfrac{1}{n^2}\right)\left(1+\dfrac{2^2}{n^3}\right)\left(1+\dfrac{3^2}{n^2}\right) + \ldots + \left(1+\dfrac{n^2}{n^2}\right)\right\}^{1/n}$

$$\ln y = \frac{1}{n}\left[\ln\left(1+\frac{1}{n^2}\right) + \ln\left(1+\frac{2^2}{n^2}\right) + \ln\left(1+\frac{3^2}{n^2}\right) + \ldots + \ln\left(1+\frac{n^2}{n^2}\right)\right]$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{n}\left[\ln\left(1+\left(\frac{1}{n}\right)^2\right) + \ln\left(1+\left(\frac{2}{n}\right)^2\right)\right]$$

$$+ \ln\left(1+\left(\frac{3}{n}\right)^2\right) + \ldots + \left(1+\left(\frac{n}{n}\right)^2\right)\Bigg]$$

$$= \int_0^1 \ln(1+x^2)\,dx = [x\ln(1+x^2)]_0^1 - \int_0^1 x \times \frac{2x}{1+x^2}\,dx$$

$$= \ln 2 - 2\int_0^1 \frac{x^2}{1+x^2} = \ln 2 - 2\int_0^1 \left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \ln 2 - 2\,[x - \arctan x]_0^1 = \ln 2 - 2[1 - \arctan 1]$$

$$= \ln 2 - 2\left(1 - \frac{\pi}{4}\right) = \ln 2 - 2 + \frac{\pi}{2} = \ln 2 + \left(\frac{\pi}{2} - 2\right)\ln e$$

$$\lim_{n\to\infty} y = \ln 2 + \ln e^{\frac{\pi}{2}-2} = \ln\left(2e^{\frac{\pi}{2}-2}\right)$$

Therefore, $y = 2\,e^{(\pi-4)/2}$

15. $\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)^{1/2}\left(1+\dfrac{3}{n}\right)^{1/3}\ldots\left(1+\dfrac{n}{n}\right)^{1/n}$

Sol. $\ln y = \ln\left(1+\dfrac{1}{n}\right) + \dfrac{1}{2}\ln\left(1+\dfrac{2}{n}\right) + \ldots + \dfrac{1}{n}\ln\left(1+\dfrac{n}{n}\right)$

$$= \sum_{r=1}^{n} \frac{1}{r}\ln\left(1+\frac{r}{n}\right) = \sum_{r=1}^{n} \frac{1}{n}\cdot\frac{n}{r}\ln\left(1+\frac{r}{n}\right)$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}\left(\frac{n}{r}\ln\left(1+\frac{r}{n}\right)\right)$$

$$= \int_0^1 \frac{1}{x}\ln(1+x)\,dx = \int_0^1 \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)dx$$

$$= \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)dx = \left[x - \frac{x^2}{2^2} + \frac{x^3}{3^3} - \frac{x^4}{4^2} + \ldots\right]_0^1$$

$$= 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\right)$$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) - 2 \times \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$= \frac{\pi^2}{6} - \frac{1}{2}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{6} - \frac{\pi^2}{12} = \frac{\pi^2}{12}$$

Thus, $\lim\limits_{n\to\infty} y = e^{\frac{\pi^2}{12}}$.

Exercise Set 5.2 (Page 181)

Evaluate the following integrals (Problems 1 – 11):

1. $\displaystyle\int_0^6 f(x)\,dx$, where $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x-2 & \text{if } x \geq 2 \end{cases}$

Sol. $\displaystyle\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^6 (3x-2)\,dx$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^6 = \frac{8}{3} + (54-12) - (6-4)$$

$$= \frac{8}{3} + 42 - 2 = \frac{8}{3} + 40 = \frac{128}{3}$$

2. $\int_{-1}^{5} |x-2|\, dx$

Sol. $\int_{-1}^{5} |x-2|\, dx = \int_{-1}^{2} |x-2|\, dx + \int_{2}^{5} |x-2|\, dx$

$= \int_{-1}^{2} -(x-2)\, dx + \int_{2}^{5} (x-2)\, dx$

$= \left[-\frac{x^2}{2} + 2x\right]_{-1}^{2} + \left[\frac{x^2}{2} - 2x\right]_{2}^{5}$

$= \left[-2 + 4 - \left(\frac{-1}{2} - 2\right)\right] + \left[\frac{25}{2} - 10 - (2-4)\right]$

$= \left(2 + \frac{5}{2}\right) + \left(\frac{5}{2} + 2\right) = \frac{9}{2} + \frac{9}{2} = 9$

3. $\int_{0}^{3\pi/4} |\cos x|\, dx$

Sol. $\int_{0}^{3\pi/4} |\cos x|\, dx = \int_{0}^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{3\pi/4} |\cos x|\, dx$

$= \int_{0}^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/4} (-\cos x)\, dx$

$= [\sin x]_{0}^{\pi/2} + [-\sin x]_{\pi/2}^{3\pi/4}$

$= \left(\sin\frac{\pi}{2} - \sin 0\right) + \left[\left(-\sin\frac{3\pi}{4}\right) - \left(-\sin\frac{\pi}{2}\right)\right]$

$= [1-0] + \left(-\frac{1}{\sqrt{2}}\right) - (-1) = 1 - \frac{1}{\sqrt{2}} + 1 = 2 - \frac{1}{\sqrt{2}}$

4. $\int_{0}^{\pi} \cos^{2n+1} x\, dx$

Sol. $\int_{0}^{\pi} \cos^{2n+1} x\, dx = \int_{0}^{\pi/2} \cos^{2n+1} x\, dx + \int_{0}^{\pi/2} \cos^{2n+1} (\pi - x)\, dx$, by Theorem 5.10

$= \int_{0}^{\pi/2} \cos^{2n+1} x\, dx + \int_{0}^{\pi/2} (-\cos x)^{2n+1}\, dx$

$= \int_{0}^{\pi/2} \cos^{2n+1} x\, dx - \int_{0}^{\pi/2} \cos^{2n+1} x\, dx$ (since $(-\cos x)^{2n+1} = -\cos^{2n+1} x$))

$= 0$

Alternative Method:

$I = \int_{0}^{\pi} \cos^{2n+1} x\, dx = \int_{0}^{\pi} \cos^{2n+1} (\pi - x)\, dx$, by Theorem 5.9

$= \int_{0}^{\pi} (-\cos x)^{2n+1}\, dx = -\int_{0}^{\pi} \cos^{2n+1} x\, dx$

or $\quad 2I = 0 \Rightarrow I = 0$

5. $\int_{0}^{\pi/4} \frac{\sec^2 \theta\, dx}{\tan x - \tan \theta}, \theta > \frac{\pi}{4}$

Sol. Put $\tan x = z$ so that $\sec^2 x\, dx = dz$ or $dx = \frac{1}{1+z^2} dz$

When $x = 0,\ z = 0$ and when $x = \frac{\pi}{4}, z = 1$. Then

$I = \int_{0}^{\pi/4} \frac{\sec^2 \theta\, dx}{\tan x - \tan \theta} = \sec^2 \theta \int_{0}^{1} \frac{dz}{(z - \tan \theta)(1 + z^2)}$

$= \sec^2 \theta \int_{0}^{1} \left[\frac{\frac{1}{\sec^2\theta}}{z - \tan\theta} + \frac{-\frac{1}{\sec^2\theta} \cdot z - \frac{\tan\theta}{\sec^2\theta}}{1+z^2}\right] dz$

$= \int_{0}^{1} \left(\frac{dz}{z - \tan\theta} - \frac{z\, dz}{1+z^2} - \frac{\tan\theta}{1+z^2}\right) dz$

$= \ln\left[|z - \tan\theta|\right]_0^1 - \left[\frac{1}{2}\ln(1+z^2)\right]_0^1 - [\tan\theta \arctan z]_0^1$

$I = |1 - \tan\theta| - \ln|0 - \tan\theta| - \frac{1}{2}(\ln 2 - \ln 1) - \tan\theta \cdot \left(\frac{\pi}{4} - 0\right)$

$I = \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| - \ln\sqrt{2} - \frac{\pi}{4}\tan\theta$

Alternative Answer:

$$\frac{\tan\theta-1}{\tan\theta}=\frac{\dfrac{\sin\theta}{\cos\theta}-\dfrac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}}{\dfrac{\sin\theta}{\cos\theta}}=\frac{\sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4}}{\sin\theta\cos\frac{\pi}{4}}=\frac{\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)}{\sin\theta}$$

Therefore, $I=\ln\left|\dfrac{\sqrt{2}\sin\left(\frac{\pi}{4}-\theta\right)}{\sin\theta}\right|-\ln\sqrt{2}-\dfrac{\pi}{4}\tan\theta$

$$=\ln\sqrt{2}+\ln\left|\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\sin\theta}\right|-\ln\sqrt{2}-\frac{\pi}{4}\tan\theta$$

$$=\ln\left|\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\sin\theta}\right|-\frac{\pi}{4}\tan\theta.$$

6. $\displaystyle\int_0^{\pi/2}\tan x\ln(\sin x)\,dx$

Sol. $I=\displaystyle\int_0^{\pi/2}\tan x\ln(\sin x)\,dx=\int_0^{\pi/2}\frac{\sin x}{\cos x}\ln\sqrt{1-\cos^2 x}\,dx.$

Put $z=\cos x$ so that $dz=-\sin x\,dx$.

When $x=0, z=1$ and when $x=\frac{\pi}{2}, z=0$. Then

$$I=-\int_1^0\frac{\ln\sqrt{1-z^2}}{z}\,dz=\frac{1}{2}\int_0^1\frac{\ln(1-z^2)}{z}\,dz$$

$$=\frac{1}{2}\int_0^1\frac{1}{z}\left(-z^2-\frac{z^4}{2}-\frac{z^6}{3}-\ldots\right)dz=-\frac{1}{2}\int_0^1\left(z+\frac{z^3}{2}+\frac{z^5}{3}+\ldots\right)dz$$

$$=-\frac{1}{2}\left[\frac{z^2}{2}+\frac{z^4}{2\cdot4}+\frac{z^6}{3\cdot6}+\ldots\right]_0^1=-\frac{1}{2}\left[\frac{1}{2}+\frac{1}{2\cdot4}+\frac{1}{3\cdot6}+\ldots\right]$$

$$=-\frac{1}{2^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right]=-\frac{1}{4}\left(\frac{\pi^2}{6}\right)=-\frac{\pi^2}{24}$$

7. $\displaystyle\int_0^{2\pi}\frac{dx}{5+3\cos x}$

Sol. $\displaystyle\int_0^{2\pi}\frac{dx}{5+3\cos x}=2\int_0^{\pi}\frac{dx}{5+3\cos(2\pi-x)}$, by Theorem 5.11 (*i*)

$$=2\int_0^{\pi}\frac{dx}{5+3\cos x}=\frac{2}{\sqrt{5^2-3^2}}\left|\arccos\frac{3+5\cos x}{5+3\cos x}\right|_0^{\pi}$$

$$=\frac{1}{2}\left[\arccos\frac{3-5}{5-3}-\arccos\frac{3+5}{5+3}\right]$$

$$=\frac{1}{2}[\arccos-1)-\arccos(1)]$$

$$=\frac{1}{2}[\pi-0]=\frac{\pi}{2}$$

8. $\displaystyle\int_0^1\arctan\left(\frac{2x-1}{1+x-x^2}\right)dx$

Sol. $I=\displaystyle\int_0^1\arctan\left(\frac{2x-1}{1+x-x^2}\right)dx$

$$=\int_0^1\arctan\frac{2(1-x)-1}{1+(1-x)-(1-x)^2}\,dx,\quad\text{by Theorem 5.9}$$

$$=\int_0^1\arctan\frac{1-2x}{1+x-x^2}=-\int_0^1\arctan\frac{2x-1}{1+x-x^2}=-I$$

or $\quad 2I=0\quad$ *i.e.*, $\quad I=0$

9. $\displaystyle\int_{-\pi/4}^{\pi/4}\frac{2x^3-x}{(x^2+1)(x-1)(x+1)}\,dx$

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \frac{2x^3 - x}{(x^2+1)(x-1)(x+1)} dx$ · Then

$$I = \int_{-\pi/4}^{0} \frac{2x^3-x}{(x^2+1)(x-1)(x+1)} dx + \int_{0}^{\pi/4} \frac{2x^3-x}{(x^2+1)(x-1)(x+1)} dx, \text{ by Theorem 5.8}$$

$= I_1 + I_2$

Now putting $x = -t$ i.e., $dx = -dt$ in I_1, we have

$$I_1 = \int_{\pi/4}^{0} \frac{2(-t)^3 - (-t)}{((-t)^2+1)(-t-1)(-t+1)} \cdot -dt$$

$$\left(\text{since } t = \frac{\pi}{4} \text{ when } x = -\frac{\pi}{4} \text{ and } t = 0 \text{ when } x = 0\right)$$

$$= \int_{\pi/4}^{0} \frac{-2t^3 + t}{(t^2+1)(t+1)(t-1)} \cdot - dt = \int_{\pi/4}^{0} \frac{2t^3 - t}{(t^2+1)(t-1)(t+1)} dt$$

$$= -\int_{0}^{\pi/4} \frac{2t^3 - t}{(t^2+1)(t-1)(t+1)} dt, \qquad \text{by Theorem 5.7}$$

$$= -\int_{0}^{\pi/4} \frac{2x^3 - x}{(x^2+1)(x-1)(x+1)} dx, \qquad \text{by Theorem 5.6}$$

Thus $I = -\int_{0}^{\pi/4} \frac{2x^3 - x}{(x^2+1)(x-1)(x+1)} dx + \int_{0}^{\pi/4} \frac{2x^3-x}{(x^2+1)(x-1)(x+1)} dx = 0$

10. $\int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

Sol. By **(5.9)**, we have $I = \int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_{0}^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1 + \cos^2 (\pi - x)} dx$

$$= \int_{0}^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx = \int_{0}^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx - \int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

or $\quad 2I = \int_{0}^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx$

Put $\cos x = z$ so that $-\sin x \, dx = dz$

$z = 1$ when $x = 0$; $z = -1$ when $x = \pi$

$$2I = -\int_{1}^{-1} \frac{\pi \, dz}{1 + z^2} = \pi \int_{-1}^{1} \frac{dz}{1 + z^2} = \pi [\arctan z]_{-1}^{1}$$

$$= \pi [\arctan (1) - \arctan (-1)] = \pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= \pi \cdot \frac{\pi}{2} \quad \text{or} \quad I = \frac{\pi^2}{4}$$

11. $\int_{2}^{4} \frac{\sqrt{\ln (9 - x)}}{\sqrt{\ln (9 - x)} + \sqrt{\ln (3 + x)}} dx$

Sol. $I = \int_{2}^{4} \frac{\sqrt{\ln (9 - x)}}{\sqrt{\ln (9 - x)} + \sqrt{\ln (3 + x)}} dx \qquad (1)$

Put $9 - x = 3 + y$ or $x = 6 - y$ so that $dx = -dy$

When $x = 2, y = 4$ and when $x = 4, y = 2$

$$I = \int_{4}^{2} \frac{\sqrt{\ln (3 + y)} (-dy)}{\sqrt{\ln (3 + y)} + \sqrt{\ln (9 - y)}} = -\int_{4}^{2} \frac{\sqrt{\ln (3 + y)} \, dy}{\sqrt{\ln (3 + y)} + \sqrt{\ln (9 - y)}}$$

$$= \int_{2}^{4} \frac{\sqrt{\ln (3 + y)} \, dy}{\sqrt{\ln (9 - y)} + \sqrt{\ln (3 + y)}} = \int_{2}^{4} \frac{\sqrt{\ln (3 + x)} \, dx}{\sqrt{\ln (9 - x)} + \sqrt{\ln (3 + x)}} \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_{2}^{4} \frac{\sqrt{\ln (9 - x)} + \sqrt{\ln (3 + x)}}{\sqrt{\ln (9 - x)} + \sqrt{\ln (3 + x)}} dx$$

$$= \int_2^4 1 \cdot dx = [x]_2^4 = 4 - 2 = 2 \quad \text{or} \quad I = 1$$

In Problems 12 – 26, show that:

12. $\displaystyle\int_0^{\pi/2} \ln(\tan x)\, dx = 0$

Sol. $$\int_0^{\pi/2} \ln(\tan x)\, dx = \int_0^{\pi/2} \ln\left(\frac{\sin x}{\cos x}\right) dx = \int_0^{\pi/2} \ln(\sin x)\, dx - \int_0^{\pi/2} \ln(\cos x)\, dx$$

$$= \int_0^{\pi/2} \ln\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx - \int_0^{\pi/2} \ln(\cos x)\, dx$$

$$= \int_0^{\pi/2} \ln(\cos x)\, dx - \int_0^{\pi/2} \ln(\cos x)\, dx = 0$$

13. $\displaystyle\int_0^{\pi/2} \sin 2x \ln(\tan x)\, dx = 0$

Sol. $$\int_0^{\pi/2} \sin 2x \ln(\tan x)\, dx = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \ln \tan\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \sin(\pi - 2x) \ln \cot x\, dx$$

$$= \int_0^{\pi/2} \sin 2x \ln \cot x\, dx$$

$$= \int_0^{\pi/2} \sin 2x \ln(\tan x)^{-1}\, dx$$

$$= -\int_0^{\pi/2} \sin 2x \ln \tan x\, dx$$

i.e., $\displaystyle 2\int_0^{\pi/2} \sin 2x \ln \tan x\, dx = 0$

or $\displaystyle\int_0^{\pi/2} \sin 2x \ln \tan x\, dx = 0$ as required

14. $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\, dx = \frac{\pi^2}{4}$

Sol. $$I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\, dx = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\, dx = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x}\, dx - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$$

or $$2I = \pi \int_0^{\pi} \frac{\sin x\, dx}{1 + \cos^2 x} dx$$

Put $\cos x = t$ in the integral on R.H.S. so that $-\sin x\, dx = dt$.
When $x = 0, t = 1$ and when $x = \pi, t = -1$.

$$2I = -\pi \int_1^{-1} \frac{dt}{1 + t^2} = \pi \int_{-1}^{1} \frac{dt}{1 + t^2} = 2\pi \int_0^1 \frac{dt}{1 + t^2}$$

$$= 2\pi\, |\arctan t|_0^1 = 2\pi\left(\frac{\pi}{4}\right) = \frac{\pi^2}{2} \quad \text{or} \quad I = \frac{\pi^2}{4}$$

15. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \frac{\pi}{4}$

Sol. Let $$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx = \int_0^{\pi/2} 1 \cdot dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

Hence $I = \frac{\pi}{4}$

16. $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x\,dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)$

Sol. Let $\displaystyle I = \int_0^{\pi/2} \frac{\sin^2 x\,dx}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\sin^2 x\left(\frac{\pi}{2} - x\right)dx}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$

$$= \int_0^{\pi/2} \frac{\cos^2 x\,dx}{\cos x + \sin x}$$

$$2I = \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$= \int_0^{\pi/2} \frac{\frac{1}{\sqrt{2}}\,dx}{\frac{1}{\sqrt{2}} \cdot \sin x + \frac{1}{\sqrt{2}} \cos x} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \csc\left(x + \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}} \left| \ln \tan\left(\frac{x}{2} + \frac{\pi}{8}\right) \right|_0^{\pi/2} = \frac{1}{\sqrt{2}} \left[\ln \tan\frac{3\pi}{8} - \ln \tan\frac{\pi}{8} \right]$$

$$= \frac{1}{\sqrt{2}} \ln \frac{\tan\frac{3\pi}{8}}{\tan\frac{\pi}{8}} = \frac{1}{\sqrt{2}} \ln \frac{\sin\frac{3\pi}{8}\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}\cos\frac{3\pi}{8}} = \frac{1}{\sqrt{2}} \ln \frac{\sin\frac{\pi}{2} + \sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} \ln \frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}} \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1}$$

$$= \frac{1}{\sqrt{2}} \ln \frac{(\sqrt{2} + 1)(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)}$$

$$= \frac{1}{\sqrt{2}} \ln (\sqrt{2} + 1)^2 = \sqrt{2} \ln(\sqrt{2} + 1)$$

Hence $I = \frac{\sqrt{2}}{2} \ln(\sqrt{2} + 1) = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)$

17. $\displaystyle\int_0^{\pi} \frac{x\,dx}{1 + \sin x} = \pi$

Sol. Let $\displaystyle I = \int_0^{\pi} \frac{x\,dx}{1 + \sin x}$. Then by Theorem 5.9, we have

$$I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin(\pi - x)} = \int_0^{\pi} \frac{\pi - x}{1 + \sin x}\,dx$$

$$= \int_0^{\pi} \frac{\pi}{1 + \sin x} \cdot dx - \int_0^{\pi} \frac{x}{1 + \sin x}\,dx = \pi \int_0^{\pi} \frac{1}{1 + \sin x}\,dx - I$$

or $\displaystyle 2I = \pi \int_0^{\pi} \frac{1}{1 + \sin x}\,dx \Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1 + \sin x}\,dx$ (1)

Now $\displaystyle\int_0^{\pi} \frac{1}{1 + \sin x}\,dx = 2\int_0^{\pi/2} \frac{1}{1 + \sin x}\,dx$, by Theorem 5.11

$$= 2 \int_0^{\pi/2} \frac{1}{1 + \sin\left(\frac{\pi}{2} - x\right)}\,dx \qquad \text{by Theorem 5.9}$$

$$= 2\int_0^{\pi/2}\frac{1}{1+\cos x}dx = 2\int_0^{\pi/2}\frac{1}{2\cos^2\frac{x}{2}}dx = \int_0^{\pi/2}\sec^2\frac{x}{2}dx$$

$$= \int_0^{\pi/4}(\sec^2 t)\cdot 2dt \qquad \text{(by setting } \frac{x}{2} = t\text{, then } dx = 2dt\text{. When } x = \frac{\pi}{2},$$

$$t = \frac{\pi}{4}\text{, when } x = 0, t = 0$$

$$= 2\left[\tan t\right]_0^{\pi/4} = 2[1-0] = 2.$$

Thus $I = \int_0^{\pi}\frac{x\,dx}{1+\sin x} = \frac{\pi}{2}\cdot 2 = \pi$ (by using (1)

18. $\int_0^{\pi/2}\left(\frac{\theta}{\sin\theta}\right)^2 d\theta = \pi\ln 2$

Sol. $\int_0^{\pi/2}\left(\frac{\theta}{\sin\theta}\right)^2 d\theta = \int_0^{\pi/2}\theta^2(\operatorname{cosec}^2\theta)\,d\theta = [\theta^2(-\cot\theta)]_0^{\pi/2} - \int_0^{\pi/2}(-\cot\theta)\cdot 2\theta\,d\theta$

$$= 0 + 2\int_0^{\pi/2}\theta\left(\frac{\cos\theta}{\sin\theta}\right)d\theta \quad \left(\because \left(\frac{\pi}{2}\right)^2\cot\frac{\pi}{2} = 0 \text{ and } \lim_{\theta\to 0}\frac{\theta^2\cos\theta}{\sin\theta} = 0\right)$$

$$= 2[\theta\cdot\ln\sin\theta]_0^{\pi/2} - 2\int_0^{\pi/2}(\ln\sin\theta)\cdot 1\,d\theta$$

$$= 0 - 2\int_0^{\pi/2}\ln\sin\theta\,d\theta \left(\because \frac{\pi}{2}\sin\frac{\pi}{2} = 0 \text{ and } \lim_{\theta\to 0}\frac{\ln\sin\theta}{1/\theta} = \lim_{\theta\to 0}\left(-\frac{\theta^2\cos\theta}{\sin\theta}\right) = 0\right)$$

$$= -2\left(-\frac{\pi}{2}\ln 2\right) = \pi\ln 2, \text{ (see Example 9)}$$

19. $\int_0^{\pi/2}\ln(\tan\theta + \cot\theta)\,d\theta = \pi\ln 2$

Sol. Let $I = \int_0^{\pi/2}\ln(\tan\theta+\cot\theta)\,d\theta = \int_0^{\pi/2}\ln\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)d\theta$

$$= \int_0^{\pi/2}\ln\frac{1}{\sin\theta\cos\theta}d\theta = -\int_0^{\pi/2}\ln\sin\theta\,d\theta - \int_0^{\pi/2}\ln\cos\theta\,d\theta$$

$$= -\int_0^{\pi/2}\ln\sin\theta\,d\theta - \int_0^{\pi/2}\ln\cos\left(\frac{\pi}{2}-\theta\right)d\theta$$

$$= -2\int_0^{\pi/2}\ln\sin\theta\,d\theta = -2\left(\frac{-\pi}{2}\ln 2\right) = \pi\ln 2.$$

20. $\int_0^{\pi}x\ln(\sin x)\,dx = \frac{\pi^2}{2}\ln\left(\frac{1}{2}\right)$

Sol. Let $I = \int_0^{\pi}x\ln\sin x\,dx = \int_0^{\pi}(\pi-x)\ln\sin(\pi-x)\,dx$

$$= \int_0^{\pi}(\pi-x)\ln\sin x\,dx = \int_0^{\pi}\pi\ln\sin x\,dx - \int_0^{\pi}x\ln\sin x\,dx$$

or $\quad 2I = \int_0^{\pi}\pi\ln\sin x\,dx = \pi\int_0^{\pi}\ln\sin x\,dx$

$$= 2\pi\int_0^{\pi/2}\ln\sin x\,dx, \text{ by Theorem 5.11 } (i)$$

$$= 2\pi\cdot\left(-\frac{\pi}{2}\ln 2\right) = \pi^2\cdot(-1)\ln(2) = \pi^2\ln\frac{1}{2}$$

or $\quad I = \frac{\pi^2}{2}\ln\frac{1}{2}$

21. $\int_0^{1}\frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$

Sol. $I = \int_0^{1}\frac{\ln(1+x)}{1+x^2}dx$

Put $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\cdot \sec^2\theta\, d\theta = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

$$= \int_0^{\pi/4} \ln\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right], \text{ by Theorem 5.9}$$

$$= \int_0^{\pi/4} \ln\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta = \int_0^{\pi/4} \ln\left[\frac{2}{1+\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \ln 2\, d\theta - \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

or $$2\int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta = \ln 2 \int_0^{\pi/4} 1\cdot d\theta = \ln 2\,[\theta]_0^{\pi/4} = \frac{\pi}{4}\ln 2$$

or $$I = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta = \frac{\pi}{8}\ln 2$$

22. $$\int_0^{\pi/2} \sin x \ln(\sin x)\, dx = \ln\left(\frac{2}{e}\right)$$

Sol. $$I = \int_0^{\pi/2} \sin x \ln(\sin x)\, dx$$

$$= \int_0^{\pi/2} \ln\sqrt{1-\cos^2 x}\cdot \sin x\, dx$$

Put $z = \cos x$ so that, $dz = -\sin x\, dx$

When $x = 0$, $z = 1$ and when $x = \frac{\pi}{2}$, $z = 0$. Then

$$I = -\int_1^0 \ln\sqrt{1-z^2}\, dz = \frac{1}{2}\int_0^1 \ln(1-z^2)\, dz$$

$$= -\frac{1}{2}\int_0^1 \left(z^2+\frac{z^4}{2}+\frac{z^6}{3}+\ldots\right) dz = -\frac{1}{2}\left[\frac{z^3}{3}+\frac{z^5}{2\cdot 5}+\frac{z^7}{3\cdot 7}+\ldots\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{3}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\ldots\right] = -\left[\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}+\ldots\right]$$

$$= -\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\ldots\right]$$

$$= -\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots$$

$$= \left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots\right]-1$$

$$= \ln 2 - 1 = \ln 2 - \ln e = \ln\frac{2}{e}$$

23. $$\int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}\, dx = \frac{\pi}{4}$$

Sol. Let $$I = \int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}\, dx = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin x}{\cos x+\sin x}\, dx$$

or $$2I = \int_0^{\pi/2} \frac{\sin x+\cos x}{\cos x+\sin x}\, dx = \int_0^{\pi/2} 1\cdot dx = \frac{\pi}{2}$$

or $$I = \int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}\, dx = \frac{\pi}{4}$$

24. $$\int_0^{\pi} \frac{x\sin x}{1+\sin x}\, dx = \frac{\pi^2}{2}-\pi$$

Sol. Let $$I = \int_0^{\pi} \frac{x\sin x\, dx}{1+\sin x} = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)\, dx}{1+\sin(\pi-x)}$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x\, dx}{1+\sin x} = \int_0^{\pi} \frac{\pi\sin x}{1+\sin x}\, dx - \int_0^{\pi} \frac{x\sin x}{1+\sin x}\, dx$$

or $2I = \pi \int_0^\pi \frac{\sin x\, dx}{1+\sin x} = \pi \int_0^\pi \left(1 - \frac{1}{1+\sin x}\right) dx$

$$= \pi \int_0^\pi 1 \cdot dx - \pi \int_0^\pi \frac{dx}{1+\sin x} = \pi [x]_0^\pi - \pi \int_0^\pi \frac{dx}{1+\sin x}$$

$$= \pi \cdot (\pi - 0) - \pi \cdot \int_0^\pi \frac{dx}{1+\sin x}$$

$$= \pi^2 - \pi \cdot 2 \qquad \left(\int_0^\pi \frac{dx}{1+\sin x} = 2 \quad \text{by Problem 17}\right)$$

Therefore $I = \frac{\pi^2}{2} - \pi$.

25. $\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x} = 0$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x} dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)} dx$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x \sin x} dx = -I$$

or $2I = 0$ *i.e.,* $I = 0$

26. $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{1+\sin x \cos x} = \frac{\pi}{3\sqrt{3}}$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin^2 x\, dx}{1+\sin x \cos x}$

$$= \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right) dx}{1+\sin\left(\frac{\pi}{2}-x\right) \text{cod}\left(\frac{\pi}{2}-x\right)} = \int_0^{\pi/2} \frac{\cos^2 x}{1+\sin x \cos x} dx$$

Therefore, $2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{1+\sin x \cos x} dx = \int_0^{\pi/2} \frac{dx}{1+\frac{1}{2}\sin 2x}$

$$2I = \int_0^{\pi/4} \frac{dx}{1+\frac{1}{2}\sin 2x} + \int_{\pi/4}^{\pi/2} \frac{dx}{1+\frac{1}{2}\sin 2x}$$

Putting $x = \frac{\pi}{2} - z$ in $\int_{\pi/4}^{\pi/2} \frac{dx}{1+\frac{1}{2}\sin 2x}$, we can prove that

$$\int_{\pi/4}^{\pi/2} \frac{dx}{1+\frac{1}{2}\sin 2x} = \int_0^{\pi/4} \frac{dx}{1+\frac{1}{2}\sin 2x}$$

Therefore $2I = 2\int_0^{\pi/4} \frac{dx}{1+\frac{1}{2}\sin 2x}$ or $I = \int_0^{\pi/4} \frac{dx}{1+\frac{1}{2}\sin 2x}$

Now we put $\tan x = t$ so that $\sec^2 x\, dx = dt$ or $dx = \frac{1}{1+t^2} dt$ and $\sin 2x = \frac{2t}{1+t^2}$. When $x = 0$, $t = 0$ and when $x = \frac{\pi}{4}$, $t = 1$.

Thus $I = \int_0^1 \frac{1}{1+\frac{1}{2}\cdot\frac{2t}{1+t^2}} \cdot \frac{1}{1+t^2} dt = \int_0^1 \frac{dt}{t^2+t+1}$

$$I = \int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dt = \frac{2}{\sqrt{3}}\left[\arctan\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$$

$$I = \frac{2}{\sqrt{3}}\left[\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right] = \frac{2}{\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{2}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

27. Let f and g be integrable on $[a, b]$ and suppose that $f(x) \le g(x)$ for all $x \in [a, b]$. Show that $\int_a^b f(x)\,dx \le \int_a^a g(x)\,dx$.

Sol. Let F and G be antiderivatives of f and g respectively. Then $F' = f$ and $G' = g$ and $-F + G$ is antiderivative of $-f + g$.

We first show that if $f(x) \ge 0$, then $\int_a^b f(x) \ge 0$.

As $F'(x) = f(x) \ge 0$ on $[a, b]$, so, F is an increasing function on $[a, b] \Rightarrow F(b) \ge F(a)$. Therefore,

$$\int_a^b f(x)\,dx = F(b) - F(a) \ge 0$$

Now
$$\int_a^b [g(x) - f(x)]\,dx = [G(x) - F(x)]_a^b$$
$$= G(b) - G(a) - [F(b) - F(a)]$$
$$= \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$

But $g(x) - f(x) \ge 0$. Therefore,

$$\int_a^b [g(x) - f(x)]\,dx = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \ge 0$$

$$\Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Alternative Method:

Let $P = [a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b]$ be a partition of $[a, b]$. Since $f(x) \le g(x)$ on $[a, b]$, we have $f(x) \le g(x)$ on each subinterval Δx_i. Hence $\sum_{i=1}^{n} f(c_i)\,\Delta x_i \le \sum_{i=1}^{n} g(c_i)\,\Delta x_i$, $c_i \in \Delta x_i$

Taking limits as $n \to \infty$, $\|P\| \to 0$, we get $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Exercise Set 5.3 (Page 189)

Determine whether the following improper integral converge. Evaluate the integrals that converge (Problems 1 – 33):

1. $\int_0^\infty e^{-x}\,dx$

Sol. $\int_0^t e^{-x}\,dx = [-e^{-x}]_0^t = -e^{-t} + 1$

Therefore, $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx$

$= \lim_{t\to\infty}(-e^{-t} + 1) = 1$ and the given integral converges.

2. $\int_0^\infty e^{-x}\sin x\,dx$

Sol. $\int e^{-x}\sin x\,dx = e^{-x}(-\cos x) - \int(-\cos x)(-e^{-x})\,dx$

$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx$

$= -e^{-x}\cos x - [e^{-x}(\sin x) - \int \sin x\cdot(-e^{-x})\,dx$

$= -e^{-x}\cos - e^{-x}\sin x - \int e^{-x}\sin x\,dx$

Therefore, $\int e^{-x}\sin x\,dx = \frac{-e^{-x}}{2}(\sin x + \cos x)$

and $\int_0^\infty e^{-x}\sin x\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\sin x\,dx$

$= \lim_{t\to\infty}\left[\frac{-e^{-x}}{2}(\sin x + \cos x)\right]_0^t = \lim_{t\to\infty}\left[-\frac{e^{-t}(\sin t + \cos t)}{2} + \frac{1}{2}\right]$

$= \frac{1}{2}$ Thus the given integral converges

3. $\int_{-\infty}^{0}\frac{dx}{1+x^2}$

Sol. $\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{1+x^2} = \lim_{t\to-\infty} [\arctan t]_t^0 = \lim_{t\to-\infty} [-\arctan t]$

$= -(\lim_{t\to-\infty} \arctan t) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

Thus the given integral converges.

4. $\int_0^\infty e^{-2x}\cos 2x\, dx$

Sol. $\int_0^\infty e^{-2x}\cos 2x\, dx = \frac{e^{-2x}}{8}[-2\cos 2x + 2\sin 2x]$, by Example 13 Page 140.

Hence $\int_0^\infty e^{-2x}\cos 2x\, dx = \lim_{t\to\infty} \int_0^t e^{-2x}\cos 2x\, dx$

$= \lim_{t\to\infty}\left[\frac{e^{-2x}}{8}(-2\cos 2x + 2\sin 2x)\right]_0^t$

$= \lim_{t\to\infty}\left[\frac{e^{-2t}}{8}(-2\cos 2t + 2\sin 2t) + \frac{1}{4}\right]$

$= \frac{1}{4}$. (Since $\lim_{t\to\infty} e^{-2t} = \lim_{t\to\infty} \frac{1}{e^{2t}} = 0$)

Thus the given integral converges.

5. $\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$

Sol. $\int_{-\infty}^{0} \frac{dx}{(2x-1)^3} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{(2x-1)^3} = \lim_{t\to-\infty}\left[\frac{-1}{4(2x-1)^2}\right]_t^0$

$= \lim_{t\to-\infty}\left[-\frac{1}{4} + \frac{1}{4(2t-1)^2}\right] = -\frac{1}{4}$

Thus the given integral converges.

6. $\int_{-\infty}^{2} e^{2x}\, dx$

Sol. $\int_{-\infty}^{2} e^{2x}\, dx = \lim_{t\to-\infty} \int_t^2 e^{2x}\, dx = \lim_{t\to-\infty}\left[\frac{1}{2}e^{2x}\right]_t^2 = \lim_{t\to-\infty}\left[\frac{1}{2}e^4 - \frac{1}{2}e^{2t}\right] = \frac{e^4}{2}$

Thus the given integral converges.

7. $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Sol. $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$

Now, $\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to-\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t\to-\infty} [\arctan x]_0^t = \frac{\pi}{2}$

And $\int_{-\infty}^{0} \frac{dx}{1+x^2} = \frac{\pi}{2}$, by **Problem 3.**

Therefore, $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ and the given integral converges.

8. $\int_{-\infty}^{\infty} \frac{x\, dx}{\sqrt{x^2+2}}$

Sol. $\int_{-\infty}^{\infty} \frac{x\, dx}{\sqrt{x^2+2}} = \int_{-\infty}^{0} \frac{x\, dx}{\sqrt{x^2+2}} + \int_0^\infty \frac{x\, dx}{\sqrt{x^2+2}}$

Now, $\int_{-\infty}^{0} \frac{x\, dx}{\sqrt{x^2+2}} = \lim_{t\to-\infty} \int_t^0 \frac{x\, dx}{\sqrt{x^2+2}} = \lim_{t\to-\infty} [\sqrt{x^2+2}]_t^0$

$= \lim_{t\to-\infty} [\sqrt{2} - \sqrt{t^2+2}] = -\infty$

Similarly, $\int_0^\infty \frac{x\, dx}{\sqrt{x^2+2}} = \infty$. Hence, $\int_{-\infty}^{\infty} \frac{x\, dx}{\sqrt{x^2+2}}$ diverges.

9. $\int_{-\infty}^{\infty} x^3\, dx$

Sol. $\int_{-\infty}^{\infty} x^3\,dx = \int_{-\infty}^{0} x^3\,dx + \int_{0}^{\infty} x^3\,dx$

$$\int_{-\infty}^{0} x^3\,dx = \lim_{t\to-\infty}\int_{t}^{0} x^3\,dx = \lim_{t\to-\infty}\left[\frac{x^4}{4}\right]_t^0 = -\lim_{t\to-\infty}\left(\frac{t^4}{4}\right) = -\infty$$

$$\int_{0}^{\infty} x^3\,dx = \lim_{t\to\infty}\int_{0}^{t} x^3\,dx = \lim_{t\to\infty}\left[\frac{x^4}{4}\right]_t^0 = \lim_{t\to\infty}\left(\frac{t^4}{4}\right) = \infty$$

Thus the given integral diverges.

10. $\int_{-\infty}^{\infty} \frac{x}{(x^4+1)}\,dx$

Sol. ln $\int_{-\infty}^{\infty} \frac{x}{x^4+1}\,dx$, put $x^2 = z$ so that $x\,dx = \frac{1}{2}dz$, then

$$\int \frac{x}{x^4+1}\,dx = \frac{1}{2}\int \frac{dz}{1+z^2} = \frac{1}{2}\arctan z = \frac{1}{2}\arctan x^2$$

Now $\int_{\infty}^{-\infty} \frac{x}{1+x^4}\,dx = \int_{-\infty}^{0} \frac{x\,dx}{1+x^4} + \int_{0}^{\infty} \frac{x\,dx}{1+x^4}$.

$$= \lim_{t\to-\infty}\int_{t}^{0} \frac{x\,dx}{1+x^4} + \lim_{t\to\infty}\int_{0}^{t} \frac{x\,dx}{1+x^4}$$

$$= \lim_{t\to-\infty}\left[\frac{1}{2}\arctan x^2\right]_t^0 + \lim_{t\to\infty}\left[\frac{1}{2}\arctan x^2\right]_0^t$$

$$= \lim_{t\to-\infty}\left[0 - \frac{1}{2}\arctan t^2\right] + \lim_{t\to\infty}\left(\frac{1}{2}\arctan t^2 - 0\right)$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right) = -\frac{\pi}{4} + \frac{\pi}{4} = 0$$

Thus the given integral converges.

11. $\int_{0}^{1} \frac{dx}{x}$

Sol. $\int_{0}^{1} \frac{dx}{x} = \lim_{t\to0^+}\int_{t}^{1} \frac{dx}{x} = \lim_{t\to0^+}[\ln x]_t^1 = \lim_{t\to0^+}[-\ln t] = \infty$

Thus the given integral diverges.

12. $\int_{0}^{a} \frac{dx}{x\sqrt{a^2-x^2}}$

Sol. $\int_{0}^{a} \frac{dx}{x\sqrt{a^2-x^2}} = \int_{0}^{a/2} \frac{dx}{x\sqrt{a^2-x^2}} + \int_{a/2}^{0} \frac{dx}{x\sqrt{a^2-x^2}}$

Now $\int_{0}^{a/2} \frac{dx}{x\sqrt{a^2-x^2}} = \lim_{t\to0^+}\int_{t}^{a/2} \frac{dx}{x\sqrt{a^2-x^2}} = \lim_{t\to0^+}\left[-\frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}\right]_t^{a/2}$

$$= \lim_{t\to0^+}\left[-\frac{1}{a}\ln(2+\sqrt{3}) + \frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-t^2}}{t}\right)\right]$$

$$= -\frac{1}{a}\ln(2+\sqrt{3}) + \frac{1}{a}\cdot\infty \quad \left(\begin{array}{l}\text{As } \dfrac{a+\sqrt{a^2-t^2}}{t} \to \infty \text{ when } t\to0^+ \\ \text{so } \ln\dfrac{a+\sqrt{a^2+t^2}}{t} \to \infty \text{ when } t\to0^+\end{array}\right)$$

$= \infty$

And $\int_{a/2}^{0} \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln(2+\sqrt{3})$ (The details are left for the students)

Thus $\int_{0}^{a} \frac{dx}{x\sqrt{a^2-x^2}}$ is not finite and it diverges.

13. $\int_{0}^{1} \frac{dx}{(x-1)^2}$

Sol. $\int_{0}^{1} \frac{dx}{(x-1)^2} = \lim_{t\to1^-}\int_{0}^{t} \frac{dx}{(x-1)^2} = \lim_{t\to1^-}\left[\frac{-1}{x-1}\right]_0^t$

$$= \lim_{t\to 1^-}\left[\frac{-1}{t-1}-1\right] = \infty$$

Hence the given integral diverges.

14. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}[\arcsin x]_0^t$

$$= \lim_{t\to 1^-}[\arcsin t - \arcsin 0] = \arcsin 1 - 0 = \frac{\pi}{2}.$$

Thus the given integral converges.

15. $\displaystyle\int_0^e x^2 \ln x\, dx$

Sol. $\displaystyle\int_0^e x^2 \ln x\, dx = \lim_{t\to 0^+}\int_t^e x^2 \ln x\, dx = I$

Now $\displaystyle\int (\ln x)\, x^2\, dx = (\ln x)\frac{x^3}{3} - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3}$

$$I = \lim_{t\to 0^+}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_t^e = \lim_{t\to 0^+}\left[\frac{e^3}{3} - \frac{e^3}{9} - \frac{t^3}{3}\ln t + \frac{t^3}{9}\right]$$

$$= \frac{2e^3}{9} \quad \left(\text{since } \lim_{t\to 0^+}\frac{\ln t}{\frac{1}{t^3}} = \lim_{t\to 0^+}\left(\frac{\frac{1}{t}}{-3\times\frac{1}{t^4}}\right) = \left(\lim_{t\to 0^+}\frac{t^3}{-3}\right) = 0\right)$$

Thus the given integral converges.

16. $\displaystyle\int_{-1}^8 \frac{dx}{x^{1/3}}$

Sol. $\displaystyle\int_{-1}^8 \frac{dx}{x^{1/3}} = \int_{-1}^0 \frac{dx}{x^{1/3}} + \int_0^8 \frac{dx}{x^{1/3}} = \lim_{t\to 0^-}\int_{-1}^t \frac{dx}{x^{1/3}} + \lim_{t\to 0^+}\int_t^8 \frac{dx}{x^{1/3}}$

$$= \lim_{t\to 0^-}\left[\frac{3}{2}x^{2/3}\right]_{-1}^t + \lim_{t\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_t^8$$

$$= \lim_{t\to 0^-}\left[\frac{3}{2}t^{2/3} - \frac{3}{2}(-1)^{2/3}\right] + \lim_{t\to 0^+}\left[\frac{3}{2}(8)^{2/3} - \frac{3}{2}t^{2/3}\right]$$

$$= -\frac{3}{2} + 6 = \frac{9}{2}$$

Hence the given integral converges.

17. $\displaystyle\int_{-2}^2 \frac{dx}{x}$

Sol. $\displaystyle\int_{-2}^2 \frac{dx}{x} = \int_{-2}^0 \frac{dx}{x} + \int_0^2 \frac{dx}{x} = \lim_{t\to 0^-}\int_{-2}^t \frac{dx}{x} + \lim_{t\to 0^+}\int_t^2 \frac{dx}{x}$

$$= \lim_{t\to 0^-}[\ln|x|]_{-2}^t + \lim_{t\to 0^+}[(\ln|x|)]_t^2$$

But both the limits are not finite and so the given integral diverges.

18. $\displaystyle\int_0^3 \frac{dx}{x^2+2x-3}$

Sol. $\displaystyle\int_0^3 \frac{dx}{(x+3)(x-1)} = \int_0^1 \frac{dx}{(x+3)(x-1)} + \int_1^3 \frac{dx}{(x+3)(x-1)}$

Now $\displaystyle\int_0^1 \frac{dx}{(x+3)(x-1)} = \lim_{t\to 1^-}\int_0^t \frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+3}\right)dx$

$$= \lim_{t\to 1^-}\left(\left[\frac{1}{4}\ln|x-1|\right]_0^t - \left[\frac{1}{4}\ln|x+3|\right]_0^t\right)$$

$$= \lim_{t\to 1^-}\left(\frac{1}{4}\ln|1-t|\right) - \frac{1}{4}\ln\frac{4}{3} \text{ which is not finite.}$$

Hence $\displaystyle\int_0^3 \frac{dx}{x^2+2x-3}$ diverges.

19. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$

Sol. $\displaystyle\int_0^{\pi/2}\frac{\cos x}{\sqrt{1-\sin x}}\,dx = \lim_{t\to(\frac{\pi}{2})^-}\int_0^t(1-\sin x)^{-1/2}\cos x\,dx$

$\displaystyle = \lim_{t\to\frac{\pi}{2}^-}[-2\sqrt{1-\sin x}]_0^t = \lim_{t\to\frac{\pi}{2}^-}[-2\sqrt{1-\sin t}+2] = 2$

Thus the given integral converges.

20. $\displaystyle\int_0^2\frac{x}{x^2-5x+6}\,dx$

Sol. $\displaystyle\int_0^2\frac{x}{x^2-5x+6}\,dx = \int_0^2\left(\frac{3}{x-3}-\frac{2}{x-2}\right)dx = 3\int_0^2\frac{dx}{x-3}-2\int_0^2\frac{dx}{x-2}$

Now $\displaystyle\int_0^2\frac{dx}{x-2} = \lim_{t\to2^-}\int_0^t\frac{dx}{x-2} = \lim_{t\to2^-}[\ln(x-2)]_0^t$

$\displaystyle = \lim_{t\to2^-}[\ln(t-2)-\ln(-2)] = \lim_{t\to2^-}\ln\left(\frac{2-t}{2}\right)$

$\displaystyle = \lim_{t\to2^-}\ln\left(1-\frac{t}{2}\right)$ which is not finite.

Hence the given integral diverges.

21. $\displaystyle\int_0^\infty\frac{\ln(1+x^2)}{1+x^2}\,dx$

Sol. $\displaystyle I = \int_0^\infty\frac{\ln(1+x^2)}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t\frac{\ln(1+x^2)}{1+x^2}\,dx$

To evaluate $\displaystyle\int_0^t\frac{\ln(1+x^2)}{1+x^2}\,dx$, we put $x=\tan\theta$ so that

$dx=\sec^2\theta\,d\theta$ When $x=0$, $\theta=0$ and $\theta=\arctan t$ when $x=t$

Now $\displaystyle\int_0^t\frac{\ln(1+x^2)}{1+x^2}\,dx = \int_0^{\arctan t}\frac{\ln(\sec^2\theta)}{\sec^2\theta}\sec^2\theta\,d\theta$ $\left(\begin{array}{l}\text{since } 1+x^2\\ =1+\tan^2\theta=\sec^2\theta\end{array}\right)$

and $\displaystyle I = \lim_{t\to\infty}\int_0^t\frac{\ln(1+x^2)}{1+x^2}\,dx = (-2)\lim_{t\to\infty}\int_0^{\arctan t}\ln(\cos\theta)\,d\theta$

$\displaystyle = -2\int_0^{\pi/2}\ln(\cos\theta)\,d\theta = -2\left(-\frac{\pi}{2}\ln 2\right)$ (by Example 9 Page 179)

$= \pi\ln 2.$

Thus the given integral converges.

22. $\displaystyle\int_0^\infty\frac{x\,dx}{(1+x)(1+x^2)}\,dx$

Sol. $\displaystyle I = \int_0^\infty\frac{x\,dx}{(1+x)(1+x^2)} = \lim_{t\to\infty}\int_0^t\frac{x\,dx}{(1+x)(1+x^2)}$

To evaluate $\displaystyle\int_0^t\frac{x\,dx}{(1+x)(1+x^2)}$, we put $x=\tan\theta$ so that $dx=\sec^2\theta\,d\theta$

When $x=0$, $\theta=0$ and $\theta=\arctan t$ when $x=t$

Now $\displaystyle\int_0^t\frac{x\,dx}{(1+x)(1+x^2)} = \int_0^{\arctan t}\frac{\tan\theta}{(1+\tan\theta)\sec^2\theta}\cdot\sec^2\theta\,d\theta$

(since $1+x^2=1+\tan^2\theta=\sec^2\theta$)

$\displaystyle = \int_0^{\arctan t}\frac{\sin\theta}{\sin\theta+\cos\theta}$

and $\displaystyle I = \lim_{t\to\infty}\int_0^t\frac{x\,dx}{(1+x)(1+x^2)} = \lim_{t\to\infty}\int_0^{\arctan t}\frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta$

$\displaystyle = \int_0^{\pi/2}\frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta = \int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)}\,d\theta$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \sin\theta} d\theta$$

Hence $2I = \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta} d\theta = \int_0^{\pi/2} 1 \cdot d\theta = [\theta]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

i.e., $I = \frac{\pi}{4}$ and the given integral converges.

23. $\int_{-\infty}^{0} \frac{e^x}{1+e^x} dx$

Sol. $I = \int_{-\infty}^{0} \frac{e^x}{1+e^x} dx = \lim_{t \to -\infty} \int_t^0 \frac{e^x}{1+e^x} dx$

To evaluate $\int_t^0 \frac{e^x}{1+e^x} dx$, we put $e^x = z$ so that $e^x\, dx = dz$.

When $x = 0$, $z = 1$ and $z = e^t$ when $x = t$

Now $\int_t^0 \frac{e^x}{1+e^x} = \int_{e^t}^{1} \frac{dz}{1+z} = [\ln(1+z)]_{e^t}^{1} = \ln 2 - \ln(1+e^t)$

and $\lim_{t\to-\infty} \int_t^0 \frac{e^x}{1+e^x} dx = \lim_{t\to-\infty} [\ln 2 - \ln(1+e^t)]$

$= \ln 2 - 0 = \ln 2$, because $\lim_{t\to-\infty} (1+e^t) = 1$

Thus the given integral converges.

24. $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Sol. $I = \int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\left[\int_0^{\infty} \frac{dx}{x^2+b^2} - \int_0^{\infty} \frac{dx}{x^2+a^2}\right]$

Let $I_1 = \int_0^{\infty} \frac{dx}{x^2+b^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{x^2+b^2}$

We put $x = b\tan\theta$ so that $dx = b\sec^2\theta\, d\theta$.

When $x = 0$, $\theta = 0$ and $\theta = \arctan\frac{t}{b}$ when $x = t$.

Now $\int_0^t \frac{dx}{x^2+b^2} = \int_0^{\arctan\frac{t}{b}} \frac{b\sec^2\theta\, d\theta}{b^2\sec^2\theta} \cdot \left(\begin{array}{l}\text{because } x^2+b^2 \\ = b^2\tan^2\theta + b^2 = b^2\sec^2\theta\end{array}\right)$

$= \frac{1}{b} \int_0^{\arctan\frac{t}{b}} 1 \cdot d\theta = \frac{1}{b}[\theta]_0^{\arctan\frac{t}{b}} = \frac{1}{b}\left(\arctan\frac{t}{b} - 0\right)$

and $\lim_{t\to\infty} \int_0^t \frac{dx}{x^2+b^2} = \frac{1}{b}\lim_{t\to\infty}\left(\arctan\frac{t}{b}\right) = \frac{1}{b}\cdot\frac{\pi}{2} = \frac{\pi}{2b}$

Similarly, $\lim_{t\to\infty} \int_0^t \frac{dx}{x^2+b^2} = \frac{\pi}{2a}$

Thus $I = \frac{1}{a^2-b^2}\left[\frac{\pi}{2b} - \frac{\pi}{2a}\right] = \frac{1}{a^2-b^2}\cdot\frac{\pi}{2}\left(\frac{a-b}{ab}\right) = \frac{\pi}{2ab\,(a+b)}$

and the given integral converges.

25. $\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} dx$

Sol. $\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} dx = \lim_{t\to\frac{\pi}{2}}\left[-\int_{\pi/4}^{t} (\cos x)^{-1/2}(-\sin x)\, dx\right]$

$= \lim_{t\to\frac{\pi}{2}} \int_t^{\pi/4} (\cos x)^{-1/2}(-\sin x)\, dx = \lim_{t\to\frac{\pi}{2}} [2(\cos x)^{1/2}]_t^{\pi/4}$

$= \lim_{t\to\frac{\pi}{2}}\left[2\left(\frac{1}{\sqrt{2}}\right)^{1/2} - (\cos t)^{1/2}\right]$

$= 2 \cdot \frac{1}{2^{1/4}} - 0$, because $\sqrt[4]{\cos t} \to 0$ when $t \to \frac{\pi}{2}$

$= 2^{3/4}$. Thus the given integral converges.

26. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

Sol. $I = \int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx$

Now $\int xe^{-x^2}\,dx = \frac{1}{2}\int e^{-z}\,dz$, by putting $x^2 = z$

$$= -\frac{1}{2}e^{-z} = -\frac{1}{2}e^{-x^2}$$

Therefore, $I = \lim_{t \to -\infty} \frac{1}{2}[-e^{-x^2}]_t^0 + \lim_{t \to \infty} \frac{1}{2}[-e^{-x^2}]_0^t$

$$= \lim_{t \to -\infty} \frac{1}{2}[-1 + e^{-t^2}] + \lim_{t \to \infty} \frac{1}{2}[-e^{-t^2} + 1]$$

$$= \frac{1}{2}[-1 + 0 + 0 + 1] = 0$$

Thus the given integral converges.

27. $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 6x + 12}$

Sol. $I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 6x + 12} = \int_{-\infty}^{0} \frac{dx}{(x+3)^2 + (\sqrt{3})^2} + \int_{0}^{\infty} \frac{dx}{(x+3)^2 + (\sqrt{3})^2}$

Now $\int \frac{dx}{(x+3)^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}} \arctan\left(\frac{x+3}{\sqrt{3}}\right)$

Therefore, $I = \frac{1}{\sqrt{3}} \lim_{t \to -\infty} \left[\arctan \frac{x+3}{\sqrt{3}}\right]_t^0 + \lim_{t \to \infty} \left[\arctan \frac{x+3}{\sqrt{3}}\right]_0^t$

$= \frac{1}{\sqrt{3}} \lim_{t \to -\infty} \left[\arctan \sqrt{3} - \arctan \frac{t+3}{\sqrt{3}}\right]$

$+ \frac{1}{\sqrt{3}} \lim_{t \to \infty} \left[\arctan \frac{t+3}{\sqrt{3}} - \arctan \sqrt{3}\right]$

$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{3} - \left(-\frac{\pi}{2}\right)\right] + \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{3}\right]$

$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{3} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{3}\right] = \frac{1}{\sqrt{3}} \cdot \pi = \frac{\pi}{\sqrt{3}}$

The given integral converges.

28. $\int_{-1}^{1} \frac{dx}{x^2}$

Sol. $I = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2} = \lim_{t \to 0^-} \int_{-1}^{t} \frac{dx}{x^2} + \lim_{t \to 0^+} \int_{t}^{1} \frac{dx}{x^2}$

As $\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-1}}{-1} = -\frac{1}{x}$, so

$$I = \lim_{t \to 0^-} \left[-\frac{1}{x}\right]_{-1}^{t} + \lim_{t \to 0^+} \left[-\frac{1}{x}\right]_{t}^{1}$$

$$= \lim_{t \to 0^-} \left[-\frac{1}{t} - 1\right] + \lim_{t \to 0^+} \left[-1 + \frac{1}{t}\right] = \infty$$

Thus the given integral diverges.

29. $\int_{2}^{\infty} \frac{dx}{x(\ln x)^3}$

Sol. $I = \int_{2}^{\infty} \frac{dx}{x(\ln x)^3} = \lim_{t \to \infty} \int_{2}^{t} \frac{dx}{x(\ln x)^3} = \lim_{t \to \infty} \left[-\frac{1}{2(\ln x)^2}\right]_2^t$

$= \lim_{t \to \infty} \left[-\frac{2}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2}\right] = \frac{1}{2(\ln 2)^2}$

Thus the given integral converges.

30. $\int_{0}^{\infty} xe^{-x}\,dx$

Sol. $\int_{0}^{\infty} xe^{-x}\,dx = \lim_{t \to \infty} \int_{0}^{t} xe^{-x}\,dx$

$= \lim_{t \to \infty} [-xe^{-x} - e^{-x}]_0^t$ (integrating by parts)

$= \lim_{t \to \infty} [te^{-t} - e^{-t} + 1] = \lim_{t \to \infty} \left[\frac{-(t+1)}{e^t} + 1\right] = 0 + 1 = 1$

Thus the given integral converges.

31. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)}$

Sol. $\displaystyle\int \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)} = \int \frac{2z\,dz}{z\,(z+1)}$, (on putting $\sqrt{x} = z$)

$$= 2\int \frac{dz}{z+1} = 2\ln|z+1| = 2\ln(\sqrt{x}+1)$$

$$\int_1^\infty \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)} = \lim_{t\to\infty} \left[2\ln(\sqrt{x}+1)\right]_1^t$$

$$= \lim_{t\to\infty}\left[2\ln\left(\frac{\sqrt{t}+1}{2}\right)\right] = \infty$$

Thus the given integral diverges.

32. $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Sol. $\displaystyle I = \int_0^1 e^{\sqrt{x}}\cdot\frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 e^{\sqrt{x}}\cdot\frac{1}{\sqrt{x}}\,dx$

As $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}}$, so

$$I = \lim_{t\to 0^+}\left[2e^{\sqrt{x}}\right]_t^1 = \lim_{t\to 0^+}(2e - 2e^{\sqrt{t}}) = 2e - 2 = 2(e-1)$$

Thus the given integral converges.

33. $\displaystyle\int_0^\infty \frac{x^3}{x^3+1}\,dx$

Sol. $\displaystyle I = \int_0^\infty \frac{x^3}{x^3+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^3}{x^3+1}\,dx$

Now $\displaystyle\int \frac{x^3}{x^3+1}\,dx = \int\left(1 - \frac{1}{x^3+1}\right)dx$

$$= \int\left(1 - \frac{1}{(x+1)(x^2-x+1)}\right)dx$$

$$= \int\left(1 - \frac{1}{3(x+1)} + \frac{x-2}{3(x^2-x+1)}\right)dx$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\int\frac{2x-1-3}{x^2-x+1}\,dx$$

As $\displaystyle\int\frac{2x-1-3}{x^2-x+1}\,dx = \int\frac{2x-1}{x^2-x+1} - \int\frac{3\,dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \ln(x^2-x+1) - \frac{3\times 2}{\sqrt{3}}\arctan\frac{(2x-1)}{\sqrt{3}},\ \text{so}$$

$\displaystyle\int\frac{x^3}{x^3+1}\,dx = x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) - \frac{1}{\sqrt{3}}\arctan\frac{(2x-1)}{\sqrt{3}}$ and

$$I = \lim_{t\to\infty}\left(\left[x - \frac{1}{3}\ln(x+1) + \frac{1}{6}\ln(x^2-x+1) - \frac{1}{3}\operatorname{arc}\left(\frac{2x-1}{3}\right)\right]_0^t\right)$$

which does not exist. Hence the given integral diverges.

34. Let $I_n = \displaystyle\int_0^\infty x^n e^{-x}\,dx$, where n is a positive integer. Prove that $I_n = nI_{n-1}$. Hence show that $I_n = n!$.

Sol. $\displaystyle I_n = \int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^n e^{-x}\,dx$

Integrating $\int x^n e^{-x}\,dx$ by parts, we have

$$\int x^n e^{-x}\,dx = x^n(-e^{-x}) - \int(-e^{-x})\cdot nx^{n-1}\,dx = -\frac{x^n}{e^x} + n\int x^{n-1}\cdot e^{-x}\,dx$$

Now $\displaystyle I_n = \lim_{t\to\infty}\left[-\frac{x^n}{e^x} + n\int_0^t x^{n-1}e^{-x}\,dx\right]$

$$= \lim_{t\to\infty}\left[-\frac{x^n}{e^x}\right]_0^t + n\lim_{t\to\infty}\int_0^t x^{n-1}\cdot e^{-x}\,dx$$

$$= \lim_{t\to\infty}\left[-\frac{t^n}{e^t}\right] + n\int_0^\infty x^{n-1}\cdot e^{-x}\,dx$$

Applying L'Hospital's rule successively, we have

$$\lim_{t\to\infty}\left[-\frac{t^n}{e^t}\right] = -\lim_{t\to\infty}\left(\frac{n(n-1)(n-2)\ldots..1}{e^t}\right) = 0$$

Thus $I_n = 0 + n\,I_{n-1} = n\,I_{n-1}$

Continuing in this way, $I_{n-1} = (n-1)\,I_{n-2}\,, I_{n-1} = (n-2)\,I_{n-3}\,\cdots$

Hence, $I_n = n(n-1)(n-2)\ldots I_1,$ where $I_1 = \int_0^\infty x\,e^{-x}\,dx = 1$ by Problem 30.

$$= n(n-1)(n-2)\ldots 1 = n!$$

35. Evaluate $\int_1^5 [x]\,dx$, where $[x]$ denotes the greatest integer less than or equal to x.

Sol. The function $[x]$is defined as under

$$\begin{aligned} [x] &= 1 \quad \text{if} \quad 1 \le x < 2 \\ &= 2 \quad \text{if} \quad 2 \le x < 3 \\ &= 3 \quad \text{if} \quad 3 \le x < 4 \\ &= 4 \quad \text{if} \quad 4 \le x < 5 \\ &= 5 \quad \text{if} \quad 5 \le x = 5 \end{aligned}$$

Therefore, $I = \lim\limits_{\varepsilon\to 0} \int_1^{2-\varepsilon} 1\cdot dx + \lim\limits_{\varepsilon\to 0}\int_2^{3-\varepsilon} 2dx + \lim\limits_{\varepsilon\to 0}\int_3^{4-\varepsilon} 3dx$

$$+ \lim_{\varepsilon\to 0}\int_4^{5-\varepsilon} 4dx + \lim_{\varepsilon\to 0}\int_{5-\varepsilon}^{5} 5dx$$

$$= \lim_{\varepsilon\to 0}[x]_1^{2-\varepsilon} + \lim_{\varepsilon\to 0}[2x]_2^{3-\varepsilon} + \lim_{\varepsilon\to 0}[3x]_3^{4-\varepsilon}$$

$$+ \lim_{\varepsilon\to 0}[4x]_4^{5-\varepsilon} + \lim_{\varepsilon\to 0}[5x]_{5-\varepsilon}^{5}$$

$$= \lim_{\varepsilon\to 0}[(2-\varepsilon-1) + (2(3-\varepsilon)-2.2) + (3(4-\varepsilon)-3.3)$$

$$+ (4(5-\varepsilon)-4.4) + (515) - 5(5-\varepsilon)]$$

$$= 1 + 2 + 3 + 4 + 0 = 10$$

Exercise Set 5.4 (Page 200)

Evaluate (Problems 1 – 21):

1. $\int \frac{\sec^4 x}{\tan^5 x}\,dx$

Sol. Put $\tan x = z$ so that $\sec^2 x\,dx = dz$

$$I = \int \frac{\sec^2 x}{\tan^5 x}\cdot \sec^2 x\,dx = \int \frac{1+z^2}{z^5}\,dz = \int\left(\frac{1}{z^5} + \frac{1}{z^3}\right)dz$$

$$= -\frac{1}{4z^4} - \frac{1}{2z^2} = -\frac{1}{4\tan^4 x} - \frac{1}{2\tan^2 x}$$

2. $\int \sin^2 x \cos^4 x\,dx$

Sol. We connect $\int \sin^p x \cos^q x\,dx$ with $\int \sin^p x \cos^{q-2} x\,dx$.

Here, $P = \sin^{p+1} x \cos^{q-1} x$

$$\begin{aligned} \frac{dP}{dx} &= (p+1)\sin^p x\cos^q x - (q-1)\cos^{q-2}x\sin^{p+2}x \\ &= (p+1)\sin^p x\cos^q x - (q-1)\cos^{q-2}x\sin^p(1-\cos^2 x) \\ &= (p+1)\sin^p x\cos^q x + (q-1)\sin^p x\cos^q x \\ &\qquad -(q-1)\sin^p x\cos^{q-2}x \\ &= (p+q)\sin^p x\cos^q x - (q-1)\sin^p x\cos^{q-2}x \end{aligned}$$

$$P = (p+q)\int\sin^p x\cos^q x\,dx - (q-1)\int\sin^p x\cos^{q-2}x\,dx$$

Hence

$$\int\sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x\cos^{q-1}x}{p+q} + \frac{q-1}{p+q}\int\sin^p x\cos^{q-2}x\,dx$$

Put $p = 2, q = 4$ in this formula to get

$$I = \int\sin^2 x\cos^4 x\,dx = \frac{\sin^3 x\cos^3 x}{6} + \frac{1}{2}\int\sin^2 x\cos^2 x\,dx$$

$$\int\sin^2 x\cos^2 x\,dx = \frac{\sin^3 x\cos x}{4} + \frac{1}{4}\int\sin^2 x\cos^0 x\,dx$$

$$= \frac{1}{4}\sin^3 x\cos x + \frac{1}{4}\int\frac{1-\cos 2x}{2}\,dx$$

$$= \frac{1}{4}\sin^3 x\cos x + \frac{x}{8} - \frac{\sin 2x}{16}$$

Therefore,

$$I = \frac{1}{6}\sin^3 x\cos^3 x + \frac{1}{8}\sin^3 x\cos x + \frac{x}{16} - \frac{\sin x\cos x}{16}$$

3. $\int \sin^6 x \cos^2 x\, dx$

Sol. We have the reduction formula

$$\int \sin^p x\cos^q x\, dx = -\frac{\sin^{p-1}x\cos^{q+1}x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2}\cos^q x\, dx$$

Put $p = 6, q = 2$. Then

$$I = \int \sin^6 x\cos^2 x\, dx = -\frac{1}{8}\sin^5 x\cos^3 x + \frac{5}{8}\int \sin^4 x\cos^2 x\, dx,$$

$$\int \sin^4 x\cos^2 x\, dx = -\frac{1}{6}\sin^3 x\cos^3 x + \frac{1}{2}\int \sin^2 x\cos^2 x\, dx,$$

$$\int \sin^2 x\cos^2 x\, dx = -\frac{1}{4}\sin x\cos^3 x + \frac{1}{4}\int \cos^2 x\, dx$$

$$= -\frac{1}{4}\sin x\cos^3 x + \frac{1}{4}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right)$$

$$I = -\frac{1}{8}\sin^5 x\cos^3 x + \frac{5}{8}\left[-\frac{1}{6}\sin^3 x\cos^3 x + \frac{1}{2}\left(-\frac{1}{4}\sin x\cos^3 x\right) + \frac{1}{8}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right)\right]$$

$$= -\frac{1}{8}\sin^5 x\cos^3 x - \frac{5}{48}\sin^3 x\cos^3 x - \frac{5}{64}\sin x\cos^3 x + \frac{5}{64}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right)$$

4. $\int \sin^{1/2} x\cos^3 x\, dx$

Sol. Put $\sin x = z^2$ so that $\cos x\, dx = 2z\, dz$. Then

$$I = \int \sin^{1/2} x\cos^3 x\, dx = \int \sin^{1/2} x\cos^2 x \cdot \cos x\, dx$$

$$= \int z(1 - z^4)\cdot 2z\, dz$$

$$= \int 2(z^2 - z^6)\, dz = \frac{2}{3}z^3 - \frac{2}{7}z^7 = \frac{2}{3}\sin^{3/2} x - \frac{2}{7}\sin^{7/2} x$$

5. $\int \sec^2 x\csc^3 x\, dx = \int \csc^3 x \cdot \sec^2 x\, dx$

Sol. Integrate by parts (taking $\csc^3 x$ as first function). Then

$$I = \int \csc^3 x \cdot \sec^2 x\, dx = \csc^3 x\tan x - \int (-3\csc^3 x\cot x)\cdot \tan x\, dx$$

Now $\int \csc^3 x\, dx = -\frac{1}{2}\csc x\cot x + \frac{1}{2}\int \csc x\, dx,$

(by the reduction formula)

$$= -\frac{1}{2}\csc x\cot c + \frac{1}{2}\ln|\csc x - \cot x|$$

$$I = \csc^3 x\tan x + 3\left[-\frac{1}{2}\csc x\cot x + \frac{1}{2}\ln|\csc x - \cot x|\right]$$

$$= \csc^3 x\tan x - \frac{3}{2}\csc x\cot x + \frac{3}{2}\ln[\csc x - \cot x]$$

6. $\int \tan^3 x\sec^5 x\, dx$

Sol. Put $\sec x = z$, so that $\sec x\tan x\, dx = dz$. Then

$$I = \int \tan^3 x\sec^5 x\, dx = \int \tan^2 x\cdot\sec^4 x(\sec x\tan x)\, dx$$

$$= \int (z^2 - 1)z^4\, dz = \frac{z^7}{7} - \frac{z^5}{5} = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x$$

7. $\int \cot^5 x\csc^4 x\, dx$

Sol. Put $\cot x = z$, so that $-\csc^2 x\, dx = dz$. Then

$$I = \int \cot^5 x\csc^4 x\, dx = -\int \cot^5 x\csc^2 x(-\csc^2 x)\, dx$$

$$= -\int z^5(1 + z^2)\, dz = -\frac{z^6}{6} - \frac{z^8}{8} = -\frac{1}{6}\cot^6 x - \frac{1}{8}\cot^8 x$$

8. $\int \frac{\sin^2 x}{\cos^5 x}\, dx$

Sol. $I = \int \frac{\sin^2 x}{\cos^5 x}\, dx = \int \tan^2 x\sec^3 x\, dx = \int (-1 + \sec^2 x)\sec^3 x\, dx$

$$= \int \sec^5 x\, dx - \int \sec^3 x\, dx$$

$$\int \sec^5 x\, dx = \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\int \sec^3 x\, dx \text{ (by the reduction formula)}$$

$$I = \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\int \sec^3 x\, dx - \int \sec^3 x\, dx$$

$$= \frac{\sec^3 x\tan x}{4} - \frac{1}{4}\int \sec^3 x\, dx$$

Now $\int \sec^3 x\, dx = \frac{\sec x\tan x}{2} + \frac{1}{2}\int \sec x\, dx$

(by the reduction formula)

$$= \frac{\sec x\tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x|$$

$$\text{Thus } I = \frac{\sec^3 x \tan x}{4} - \frac{1}{4}\left[\frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x|\right]$$

$$= \frac{\sec^3 x \tan x}{4} - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x|$$

9. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^4 x\, dx$

Sol. By the reduction formula $\int \cot^4 x\, dx = -\frac{\cot^3 x}{3} - \int \cot^2 x\, dx$

$$\int \cot^4 x\, dx = -\frac{\cot^3 x}{3} - \int (\operatorname{cosec}^2 x - 1)\, dx = -\frac{\cot^3 x}{3} + \cot x + x$$

$$\int_{\pi/4}^{\pi/2} \cot^4 x\, dx = \left[-\frac{\cot^3 x}{3} + \cot x + x\right]_{\pi/4}^{\pi/2} = \frac{\pi}{2} - \left[-\frac{1}{3} + 1 + \frac{\pi}{4}\right] = \frac{\pi}{4} - \frac{2}{3}$$

10. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x\, dx$

Sol. Put $\csc x = z$, so that $-\csc x \cot x\, dx = dz$

$$I = \int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x\, dx = -\int_{\pi/4}^{\pi/2} (\csc^2 x - 1)\csc^2 x\,(-\csc x \cot x)\, dx$$

$$= -\int_{\sqrt{2}}^{1} (z^2 - 1) z^2\, dz = \int_{1}^{\sqrt{2}} (z^4 - z^2)\, dz$$

$$= \left[\frac{z^5}{5} - \frac{z^3}{3}\right]_1^{\sqrt{2}} = \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= \sqrt{2}\left(\frac{4}{5} - \frac{2}{3}\right) - \left(-\frac{2}{15}\right) = \frac{2}{15}(\sqrt{2} + 1)$$

11. $\displaystyle\int_0^{\pi/2} \tan^5\left(\frac{x}{2}\right) dx$

Sol. Put $\frac{x}{2} = z$, so that $dx = 2\, dz$. Then

$$I = \int_0^{\pi/2} \tan^5\left(\frac{x}{2}\right) dx = 2\int_0^{\pi/4} \tan^5 z\, dz$$

Now $\int \tan^5 z\, dz = \frac{\tan^4 z}{4} - \int \tan^3 z dz = \frac{\tan^4 z}{4} - \left[\frac{\tan^2 z}{2} - \int \tan z dz\right]$

$$= \frac{\tan^4 z}{4} - \frac{\tan^2 z}{2} + \ln \sec z$$

$$\text{Thus } I = 2\left[\frac{\tan^4 z}{4} - \left(\frac{\tan^2 z}{2}\right) + \ln \sec z\right]_0^{\pi/4}$$

$$= 2\left[\left(\frac{1}{4} - \frac{1}{2} + \ln\sqrt{2}\right) - 0\right] \text{ because } \ln \sec 0 = \ln 1 = 0$$

$$= 2\left(-\frac{1}{4}\right) + 2\log\sqrt{2} = -\frac{1}{2} + \ln 2$$

12. $\displaystyle\int_0^a (a^2 - x^2)^{5/2}\, dx$

Sol. Put $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$. Then

$$I = \int_0^a (a^2 - x^2)^{5/2}\, dx = \int_0^{\pi/2} (a^2 - a^2\sin^2\theta)^{5/2} \cdot a\cos\theta\, d\theta$$

$$= a^6 \int_0^{\pi/2} \cos^6\theta\, d\theta = a^6 \cdot \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}, \quad \text{(Wallis' formula)}$$

$$= \frac{5\pi a^6}{32}$$

13. $\displaystyle\int_0^{\pi} \frac{\sin^4 x}{(1+\cos x)^2}\, dx$

Sol. $\displaystyle\int_0^{\pi} \frac{\sin^4 x}{(1+\cos x)^2}\, dx = \int_0^{\pi} \frac{(\sin^2 x)^2}{(1+\cos x)^2}\, dx = \int_0^{\pi} \frac{(1-\cos x)^2(1+\cos x)^2}{(1+\cos x)^2}\, dx$

$$= \int_0^{\pi} (1-\cos x)^2\, dx = \int_0^{\pi} (1 - 2\cos x + \cos^2 x)\, dx$$

$$= \int_0^{\pi} \left(1 - 2\cos x + \frac{1+\cos 2x}{2}\right) dx$$

$$= \left[\frac{3}{2}x - 2\sin x + \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{3}{2}\pi.$$

14. $\displaystyle\int_0^{\pi/4} \sin^4 2x\, dx$

Sol. $\int_0^{\pi/4}\sin^4 2x\,dx = \int_0^{\pi/2}\sin^4 t\cdot\frac{dt}{2}$, putting $2x = t$ so that, $2\,dx = dt$

$$= \frac{1}{2}\int_0^{\pi/2}\sin^4 t\,dt = -\frac{1}{2}\cdot\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} = \frac{3\pi}{32}$$

15. $\int_0^{\pi/6}\sin^6 3x\,dx$

Sol. Put $3x = \theta$ so that $3\,dx = d\theta$ or $dx = \frac{1}{3}d\theta$

Now, when $x = 0$, $\theta = 0$ and when $x = \frac{\pi}{6}$, $\theta = \frac{\pi}{2}$

Therefore, $\int_0^{\pi/6}\sin^6 3x\,dx = \int_0^{\pi/2}(\sin^6\theta).\frac{1}{3}d\theta = \frac{1}{3}\int_0^{\pi/2}\sin^6\theta\,d\theta$

$$= \frac{1}{3}\cdot\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{5\pi}{96} \text{ (By Wallis' formula)}$$

16. $\int_0^{\pi/8}\sin^5 4x\cos^4 4x\,dx$

Sol. Put $4x = z$ so that $dx = \frac{dz}{4}$

When $x = 0$, $z = 0$ and when $x = \frac{\pi}{8}, z = \frac{\pi}{2}\cdot$ Then

$$I = \int_0^{\pi/8}\sin^5 4x\cos^4 4x\,dx = \frac{1}{4}\int_0^{\pi/2}\sin^5 z\cos^4 z\,dz$$

Here p is odd and q is even, so

Using $\int_0^{\pi/2}\sin^p x\cos^q x\,dx = \frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots}$, we have

$$\frac{1}{4}\int_0^{\pi/2}\sin^5 z\cos^4 z\,dz = \frac{1}{4}\left[\frac{(5-1)(5-3)(4-1)(4-3)}{(5+4)(5+4-2)(5+4-4)(5+4-6)}\right]$$

$$= \frac{1}{4}\left(\frac{4}{9}\cdot\frac{2}{7}\cdot\frac{3}{5}\cdot\frac{1}{3}\right) = \frac{2}{315}$$

17. $\int_0^{\pi/4}\cos^2 2x\,dx$

Sol. Put $2x = \theta$ so that $dx = \frac{1}{2}d\theta$

When $x = 0$, $\theta = 0$ and when $x = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$

Therefore, $\int_0^{\pi/4}\cos^2 2x\,dx = \int_0^{\pi/2}(\cos^2\theta)\cdot\frac{1}{2}d\theta = \frac{1}{2}\int_0^{\pi/2}\cos^2\theta\,d\theta$

$$= \frac{1}{2}\cdot\left(\frac{2-1}{2}\cdot\frac{\pi}{2}\right) = \frac{\pi}{8} \quad \text{(Here } n = 2\text{)}$$

18. $\int_0^{\pi/6}\cos^3 3x\,dx$

Sol. Put $3x = \theta$ so that $dx = \frac{1}{3}d\theta$

When $x = 0$, $\theta = 0$ and when $x = \frac{\pi}{6}$, $\theta = \frac{\pi}{2}$

Therefore, $\int_0^{\pi/6}\cos^3 3x\,dx = \int_0^{\pi/2}\cos^3\theta\frac{1}{3}d\theta$

$$= \frac{1}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta = \frac{1}{3}\cdot\left(\frac{3-1}{3}\right), \text{ (by Wallis' formula)}$$

$$= \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$$

19. $\int_0^{\pi/3}\sin^2 6x\cos^4 3x\,dx$

Sol. $I = \int_0^{\pi/3}(2\sin 3x\cos 3x)^2\cos^4 3x\,dx$

$$= 4\int_0^{\pi/3}\sin^2 3x\cos^6 3x\,dx$$

Put $3x = z$, so that $dx = \frac{dz}{3}$. When $x = 0, z = 0$

and when $x = \frac{\pi}{3}, z = \pi$

$I = \frac{4}{3}\int_0^{\pi}\sin^2 z\cos^6 z\,dz = \frac{8}{3}\int_0^{\pi/2}\sin^2 z\cos^6 z\,dz$, by Theorem5.11 (i)

Using $\int_0^{\pi/2} \sin^p x \cos^q x\,dx = \frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots}\cdot\frac{\pi}{2}$,

we get $\frac{8}{3}\int_0^{\pi/2}\sin^2 z\cos^6 z\,dz = \frac{8}{3}\cdot\left(\frac{1\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2}\right) = \frac{5}{96}\pi$

20. $\int_{\pi/3}^{\pi/2}\frac{\cos^2 x}{\sin x}\,dx$

Sol. Put $\cos x = z$, then $-\sin x\,dx = dz$

or $-\sin^2 x\cdot\frac{1}{\sin x}dx = dz \Rightarrow \frac{1}{\sin x}dx = -\frac{1}{1-z^2}dz$

When $x = \frac{\pi}{3}, z = \frac{1}{2}$ and when $x = \frac{\pi}{2}, z = 0$

$$I = \int_{\pi/3}^{\pi/2}\frac{\cos^2 x}{\sin x}dx = \int_{1/2}^{0} z^2\cdot\left(-\frac{1}{1-z^2}\right)dz = \int_{1/2}^{0}\left(1-\frac{1}{1-z^2}\right)dz$$

$$= \int_{1/2}^{0}\left[1-\frac{1}{2}\left(\frac{1}{1+z}+\frac{1}{1-z}\right)\right]dz = [z]_{1/2}^{0} - \frac{1}{2}\left[\ln\left(\frac{1+z}{1-z}\right)\right]_{1/2}^{0}$$

$$= \left(0-\frac{1}{2}\right)-\frac{1}{2}\left[\ln 1 - \ln\left(\frac{3/2}{1/2}\right)\right] = -\frac{1}{2}+\frac{1}{2}\ln 3 = -\frac{1}{2}+\ln\sqrt{3}$$

21. $\int_0^1\frac{x^6\,dx}{\sqrt{1-x^2}}$

Sol. Put $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$

When $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{\pi}{2}$

Therefore, $\int_0^1\frac{x^6\,dx}{\sqrt{1-x^2}} = \int_0^{\pi/2}\frac{\sin^6\theta\cdot\cos\theta\,d\theta}{\cos\theta} = \int_0^{\pi/2}\sin^6\theta\,d\theta$

$$= \frac{5\cdot3\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{5\pi}{32} \quad \text{(By Walls formula)}$$

22. Show that

$$\int\sec^{2n+1}x\,dx = \frac{\sec^{2n-1}x\tan x}{2n} + \left(1-\frac{1}{2n}\right)\int\sec^{2n-1}x\,dx.$$

Sol. $\int\sec^{2n+1}x\,dx = \int\sec^{2n-1}x\cdot\sec^2 x\,dx$

$= \sec^{2n-1}x\cdot\tan x - \int\tan x\cdot(2n-1)\sec^{2n-2}x\cdot\sec x\tan x\,dx$

(Integrating by parts)

$= \sec^{2n-1}x\cdot\tan x - (2n-1)\int\sec^{2n-1}x\tan^2 x\,dx$

$= \sec^{2n-1}x\cdot\tan x - (2n-1)\int\sec^{2n-1}x(\sec^2 x - 1)\,dx$

$= \sec^{2n-1}x\,.\tan x - (2n-1)\int\sec^{2n+1}x\,dx + (2n-1)\int\sec^{2n-1}x\,dx$

Therefore, $2n\int\sec^{2n+1}x\,dx = \sec^{2n-1}x\cdot\tan x - (2n-1)\int\sec^{2n-1}x\,dx$

or $\int\sec^{2n+1}x\,dx = \frac{\sec^{2n-1}x\tan x}{2n} + \left(1-\frac{1}{2n}\right)\int\sec^{2n-1}x\,dx$

23. Obtain a reduction formula for $\int\frac{dx}{(a^2+x^2)^n}$, where n is an integer.

Show that $\int_0^\infty\frac{dx}{(1+x^2)^5} = \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\frac{\pi}{2}$

Sol. $\int\frac{dx}{(a^2+x^2)^n} = \int(a^2+x^2)^{-n}\cdot 1dx$

$= (a^2+x^2)^{-n}\cdot x - \int x\cdot(-n)(a^2+x^2)^{-n-1}\,2x\,dx$ (Integrating by parts)

$= x(a^2+x^2)^{-n} + 2n\int x^2(a^2+x^2)^{-n-1}dx$

$= x(a^2+x^2)-n + 2n\int(a^2+x^2-a^2)(a^2+x^2)^{-n-1}$

(Writing x^2 as $a^2+x^2-a^2$)

$= x(a^2+x^2)^{-n} + 2n\int(a^2+x^2)^{-n}dx - 2na^2\int(a^2+x^2)^{-n-1}dx$

or $2na^2\int\frac{dx}{(a^2+x^2)^{n+1}} = x(a^2+x^2)^{-n} + (2n-1)\int\frac{dx}{(a^2+x^2)^n}$

Changing n into $n-1$, we get

$$2(n-1)a^2\int\frac{dx}{(a^2+x^2)^n} = \frac{x}{(a^2+x^2)^{n-1}} + (2n-3)\int\frac{dx}{(a^2+x^2)^{n-1}}$$

Dividing both sides by $2\,(n-1)\,a^2$, we get

$$\int\frac{dx}{(a^2+x^2)^n}=\frac{x}{2\,(n-1)a^2\,(a^2+x^2)^{n-1}}+\frac{2n-3}{2a^2(n-1)}\int\frac{dx}{(a^2+x^2)^{n-1}}$$

Integrating between the limits 0 to ∞

$$\int_0^\infty\frac{dx}{(a^2+x^2)^n}=\frac{2n-3}{2a^2\,(n-1)}\int_0^\infty\frac{dx}{(a^2+x^2)^{n-1}}$$

Taking $a=1$, and $n=5$, we have

$$\int_0^\infty\frac{dx}{(1+x^2)^5}=\frac{7}{2\cdot 4}\int_0^\infty\frac{dx}{(1+x^2)^4} \qquad (1)$$

$$\int_0^\infty\frac{dx}{(1+x^2)^4}=\frac{5}{2\cdot 3}\int_0^\infty\frac{dx}{(1+x^2)^3} \qquad (2)$$

$$\int_0^\infty\frac{dx}{(1+x^2)^3}=\frac{3}{2\cdot 2}\int_0^\infty\frac{dx}{(1+x^2)^2} \qquad (3)$$

$$\int_0^\infty\frac{dx}{(1+x^2)^2}=\frac{1}{2\cdot 1}\int_0^\infty\frac{dx}{(1+x^2)} \qquad (4)$$

$$\int_0^\infty\frac{dx}{1+x^2}=\lim_{n\to\infty}\left|\arctan x\right|_0^\infty=\frac{\pi}{2} \qquad (5)$$

Multiplying vertically (1) to (5), we get

$$\int_0^\infty\frac{dx}{(1+x^2)^5}=\frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2}=\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi}{2}$$

24. If I_n denotes $\int_0^1 x^p(1-x^q)^n\,dx$, where p, q and n are positive, prove that $(qn+p+1)\,I_n=qnI_{n-1}$. Evaluate I_n when n is a positive integer.

Sol. $\int x^p\,(1-x^q)^n\,dx=(1-x^q)^n\,\dfrac{x^{p+1}}{p+1}-\displaystyle\int\frac{x^{p+1}}{p+1}\cdot n\,(1-x^q)^{n-1}\,(-q)x^{q-1}\,dx$

$$=\frac{x^{p+1}\,(1-x^q)^n}{p+1}+\frac{qn}{p+1}\int x^{p+q}\,(1-x^q)^{n-1}\,dx$$

$$=\frac{x^{p+1}\,(1-x^q)^n}{p+1}-\frac{qn}{p+1}\int x^p\,(1-x^q-1)\,(1-x^q)^{n-1}\,dx$$

$$=\frac{x^{p+1}\,(1-x^q)^n}{p+1}-\frac{qn}{p+1}\int x^p\,[(1-x^q)^n-(1-x^q)^{n-1}]\,dx$$

$$=\frac{x^{p+1}\,(1-x^q)^n}{p+1}-\frac{qn}{p+1}\int x^p\,(1-x^q)^n\,dx+\frac{qn}{p+1}\int x^p\,(1-x^q)^{n-1}\,dx$$

Therefore, $\left(1+\dfrac{qn}{p+1}\right)\displaystyle\int x^p\,(1-x^p)^n\,dx$

$$=\frac{x^{p+1}\,(1-x^q)^n}{p+1}+\frac{qn}{p+1}\int x^p\,(1-x^q)^{n-1}\,dx$$

or $\quad (qn+p+1)\int x^p\,(1-x^q)^n\,dx$

$$=x^{p+1}\,(1-x^q)^n+qn\int x^p\,(1-x^q)^{n-1}\,dx$$

Integrating between the limits 0 and 1, we have

$$(qn+p+1)\int_0^1 x^p\,(1-x^q)^n\,dx$$

$$=\left[x^{p+1}\,(1-x^q)^n\right]_0^1+qn\int_0^1 x^p\,(1-x^q)^{n-1}\,dx$$

Thus $\quad (qn+p+1)I_n=qn\,I_{n-1}$ or $I_n=\dfrac{qn}{qn+p+1}I_{n-1}$

Now $I_{n-1}=\dfrac{q\,(n-1)}{q\,(n-1)+p+1}I_{n-2}$

$$I_{n-2}=\frac{q\,(n-2)}{q\,(n-2)+p+1}I_{n-3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_3=\frac{3q}{3q+p+1}I_2 \quad ; \qquad I_2=\frac{2q}{2q+p+1}I_1$$

$$I_1=\frac{q}{q+p+1}I_0\,; \qquad I_0=\int_0^1 x^p\,dx=\frac{1}{p+1}$$

Multiplying vertically, we have

$$I_n=\frac{q^n\cdot n!}{(qn+p+1)\{q(n-1)+p+1\}\,\{q(n-2)+p+1\}\ldots(2q+p+1)\,(q+p+1)\,(p+1)}$$

which is the required value.

25. Obtain a reduction formula for $\int \frac{x^n}{\sqrt{1-x^2}}\,dx$ and hence evaluate

$\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$

Sol. We connect $\int x^m (1-x^2)^{-1/2}\,dx$ with $\int x^{m-2}(1-x^2)^{-1/2}\,dx$

Here $P = x^{m-1}(1-x^2)^{1/2}$

$$\frac{dP}{dx} = (m-1)x^{m-2}(1-x^2)^{1/2} + x^{m-1}\cdot\frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= (m-1)x^{m-2}(1-x^2)^{1/2} - x^m(1-x^2)^{-1/2}$$

Integrating, we get

$P = (m-1)\int x^{m-2}(1-x^2)^{1/2}\,dx - \int x^m(1-x^2)^{-1/2}\,dx$ or

$$x^{m-1}(1-x^2)^{1/2} = (m-1)\int x^{m-2}(1-x^2)^{-1/2}(1-x^2)\,dx - \int x^m(1-x)^{-1/2}dx$$

$$= (m-1)\int x^{m-2}(1-x^2)^{-1/2}dx - (m-1)\int x^m(1-x^2)^{-1/2}dx - \int x^m(1-x^2)^{-1/2}dx$$

$$= (m-1)\int x^{m-2}(1-x^2)^{-1/2}\,dx - m\int x^m(1-x^2)^{-1/2}\,dx$$

or $\quad m\int x^m(1-x^2)^{-1/2}\,dx$

$$= -x^{m-1}(1-x^2)^{1/2} + (m-1)\int x^{m-2}(1-x^2)^{-1/2}\,dx$$

Therefore,

$$\int x^m(1-x^2)^{-1/2}\,dx = \frac{-x^{m-1}(1-x^2)^{1/2}}{m} + \frac{m-1}{m}\int \frac{x^{m-2}}{\sqrt{1-x^2}}\,dx$$

or $$\int \frac{x^m}{\sqrt{1-x^2}}\,dx = \frac{-x^{m-1}(1-x^2)^{1/2}}{m} + \frac{m-1}{m}\int \frac{x^{m-2}}{\sqrt{1-x^2}}\,dx$$

Putting $m = 3$ in the above equation, we have

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = -\frac{x^2(1-x^2)^{1/2}}{3} + \frac{2}{3}\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= -\frac{x^2(1-x^2)^{1/2}}{3} + \frac{1}{3}\int(-2x)(1-x^2)^{-1/2}\,dx$$

$$= -\frac{x^2(1-x^2)^{1/2}}{3} + \frac{1}{3}\,\frac{(1-x^2)^{1/2}}{1/2}$$

$$= -\frac{x^2(1-x^2)^{1/2}}{3} + \frac{2}{3}\sqrt{1-x^2}$$

26. Calculate the value of $\int_0^{2a} x^m\sqrt{2ax-x^2}\,dx$, n being a positive integer.

Hence or otherwise calculate the values of

(*i*) $\int_0^{2a} x\sqrt{2ax-x^2}\,dx$

(*ii*) $\int_0^{2a} x^4\sqrt{2ax-x^2}\,dx$

Sol. $\int_0^{2a} x^m\sqrt{2ax-x^2}\,dx = \int_0^{2a} x^{m+1/2}(2a-x)^{1/2}\,dx = I_m$ (say)

We connect $\int x^{m+1/2}(2a-x)^{1/2}\,dx$ with $\int x^{m-1/2}(2a-x)^{1/2}\,dx$

Here $P = x^{m+1/2}(2a-x)^{3/2}$

$$\frac{dP}{dx} = \left(m+\frac{1}{2}\right)x^{m-1/2}(2a-x)^{3/2} + x^{m+1/2}\cdot\frac{3}{2}(2a-x)^{1/2}(-1)$$

$$= \left(m+\frac{1}{2}\right)x^{m-1/2}(2a-x)^{1/2}(2a-x) - \frac{3}{2}x^{m+1/2}(2a-x)^{1/2}$$

$$= 2a\left(m+\frac{1}{2}\right)x^{m-1/2}(2a-x)^{1/2} - \left(m+\frac{1}{2}\right)x^{m+1/2}(2a-x)^{1/2}$$

$$-\frac{3}{2}x^{m+1/2}(2a-x)^{1/2}$$

$$= 2a\left(m+\frac{1}{2}\right)x^{m-1/2}(2a-x)^{1/2} - (m+2)x^{m+1/2}(2a-x)^{1/2}\,dx$$

Integrating, we have

$$P = x^{m+1/2}(2a-x)^{3/2}$$

$$= a(2m+1)\int x^{m-1/2}(2a-x)^{1/2}\,dx - (m+2)\int x^{m+1/2}(2a-x)^{1/2}\,dx$$

or $\quad (m+2)\int x^{m+1/2}(2a-x)^{1/2}\,dx$

$$= -x^{m+1/2}(2a-x)^{3/2} + a(2m+1)\int x^{m-3/2}(2a-x)^{3/2}dx$$

or $$(m+2)\int_0^{2a} x^{m+1/2}(2a-x)^{1/2}dx = 0 + a(2m+1)\int_0^{2a} x^{m-1/2}(2a-x)^{1/2}dx$$

Thus $(m+2)I_m = a(2m+1)I_{m-1}$ or $I_m = \frac{(2m+1)a}{m+2}I_{m-1}$ (1)

Similarly, $I_{m-1} = \frac{(2m-1)a}{m+1}I_{m-2}$ (2)

$$I_{m-2} = \frac{(2m-3)a}{m} I_{m-3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_3 = \frac{7a}{5} I_2 \;,\; I_2 = \frac{5a}{4} I_1 \;,\; I_1 = \frac{3a}{3} I_0$$

$$I_0 = \int_0^{2a} x^{1/2}(2a-x)^{1/2}$$

$$= \int_0^{2a} \sqrt{2ax - x^2}\, dx. \text{ Putting } x = 2a\sin^2\theta, \text{ so that}$$

$$dx = 4a\sin\theta\cos\theta\, d\theta, \text{ we have}$$

$$I_0 = \int_0^{\pi/2} \sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta} \cdot (4a\sin\theta\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{4a^2\sin^2\theta(1-\sin^2\theta)} \cdot 4a\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} 2a\sin\theta\cos\theta \cdot 4a\sin\theta\cos\theta\, d\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta = 8a^2 \frac{1\cdot 1}{4\cdot 2}\frac{\pi}{2} = \frac{a^2\pi}{2}$$

Multiplying vertically, we get

$$I_m = \frac{(2m+1)(2m-1)\ldots 5\cdot 3}{(m+2)(m+1)\ldots 4.\,3} a^{m+2} \cdot \frac{\pi}{2} \qquad \text{(A)}$$

(i) Putting $m = 1$ in (A), we have

$$I_1 = \int_0^{2a} x\sqrt{2ax - x^2}\, dx = \frac{3}{3}\cdot a^3 \frac{\pi}{2} = \frac{a^3\pi}{2}$$

(ii) Putting $m = 4$ in (A), we have

$$I_4 = \int_0^{2a} x^4\sqrt{2ax - x^2}\, dx = \frac{9\cdot 7\cdot 5\cdot 3}{6\cdot 5\cdot 4\cdot 3}\cdot a^6 \cdot \frac{\pi}{2} = \frac{21a^6\pi}{16}$$

27. If $\quad I_n = \int x^n (a^2 - x^2)^{1/2}\, dx$, prove that

$$I_n = \frac{x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \frac{n-1}{n+2} a^2 I_{n-2}$$

Hence evaluate $\displaystyle\int_0^a x^4\sqrt{a^2 - x^2}\, dx.$

Sol. Connect $\int x^n (a^2-x^2)^{1/2}\, dx$ with $\int x^{n-2}(a^2-x^2)^{1/2}\, dx$

Here $P = x^{n-1}(a^2 - x^2)^{3/2}$

$$\frac{dP}{dx} = (n-1)x^{n-2}(a^2-x^2)^{3/2} + x^{n-1}\cdot\frac{3}{2}(a^2-x^2)^{1/2}(-2x)$$

$$= (n-1)x^{n-2}(a^2-x^2)^{1/2}(a^2-x^2) - 3x^n(a^2-x^2)^{1/2}$$

$$= (n-1)a^2x^{n-2}(a^2-x^2)^{1/2} - (n-1)x^n(a^2-x^2)^{1/2} - 3x^n(a^2-x^2)^{1/2}$$

$$= (n-1)a^2x^{n-2}(a^2-x^2)^{1/2} - (n-1+3)x^n(a^2-x^2)^{1/2}$$

Integrating, we get

$$P = x^{n-1}(a^2-x^2)^{3/2} = (n-1)a^2\int x^{n-2}(a^2-x^2)^{1/2}dx - (n+2)\int x^n(a^2-x^2)^{1/2}dx$$

or $\quad (n+2)\int x^n(a^2-x^2)^{1/2}\, dx$

$$= -x^{n-1}(a^2-x^2)^{3/2} + (n-1)a^2\int x^{n-2}(a^2-x^2)^{1/2}\, dx$$

Therefore, $\displaystyle I_n = \frac{-x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \frac{n-1}{n+2}a^2 I_{n-2}$

$$I_4 = \frac{-x^3(a^2-x^2)^{3/2}}{6} + \frac{3}{6}\cdot a^2 I_2 \text{ and } I_2 = -\frac{x(a^2-x^2)^{1/2}}{4} + \frac{1}{4}a^2 I_0$$

Hence $\displaystyle\int_0^a x^4(a^2-x^2)^{1/2}\, dx = \frac{1}{2}a^2\int_0^a x^2(a^2-x^2)^{1/2}\, dx$

$$= \frac{1}{2}a^2\cdot\left(\frac{1}{4}a^2\int_0^a\sqrt{a^2-x^2}\, dx\right)$$

$$= \frac{a^4}{8}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arcsin\frac{x}{a}\right]_0^a$$

$$= \frac{a^4}{8}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^6}{32}$$

28. Prove that $\displaystyle\int x^m(\ln x)^n\, dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\, dx.$

Hence calculate

(i) $\displaystyle\int x^m(\ln x)^3\, dx$ $\qquad$ *(ii)* $\displaystyle\int_0^1 x^m(\ln x)^n\, dx$

Sol. $\displaystyle\int x^m(\ln x)^n\, dx = (\ln x)^n\cdot\frac{x^{m+1}}{m+1} - \int\frac{x^{m+1}}{m+1}\cdot n(\ln x)^{n-1}\cdot\frac{1}{x}\, dx$

(Integrating by parts)

$$= \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\, dx \qquad (1)$$

(i) Putting $n = 3$ in (1), we get

$$\int x^m (\ln x)^3\, dx = \frac{x^{m+1}(\ln x)^3}{m+1} - \frac{3}{m+1}\int x^m (\ln x)^2\, dx$$

$$= \frac{x^{m+1}(\ln x)^3}{m+1} - \frac{3}{m+1}\left[\frac{x^{m+1}(\ln x)^2}{m+1} - \frac{2}{m+1}\int x^m \ln x\, dx\right]$$

$$= \frac{x^{m+1}(\ln x)^3}{m+1} - \frac{3x^{m+1}(\ln x)^2}{(m+1)^2} + \frac{6}{(m+1)^2}\left[\ln x \cdot \frac{x^{m+1}}{m+1} - \int \frac{x^m}{m+1}\, dx\right]$$

$$= \frac{x^{m+1}}{m+1}\left[(\ln x)^3 - \frac{3(\ln x)^2}{m+1} + \frac{6\ln x}{(m+1)^2} - \frac{6}{(m+1)^3}\right]$$

(ii) Again, from (1), we have

$$\int_0^1 x^m (\ln x)^n\, dx = -\frac{n}{m+1}\int_0^1 x^m (\ln x)^{n-1}\, dx$$

i.e., $I_{m,n} = -\frac{n}{m+1} I_{m,n-1}$

$$I_{m,n-1} = -\frac{n-1}{m+1} I_{m,n-2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_{m,2} = -\frac{2}{m+1} I_{m,1}\ ;$$

$$I_{m,1} = -\frac{1}{m+1} I_{m,0}\ ;$$

$$I_{m,0} = \int_0^1 x^m\, dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}$$

Multiplying vertically, we get $I_{m,n} = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$

29. Prove that

$$\int_0^{\pi/2} \cos^m x \sin nx\, dx = \frac{1}{m+n} + \frac{m}{m+n}\int_0^{\pi/2} \cos^{m-1} x \sin(n-1)x\, dx$$

Hence evaluate $\int_0^{\pi/2} \cos^5 x \sin 3x\, dx$.

Sol. $\int \cos^m x \sin nx\, dx$

$$= \cos^m x\left(-\frac{\cos nx}{n}\right) - \int m\cos^{m-1} x\,(-\sin x)\left(-\frac{\cos nx}{n}\right) dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \sin x \cos nx\, dx \qquad (1)$$

Since $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$, we have
$\cos nx \sin x = \sin nx \cos x - \sin(n-1)x$.

Putting this value of $\cos nx \sin x$ into (1), we get

$$\int \cos^m x \sin nx\, dx = -\frac{\cos^m x \cos nx}{n}$$

$$-\frac{m}{n}\int \cos^{m-1} x\,[\sin nx \cos x - \sin(n-1)x]\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx\, dx$$

$$+ \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\, dx$$

Transposition yields $\left(1 + \frac{m}{n}\right)\int \cos^m x \sin nx\, dx$

$$= -\frac{\cos^m x \cos nx}{n} + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\, dx$$

or $\int \cos^m x \sin nx\, dx$

$$= -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \sin(n-1)x\, dx$$

Hence $\int_0^{\pi/2} \cos^m x \sin nx\, dx$

$$= \left[-\frac{\cos^m x \cos nx}{m+n}\right]_0^{\pi/2} + \frac{m}{m+n}\int_0^{\pi/2} \cos^{m-1} x \sin(n-1)x\, dx$$

$$= \frac{1}{m+n} + \frac{n}{m+n}\int_0^{\pi/2} \cos^{m-1} x \sin(n-1)x\, dx.$$

Putting $m = 5$ and $n = 3$ in the above formula we have

$$\int_0^{\pi/2} \cos^5 x \sin 3x\, dx = \frac{1}{5+3} + \frac{5}{5+3}\int_0^{\pi/2} \cos^4 x \sin 2x\, dx$$

$$= \frac{1}{8} + \frac{5}{8}\int_0^{\pi/2} \cos^4 x \sin 2x\, dx \text{ and}$$

$$\int_0^{\pi/2} \cos^4 x \sin 2x\, dx = \frac{1}{4+2} + \frac{4}{4+2}\int_0^{\pi/2} \cos^3 x \sin x\, dx$$

$$= \frac{1}{6} + \frac{2}{3}\left[-\frac{\cos^4 x}{4}\right]_0^{\pi/2} = \frac{1}{6} + \frac{2}{3}\left[-\left(0 - \frac{1}{4}\right)\right] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Thus $\int_0^{\pi/2} \cos^5 x \sin 3x \, dx = \frac{1}{8} + \frac{5}{8} \cdot \frac{1}{3} = \frac{3+5}{24} = \frac{1}{3}$

30. Find a reduction formula for $\int \frac{x^n}{\sqrt{ax^2 + 2bx + c}} dx$.

Sol. We connect the given integral with $\frac{x^{n-2}}{\sqrt{ax^2 + 2bx + c}} dx$

Hence $P = x^{n-2+1}(ax^2 + 2bx + c)^{-1/2+1} = x^{n-1}(ax^2 + 2bx + c)^{1/2}$

$$\frac{dP}{dx} = (n-1)\, x^{n-2} \cdot (ax^2 + 2bx + c)^{1/2} + x^{n-1} \cdot \frac{2ax + 2b}{2\sqrt{ax^2 + 2bx + c}}$$

$$= \frac{(n-1)x^{n-2}(ax^2 + 2bx + c) + x^{n-1}(ax + b)}{\sqrt{ax^2 + 2bx + c}}$$

$$= \frac{(n-1)(ax^n + 2bx^{n-1} + cx^{n-2}) + ax^n + bx^{n-1}}{\sqrt{ax^2 + 2bx + c}}$$

$$= \frac{an\, x^n}{\sqrt{ax^2 + 2bx + c}} + \frac{b(2n-1)x^{n-1}}{\sqrt{ax^2 + 2bx + c}} + \frac{c(n-1)x^{n-2}}{\sqrt{ax^2 + 2bx + c}}$$

Integrating, we obtain

$$P = x^{n-1}\sqrt{ax^2 + 2bx + c} = an \int \frac{x^n}{\sqrt{ax^2 + 2bx + c}} dx + \int \frac{b(2n-1)x^{n-1}}{\sqrt{ax^2 + 2bx + c}} dx + \int \frac{c(n-1)x^{n-2}}{\sqrt{ax^2 + 2bx + c}} dx$$

or $\int \frac{x^n}{\sqrt{ax^2 + 2bx + c}} dx$

$$= \frac{x^{n-1}\sqrt{ax^2 + 2bx + c}}{an} - \frac{b(2n-1)}{an} \int \frac{x^{n-1}}{\sqrt{ax^2 + 2bx + c}} dx - \frac{c(n-1)}{an} \int \frac{x^{n-2}}{\sqrt{ax^2 + 2bx + c}} dx$$

is the required reduction formula.

Exercise Set 5.5 (Page 211)

In each of Problems (1 – 12), use the trapezoidal rule to approximate the given integral:

1. $\int_1^4 \frac{dx}{x} = \ln 4$ with $n = 3$

Sol. Here length of each subinterval is $\frac{4-1}{3} = 1$. The interval [1, 4] is partitioned into subintervals by the points

$x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4$

The corresponding function values are as under:

n	x_n	$f(x_n)$
0	1	1
1	2	$\frac{1}{2}$
2	3	$\frac{1}{3}$
3	4	$\frac{1}{4}$

Substituting into the trapezoidal rule, we have

$$\int_1^4 \frac{dx}{x} \approx \frac{4-1}{3}\left[\frac{1}{2}(1) + \frac{1}{2} + \frac{1}{3} + \frac{1}{2}\left(\frac{1}{4}\right)\right] = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = 1 + \frac{11}{24} = \frac{35}{24}$$

≈ 1.4583 By calculator, $\ln 4 \approx 1.3863$

2. $\int_0^{\pi/3} \cos x \, dx = \frac{\sqrt{3}}{2}$ with $n = 4$

Sol. Length of each subinterval is $\frac{\frac{\pi}{3} - 0}{4} = \frac{\pi}{12}$

The interval $\left[0, \frac{\pi}{2}\right]$ is partitioned by the points

$x_0 = 0,\ x_1 = \frac{\pi}{12},\ x_2 = \frac{2\pi}{12} = \frac{\pi}{6},\ x_3 = \frac{3\pi}{12} = \frac{\pi}{4},\ x_4 = \frac{\pi}{3}$

$f(x_0) = \cos 0 = 1, f(x_1) = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$f(x_2) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, f(x_3) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $f(x_4) = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\int_0^{\pi/3} \cos x\, dx \approx \frac{\pi}{12}\left[\frac{1}{2}(1) + \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} + \frac{1}{2}\left(\frac{1}{2}\right)\right]$$

$$\approx \frac{\pi}{12}\left[\frac{1}{2} + \frac{1}{4} + \frac{\sqrt{3}+1+\sqrt{2}\cdot\sqrt{3}+2}{2\sqrt{2}}\right]$$

$$\approx \frac{\pi}{12}\left[0.75 + \frac{\sqrt{3}+\sqrt{6}+3}{2\sqrt{2}}\right]$$

$$\approx \frac{\pi}{12}\left[0.75 + \frac{1.7321 + 2.4495 + 3}{2.8284}\right] = \frac{\pi}{12}\left[0.75 + \frac{7.1816}{2.8284}\right]$$

$$\approx 0.2618\,[0.75 + 2.5391] = 0.2618\,(3.2891)$$

$$\approx 0.8611 \qquad \text{By calculator } \frac{\sqrt{3}}{2} \approx 0.8660$$

3. $\int_0^2 e^{-x^2}\, dx$ with $n = 4$

Sol. Length of each subinterval $= \frac{2-0}{4} = \frac{1}{2}$

The interval [0, 2] is partitioned into subintervals by the points

$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1, x_3 = \frac{3}{2},\ x_4 = 2$$

$$f(x_0) = 1, f(x_1) = e^{-1/4} \approx 0.7788, f(x_2) = e^{-1} = 0.3679$$

$$f(x_3) = e^{-9/4} = 0.1054,\ f(x_4) = e^{-4} = 0.0183$$

$$\int_0^2 e^{-x^2}\, dx \approx \frac{2-0}{4}\left[\frac{1}{2}(1) + 0.7788 + 0.3679 + 0.1054 + \frac{1}{2}(0.01830)\right]$$

$$\approx \frac{1}{2}(1.7613) \approx .8807 \approx 0.88$$

4. $\int_0^4 x^2\, dx$ with $n = 8$

Sol. Length of subinterval $= \frac{4-0}{8} = \frac{1}{2}$

Points of the partitions are

$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2,\ x_5 = \frac{5}{2}$$

$$x_6 = 3,\ x_7 = \frac{7}{2},\ x_8 = 4.$$

$$\int_0^4 x^2\, dx \approx \frac{1}{2}\left[\frac{1}{2}f(x_0) + f(x_1) + \ldots + f(x_7) + \frac{1}{2}f(x_8)\right]$$

$$\approx \frac{1}{2}\left[0 + 0.25 + 1 + 2.25 + 4 + 6.25 + 9 + 12.25 + \frac{16}{2}\right] = \frac{43}{2}$$

$$\approx 21.5$$

5. $\int_0^\pi \sin x\, dx$ with $n = 6$

Sol. Length of each subinterval $= \frac{\pi}{6}$

Points of the partition of $[0, \pi]$ are

$$x_0 = 0,\ x_1 = \frac{\pi}{6},\ x_2 = \frac{\pi}{3},\ x_3 = \frac{\pi}{2},\ x_4 = \frac{2\pi}{3},\ x_5 = \frac{5\pi}{6},\ x_6 = \pi$$

$$f(x_0) = 0,\ f(x_1) = \sin\frac{\pi}{6} = \frac{1}{2},\ f(x_2) = \frac{\sqrt{3}}{2},\ f(x_3) = 1,$$

$$f(x_4) = \frac{\sqrt{3}}{2},\ f(x_5) = \frac{1}{2},\ f(x_6) = 0$$

$$\int_0^\pi \sin x\, dx \approx \frac{\pi}{6}\left[\frac{1}{2}(0) + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2} + \frac{1}{2}(0)\right]$$

$$\approx \frac{\pi}{6}(2 + \sqrt{3}) \approx \pi\left(\frac{3.732}{6}\right) \approx (3.14159)\,(0.6228) \approx 1.9541$$

6. $\int_0^2 \frac{dx}{1+x^3}$ with $n = 4$

Sol. Length of each interval $= \frac{2-0}{4} = \frac{1}{2}$

Points of a subdivision of [0, 2] are

$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$$

$$\int_0^2 \frac{dx}{1+x^3} \approx \frac{1}{2}\left[\frac{1}{2}(1) + \frac{8}{9} + \frac{1}{2} + \frac{8}{35} + \frac{1}{2}\left(\frac{1}{9}\right)\right]$$

$$\approx \frac{1}{2}[0.5 + 0.88889 + 0.5 + 0.22857 + 0.05556] = \frac{1}{2}(2.17302)$$

$$\approx 1.0865$$

7. $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ with $n = 4$

Sol. Length of each subinterval $= \frac{1-0}{4} = \frac{1}{4}$

Points of a subdivision of [0, 1] are

$x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 = 1$

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} \approx \frac{1}{4}\left[\frac{1}{2}f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + \frac{1}{2}f(1)\right]$$

$$\approx \frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{2}\right) + \frac{4}{\sqrt{63}} + \frac{2}{\sqrt{15}} + \frac{4}{\sqrt{55}} + \frac{1}{2}\frac{1}{\sqrt{3}}\right]$$

$$\approx \frac{1}{4}\left[\frac{1}{4} + \frac{4}{21}\cdot\sqrt{7} + \frac{2}{15}\cdot\sqrt{15} + \frac{4}{55}\cdot\sqrt{55} + \frac{1}{6}\cdot\sqrt{3}\right]$$

$$\approx \frac{1}{4}(0.2500+0.5040+0.5164+0.5394+0.2887) = \frac{1}{4}(2.0985)$$

$$\approx 0.5246$$

8. $\int_{-2}^2 (2x^2 + 1)\,dx$ with $n = 4$

Sol. Length of each subinterval $= \frac{2-(-2)}{4} = 1$

Points of subdivision of [−2, 2] are

$x_0 = -2,\ x_1 = -1\ \ x_2 = 0,\ x_3 = 1,\ x_4 = 2$

$$\int_{-2}^2 (2x^2 + 1)\,dx \approx 1\cdot\left[1.\frac{1}{2}f(-2) + f(-1) + f(0) + f(1) + \frac{1}{2}f(2)\right]$$

$$\approx \frac{1}{2}(9) + 3 + 1 + 3 + \frac{9}{2} = 16.$$

9. $\int_1^5 \frac{dx}{x^2}$ with $n = 4$

Sol. Length of each subinterval $= \frac{5-1}{4} = 1$

Points of subdivision of [1, 5] are

$x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4,\ x_4 = 5$

$$\int_5^1 \frac{dx}{x^2} \approx 1\cdot\left[\frac{1}{2}f(1) + f(2) + f(3) + f(4) + \frac{1}{2}f(5)\right]$$

$$\approx \frac{1}{2}(1) + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{2}\left(\frac{1}{25}\right)$$

$$\approx 0.5000 + 0.2500 + 0.1111 + 0.0625 + 0.0200 = 0.9436$$

10. $\int_0^1 e^{-x}\,dx$ with $n = 6$

Sol. Length of each subinterval $= \frac{1-0}{6} = \frac{1}{6}$

Points of subdivision of [0, 1] are

$x_0 = 0,\ x_1 = \frac{1}{6},\ x_2 = \frac{1}{3},\ x_3 = \frac{1}{2},\ x_4 = \frac{2}{3},\ x_5 = \frac{5}{6},\ x_6 = 1$

$$\int_0^1 e^{-x}\,dx \approx \frac{1}{6}\left[\frac{1}{2}f(0) + f\left(\frac{1}{6}\right) + f\left(\frac{1}{3}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{2}{3}\right) + f\left(\frac{5}{6}\right) + \frac{1}{2}f(1)\right]$$

$$\approx \frac{1}{6}\left[\frac{1}{2}\left(\frac{1}{1}\right) + \frac{1}{1.18136} + \frac{1}{1.3956} + \frac{1}{1.6487} + \frac{1}{1.9477} + \frac{1}{2}\left(\frac{1}{2.71828}\right)\right]$$

$$\approx \frac{1}{6}[0.5+0.8465+0.7165+0.6065+0.5134+0.4346+0.1839] = \frac{1}{6}[3.8014]$$

$$\approx 0.6336 \quad \left(\text{Note: } \int_0^1 e^{-x}\,dx = 1 - \frac{1}{e} \approx 1 - 0.3679 = 0.6321\right)$$

11. $\int_1^2 \ln x\,dx$ with $n = 4$

Sol. Length of each subinterval $= \frac{2-1}{4} = \frac{1}{4}$

Points of subdivision of [1, 2] are

$x_0 = 1,\ x_1 = \frac{5}{4},\ x_2 = \frac{6}{4} = \frac{3}{2},\ x_3 = \frac{7}{4},\ x_4 = 2$

$$\int_1^2 \ln x\,dx \approx \frac{1}{4}\left[\frac{1}{2}f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{7}{4}\right) + \frac{1}{2}f(2)\right]$$

$$\approx \frac{1}{4}\left[\frac{1}{2}\ln(1) + \ln(1.25) + \ln(1.5) + \ln(1.75) + \frac{1}{2}\ln 2\right]$$

$$\approx \frac{1}{4}\left[\frac{1}{2}(0) + 0.2231 + 0.4055 + 0.5596 + \frac{1}{2}(0.6931)\right]$$

$$\approx \frac{1.5348}{4} = 0.3837$$

$$\left(\text{Note: } \int_1^2 \ln x\, dx = 2\ln 2 - 1 = \ln 4 - 1 \approx 1.3863 - 1 = .3863\right)$$

12. $\displaystyle\int_0^2 \frac{1}{\sqrt{1+x^2}}\, dx$ with $n = 4$

Sol. Length of each subinterval $= \dfrac{2-0}{4} = \dfrac{1}{2}$

Points of subdivision of $[0, 2]$ are

$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$$

$$\int_0^2 \frac{1}{\sqrt{1+x^2}}\, dx \approx \frac{1}{2}\left[\frac{1}{2}f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + \frac{1}{2}f(2)\right]$$

$$\approx \frac{1}{2}\left[\frac{1}{2}(1) + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{13}} + \frac{1}{2}\left(\frac{1}{\sqrt{5}}\right)\right]$$

$$\approx \frac{1}{2}\left[\frac{1}{2}(1) + 0.9444 + 0.7071 + 0.5547 + \frac{1}{2}(0.4472)\right] = \frac{2.9298}{2}$$

$$\approx 1.4649 \quad \left(\text{Note: } \int_0^2 \frac{1}{\sqrt{1+x^2}}\, dx = \ln(\sqrt{5}+2) \approx \ln 4.2361 \approx 1.4436\right)$$

13. Use Simpson's rule to approximate the integrals of Problems 3, 4, 10, 11 and 12.

Sol.

(3) $\displaystyle\int_0^2 e^{-x^2}\, dx$ with $n = 4$

Length of each subinterval is $\frac{1}{2}$ · Using the function values as in Problem 3, and substituting into Simpson's rule, we have

$$\int_2^0 e^{-x^2}\, dx \approx \frac{2-0}{3\times 4}[1 + 4(0.7788) + 2(0.3679) + 4(0.1055) + 0.0183]$$

$$\approx \frac{1}{6}[1 + 3.1152 + 0.7358 + 0.4220 + 0.0183] = \frac{1}{6}(5.2913)$$

$$\approx 0.8819$$

(4) $$\int_0^4 x^2\, dx \approx \frac{4-0}{3\times 8}[0 + 4(0.25) + 2(1) + 4(2.25) + 2(4) + 4(6.25)$$

$$+ 2(9) + 4(12.25) + 16]$$

$$\approx \frac{1}{6}[1 + 2 + 9 + 8 + 25 + 18 + 49 + 16] = \frac{128}{6}$$

$$\approx 21.3333$$

(10) $\displaystyle\int_0^1 e^{-x}\, dx$ with $n = 6$

By Simpson's rule, we have

$$\int_0^1 e^{-x}\, dx \approx \frac{1-0}{3\times 6}[1 + 4(0.8465) + 2(0.7165) + 4(0.6065)$$

$$+ 2(0.5134) + 4(0.4346) + 0.3679]$$

$$\approx \frac{1}{18}[1+3.3360+1.4330+2.4260+1.0268+1.7384+0.3679] = \frac{1}{18}(11.3781)$$

$$\approx 0.6321$$

(11) $\displaystyle\int_1^2 \ln x\, dx$ with $n = 4$

By Simpson's rule, we have

$$\int_1^2 \ln x\, dx \approx \frac{1-0}{3\times 4}[0 + 4(0.2231) + 2(0.4055) + 4(0.5596) + 0.6931]$$

$$\approx \frac{1}{12}[0.8924 + 0.8110 + 2.2384 + 0.6931] = \frac{1}{12}[4.6349]$$

$$\approx 0.3862$$

(12) $\displaystyle\int_0^2 \frac{dx}{\sqrt{1+x^3}}$ with $n = 4$

By Simpson's rule, the given integral

$$\int_0^2 \frac{dx}{\sqrt{1+x^3}} \approx \frac{2-0}{3\times 4}[1 + 4(0.9444) + 2(0.7071) + 4(0.5547) + 0.4472]$$

$$\approx \frac{1}{6}[1 + 3.7776 + 1.4142 + 2.2188 + 0.4472] = \frac{1}{6}(8.8578)$$

$$\approx 1.4763$$

Find a bound on the error in approximating the given integral using (i) the trapezoidal rule (ii) Simpson's rule. (Problems 14 – 16):

14. $\displaystyle\int_{-1}^2 x^5\, dx$ with $n = 10$

Sol.

(i) Here $f(x) = x^5$, $f'(x) = 5x^4$, $f''(x) = 20x^3$

$f'''(x) = 60x^2$, $f^{(4)}(x) = 120x$

Now max $|f''(x) = 20x^3|$ on $[-1, 2]$ is attained at $x = 2$ and the maximum value $M = 20 \times 8 = 160$

$$\text{Maximum error} = \frac{(b-a)^3 M}{12n^2} = \frac{[2-(-1)]^3 M}{12 \cdot 10^2} = \frac{27 \times 160}{12 \times 100} = \frac{36}{10} = 3.6$$

(ii) For Simpson's rule,

$$\begin{aligned} M &= \max. |f^{(4)}(x)| \\ &= \max. |120x| \quad \text{on } [-1, 2] \\ &= 240 \end{aligned}$$

$$\text{Maximum error} = \frac{M(b-a)^5}{180n^4} = \frac{240 \times 3^5}{180 \times 10^4}$$

$$= \frac{240 \times 243}{180 \times 10000} = \frac{324}{10000} = 0.0324$$

15. $\displaystyle\int_1^3 \frac{dx}{x}$ with $n = 10$

Sol. $\displaystyle\int_1^3 \frac{dx}{x}$ with $n = 10$

(i) $f(x) = \dfrac{1}{x}$, $f'(x) = \dfrac{-1}{x^2}$, $f''(x) = \dfrac{2}{x^3}$

$f'''(x) = \dfrac{-6}{x^4}$, $f^{(4)}(x) = \dfrac{24}{x^5}$

Now max. $|f''(x)| = \max \left|\dfrac{2}{x^3}\right| = 2$ on $[1, 3]$

$$\text{Maximum error} = \frac{M(b-a)^3}{12n^2}$$

$$= \frac{2 \times 2^3}{12 \times 100} = \frac{1}{3}\left(\frac{4}{100}\right) = \frac{1}{3}(.04) \approx 0.01333$$

(ii) Max. $|f^{(4)}(x)| = \max \left|\dfrac{24}{x^5}\right| = 24$ on $[1, 3]$

$$\text{Maximum error} = \frac{M(b-a)^5}{180n^4} = \frac{24 \times 2^5}{180 \times 10^4}$$

$$= \frac{2 \times 32}{15 \times 10000} = \frac{1}{15}(.0064) \approx .00043$$

16. $\displaystyle\int_0^2 \frac{dx}{\sqrt{1+x}}$ with $n = 8$

Sol.

(i) $f(x) = (1+x)^{-1/2}$, $f'(x) = -\dfrac{1}{2}(1+x)^{-3/2}$

$f''(x) = \left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)(1+x)^{-5/2}$, $f'''(x) = \left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)\left(-\dfrac{5}{2}\right)(1+x)^{-7/2}$

$f^{(4)}(x) = \left(-\dfrac{1}{2}\right)\cdot\left(-\dfrac{3}{2}\right)\cdot\left(-\dfrac{5}{2}\right)\cdot\left(-\dfrac{7}{2}\right)(1+x)^{-9/2}$

$$M = \max. |f''(x)| = \max_{x \varepsilon [0, 2]} \left|\frac{3}{4} \cdot \frac{1}{(1+x)^{5/2}}\right| = \frac{3}{4}$$

$$\text{Maximum error} = \frac{M(b-a)^3}{12n^2} = \frac{\frac{3}{4} \cdot 2^3}{12(8)^2}$$

$$= \frac{3}{4} \times \frac{1}{12 \times 8} = \frac{1}{128} = 0.0078125$$

(ii) $M = \max. |f^{(4)}(x)| = \displaystyle\max_{x \in [0,2]} \left|\frac{105}{16} \cdot \frac{1}{(1+x)^{9/2}}\right| = \frac{105}{16}$

$$\text{Maximum error} = \frac{M(b-a)^5}{180n^4}$$

$$= \frac{105 \times 32}{16 \times 180 \times 64 \times 64} = \frac{7}{192 \times 128} \approx 0.0002848$$

17. With $n = 8$, find the area under the semicircle $y = \sqrt{4 - x^2}$ and above the x-axis by (i) the trapezoidal rule (ii) Simpson's rule.

Sol. Required area

$$A = \int_{-2}^{2} \sqrt{4 - x^2}\, dx \quad \text{with} \quad n = 8$$

$$\text{Length of each subinterval} = \frac{2-(-2)}{8} = \frac{1}{2}$$

Points of subdivision of $[-2, 2]$ are

$x_0 = -2$, $x_1 = -\dfrac{3}{2}$, $x_2 = -1$, $x_3 = -\dfrac{1}{2}$, $x_4 = 0$,

$x_5 = \dfrac{1}{2}$, $x_6 = 1$, $x_7 = \dfrac{3}{2}$, $x_8 = 2$

$f(x_0) = f(-2) = 0$, $f(x_1) = f\left(-\dfrac{3}{2}\right) \approx 1.32287$

$f(x_2) = f(-1) \approx 1.73210,\ f(x_3) = f\left(-\frac{1}{2}\right) \approx 1.9365$

$f(x_4) = f(0) = 2,\ f(x_5) = f\left(\frac{1}{2}\right) = 1.9365$

$f(x_6) = f(1) = 1.7321,\ f(x_7) = f\left(\frac{3}{2}\right) = 1.3229$

$f(x_8) = f(2) = 0$

(i) By the trapezoidal rule

$$A \approx \frac{1}{2}\left[\frac{1}{2}(0)+1.3229-1.7321+1.9365+2+1.9365+1.7321+1.3229+\frac{1}{2}(0)\right],$$

$$\approx \frac{1}{2}(11.9830) \approx 5.9915$$

(ii) By Simpson's rule,

$$A \approx \frac{1}{6}[0 + 4(1.3229) + 2(1.7321) + 4(1.9365) + 2(2) + 4(1.9365) + 2(1.7321) + 4(1.3229)+0]$$

$$\approx \frac{1}{6}[5.2916+3.4642+7.7460+4+7.7460+3.4642+5.2916] = \frac{37.0036}{6}$$

$$\approx 6.1673$$

Actual area $= 2\int_0^2 \sqrt{4-x^2}\,dx$

Put $x = 2\sin\theta$ so that $dx = 2\cos\theta\, d\theta$

When $x = 0,\ \theta = 0$ and when $x = 2,\ \theta = \frac{\pi}{2}$

$$\text{Area} = 2\int_0^{\pi/2} (2\cos\theta)\, 2\cos\theta\, d\theta = 8\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 8\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = 8\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 8\left[\left(\frac{\pi}{4}+0\right)-0\right] = 8\left(\frac{\pi}{4}\right) = 2\pi \approx 2(3.14159) \approx 6.2832$$

18. A reading of the velocity of a ship was made every quarter hour as shown below:

Time t (hours)	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
Velocity $v(t)$ (mph)	19.5	24.3	34.2	40.5	38.4	26.2	18	16	8

Estimate the distance travelled by the ship during the 2-hour period.

Sol. The total distance travelled by the ship during the 2-hour period is $\int_0^2 v(t)\,dt$.

We approximate this integral by the trapezoidal rule.

Here length of each subinterval is $\frac{2-0}{8} = \frac{1}{4}$. The points of subdivision and the corresponding values are as in the table above. Substituting into the trapezoidal rule, we have

$$\int_0^2 v(t)dt \approx \frac{1}{4}\left[\frac{1}{2}(19.5)+24.3+34.2+40.5+38.4 + 26.2 + 18 + 16 + \frac{1}{2}(8)\right]$$

$$\approx \frac{1}{4}[9.75+24.30+34.20+40.50+38.40+26.20+18.00+16.00+4.00] = \frac{1}{4}(211.35)$$

$$\approx 52.8375 \approx 52.84$$

The total distance travelled by the ship $\approx$ 52.84 miles.

Chapter 6

PLANE CURVES - I

Exercise Set 6.1 (Page 234)

Examine whether each of the given equations represents two straight lines. If so, find the equation of each straight line (Problems 1 – 5):

1. $10xy + 8x - 15y - 12 = 0$

Sol. Here $a = 0, b = 0, h = 5, g = 4, f = -\frac{15}{2}, c = -12,$

Now, $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 0 & 5 & 4 \\ 5 & 0 & -\frac{15}{2} \\ 4 & -\frac{15}{2} & -12 \end{vmatrix} = -5(-60 + 30) + 4\left(-\frac{75}{2}\right)$$

$$= 150 - 150 = 0$$

Hence the given equation represents two straight lines.

$10xy + 8x - 15y - 12 = 0$, my be written as

$2x(5y + 4) - 3(5y + 4) = 0$

or $(2x - 3)(5y + 4) = 0$

Thus $2x - 3 = 0$, $5y + 4 = 0$ are the lines represented by the given equation.

2. $2x^2 - xy + 5x - 2y + 2 = 0$

Sol. Here, $a = 2, b = 0, h = -\frac{1}{2}, g = \frac{5}{2}, f = -1, c = 2,$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & 0 & -1 \\ \frac{5}{2} & -1 & 2 \end{vmatrix} = 2(-1) + \frac{1}{2}\left(-1 + \frac{5}{2}\right) + \frac{5}{2}\left(\frac{1}{2}\right)$$

$$= -2 + \frac{3}{4} + \frac{5}{4} = -2 + 2 = 0$$

Thus the given equation represents two straight lines.

Now to obtain an equation of each line, we solve the given equation as a quadratic in x. The equation may be written as

$2x^2 + (5 - y)x + (2 - 2y) = 0$

$$x = \frac{-(5-y) \pm \sqrt{(5-y)^2 - 8(2-2y)}}{4} = \frac{(y-5) \pm \sqrt{y^2+6y+9}}{4}$$

$$= \frac{(y-5) \pm \sqrt{(y+3)^2}}{4} = \frac{(y-5) \pm (y+3)}{4}$$

$$= \frac{y-5+y+3}{4}, \frac{y-5-y-3}{4} = \frac{y-1}{2}, -2$$

Equations of the lines are

$$2x - y + 1 = 0 \quad \text{and} \quad x + 2 = 0$$

3. $6x^2 - 17xy - 3y^2 + 22x + 10y - 8 = 0$

Sol. Here $a = 6, b = -3, h = \frac{-17}{2}, g = 11, f = 5, c = -8,$

and $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 6(-3)(-8) + 2 \times 5 \times 11 \times \left(\frac{-17}{2}\right) - 6(5)^2 - (-3)(11)^2 - (-8)\left(\frac{-17}{2}\right)^2$$

$$= 144 - 55 \times 17 - 6 \times 25 + 3 \times 121 + 8 . \frac{289}{4}$$

$$= 144 - 935 - 150 + 363 + 578 = 1085 - 1085 = 0$$

Thus the given equation represents a pair of lines. The equation may be written as

$$6x^2 + (22 - 17y)\,x + (-3y^2 + 10y - 8) = 0$$

$$x = \frac{-(22-17y) \pm \sqrt{(22-17y)^2 - 24(-3y^2+10y-8)}}{12}$$

$$= \frac{-(22-17y) \pm \sqrt{484 + 289y^2 - 748y + 72y^2 - 240y + 192}}{12}$$

$$= \frac{-(22-17y) + \sqrt{361y^2 - 988y + 676}}{12}$$

$$= \frac{(-22+17y) \pm (19y-26)}{12} = -4 + 3y, \frac{2-y}{6}$$

Equations of the lines are

$$x - 3y + 4 = 0 \quad \text{and} \quad 6x + y - 2 = 0$$

4. $10x^2 - 23xy - 5y^2 - 29x + 32y + 21 = 0$

Sol. Here $a = 10, b = -5, h = \frac{-23}{2}, g = \frac{-29}{2}, f = 16, c = 21,$

Now, $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= -10 \times (-5) \times 21 + 2\left(16 \times \frac{-29}{2} \times \frac{-23}{2}\right) - 10 \times (16)^2 - (-5)\left(\frac{-29}{2}\right)^2 - 21\left(\frac{-23}{2}\right)^2$$

$$= -1050 + 5336 - 2560 + \frac{4205}{4} - \frac{11109}{4}$$

$$= 1726 + \frac{4205 - 11109}{4} = 1726 + \frac{-6904}{4} = 1726 - 1726 = 0$$

Hence the equation represents a pair of lines. The given equation can be written as

$$10x^2 - (23y + 29)x + (-5y^2 + 32y + 21) = 0$$

$$x = \frac{(23y+29) \pm \sqrt{(23y+29)^2 - 40(-5y^2+32y+21)}}{20}$$

$$= \frac{(23y+29) \pm \sqrt{529y^2 + 1334y + 841 + 200y^2 - 1280y - 840}}{20}$$

$$= \frac{(23y+29) \pm \sqrt{729y^2 + 54y + 1}}{20} = \frac{(23y+29) \pm (27y+1)}{20}$$

$$= \frac{5y+3}{2}, \frac{-y+7}{5}$$

Equations of the lines are

$$2x - 5y - 3 = 0 \quad \text{and} \quad 5x + y - 7 = 0$$

5. $6x^2 - 15y^2 - xy + 16x + 24y = 0$

Sol. Here $a = 6, b = -15, h = \frac{-1}{2}, g = 8, f = 12, c = 0,$

and $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 0 - 96 - 6 \times 144 + 15 \times 64 - 0$$

$$= -96 - 864 + 960 = -960 + 960 = 0$$

The given equation represents a pair of lines. It may be written as

$$6x^2 + (-y + 16)x + (-15y^2 + 24y) = 0$$

$$x = \frac{-(16-y) \pm \sqrt{(16-y)^2 - 24(-15y^2+24y)}}{12}$$

$$= \frac{(y-16) \pm \sqrt{256 + y^2 - 32y + 360y^2 - 576y}}{12}$$

$$= \frac{(y-16) \pm \sqrt{361y^2 - 608y + 256}}{12} = \frac{(y-16) \pm (19y-16)}{12}$$

$$= \frac{20y-32}{12}, \frac{-18y}{12} \qquad i.e., \quad x = \frac{5y-8}{3}, \frac{-3y}{2}$$

Equations of the lines are

$$3x - 5y + 8 = 0 \quad \text{and} \quad 2x + 3y = 0$$

For what value of λ will each of the following equations represent a pair of straight lines? (Problems 6 – 8):

6. $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$

Sol. Here $a = \lambda, h = -5, b = 12, g = \frac{5}{2}, f = -8, c = -3.$

The given equation represents two straight lines if

$$\begin{vmatrix} \lambda & -5 & 5/2 \\ -5 & 12 & -8 \\ 5/2 & -8 & -3 \end{vmatrix} = 0$$

or $\quad \lambda(-36-64)+5(15+20)+\frac{5}{2}(40-30)=0$

or $\quad -100\lambda+175+25=0$

or $\quad -100\lambda=-200 \quad$ *i.e.,* $\quad \lambda=2$

7. $\lambda xy+5x+3y+2=0 \qquad (1)$

Sol. Here $a=0, b=0, h=\frac{\lambda}{2}, g=\frac{5}{2}, f=\frac{3}{2}, c=2,$

(1) represents two straight lines if

$$\begin{vmatrix} 0 & \frac{\lambda}{2} & \frac{5}{2} \\ \frac{\lambda}{2} & 0 & \frac{3}{2} \\ \frac{5}{2} & \frac{3}{2} & 2 \end{vmatrix} = 0 \quad \text{or} \quad -\frac{\lambda}{2}\left(\lambda-\frac{15}{4}\right)+\frac{5}{2}\times\frac{3\lambda}{4}=0$$

or $\quad -\frac{\lambda^2}{2}+\frac{15\lambda}{8}+\frac{15\lambda}{8}=0 \quad$ or $\quad -\frac{\lambda^2}{2}+\frac{30\lambda}{8}=0$

or $\quad -2\lambda^2+15\lambda=0 \quad$ or $\quad \lambda(-2\lambda+15)=0$

i.e., $\quad \lambda=0, \quad \frac{15}{2}$

If $\lambda=0$, the given equation is linear and represents a straight line.

Hence (1) represents a pair of lines if $\lambda=\frac{15}{2}$.

8. $4x^2-9y^2-2(8+\lambda)x-18y=29+2\lambda.$

Sol. The given equation can be written as

$$4x^2-9y^2-2(8+\lambda)x-18y-(29+2\lambda)=0 \qquad (1)$$

Here $a=4, b=-9, h=0, g=-(8+\lambda), f=-9, c=-(29+2\lambda),$

(1) will represent a pair of lines if

$$\begin{vmatrix} 4 & 0 & -(8+\lambda) \\ 0 & -9 & -9 \\ -(8+\lambda) & -9 & -(29+2\lambda) \end{vmatrix} = 0$$

or $\quad 4[9(29+2\lambda)-9\times 9]-(8+\lambda)[0-(-9)\times(-(8+\lambda))]=0$

or $\quad 36(20+2\lambda)+9(8+\lambda)^2=0 \Rightarrow 4(20+2\lambda)+(8+\lambda)^2=0$

or $\quad 80+8\lambda+64+\lambda^2+16\lambda=0$

or $\quad \lambda^2+24\lambda+144=0 \quad$ or $\quad (\lambda+12)^2=0$

Thus $\lambda=-12$

Find the angle between each of the following pairs of lines (Problem 9 – 13):

9. $x^2-2xy\tan\theta-y^2=0$

Sol. Here $a=1, b=-1$ so that $a+b=0$

Thus the two lines are perpendicular.

10. $3x^2+7xy+2y^2=0$

Sol. Here $a=3, b=2, h=\frac{7}{2}$

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{\frac{49}{4}-3\times 2}}{3+2}=\frac{2}{5}\times\sqrt{\frac{25}{4}}=\frac{2}{5}\times\frac{5}{2}=1$$

Thus $\quad \theta=45^\circ$

11. $11x^2+16xy-y^2=0$

Sol. Here $a=11, h=8, b=-1$

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{64+11}}{11-1}=\frac{2\times 5\sqrt{3}}{10}=\sqrt{3}$$

Hence $\quad \theta=60^\circ$

12. $x^2+4xy+y^2-6x-3=0$

Sol. Here $a=1, b=1, h=2$

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{4-1}}{1+1}=\frac{2\sqrt{3}}{2}=\sqrt{3}$$

Thus $\quad \theta=60^\circ$

13. $6x^2+xy-y^2-21x-8y+9=0$

Sol. Here $a=6, b=-1, h=\frac{1}{2}$

$$\tan\theta=\frac{2\sqrt{\frac{1}{4}+6}}{6+(-1)}=\frac{2\times\frac{5}{2}}{5}=1$$

Thus $\theta=45^\circ$

14. Show that $16xy-6x+8y-3=0$ represents a pair of straight lines. Also prove that this together with the coordinate axes form a rectangle and find the area enclosed by the rectangle.

Sol. $16xy-6x+8y-3=0 \qquad (1)$

Here $a=0, b=0, h=8, g=-3, f=4, c=-3$, so

$$\begin{vmatrix} 0 & 8 & -3 \\ 8 & 0 & 4 \\ -3 & 4 & -3 \end{vmatrix} = -8(-24+12)+(-3)(32-0)=96-96=0$$

Thus (1) represents a pair of straight lines.

(1) can be written as

$$2x(8y-3)+1\,.\,(8y-3)=0 \;\Rightarrow\; (2x+1)(8y-3)=0$$

i.e., equations of lines are $2x+1=0$ (2)

$8y-3=0$ (3)

As $a+b=0$, so these lines are perpendicular.

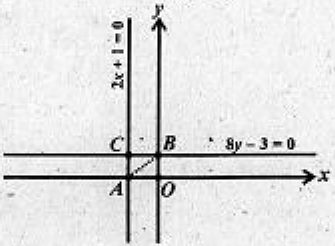
(2) is perpendicular to the x-axis and cuts it at $A\left(-\frac{1}{2}, 0\right)$

(3) is perpendiculars to the y-axis and cuts it at $B\left(0, \frac{3}{8}\right)$

Thus these lines together with the coordinates axes form a rectangle.

Let C be the point of intersection of (2) and (3).

Then C is $\left(-\frac{1}{2}, \frac{3}{8}\right)$.



Area of rectangle $AOBC = 2 \times$ area of the triangle AOB.

$$= 2\,.\left(\frac{1}{2}\begin{vmatrix} -\frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 \\ 0 & \frac{3}{8} & 1 \end{vmatrix}\right)$$

$$= -\frac{1}{2}\left(0-\frac{3}{8}\right)+1(0)=\frac{3}{16}$$

Thus the required area is $\frac{3}{16}$ square units.

15. Show that an equation of the rectangular hyperbola $x^2-y^2=1$ referred to its asymptotes as axes is $x'y'=-\frac{1}{2}$.

Sol. Equation of the asymptotes of the hyperbola

$x^2-y^2=1$ (1)

is $x^2-y^2=0$

i.e., $x-y=0$ and $x+y=0$ are asymptotes of (1)

These are the new axes. The line $x-y$ or $y=x$ makes an angle of 45° with the x-axis. Therefore the new axes are obtained by rotating the original axes through an angle of 45°. Equations of transformation are

$$x = x'\cos 45^\circ - y'\cos 45^\circ$$
$$y = x'\sin 45^\circ + y'\cos 45^\circ$$

i.e., $x=\frac{x'-y'}{\sqrt{2}}$, $y=\frac{x'+y'}{\sqrt{2}}$

Substituting into the equation (1) of the hyperbola, we get

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2-\left(\frac{x'+y'}{\sqrt{2}}\right)^2=1$$

or $[(x'-y')+(x'+y')]\,[(x'-y')-(x'+y')]=2$

or $(2x')(-2y')=2 \;\Rightarrow\; -4x'y'=2$

or $x'y'=-\frac{1}{2}$ as required.

Analyze and graph the conic represented by each of the following equations (Problems 16 – 25):

16. $\sqrt{x}+\sqrt{y}=1$

Sol. Squaring both sides, we have

$x+y+2\sqrt{xy}=1$

or $x+y-1=-2\sqrt{xy}$

Again squaring, we obtain

$x^2+2xy+y^2-2x-2y+1=4xy$

or $x^2-2xy+y^2-2x-2y+1=0$ (1)

Here $a=1=b$ and $a-b=0$, so $2\theta=90^\circ \Rightarrow \theta=45^\circ$

To remove the product term xy, we rotate the axes through an angle of 45°. Equations of transformation are

$$x=\frac{x'-y'}{\sqrt{2}},\quad y=\frac{x'+y'}{\sqrt{2}}$$

Substituting into (1), we get

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2-2\frac{(x'-y')(x'+y')}{2}+\left(\frac{x'+y'}{\sqrt{2}}\right)^2-2\left(\frac{x'-y'}{\sqrt{2}}\right)-2\left(\frac{x'+y'}{\sqrt{2}}\right)+1=0$$

or $2y'^2=2\sqrt{2}\,x'-1$

or $y'^2=\sqrt{2}\left(x'-\frac{1}{2\sqrt{2}}\right)$

or $Y^2=\sqrt{2}\,X$, where $y'=Y$ and $x'-\frac{1}{2\sqrt{2}}=X$

This represents a parabola.

Axis: $Y=0 \;\Rightarrow\; y'=0 \Rightarrow \frac{-x+y}{\sqrt{2}}=0$ or $x-y=0$

Focus: $X=\frac{\sqrt{2}}{4}=\frac{1}{2\sqrt{2}}$, $Y=0$

i.e., $x'-\frac{1}{2\sqrt{2}}=\frac{1}{2\sqrt{2}} \Rightarrow x'=\frac{1}{\sqrt{2}}$ and $y'=0$

or $x' = \dfrac{x+y}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ and $\dfrac{-x+y}{\sqrt{2}} = 0$

i.e., $\left.\begin{aligned} x+y&=1\\ x-y&=0 \end{aligned}\right\} \Rightarrow x = \dfrac{1}{2},\ y = \dfrac{1}{2}$ (with "and" before the second line)

Thus $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ is focus of the given parabola.

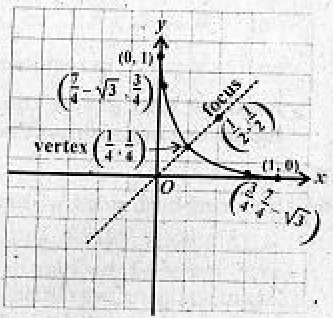
Vertex: $X = 0, Y = 0$

i.e., $x' - \dfrac{1}{2\sqrt{2}} = 0$

$\Rightarrow x' = \dfrac{1}{2\sqrt{2}}$ and $y' = 0$

or $\dfrac{x+y}{\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$

$\Rightarrow \quad x + y = \dfrac{1}{2}$ and $\dfrac{-x+y}{\sqrt{2}} = 0$



These equations yield $x = \dfrac{1}{4}, y = \dfrac{1}{4}$

i.e., $\left(\dfrac{1}{4}, \dfrac{1}{4}\right)$ is the vertex. The graph of the parabola is as shown.

17. $xy = 1$ (1)

Sol. In order to eliminate the xy term, let the axes be rotated through an angle θ where

$$\tan 2\theta = \frac{2h}{a-b}$$

Here $a - b = 0$, therefore $2\theta = 90^\circ \quad \Rightarrow \quad \theta = 45^\circ$

Equations of transformation are

$$x = x'\cos 45^\circ - y'\sin 45^\circ = \frac{x'-y'}{\sqrt{2}}$$

$$y = y'\sin 45^\circ + y'\cos 45^\circ = \frac{x'+y'}{\sqrt{2}}$$

Substituting into (1), we have

$$\frac{x'^2 - y'^2}{2} = 1 \qquad i.e., \quad \frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

which is standard form of a rectangular hyperbola. Its centre is (0, 0), *i.e.*,

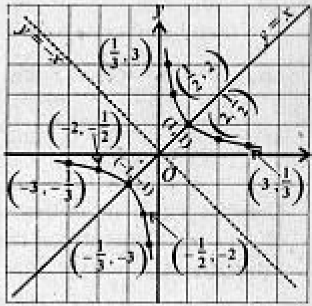
$$x' = 0 \Rightarrow x\cos 45^\circ + y\sin 45^\circ = \frac{x+y}{\sqrt{2}} \Rightarrow 0 \Rightarrow x + y = 0$$

and $y' = 0 \Rightarrow -x\cos 45^\circ + y\sin 45^\circ = \dfrac{-x+y}{\sqrt{2}} = 0 \Rightarrow -x + y = 0$

Solving the equations, we get $x = 0, y = 0$. Thus the centre of (1) is (0, 0) referred to xy-system.

Vertices: $x' = \pm\sqrt{2}$, $y' = 0$ *i.e.* $\pm\sqrt{2} = \dfrac{x+y}{\sqrt{2}}$

$\Rightarrow x + y = \pm 2, 0 = \dfrac{-x+y}{\sqrt{2}} \Rightarrow -x + y = 0$



Solving $x + y = 2$
and $-x + y = 0$,
we get $x = 1, y = 1$
Solving $x + y = -2$
and $-x + y = 0$, we get
$x = -1, y = -1$

The vertices are (1, 1) and (–1, –1) in the original system of axis.

Equation of the transverse axis is

$y' = 0$ *i.e.*, $x = y$

and conjugate axis is $x' = 0$

i.e., $x = -y$

Joint equation of the asymptotes is $\dfrac{x'^2}{2} - \dfrac{y'^2}{2} = 0$

or $x' + y' = 0$ and $x' - y' = 0$ are asymptotes referred to the $x'y'$-system.

i.e., $\dfrac{x+y}{\sqrt{2}} + \dfrac{-x+y}{\sqrt{2}} = 0 \Rightarrow y = 0$

and $\dfrac{x+y}{\sqrt{2}} - \dfrac{-x+y}{\sqrt{2}} = 0 \Rightarrow x = 0$

Thus $y = 0$ and $x = 0$ are asymptotes in the xy-system.

The graph of the conic is as shown.

18. $xy + x - 2y + 3 = 0$ (1)

Sol. Here $a = 0 = b$ so the axes should be rotated through an angle of 45° to eliminate the products term xy.

$x = \dfrac{x'-y'}{\sqrt{2}}, \ y = \dfrac{x'+y'}{\sqrt{2}}$ are the equations of transformation.

Substituting into (1), we have

$$\frac{x'-y'}{\sqrt{2}} \cdot \frac{x'+y'}{\sqrt{2}} + \frac{x'-y'}{\sqrt{2}} - 2 \cdot \frac{x'+y'}{\sqrt{2}} + 3 = 0$$

or $x'^2 - y'^2 + \sqrt{2}(x' - y') - 2\sqrt{2}(x' + y') + 6 = 0$

i.e., $x'^2 - y'^2 - \sqrt{2}x' - 3\sqrt{2}y' + 6 = 0$

or $\left(x'^2 - \sqrt{2}x' + \dfrac{1}{2}\right) - \left(y'^2 + 3\sqrt{2}y' + \dfrac{9}{2}\right) = -6 - 4$

or $\left(x'-\dfrac{1}{\sqrt{2}}\right)^2-\left(y'+\dfrac{3}{\sqrt{2}}\right)^2=-10$ (2)

Let $X=x'-\dfrac{1}{\sqrt{2}}$, $Y=y'+\dfrac{3}{\sqrt{2}}$, so that (2) becomes

$\dfrac{Y^2}{10}-\dfrac{X^2}{10}=1$. This is a rectangular hyperbola with transverse axis $X=0$ and conjugate axis $Y=0$.

i.e., $x'-\dfrac{1}{\sqrt{2}}=0$ and $y'+\dfrac{3}{\sqrt{2}}=0$

or $\dfrac{x+y}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}=0$ and $\dfrac{-x+y}{\sqrt{2}}+\dfrac{3}{\sqrt{2}}=0$

or $x+y-1=0$ and $-x+y+3=0$

Centre: The point of intersection of $x+y-1=0$ and $-x+y+3=0$ is $(2,-1)$, *i.e.*, the centre of the conic is $(2,-1)$.

Vertices: $Y=\pm\sqrt{10}\,, X=0$

i.e., $y'+\dfrac{3}{\sqrt{2}}=\pm\sqrt{10}$, $x'-\dfrac{1}{\sqrt{2}}=0$

or $\dfrac{-x+y}{\sqrt{2}}+\dfrac{3}{\sqrt{2}}=\pm\sqrt{10}\,,\dfrac{x+y}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}=0$

$x+y-1=0$

$-x+y+3=\pm\sqrt{10}\times\sqrt{2}=\pm 2\sqrt{5}$

Adding the above equations, we get $2y+2=\pm 2\sqrt{5}$

or $y=-1\pm\sqrt{5}$

If $y=-1+\sqrt{5}$, then $x=1-y$

$=1+1-\sqrt{5}=2-\sqrt{5}$

If $y=-1-\sqrt{5}$, then $x=1-y$

$=1+1+\sqrt{5}=2+\sqrt{5}$

Thus the vertices are

$(2-\sqrt{5}\,,-1+\sqrt{5}),\ (2+\sqrt{5}\,,-1-\sqrt{5})$

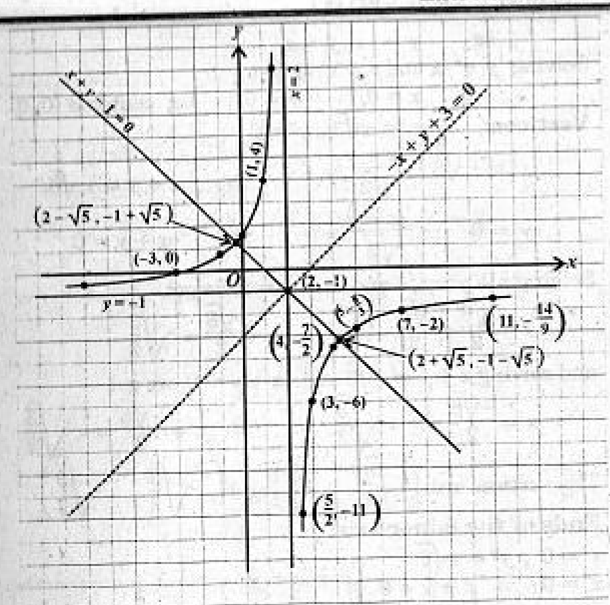
Asymptotes: $Y^2-X^2=0$

or $(Y-X)(Y+X)=0$

or $\left(y'+\dfrac{3}{\sqrt{2}}-x'+\dfrac{1}{\sqrt{2}}\right)\left(y'+\dfrac{3}{\sqrt{2}}+x'-\dfrac{1}{\sqrt{2}}\right)=0$

i.e., $\dfrac{-x+y}{\sqrt{2}}+\dfrac{3}{\sqrt{2}}-\dfrac{x+y}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}=0 \Rightarrow -2x+4=0$

$\Rightarrow x-2=0$



or $\dfrac{-x+y}{\sqrt{2}}+\dfrac{3}{\sqrt{2}}+\dfrac{x+y}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}=0 \Rightarrow 2y+2=0 \Rightarrow y+1=0$

Thus $x-2=0$ and $y+1=0$ are asymptotes.

The graph of the conic is as shown above.

10. $5x^2-2xy+5y^2-12=0$ (1)

Sol. Since $a-b=5-5=0$, the axes are to be rotated through an angle of 45° to remove the xy term.

$$5\left(\frac{x'-y'}{\sqrt{2}}\right)^2-2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right)-5\left(\frac{x'+y'}{\sqrt{2}}\right)^2-12=0$$

$$\frac{5}{2}(x'^2+y'^2-2x'y')-(x'^2-y'^2)+\frac{5}{2}(x'^2+y'^2+2x'y')-12=0$$

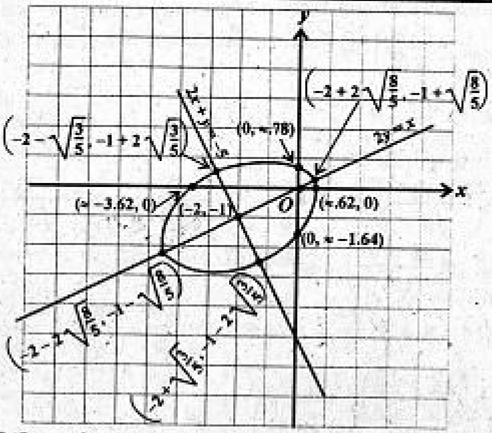
$$5x'^2+5y'^2-x'^2+y'^2=12 \quad\Rightarrow\quad 4x'^2+6y'^2=12$$

or $\dfrac{x'^2}{3}+\dfrac{y'^2}{2}=1$ which is an ellipse.

Major axis: $y'=0$ *i.e.,* $\dfrac{-x+y}{\sqrt{2}}=0$ or $y=x$

Minor axis: $x'=0$ *i.e.,* $\dfrac{x+y}{\sqrt{2}}=0$ or $y=-x$

Centre: $x'=0\,,\ y'=0$



Ends of the minor axis:

$X = 0\ ,\ \ Y = \pm\sqrt{3}$

$\Rightarrow\quad x' + \sqrt{5} = 0, y' = \pm\sqrt{3}$

or $\quad \dfrac{2x+y}{\sqrt{5}} + \sqrt{5} = 0\quad ,\quad \dfrac{-x+2y}{\sqrt{5}} = \pm\sqrt{3}$

or $\quad 2x + y + 5 = 0\quad ,\quad -x + 2y = \pm\sqrt{15}$

Solving the equations, we get

$$\left(-2-\sqrt{\frac{3}{5}}, -1+2\sqrt{\frac{3}{5}}\right) \text{ and } \left(-2+\sqrt{\frac{3}{5}}, -1-2\sqrt{\frac{3}{5}}\right)$$

as the end points of minor axis.

The graph of the curve is as shown above.

21. $x^2 - 2xy + y^2 - 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0$ (1)

Sol. Here, $a = 1\ , b = 1\ , h = -1$

$ab - h^2 = 0$. Hence (1) is a parabola.

Since $a - b = 0$, the angle of ration is $\theta = 45^\circ$

Equations of transformation are

$$\left.\begin{aligned} x &= \frac{x'-y'}{\sqrt{2}} \\ y &= \frac{x'+y'}{\sqrt{2}} \end{aligned}\right\}, \quad x' = \frac{x+y}{\sqrt{2}},\quad y' = \frac{-x+y}{\sqrt{2}}$$

Transformed equation (1) is

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + \left(\frac{x'+y'}{\sqrt{2}}\right)^2$$

$$-2\sqrt{2}\left(\frac{x'-y'}{\sqrt{2}}\right) - 2\sqrt{2}\left(\frac{x'+y'}{\sqrt{2}}\right) + 2 = 0$$

$$\frac{x'^2+y'^2}{2} - x'y' - (x'^2 - y'^2) + \frac{x'^2+y'^2}{2} + x'y' - 4x' + 2 = 0$$

or $\quad 2y'^2 - 4x' + 2 = 0$ or $y'^2 = 2\left(x' - \dfrac{1}{2}\right)$

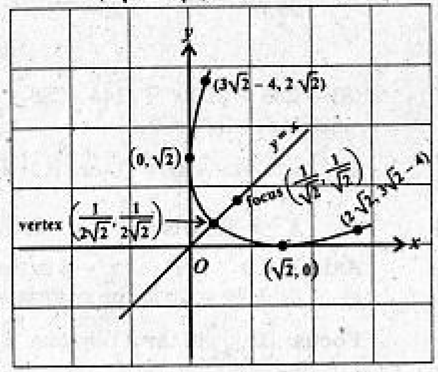
or $\quad Y^2 = 2X$, where $Y = y', X = x' - \dfrac{1}{2}$. This is a parabola.

Axis: $Y = 0 \Rightarrow y' = 0 \Rightarrow y' = \dfrac{-x+y}{\sqrt{2}} = 0\quad$ *i.e.,* $x - y = 0$.

Vertex: $X = 0\ , Y = 0\qquad$ *i.e.,* $\quad x' = \dfrac{1}{2}, y' = 0$ in $x'y'$-system

or $\quad \dfrac{x+y}{\sqrt{2}} = \dfrac{1}{2} \Rightarrow x + y = \dfrac{1}{\sqrt{2}}\Bigg],\ \dfrac{-x+y}{\sqrt{2}} \Rightarrow -x + y = 0$

Solving the equations, we get $\left(\dfrac{1}{2\sqrt{2}}, \dfrac{1}{2\sqrt{2}}\right)$ as vertex in the xy-system.



Focus: $X = \dfrac{1}{2}, Y = 0$

i.e., $\quad x' - \dfrac{1}{2} = \dfrac{1}{2}$

and $\quad y' = 0$

$\Rightarrow x' = 1$ and $y' = 0$

in $x'y'$-system

$x' = 1 \Rightarrow \dfrac{x+y}{\sqrt{2}} = 1,$

$y = 0 \Rightarrow \dfrac{-x+y}{\sqrt{2}} = 0$

or $\quad x + y = \sqrt{2}$

and $x + y = 0$

Solving the equations, we get

$$x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, \textit{i.e.,} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ is focus.}$$

The graph is as shown above.

22. $9x^2 + 24xy + 16y^2 - 125y = 0$

Sol. $9x^2 + 24xy + 16y^2 - 125y = 0$ (1)

Here $a = 9, b = 16, h = 12$

$ab - h^2 = 0$. Thus (1) represents a parabola

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2h}{a-b} = \frac{24}{-7}$$

i.e., $12\tan^2\theta - 7\tan\theta - 12 = 0$

or $\tan\theta = \dfrac{7 \pm \sqrt{49+576}}{24} = \dfrac{7 \pm 25}{24} = \dfrac{4}{3}, -\dfrac{3}{4}$

We choose θ such that $\tan\theta = \frac{4}{3}$ and so $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$

Equations of transformation are

$$x = x'\cos\theta - y'\sin\theta = \frac{3x'-4y'}{5}$$

$$y = x'\sin\theta + y'\cos\theta = \frac{4x'+3y'}{5}$$

$$x' = \frac{3x+4y}{5}, \; y' = \frac{-4x+3y}{5}$$

Substituting for x, y into (1), we have

$$\frac{9}{25}(3x'-4y')^2 + \frac{24}{25}(3x'-4y')(4x'+3y') + \frac{16}{25}(4x'+3y')^2$$
$$-\frac{625}{25}(4x'+3y') = 0$$

$(81+288+256)x' + (144-288+144)y'^2 + (-216-168+384)x'y'$
$-625(4x'+3y') = 0$

i.e., $x'^2 - 4x' - 3y' = 0$ or $(x'-2)^2 = 3\left(y' + \frac{4}{3}\right)$

or $X^2 = 3Y$, where $X = x'-2$, $Y = y' + \frac{4}{3}$

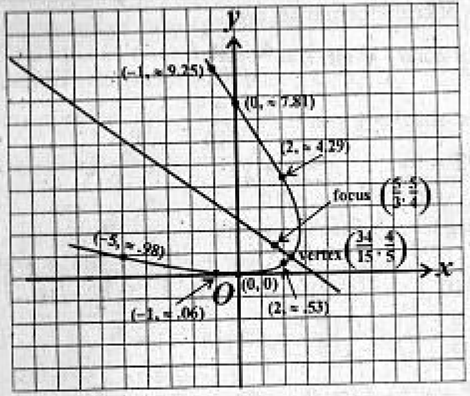
Axis: $X = 0$ or $x' = 2$ in the $x'y'$-system
or $3x + 4y = 10$ in the xy-system

Focus: $\left(0, \frac{3}{4}\right)$ in the XY-system. *i.e.,* $x' - 2 = 0$, $y' + \frac{4}{3} = \frac{3}{4}$

or $x' = 2, y' = -\frac{7}{12}$ in the $x'y'$ system

$$\left.\begin{aligned} &\text{or} \quad \frac{3x+4y}{5} = 2 \quad \Rightarrow 3x+4y = 10 \\ &\text{and} \quad \frac{-4x+3y}{5} = \frac{-7}{12} \Rightarrow -4x+3y = -\frac{35}{\sqrt{12}} \end{aligned}\right\} \text{in the } xy\text{-system}$$

Solving these equations, we get $y = \frac{5}{4}, x = \frac{5}{3}$



Thus focus is $\left(\frac{5}{3}, \frac{5}{4}\right)$ in the xy-system

Vertex: (0, 0) in the XY-system

or $\left(2, \frac{-4}{3}\right)$ in the $x'y'$-system

$$\left.\begin{aligned} &\text{i.e.,} \quad \frac{3x+4y}{5} = 2 \Rightarrow 3x+4y = 10, \\ &\quad \frac{-4x+3y}{5} = \frac{-4}{3} \Rightarrow -4x+3y = -\frac{-20}{3} \end{aligned}\right\} \text{in the } xy\text{-system}$$

Solving these equations, we get $y = \frac{4}{3}, x = \frac{34}{15}$

Thus vertex is $\left(\frac{34}{15}, \frac{4}{5}\right)$ in the xy-system

The graph of the conic is as shown above.

28. $2x^2 + 6xy + 10y^2 - 11 = 0$ (1)

Sol. Here $a = 2, b = 10, h = 3$

$h^2 - ab = 9 - 20 < 0$

Hence the conic is an ellipse.

i.e., $\tan 2\theta = \dfrac{2h}{a-b} = \dfrac{6}{2-10}$ *i.e.,* $\dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{6}{-8} = \dfrac{-3}{4}$

or $3\tan^2\theta - 8\tan\theta - 3 = 0$

or $\tan\theta = \dfrac{8 \pm \sqrt{64+36}}{6} = 3, -\dfrac{1}{3}$

We choose θ such that $\tan\theta = 3$

Therefore, $\sin\theta = \dfrac{3}{\sqrt{10}}, \cos\theta = \dfrac{1}{\sqrt{10}}$

Equations of transformation are

$$x = x'\cos\theta - y'\sin\theta = \frac{x'-3y'}{\sqrt{10}}$$

$$y = x'\sin\theta + y'\cos\theta = \frac{3x'+y'}{\sqrt{10}}$$

and $x' = \dfrac{x+3y}{\sqrt{10}}$, $y' = \dfrac{-3x+y}{\sqrt{10}}$

Substituting for x, y into (1), we have

$$2\left(\frac{x'-3'}{\sqrt{10}}\right)^2 + 6\left(\frac{x'-3y'}{\sqrt{10}}\right)\left(\frac{3x'+y'}{\sqrt{10}}\right) + 10\left(\frac{3x'+y'}{\sqrt{10}}\right)^2 - 11 = 0$$

or $\left(\frac{1}{5}+\frac{9}{5}+9\right)x'^2 + \left(\frac{9}{5}-\frac{9}{5}+1\right)y'^2 - 11 = 0$

or $11x'^2 + y'^2 - 11 = 0$ *i.e.,* $\dfrac{x'^2}{1} + \dfrac{y'^2}{(\sqrt{11^2})} = 1$

The major axis of the ellipse lies along the y'-axis

Centre: $x' = 0$, $y' = 0$ *i.e.,* $x + 3y = 0$ and $-3x + y = 0$

which give $x = 0$, $y = 0$

Foci $= (0, \mp c)$ (where $c = \sqrt{11-1} = \sqrt{10}$) , in $x'y'$-system

i.e., $\dfrac{x+3y}{\sqrt{10}} = 0$

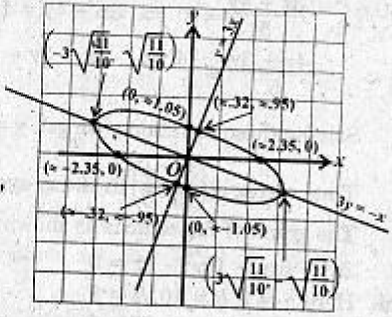
and $\dfrac{-3x+y}{\sqrt{10}} = \mp\sqrt{10}$

or $x + 3y = 0$

and $-3x + y = \mp 10$

Solving the above equations, we get

$(3, -1)$ and $(-3, 1)$ as foci.



Major axis: $x' = 0$,

i.e., $x + 3y = 0$

Minor axis: $y' = 0$,

i.e., $3x - y - 0$

Vetrices: $A' = (0, -\sqrt{11})$, $A = (0, \sqrt{11})$ in the $x'y'$-system

To find A', we have

or $x + 3y = 0$ and $\dfrac{-3x+y}{\sqrt{10}} = -\sqrt{11}$ in the xy-system

which give $A' = \left(3\sqrt{\dfrac{11}{10}}, -\sqrt{\dfrac{11}{10}}\right)$

Similarly, $x^3 + 3y = 0$ and $\dfrac{-3x+y}{\sqrt{10}} = \sqrt{11}$ in the xy-system

which yield $A = \left(-3\sqrt{\dfrac{11}{10}}, \sqrt{\dfrac{11}{10}}\right)$

The graph is as shown above.

24. $x^2 - 4xy + 4y^2 + 5\sqrt{5}\,y + 1 = 0$ (1)

Sol. Here $\tan 2\theta = \dfrac{2h}{a-b} = \dfrac{-4}{1-4} = \dfrac{-4}{-3} = \dfrac{4}{3}$

or $\dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{4}{3} \Rightarrow 3\tan\theta = 2 - 2\tan^2\theta$

or $2\tan^2\theta + 3\tan\theta - 2 = 0$

$\tan\theta = \dfrac{-3\pm\sqrt{9+16}}{4} = \dfrac{-3\pm 5}{4}$

i.e., $\tan\theta = \dfrac{1}{2}$, -2

We take $\tan\theta = \dfrac{1}{2}$ so that $\sin\theta = \dfrac{1}{\sqrt5}$, $\cos\theta = \dfrac{2}{\sqrt5}$

Equations of transformation are

$$x = \frac{2x'-y'}{\sqrt5}, \quad y = \frac{x'+2y'}{\sqrt5}$$

Transformed equation (1) is

$$\left(\frac{2x'-y'}{\sqrt5}\right)^2 - 4\left(\frac{2x'-y'}{\sqrt5}\right)\left(\frac{x'+2y'}{\sqrt5}\right) + 4\left(\frac{x'+2y'}{\sqrt5}\right)^2 + 5\sqrt5\left(\frac{x'+2y'}{\sqrt5}\right) + 1 = 0$$

$$\left(\frac45 - \frac85 + \frac45\right)x'^2 + \left(\frac15 + \frac85 + \frac{16}{5}\right)y'^2 + \left(-\frac45 - \frac{12}{5} + \frac{16}{5}\right)x'y' + 5x' + 10y' + 1 = 0$$

or $5y'^2 + 10y' = -5x' - 1 \Rightarrow y'^2 + 2y' = -x' - \dfrac15$

or $(y'+1)^2 = -x' - \dfrac15 + 1 = -\left(x' - \dfrac45\right)$

Now $Y^2 = -X$ where $X = x' - \dfrac45$ $Y = y' + 1$

This is a parabola.

Vertex: $X = 0$, $Y = 0$

$X = 0 \Rightarrow x' - \dfrac45 = 0 \Rightarrow x' = \dfrac45$

And $Y = 0 \Rightarrow y' + 1 = 0 \Rightarrow y' = -1$

$x' = \dfrac45 \Rightarrow \dfrac{2x+y}{\sqrt5} = \dfrac45$

and $y' = -1 \quad \Rightarrow \quad \dfrac{-x+2y}{\sqrt{5}} = -1$

i.e., $2x + y = \dfrac{4}{\sqrt{5}}$ and $-x + 2y = -\sqrt{5}$

Solving these equations, we have $x = \dfrac{13}{5\sqrt{5}}$, $y = \dfrac{-6}{5\sqrt{5}}$

Axis: $Y = 0$

$$Y = 0 \quad \Rightarrow \quad y' + 1 = 0 \quad \Rightarrow \quad \frac{-x+2y}{\sqrt{5}} = -1$$

i.e., $-x + 2y = -\sqrt{5}$ in the xy-system.

Focus: $X = -\dfrac{1}{4}$, $Y = 0$, *i.e.*, $x' - \dfrac{4}{\sqrt{5}} = -\dfrac{1}{4}$

and $y' = -1$

or $\dfrac{2x+y}{\sqrt{5}} = \dfrac{4}{\sqrt{5}} - \dfrac{1}{4} \quad \Rightarrow \quad 2x + y = 4 - \dfrac{\sqrt{5}}{4}$

and $-x + 2y = -\sqrt{5}$

Solving the equations, we get

$$x = \frac{8}{5} + \frac{1}{2\sqrt{5}}, \quad y = \frac{4}{5} - \frac{9}{4\sqrt{5}}$$

25. $16x^2 - 24xy + 9y^2 + 100x - 200y + 100 = 0$ (1)

Sol. Here $\tan 2\theta = \dfrac{2h}{a-b} = \dfrac{-24}{16-9} = \dfrac{-24}{7}$

i.e., $\dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{-24}{7}$ or $\dfrac{\tan\theta}{1-\tan^2\theta} = \dfrac{-12}{7}$

or $12\tan^2\theta - 7\tan\theta - 12 = 0$

$$\tan\theta = \frac{7 \pm \sqrt{49+576}}{24} = \frac{7+25}{24} = \frac{4}{3}, -\frac{3}{4}$$

We take $\tan\theta = \dfrac{4}{3}$ so that $\sin\theta = \dfrac{4}{5}$, $\cos\theta = \dfrac{3}{5}$

Equations of transformation are

$$x = \frac{3x' - 4y'}{5}, \quad y = \frac{4x' + 3y'}{5}$$

Substituting into (1), we get

$$16\left(\frac{3x'-4y'}{5}\right)^2 - 24\left(\frac{3x'-4y'}{5}\right)\left(\frac{4x'+3y'}{5}\right)$$
$$+ 9\left(\frac{4x'-3y'}{5}\right)^2 + 100\left(\frac{3x'-4y'}{5}\right)$$
$$- 200\left(\frac{4x'+3y'}{5}\right) + 100 = 0$$

or $\left(\dfrac{144}{25} - \dfrac{288}{25} + \dfrac{144}{25}\right)x'^2 + \left(\dfrac{256}{25} + \dfrac{288}{25} + \dfrac{81}{25}\right)y'^2$

$+ (60 - 160)x' + (-80 - 120)y' + 100 = 0$

or $25y' - 100x' - 200y' + 100 = 0$

or $25x'^2 - 200y' = 100(x - 1)$

or $y'^2 - 8y' + 16 = 4(x - 1) = 16$

or $(y' - 4)^2 = 4(x' + 3)$

or $Y^2 = 4X$

where $Y = y' - 4$, $X = x' + 3$

This is a parabola with vertex $X = 0$, $Y = 0$

i.e., $x' = -3$, $y' = 4$ in the $x'y'$-system.

But $x = \dfrac{3x' - 4y'}{5} = \dfrac{-9-16}{5} = -5$

$y = \dfrac{4x' + 3y'}{5} = \dfrac{-12+12}{5} = 0$

Thus $(-5, 0)$ is the vertex in the xy-system.

Axis: $Y = 0$ *i.e.*, $y' = 4$

or $\dfrac{-4x+3y}{5} = 4$ or $-4x + 3y = 20 \Rightarrow 4x - 3y + 20 = 0$

Focus: $Y = 0$, $X = 1$

or $y' = 4$ and $x' + 3 = 1 \quad \Rightarrow \quad x' = -2$

i.e., $(-2, 4)$ in the $x'y'$- system. or $x = \dfrac{-6-16}{5}$, $y = \dfrac{-8+12}{5}$

i.e., $\left(-\dfrac{22}{5}, \dfrac{4}{5}\right)$ in the xy-system.

Exercise Set 6.2 (Page 245)

Find equations of tangent and normal to each of the following curves at the indicated point (Problems 1 – 4):

1. $y^2 = 4ax$ at $(a, -2a)$

Sol. $y^2 = 4ax$ (1)

Differentiating (1) w.r.t, x, we have

$$2y\frac{dy}{dx} = 4a \quad \text{or} \quad \frac{dy}{dx} = \frac{2a}{y}$$

$\left(\dfrac{dy}{dx}\right)_{(a,-2a)} = \dfrac{2a}{-2a} = -1$ which is the slope of the tangent at $(a, -2a)$.

Slope of the normal at $(a, -2a) = -\dfrac{1}{-1} = 1$

Hence equation of the tangent to (1) at $(a, -2a)$ is

$$y-(-2a) = -1\,(x-a) \quad \text{or} \quad y + 2a = -x + a$$

i.e., $x + y + a = 0$

Equation of the normal to (1) at $(a, -2a)$ is

$$y-(-2a) = 1\,(x-a) \quad \text{or} \quad y + 2a = x - a$$

or $x - y - 3a = 0$

2. $xy = c^2$ at $\left(cp, \dfrac{c}{p}\right)$

Sol. $xy = c^2$ (1)

Differentiating (1) w.r.t. x, we get

$$x\frac{dy}{dx} + y = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\left(\frac{dy}{dx}\right)_{\left(cp,\,\frac{c}{p}\right)} = -\frac{c/p}{cp} = -\frac{1}{p^2}$$

which is the slope of the tangent at $\left(cp, \dfrac{c}{p}\right)$.

Slope of the normal at $\left(cp, \dfrac{c}{p}\right) = p^2$

Hence equation of the required tangent is

$$\left(y - \frac{c}{p}\right) = -\frac{1}{p^2}.(x - cp)$$

or $p^2\left(y - \dfrac{c}{p}\right) = -(x - cp) \Rightarrow p^2y - cp = -x + cp$

or $x + p^2y = 2cp$

Equation of the normal is $\left(y - \dfrac{c}{p}\right) = p^2\,(x - cp)$

or $py - c = p^3x - cp^4$ or $p^3x - py = c(p^4 - 1)$

3. $x(x^2 + y^2) - ay^2 = 0$ at $x = \dfrac{a}{2}$

Sol. $x^3 + xy^2 - ay^2 = 0$

Let $f(x, y) = x^3 + xy^2 - ay^2 = 0$.

When $x = \dfrac{a}{2}$, $f\left(\dfrac{a}{2}, y\right) = \left(\dfrac{a}{2}\right)^3 + \dfrac{a}{2}\,(y^2) - ay^2 = 0$

or $\dfrac{a^3}{8} - \dfrac{a}{2}y^2 = 0$ or $y^2 = \dfrac{a^2}{4}$ or $y = \pm\dfrac{a}{2}$

Hence the points at which the tangents and normal are required are

$$\left(\frac{a}{2}, \frac{a}{2}\right), \left(\frac{a}{2}, \frac{-a}{2}\right)$$

Now $\dfrac{\partial f}{\partial x} = 3x^2 + y^2$ and $\dfrac{\partial f}{\partial y} = 2xy - 2ay$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2 + y^2}{2xy - 2ay}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{a}{2},\frac{a}{2}\right)} = -\frac{3\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}{2\left(\frac{a}{2}\right)^2 - 2a\left(\frac{a}{2}\right)} = 2$$

which is the slope of the tangent to the curve at $\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$

Equation of the tangent at $\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$ is

$$y - \frac{a}{2} = 2\left(x - \frac{a}{2}\right)$$

or $y - \dfrac{a}{2} = 2x - a \Rightarrow 2x - y - a + \dfrac{a}{2} = 0$

or $4x - 2y - a = 0$.

Slope of the normal at $\left(\dfrac{a}{2}, \dfrac{a}{2}\right) = -\dfrac{1}{2}$

Equation of the normal at $\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$ is

$$y - \frac{a}{2} = -\frac{1}{2}\left(x - \frac{a}{2}\right) \Rightarrow 2y - a = -x + \frac{a}{2}$$

or $2x + 4y = 3a$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{a}{2},-\frac{a}{2}\right)} = -\frac{3\left(\frac{a}{2}\right)^2 + \left(\frac{-a}{2}\right)^2}{2\left(\frac{a}{2}\right)\left(\frac{-a}{2}\right) - 2a\left(\frac{-a}{2}\right)} = -\frac{a^2}{-\frac{a^2}{2} + a^2} = -2$$

which is the slope of the tangent to the curve at $\left(\dfrac{a}{2}, \dfrac{-a}{2}\right)$

Equation of the tangent at $\left(\dfrac{a}{2}, \dfrac{-a}{2}\right)$ is

$$y + \frac{a}{2} = -2\left(x - \frac{a}{2}\right) \Rightarrow 2y + a = -2\,(2x - a)$$

or $4x + 2y - a = 0$

Slope of the normal at $\left(\dfrac{a}{2}, \dfrac{-a}{2}\right) = \dfrac{1}{2}$

Equation of the normal at $\left(\frac{a}{2}, \frac{-a}{2}\right)$ is

$$y + \frac{a}{2} = \frac{1}{2}\left(x - \frac{a}{2}\right) \Rightarrow y + \frac{a}{2} = \frac{x}{2} - \frac{a}{4}$$

or $\quad 2x - 4y = 3a$

4. $c^2(x^2 + y^2) = x^2y^2$ at $\left(\frac{c}{\cos\theta}, \frac{c}{\sin\theta}\right)$

Sol. $c^2(x^2 + y^2) = x^2y^2$ or $\frac{c^2}{x^2} + \frac{c^2}{y^2} = 1$

Its parametric equations are $x = \frac{c}{\cos\theta}$, $y = \frac{c}{\sin\theta}$

$$\frac{dx}{d\theta} = \frac{c\sin\theta}{\cos^2\theta}, \quad \frac{dy}{d\theta} = -\frac{c\cos\theta}{\sin^2\theta}$$

$$\frac{dy}{dx} = \left(-\frac{c\cos\theta}{\sin^2\theta}\right)\left(\frac{\cos^2\theta}{c\sin\theta}\right) = -\frac{\cos^3\theta}{\sin^3\theta} = -\cot^3\theta$$ which is

the slope of the tangent to the curve at the point $\left(\frac{c}{\cos\theta}, \frac{c}{\sin\theta}\right)$

Equation of the tangent is

$$y - \frac{c}{\sin\theta} = -\cot^3\theta\left(x - \frac{c}{\cos\theta}\right)$$

or $$y - \frac{c}{\sin\theta} = -\frac{\cos^3\theta}{\sin^3\theta}\left(x - \frac{c}{\cos\theta}\right)$$

$$= -x\frac{\cos^3\theta}{\sin^3\theta} + \frac{c\cos^2\theta}{\sin^3\theta}$$

or $\quad y\sin^3\theta - c\sin^2\theta = -x\cos^3\theta + c\cos^2\theta$

or $\quad x\cos^3\theta + y\sin^3\theta = c$

Slope of the normal at the given point is $\tan^3\theta$.

Equation of the normal is

$$y - \frac{c}{\sin\theta} = \tan^3\theta\left(x - \frac{c}{\cos\theta}\right)$$

$$= \frac{\sin^3\theta}{\cos^3\theta}\left(x - \frac{c}{\cos\theta}\right)$$

or $$y\cos^3\theta - \frac{c\cos^3\theta}{\sin\theta} = x\sin^3\theta - \frac{c\sin^3\theta}{\cos\theta}$$

or $$x\sin^3\theta - y\cos^3\theta = \frac{c\sin^3\theta}{\cos\theta} - \frac{c\cos^3\theta}{\sin\theta} = \frac{c\sin^4\theta - c\cos^4\theta}{\sin\theta\cos\theta}$$

$$= \frac{c(\sin^2\theta - \cos^2\theta)}{\sin\theta\cos\theta} = -\frac{2c\cos 2\theta}{\sin 2\theta} = -2c\cot 2\theta$$

or $\quad x\sin^3\theta - y\cos^3\theta + 2c\cot 2\theta = 0$

Find the points where the tangent is parallel to the x-axis and where it is parallel to the y-axis for each of the given curves (Problems 5 – 7):

5. $x^3 + y^3 = a^3$ (1)

Sol. Differentiating (1) w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x^2}{y^2}$$

For tangent to be parallel to the x-axis, $\frac{dy}{dx} = 0$

i.e., $x^2 = 0$ or $x = 0$

Putting $x = 0$ in (1), we get $y^3 = a^3 \Rightarrow y = a$

The tangent is parallel to the x-axis at $(0, a)$.

For tangent to be parallel to the y-axis or perpendicular to the x-axis,

$\frac{dy}{dx} = \infty$ which requires $y^2 = 0$ *i.e.,* $y = 0$

Putting $y = 0$ in (1), we have $x^3 = a^3 \Rightarrow x = a$

The tangent is perpendicular to the x-axis at $(a, 0)$.

6. $x^3 + y^3 = 3axy$ (1)

Sol. Differentiating (1) w.r.t. x, we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$$

or $\quad (y^2 - ax)\frac{dy}{dx} = ay - x^2$ or $\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$

For tangent to be parallel to the x-axis, $\frac{dy}{dx} = 0$

Therefore, $ay - x^2 = 0 \Rightarrow y = \frac{x^2}{a}$ (2)

Solving (1) and (2), we have

$$x^3 + \left(\frac{x^2}{a}\right)^3 = 3ax\left(\frac{x^2}{a}\right) \Rightarrow \frac{x^6}{a^3} = 2x^3$$

or $\quad x^3 = 2a^3$ which gives $x = 2^{1/3}a$

Putting $x = 2^{1/3}a$ in (2), we get

$$y = \frac{(2^{1/3}a)^2}{a} = \frac{2^{2/3}a^2}{a} = 2^{2/3}a$$

Hence the tangent is parallel to the x-axis at

$$(2^{1/3}a, 2^{2/3}a) = (\sqrt[3]{2}a, \sqrt[3]{4}\,a)$$

Now for tangent to be parallel to the y-axis, $\frac{dy}{dx} = \infty$

which requires $y^2 - ax = 0$ or $x = \dfrac{y^2}{a}$ (3)

Solving (1) and (3), we have

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3a\left(\frac{y^2}{a}\right)y \quad \text{or} \quad \frac{y^6}{a^3} + y^3 = 3y^3$$

i.e., $y^3 = 2a^3$ which gives $y = 2^{1/3}a$

Putting $y = 2^{1/3}a$ in (3), we get $x = 2^{2/3}a$

Hence the tangent is parallel to the y-axis at

$$(2^{2/3}a, 2^{1/3}a) = (\sqrt[3]{4}\,a, \sqrt[3]{2}\,a)$$

7. $25x^2 + 12xy + 4y^2 = 1$ (1)

Sol. $f(x, y) = 25x^2 + 12xy + 4y^2 - 1 = 0$

$$\frac{\partial f}{\partial x} = 50x + 12y, \quad \frac{\partial f}{\partial y} = 12x + 8y$$

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = -\frac{50x + 12y}{12x + 8y} = -\frac{25x + 6y}{6x + 4y}$$

For tangent to be the parallel to the x-axis, $\dfrac{dy}{dx} = 0$

which implies $25x + 6y = 0$ or $x = -\dfrac{6}{25}y$ (2)

Putting $x = \dfrac{-6}{25}y$ in (1), we get

$$25\left(-\frac{6}{25}y\right)^2 + 12\left(-\frac{6}{25}y\right)y + 4y^2 = 1$$

or $\quad \dfrac{36}{25}y^2 - \dfrac{72}{25}y^2 + 4y^2 = 1 \Rightarrow 36y^2 - 72y^2 + 100y^2 = 25$

or $\quad 64y^2 = 25 \quad \Rightarrow \quad y = \pm\dfrac{5}{8}$

Putting $y = \dfrac{5}{8}$ in (2), we obtain $x = -\dfrac{6}{25} \times \dfrac{5}{8} = -\dfrac{3}{20}$

Putting $y = -\dfrac{5}{8}$ in (2), we have

$$x = -\frac{6}{25} \times \left(-\frac{5}{8}\right) = \frac{3}{20}$$

Hence tangent is parallel to the x-axis at

$$\left(-\frac{3}{20}, \frac{5}{8}\right) \text{ and } \left(\frac{3}{20}, -\frac{5}{8}\right)$$

Now for tangent to be parallel to the y-axis, $\dfrac{dy}{dx} = \infty$

which requires that $6x + 4y = 0$ or $y = -\dfrac{3}{2}x$ (3)

Putting $y = -\dfrac{3}{2}x$ in (1), we obtain

$$25x^2 + 12x\left(-\frac{3}{2}x\right) + 4\left(-\frac{3}{2}x\right)^2 = 1$$

or $\quad 25x^2 - 18x^2 + 9x^2 = 1$ or $16x^2 = 1 \Rightarrow x = \pm\dfrac{1}{4}$

Putting $x = \dfrac{1}{4}$ in (3), we have $y = -\dfrac{3}{2}\left(\dfrac{1}{4}\right) = -\dfrac{3}{8}$

and putting $x = -\dfrac{1}{4}$ in (3), we have

$$y = -\frac{3}{2}\left(-\frac{1}{4}\right) = \frac{3}{8}$$

Hence tangents are parallel to the y-axis at

$$\left(\frac{1}{4}, -\frac{3}{8}\right) \text{ and } \left(-\frac{1}{4}, \frac{3}{8}\right).$$

8. If $p = x\cos\theta + y\sin\theta$ touches the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1,$$

prove that $p^n = (a\cos\theta)^n + (b\sin\theta)^n$.

Sol. $f(x, y) = \left(\dfrac{x}{a}\right)^{\frac{n}{n-1}} + \left(\dfrac{y}{b}\right)^{\frac{n}{n-1}} - 1 = 0$ (1)

$$\frac{\partial f}{\partial x} = \frac{n}{n-1}\left(\frac{x}{a}\right)^{\frac{n}{n-1}-1}\left(\frac{1}{a}\right) = \frac{n}{(n-1)a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}$$

$$\frac{\partial f}{\partial y} = \frac{n}{n-1}\left(\frac{x}{b}\right)^{\frac{n}{n-1}-1}\left(\frac{1}{b}\right) = \frac{n}{(n-1)b}\left(\frac{x}{b}\right)^{\frac{1}{n-1}}$$

Equation of the tangent is $\quad Y - y = -\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}(X - x)$

or $\quad (X - x)\dfrac{\partial f}{\partial f} + (Y - y)\dfrac{\partial f}{\partial y} = 0$

Substituting the values, we have

$$\frac{(X-x)}{a}\cdot\frac{n}{n-1}\cdot\left(\frac{x}{a}\right)^{\frac{1}{n-1}}+\frac{(Y-y)}{b}\cdot\frac{n}{(n-1)}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}=0$$

or $$\frac{X-x}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}+\frac{Y-y}{b}\left(\frac{x}{b}\right)^{\frac{1}{n-1}}=0$$

or $$\frac{X}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}-\left(\frac{x}{b}\right)^{\frac{n}{n-1}}+\frac{Y}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}-\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=0$$

or $$\frac{X}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}+\frac{Y}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}=\left(\frac{x}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=1 \quad \text{(From (1))}$$

This is identical to $X\cos\theta+Y\sin\theta=p$

and so, on comparing coefficients of like terms, we get

$$\frac{\frac{1}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}}{\cos\theta}=\frac{\frac{1}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}}{\sin\theta}=\frac{1}{p}$$

or $$\left(\frac{x}{a}\right)^{\frac{1}{n-1}}=\frac{a\cos\theta}{p} \tag{2}$$

and $$\left(\frac{y}{b}\right)^{\frac{1}{n-1}}=\frac{b\sin\theta}{p} \tag{3}$$

Raising both the sides in (2) and (3) to the power n, we have

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}}=\frac{(a\cos\theta)^n}{p^n} \tag{4}$$

$$\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=\frac{(b\sin\theta)^n}{p^n} \tag{5}$$

Adding (4) and (5), we get

$$\frac{(a\cos\theta)^n}{p^n}+\frac{(b\sin\theta)^n}{p^n}=\left(\frac{x}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=1$$

or $(a\cos\theta)^n+(b\sin\theta)^n=p^n$.

9. The tangent at any point on the curve $x^3+y^3=2a^3$ makes intercepts p and q on the coordinate axes. Show that

$$p^{-3/2}+q^{-3/2}=2^{-1/2}a^{-3/2}$$

Sol. $f(x,y)=x^3+y^3-2a^3=0$ (1)

$$\frac{\partial f}{\partial x}=3x^2,\ \frac{\partial f}{\partial y}=3y^2$$

Equation of the tangent at any point on the curve is

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0$$

Substituting the values, we have

$(X-x)\,3x^2+(Y-y)\,3y^2=0$

or $Xx^2+Yy^2=x^3+y^3$

or $Xx^2+Yy^2=2a^3$, (using (1)) (2)

The tangent cuts off length p on the X-axis.

Putting $X=p$ and $Y=0$ into (2), we get

$$px^2+0=2a^3 \Rightarrow x^2=\frac{2a^3}{p} \text{ or } x=\left(\frac{2a^3}{p}\right)^{1/2} \tag{3}$$

Also the tangent cuts off length q on the Y-axis.

Putting $X=0$ and $Y=q$ in (2), we have

$$0+qy^2=2a^3 \Rightarrow y^2=\frac{2a^3}{q} \Rightarrow y=\left(\frac{2a^3}{q}\right)^{1/2} \tag{4}$$

Raising both the sides of (3) and (4) to the power 3 and adding the results, we get

$$x^3+y^3=\left(\frac{2a^3}{p}\right)^{3/2}+\left(\frac{2a^3}{q}\right)^{3/2}=\frac{2^{3/2}a^{9/2}}{p^{3/2}}+\frac{2^{3/2}a^{9/2}}{q^{3/2}}$$

or $2a^3=2^{3/2}a^{9/2}(p^{-3/2}+q^{-3/2})$

or $2^{-1/2}a^{-3/2}=p^{-3/2}+q^{-3/2}$ which is the required condition.

Find the angle of intersection of the given curves (Problems 10 – 12):

Note: The angle of intersection of two curves at a point of intersection of the curves is the angle between the tangents to the two curves at that point.

10. The parabolas $y^2=4ax$ and $x^2=4ay$ at the point other than (0, 0).

Sol. $y^2=4ax$ (1)

$x^2=4ay$ (2)

$$x=\frac{y^2}{4a} \tag{3}$$

From (1), putting $x=\frac{y^2}{4a}$ in (2), we get

$$\left(\frac{y^2}{4a}\right)^2=4ay \quad\text{or}\quad y^4=64a^3y$$

i.e., $y(y^3-64a^3)=0$ or $y(y-4a)(y^2+4ay+16a^2)=0$

or $y=0, 4a$ (real values)

Putting $y=0$ in (3), we have $x=0$

Putting $y = 4a$ in (3), we get $x = \frac{16a^2}{4a} = 4a$

Therefore, the points of intersection of (1) and (2) are (0, 0) and (4a, 4a).

Now $y^2 = 4ax$ gives

$$2y\frac{dy}{dx} = 4a$$

or $\frac{dy}{dx} = \frac{2a}{y}$

or $\left(\frac{dy}{dx}\right)_{(4a,\,4a)} = \frac{2a}{4a}$

$$= \frac{1}{2} = m_1 \text{ (say)}$$

And $x^2 = 4ay$ gives

$$2x = 4a\frac{dy}{dx}$$

or $\frac{dy}{dx} = \frac{x}{2a}$

or $\left(\frac{dy}{dx}\right)_{(4a,\,4a)} = \frac{4a}{2a}$

$$= 2 = m_2 \text{ (say)}$$

The angle of intersection θ is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} \qquad (1)$$

$$= \frac{2 - \frac{1}{2}}{1 + 2\frac{1}{2}} = \frac{\frac{3}{2}}{2} = \frac{3}{4} \quad i.e., \quad \theta = \text{arc}\tan\left(\frac{3}{4}\right)$$

11. $x^2 - y^2 = a^2 \quad , \quad x^2 + y^2 = a^2\sqrt{2}$

Sol. $x^2 - y^2 = a^2 \qquad (1)$

$x^2 + y^2 = \sqrt{2}\,a^2 \qquad (2)$

Adding (1) and (2), we get

$$2x^2 = (\sqrt{2} + 1)a^2 \text{ or } x^2 = \left(\frac{\sqrt{2}+1}{2}\right)a^2 \Rightarrow x = \pm\sqrt{\frac{\sqrt{2}+1}{2}}\,a$$

From (1), $y^2 = x^2 - a^2 = \frac{\sqrt{2}+1}{2}a^2 - a^2 = \frac{\sqrt{2}-1}{2}a^2$

or $y = \pm\sqrt{\frac{\sqrt{2}-1}{2}}\,a$

Thus the four points of intersection are

$$\left(\pm\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,\,\pm\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$$

Let θ be the angle from l_1 to l_2 where l_1 is the tangent to (1) and l_2 is the tangent to (2) at (x_1, y_1) which is any point of intersection.

From (1), $2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

and $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{x_1}{y_1} = m_1$ (say)

From (2), $\frac{dy}{dx} = -\frac{x}{y}$ and $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{x_1}{y_1} = m_2$ (say)

$$\text{Now } \tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{-\frac{x_1}{y_1} - \frac{x_1}{y_1}}{1 + \left(-\frac{x_1}{y_1}\right)\left(\frac{x_1}{y_1}\right)} = \frac{-\frac{2x_1}{y_1}}{1 - \frac{x_1^2}{y_1^2}}$$

$$= \frac{2x_1 y_1}{x_1^2 - y_1^2} = \frac{2x_1 y_1}{a^2}, \text{ since } \quad x_1^2 - y_1^2 = a^2$$

As $2\left(\sqrt{\frac{\sqrt{2}+1}{2}}\,a\right)\left(\sqrt{\frac{\sqrt{2}+1}{2}}\,a\right) = \frac{2\sqrt{(\sqrt{2}+1)(\sqrt{2}-1)}}{2}a^2 = a^2$

and $2\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\right)\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\right) = a^2$, so

$\tan\theta = \frac{a^2}{a^2} = 1$ *i.e.*, $\theta = 45^\circ$ at the points $\left(\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$

and $\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,-\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$,

But $2x_1y_1 = -a^2$ for the points $\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$

and $\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,-\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$, and so $\tan\theta = \frac{-a^2}{a^2} = -1$

i.e., $\theta = 135^\circ$ at these points.

12. $y^2 = ax \quad$ and $\quad x^3 + y^3 = 3axy$

Sol. $y^2 = ax \qquad (1)$

and $x^3 + y^3 = 3axy \qquad (2)$

From (1), $x = \frac{y^2}{a} \qquad (3)$

Putting this value of x into (2), we get

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3a\left(\frac{y^2}{a}\right).y$$

or $y^6 - 2a^3y^3 = 0 \Rightarrow y^3(y^3 - 2a^3) = 0 \Rightarrow y = 0\,,\,2^{1/3}a$

or $\frac{y^6}{a^3} + y^3 = 3y^3 \quad$ or $\quad y^3 + a^3 = 3a^3 \quad$ or $\quad y = 2^{1/3}a$

Now for $y = 0$ in (3), we get $x = 0$ and putting $y = 2^{1/3}a$ in (3), we have

$$x = \frac{(2^{1/3}\, a)^2}{a} = 2^{2/3}\, a.$$

The point of intersection of (1) and (2), say P, is

$$P(2^{2/3}\, a,\ 2^{1/3}\, a)$$

Differentiating (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = a$$

or $\quad \dfrac{dy}{dx} = \dfrac{a}{2y}$

i.e., $\quad \left(\dfrac{dy}{dx}\right)_P = \dfrac{a}{2.2^{1/3}\, a} = \dfrac{1}{2^{4/3}}$

i.e., $\quad \tan\theta_1 = \dfrac{1}{2^{4/3}} = m_1,$

where θ_1 is the angle between the tangent to (1) and the x-axis at the point of intersection.

Differentiating (1) w.r.t. x, we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\,\frac{dy}{dx} + y\right]$$

or $\quad (3y^2 - 3ax)\dfrac{dy}{dx} = 3ay - 3x^2$

or $\quad \dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$

i.e., $\quad \left(\dfrac{dy}{dx}\right)_P = \dfrac{a\,.\,2^{1/3}a - (2^{2/3}\, a)^2}{(2^{1/3}\, a)^2 - a.\,2^{2/3}\, a}$

$$= \frac{2^{1/3}\, a^2 - 2^{4/3}\, a^2}{2^{2/3}\, a^2 - 2^{2/3}\, a^2}$$

$$= \frac{2^{1/3}\, a^2 - 2^{4/3}\, a^2}{0}$$

i.e., $\quad \theta_2 = 90°,$

where θ_2, is the angle between the tangent to (2) and the x-axis at the point of intersection.

If θ is the required angle, then $\theta = 90° - \theta_1$

i.e., $\quad \tan\theta = \tan(90° - \theta_1)$

$$= \cot\theta_1 = \frac{1}{\tan\theta_1} = 2^{4/3}$$

Hence $\quad \theta = \arctan(2^{4/3})$

13. Find the condition that the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ should intersect orthogonally.

Sol. Let (x_1, y_1) be the point of intersection of the given curves

$$ax^2 + by^2 = 1 \tag{1}$$

and $\quad a_1x^2 + b_1y^2 = 1 \qquad (2)$

Differentiating (1) w.r.t, x, we have

$$2ax + 2by\frac{dy}{dx} = 0$$

or $\quad \dfrac{dy}{dx} = -\dfrac{ax}{by} \quad$ *i.e.,* $\quad \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = -\dfrac{ax_1}{by_1} = m_1 \quad$ (say)

Similarly, from (2), $\dfrac{dy}{dx} = -\dfrac{a_1 x}{b_1 x}$

i.e., $\quad \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \dfrac{-a_1x_1}{b_1y_1} = m_2 \quad$ (say)

Since the curves cut orthogonally, we have

$$m_1\, m_2 = -1$$

i.e., $\quad \left(-\dfrac{ax_1}{by_1}\right)\left(-\dfrac{a_1 x_1}{b_1 y_1}\right) = -1$

or $\quad \dfrac{aa_1\, x_1^{\,2}}{bb_1\, y_1^{\,2}} = -1 \qquad (3)$

Now, from (1) and (2), by subtraction, we get

$$(a - a_1)\, x^2 + (b - b_1)\, y^2 = 0$$

or $\quad \dfrac{x_1^{\,2}}{y_1^{\,2}} = -\dfrac{b - b_1}{a - a_1} \quad$ at (x_1, y_1)

Substituting this value into (3), we have

$$\frac{aa_1\,(b - b_1)}{bb_1\,(a - a_1)} = 1 \quad \text{or} \quad aa_1\,(b - b_1) = bb_1\,(a - a_1)$$

Dividing both sides by aa_1bb_1, we get

$$\frac{b - b_1}{bb_1} = \frac{a - a_1}{aa_1} \qquad \text{or} \qquad \frac{1}{b_1} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{a}$$

or $\quad \dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a_1} - \dfrac{1}{b_1} \quad$ is the required condition.

14. Show that the pedal equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{is} \qquad \frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$$

Sol. The given equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad (1)$

Let (x_1, y_1) lie on (1). Then $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1 \qquad (2)$

Equation of the tangent to (1) at (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \textit{i.e.,} \quad \frac{x_1}{a^2}x + \frac{y_1}{b^2}y - 1 = 0$$

$$p = \frac{|-1|}{\sqrt{\dfrac{x_1^2}{a^4} + \dfrac{y_1^2}{b^4}}} \quad \text{or} \quad \frac{1}{p^2} = \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} \tag{3}$$

Also $x_1^2 + y_1^2 = r^2$ (4)

Now we eliminate x_1^2, y_1^2 from (2), (3) and (4).

From (4) and (2), we have

$$x_1^2 + y_1^2 - r^2 = 0 \quad \text{and} \quad b^2x_1^2 + ay_1^2 - a^2b^2 = 0$$

Therefore, $\dfrac{x_1^2}{-a^2b^2 + a^2r^2} = \dfrac{y_1^2}{-b^2r^2 + a^2b^2} = \dfrac{1}{a^2 - b^2}$

or $x_1^2 = \dfrac{a^2(r^2 - b^2)}{a^2 - b^2}$; $y_1^2 = \dfrac{-b^2(r^2 - a^2)}{a^2 - b^2}$

Substituting these values into (3), we get

$$\frac{1}{p^2} = \frac{r^2 - b^2}{a^2(a^2 - b^2)} - \frac{r^2 - a^2}{b^2(a^2 - b^2)}$$

$$= \frac{b^2r^2 - b^4 - a^2r^2 + a^4}{a^2b^2(a^2 - b^2)} = \frac{a^4 - b^4}{a^2b^2(a^2 - b^2)} - \frac{r^2(a^2 - b^2)}{a^2b^2(a^2 - b^2)}$$

$$= \frac{a^2 + b^2}{a^2b^2} - \frac{r^2}{a^2b^2} = \frac{1}{b^2} + \frac{1}{a^2} - \frac{r^2}{a^2b^2}$$

is the required pedal equation.

15. Show that the pedal equation of the curve

$$c^2(x^2 + y^2) = x^2y^2 \text{ is } \frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$$

Sol. The given equation is $c^2(x^2 + y^2) = x^2y^2$ (1)

Divide through by $c^2x^2y^2$ to have from (1)

$$\frac{1}{x^2y^2}(x^2 + y^2) = \frac{1}{c^2} \quad \text{or} \quad \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{c^2} \tag{2}$$

Now differentiating (2) w.r.t. x, we get

$$-\frac{2}{x^3} - \frac{2}{y^3}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{y^3}{x^3}$$

If (x_1, y_1) lies on (1), then

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{y_1^3}{x_1^3} \text{ and also } c^2(x_1^2 + y_1^2) = x_1^2y_1^2$$

Equation of the tangent to (1) at (x_1, y_1) is

$$y - y_1 = -\frac{y_1^3}{x_1^3}(x - x_1) \quad \text{or} \quad -\frac{y}{y_1^3} + \frac{y_1}{y_1^3} = \frac{x}{x_1^3} - \frac{x_1}{x_1^3}$$

or $-\dfrac{y}{y_1^3} + \dfrac{1}{y_1^2} = \dfrac{x}{x_1^3} - \dfrac{1}{x_1^2}$ or $\dfrac{x}{x_1^3} + \dfrac{y}{y_1^3} - \dfrac{1}{x_1^2} - \dfrac{1}{y_1^2} = 0$

$$p = \frac{\left|-\left(\dfrac{1}{x_1^2} + \dfrac{1}{y_1^2}\right)\right|}{\sqrt{\dfrac{1}{x_1^6} + \dfrac{1}{y_1^6}}} = \frac{\dfrac{x_1^2 + y_1^2}{x_1^2y_1^2}}{\sqrt{\dfrac{x_1^6 + y_1^6}{x_1^6y_1^6}}}$$

$$p^2 = \frac{(x_1^2 + y_1^2)^2}{x_1^4y_1^4} \times \frac{x_1^6y_1^6}{x_1^6 + y_1^6} = \frac{x_1^2y_1^2(x_1^2 + y_1^2)^2}{(x_1^2 + y_1^2)^3 - 3x_1^2y_1^2(x_1^2 + y_1^2)}$$

$$= \frac{c^2r^2 \cdot r^4}{r^6 - 3c^2r^2 \cdot r^2} = \frac{c^2r^2}{r^2 - 3c^2}, \text{ since } x_1^2 + y_1^2 = r^2$$

or $\dfrac{1}{p^2} = \dfrac{r^2 - 3c^2}{c^2r^2} = \dfrac{1}{c^2} - \dfrac{3}{r^2} \Rightarrow \dfrac{1}{p^2} + \dfrac{3}{r^2} = \dfrac{1}{c^2}$

which is the required pedal equation.

16. Show that from any point three normals can be drawn to a parabola $y^2 = 4ax$ and the sum of the slopes of the three normals is zero.

Sol. Equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3. \quad \text{(Example 7)}$$

If it passes through a given point (h, k), then

$$k = mh - 2am - am^3$$

or $am^3 + 0m^2 - m(2a - h) + k = 0$ (1)

This is a cubic in m which gives the slopes of the three normals which pass through the point (h, k).

Hence three normals, can be drawn from any point (h, k) to the parabola $y^2 = 4ax$.

If m_1, m_2, m_3 are roots of the cubic (1), then

$$m_1 + m_2 + m_3 = \frac{-0}{a} = 0$$

Hence the sum of the slopes of the three normals to the parabola $y^2 = 4ax$ drawn from any point is zero.

17. Show that the tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix.

Sol. Let $y^2 = 4ax$ be equation of a parabola. If t_1, t_2 are the extremities of a focal chord then $t_1t_2 = -1$. Also tangents at t_1, t_2 are

$$t_1y = x + at_1^2 \tag{1}$$

and $t_2 y = x + at_2^2$ (2)

The product of the slopes of (1) and (2)

$$= \frac{1}{t_1} \cdot \frac{1}{t_2}$$

$$= -1, \text{ (since } t_1 t_2 = -1)$$

Therefore, the tangents to (1) and (2) are perpendicular to each other. The point of intersection of (1) and (2) is

$[at_1 t_2, a(t_1 + t_2)]$

i.e., $[-a, a(t_1 + t_2)]$

which clearly lies on the directrix $x = -a$.

Hence the result.

18.(*a*) Show that the tangent at the vertex of a diameter of a parabola is parallel to the chords bisected by the diameter.

Sol.

(*a*) Let an equation of the parabola be $y^2 = 4ax$ and m be the slope of the parallel chords.

Equation of the diameter bisecting the chords is

$$y = \frac{2a}{m} \qquad (1)$$

Let (x_1, y_1) be the vertex of the diameter.

Since (x_1, y_1) lies on (1), therefore

$$y_1 = \frac{2a}{m} \quad \text{or} \quad m = \frac{2a}{y_1} \qquad (2)$$

The equation of the tangent at the vertex (x_1, y_1) is

$yy_1 = 2a(x + x_1)$.

The slope of the tangent is $= \frac{2a}{y_1} = m =$ the slope of the parallel chords.

Hence the tangent is parallel to the chords.

18.(*b*) Prove that the tangents at the ends of any chord of a parabola meet on the diameter which bisects the chord.

Sol. Let an equation of the parabola be $y^2 = 4ax$ and let equation of a chord PQ be $y = mx + c$. Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. Equations of the tangents at P and Q are respectively.

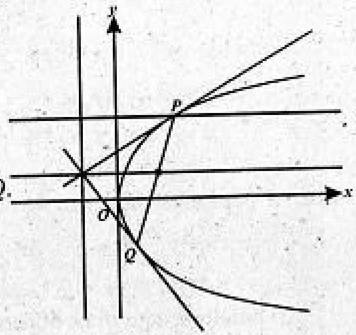
$$yy_1 = 2a(x + x_1)$$
$$yy_2 = 2a(x + x_2)$$

Ordinate of the point of intersection of these tangents is

$$y = \frac{2a(x_1 - x_2)}{y_1 - y_2} \qquad (1)$$

But $m = \frac{y_1 - y_2}{x_1 - x_2} =$ slope of PQ.

Therefore, (1) is $y = \frac{2a}{m}$



By Example 10, $y = \frac{2a}{m}$ is equation of the diameter bisecting the chords parallel to PQ. Thus the tangents at P and Q meet on this diameter.

19. Find the condition that the straight line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find the coordinates of the point of contact.

Sol. Let us assume that $lx + my + n = 0$ touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

The tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

or $b^2 x_1 x + a^2 y_1 y - a^2 b^2 = 0$. This must be identical to

$lx + my + n = 0$

Therefore, $\frac{b^2 x_1}{l} = \frac{a^2 y_1}{m} = \frac{-a^2 b^2}{n}$

or $x_1 = \frac{-a^2 l}{n}$ and $y_1 = \frac{-b^2 m}{n}$

The point of contact is $\left(\frac{-a^2 l}{n}, \frac{-b^2 m}{n}\right)$

It lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so

$\frac{a^4 l^2}{n^2 a^2} + \frac{b^4 m^2}{n^2 b^2} = 1$ *i.e.*, $a^2 l^2 + b^2 m^2 = n^2$ is the required condition.

20. Show that the condition that normal at the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) on the ellipse $\frac{x_2}{a_2} + \frac{y^2}{b^2} = 1$ may be concurrent is

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0$$

Sol. Equation of the normal at (h, k) on the ellipse can be written as

$$\frac{a^2x}{h} - \frac{b^2y}{k} = a^2 - b^2, \text{ (Example 12)}$$

or $\quad a^2kx - b^2hy = (a^2 - b^2)\,hk.$

Therefore, equations of the three normals at (x_1, y_1), (x_2, y_2), (x_3, y_3) are

$$a^2y_1 - b^2x_1y - (a^2 - b^2)\,x_1y_1 = 0 \qquad (1)$$
$$a^2y_2 - b^2x_2y - (a^2 - b^2)\,x_2y_2 = 0 \qquad (2)$$
and $$a^2y_3 - b^2x_3y - (a^2 - b^2)\,x_3y_3 = 0 \qquad (3)$$

Solving (2) and (3), we have

$$x = \frac{(a^2 - b^2)\,x_2x_3(y_3 - y_2)}{a^2(x_2y_3 - x_3y_2)}, \; y = \frac{(a^2 - b^2)\,y_2y_3(x_3 - x_2)}{b^2(x_2y_3 - x_3y_2)}$$

Putting these values of x and y into (1), we get the condition of concurrency of normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1), (x_2, y_2) and (x_3, y_3). Hence

$$a^2y_1\left(\frac{(a^2 - b^2)\,x_2x_3(y_3 - y_2)}{a^2(x_2y_3 - x_3y_2)}\right) - b^2x_1\left(\frac{(a^2 - b^2)\,y_2y_3(x_3 - x_2)}{a^2(x_2y_3 - x_3y_2)}\right)$$
$$- (a^2 - b^2)x_1y_1 = 0$$

$$x_1(y_2x_3y_3 - y_3x_2y_2) - y_1(x_2x_3y_3 - x_3x_2y_2) + x_1y_1(x_2y_3 - x_3y_2) = 0$$

or $$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0$$

21. If a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with centre C, meets the major and minor axes in T and t, prove that $\frac{a^2}{CT^2} + \frac{b^2}{Ct^2} = 1$

Sol. Tangent to the ellipse at the point 'θ' is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

This meets the major axis where $y = 0$

i.e., $\frac{x}{a}\cos\theta = 1 \quad$ or $\quad x = \frac{a}{\cos\theta}$

or $\quad CT = \frac{a}{\cos\theta} \quad$ or $\quad \cos\theta = \frac{a}{CT} \qquad (1)$

Similarly, (1) meets the minor axis where $x = 0$

i.e., $\frac{y}{b}\sin\theta = 1 \quad$ or $\quad y = \frac{b}{\sin\theta}$

i.e., $Ct = \frac{b}{\sin\theta} \quad$ or $\quad \sin\theta = \frac{b}{Ct} \qquad (2)$

Squaring (1) and (2) and adding the results, we get

$$\cos^2\theta + \sin^2\theta = 1 = \frac{a^2}{CT^2} + \frac{b^2}{Ct^2} \text{ as required.}$$

22. Show that the locus of the point of intersection of tangents at two points on the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x_2}{a_2} + \frac{y_2}{b_2} = \sec^2\lambda$, where 2λ is the difference of the eccentric angles of the two points.

Sol. Let the two points on the ellipse be

$(a\cos\alpha, b\sin\alpha)$, $(a\cos\beta$, be $\sin\beta)$

Equations of the tangents at these point are

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$$

and $$\frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta = 1$$

The point of intersection of the tangents is

$$x = \frac{a(\sin\alpha - \sin\beta)}{\sin(\alpha - \beta)} = \frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

$$y = \frac{-b(\cos\alpha - \cos\beta)}{\sin(\alpha - \beta)} = \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \qquad (1)$$

Now $\alpha - \beta = 2\lambda$ (Given)

Squaring the equations in (1) and adding the results, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)} = \frac{1}{\cos^2\lambda} = \sec^2\lambda$$

23. Show that locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle and product of the lengths of perpendiculars is equal to the square of the semi-minor axis.

Sol. Let an equation of an ellipse be $\frac{x^2}{a^2}+\frac{x^2}{b^2}=1$.

Equation of a tangent to the ellipse, in the m – form, is

$$y = mx + \sqrt{a^2m^2+b^2} \qquad (1)$$

Slope of any line perpendicular to it is $-\frac{1}{m}$

Equation of the perpendicular F_1P from F_1 $(-ae, 0)$ on the tangent (1) is

$$y - 0 = -\frac{1}{m}(x + ae) \qquad (2)$$

To find the locus of P, the foot of the perpendicular [*i.e.*, point of intersection of (1) and (2)], we have to eliminate m between (1) and (2). (1) and (2) may be written as

$$y - mx = \sqrt{a^2m^2+b^2} \qquad (3)$$

and $\quad x + my = -ae \qquad (4)$

Squaring (3) and (4), we have

$$y^2 - 2mxy + m^2x^2 = a^2m^2 + b^2 \qquad (5)$$

and $\quad x^2 + 2mxy + m^2y^2 = a^2e^2 \qquad (6)$

Adding (5) and (6), we get

$$\begin{aligned} x^2(1+m^2) + y^2(1+m^2) &= a^2m^2 + b^2 + a^2e^2 \\ &= a^2m^2 + a^2(1-e^2) + a^2e^2 \\ &= a^2(1+m^2) \end{aligned}$$

or $\quad x^2 + y^2 = a^2$, which is the required locus.

This is auxiliary circle of the given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Similarly, the locus of Q, the foot of the perpendicular from the other focus F_2 on the tangent (1), is the auxiliary circle.

Now we prove that $|F_1P| \cdot |F_2Q| = CB^2$.

Equation of the tangent is $y = mx + \sqrt{a^2m^2+b^2}$

or $\quad mx - y + \sqrt{a^2m^2+b^2} = 0 \qquad (1)$

$|F_1P|$ = perpendicular from $(-ae, 0)$ on (1)

$$= \frac{|-aem + \sqrt{a^2m^2+b^2}|}{\sqrt{1+m^2}}$$

and $\quad |F_2Q|$ = perpendicular from $(ae, 0)$ on (1)

$$= \frac{|aem + \sqrt{a^2m^2+b^2}|}{\sqrt{1+m^2}}$$

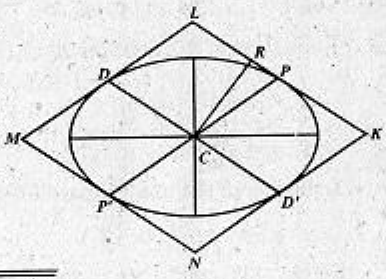
$$\begin{aligned} |F_1P| \cdot |F_2Q| &= \frac{a^2m^2 + b^2 - a^2e^2m^2}{1+m^2} = \frac{a^2m^2(1-e^2)+b^2}{1+m^2} \\ &= \frac{m^2b^2+b^2}{1+m^2} = \frac{b^2(1+m^2)}{1+m^2} = b^2 = CB^2. \end{aligned}$$

24. Prove that the area enclosed by the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant.

Sol. Let $P'CP$ and $D'CD$ be conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then

$P = (a\cos\theta, b\sin\theta)$ and $D = (-a\sin\theta, b\cos\theta)$.

Let $KLMN$ be the parallelogram formed by the tangents at P, D, P' and D'.



$$\begin{aligned} \text{Area } KLMN &= 4 \text{ area } CPLD \\ &= 4\,|CR| \cdot |PL| \\ &= 4\,|CR| \cdot |CD| \end{aligned}$$

where $|CR|$ = length of the perpendicular from C on the tangent at P.

Equation of the tangent at P is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta - 1 = 0$$

Therefore, $|CR| = \dfrac{|-1|}{\sqrt{\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}}}$

$$= \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} = \frac{ab}{|CD|},$$

since $\quad |CD| = \sqrt{(-a\sin\theta - 0)^2 + (b\cos\theta - 0)^2}$

Hence, $\quad$ area $KLMN = 4\,|CR| \cdot |CD| = 4 \cdot \frac{ab}{|CD|} \cdot |CD|$

$= 4ab$, which is a constant

25. The hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ are said to be conjugate to each other. If e and e' are eccentricities of a hyperbola and its **conjugate**, prove that $\frac{1}{e^2}+\frac{1}{e'^2}=1$

Sol. The eccentricity e of the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by $b^2 = a^2 (e^2 - 1)$ *i.e.*, $\frac{b^2}{a^2} = e^2 - 1$ (1)

The eccentricity e' of the conjugate hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is given by $a^2 = b^2 (e'^2 - 1)$ *i.e.*, $\frac{a^2}{b^2} = e'^2 - 1$ (2)

Multiplying (1) and (2) together, we have

$$(e^2 - 1)(e'^2 - 1) = 1 \quad \text{or} \quad e^2 e'^2 = e^2 + e'^2$$

or $\frac{1}{e^2} + \frac{1}{e'^2} = 1.$

26. Show that the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the lines drawn from any point on the hyperbola parallel to the asymptotes form a parallelogram of constant area $\frac{ab}{2}$.

Sol. Let $P(x_1, y_1)$ be any point on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Equations of the asymptotes of the hyperbola are

$$y = \frac{b}{a} x \tag{1}$$

and $y = -\frac{b}{a} x$ (2)

Equation of the line PR parallel to (2) is

$$y - y_1 = -\frac{b}{a}(x - x_1) \tag{3}$$

Solving (1) and (3), we have

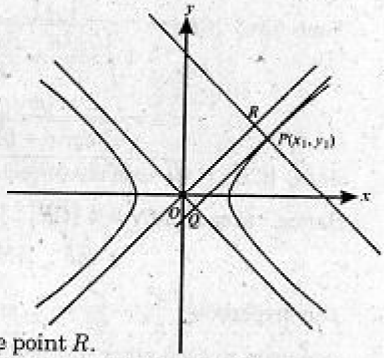
$$\frac{b}{a} x - y_1 = -\frac{b}{a} x + \frac{b}{a} x_1$$

$$\Rightarrow \frac{2b}{a} x = y_1 + \frac{b}{a} x_1$$

or $x = \frac{a}{2b}\left(y_1 + \frac{b}{a} x_1\right)$

and so $y = \frac{b}{a} \frac{a}{2b}\left(y_1 + \frac{b}{a} x_1\right)$

$$= \frac{1}{2}\left(y_1 + \frac{b}{a} x_1\right)$$



These are the coordinates of the point R.

The line PQ parallel to (1) meets the asymptotes (2) at Q.

We need the area of the parallelogram $OQPR$.

Required area = 2 area $\Delta\, OPR$.

$$= \begin{vmatrix} 0 & 0 & 1 \\ x_1 & y_1 & 1 \\ \frac{1}{2}\left(y_1 + \frac{bx_1}{a}\right) & \frac{a}{2b}\left(y_1 + \frac{b}{a} x_1\right) & 1 \end{vmatrix}$$

$$= \frac{x_1}{2a}(ay_1 + bx_1) - \frac{ay_1(ay_1 + bx_1)}{2ab}$$

$$= \frac{1}{2ab}(ab\, x_1 y_1 + b^2 x_1^2 - a^2 y_1^2 - ab\, x_1 y_1) = \frac{1}{2ab}[b^2 x_1^2 - a^2 y_1^2]$$

$$= \frac{1}{2ab}(a^2 b^2), \text{ since } \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$$= \frac{ab}{2}$$

27. Show that the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at the point 't'' such that $t^3 t' = -1$.

Note: The point $\left(ct, \frac{c}{t}\right)$ on $xy = c^2$ is referred to as the point 't'.

Sol. Equation of the normal at the point 't' is

$$t^3 x - ty + c - ct^4 = 0$$

Let this normal cut the hyperbola again at the point 't''. Then

$\left(ct', \frac{c}{t'}\right)$ lies on this normal

Hence $t^3 ct' - t\frac{c}{t'} + c - ct^4 = 0$

or $t'^2 t^3 - t + t' - t' t^4 = 0$

or $t'(t^3 t' + 1) - t(1 + t' t^3) = 0$

or $(t' - t)(t^3 t' + 1) = 0$

But $t \neq t'$, so $t^3 t' + 1 = 0 \Rightarrow t^3 t' = -1$

28. Prove that if P is any point on a hyperbola with foci F_1 and F_2, then the tangent at P bisects the angle $F_1 P F_2$.

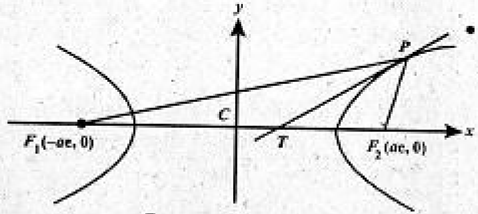
Sol. Let the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

Let $P(a \sec\theta, b \tan\theta)$ be any point on (1). Foci of (1) are $F_2(ae, 0)$ and $F_1(-ae, 0)$.

Equation of the tangent at P is

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$$



It cuts the x-axis at $T\left[\dfrac{a}{\sec\theta}, 0\right]$ i.e., $(a\cos\theta, 0)$

$$(PF_2)^2 = (a\sec\theta - ae)^2 + (b\tan\theta)^2$$
$$= a^2\sec^2\theta + a^2e^2 + b^2\tan^2\theta - 2a^2e\sec\theta$$
$$= a^2\sec^2\theta + (a^2+b^2) + b^2\tan^2\theta - 2a^2e\sec\theta$$
$$= a^2\sec^2\theta + a^2 + b^2\sec^2\theta - 2a^2e\sec\theta$$
$$= (a^2+b^2)\sec^2\theta + a^2 - 2a^2e\sec\theta$$
$$= a^2e^2\sec^2\theta + a^2 - 2a^2e\sec\theta$$
$$= (ae\sec\theta - a)^2 = a^2(e\sec\theta - 1)^2$$

Similarly $(PF_1)^2 = a^2(-e\sec\theta - 1)^2$ (changing e into $-e$)

$$= a^2(e\sec\theta + 1)^2$$

$$\frac{(PF_2)^2}{(PF_1)^2} = \frac{a^2(e\sec\theta - 1)^2}{a^2(e\sec\theta + 1)^2} = \frac{(e-\cos\theta)^2}{(e+\cos\theta)^2}$$

$$\frac{|PF_2|}{|PF_1|} = \frac{|e-\cos\theta|}{|e+\cos\theta|} \qquad (2)$$

Further, $(F_2T)^2 = (a\cos\theta - ae)^2 = a^2(\cos\theta - e)^2$

$$(F_1T)^2 = (a\cos\theta + ae)^2 = a^2(\cos\theta + e)^2$$

$$\frac{|F_2'T|}{|F_1T|} = \frac{|e-\cos\theta|}{|e+\cos\theta|} \qquad (3)$$

From (2) and (3), it is clear that the tangent divides F_2F_1 internally in the ratio of PF_2 and PF_1.

Hence PT bisects the angle F_2PF_1.

29. Find an equation of a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ in the form $\dfrac{x}{a}\cosh\theta - \dfrac{y}{b}\sinh\theta = 1$. Show that the product of lengths of perpendiculars on it from the foci is constant.

Sol. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (1)

The point $x = a\cosh\theta, y = b\sinh\theta$ lies on (1). Thus any point on (1) is $P(a\cosh\theta, b\sinh\theta)$. Equation of the tangent to (1) at the point P is

$$\frac{x\,.\,a\cosh\theta}{a^2} - \frac{y\,.\,b\sinh\theta}{b^2} = 1 \text{ or } \frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta = 1 \qquad (2)$$

as desired.

The foci of the hyperbola (1) are $F_2(a, e, 0)$, $F'_1(-ae, 0)$.

Let F_2Q, F'_1R be perpendiculars from F_2 and F'_1 to (2). Then

$$|F_2Q| = \frac{|e\cosh\theta - 1|}{\sqrt{\dfrac{\cosh^2\theta}{a^2} + \dfrac{\sinh^2\theta}{b^2}}}$$

$$|F_1R| = \frac{|-e\cosh\theta - 1|}{\sqrt{\dfrac{\cosh^2\theta}{a^2} + \dfrac{\sinh^2\theta}{b^2}}} = \frac{|e\cosh\theta + 1|}{\sqrt{\dfrac{\cosh^2\theta +}{a^2} + \dfrac{\sinh^2\theta}{b^2}}}$$

$$F_2Q.F_1R = \frac{(e^2\cosh^2\theta - 1)\,a^2b^2}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$

$$= \frac{a^2b^2\left(\dfrac{a^2+b^2}{a^2}\right)\cosh^2\theta - a^2b^2}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$

$$= \frac{a^2b^2\cosh^2\theta + b^4\cosh^2\theta - a^2b^2}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$

$$= \frac{a^2b^2(\cosh^2\theta - 1) + b^4\cosh^2\theta}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$

$$= \frac{b^2(a^2\sinh^2\theta + b^2\cosh^2\theta)}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$

$= b^2$, which is a constant.

30. Find an equation of a normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ in the form $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 + b^2$. Prove that the normal is external bisector of the angle between the focal distances of its foot.

Sol. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (1)

Differentiating (1) w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = \frac{2x}{a^2}\times\frac{b^2}{2y} = \frac{b^2x}{a^2y}$$

At $P(a\sec\theta, b\tan\theta)$,

$$\frac{dy}{dx} = \frac{b^2a\sec\theta}{a^2b\tan\theta} = \frac{b}{a\sin\theta}$$

Thus slope of the normal is $-\dfrac{a\sin\theta}{b}$

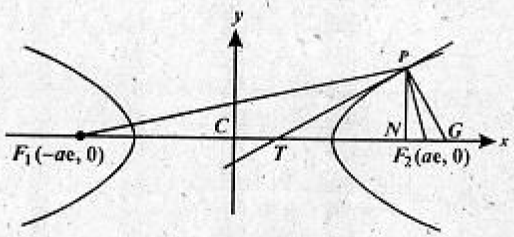
Equation of the normal at P is

$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

or $\quad by - b^2\tan\theta = -a\sin\theta(x - a\sec\theta)$

$$by\cot\theta - b^2 = -a^x\cos\theta + a^2$$

$$ax\cos\theta + by\cot\theta = a^2 + b^2 \quad \text{or} \quad \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$



Suppose that the normal at P ($a\sec\theta$, $b\tan\theta$) meets the transverse axis in G. Then coordinates of G are

$$\left(\frac{(a^2+b^2)\sec\theta}{a}, 0\right)$$

As $\dfrac{a^2+b^2}{a}\sec\theta = \dfrac{a^2e^2}{a}\sec\theta = ae^2\sec\theta$, so it is $(ae^2\sec\theta, 0)$

Now $(F_2P)^2 = (a\sec\theta - ae)^2 + b^2\tan^2\theta$

$$= (a\sec\theta - ae)^2 + b^2(\sec^2\theta - 1)$$
$$= a^2\sec^2\theta - 2a^2e\sec\theta + a^2e^2 + b^2\sec^2\theta - b^2$$
$$= (a^2+b^2)\sec^2\theta - 2a^2e\sec\theta + a^2e^2 - b^2$$
$$= a^2e^2\sec^2\theta - 2a^2e\sec\theta + a^2 = [a(e\sec\theta - 1)]^2$$

$$(F_2G)^2 = (ae^2\sec\theta - ae)^2 = [ae(e\sec\theta - 1)]^2$$
$$= e^2[a(e\sec\theta - 1)]^2$$

Therefore, $(F_2G)^2 = e^2(F_2P)^2 \Rightarrow |F_2G| = e|F_2P|$

Similarly, $|F_1G| = e|F_1P|$

Therefore, $\dfrac{|F_2G|}{|F_1G|} = \dfrac{|F_2P|}{|F_1P|}$

Hence the normal PG is external bisector of the angle F_1PF_2.

Since the tangent PT at P is perpendicular to the normal PG, therefore, PT bisects the interior angle F_1PF_2.

Exercise Set 6.3 (Page 250)

In Problems 1 – 8, express the given equations in rectangular coordinates:

1. $r^2 = a^2\sin 2\theta$

Sol. $r^2 = a^2\sin 2\theta$ or $r^2 = 2a^2\sin\theta\cos\theta$

Multiplying both sides by r^2, we have

$$r^4 = 2a^2(r\sin\theta)(r\cos\theta) \quad \text{or} \quad (x^2+y^2)^2 = 2a^2xy$$

2. $r^4\sin 4\theta = a^4$

Sol. $r^4\sin 4\theta = a^4$ or $r^4(2\sin 2\theta\cos 2\theta) = a^4$

i.e., $\quad 4r^4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) = a^4$

or $\quad 4(r\cos\theta)(r\sin\theta)(r^2\cos^2\theta - r^2\sin^2\theta) = a^4$

or $\quad 4xy(x^2 - y^2) = a^4$

3. $r^2 = a^2\cos 2\theta$

Sol. $r^2 = a^2\cos 2\theta$ or $r^2 = a^2(\cos^2\theta - \sin^2\theta)$

Multiplying both sides by r^2, we have

$$r^4 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$$
$$= a^2[(r\cos\theta)^2 - (r\sin\theta)^2]$$

or $\quad (x^2+y^2)^2 = a^2(x^2 - y^2)$

4. $r = 2a\sin\theta\tan\theta$

Sol. $r = 2a\sin\theta\tan\theta$ or $r^2 = 2a(r\sin\theta)\dfrac{(r\sin\theta)}{r\cos\theta}$

i.e., $\quad (x^2+y^2) = \dfrac{2ay^2}{x} \Rightarrow x(x^2+y^2) = 2ay^2$

5. $r = 1 - \cos\theta$

Sol. $r = 1 - \cos\theta$ (1)

Multiplying both sides of (1) by r, we get

$$r^2 = r - r\cos\theta \quad \text{or} \quad r^2 + r\cos\theta = r$$

i.e., $\quad x^2 + y^2 + x = \sqrt{x^2+y^2}$

or $\quad (x^2 + y^2 + x)^2 = x^2 + y^2$

6. $r^2(4\sin^2\theta - 9\cos^2\theta) = 36$

Sol. $r^2(4\sin^2\theta - 9\cos^2\theta) = 36$ or $4r^2\sin^2\theta - 9r^2\cos^2\theta = 36$

i.e., $\quad 4y^2 - 9x^2 = 36$ or $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

7. $r = \dfrac{8}{2 - \cos\theta}$

Sol. $r = \dfrac{8}{2-\cos\theta}$ can be written as $2r - r\cos\theta = 8$

i.e., $2\sqrt{x^2+y^2} - x = 8 \Rightarrow 2\sqrt{x^2+y^2} = x + 8$

or $4(x^2+y^2) = x^2 + 16x + 64$

or $3x^2 + 4y^2 - 16x - 64 = 0$

8. $r = 2\sin\theta + 3\cos\theta$

Sol. $r = 2\sin\theta + 3\cos\theta$

Multiplying both sides by r, we get

$$r^2 = 2r\sin\theta + 3r\cos\theta$$

or $x^2 + y^2 = 2y + 3x$

Transform the given equation in polar coordinates (Problems 9 – 15).

9. $xy = a$

Sol. $xy = a$

Putting $x = r\cos\theta, y = r\sin\theta$, we have

$$(r\cos\theta)(r\sin\theta) = a$$

or $r^2\sin\theta\cos\theta = a \Rightarrow r^2 = a\sec\theta\csc\theta$

10. $y^2 = 4x$

Sol. $y^2 = 4x$

Putting $x = r\cos\theta, y = r\sin\theta$, we have

$$(r\sin\theta)^2 = 4r\cos\theta$$

or $r^2\sin^2\theta - 4r\cos\theta = 0 \Rightarrow r\sin^2\theta - 4\cos\theta = 0$

or $r = 4\cot\theta\csc\theta$

11. $y = \dfrac{x}{x+1}$

Sol. $y = \dfrac{x}{x+1}$ can be written as $xy + y = x$ and its polar form is

$$(r\cos\theta)(r\sin\theta) + r\sin\theta = r\cos\theta$$

Dividing both sides by r, we have

$$r\cos\theta\sin\theta + \sin\theta = \cos\theta$$

$\Rightarrow$ $r\sin\theta\cos\theta = \cos\theta - \sin\theta$

or $r = \dfrac{\cos\theta - \sin\theta}{\sin\theta\cos\theta} = \csc\theta - \sec\theta$

12. $x^2 + y^2 - 8x + 6y + 7 = 0$

Sol. $x^2 + y^2 - 8x + 6y + 7 = 0$

or $(r\cos\theta)^2 + (r\sin\theta)^2 - 8r\cos\theta + 6r\sin\theta + 7 = 0$

or $r^2 - 8r\cos\theta + 6r\sin\theta + 7 = 0$

$\Rightarrow$ $r^2 - 2r(4\cos\theta - 3\sin\theta) + 7 = 0$

13. $(x^2+y^2)\,y^2 = a^2x^2$

Sol. $(x^2+y^2)\,y^2 = a^2x^2$ (1)

Putting $x = r\cos\theta, y = r\sin\theta$ in (1) we have

$$r^2 \,.\, r^2\sin^2\theta = a^2r^2\cos^2\theta$$

Dividing both sides by r^2, we have

$$r^2\sin^2\theta = a^2\cos^2\theta$$

$\Rightarrow$ $r^2 = a^2\cot^2\theta$

14. $x^3 + 4x^2 + xy^2 - 4y^2 = 0$

Sol. $x^3 + 4x^2 + xy^2 - 4y^2 = 0$ can be written as

$$x(x^2+y^2) + 4(x^2-y^2) = 0$$

Putting $x = r\cos\theta, y = r\sin\theta$, we have

$$r\cos\theta(r^2) + 4(r^2\cos^2\theta - r^2\sin^2\theta) = 0$$

or $r^2[r\cos\theta + 4(\cos^2\theta - \sin^2\theta)] = 0$

$\Rightarrow$ $r\cos\theta + 4\cos 2\theta = 0$

or $r + 4\cos 2\theta\sec\theta = 0$

15. $x^4 + 2x^2y^2 + y^4 - 6x^2y + 2y^3 = 0$

Sol. $x^4 + 2x^2y^2 + y^4 - 6x^2y + 2y^3 = 0$ can be written as

$$(x^2+y^2)^2 - 2y(3x^2 - y^2) = 0$$

or $r^4 - 2r\sin\theta(3r^2\cos^2\theta - r^2\sin^2\theta) = 0$

Dividing both sides by r^3, we get

$$r - 2\sin\theta(3\cos^2\theta - \sin^2\theta) = 0.$$

Exercise Set 6.4 (Page 256)

In Problems 1 – 6, identify and graph the given polar equations:

1. $r = \dfrac{4}{1+\cos\theta}$

Sol. $r = \dfrac{4}{1+\cos\theta}$ can be written as $\dfrac{4}{r} = 1 + \cos\theta$

Comparing it with $\dfrac{l}{r} = 1 + e\cos\theta$, we have

$l = 4$ = length of the semi latusrectum, $e = 1$

Hence, the conic is a parabola.

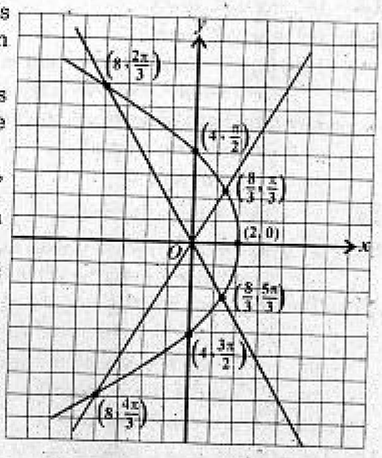
Some points on the graph in polar form are

$$(2, 0), \left(\frac{8}{3}, \frac{\pi}{3}\right), \left(4, \frac{\pi}{2}\right), \left(8, \frac{2\pi}{3}\right), \left(8, \frac{4\pi}{3}\right) \left(4, \frac{3\pi}{2}\right) \text{ and } \left(\frac{8}{3}, \frac{5\pi}{3}\right).$$

The graph is as shown,

$y^2 = -8\ (x - 2)$ is equation of the conic in cartesian system.

(2, 0) is vertex and (0, 0) is focus. Points shown in the graph are (2, 0), $\left(\frac{4}{3}, \pm\frac{4\sqrt{3}}{3}\right)$, $(0, \pm 4)$ and $(-4, \pm 4\sqrt{3})$ in cartesian system.



Note: The conic is symmetric about the initial line. The upper half of the conic can be described by taking some special values of θ from 0 to $\frac{2\pi}{3}$

Using symmetry, the lower half can be traced.

2. $r = \dfrac{10}{1 - \sin\theta}$

Sol. $r = \dfrac{10}{1 - \sin\theta}$ can be written as $\dfrac{10}{r} = 1 - \sin\theta$

or $\dfrac{10}{r} = r - \cos\left(\dfrac{\pi}{2} - \theta\right)$

Comparing it with $\dfrac{l}{r} = 1 - e\cos\theta$, we have $l = 10$, $e = 1$

Hence the conic is a parabola.

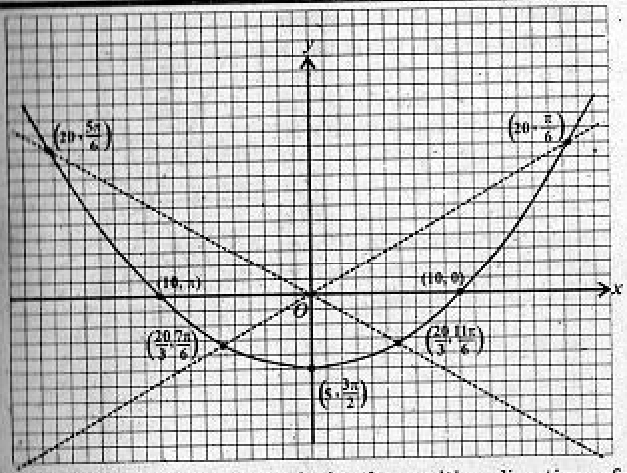
We plot the following points (in polar form)

$(10, 0), \left(20, \frac{\pi}{6}\right), \left(20, \frac{5\pi}{6}\right), (10, \pi), \left(\frac{20}{3}, \frac{7\pi}{6}\right), \left(5, \frac{3\pi}{2}\right) \left(\frac{20}{3}, \frac{11\pi}{6}\right)$.

Now joining the plotted points smoothly, we get the graph of the given conic.

Equation of the conic in cartesian coordinate system is $x^2 = 20(y + 5)$. The vertex of the parabola is (0, –5) and its focus is (0, 0). The points shown in the graph in cartesian system are

$(\pm 10, 0), (\pm 10\sqrt{3}, 10), \left(\pm\frac{10\sqrt{3}}{3}, -\frac{10}{3}\right), (0, -5)$.



The orientation of the curve is in the positive direction of the y-axis.

Note: The conic is symmetric about the y-axis.

3. $r = \dfrac{8}{3 - \cos\theta}$

Sol. $r = \dfrac{8}{3 - \cos\theta}$ can be written as

$\dfrac{8}{r} = 3 - \cos\theta = 3\left(1 - \dfrac{1}{3}\cos\theta\right)$

or $\dfrac{\frac{8}{3}}{r} = 1 - \dfrac{1}{3}\cos\theta$

Comparing it with $\dfrac{l}{r} = 1 - e\cos\theta$, we have

$l = \dfrac{8}{3}$; $e = \dfrac{1}{3}$ which is less than 1

Hence the conic is an ellipse.

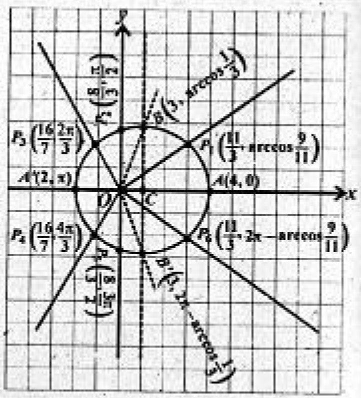
We plot the following points (in polar form)

$A(4, 0), P_1\left(\frac{11}{3}, \arccos\frac{9}{11}\right), B\left(3, \arccos\frac{1}{3}\right), P_2\left(\frac{8}{3}, \frac{\pi}{2}\right)$,

$P_3\left(\frac{16}{7}, \frac{2\pi}{3}\right), A'(2, \pi), P_4\left(\frac{16}{7}, \frac{4\pi}{3}\right), P_5\left(\frac{8}{3}, \frac{3\pi}{2}\right)$,

$B'\left(3, 2\pi - \arccos\frac{1}{3}\right), P_6\left(\frac{11}{3}, 2\pi - \arccos\frac{9}{11}\right)$

Joining the plotted points smoothly, we get the graph of the given conic.



Equation of the conic in cartesian coordinate system is

$$\frac{(x-1)^2}{9} + \frac{y^2}{8} = 1$$

The centre of the conic is $C(1, 0)$ and vertices are $A(4, 0)$ and $A'(-2, 0)$. The points shown in the graph in cartesian system are $A(4, 0)$, $P_1\left(3, \frac{2\sqrt{10}}{3}\right)$, $B(1, 2\sqrt{2})$, $P_2\left(0, \frac{8}{3}\right)$, $P_3\left(-\frac{8}{7}, \frac{8\sqrt{3}}{7}\right)$, $A'(-2, 0)$, $P_4\left(-\frac{8}{7}, \frac{-8\sqrt{3}}{7}\right)$, $P_5\left(0, -\frac{8}{3}\right)$, $B'(1, -2\sqrt{2})$, $P_6\left(\frac{11}{3}, \frac{-2\sqrt{10}}{3}\right)$

4. $r = \dfrac{9}{2 + \sin\theta}$

Sol. $r = \dfrac{9}{2 + \sin\theta}$ can be written as

$$\frac{9}{r} = 2 + \sin\theta = 2 - \cos\left(\frac{\pi}{2} + \theta\right) = 2\left\{1 - \frac{1}{2}\cos\left(\frac{\pi}{2} + \theta\right)\right\}$$

$$\frac{\frac{9}{2}}{r} = 1 - \frac{1}{2}\cos\left(\frac{\pi}{2} + \theta\right)$$

Comparing it with $\frac{l}{r} = 1 - e\cos\theta$, we have

$l = \frac{9}{2}$ = length of the semi-latusrectum, $e = \frac{1}{2}$ which is less than 1

Hence the conic is an ellipse.

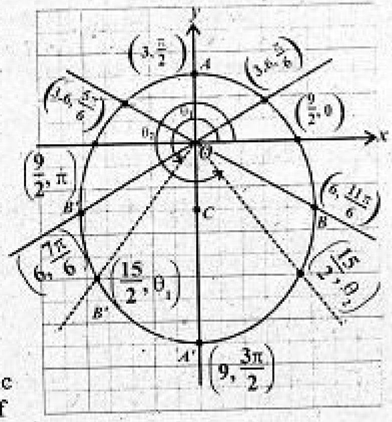
We plot the following points (in polar form)

$\left(\frac{9}{2}, 0\right), \left(3.6, \frac{\pi}{6}\right), \left(3, \frac{\pi}{2}\right), \left(3.6, \frac{5\pi}{6}\right)$

$\left(\frac{9}{2}, \pi\right), \left(6, \frac{7\pi}{6}\right), \left(\frac{15}{2}, \theta_1\right), \left(9, \frac{3\pi}{2}\right)$,

$\left(\frac{15}{2}, \theta_2\right), \left(6, \frac{11\pi}{6}\right)$ where $\tan\theta_1 = \frac{4}{3}$ (θ_1 is in the 3rd quadrant) and $\tan\theta_2 = -\frac{4}{3}$ (θ_2 is in the 4rth quadrant).

Joining the plotted points smoothly we get the graph of the given conic.



Equation of the conic in cartesian coordinate system is $\frac{x^2}{27} + \frac{(y+3)^2}{36} = 1$.

The cente of the conic is $(0, -3)$.

A, A', B and B' are $(0, 3)$, $(0, -9)$, $(3\sqrt{3}, -3)$ and $(-3\sqrt{3}, -3)$ in the cartesian system respectively.

Note: The conic is symmetric about the y-axis. The left half of the conic can be traced by taking some special values of θ from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. Using symmetry, the other half can be traced.

5. $r = \dfrac{6}{1 - 2\sin\theta}$

Sol. $r = \dfrac{6}{1 - 2\sin\theta}$ can be written as $\frac{6}{r} = 1 - 2\sin\theta = 1 - 2\cos\left(\frac{\pi}{2} - \theta\right)$

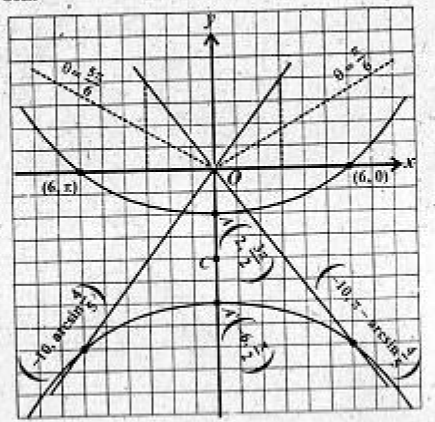
Comparing it with $\frac{l}{r} = 1 - e\cos\theta$, we have

$l = 6$, $e = 2$ which is greater than 1

Hence the conic is a hyperbola.

We plot the following points (in polar form)

$(6, 0), \left(-10, \arcsin\frac{4}{5}\right)$, $\left(-6, \frac{\pi}{2}\right), \left(-10, \pi - \arcsin\frac{4}{5}\right)$, $(6, \pi), \left(2, \frac{3\pi}{2}\right)$



Joining the plotted points smoothly, we get the graph of the given conic. Equation of the conic in cartesian coordinate system is

$$\frac{(y+4)^2}{4} - \frac{x^2}{12} = 1$$

The centre of the conic is $(0, -4)$. A and A' are $(0, -2)$ and $(0, -6)$ respectively.

6. $r = \dfrac{10}{2 + 3\cos\theta}$

Sol. $r = \dfrac{10}{2 + 3\cos\theta}$ can be written as

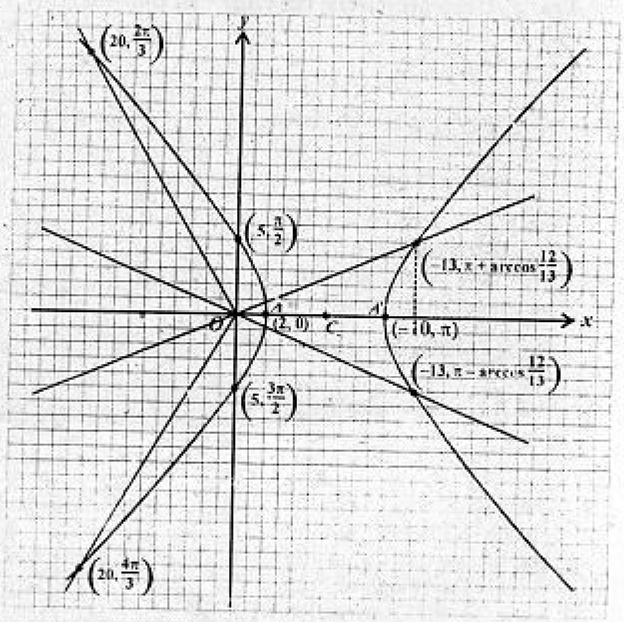
$$\frac{10}{r} = 2 + 3\cos\theta = 2\left(1 + \frac{3}{2}\cos\theta\right)$$

Thus $\dfrac{5}{r} = 1 + \dfrac{3}{2}\cos\theta$

Comparing it with $\dfrac{l}{r} = 1 + e\cos\theta$, we have

$$l = 5\ ,\ e = \frac{3}{2} \text{ which is greater than 1}$$

Hence the conic is a hyperbola.



We plot the following points (in polar form),

$(2, 0)$, $\left(5, \dfrac{\pi}{2}\right)$, $\left(20, \dfrac{2\pi}{3}\right)$, $\left(-13, \pi - \arccos\dfrac{12}{13}\right)$, $(-10, \pi)$,

$\left(-13, \pi + \arccos\dfrac{12}{13}\right)$, $\left(20, \dfrac{4\pi}{3}\right)$, $\left(5, \dfrac{3\pi}{2}\right)$

Joining the plotted points smoothly, we get the graph of the given conic.

Equation of the conic in the cartesian coordinate system is

$$\frac{(x-6)^2}{16} - \frac{y^2}{20} = 1$$

The centre of the conic is $(6, 0)$.

A and A' are $(2, 0)$ and $(10, 0)$ in cartesian system respectively.

7. Show that in any conic the sum of the reciprocals of the segments of any focal chord is constant.

Sol. Let PQ be any focal chord of the conic $\dfrac{l}{r} = 1 - e\cos\theta$, making angle α with the polar line along the positive direction of the x-axis.

Then P is (FP, α) and Q is $(FQ, \pi + \alpha)$.

Now, $\dfrac{l}{FP} = 1 - e\cos\alpha \Rightarrow \dfrac{1}{FP} = \dfrac{1 - e\cos\alpha}{l}$

and $\dfrac{l}{FQ} = 1 - e\cos(\pi + \alpha) = 1 + e\cos\alpha \Rightarrow \dfrac{1}{FQ} = \dfrac{1 + e\cos\alpha}{l}$

Hence $\dfrac{1}{FP} + \dfrac{1}{FQ} = \dfrac{1 - e\cos\alpha}{l} + \dfrac{1 + e\cos\alpha}{l}$

$$= \frac{2}{l}, \text{ which is constant.}$$

8. If PFP' QFQ' are two perpendicular focal chords of a conic, prove that $\dfrac{1}{|PF|\,.\,|FP'|} + \dfrac{1}{|QF|\,.\,|FQ'|}$, is constant.

Sol. Let P be (FP, α)

Then P' is $(FP', \pi + \alpha)$, Q is $\left(FQ, \dfrac{\pi}{2} + \alpha\right)$ and Q' is $\left(FQ', \dfrac{3\pi}{2} + \alpha\right)$

Now $\dfrac{1}{|PF|} = \dfrac{1 - e\cos\alpha}{l}$, $\dfrac{1}{|FP'|} = \dfrac{1 - e\cos(\pi + \alpha)}{l} = \dfrac{1 + e\cos\alpha}{l}$

Thus $\dfrac{1}{|PF|\,.\,|FP'|} = \dfrac{1 - e^2\cos^2\alpha}{l^2}$

$$\frac{1}{|QF|} = \frac{1 - e\cos\left(\frac{\pi}{2} + \alpha\right)}{l} = \frac{1 + e\sin\alpha}{l}$$

$$\frac{1}{|FQ'|} = \frac{1 - e\cos\left(\frac{3\pi}{2} + \alpha\right)}{l} = \frac{1 - e\sin\alpha}{l}$$

Thus $\dfrac{1}{|QF|\,.\,|FQ'|} = \dfrac{1 - e^2\sin^2\alpha}{l^2}$

$$\frac{1}{|PF|\,.\,|FP'|} + \frac{1}{|QF|\,.\,|FQ'|} = \frac{1 - e^2\cos^2\alpha}{l^2} + \frac{1 - e^2\sin^2\alpha}{l^2}$$

$$= \frac{2 - e^2}{l^2}, \text{ which is constant}$$

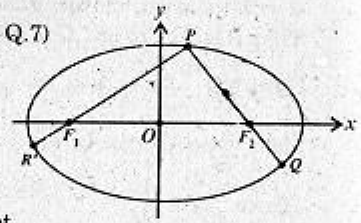
9. If PF_2Q, PF_1R be two chords of an ellipse through the foci F_2, F_1, show that

$$\frac{|PF_2|}{|F_2Q|} + \frac{|PF_1|}{|F_1R|}$$

is independent of the position of P.

Sol. As PF_2Q is a focal chord, so

$$\frac{1}{|PF_2|} + \frac{1}{|F_2Q|} = \frac{2}{l} \text{ (See Q.7)}$$



or $\quad 1 + \dfrac{|PF_2|}{|F_2Q|} = \dfrac{2|PF_2|}{l}$

Similarly $1 + \dfrac{|PF_1|}{|F_1R|} = \dfrac{2|PF_1|}{l}$

Adding the above results, we get

$$2 + \frac{|PF_2|}{|F_2Q|} + \frac{|PF_1|}{|F_1R|} = \frac{2|PF_2|}{l} + \frac{2|PF_1|}{l} = \frac{2(|PF_2| + |PF_1|)}{l}$$

$$= \frac{2(2a)}{l} = \frac{4a}{l}$$

or $\quad \dfrac{|PF_2|}{|F_2Q|} + \dfrac{|PF_1|}{|F_1R|} = \dfrac{4a}{l} - 2$ which is independent of the position of P.

Express each of the given equations in polar form and find the eccentricity and equation of the directrix. (Problems 10 – 12):

10. $y^2 = 4 - 4x$

Sol. The equation in polar coordinates is

$$r^2\sin^2\theta = 4 - 4r\cos\theta$$

or $\quad r^2\sin^2\theta + 4r\cos\theta - 4 = 0$

$$r = \frac{-4\cos\theta \pm \sqrt{16\cos^2\theta + 16\sin^2\theta}}{2\sin^2\theta} = \frac{-4\cos\theta \pm 4}{2(1 - \cos^2\theta)}$$

$$= \frac{4 - 4\cos\theta}{2(1 - \cos^2\theta)}, \text{ neglecting the -ve sign.}$$

$$= \frac{4(1 - \cos\theta)}{2(1 - \cos\theta)(1 + \cos\theta)} = \frac{2}{1 + \cos\theta}$$

This is a parabola with eccentricity $e = 1$

Equation of the directrix is $x = 2$ or $r\cos\theta = 2 \Rightarrow r = 2\sec\theta$

11. $3y^2 - 16y - x^2 + 16 = 0$

Sol. The given equation in polar form is

$$3r^2\sin^2\theta - 16r\sin\theta - r^2\cos^2\theta + 16 = 0$$

or $\quad r^2(3\sin^2\theta - \cos^2\theta) - 16r\sin\theta + 16 = 0$

$$r = \frac{16\sin\theta \pm \sqrt{256\sin^2\theta - 192\sin^2\theta + 64\cos^2\theta}}{2(3\sin^2\theta - \cos^2\theta)}$$

$$= \frac{16\sin\theta \pm 8}{2(4\sin^2\theta - 1)}$$

or $\quad r = \dfrac{8(2\sin\theta - 1)}{2(2\sin\theta - 1)(2\sin\theta + 1)}$ (neglecting the +ve sign)

$$= \frac{4}{2\sin\theta + 1} = \frac{4}{1 + 2\sin\theta}$$

This is a hyperbola with $e = 2$

Equation of directrix is $y = \dfrac{4}{2} = 2$

i.e., $r\sin\theta = 2 \Rightarrow r = 2\csc\theta$

12. $8x^2 + 9y^2 + 4x - 4 = 0$

Sol. The equation in polar form is

$$8r^2\cos^2\theta + 9r^2\sin^2\theta + 4r\cos\theta - 4 = 0$$

or $\quad r^2(8\cos^2\theta + 9\sin^2\theta) + 4r\cos\theta - 4 = 0$

$$r = \frac{-4\cos\theta \pm \sqrt{16\cos^2\theta + 128\cos^2\theta + 144\sin^2\theta}}{2(8\cos^2\theta + 9\sin^2\theta)}$$

$$= \frac{-4\cos\theta \pm 12}{2(8\cos^2\theta + 9 - 9\cos^2\theta)} = \frac{-2\cos\theta \pm 6}{9 - \cos^2\theta}$$

or $\quad r = \dfrac{6 - 2\cos\theta}{(3 - \cos\theta)(3 + \cos\theta)}$ (neglecting –ve sign)

$$= \frac{2}{3 + \cos\theta} = \frac{\frac{2}{3}}{1 + \frac{1}{3}\cos\theta}$$

This is an ellipse with $e = \frac{1}{3}$

Equation of directrix is $x = \frac{2/3}{1/3} = 2$

i.e., $r \cos \theta = 2 \Rightarrow r = 2 \sec \theta$.

Exercise Set 6.5 (Page 261)

Sketch the graph of each of the given curves:

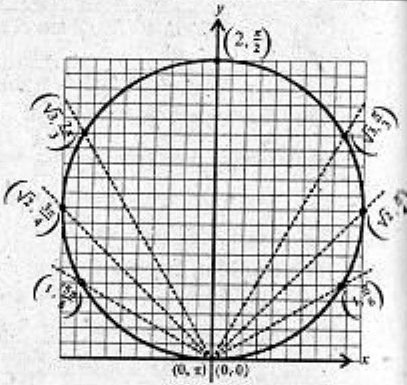
1. $r = 2 \sin \theta$

Sol. If (r, θ) is replaced by $(-r, -\theta)$ in $r = 2 \sin \theta$, the equation remains unchanged. Hence the curve is symmetric about the y-axis.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

The graph is a circle of radius 1 and passes through the pole having its centre (0, 1) on the y-axis. The graph of the curve is as shown.



2. $r = 3 \cos \theta$

Sol. Replacing (r, θ) by $(r, -\theta)$ in $r = 3 \cos \theta$, we have

$$r = 3 \cos(-\theta) = 3 \cos \theta$$

i.e., the equation remains unchanged, so the curve is symmetric about the initial line (the x-axis).

Table of values

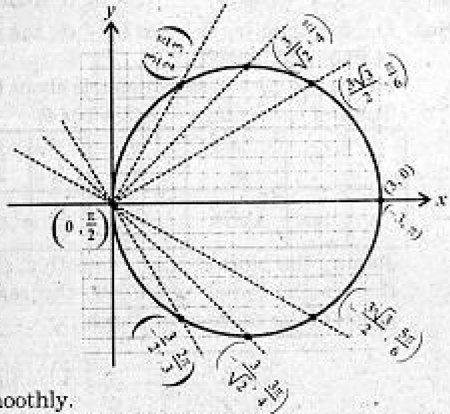
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	3	$\frac{3\sqrt{3}}{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3}{2}$	0	$-\frac{3}{2}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3\sqrt{3}}{2}$	-3

The graph of the given equation is a circle of radius $\frac{3}{2}$ having its centre $\left(\frac{3}{2}, 0\right)$ on the initial line.

It passes through the pole O. The graph of the curve is as shown.

Note: The upper half of the curve can be traced by joining the plotted points shown above the initial line smoothly.

Using symmetry other half of the curve can be traced.



3. $r = a(1 - \sin \theta)$ $\quad a > 0$ **(Cardioid)**

Sol. On changing (r, θ) to $(r, \pi - \theta)$, the equation $r = a(1 - \sin \theta)$, remains unchanged.

Hence the curve is symmetric about the y-axis.

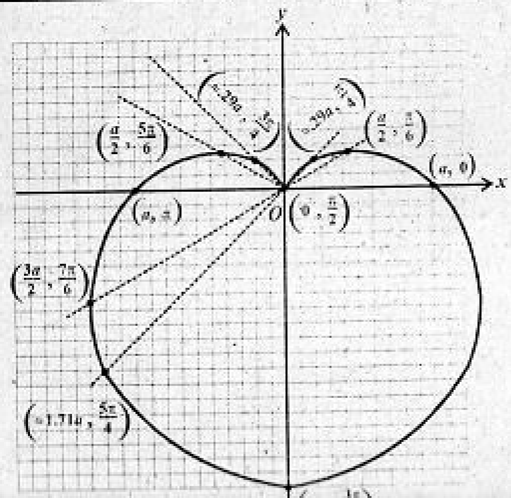
Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	a	$\frac{a}{2}$	$0.29a$	0	$0.29a$	$\frac{a}{2}$	a	$\frac{3a}{2}$	$1.71a$	$2a$

Joining the plotted points from $\theta = \frac{\pi}{2}$ to $\theta = \frac{3\pi}{2}$ smoothly, we get the left half of the curve.

Using symmetry the remaining part is traced.

The graph of the curve is as shown.



4. $r = a(\pm 1 + \cos\theta)$ $a > 0$ **(cardioids)**

Sol. On changing (r, θ) by to $(r, -\theta)$, the equations $r = a(\pm 1 + \cos\theta)$, remain unchanged.

Hence the curve are symmetric about the initial line $\theta = 0$

Table of values for $r = a(1 + \cos\theta)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
r	$2a$	$1.87a$	$1.71a$	$\frac{a}{2}$	a	$\frac{a}{2}$	$0.29a$	0

Joining the plotted points smoothly, the upper half of the curve is the traced. Using symmetry, the remaining part is traced. The graph of the curve is as shown.

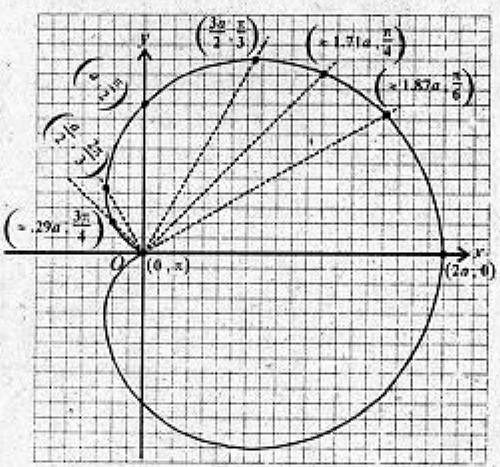
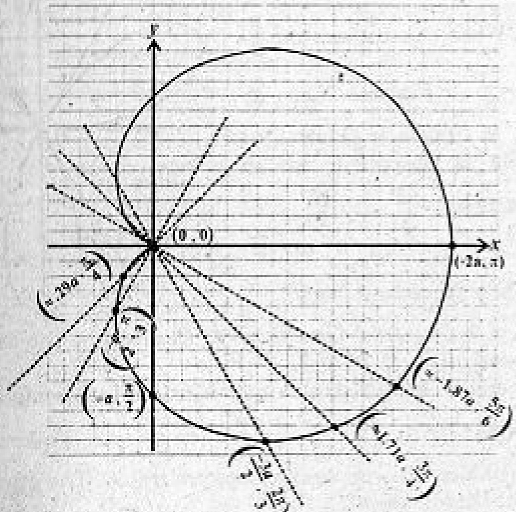


Table of values for $r = a(-1 + \cos\theta)$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	$-0.29a$	$\frac{-a}{2}$	$-a$	$\frac{-3a}{2}$	$-1.71a$	$-1.87a$	$-2a$

The lower half of the curve is traced with help of the plotted points. Using symmetry, the remaining part is traced. The graph of the curve is as shown.



5. $r = a\sin 3\theta$, $a > 0$ **(three-leafed rose)**

Sol.

Table of values

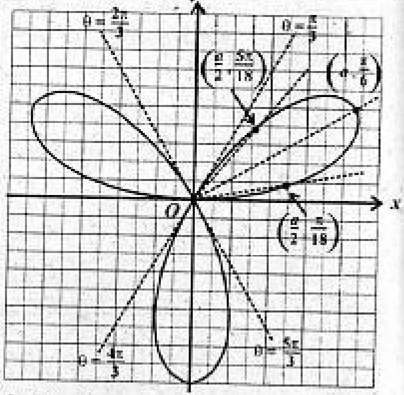
θ	0	$\frac{\pi}{18}$	$\frac{\pi}{6}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{\pi}{2}$	$\frac{11\pi}{18}$	$\frac{2\pi}{3}$
r	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0

θ	$\frac{2\pi}{3}$	$\frac{13\pi}{18}$	$\frac{5\pi}{6}$	$\frac{17\pi}{18}$	π
r	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0

The curve is symmetric about the y-axis. As θ increases from 0 to $\frac{\pi}{6}$, r increases from 0 to a. As θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r decreases from a to 0.

The loop in the first quadrant is described.

As θ increases from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, r decreases from 0 to $-a$ to 0. When θ increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, r increase from $-a$ to 0. The loop below the pole is described. As θ increases $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$, r increases from 0 to a. When θ increases from $\frac{5\pi}{6}$ to π, r decreases from a to 0. The loop in the 2nd quadrant is described.



The loop in the 1st quadrant is traced by joining the plotted the points (0, 0), $\left(\frac{a}{2}, \frac{\pi}{18}\right)$, $\left(a, \frac{\pi}{6}\right)$, $\left(\frac{a}{2}, \frac{5\pi}{18}\right)$ etc. Following the above procedure, the remaining loops are traced. The graph of the curve is as shown. above.

6. $r = a\cos 3\theta$, $a > 0$ **(three-leafed rose)**

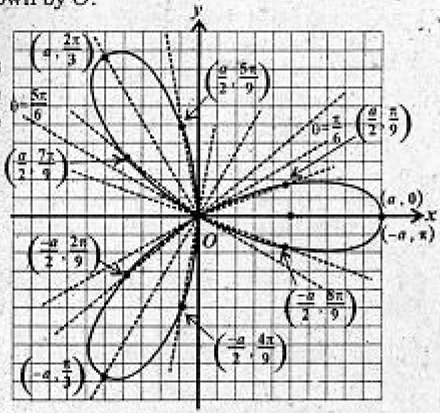
Sol. It is symmetric about the initial line. As θ increases from 0 to $\frac{\pi}{6}$, r decreases a to 0. The half loop in the 1st quadrant is traced. As θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r is negative and varies from 0 to $-a$. When θ increases from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, r varies from $-a$ to 0. Thus the loop in the 3rd quadrant is described. As θ increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, r varies from 0 to a. When θ increases from $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$, r varies from a to 0. Thus the loop in the 2nd quadrant is described. As θ increases from $\frac{5\pi}{6}$ to π, r varies from 0 to $-a$. Thus half loop in the 4th quadrant is described.

Table of values

θ	0	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{\pi}{3}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$	$\frac{5\pi}{9}$	$\frac{2\pi}{3}$	$\frac{7\pi}{9}$	$\frac{5\pi}{6}$	$\frac{8\pi}{9}$	π
r	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$

$\left(0, \frac{\pi}{6}\right)$, $\left(0, \frac{\pi}{2}\right)$, $\left(0, \frac{5\pi}{6}\right)$ are shown by O.

Joining the plotted points smoothly, three loops are traced. The graph of the curve is as shown.



7. $r = a\cos 2\theta$, $a > 0$ **(four-leafed rose)**

Sol. The curve is symmetric about the initial line.

Table of values

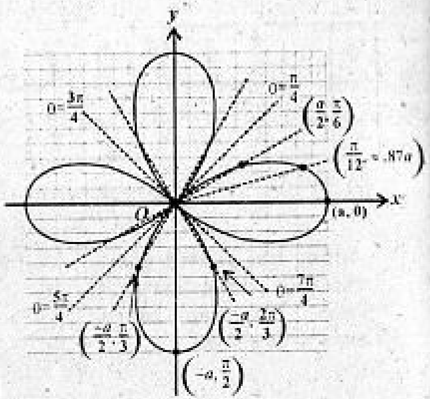
θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$
r	a	$\approx .87a$	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0

θ	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a

As θ increases from 0 to $\frac{\pi}{4}$, r decreases from a to 0. The half loop in the 1st quadrant is described.

As θ increases from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r decreases from 0 to $-a$. When θ increases from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, r increases from $-a$ to 0. Thus the loop below, the pole is described. Other details are left for the reader.

$\left(0, \frac{\pi}{4}\right)$, $\left(0, \frac{3\pi}{4}\right)$, $\left(0, \frac{5\pi}{4}\right)$, $\left(0, \frac{7\pi}{4}\right)$ are shown by O.



8. $r = a\cos 5\theta$, $a > 0$ **(five-leafed rose)**

Sol.

Table of values

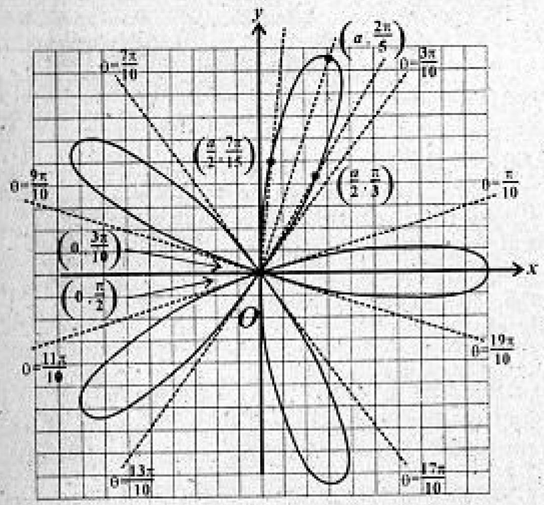
θ	0	$\frac{\pi}{15}$	$\frac{\pi}{10}$	$\frac{2\pi}{15}$	$\frac{\pi}{5}$	$\frac{4\pi}{15}$	$\frac{3\pi}{10}$	$\frac{\pi}{3}$	$\frac{2\pi}{5}$	$\frac{7\pi}{15}$	$\frac{\pi}{2}$
r	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0

half loop in the 1st quadrant; loop in the 3rd quadrant; loop in the 1st quadrant

θ	$\frac{\pi}{2}$	$\frac{8\pi}{15}$	$\frac{3}{5}$	$\frac{2\pi}{3}$	$\frac{7\pi}{10}$	$\frac{11\pi}{15}$	$\frac{4\pi}{5}$	$\frac{13\pi}{15}$	$\frac{9\pi}{10}$	$\frac{14\pi}{15}$	π
r	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$

loop in the 4th quadrant; loop in the 2nd quadrant; half loop in the 4th quadrant

The loop between the angle $\theta = \frac{3\pi}{10}$ and $\theta = \frac{\pi}{2}$ is plotted. Other loops are shown in the graph. Readers are advised to write the details.



9. $r^2 = a\sin 2\theta$ $\quad a > 0$ **(lemniscate)**

Sol. Since $r^2 = (-r^2)$, the graph is symmetric about the pole. As $r^2 \geq 0$, the function is defined for those values of θ for which $\sin 2\theta \geq 0$. If $\theta \in [0, 2\pi]$ then $\sin 2\theta \geq 0$ if and only if θ is in $\left[0, \frac{\pi}{2}\right]$ or in $\left[\pi, \frac{3\pi}{2}\right]$.

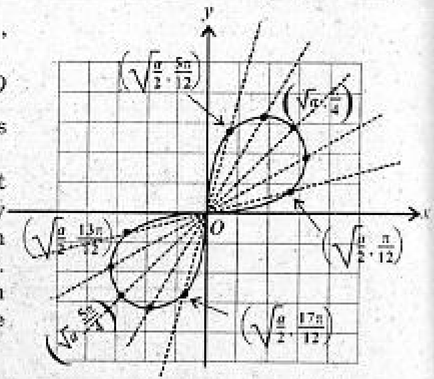
Table of values

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$r = \sqrt{a\sin 2\theta}$	0	$\sqrt{\frac{a}{2}}$	$\sqrt{a \cdot \frac{\sqrt{3}}{2}}$	$\sqrt{a}$	$\sqrt{a \cdot \frac{\sqrt{3}}{2}}$	$\sqrt{\frac{a}{2}}$	0

θ	π	$\frac{13\pi}{12}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$
$r = \sqrt{a\sin 2\theta}$	0	$\sqrt{\frac{a}{2}}$	$\sqrt{a \cdot \frac{\sqrt{3}}{2}}$	$\sqrt{a}$	$\sqrt{a \cdot \frac{\sqrt{3}}{2}}$	$\sqrt{\frac{a}{2}}$	0

The points $(0, 0)$, $\left(0, \frac{\pi}{2}\right)$, $(0, \pi)$, $\left(0, \frac{3\pi}{2}\right)$ are shown by O (the pole). The graph is as shown.

Note: The loop in the Ist quadrant can be drawn by joining the plotted points in the first quadrant smoothly. Using symmetry the loop in the 3rd quadrant can be traced.



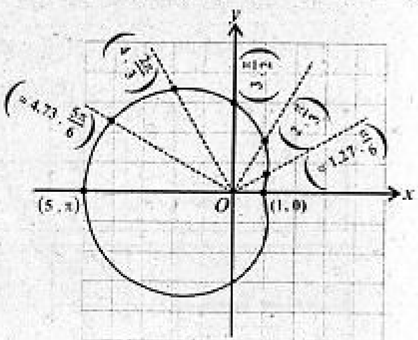
10. $r = 3 - 2\cos\theta$ **(limacon)**

Sol. On changing (r, θ) into $(r, -\theta)$, the equation of the curve remains unchanged. Thus curve is symmetric about the initial line.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	1	≈ 1.27	2	3	4	≈ 4.73	5

The upper half of the curve is drawn by joining the plotted points smoothly.
Using symmetry remaining part is traced.
The graph of the curve is as shown.



11. $r = 3 + 4\cos\theta$ **(limacon)**

Sol. On changing (r, θ) into $(r, -\theta)$ the equation of the curve remains unchanged. Thus the curve is symmetric about the initial line.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{36}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
r	7	$3+2\sqrt{3}$	5	3	1	$3-2\sqrt{3}$	-1	$3-2\sqrt{3}$	1	3	5	7

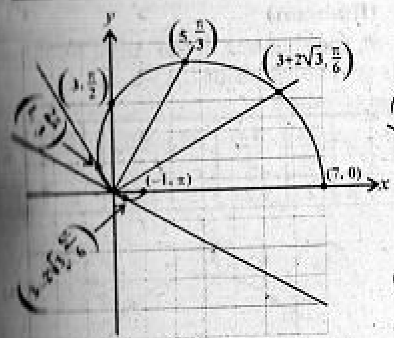


Fig. (1)

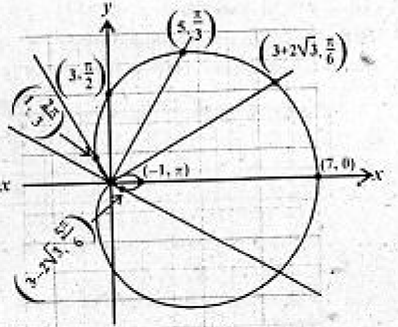


Fig. (2)

As θ varies from 0 to π, the half of the curve is described as shown in the figure (1). Using symmetry, the other half of the curve is traced. The graph of the curve is as shown in the figure (2).

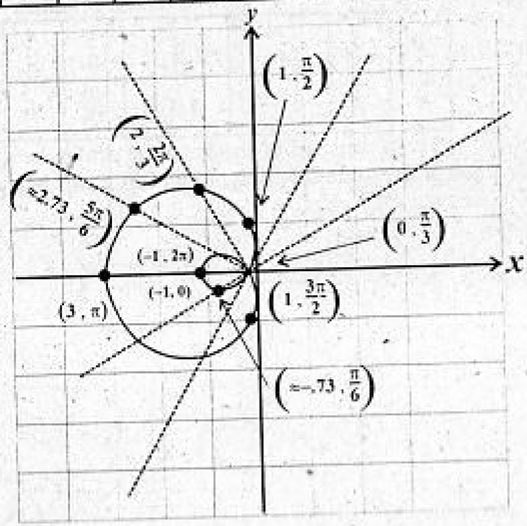
12. $r = 1 - 2\cos\theta$ **(limacon)**

Sol. On changing (r, θ) into $(r, -\theta)$, the equation of the curve remains unchanged. Therefore the curve is symmetric about the initial line.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
r	-1	-0.73	0	1	2	2.73	3	1	-1

The half part of the curve is drawn by joining the plotted points starting from $(-1, 0)$ to $(3, \pi)$ smoothly.
Other half is traced by using symmetry.
The graph of the curve is as shown.



13. $r = 3 + 2\sin\theta$ **(limacon)**

Sol. If (r, θ) are replaced by $(r, \pi, -\theta)$, the equation of the curve remains unchanged. Hence the curve is symmetric about the y-axis.

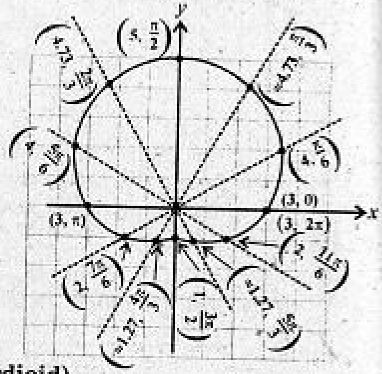
Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	3	4	4.73	5	4.73	4	3	2	1.27	1	1.27	2	3

The plotted points are joined smoothly drawn the required graph.

The graph of the curve is as shown.

Note: The half curve can be drawn by joining the plotted points starting from $\left(5, \frac{\pi}{2}\right)$ to $\left(1, \frac{3\pi}{2}\right)$ smoothly. Using symmetry, other half can be traced.



14. $r = -a(1 + \cos\theta)$ $a > 0$ **(cardioid)**

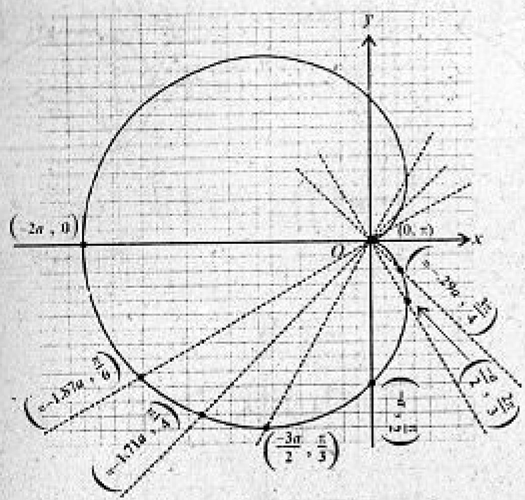
Sol. Since $\cos(-\theta) = \cos\theta$, the curve is symmetric about the initial line.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
r	$-2a$	$-1.87a$	$-1.71a$	$\frac{-3a}{2}$	$-a$	$\frac{-a}{2}$	$-0.29a$	0

The lower half of the graph is drawn by joining the plotted points of the table smoothly, Remaining part is traced by using symmetry.

The graph of the curve is as shown.



15. $r = a\sin\frac{\theta}{2}, a > 0$

Sol. Since $a\sin\left(\pi - \frac{\theta}{2}\right) = a\sin\left(\frac{\theta}{2}\right) = r$, the curve is symmetric about the y-axis.

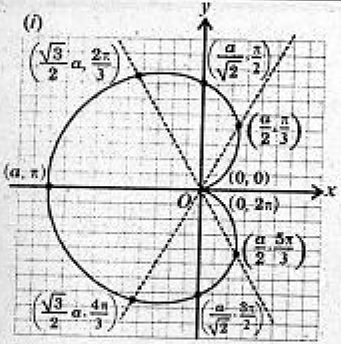
(i)

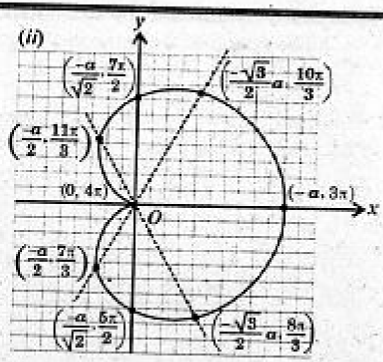
Table of values

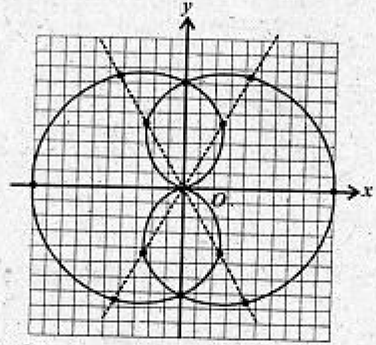
θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
r	0	$\frac{a}{2}$	$\frac{a}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}a$	a	$\frac{\sqrt{3}}{2}a$	$\frac{a}{\sqrt{2}}$	$\frac{a}{2}$	0

(ii)

θ	$\frac{7\pi}{3}$	$\frac{5\pi}{2}$	$\frac{8\pi}{3}$	3π	$\frac{10\pi}{3}$	$\frac{7\pi}{2}$	$\frac{11\pi}{3}$	4π
r	$\frac{-a}{2}$	$\frac{-a}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}a$	$-a$	$\frac{-\sqrt{3}}{2}a$	$\frac{-a}{\sqrt{2}}$	$\frac{-a}{2}$	-0







The first graph is drawn by joining the plotted points of the table (*i*) smoothly. The second graph is got by joining the plotted points of the table (*ii*), smoothly. The combined graph is shown in the last figure.

Note: After drawing the graph according to the table (*i*), other part of the graph can be traced by using symmetry.

Exercise Set 6.6 (Page 266)

Find ψ for each of the given curves (Problems 1 – 4):

1. $r = a(1 - \cos\theta)$ (1)

Sol. Taking ln of both sides of (1), we get

$$\ln r = \ln a + \ln(1 - \cos\theta)$$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}(\sin\theta) = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Therefore, $r\dfrac{d\theta}{dr} = \tan\dfrac{\theta}{2}$ and so $\tan\psi = \tan\dfrac{\theta}{2}$

Hence $\psi = \dfrac{\theta}{2}$

2. $r = -5\csc\theta$

Sol. $\dfrac{dr}{d\theta} = -5(-\csc\theta\cot\theta) = 5\csc\theta\cot\theta$

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{-5\csc\theta}{5\csc\theta\cot\theta} = -\tan\theta = \tan(\pi - \theta)$$

Hence $\psi = \pi - \theta$

3. $\dfrac{2a}{r} = 1 + \sin\theta$

Sol. $\dfrac{2a}{r} = 1 + \sin\theta \Rightarrow r = \dfrac{2a}{1+\sin\theta}$, so

$$\frac{dr}{d\theta} = \frac{-2a\cos\theta}{(1+\sin\theta)^2}$$

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{2a}{1+\sin\theta}\left(-\frac{(1+\sin\theta)^2}{2a\cos\theta}\right) = -\frac{1+\sin\theta}{\cos\theta}$$

$$= -\frac{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)} = -\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}$$

$$= -\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \tan\left(\pi - \left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right)$$

Hence $\psi = \dfrac{3\pi}{4} - \dfrac{\theta}{2}$

4. $r = \dfrac{3}{2(1-\cos\theta)}$

Sol. $r = \dfrac{3}{2(1-\cos\theta)} = \dfrac{3}{2}(1-\cos\theta)^{-1}$.

$$\frac{dr}{d\theta}=\frac{3}{2}(-1)\,.\,(1-\cos\theta)^{-2}\times-(-\sin\theta)=\frac{3}{2}\left[\frac{-\sin\theta}{(1-\cos\theta)^2}\right]$$

$$\tan\psi=r.\frac{d\theta}{dr}=\frac{3}{2}.\frac{1}{1-\cos\theta}\times\frac{2}{3}\left[-\frac{(1-\cos\theta)^2}{\sin\theta}\right]$$

$$=-\frac{1-\cos\theta}{\sin\theta}=-\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=-\tan\frac{\theta}{2}$$

i.e., $\tan\psi=\tan\left(\pi-\frac{\theta}{2}\right)\Rightarrow\psi=\pi-\frac{\theta}{2}$

Find the measure of the angle of intersection of the given curves (Problems 5 – 10):

5. $r=1$ and $r=2\sin\theta$

Sol. $r=1$ (1)

and $r=2\sin\theta$ (2)

Solving (1) and 2), we get

$$1=2\sin\theta \text{ or } \sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6},\ \frac{5\pi}{6}$$

Thus the points of intersection of (1) and (2) are $\left(1,\frac{\pi}{6}\right)$ and $\left(1,\frac{5\pi}{6}\right)$

From (1), $\frac{dr}{d\theta}=0$ and $\tan\psi_1=\frac{r}{dr/d\theta}=\frac{1}{0}\Rightarrow\psi_1=\frac{\pi}{2}$

From (2), $\frac{dr}{d\theta}=2\cos\theta$ and $\tan\psi_2=\frac{r}{dr/d\theta}=\frac{2\sin\theta}{2\cos\theta}=\tan\theta$

$\Rightarrow\quad\psi_2=\theta$

Now at $\left(1,\frac{\pi}{6}\right)$, $\psi_1=\frac{\pi}{2}$ and $\psi_2=\frac{\pi}{6}$

Therefore $\psi_1-\psi_2=\frac{\pi}{2}-\frac{\pi}{6}=\frac{3\pi-\pi}{6}=\frac{\pi}{3}$

At $\left(1,\frac{5\pi}{6}\right)$, $\psi_1=\frac{\pi}{2}$ and $\psi_2=\frac{5\pi}{6}$, therefore

$$\psi_2-\psi_1=\frac{5\pi}{6}-\frac{\pi}{2}=\frac{5\pi-3\pi}{6}=\frac{\pi}{3}$$

6. $r=a\theta$ and $r\theta=a$

Sol. $r=a\theta$ (1)

and $r\theta=a$ (2)

Solving (1) and (2), we get

$$a\theta\,.\,\theta=a \quad\text{or}\quad \theta^2=1 \quad\text{or}\quad \theta=\pm1$$

From (1), we have

$$\ln r=\ln a+\ln\theta$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{1}{\theta}\Rightarrow r.\frac{d\theta}{dr}=\theta$$

i.e., $\tan\psi_1=\theta$

From (2), we get

$$\ln r+\ln\theta=\ln a$$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta}+\frac{1}{\theta}=0 \text{ or } \frac{1}{r}\frac{dr}{d\theta}=-\frac{1}{\theta}$$

i.e, $r.\frac{d\theta}{dr}=-\theta\Rightarrow\tan\psi_2=-\theta$

At $\theta=1$, $\tan\psi_1=1$ and $\tan\psi_2=-1$

Thus $\psi_1=\frac{\pi}{4}$ and $\psi_2=\frac{3\pi}{4}$

The required angle is $\psi_2-\psi_1=\frac{3\pi}{4}-\frac{\pi}{4}=\frac{\pi}{2}$

At $\theta=-1$, $\tan\psi_1=-1$ and $\tan\psi_2=1$

Thus $\psi_1=\frac{3\pi}{4}$, $\psi_2=\frac{\pi}{4}$

The required angle is $\psi_1-\psi_2=\frac{3\pi}{4}-\frac{\pi}{4}=\frac{\pi}{2}$

7. $r=\frac{a\theta}{1+\theta}$ and $r=\frac{a}{1+\theta^2}$

Sol. $r=\frac{a\theta}{1+\theta}$ (1)

and $r=\frac{a}{1+\theta^2}$ (2)

Solving (1) and (2), we have

$$\frac{a\theta}{1+\theta}=\frac{a}{1+\theta^2} \text{ or } \frac{\theta}{1+\theta}=\frac{1}{1+\theta^2}\Rightarrow\theta+\theta^3=1+\theta$$

or $\theta^3=1\quad\Rightarrow\quad\theta=1$ (3)

Taking ln of (1), we get

$$\ln r=\ln a+\ln\theta-\ln(1+\theta)$$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{1}{\theta}-\frac{1}{1+\theta}=\frac{1+\theta-\theta}{\theta(1+\theta)}=\frac{1}{\theta(1+\theta)}$$

or $r\frac{d\theta}{dr}=\theta(1+\theta)\Rightarrow\tan\psi_1=\theta(1+\theta)$

At $\theta=1$, $\tan\psi_1=1(1+1)=2$ (4)

Taking logarithm of both sides of (2), we have

$$\ln r=\ln a-\ln(1+\theta^2)$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{2\theta}{1+\theta^2} \quad \text{or} \quad r\frac{d\theta}{dr} = -\frac{1+\theta^2}{2\theta} \Rightarrow \tan\psi_2 = -\frac{1+\theta^2}{2\theta}$$

At $\theta = 1, \tan\psi_2 - \frac{1+(1)^2}{2(1)} = -\frac{2}{2} = -\frac{2}{2} = -1$

If β is the angle between two curves, then

$$\tan\beta = \frac{\tan\psi_1 - \tan\psi_2}{1+\tan\psi_1\tan\psi_2} = \frac{2+1}{1-2} = \frac{3}{-1} = -3$$

i.e., $\beta = \arctan(-3)$

8. $r = ae^{\theta}$ and $re^{\theta} = b$

Sol.

$r = ae^{\theta}$ (1)

Taking logarithms

$\ln r = \ln a + \ln e^{\theta}$

$= \ln a + \theta$

Differentiating w.r.t., 'θ'

$\frac{1}{r}\frac{dr}{d\theta} = 1 \Rightarrow \frac{rd\theta}{dr} = 1$

i.e., $\tan\psi_1 = 1$

or $\psi_1 = \frac{\pi}{4}$

$re^{\theta} = b$ (2)

Taking logarithms

$\ln r + \ln e^{\theta} = \ln b$

$\ln r + \theta = \ln b$

Differentiating w.r.t. 'θ'

$\frac{1}{r}\frac{dr}{d\theta} + 1 = 0 \Rightarrow r\frac{d\theta}{dr} = -1$

i.e., $\tan\psi_2 = -1$

or $\psi_2 = \frac{3\pi}{4}$

At the point of intersection $\left(\sqrt{ab}, \ln\sqrt{\frac{b}{a}}\right)$ of (1) and (2), the required angle $= \psi_2 - \psi_1 = \frac{3\pi}{4} - \left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

9. $r = a(1-\cos\theta)$ and $r = \frac{a}{1-\cos\theta}$

Sol. $r = a(1-\cos\theta)$ (1)

and $r = \frac{a}{1-\cos\theta}$ (2)

Solving (1) and (2), we have

$$a(1-\cos\theta) = \frac{a}{1-\cos\theta} \Rightarrow (1-\cos\theta)^2 = 1$$

or $1-\cos\theta = \pm 1$

or $1-\cos\theta = 1 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

or $1-\cos\theta = -1 \Rightarrow \cos\theta = 2$ which is not possible.

The points of intersection of (1) and (2) are $\left(a, \frac{\pi}{2}\right)$ and $\left(a, \frac{3\pi}{2}\right)$

From (1) $\ln r = \ln a + \ln(1-\cos\theta)$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta}(-(-\sin\theta) = \frac{\sin\theta}{1-\cos\theta}$$

or $$r\cdot\frac{d\theta}{dr} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} \Rightarrow \tan\psi_1 = \tan\frac{\theta}{2} \Rightarrow \psi_1 = \frac{\theta}{2}$$

From (2), $\ln r = \ln a - \ln(1-\cos\theta)$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{1-\cos\theta}, (-(-\sin\theta) = -\frac{\sin\theta}{1-\cos\theta}$$

or $$r\frac{d\theta}{dr} = -\frac{1-\cos\theta}{\sin\theta} = -\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\tan\psi_2 = -\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} = \tan\left(\pi - \frac{\theta}{2}\right)$$

Thus $\tan\psi_2 = \tan\left(\pi - \frac{\theta}{2}\right) \Rightarrow \psi_2 = \left(\pi - \frac{\theta}{2}\right)$

At $\left(a, \frac{\pi}{2}\right)$, $\psi_1 = \frac{\pi/2}{2} = \frac{\pi}{4}$ and $\psi_2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Thus $\psi_2 - x_1 = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

At $\left(a, \frac{3\pi}{2}\right)$, $\psi_1 = \frac{\frac{3\pi}{2}}{2} = \frac{3\pi}{4}$ and $\psi_2 = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$

Thus $\psi_1 - \psi_2 = \frac{3\pi}{4} - \left(\frac{\pi}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2}$

10. $r = \cos 2\theta$ and $r = \sin\theta$ at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$

Sol. $r = \cos 2\theta$ (1)

and $r = \sin\theta$ (2)

Differentiating (1) w.r.t. θ, we have

$$\frac{dr}{d\theta} = -(\sin 2\theta)\,.\,2 = -2\sin 2\theta$$

$$\tan\psi_1 = r\cdot\frac{d\theta}{dr} = \cos^2\theta \times \left(-\frac{1}{2\sin 2\theta}\right) = -\frac{1}{2} \text{ at } 2\theta$$

At $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\tan\psi_1 = -\frac{1}{2}\cdot\cot\frac{\pi}{3} = -\frac{1}{2}\cdot\frac{1}{\sqrt{3}} = -\frac{1}{2\sqrt{3}}$

Differentiating (2) w.r.t. θ, we get

$$\frac{dr}{d\theta} = \cos\theta$$

$$\tan\psi_2 = r\frac{d\theta}{dr} = \frac{\sin\theta}{\cos\theta} = \tan\theta \Rightarrow \psi_2 = \theta$$

At $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\psi_2 = \frac{\pi}{6}$

If β is the angle between the two curves then

$$\tan\beta = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2\tan\psi_1} = \frac{\frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}\left(-\frac{1}{2\sqrt{3}}\right)} = \frac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{5}{6}}$$

$$= \frac{\sqrt{3}}{2}\times\frac{6}{5} = \frac{3\sqrt{3}}{5} \Rightarrow \beta = \arctan\left(\frac{3\sqrt{3}}{5}\right)$$

Find the pedal equation of each of the given curves (Problems 111 – 15):

11. $\frac{l}{r} = 1 + e\cos\theta$

Sol. Taking logarithm, we have

$$\ln l - \ln r = \ln(1 + e\cos\theta)$$

Differentiating w.r.t. θ, we have

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1 + e\cos\theta} \Rightarrow \frac{dr}{d\theta} = \frac{r(e\sin\theta)}{1 + e\cos\theta} = \frac{r^2 e\sin\theta}{l}$$

Now, $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left[\frac{dr}{d\theta}\right]^2 = \frac{1}{r^2} + \frac{1}{r^4}\,\frac{r^4 e^2\sin^2\theta}{l^2}$

$$= \frac{1}{r^2} + \frac{e^2\sin^2\theta}{l^2} = \left[\frac{1 + e\cos\theta}{l}\right]^2 + \frac{e^2\sin^2\theta}{l^2}$$

$$= \frac{1 + 2e\cos\theta + e^2\cos^2\theta + e^2\sin^2\theta}{l^2} = \frac{1 + 2e\cos\theta + e^2}{l^2}$$

$$= \frac{1}{l^2}\left[1 + e^2 + 2\left(\frac{l}{r} - 1\right)\right], \text{ since } e\cos\theta = \frac{l}{r} - 1.$$

$$= \frac{1}{l^2}\left[e^2 - 1 + \frac{2l}{r}\right] = \frac{e^2 - 1}{l^2} + \frac{2}{lr}$$

which is the required pedal equation.

12. $r = a\theta$

Sol. $\frac{dr}{d\theta} = a$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4}a^2$$

$$= \frac{1}{r^2} + \frac{a^2}{r^4} \text{ which is the required pedal equation.}$$

13. $r = a\sin m\theta$

Sol. $\ln r = \ln a + \ln\sin m\theta$ (1)

Differentiating (1) w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{m\cos m\theta}{\sin m\theta} \text{ or } \frac{dr}{d\theta} = \frac{m\,r\cos m\theta}{\sin m\theta}$$

Now $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$$= \frac{1}{r^2} + \frac{1}{r^4}\,\frac{m^2r^2\cos^2 m\theta}{\sin^2 m\theta} = \frac{1}{r^2} + \frac{m^2}{r^2}\left[\frac{1 - \frac{r^2}{a^2}}{\frac{r^2}{a^2}}\right]$$

$$= \frac{1}{r^2} + \frac{m^2}{r^2}\left\{\frac{a^2 - r^2}{r^2}\right\} = \frac{1}{r^2} + \frac{m^2}{r^4}(a^2 - r^2)$$

or $\frac{1}{p^2} = \frac{1}{r^4}[r^2 + m^2(a^2 - r^2)]$

i.e., $r^4 = p^2[a^2m^2 + r^2(1 - m^2)]$

which is the required pedal equation.

14. $r = a + b\cos\theta$

Sol. Taking logarithm on both sides, we get

$$\ln r = \ln(a + b\cos\theta)$$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-b\sin\theta}{a + b\cos\theta}$$

$$\frac{dr}{d\theta} = -\frac{b\,r\sin\theta}{a + b\cos\theta} = -\frac{b\,r\sin\theta}{r} = -b\sin\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4}b^2\sin^2\theta$$

$$= \frac{1}{r^4}[r^2 + b^2(1 - \cos^2\theta)] = \frac{1}{r^4}[r^2 + b^2 - b^2\cos^2\theta]$$

$$= \frac{1}{r^4}[r^2 + b^2 - (r - a)^2] = \frac{1}{r^4}[r^2 + b^2 - r^2 - a^2 + 2ar]$$

$$= \frac{1}{r^4}[b^2 - a^2 + 2ar].$$

Thus $r^4 = (b^2 - a^2 + 2ar)\,p^2$ is the required pedal equation.

15. $r = a(1 - \sin\theta)$

Sol. $r = a(1 - \sin\theta) \;\Rightarrow\; r - a = -a\sin\theta$ (1)

$$\frac{dr}{d\theta} = -a\cos\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{a^2\cos^2\theta}{r^4} = \frac{1}{r^4}[r^2 + a^2(1 - \sin^2\theta)]$$

i.e, $\dfrac{1}{p^2} = \dfrac{1}{r^4}[r^2 + a^2 - a^2\sin^2\theta]$ (2)

For the pedal equation, we have to eliminate θ between (1) and (2). From (1),

$$a^2\sin^2\theta = (r - a)^2$$

Thus $\dfrac{1}{p^2} = \dfrac{1}{r^4}[r^2 + a^2 - (r - a)^2] = \dfrac{1}{r^4}(2ar) = \dfrac{2a}{r^3}$

i.e., $r^3 = 2p^2 a$ is the required pedal equation.

16. Show that the curves $r^m = a^m\cos m\theta$ and $r^m = a^m\sin m\theta$ cut each other orthogonally.

Sol. $r^m = a^m\cos m\theta$ and $r^m = a^m\sin m\theta$

Taking ln, we have

$m\ln r = m\ln a + \ln\cos m\theta$

Differentiating w.r.t. θ, we get

$m\cdot\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{1}{\cos m\theta}(-m\sin m\theta)$,

$\dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{\sin m\theta}{\cos m\theta} = -\tan m\theta$

or $r.\dfrac{d\theta}{dr} = -\cot m\theta$

i.e., $\tan\psi_1 = \tan\left(\dfrac{\pi}{2} + m\theta\right)$

$\Rightarrow$ $\psi_1 = \dfrac{\pi}{2} + m\theta$

Taking ln, we get

$m\ln r = m\ln a + \ln\sin m\theta$

Differentiating w.r.t. θ, we have

$m\cdot\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{1}{\sin m\theta}(m\cos m\theta)$

or $\dfrac{1}{r}\dfrac{dr}{d\theta} = \cot m\theta$

or $r.\dfrac{d\theta}{dr} = \tan m\theta$

i.e, $\tan\psi_2 = \tan m\theta$

$\Rightarrow$ $\psi_2 = m\theta$

Thus the angle between the two curves is $\psi_1 - \psi_2 = \dfrac{\pi}{2}$

Hence the two curves cut each other orthogenally.

17. Show that the tangents to the cardioid $r = a(1 + \cos\theta)$ at the points $\theta = \dfrac{\pi}{3}$ and $\theta = \dfrac{2\pi}{3}$ are respectively parallel and perpendicular to the initial line.

Sol. $r = a(1 + \cos\theta)$ (1)

Taking logarithm, we have

$$\ln r = \ln a + \ln(1 + \cos\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1 + \cos\theta}$$

or $$r\frac{d\theta}{dr} = -\frac{1 + \cos\theta}{\sin\theta} = -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

i.e., $\tan\psi = -\cot\dfrac{\theta}{2} = \tan\left[\dfrac{\pi}{2} + \dfrac{\theta}{2}\right] \Rightarrow \psi = \dfrac{\pi}{2} + \dfrac{\theta}{2}$

(i) At $\theta = \dfrac{\pi}{3}$, $\psi = \dfrac{\pi}{2} + \dfrac{\pi}{6} = \dfrac{3\pi + \pi}{6} = \dfrac{4\pi}{6} = \dfrac{2\pi}{3}$

Now $\alpha = \psi + \theta = \dfrac{2\pi}{3} + \dfrac{\pi}{3} = \pi$

Thus the tangent to (1) at $\theta = \dfrac{\pi}{3}$ is parallel to the initial line.

(ii) At $\theta = \dfrac{2\pi}{3}$, $\psi = \dfrac{\pi}{2} + \dfrac{\pi}{3} = \dfrac{5\pi}{6}$

$$\alpha = \psi + \theta = \frac{5\pi}{6} + \frac{2\pi}{3} = \frac{9\pi}{6} = \frac{3\pi}{2}$$

Therefore, the tangent to (1) at $\theta = \dfrac{2\pi}{3}$ is perpendicular to the initial line.

18. Show that $\tan\psi = \dfrac{x\dfrac{dy}{dx} - y}{y\dfrac{dy}{dx} + x}$

Sol. Let the initial line Ox be taken as the x-axis. Let the polar coordinates of a point P on the curve $r = f(\theta)$ be (r, θ) and suppose that the cartesian coordinates of P are (x, y). Then

$$x = r\cos\theta,\; y = r\sin\theta$$

and $\tan\theta = \dfrac{y}{x}$ (1)

If α is the angle which the tangent at P makes with Ox, then

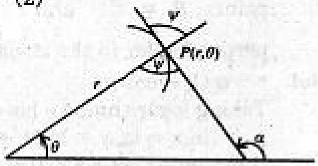
$$\frac{dy}{dx} = \tan \alpha \qquad (2)$$

From the figure, we have $\psi = \alpha - \theta$

Hence $\tan \psi = \tan (\alpha - \theta)$

$$= \frac{\tan \alpha - \tan \theta}{1 + \tan \alpha \tan \theta}$$

$$= \frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{dy}{dx} \cdot \frac{y}{x}} = \frac{x\frac{dy}{dx} - y}{x + y\frac{dy}{dx}}, \text{ from (1) and (2)}$$



19. Show that at any point of the lemeniscate $r^2 = a^2 \cos 2\theta, 0 \le \theta \le \frac{\pi}{4}$, the measure of the angle between the radius vector and the outward-pointed normal is 2θ.

Sol. $r^2 = a^2 \cos 2\theta$

Taking logarithms, we have

$$2 \ln r = 2 \ln a + \ln \cos 2\theta$$

Differentiating w.r.t. θ, we get

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{-2 \sin 2\theta}{\cos 2\theta} \Rightarrow r\frac{d\theta}{dr} = -\cot 2\theta$$

or $\quad \tan \psi = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$

Thus $\psi = \frac{\pi}{2} + 2\theta$

The tangent makes an angle $\frac{\pi}{2} + 2\theta$ with the radius vector. As the normal is perpendicular to the tangent, it will make an angle 2θ with the radius vector.

20. Find an equation (in rectangular coordinates) of the line tangent to:

(*i*) $r = \sin 2\theta$ at $\left(1, \frac{\pi}{4}\right)$ (*ii*) $r = 1 + \cos \theta$ at $\left(1, \frac{\pi}{2}\right)$

Sol.

(*i*) $r = \sin 2\theta$

$$\frac{dr}{d\theta} = (\cos 2\theta) \cdot 2 = 2 \cos 2\theta \quad \text{and}$$

at $\left(1, \frac{\pi}{4}\right), \frac{dr}{d\theta} = 2 \cos \frac{\pi}{2} = 2(0) = 0$

Putting $\frac{dr}{d\theta} = 0, r = 1, \theta = \frac{\pi}{4}$ in $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin \theta + r \cos \theta}{\frac{dr}{d\theta}\cos \theta - r \sin \theta}$, we have

$$\frac{dy}{dx} = \frac{0 \cdot \sin\frac{\pi}{4} + 1 \cdot \cos\frac{\pi}{4}}{0 \cdot \cos\frac{\pi}{4} - 1 \cdot \sin\frac{\pi}{4}} = \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1$$

Now at $\left(1, \frac{\pi}{4}\right)$, $x = 1 \cdot \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $y = 1 \cdot \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Thus the equation of the line tangent to (*i*) at $\left(1, \frac{\pi}{4}\right)$ is

$$y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right) \text{ in rectangular coordinates}$$

$$\sqrt{2}y - 1 = -(\sqrt{2}x - 1) \Rightarrow \sqrt{2}\,x + \sqrt{2}y - 2 = 0$$

or $\quad x + y - \sqrt{2} = 0$

(*ii*) $r = 1 + \cos \theta$

$\frac{dr}{d\theta} = -\sin \theta$ and at $\left(1, \frac{\pi}{2}\right), \frac{dr}{d\theta} = -\sin\frac{\pi}{2} = -1$

Putting $\frac{dr}{d\theta} = -1, r = 1, \theta = \frac{\pi}{2}$ in

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin \theta + r \cos \theta}{\frac{dr}{d\theta}\cos \theta - r \sin \theta}, \text{ we have}$$

$$\frac{dy}{dx} = \frac{(-1)\sin\frac{\pi}{2} + 1 \cdot \cos\frac{\pi}{2}}{(-1)\cos\frac{\pi}{2} - 1 \cdot \sin\frac{\pi}{2}} = \frac{-1}{-1} = 1$$

Now at $\left(1, \frac{\pi}{2}\right)$, $x = 1 \cdot \cos\frac{\pi}{2} = 1 \cdot 0 = 0$ and $y = 1 \cdot \sin \cdot \frac{\pi}{2} = 1$

Thus an equation of the line tangent to (*ii*) at $\left(1, \frac{\pi}{2}\right)$ in rectangular coordinates is

$$y - 1 = 1(x - 0) \Rightarrow x - y + 1 = 0.$$

Exercise Set 6.7 (Page 275)

Find parametric equations of the given curves (Problems 1 – 3):

1. $r = a \sin 2\theta,\ 0 \le \theta \le 2\pi$

Sol. $r = a \sin 2\theta$ (1)

We know that $x = r\cos\theta,\ y = r\sin\theta$

Writing the value of r from (1) in the above equations, we have

$$x = a\sin 2\theta \cos\theta,\ y = a \sin 2\theta \sin\theta$$

as the parametric equations of (1), θ being the parameter.

2. $r = \theta$

Sol. Putting $r = \theta$ in $x = r\cos$ and $y = r\sin\theta$, we have

$$x = \theta\cos\theta,\ y = \theta\sin\theta$$

which are the desired parametric equations.

3. $r = 2 + 3\sin\theta.$

Sol. Parametric equations of the given limacon are

$$x = r\cos\theta = (2 + 3\sin\theta)\cos\theta,$$
$$y = r\sin\theta = (2 + 3\sin\theta)\sin\theta$$

4. Show that the equations $x = a + r\cos\theta,\ y = b + r\sin\theta$ are parametric equations for a circle with centre (a, b) and radius $|r|$.

Sol. From the given equations, we have

$$x - a = r\cos\theta$$
$$y - b = r\sin\theta$$

Squaring the above equations and adding the results, we get

$$(x - a)^2 + (y - b)^2 = r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2$$

which is a circle with centre (a, b) and radius $|r|$.

5. Show that the curve whose parametric equations are

$$\left.\begin{aligned} x &= a\cos\theta + h \\ y &= b\sin\theta + k \end{aligned}\right\},\quad 0 \le \theta \le 2\pi$$

is an ellipse with centre (h, k).

Sol. $x - h = a\cos\theta$ or $\dfrac{x-h}{a} = \cos\theta$

$y - k = b\sin\theta$ or $\dfrac{y-k}{b} = \sin\theta$

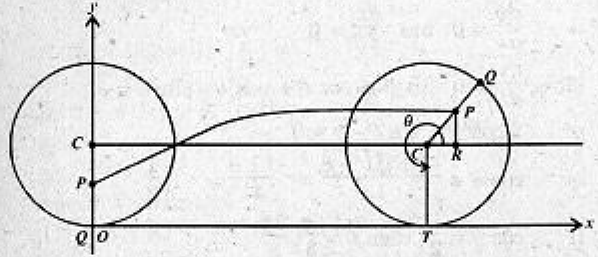
Squaring these equations and adding the results, we have

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

which is an ellipse with centre at (h, k).

6. A wheel of radius a rolls on a straight line without slipping or sliding. Let P be a fixed point on the wheel, at a distance b from the centre of the wheel. Find parametric equations of the curve described by the point P. The curve is called a **trochoid**. Hence deduce parametric equations for a cycloid.

Sol. Let x-axis be taken the line on which the wheel rolls. In the initial position, let the centre C of the wheel be on the y-axis and the point P is on the y-axis below C. Let the point Q on the rim of the wheel that lies on the radial line CP coincide with the origin O.



Let the wheel turn through an angle θ (the radial line CQ turns through an angle θ) in moving from its initial position to some general position as shown in the figure. Let $\angle RCP = \phi$.

Then $\theta + \phi = \dfrac{3\pi}{2}$ (1)

In the general position, let P have coordinates (x, y).

Now $x = OT + CR = a\theta + b\cos\phi$ (2)

$y = TC + RP = a + b\sin\phi$ (3)

Substituting the value of ϕ from (1) into (2) and (3), we get

$$x = a\theta + b\cos\left(\frac{3\pi}{2} - \theta\right) = a\theta - b\sin\theta \quad (4)$$

$$y = a + b\sin\left(\frac{3\pi}{2} - \theta\right) = a - b\cos\theta \quad (5)$$

as parametric equations of the trochoid.

For a cycloid, the point P is on the rim of the wheel so that $b = a$ and from (4)(and (5), we have

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$

as equations of the cycloid.

7. Find the points at which $r = 1 + \cos\theta$ has horizontal and vertical tangents.

Sol. Parametric equations of the curve are

$$x = (1 + \cos\theta)\cos\theta = \cos\theta + \cos^2\theta$$
$$y = (1 + \cos\theta)\sin\theta = \sin\theta + \sin\theta\cos\theta$$
$$\frac{dx}{d\theta} = -\sin\theta - 2\sin\theta\cos\theta$$
$$\frac{dy}{d\theta} = \cos\theta + \cos^2\theta - \sin^2\theta$$

Therefore, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

The curve has horizontal tangent at the points where

$$\frac{dy}{d\theta} = 0 \text{ but } \frac{dx}{d\theta} \neq 0$$

Now, $\frac{dy}{d\theta} = 0$ implies $\cos\theta + \cos^2\theta - \sin^2\theta = 0$

or $\quad 2\cos^2\theta + \cos\theta - 1 = 0$

or $\quad \cos\theta = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = -1, \frac{1}{2}$

If $\quad \cos\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

If $\quad \cos\theta = -1, \theta = \pi$

But $\frac{dx}{d\theta} = 0$ for $\theta = \pi$ and it is non-zero for $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Thus, tangents are horizontal at

$$\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right)$$

Tangents are vertical at points where $\frac{dx}{d\theta} = 0$ but $\frac{dy}{d\theta} \neq 0$

Now $\frac{dx}{d\theta} = 0$ implies $-\sin\theta - 2\sin\theta\cos\theta = 0$

or $\quad \sin\theta(1 + 2\cos\theta) = 0 \Rightarrow \sin\theta = 0$ implies $\theta = 0, \pi$,

and $\quad 1 + 2\cos\theta = 0$ gives $\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$\frac{dy}{d\theta} \neq 0 \text{ at } \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

Hence, tangents are vertical at

$$(2, 0), \left(\frac{1}{2}, \frac{2\pi}{3}\right), \left(\frac{1}{2}, \frac{4\pi}{3}\right).$$

8. Find the points on the curve

$$x(t) = t^2 + 4, \; y(t) = 3t^2 - 6t + 2$$

where tangents are horizontal and vertical.

Sol. $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 6t - 6$

Hence $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6(t-1)}{2t} = \frac{3(t-1)}{t}$

For vertical tangents, $\frac{dx}{dt} = 0$ but $\frac{dy}{dt} \neq 0$

$$\frac{dx}{dt} = 0 \text{ implies } t = 0 \text{ and } \frac{dy}{dt} \neq 0 \text{ at } t = 0$$

The curve has vertical tangent at $x(0) = 4, y(0) = 2$.

i.e., at (4, 2).

For horizontal tangents $\frac{dy}{dt} = 0$ but $\frac{dx}{dt} \neq 0$

Now $\frac{dy}{dt} = 0$ gives $3(t-1) = 0$ or $t = 1$

$$\frac{dx}{dt} \neq 0 \text{ at } t = 1$$

The tangent is horizontal at

$x(1) = 5, y(1) = -1$ *i.e.*, at $(5, -1)$.

Find equations of the tangent and normal to each of the given curve at the indicated point (Problem 9 – 11):

9. $x = 2a\cos\theta - a\cos 2\theta, \; y = 2a\sin\theta - a\sin 2\theta$ at $\theta = \frac{\pi}{2}$

Sol. $\frac{dx}{d\theta} = -2a\sin\theta + 2a\sin 2\theta$

and $\frac{dy}{d\theta} = 2a\cos\theta - 2a\cos 2\theta$

Therefore, $\frac{dy}{dx} = \frac{2a\cos\theta - 2a\cos 2\theta}{2a\sin 2\theta - 2a\sin\theta} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$

$$= \frac{2\sin\frac{\theta}{2}\sin\frac{3\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{3\theta}{2}} = \tan\frac{3\theta}{2}$$

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = \tan\frac{3}{2}\left[\frac{\pi}{2}\right] = \tan\frac{3\pi}{4} = -1$$

At $\quad \theta = \frac{\pi}{2}$

$$x = 2a\cos\frac{\pi}{2} - a\cos 2\left[\frac{\pi}{2}\right] = a$$

$$y = 2a\sin\frac{2\pi}{2} - a\sin 2\left(\frac{\pi}{2}\right) = 2a$$

Hence equation of the tangent at $\theta = \frac{\pi}{2}$ is

$$y - 2a = -(x - a)$$

i.e., $x + y = 3a$.

Slope of the normal at $\theta = \frac{\pi}{2}$ is 1

Equation of the normal at $(a, 2a)$ is $y - 2a = x - a$

or $x - y + a = 0$.

10. $x = \dfrac{2at^2}{1+t^2}, y = \dfrac{2at^3}{1+t^2}$ at $t = \dfrac{1}{2}$

Sol. At $t = \frac{1}{2}$, we have $x = \dfrac{2a \cdot \frac{1}{4}}{1 + \frac{1}{4}} = \dfrac{\frac{a}{2}}{\frac{5}{4}} = \dfrac{2a}{5}$ and $y = \dfrac{2a \cdot \left(\frac{1}{8}\right)}{1 + \frac{1}{4}} = \dfrac{\frac{a}{4}}{\frac{5}{4}} = \dfrac{a}{5}$

Now, $x = \dfrac{2at^2}{1+t^2} = 2a\left[1 - \dfrac{1}{1+t^2}\right]$

$$\frac{dx}{dt} = 2a\left[\frac{2t}{(1+t^2)^2}\right] = \frac{4at}{(1+t^2)^2} \quad (1)$$

$$y = \frac{2at^3}{1+t^2} = 2a\left[t - \frac{t}{1+t^2}\right]$$

$$\frac{dy}{dt} = 2a\left[1 - \frac{(1+t^2) - t(2t)}{(1+t^2)^2}\right]$$

$$= 2a\left[1 - \frac{1-t^2}{(1+t^2)^2}\right] = 2a\left[\frac{(1+t^2)^2 - 1 + t^2}{(1+t^2)^2}\right]$$

$$= 2a\frac{3t^2 + t^4}{(1+t^2)^2} = \frac{2at^2(3+t^2)}{(1+t^2)^2} \quad (2)$$

Dividing (2) by (1), we have

$$\frac{dy}{dx} = \frac{2at^2(3+t)^2}{(1+t^2)^2} \times \frac{(1+t^2)}{4at} = \frac{t(3+t^2)}{2}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{1}{2}} = \frac{\frac{1}{2}\left(3 + \frac{1}{4}\right)}{2} = \frac{13}{16}$$

Equation of the required tangent is

$$y - \frac{a}{5} = \frac{13}{16}\left[x - \frac{2a}{5}\right] = \frac{13x}{16} - \frac{13a}{40}$$

$$\frac{13x}{16} - y = \frac{13a}{40} - \frac{a}{5} = \frac{a}{8}$$

or $13x - 16y = 2a$.

Slope of the normal at $t = \frac{1}{2}$ *i.e.*, at the point $\left(\frac{2a}{5}, \frac{a}{5}\right)$ is $\frac{-16}{13}$

Equation of the normal at this point is

$$y - \frac{a}{5} = \frac{-16}{13}\left(x - \frac{2a}{5}\right)$$

$$13y - \frac{13a}{5} = -16x + \frac{29a}{5}$$

or $\quad 16x + 13y = \dfrac{32a}{5} + \dfrac{13a}{5}$ *i.e.*, $16x + 13y = 9a$

11. $x = (t-1)^{3/2}, \; y = 3t$ at $t = 5$

Sol. $\dfrac{dx}{dt} = \dfrac{3}{2}(t-1)^{1/2}, \; \dfrac{dy}{dt} = 3$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{\frac{3}{2}(t-1)^{1/2}} = \frac{2}{\sqrt{t-1}}$$

$$\left.\frac{dy}{dx}\right|_{t=5} = \frac{2}{\sqrt{5-1}} = 1$$

The point $P(x, y)$ at $t = 5$ is $P(8, 15)$

Equation of the tangent at P is

$$y - 15 = 1(x - 8) \text{ or } x - y + 7 = 0$$

Equation of the normal at P is

$$y - 15 = -1(x - 8)$$

or $\quad x + y - 23 = 0$

12. Show that the normal at any point of the curve

$$x = a\cos\theta + a\theta\sin\theta, y = a\sin\theta - a\theta\cos\theta$$

is at a constant distance from the origin.

Sol. $\dfrac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$

$$\frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$$

Therefore, $\dfrac{dy}{dx} = \dfrac{a\theta\sin\theta}{a\theta\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$

Slope of the normal $= -\dfrac{\cos\theta}{\sin\theta}$

Equation of the normal is

$$y - a\sin\theta + a\theta\cos\theta = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta - a\theta\sin\theta)$$

or $\quad y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta$

$$= -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

or $\quad x\cos\theta + y\sin\theta - a = 0 \qquad (1)$

Length of the perpendicular from (0, 0) to (1)

$$= \frac{|-a|}{\sqrt{\cos^2\theta+\sin^2\theta}} = a \quad \text{which is constant}$$

13. Prove that an equation of the normal to the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ can be written in the form

$$x\sin t - y\cos t + a\cos t = 0,\ t \text{ being parameter.}$$

Sol. $x^{2/3}+y^{2/3}=a^{2/3} \qquad (1)$

Differentiating (1) w.r.t. x, we have

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

Slope of the normal $= \dfrac{x^{1/3}}{y^{1/3}}$

Let this slope be denoted by $\tan t$, then

$$\frac{x^{1/3}}{y^{1/3}} = \tan t$$

or $\quad x^{1/3} = y^{1/3}\tan t \qquad (2)$

Putting this value of $x^{1/3}$ in (1), we get

$$y^{2/3}\tan^2 t + y^{2/3} = a^{2/3}$$

or $\quad y^{2/3}(1+\tan^2 t) = a^{2/3} \Rightarrow y^{2/3}\sec^2 t = a^{2/3}$

or $\quad \dfrac{y^{2/3}}{a^{2/3}} = \cos^2 t \Rightarrow \dfrac{y}{a} = \cos^3 t$ or $y = a\cos^3 t$

From (2), $x^{1/3} = (a\cos^3 t)^{1/3}\tan t = a^{1/3}\cos t\dfrac{\sin t}{\cos t} = a^{1/3}\sin t$

or $\quad x = a\sin^3 t$

Thus we have to find equation of the normal at $(a\sin^3 t, a\cos^3 t)$

Slope of the normal $= \dfrac{x^{1/3}}{y^{1/3}} = \dfrac{a^{1/3}\sin t}{a^{1/3}\cos t} = \dfrac{\sin t}{\cos t}$

Hence the equation of the normal is

$$y - a\cos^3 t = \frac{\sin t}{\cos t}(x - a\sin^3 t)$$

or $\quad y\cos t - a\cos^4 t = x\sin t - a\sin^4 t$

or $\quad x\sin t - y\cos t = -a(\cos^2 t - \sin^2 t) = -\cos 2t$

$\quad x\sin t - y\operatorname{cost} t + a\cos 2t = 0$ as required.

14. Show that the pedal equation of the curve

$$x = a\cos^3\theta,\ y = a\sin^3\theta$$

is $\quad r^2 = a^2 - 3p^2$

Sol. $x = a\cos^3\theta$ and $y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad\text{and}\quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta}$$

Equation of the tangent at $(a\cos^3\theta, a\sin^3\theta)$ is

$$y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$$

or $\quad y\cos\theta - a\sin^3\theta\cos\theta = -x\sin\theta + a\sin\theta\cos^3\theta$

or $\quad x\sin\theta + y\cos\theta - a\sin\theta\cos\theta(\sin^2\theta+\cos^2\theta) = 0$

or $\quad x\sin\theta + y\cos\theta - a\sin\theta\cos\theta = 0$

$$p = \frac{|-a\sin\theta\cos\theta|}{\sqrt{\sin^2\theta+\cos^2\theta}} \quad\text{or}\quad p^2 = a^2\sin^2\theta\cos^2\theta \qquad (1)$$

Now $r^2 = x^2 + y^2 = a^2\cos^6\theta + a^2\sin^6\theta$

$= a^2(\cos^2\theta+\sin^2\theta)(\cos^4\theta - \sin^2\theta\cos^2\theta + \sin^4\theta)$

$= a^2[(\cos^2\theta+\sin^2\theta)^2 - 3\sin^2\theta\cos^2\theta]$

$= a^2(1 - 3\sin^2\theta\cos^2\theta)$

or $\quad 3a^2\sin^2\theta\cos^2\theta = a^2 - r^2 \qquad (2)$

Using (2), we get from (1)

$$3p^2 = a^2 - r^2 \quad\text{or}\quad r^2 = a^2 - 3p^2$$

i.e., $\quad r^2 = a^2 - 3p^2$

which is the required pedal equation.

15. Prove that the pedal equation of the curve

$$x = 2a\cos\theta - a\cos 2\theta,\ y = 2a\sin\theta - a\sin 2\theta$$

is $\quad 9(r^2 - a^2) = 8p^2$

Sol. From $x = 2a\cos\theta - a\cos 2\theta$, we have

$$\frac{dx}{d\theta} = -2a\sin\theta + 2a\sin 2\theta = 2a(\sin 2\theta - \sin\theta)$$

$$= 4a\sin\frac{\theta}{2}\cos\frac{3\theta}{2} \qquad (1)$$

From $y = 2a\sin\theta - a\sin 2\theta$, we get

$$\frac{dy}{d\theta} = 2a\cos\theta - 2a\cos 2\theta = 2a(\cos\theta - \cos 2\theta)$$

$$= 4a\sin\frac{\theta}{2}\sin\frac{3\theta}{2} \qquad (2)$$

Dividing (2) by (1), we have

$$\frac{dy}{dx} = \frac{4a\sin\frac{\theta}{2}\sin\frac{3\theta}{2}}{4a\sin\frac{\theta}{2}\cos\frac{3\theta}{2}} = \frac{\sin\frac{8\theta}{2}}{\cos\frac{3\theta}{2}}$$

Equation of the tangent at $(2a\cos\theta - a\cos 2\theta, 2a\sin\theta - a\sin 2\theta)$ is

$$y - 2a\sin\theta + a\sin 2\theta = \frac{\sin\frac{3\theta}{2}}{\cos\frac{3\theta}{2}}(x - 2a\cos\theta + a\cos 2\theta)$$

or $$y\cos\frac{3\theta}{2} - 2a\sin\theta\cos\frac{3\theta}{2} + a\sin 2\theta\cos\frac{3\theta}{2}$$
$$= x\sin\frac{3\theta}{2} - 2a\cos\theta\sin\frac{3\theta}{2} + a\cos 2\theta\sin\frac{3\theta}{2}$$

or $$x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 2a\left\{\sin\frac{3\theta}{2}\cos\theta - \sin\theta\cos\frac{3\theta}{2}\right\}$$
$$- a\left\{\sin 2\theta\cos\frac{3\theta}{2} - \cos 2\theta\sin\frac{3\theta}{2}\right\} = 0$$

or $$x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 2a\sin\frac{\theta}{2} - a\sin\frac{\theta}{2} = 0$$

or $$x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 3a\sin\frac{\theta}{2} = 0$$

$$p = \frac{\left|-3a\sin\frac{\theta}{2}\right|}{\sqrt{\sin^2\frac{3\theta}{2} + \cos^2\frac{3\theta}{2}}} \text{ or } p^2 = 9a^2\sin^2\frac{\theta}{2} \quad (3)$$

Now, $r^2 = x^2 + y^2$
$$= (2a\cos\theta - a\cos 2\theta)^2 + (2a\sin\theta - a\sin 2\theta)^2$$
$$= 4a^2\cos^2\theta + a^2\cos^2 2\theta - 4a^2\cos 2\theta\cos\theta$$
$$+ 4a^2\sin^2\theta + a^2\sin^2 2\theta - 4a^2\sin 2\theta\sin\theta$$
$$= 4a^2 + a^2 - 4a^2[\cos 2\theta\cos\theta + \sin\theta\sin 2\theta]$$
$$= 5a^2 - 4a^2\cos\theta$$
$$= 5a^2 - 4a^2\left[1 - 2\sin^2\frac{\theta}{2}\right] = a^2 + 8a^2\sin^2\frac{\theta}{2}$$

or $$r^2 - a^2 = 8a^2\sin^2\frac{\theta}{2} \quad (4)$$

From (3), $\sin^2\frac{\theta}{2} = \frac{p^2}{9a^2}$ (5)

From (4) and (5), we have
$$r^2 - a^2 = 8a^2\left(\frac{p^2}{9a^2}\right) = \frac{8p^2}{9}$$

or $9(r^2 - a^2) = 8p^2$ which is the required equation.

16. Show that the pedal equation of the curve
$$x = ae^\theta(\sin\theta - \cos\theta), y = ae^\theta(\sin\theta + \cos\theta)$$

is $r = \sqrt{2}p$.

Sol. $x = ae^\theta(\sin\theta - \cos\theta)$

$$\frac{dx}{d\theta} = ae^\theta(\sin\theta - \cos\theta) + ae^\theta(\cos\theta + \sin\theta) = 2ae^\theta\sin\theta \quad (1)$$

$y = ae^\theta(\sin\theta + \cos\theta)$

$$\frac{dy}{d\theta} = ae^\theta(\sin\theta + \cos\theta) + ae^\theta(\cos\theta - \sin\theta) = 2ae^\theta\cos\theta \quad (2)$$

From (1) and (2), we have
$$\frac{dy}{dx} = \frac{2a\,e^\theta\cos\theta}{2a\,e\sin\theta} = \frac{\cos\theta}{\sin\theta}$$

Equation of the tangent at $(ae^\theta(\sin\theta - \cos\theta), ae^\theta(\sin\theta + \cos\theta)$ is
$$y - ae^\theta(\sin\theta + \cos\theta) = \frac{\cos\theta}{\sin\theta}[x - ae^\theta(\sin\theta - \cos\theta)]$$

or $$y\sin\theta - ae^\theta(\sin^2\theta + \sin\theta\cos\theta)$$
$$= x\cos\theta - ae^\theta(\cos\theta\sin\theta - \cos^2\theta)$$

or $$y\sin\theta - ae^\theta\sin^2\theta = x\cos\theta + ae^\theta\cos^2\theta$$

or $$x\cos\theta - y\sin\theta + ae^\theta = 0$$

$$p = \frac{ae^\theta}{\sqrt{\cos^2\theta + \sin^2\theta}} = ae^\theta \quad (3)$$

Now $$r^2 = x^2 + y^2 = a^2e^{2\theta}[(\sin\theta - \cos\theta)^2 + (\sin\theta + \cos\theta)^2]$$
$$= a^2e^{2\theta}[2(\sin^2\theta + \cos^2\theta)] = 2a^2e^{2\theta} \quad (4)$$

Squaring (3), we have
$$p^2 = a^2e^{2\theta} \quad (5)$$

From (4) and (5), we get
$r^2 = 2p^2 \Rightarrow r = \sqrt{2}p$ which is the required pedal equation

17. Prove that the pedal equation of the curve
$$x = a(3\cos\theta - \cos^3\theta), y = a(3\sin\theta - \sin^3\theta)$$
is $3p^2(7a^2 - r^2) = (10a^2 - r^2)^2$.

Sol. $x = a(3\cos\theta - \cos^3\theta)$

$$\frac{dx}{d\theta} = a[-3\sin\theta + 3\cos^2\theta\sin\theta]$$
$$= -3a\sin\theta(1 - \cos^2\theta) = -3a\sin^3\theta$$

$y = a(3\sin\theta - \sin^3\theta)$

$$\frac{dy}{d\theta} = a[3\cos\theta - 3\sin^2\theta\cos\theta]$$
$$= 3a\,\mathrm{cso}\,\theta(1 - \sin^2\theta) = 3a\cos^3\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\cos^3\theta}{-3a\sin^3\theta} = -\frac{\cos^3\theta}{\sin^3\theta}$$

Equation of the tangent at $(a(3\cos\theta - \cos^3\theta), a(3\sin\theta - \sin^3\theta))$ is

$$y - a(3\sin\theta - \sin^3\theta) = -\frac{\cos^3\theta}{\sin^3\theta}[x - a(3\cos\theta - \cos^3\theta)]$$

or $\quad y\sin^3\theta - a\sin^3\theta(3\sin\theta - \sin^3\theta)$

$= -x\cos^3\theta + a\cos^3\theta(3\cos\theta - \cos^3\theta)$

or $\quad y\sin^3\theta - 3a\sin^4\theta + a\sin^6\theta = -x\cos^3\theta + 3a\cos^4\theta - a\cos^6\theta$

or $\quad x\cos^3\theta + y\sin^3\theta - 3a(\sin^4\theta + \cos^4\theta + a(\sin^6\theta + \cos^6\theta) = 0$

Now $\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta$

$= 1 - 2\sin^2\theta\cos^2\theta$

and $\sin^6\theta + \cos^6\theta = (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$

$= 1 - 3\sin^2\theta\cos^2\theta$

Hence equation of the tangent can be written as

$x\cos^3\theta + y\sin^3\theta - 3a(1 - 2\sin^2\theta\cos^2\theta) + a(1 - 3\sin^2\theta\cos^2\theta) = 0$

or $\quad x\cos^3\theta + y\sin^3\theta - (2a - 3a\sin^2\theta\cos^2\theta) = 0 \qquad (1)$

Length of the perpendicular from the origin to (1) is

$$p = \frac{|-(2a - 3a\sin^2\theta\cos^2\theta)|}{\sqrt{\cos^6\theta + \sin^6\theta}} = \frac{|-(2a - 3a\sin^2\theta\cos^2\theta)|}{\sqrt{1 - 3\sin^2\theta\cos^2\theta}}$$

or $$p^2 = \frac{(2a - 3a\sin^2\theta\cos^2\theta)^2}{1 - 3\sin^2\theta\cos^2\theta} \qquad (2)$$

Now $r^2 = x^2 + y^2 = a^2(3\cos\theta - \cos^3\theta)^2 + a^2(3\sin\theta - \sin^3\theta)^2$

$= a^2(9\cos^2\theta + \cos^6\theta - 6\cos^4\theta) + a^2[9\sin^2\theta + \sin^6\theta - 6\sin^4\theta]$

$= a^2[9(\cos^2\theta + \sin^2\theta) + (\cos^6\theta + \sin^6\theta) - 6(\cos^4\theta + \sin^4\theta)]$

$= a^2[9 + 1 - 3\sin^2\theta\cos^2\theta - 6(1 - 2\sin^2\theta\cos^2\theta)]$

$= a^2[10 - 3\sin^2\theta\cos^2\theta - 6 + 12\sin^2\cos^2\theta]$

$= a^2[4 + 9\sin^2\theta\cos^2\theta] = 4a^2 + 9a^2\sin^2\theta\cos^2\theta$

or $\quad 9a^2\sin^2\theta\cos^2\theta = r^2 - 4a^2$ or $\sin^2\theta\cos^2\theta = \dfrac{r^2 - 4a^2}{9a^2}$

Putting this value in (2), we get

$$p^2 = \frac{\left(2a - \frac{r^2 - 4a^2}{3a}\right)^2}{1 - \frac{r^2 - 4a^2}{3a^2}} = \frac{(10a^2 - r^2)^2}{9a^2} \times \frac{3a^2}{7a^2 - r^2}$$

$$= \frac{(10a^2 - r^2)^2}{7a^2 - r^2} \times \frac{1}{3} \text{ or } 3p^2(7a^2 - r^2) = (10a^2 - r^2)^2$$

18. If $x = a\cos g(t)$, $y = b\sin g(t)$, prove that

$$xy^2\frac{d^2y}{dx^2} = b^2\frac{dy}{dx}$$

Sol. $x = a\cos g(t)$ gives

$$\frac{dx}{dt} = (-a\sin g(t))\,g'(t)$$

$y = b\sin g(t)$ gives

$$\frac{dy}{dt} = (b\cos g(t))\,.\,g'(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{b}{a}\,\frac{\cos g(t)}{\sin g(t)} = \frac{b\,.\,\frac{x}{a}}{a\,.\,\frac{y}{b}} = -\frac{b^2}{a^2}\cdot\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y\,.\,1 - x\frac{dy}{dx}}{y^2}\right] = -\frac{b^2}{a^2}\left[\frac{y - x\left(-\frac{b^2}{a^2}\frac{x}{y}\right)}{y^2}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{a^2y^2 + b^2x^2}{a^2y^3}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{a^2b^2\sin^2 g(t) + b^2a^2\cos^2 g(t)}{a^2y^3}\right]$$

$$= -\frac{b^2}{a^2}\cdot\frac{a^2b^2}{a^2y^3} = -\frac{b^4}{a^2y^3}$$

Now, $$xy^2\frac{d^2y}{dx^2} = xy^2\left(\frac{-b^4}{a^2y^3}\right) = -\frac{b^4}{a^2}\frac{x}{y}$$

$$= b^2\left(-\frac{b^2}{a^2}\frac{x}{y}\right) = b^2\frac{dy}{dx} \text{ as required.}$$

Chapter 7

PLANE CURVES II

Exercise Set 7.1 (Page 284)

Find equations of the asymptotes of the following curves:

1. $y = \dfrac{(x-2)^2}{x^2}$

Sol. The equation can be written as

$$x^2 y = (x-2)^2$$

or $$x^2 (y-1) = -4x + 4 \tag{1}$$

We find the asymptotes parallel to coordinate axes.

The highest power of y in (1) is y and its coefficient is x^2.

Asymptotes parallel to the y-axis are

$$x^2 = 0$$

i.e., two coincident asymptotes $x = 0$.

Coefficient of highest power of x, *i.e.*, of x^2 in (1) is $y - 1$. Therefore,

$y - 1 = 0$ is an asymptotes parallel to the x-axis.

Hence the required asymptotes are

$$x = 0 \quad \text{and} \quad y = 1.$$

2. $x^2 y^2 = 12(x-3)$

Sol. Asymptotes parallel to the y-axis are

$$x^2 = 0 \quad i.e., \quad x = 0$$

Asymptotes parallel to the x-axis are

$$y^2 = 0 \quad \text{or} \quad y = 0$$

3. $2xy = x^2 + 3$

Sol. Asymptote parallel to the y-axis is

$$x = 0.$$

There is no asymptote parallel to the x-axis.

For oblique asymptotes, equation of the curve can be written as

$$2xy - x^2 - 3 = 0$$

Here, putting $y = m$ and $x = 1$, we have

$$\phi_2(m) = 2m - 1 = 0 \text{ gives } m = \frac{1}{2}$$

$$\phi'_2(m) = 2$$

$$\phi_1(m) = 0$$

To find c we apply the formula

$$c\phi'_2(m) + \phi_1(m) = 0$$

or $2c = 0$ or $c = 0$

Hence an oblique asymptote is $y = \frac{1}{2}x$.

4. $x^2(x-y)^2 + a^2(x^2-y^2) = a^2xy$ (1)

Sol. The equation is

$$x^4 + x^2y^2 + 2x^3 + a^2x^2 - a^2y^2 - a^2xy = 0$$

Coefficient of highest power of y *i.e.*, of y^2 in (1) is $x^2 - a^2$

Hence $x^2 - a^2 = 0$ or $x = \pm a$

are asymptote parallel to the y-axis.

There is no asymptote parallel to the x-axis.

Now, for oblique asymptotes,

$\phi_4(m) = 1 + m^2 - 2m = 0$ gives $m = 1, 1$

$\phi'_4(m) = 2m - 2$

$\phi''_4(m) = 2$

$\phi_3(m) = 0$

$\phi'_3(m) = 0$

$\phi_2(m) = a^2(1-m^2) - a^2m$

To find c, we apply the formula

$$\frac{c^2}{2!}\phi''_4(m) + c\,\phi'_3(m) + \phi_2(m) = 0$$

or $c^2 + a^2(1-m^2) - a^2m = 0$

Putting $m = 1$, $c^2 = a^2$ or $c = \pm a$

Hence the oblique asymptotes are $y = x \pm a$.

5. $(x-y)^2(x^2+y^2) - 10(x-y)x^2 + 12y^2 + 2x + y = 0$ (1)

Sol. Here coefficient of x^4 is 1 and that of y^4 is 1 and so (1) has no asymptotes parallel to the coordinate axes.

For oblique asymptotes, we have

$\phi_4(m) = (1-m)^2(1+m^2) = 0$ gives $m = 1, 1$

and the other two values of m are imaginary.

$\phi_4(m) = (1-2m+m^2)(1+m^2)$

$= m^4 - 2m^3 + 2m^2 - 2m + 1$

$\phi'_4(m) = 4m^3 - 6m^2 + 4m - 2$

$\phi''_4(m) = 12m^3 - 12m + 4$

$\phi_3(m) = -10(1-m) = 10m - 10$

$\phi'_3(m) = 10$

$\phi_2(m) = 12m^2$

To find c, we use

$$\frac{c^2}{2!}\phi''_4(m) + c\phi'_3(m) + \phi_2(m) = 0$$

or $c^2(6m^2 - 6m + 2) + 10c + 12m^2 = 0$

Putting $m = 1$, the above equation becomes

$2c^2 + 10c + 12 = 0$

or $c^2 + 5c + 6 = 0$

or $(c+2)(c+3) = 0$

or $c = -2, -3$

Hence the asymptotes are

$y = x - 2, y = x - 3$

6. $x^2y + xy^2 + xy + y^2 + 3x = 0$ (1)

Sol. Coefficient of the highest power of y in (1) is $x + 1$. Therefore,

$x + 1 = 0$

is an asymptote parallel to y-axis

For oblique asymptotes, we have

$\phi_3(m) = m + m^2 = 0$ gives $m = 0, -1$

$\phi'_3(m) = 1 + 2m$

$\phi_2(m) = m + m^2$

To find c, we apply the formula

$c\phi'_3(m) + \phi_2(m) = 0$ or $c(1+2m) + m + m^2 = 0$ (2)

Putting $m = 0$ in (2), we have $c = 0$.

Thus $y = 0$ is an asymptote parallel to the x-axis.

Putting $m = -1$ in (2), we get

$c(-1) - 1 + 1 = 0$ or $c = 0$

Therefore, $y = -x$ is an oblique asymptote.

7. $(x-y+1)(x-y-2)(x+y) = 8x - 1$

Sol. The equation can be written as

$[(x-y)^2 - (x-y) - 2](x+y) = 8x - 1$

or $(x-y)^2(x+y) - (x-y)(x+y) - 2(x+y) - 8x + 1 = 0$

or $(x-y)^2(x+y) - (x-y^2) - 10x - 2y + 1 = 0$

There are no asymptotes parallel to the coordinate axes.

Here $\phi_3(m) = (1-m)^2(1+m) = 0$ gives $m = 1, 1, -1$

$\phi_3(m) = (1-2m+m^2)(1+m) = m^3 - m^2 - m + 1$

$\phi'_3(m) = 3m^2 - 2m - 1$

$\phi''_3(m) = 6m - 2$

$\phi_2(m) = -1 + m^2$

$\phi_1(m) = -10 - 2m$

To find c when $m = -1$, we use

$c\phi'_3(m) + \phi_2(m) = 0$

or $c(3m^2 - 2m - 1) + (-1 + m^2) = 0$

(1)

Putting $m = -1$ in (1), we get $c(3 + 2 - 1) + 0 = 0$

or $4c = 0$ or $c = 0$

Therefore, $y = -x$ is an asymptote

To find c for $m = 1, 1$, we apply the formula

$\frac{c^2}{2!}\phi''_3(m) + c\phi'_2(m) + \phi_1(m) = 0$

or $\frac{c^2}{2}(6m - 2) + c(2m) - 10 - 2m = 0$

Putting $m = 1$, the above equation becomes

$\frac{c^2}{2}(4) + 2c - 12 = 0$

or $c^2 + c - 6 = 0$ or $(c + 3)(c - 2) = 0$

Thus $c = 2, -2$

Hence $y = x + 2$ and $y = x - 3$ are asymptotes.

8. $y^3 + x^2y + 2xy - y + 1 = 0$

Sol. Here $\phi_3(m) = m^3 + m + 2m^2$

$\phi_3(m) = 0$ gives

$m^3 + 2m^2 + m = 0$

or $m(m^2 + 2m + 1)$

i.e., $m = 0, -1, -1$

$\phi'_3(m) = 3m^2 + 1 + 4m$

$\phi''_3(m) = 6m + 4$

$\phi_2(m) = 0, \phi'_2(m) = 0, \phi_1(m) = -m$

To find c, when m has two equal values, we use

$\frac{c^2}{2!}\phi''_3(m) + c\phi'_2(m) + \phi_1(m) = 0$

or $\frac{c^2}{2}(6m + 4) + 0 - m = 0$ or $c^2(3m + 2) - m = 0$ (1)

Putting $m = -1$ in (1), we have $-c^2 + 1 = 0$

or $c^2 = 1$ or $c = \pm 1$

Asymptotes parallel to each other are

$y = -x + 1$ and $y = -x - 1$

To find c when $m = 0$, we apply the formula

$c\phi'_3(m) + \phi_2(m) = 0$

or $c(3m^2 + 1 + 4m) + 0 = 0$ *i.e.,* $c = 0$

Therefore $y = 0$ is an asymptote.

Hence the required asymptotes are

$y = 0, y + x = \pm 1$

9. $y(x - y)^2 = x + y$

Sol. The given equation can be written as

$y(x - y)^2 - x - y = 0$

Here $\phi_3(m) = m(1 - m)^2$

$\phi_3(m) = 0$ gives $m = 1, 1, 0$

Again, $\phi_3(m) = m(m^2 - 2m + 1) = 3m^2 - 2m^2 + m$

$\phi'_3(m) = 3m^2 - 4m + 1$

$\phi''_3(m) = 6m - 4$

$\phi_2(m) = 0, \phi'_2(m) = 0$

$\phi_1(m) = -1 - m$

To find c, when $m = 1, 1$, we apply the formula

$\frac{c^2}{2!}\phi''_3(m) + c\phi'_2(m) + \phi_1(m) = 0$

or $\frac{c^2}{2}(6m - 4) + 0 - 1 - m = 0$

or $c^2(3m - 2) - 1 - m = 0$ (1)

Putting $m = 1$ in (1), we have $c^2 - 2 = 0$

or $c = \pm\sqrt{2}$

The corresponding asymptotes are

$y = x \pm \sqrt{2}$

To find c when $m = 0$, we use

$c\phi'_3(m) + \phi_2(m) = 0$

or $c(3m^2 - 4m + 1) = 0$ *i.e.,* $c = 0$

The corresponding asymptote is $y = 0$

10. $x^2y^2(x^2 - y^2)^2 = (x^2 + y^2)^3$ (1)

Sol. Coefficient of the highest power of x *i.e.,* of x^6 in (1) is

$y^2 - 1$

and coefficient of the highest power of y *i.e.,* of y^6 in (1) is

$x^2 - 1$

Thus asymptotes parallel to the coordinate axes are

$x = \pm 1,$ $y = \pm 1$

For oblique asymptotes, we have

$\phi_8(m) = m^2(1-m^2)^2$

$\phi_8(m) = 0$, gives $m = 0, 0, 1, 1, -1, -1.$

Again, $\phi_8(m) = m^2(m^4 - 2m^2 + 1)$

$= m^6 - 2m^4 + m^2$

$\phi'_8(m) = 6m^5 - 8m^3 + 2m$

$\phi''_8(m) = 30\, m^4 - 2m + 2$

$\phi_7(m) = 0$

$\phi'_7(m) = 0$

$\phi_6(m) = -(1+m^2)^3$

To find c for equal values of m, we apply the formula.

$$\frac{c^2}{2!}\phi''_8(m) + c\phi'_7(m) + \phi_6(m) = 0$$

or $$\frac{c^2}{2!}(30m^4 - 24m^2 + 2) - (1+m^2)^3 = 2 \qquad (2)$$

(i) When $m = 0, 0$, (2) gives $c^2 - 1 = 0$ or $c = \pm 1$

Hence $y = \pm 1$ and are the asymptotes which have already been determined.

(ii) When $m = 1, 1$, (2) gives

$$\frac{c^2}{2}(30 - 24 + 2) - (1+1)^3 = 0 \quad \text{or} \quad 4c^2 = 8$$

or $c = \pm\sqrt{2}$

Thus $y = x \pm \sqrt{2}$ are asymptotes.

(iii) When $m = -1, -1$, (2) gives

$$\frac{c^2}{2!}(30 - 24 + 2) - 8 = 0 \quad \text{or} \quad c = \pm\sqrt{2}$$

Therefore, the asymptotes are $y = -x \pm \sqrt{2}$

11. $xy^2 = (x+y)^2$

Sol. The given equation can be written as

$$xy^2 - (x + 2xy + y^2) = 0 \qquad (1)$$

Coefficient of the highest power of y *i.e.*, of y^2 in (1) is $x - 1$.

Thus $x - 1 = 0$ is an asymptote parallel to the y-axis.

Now $\phi_3(m) = m^2 = 0$ gives $m = 0, 0$

$\phi'_3(m) = 2m$

$\phi_2(m) = -(1 + 2m + m^2)$

For $m = 0$, $\phi'_3(m) = 0$ but $\phi_2(m) = -1$.

Thus $c\phi'_3(m) + \phi_2(m) = 0$ is not an identity an so there is no asymptotes corresponding to $m = 0$.

12. $xy^2 - x^2y - 3x^2 - 2xy + y^2 + x - 2y + 1 = 0 \qquad (1)$

Sol. Highest power of y in (1) is y^2 and its coefficient is

$x + 1$

An asymptote parallel to the y-axis is $x + 1 = 0$

For oblique asymptotes, we have

$\phi_3(m) - m = 0$ gives

$m(m-1) = 0$ or $m = 0, m = 1$

$\phi'_3(m) = 2m - 1$

$\phi_2(m) = -3 - 2m + m^2$

To find c we apply the formula

$c\phi'_3(m) + \phi_2(m) = 0$

i.e., $c(2m-1) - 3 - 2m + m^2 = 0 \qquad (2)$

(i) When $m = 0$, from (2), we get $-c - 3 = 0$ or $c = -3$

Thus $y = -3$ is an asymptote.

(ii) When $m = 1$, (2) gives $c(1) - 3 - 2 + 1 = 0$ or $c = 4$.

$y = x + 4$

13. $r = \dfrac{a}{\theta}$

Sol. When $r = \infty$, $\theta = 0$. There can be only one asymptote to the curve. Differentiating (1) w.r.t. r, we have

$$1 = -\frac{a}{\theta^2}\frac{d\theta}{dr} \quad \text{or} \quad \frac{d\theta}{dr} = -\frac{\theta^2}{a}$$

Therefore, $\lim\limits_{\theta\to 0} r^2\dfrac{d\theta}{dr} = \lim\limits_{\theta\to 0}\dfrac{a^2}{\theta^2}\cdot\dfrac{-\theta^2}{a} = -a\ (= p).$

Equation of an asymptote is

$p = r\sin(\alpha - \theta)$

i.e., $-a = r\sin(0 - \theta)$ or $a = r\sin\theta$

14. $r = \dfrac{a}{\sqrt{\theta}} \qquad (1)$

Sol. Here $\theta = 0$ when $r = \infty$. Differentiating (1) w.r.t. r, we get

$$1 = \frac{-1}{2}a\theta^{-3/2}\frac{d\theta}{dr} \quad \text{or} \quad \frac{d\theta}{dr} = \frac{-2}{a}\theta^{3/2}$$

or $$r^2\frac{d\theta}{dr} = \frac{a^2}{\theta}\left(\frac{-2}{a}\theta^{3/2}\right) = -2a\sqrt{\theta}$$

$$\lim_{\theta\to 0} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0}(-2a\sqrt{\theta}) = 0\ (= p)$$

Equation of an asymptote is

$$p = r\sin(\alpha - \theta) \quad i.e., \quad 0 = r\sin(0-\theta)$$
$$\sin\theta = 0 \quad i.e., \quad \theta = 0$$

15. $r = a\cos\theta + b$ (1)

Sol. When $r = \infty$, $\theta = 0, \pi, 2\pi, 3\pi \ldots$

Differentiating (1) w.r.t. r, we have

$$1 = -a\csc\theta\cot\theta\frac{d\theta}{dr} = -\frac{a\cos\theta}{\sin^2\theta}\frac{d\theta}{dr}$$

or $$\frac{d\theta}{dr} = -\frac{\sin^2\theta}{a\cos\theta}$$

or $$r^2\frac{d\theta}{dr} = -\left(\frac{a+b\sin\theta}{\sin\theta}\right)^2\cdot\frac{\sin^2\theta}{a\cos\theta} = -\frac{(a+b\sin\theta)^2}{a\cos\theta}$$

$$\lim_{\theta\to 0} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0}\frac{-(a+b\sin\theta)^2}{a\cos\theta} = -a$$

And $$\lim_{\theta\to 0} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0}\frac{-(a+b\sin\theta)^2}{a\cos\theta} = a.$$

Thus the limiting value of $r^2\dfrac{d\theta}{dr}$ is $-a$ when $\theta \to 0, 2\pi, 4\pi$ etc. and its limiting value is a when $\theta \to \pi, 3\pi, 5\pi$ etc. Therefore, there are only two asymptotes to the curve whose equations are

$$-a = r\sin(0-\theta) \quad \text{and} \quad a = r\sin(\pi - \theta).$$

But both of these equations yield the single equation

$$r\sin\theta = a$$

Hence there is only one asymptotes of the curve.

16. $r = 2a\sin\theta\tan\theta$ (1)

Sol. When $r = \infty$, $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$

Differentiating (1) w.r.t. r, we have

$$1 = [2a\sin\theta\sec^2\theta + 2a\cos\theta\tan\theta]\frac{d\theta}{dr}$$

$$= \left[\frac{2a\sin\theta}{\cos^2\theta} + 2a\sin\theta\right]\frac{d\theta}{dr} = \frac{2a\sin\theta(1+\cos^2\theta)}{\cos^2\theta}\frac{d\theta}{dr}$$

or $$\frac{d\theta}{dr} = \frac{\cos^2\theta}{2a\sin\theta(1+\cos^2\theta)}$$

or $$r^2\frac{d\theta}{dr} = \frac{4a^2\sin^4\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{2a\sin\theta(1+\cos^2\theta)} = \frac{2a\sin^2\theta}{\cos^2\theta}$$

$$\lim_{\theta\to\pi/2} r^2\frac{d\theta}{dr} = \lim_{\theta\to\pi/2}\frac{2a\sin^3\theta}{1+\cos^2\theta} = 2a$$

And $$\lim_{\theta\to 3\pi/2} r^2\frac{d\theta}{dr} = \lim_{\theta\to 3\pi/2}\frac{2a\sin^2\theta}{1+\cos^2\theta} = -2a$$

Thus $\lim r^2\dfrac{d\theta}{dr} = 2a$ when $\theta \to \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}$ etc.

$\lim r^2\dfrac{d\theta}{dr} = -2a$ when $\theta \to \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, \dfrac{11\pi}{2}$ etc.

Equations of the asymptotes are

$$2a = r\sin\left(\frac{\pi}{2} - \theta\right) \quad \text{and} \quad -2a = r\sin\left(\frac{3\pi}{2} - \theta\right)$$

$i.e.,$ $2a = r\cos\theta$ and $-2a = -r\cos\theta$.

Thus there is only one asymptote whose equation is $r\cos\theta = 2a$.

17. $r\sin 2\theta = a\cos 3\theta$

Sol. Here $r = \dfrac{a\cos 3\theta}{\sin 2\theta}$ (1)

When $r = \infty$, $2\theta = 0, \pi, 2\pi, 3\pi, \ldots$

$i.e.,$ $\theta = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi, \ldots$

Differentiating (1) w.r.t. r, we get

$$1 = \frac{\sin 2\theta(-3a\sin 3\theta) - a\cos 3\theta(2\cos 2\theta)}{\sin^2 2\theta}\frac{d\theta}{dr}$$

Therefore, $$\frac{d\theta}{dr} = \frac{-\sin^2 2\theta}{3a\sin 2\theta\sin 3\theta + 2a\cos 2\theta\cos 3\theta}$$

$$r^2\frac{d\theta}{dr} = \frac{-a^2\cos^2 3\theta}{3a\sin 2\theta\sin 3\theta + 2a\cos 2\theta\cos 3\theta}$$

$\lim r^2\cdot\dfrac{d\theta}{dr} = 0$, when $\theta \to \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ etc.

$\lim r^2\dfrac{d\theta}{dr} = \dfrac{-a^2}{2a}$ when $\theta \to 0, 2\pi, 4\pi$ etc.

$$= \frac{-a}{2}$$

$$\lim_{\theta\to\pi, 3\pi, \text{etc}} r^2\frac{d\theta}{dr} = \frac{-a^2}{-2a} = \frac{a}{2}$$

The asymptotes are

(i) $0 = r\sin\left(\dfrac{\pi}{2} - \theta\right)$

(ii) $\dfrac{-a}{2} = r\sin(0-\theta)$

(iii) $\dfrac{a}{2} = r\sin\sin(\pi - \theta)$.

On simplification, we have

(*i*) as $r\cos\theta = 0$ *i.e.*, $\theta = \frac{\pi}{2}$

(*ii*) as $\frac{a}{2} = r\sin\theta$ and

(*iii*) as $\frac{a}{2} = r\sin\theta$

Thus, there are only two distinct asymptotes viz:

$$\theta = \frac{\pi}{2} \quad \text{and} \quad \frac{a}{2} = r\sin\theta.$$

18. $r = \dfrac{a}{1-\cos\theta}$ (1)

Sol. When $r = \infty$, $\theta = 0, 2\pi, 4\pi$ etc.

Differentiating (1) w.r.t. r, we have

$$1 = \frac{-a\sin\theta}{(1-\cos\theta)^2}\cdot\frac{d\theta}{dr}$$

Therefore, $\dfrac{d\theta}{dr} = \dfrac{-(1-\cos\theta)^2}{a\sin\theta}$

$$r^2\frac{d\theta}{dr} = \frac{a^2}{(1-\cos\theta)^2}\cdot\frac{-(1-\cos\theta)^2}{a\sin\theta} = \frac{-a}{\sin\theta}$$

$$\lim_{\theta\to 0,\,2\pi,\,\text{etc.}} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0,\,2\pi,\,\text{etc.}}\left(\frac{-a}{\sin\theta}\right)$$

which diverges to $-\infty$.

Hence, there is no asymptote of the curve.

19. $r\sin n\theta = a$

Sol. $r = \dfrac{a}{\sin n\theta}$ (1)

When $r = \infty$, $n\theta = k\pi$, where k is an integer.

Differentiating (1) w.r.t. r, we get

$$1 = \frac{-na\cos n\theta}{\sin^2 n\theta}\cdot\frac{d\theta}{dr} \quad \text{or} \quad \frac{d\theta}{dr} = \frac{-\sin^2 n\theta}{na\cos n\theta}$$

$$r^2\frac{d\theta}{dr} = \frac{a^2}{\sin^2 n\theta}\left(\frac{-\sin^2 n\theta}{na\cos n\theta}\right) = \frac{-a}{n\cos n\theta}$$

$$\lim_{\theta\to k\pi/n} r^2\frac{d\theta}{dr} = \lim_{\theta\to k\pi/n}\frac{-a}{n\cos n\theta} = \frac{-a}{n}\sec k\pi,$$

Hence equation of the asymptote is

$$-\frac{a}{n}\sec k\pi = r\sin\left(\frac{k\pi}{n}-\theta\right) = -r\sin\left(\theta-\frac{k\pi}{n}\right)$$

i.e., $\dfrac{a}{n}\sec k\pi = r\sin\left(\theta-\dfrac{k\pi}{n}\right)$.

20. $r(e^\theta - 1) = a(e^\theta + 1)$

Sol. $r = \dfrac{a(e^\theta+1)}{e^\theta-1}$ (1)

Differentiating (1) w.r.t. r, we get

$$1 = \frac{a(e^\theta-1)e^\theta - a(e^\theta+1)e^\theta}{(e^\theta-1)^2}\cdot\frac{d\theta}{dr}$$

Therefore, $\dfrac{d\theta}{dr} = \dfrac{-(e^\theta-1)^2}{2a\,e^\theta}$

$$r^2\frac{d\theta}{dr} = \frac{a^2(e^\theta+1)^2}{(e^\theta-1)^2}\cdot\frac{-(e^\theta-1)^2}{2a\,e^\theta} = \frac{-a(e^\theta+1)^2}{2e^\theta}$$

$$\lim_{\theta\to 0} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0}\frac{-a(e^\theta+1)^2}{2e^\theta}, \text{ since } r\to\infty \text{ when } \theta\to 0$$

$$= \frac{-4a}{2} = -2a$$

Equation of an asymptote is

$$-2a = r\sin(0-\theta) = -r\sin\theta$$

i.e., $r\sin\theta = 2a$,

21. $r^n\sin n\theta = a^n$

Sol. $r^n = \dfrac{a^n}{\sin n\theta}$ (1)

Here, if $r = \infty$ then $n\theta = k\pi$, where k is any integer.

Differentiating (1) w.r.t. r, we get

$$nr^{n-1} = \frac{-na^n\cos n\theta}{\sin^2 n\theta}\cdot\frac{d\theta}{dr}$$

Therefore, $\dfrac{d\theta}{dr} = \dfrac{-r^{n-1}\sin^2 n\theta}{a^n\cos n\theta}$

$$r^2\frac{d\theta}{dr} = \frac{-r^{n+1}\sin^2 n\theta}{a^n\cos n\theta} = \frac{-a^n}{\sin n\theta}\cdot\frac{a}{\sin^{1/n} n\theta}\cdot\frac{\sin^2 n\theta}{a^n\cos n\theta}$$

$$= \frac{-a\sin^{(n-1)/n} n\theta}{\cos n\theta}$$

$$\lim_{\theta\to k\pi/n} r^2\frac{d\theta}{dr} = \lim_{\theta\to k\pi/n}\frac{-a\sin^{(n-1)/n} n\theta}{\cos n\theta} = 0.$$

Equation of an asymptote is $0 = r\sin\left(\dfrac{k\pi}{n}-\theta\right)$

i.e., $\dfrac{k\pi}{n}-\theta = 0$ or $\theta = \dfrac{k\pi}{n}$, where k is any integer.

22. $r^2 \sin\theta = a^2 \cos 2\theta$

Sol. $\dfrac{r^2}{a^2} = \dfrac{\cos 2\theta}{\sin\theta}$ (1)

When $r = \infty$, $\theta = 0, \pi, 2\pi$, etc.

Differentiating (1) w.r.t. r, we get

$$\frac{2r}{a^2} = \frac{\sin\theta(-2\sin 2\theta) - \cos 2\theta\cos\theta}{\sin^2\theta}\frac{d\theta}{dr}$$

Therefore, $\dfrac{d\theta}{dr} = \dfrac{-2r\sin^2\theta}{a^2(2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta)}$

$$r^2\frac{d\theta}{dr} = \frac{a^2\cos 2\theta}{\sin\theta} \times \frac{-2r\sin^2\theta}{a^2(2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta)}$$

$$= \pm\frac{2\cos 2\theta\sin\theta}{2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta} \times \sqrt{\frac{a^2\cos 2\theta}{\sin\theta}}$$

$$= \pm\frac{2a\cos 2\theta\sqrt{\sin\theta}\sqrt{\cos 2\theta}}{2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta}$$

Taking limits as $\theta \to 0, \pi, 2\pi$ etc.

$$\lim_{\theta\to 0 \text{ etc.}} r^2\frac{d\theta}{dr} = \pm\lim_{\theta\to 0\text{ etc.}}\frac{2a(\cos 2\theta)^{3/2}\sqrt{\sin\theta}}{2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta} = 0$$

Equation of an asymptote is

$0 = r\sin(0 - \theta)$ *i.e.*, $\theta = 0$.

Exercise Set 7.2 (Page 293)

Locate the points of relative extreme of each of the following curves (Problems 1 - 10):

1. $f(x) = 2x^3 - 15x^2 + 36x + 10$

Sol. Here $f(x) = 2x^3 - 15x^2 + 36x + 10$

$f'(x) = 6x2 - 30x + 36$

For points of maxima and minima

$f'(x) = 0$ gives $6(x^2 - 5x + 6) = 0$

or $(x-2)(x-3) = 0$ *i.e.*, $x = 2, x = 3$

$f''(x) = 12x - 30$

When $x = 2$, $f''(x) = 24 - 30 = -6 < 0$.

Hence f has relative maximum at $x = 2$.

When $x = 3$, $f''(x) = 36 - 30 = 6 > 0$

Therefore, f has relative minimum at $x = 3$.

2. $f(x) = 3x^4 - 4x^3 + 5$

Sol. $f'(x) = 12x^3 - 12x^2$

For extreme values,

$f'(x) = 0$ gives $12x^3 - 12x^2 = 0$

or $12x^2(x-1) = 0$ or $x = 0, x = 1$

$f''(x) = 36x^2 - 24x$

When $x = 1$, $f''(x) = 36 - 24 = 12 > 0$

Thus f has relative minimum at $x = 1$

When $x = 0$, $f''(x) = 0$. The second derivative test fails.

We apply **Theorem, 7.7**

For $x \in\,]-h, h[$, $f'(x)$ does not change sign, h being small. Thus $f'(x)$ has no relative extrema at $x = 0$.

3. $f(x) = 12x^5 - 45x^4 + 40x^3 + 6$

Sol. $f(x) = 12x^5 - 45x^4 + 40x^3 + 6$

$f'(x) = 60x^4 - 180x^3 + 120x^2$

For relative maximum and minimum

$f'(x) = 0 \Rightarrow 60x^4 - 180x^3 + 120x^2 = 0$

or $60x^2(x^2 - 3x + 2) = 0$ or $60x^2(x-2)(x-1) = 0$

i.e., $x = 0, 2, 1$

$f''(x) = 240x^3 - 540x^2 + 240x$

$= 60x(4x^2 - 9x + 4)$

When $x = 2$, $f''(x) = 240 > 0$.

Therefore, f has relative minimum at $x = 2$

When $x = 1$, $f''(x) = -60 < 0$.

Thus f has relative maximum at $x = 1$.

At $x = 0$, $f''(x) = 0$.

The second derivative test fails. We apply **Theorem 7.7.**

For $x \in\,]-h, h[$, $f(x)$ does not change sign.

Thus $f(x)$ has no extreme at $x = 0$.

4. $f(x) = (x-1)(x-2)(x-3)$

Sol. $f(x) = (x-1)(x^2 - 5x + 6) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(x) = 0$ gives $3x^2 - 12x + 11 = 0$

or $x = \dfrac{12 \pm\sqrt{144 - 132}}{6} = \dfrac{12\pm\sqrt{12}}{6} = \dfrac{6\pm\sqrt{3}}{3} = 2 \pm \dfrac{1}{\sqrt{3}}$

$f''(x) = 6x - 12$

When $x = 2 + \dfrac{1}{\sqrt{3}}$, $f''(x) = 12 + 2\sqrt{3} - 12 = 2\sqrt{3} > 0$

Therefore, f has relative minimum at $x = 2 + \dfrac{1}{\sqrt{3}}$

When $x = 2 - \frac{1}{\sqrt{3}}$, $f''(x) = 12 - 2\sqrt{3} - 12 = -2\sqrt{3} < 0$

Hence f has relative maximum at $x = 2 - \frac{1}{\sqrt{3}}$

5. $f(x) = \sin x \cos 2x$

Sol. $f(x) = \sin x \cos 2x$

$= \sin x (1 - 2\sin^2 x)$

$= \sin x - 2\sin^3 x$

$f'(x) = \cos x - 6\sin^2 x \cos x$

$= \cos x (1 - 6\sin^2 x)$

For maxima and minima, $f'(x) = 0$

$\Rightarrow \quad \cos x = 0 \quad$ or $\quad 1 - 6\sin^2 x = 0$

$\Rightarrow \quad x = \pm\frac{\pi}{2} \quad$ or $\quad 6\sin^2 x = 1 \quad$ or $\quad \sin x = \frac{1}{6}$

or $\quad \sin x = \pm\frac{1}{\sqrt{6}}$

$f''(x) = -\sin x (1 - 6\sin^2 x) + \cos x (-12 \sin x \cos x)$

When $x = \frac{\pi}{2}$, $f''(x) = 5 > 0$ and so f has relative minimum at $x = \frac{\pi}{2}$

When $x = \frac{-\pi}{2}$, $f''(x) = -5 < 0$ and so f has relative maximum at

$x = \frac{-\pi}{2}$.

When $\sin x = \frac{1}{\sqrt{6}}$

$$f''(x) = \frac{1}{\sqrt{6}}(1-1) + \cos x\left(-12\frac{1}{\sqrt{6}}\right)(\cos x) = \frac{-12}{\sqrt{6}}\cos^2 x < 0$$

Thus f has relative maximum at $x = \arcsin\left(\frac{1}{\sqrt{6}}\right)$.

When $\sin x = -\frac{1}{\sqrt{6}}$, we have

$$f''(x) = \cos x\left(-12\cos x.\left(-\frac{1}{\sqrt{6}}\right)\right) = \frac{12}{\sqrt{6}}\cos^2 x > 0$$

Therefore, f has relative minimum at $x = \arcsin\left(-\frac{1}{\sqrt{6}}\right)$.

6. $f(x) = a\sec x + b\csc x$, $(0 < a < b)$

Sol. $f(x) = a\sec x + b\csc x$, $(0 < a < b)$ (1)

$f'(x) = a\sec x\tan x - b\csc x\cot x$

$$= \frac{a}{\cos x} \times \frac{\sin x}{\cos x} - \frac{b}{\sin x} \cdot \frac{\cos x}{\sin x} = \frac{a\sin x}{\cos^2 x} - \frac{b\cos x}{\sin^2 x}$$

$$= \frac{a\sin^3 x - b\sin^3 x}{\sin^2 x\cos^2 x} \quad (2)$$

For extreme values,

$f'(x) = 0 \quad$ gives $\quad a\sin^3 x - b\cos^3 x = 0$

or $\quad a\sin^3 x = b \quad \Rightarrow \quad \frac{\sin^3 x}{\cos^3 x} = \frac{b}{a}$

$\Rightarrow \quad \tan^3 x = \frac{b}{a} \quad \Rightarrow \quad \tan x = \left(\frac{b}{a}\right)^{1/3}$

$\tan x = \frac{b^{1/3}}{a^{1/3}} \quad$ *i.e.*, $\quad \sin x = \pm\frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$

and $\quad \cos x = \pm\frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$

Since $\tan x$ is +ve, both $\sin x$ and $\cos x$ have the same sign.

Differentiating (2), we get

$$f''(x) = \frac{3a\sin^2 x\cos x + 3b\cos^2 x\sin x}{\sin^2 x\cos^2 x}$$

$+$ term involving $(a\sin^3 x - b\cos^3 x)$

$$= \frac{3(a\sin x + b\cos x)}{\sin x\cos x}$$

$+$ term involving $a(\sin^3 x - b\cos^3 x)$

When $\sin x$ and $\cos x$ are positive

$f''(x) > 0$

Thus f has relative minimum when $\sin x = \frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$.

and $\quad \cos x = \frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$.

When both $\sin x$ and $\cos x$ are negative, $f''(x) < 0$

Thus f has relative at the point x where f maximum

$\sin x = \frac{-b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}} \quad$ and $\quad \cos x = \frac{-a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$.

7. $f(x) = \sin x\cos^2 x$

Sol. $f'(x) = \cos^3 x - 2\sin^2 x\cos x$

$= \cos x(\cos^2 x - 2\sin^2 x)$

$= \cos x(1 - 3\sin^2 x)$

$f'(x) = 0 \quad$ gives $\quad \cos x = 0, \sin^2 x = \frac{1}{3}$

$\Rightarrow \quad x = \pm\frac{\pi}{2}, \sin x = \pm\frac{1}{\sqrt{3}}$

$f''(x) = -\sin x\,(1 - 3\sin^2 x) + \cos x\,(-6\sin x\cos x)$

When $x = \frac{\pi}{2}, f''(x) = 2 > 0$

Thus f has relative minimum at $x = \frac{\pi}{2}$.

When $x = \frac{-\pi}{2}$,

$f''(x) = -2 < 0$ and so f has relative maximum at $x = \frac{-\pi}{2}$.

When $\sin x = \frac{1}{\sqrt{3}}, f''(x) < 0$

and so f has relative maximum at $x = \arcsin\frac{1}{\sqrt{3}}$

When $\sin x = -\frac{1}{\sqrt{3}}, f''(x) > 0$

and so f has relative minimum at $x = \arcsin\left(\frac{-1}{\sqrt{3}}\right)$.

8. $f(x) = e^x \cos(x - a)$

Sol. $f'(x) = re^x \cos(x - a + \theta)$,

Where $r = \sqrt{2}, \theta = \frac{\pi}{4}$ i.e., $f'(x) = \sqrt{2}\,e^x \cos\left(x - a + \frac{\pi}{4}\right)$

For extreme values, $\cos\left(x - a + \frac{\pi}{4}\right) = 0$ gives $x - a + \frac{\pi}{4} = \pm\frac{\pi}{2}$

or $\quad x = a + \frac{\pi}{4}, \qquad a - \frac{3\pi}{4}$

$f''(x) = 2e^x \cos\left(x - a + \frac{\pi}{2}\right)$

Now, when $x - a = \frac{\pi}{4}, f''(x) = -2e^{a+\frac{\pi}{4}}\frac{1}{\sqrt{2}} = -\sqrt{2}\,e^{a+\frac{\pi}{4}} < 0$

Thus f has relative maximum at $x = \frac{\pi}{4} + a$.

When $x - a = -\frac{3\pi}{4}, f''(x) = \sqrt{2}\,e^{a-\frac{3\pi}{4}} > 0$.

Therefore, f has relative minimum at $x = a - \frac{3\pi}{4}$

9. $f(x) = x^x$

Sol. Let $y = f(x) = x^x \quad$ i.e., $\quad y = x^x$

Taking logarithm, we have

$\ln y = x \ln x$

Differentiating, $\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \ln x \,.\, 1 = 1 + \ln x$

or $\quad \frac{dy}{dx} = y\,(1 + \ln x) = x^x\,(1 + \ln x) \qquad (1)$

$\frac{dy}{dx} = 0 \quad$ gives $\quad x^x\,(1 + \ln x) = 0$

Since x^x cannot be zero, we have

$1 + \ln x = 0$

or $\quad \ln x = -1 \qquad$ or $\qquad x = e^{-1} = \frac{1}{e}$

Differentiating (1), we get

$\frac{d^2y}{dx^2} = x^x \,.\, \frac{1}{x}$ + terms involving $(1 + \ln x)$

Putting $x = \frac{1}{e}, \frac{d^2y}{dx^2} > 0$, as the term involving $(1 + \ln x)$ will vanish at $x = \frac{1}{e}$.

Hence $f(x) = x^x$ has a relative minimum at $x = e^{-1}$.

10. $f(x) = \frac{\ln x}{x}, 0 < x < \infty$.

Sol. Let $\quad f(x) = y = \frac{\ln x}{x}$

$$\frac{dy}{dx} = \frac{x \,.\, \frac{1}{x} - \ln x \,.\, 1}{x^2} = \frac{1 - \ln x}{x^2} \qquad (1)$$

For maxima and minima, $\frac{dy}{dx} = 0 \quad$ gives $\quad 1 - \ln x = 0$

$\Rightarrow \quad \ln x = 1 \Rightarrow x = e$

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(-\frac{1}{x}\right) - \frac{2}{x^3}(1 - \ln x) = -\frac{1}{x^3} - \frac{2}{x^3}(1 - \ln x)$$

When $x = e, \frac{d^2y}{dx^2} < 0$. Therefore, $x = e$ is a point of relative maximum.

11. Find the relative extreme of y if $r = 1 - \sin\theta$

Sol. By **(6.29)**, we have

$$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}} = \frac{(1-\sin\theta)\cos\theta - \sin\theta\cos\theta}{-\sin\theta(1-\sin\theta) - \cos^2\theta}$$

$$= \frac{\cos\theta(1-2\sin\theta)}{-\sin\theta(1-\sin\theta)-\cos^2\theta}$$

For extrema, $\frac{dy}{dx} = 0$ gives

$$\cos\theta(1-2\sin\theta) = 0$$

$\Rightarrow$ $\cos\theta = 0$ or $1 - 2\sin\theta = 0$ $\Rightarrow$ $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

Critical points are $\left(0, \frac{\pi}{2}\right), \left(2, \frac{3\pi}{2}\right), \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$

We use the first derivative test to check the nature of these critical points.

At $\theta = 91^\circ, \frac{dy}{dx} = \frac{-0.01(1-2\times0.99)}{-0.99(1-0.99)-(-0.01)^2} = \frac{+\text{ive}}{-\text{ive}} = -\text{ive}$

$\theta = 89^\circ, \frac{dy}{dx} = \frac{0.01(1-2\times0.99)}{-0.99(1-0.99)-(0.01)^2} = \frac{-\text{ive}}{-\text{ive}} = +\text{ive}$

Relative minimum at $\left(0, \frac{\pi}{2}\right)$.

At $\theta = 269^\circ, \frac{dy}{dx} = \frac{-0.01(1-2\times(-0.99))}{0.99(1+0.99)-(0.01)^2} = \frac{-\text{ive}}{+\text{ive}} = -\text{ive}$

$\theta = 271^\circ, \frac{dy}{dx} = \frac{0.01(1-2(-0.99))}{0.99(1+0.99)-(0.01)^2} = \frac{+\text{ive}}{+\text{ive}} = +\text{ive}$

Relative minimum at $\left(2, \frac{3\pi}{2}\right)$.

At $\theta = 151^\circ, \frac{dy}{dx} = \frac{-0.87(1-2\times0.48)}{-0.48(1-0.48)-(-0.87)^2} = \frac{-\text{ive}}{-\text{ive}} = +\text{ive}$

$\theta = 149^\circ, \frac{dy}{dx} = \frac{-0.85(1-2\times0.51)}{-0.51(1-0.51)-(-0.85)^2} = \frac{+\text{ive}}{-\text{ive}} = -\text{ive}$

Relative maximum at $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$.

At, $\theta = 31^\circ, \frac{dy}{dx} = \frac{0.85(1-2\times0.51)}{-0.51(1-0.51)-(0.85)^2} = \frac{-\text{ive}}{-\text{ive}} = +\text{ive}$

$\theta = 29^\circ, \frac{dy}{dx} = \frac{0.87(1-2\times0.48)}{-0.48(1-0.48)-(0.87)^2} = \frac{+\text{ive}}{-\text{ive}} = -\text{ive}$

Relation maximum at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$.

Thus y has relative maximum at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$ and relative minimum at $\left(0, \frac{\pi}{2}\right), \left(2, \frac{3\pi}{2}\right)$.

12. Find the point on the straight line $2x - 7y + 5 = 0$ that is closest to the origin.

Sol. Let $P(a, b)$ be a point on the line

$$2x - 7y + 5 = 0 \tag{1}$$

The distance p of $P(a, b)$ from the origin is

$$p = \sqrt{a^2 + b^2} \tag{2}$$

Since $P(a, b)$ lies on (1), we have

$$2a - 7b + 5 = 0 \tag{3}$$

The point $P(a, b)$ is closest to the origin if the distance p is minimum.

From (3), we get $b = \frac{(2a+5)}{7}$

Substituting this value of b into (2) and after squaring, we obtain

$$49p^2 = 53a^2 + 20a + 25$$

Differentiating w.r.t. a, we have

$$98p\frac{dp}{da} = 106a + 20 \text{ or } \frac{dp}{da} = \frac{1}{98p}(106a + 20)$$

$$\frac{dp}{da} = 0 \quad \text{implies } a = \frac{-10}{53}$$

$$\frac{d^2p}{da^2} = -\frac{1}{98p^2}\frac{dp}{da}(106a + 20) + \frac{1}{98p} \times 106$$

p is minimum for $a = -\frac{10}{53}$, since $\frac{d^2p}{da^2}$ is positive at this point.

When $a = -\frac{10}{53}, b = \frac{245}{371} = \frac{35}{53}$

Hence $P\left(\frac{-10}{53}, \frac{35}{53}\right)$ is the required point.

13. Find the extrema of the radii vectors of the curve

$$\frac{c^4}{r^2} = \frac{a^2}{\sin^2\theta} + \frac{b^2}{\cos^2\theta}; a > 0, b > 0.$$

Sol. $$\frac{c^4}{r^2} = \frac{a^2}{\sin^2\theta} + \frac{b^2}{\cos^2\theta} \tag{1}$$

Differentiating both sides w.r.t. θ, we get

$$-\frac{2c^4}{r^3}\frac{dr}{d\theta} = -\frac{2a^2}{\sin^3\theta}\cos\theta + \frac{2b^2\sin\theta}{\cos^3\theta}$$

$$\frac{dr}{d\theta} = 0 \text{ gives } -\frac{2a^2\cos\theta}{\sin^3\theta} + \frac{2b^2\sin\theta}{\cos^3\theta} = 0$$

or $\quad a^2\cos^4\theta = b^2\sin^4\theta$ or $\dfrac{\sin^4\theta}{\cos^4\theta} = \dfrac{a^2}{b^2}$ or $\tan^2\theta = \dfrac{a}{b}$

$$\sin^2\theta = \frac{a}{a+b} \text{ and } \cos^2\theta = \frac{b}{a+b}$$

Now $\dfrac{dr}{d\theta} = \dfrac{r^3}{c^4}\left(\dfrac{a^2\cos\theta}{\sin^3\theta} - \dfrac{b^2\sin\theta}{\cos^3\theta}\right) = \dfrac{r^3}{c^4}\dfrac{(a^2\cos^4\theta - b^2\sin^4\theta)}{\sin^3\theta\cos^3\theta}$

$$= \frac{r^3}{c^4\sin^3\theta\cos^3\theta}(a^2\cos^4\theta - b^2\sin^4\theta)$$

Differentiating it again, we have

$$\frac{d^2r}{d\theta^2} = \frac{r^4}{c^4\sin^3\theta\cos^3\theta}\begin{pmatrix}(-4a^2\cos^3\theta\sin\theta - 4b^2\sin^3\theta\cos\theta) \\ + \text{ terms involving } (a^4\cos^4\theta - b^2\sin^4\theta)\end{pmatrix}$$

$$= \frac{-4r^4}{c^4\sin^2\theta\cos^2\theta}(a^2\cos^2\theta + b^2\sin^2\theta) + \ldots$$

$$= \frac{-4c^4\sin^2\theta\cos^2\theta}{a^2\cos^2\theta + b^2\sin^2\theta} + \ldots, \qquad \text{using (1)}$$

$$= -\text{ive, when } \sin^2\theta = \frac{a}{a+b},\ \cos^2\theta = \frac{b}{a+b}$$

Thus r is maximum for this value of θ.

The maximum value of r is given by $\dfrac{c^4}{r^2} = \dfrac{a^2}{\dfrac{a}{a+b}} + \dfrac{b^2}{\dfrac{b}{a+b}}$

or $\quad \dfrac{c^4}{r^2} = a(a+b) + b(a+b) = (a+b) + b(a+b) = (a+b)^2$

or $\quad \dfrac{r^2}{c^4} = \dfrac{1}{(a+b)^2}$ or $r = \dfrac{c^2}{a+b}$

Find the points of inflection of each of the following curves (Problems 14 – 17):

14. $y = \dfrac{x^3 - x}{3x^2+1}$

Sol. $y = \dfrac{x^3 - x}{3x^2+1}$ (1)

$$= \frac{1}{3}x - \frac{\frac{4}{3}x}{3x^2+1} = \frac{1}{3}x - \frac{4}{3}\frac{x}{3x^2+1}$$

$$\frac{dy}{dx} = \frac{1}{3} - \frac{4}{3}\frac{3x^2+1-x(6x)}{(3x^2+1)^2} = \frac{1}{3} - \frac{4}{3}\frac{(1-3x^2)}{(3x^2+1)^2}$$

$$\frac{d^2y}{dx^2} = -\frac{4}{3}\left[\frac{(3x^2+1)^2(-6x) - (1-3x^2)\,.\,2(3x^2+1)6x}{(3x^2+1)^4}\right]$$

$$= -\frac{4}{3}\left[\frac{-6x(3x^2+1) - 12x(1-3x^2)}{(3x^2+1)^2}\right]$$

$$= -\frac{4}{3}\left[\frac{-18x^3 - 6x - 12x + 36x^3}{(3x^2+1)^3}\right]$$

$$= -\frac{4}{3}\left[\frac{18x^3 - 18x}{(3x^2+1)^2}\right] = -24\frac{(x^3-x)}{(3x^2+1)^2}$$

$$= -\frac{24x(x^2-1)}{(3x^2+1)^2} \qquad (2)$$

From (1) and (2), the possible points of inflection are

(0, 0), (1, 0), (–1, 0)

If $\quad x < 0, \dfrac{d^2y}{dx^2} < 0$ and if $x > 0, \dfrac{d^2y}{dx^2} > 0$, ($x$ is small).

Thus (0, 0) is a point of inflection.

Similarly, we find that (1, 0) and (–1, 0) are points of inflection.

15. $x = (y-1)(y-2)(y-3)$

Sol. $\dfrac{dx}{dy} = (y-2)(y-3) + (y-1)(y-3) + (y-1)(y-2)$

$$= (y^2 - 5y + 6) + (y^2 - 4y + 3) + (y^2 - 3 + 2)$$

$$= 3y^2 - 12y + 11$$

$$\frac{d^2x}{dy^2} = 6y - 12$$

$$\frac{d^2x}{dy^2} = 0 \quad \text{gives} \quad y = 2$$

When $y = 2, x = 0$

Thus (0, 2) is a possible point of inflection

If $y < 2, \dfrac{d^2x}{dy^2} < 0$ and if $y > 2, \dfrac{d^2x}{dy^2} > 0$. Thus (0, 2) is a point if inflection.

16. $y^2 = x(x+1)^2$ (1)

Sol. Differentiating (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = (x+1)^2 + x\,.\,2(x+1)$$

$$= (x+1)(x+1+2x) = (x+1)(3x+1)$$

$$\frac{dy}{dx} = \frac{(x+1)(3x+1)}{2y} = \frac{(x+1)(3x+1)}{2\sqrt{x}(x+1)}$$

$$= \frac{1}{2}\left(\frac{3x+1}{\sqrt{x}}\right) = \frac{1}{2}(3\sqrt{x} + x^{-1/2})$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{3}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}\right] = \frac{1}{4}x^{-1/2}(3 - x\,x^{-1})$$

$$\frac{d^2y}{dx^2} = 0 \qquad \text{gives} \quad x = \frac{1}{3}$$

Putting $x = \frac{1}{3}$ in (1), we have

$$y^2 = \frac{1}{3}\left(\frac{4}{3}\right)^2 = \frac{16}{27}\ ; \quad y = \pm\frac{4}{3\sqrt{3}}$$

Thus possible points of inflection are

$$\left(\frac{1}{3}, \frac{4}{3\sqrt{3}}\right) \text{ and } \left(\frac{1}{3}, -\frac{4}{3\sqrt{3}}\right)$$

If $\quad x < \frac{1}{3}, \frac{d^2y}{dx^2} < 0 \quad$ and if $\ x > \frac{1}{3}, \frac{d^2y}{dx^2} > 0.$

Thus $x = \frac{1}{3}$ gives point of inflection.

Hence $\left(\frac{1}{3}, \pm\frac{4}{3\sqrt{3}}\right)$ are points if inflection.

17. $a^2y^2 = x^2(a^2 - x^2)$ (1)

Sol. Differentiating (1) w.r.t. x, we get

$$2a^2y\frac{dy}{dx} = 2x(a^2 - x^2) - 2x^3$$

$$= 2a^2x - 2x^3 - 2x^3 = 2x(a^2 - 2x^2)$$

or $\quad \frac{dy}{dx} = \frac{2x(a^2 - 2x^2)}{2a^2 \cdot y} = \frac{x(a^2 - 2x^2)}{a \cdot x\sqrt{a^2 - x^2}} = \frac{a^2 - 2x^2}{a\sqrt{a^2 - x^2}}$

$$a\,\frac{d^2y}{dx^2} = \frac{\sqrt{a^2 - x^2}\,(-4x) - (a^2 - 2x^2) \times -x\,\sqrt{(a^2 - x^2)}}{(a^2 - x^2)}$$

$$= \frac{(a^2 - x^2)(-4x) - x(a^2 - 2x^2)}{(a^2 - x^2)^{3/2}}$$

$$= \frac{-4a^2x + 4x^3 - a^2x + 2x^3}{(a^2 - x^2)^{3/2}}$$

$$= \frac{6x^3 - 5a^2x}{(a^2 - x^2)^{3/2}} = \frac{x(6x^2 - 5a^2)}{(a^2 - x^2)^{3/2}}$$

$\frac{d^2y}{dx^2} = 0$ gives $x = 0$ and so $y = 0$.

Therefore, the possible point of inflection is (0, 0).

For $\ x < 0$ and $x > 0$, $\frac{d^2y}{dx^2}$ changes signs. Therefore, (0, 0) is a point of inflection.

18. Find a and b so that the function f given by $f(x) = ax^3 + bx^2$ has (1, 6) as a point of inflection.

Sol. $f(x) = ax^3 + bx^2$

$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

Since (1, 6) is a point of inflection

$f''(1) = 6a + 2b = 0$

or $\quad b = -a$ (1)

Moreover, $6 = f(1) = a + b$ (2)

Solving (1) and (2) simultaneously, we get

$a = -3 \quad$ and $\quad b = 9.$

19. Find the intervals in which the curves $y = 3x^5 - 40x^3 + 3x - 20$ faces (*i*) upward (*ii*) downward. Also find the points of inflection.

Sol. $y = 3x^5 - 40x^3 + 3x - 20$

$$\frac{dy}{dx} = 15x^4 - 120x^2 + 3$$

$$\frac{d^2y}{dx^2} = 60x^3 - 240x^2 = 60x(x^2 - 4)$$

Now $\frac{d^2y}{dx^2} = 0$, $\quad$ when $\ x = 0, 2, -2$

Hence we consider the intervals

]$-\infty$, -2[,]-2, 0[,]0, 2[and [2, ∞[

In the interval]$-\infty$, -2[$\quad$ *i.e.*, $\quad$ when $\ x < -2$

$\frac{d^2y}{dx^2} < 0$ and so the curve is concave downward

In the interval]-2, 0[, *i.e.*, for $\ -2 < x < 0$

$\frac{d^2y}{dx^2} > 0$ and the curve is concave upward.

In the interval]0, 2[, $\frac{d^2y}{dx^2} < 0$ and so the curve is concave downward. In the interval]2, ∞[*i.e.*, for $\quad 2 < x < \infty$.

$\frac{d^2y}{dx^2} > 0$, and so the curve is concave upward.

The possible points of inflection are

(-2, 198), (0, -20), (2, 238).

From the concavity of the curve, we find that these are all points of inflection.

20. Find the intervals in which the curve $y = (x^2 + 4x + 5)\, e^{-x}$ faces upward or downward. Also find its points of inflection.

Sol. $y = (x^2 + 4x + 5)\, e^{-x}$

$$\frac{dy}{dx} = -(x^2 + 4x + 5)e^{-x} + e^{-x}(2x + 4)$$
$$= -(x^2 + 2x + 1)e^{-x}$$
$$\frac{d^2y}{dx^2} = (x^2 + 2x + 1)e^{-x} - e^{-x}(2x + 2)$$
$$= (x^2 - 1)e^{-x}$$

Now $\frac{d^2y}{dx^2} = 0$ if $x = 1, -1.$

$\frac{d^2y}{dx^2} > 0$ for $x > 1$ and $x < -1$

Thus the curve is concave up in $]1, \infty[$ and $]-\infty, -1[$.

For $-1 < x < 1$, $\frac{d^2y}{dx^2} < 0$. Hence the curve faces down in $]-1, 1[$. The possible points of inflection are at $x = -1, 1$.

i.e., the possible points of inflection are $\left(1, \frac{10}{e}\right)$ and $(-1, 2e)$.

For $x < 1$ $\frac{d^2y}{dx^2} < 0$ and for $x > 1$, $\frac{d^2y}{dx^2} > 0$

Thus $\frac{d^2y}{dx^2}$ changes sign when passing though the point $\left(1, \frac{10}{e}\right)$ and so this is a point of inflection.

Similarly, for $x < -1$, $\frac{d^2y}{dx^2} > 0$ and for $x > -1$, $\frac{d^2y}{dx^2} < 0$.

Therefore, $\frac{d^2y}{dx^2}$ changes sign when passing though the point $(-1, 2e)$ and this is also a point of inflection.

21. Use calculus to show that $5x^2 - 20x + 81 > 0$ for all x.

Sol. $f(x) = 5x^2 - 20x + 81$

$f'(x) = 10x - 20$

$f''(x) = 10$

$f(x)$ has a stationary point at $x = 2$, $f(2) = 61$ *i.e.*, $(2, 61)$ is a critical point of $f(x)$. Since $f''(x) = 10 > 0$, $f(x)$ has absolute minimum at the point $(2, 61)$. The curve is concave up and so $f(x) > 0$ for all x.

22. Show that $x^4 - 4x^3 + 12x^2 + 40 > 0$ for all x.

Sol. $f(x) = x^4 - 4x^3 + 12x^2 + 40$

$f'(x) = 4x^3 - 12x^2 + 24x$

$= 4x(x^2 - 3x + 6)$

$f'(x) = 0 \Rightarrow x = 0$ or $x^2 - 3x + 6 = 0$

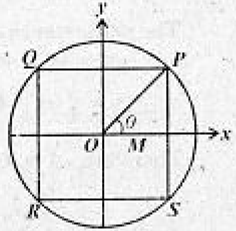
The equation $x^2 - 3x + 6 = 0$ has imaginary roots. Thus $f(x)$ has an extreme value at $x = 0$.

$f''(x) = 12x^2 - 24x + 24$

$f''(0) = 24 > 0$

Thus $f(x)$ has absolute minimum at $x = 0$. The curve is concave up and so $f(x)$ is always positive.

23. Find the dimensions of the rectangle of maximum area that can be inscribed in a circled of radius r.



Sol. Let the length PQ and breath QR of the inscribed rectangle be $2x$ and $2y$ respectively. Suppose O is the centre of the circle and OP makes an angle θ with the line Ox parallel to PQ.

Then $|OM| = x$ and $|PM| = y$.

From the right triangle OMP, we have

$x = OP\cos\theta = r\cos\theta,$

$y = OP\sin\theta = r\sin\theta,$

where $OP = r$ is the radius of the circle. Area A of the rectangle $PQRS$ is

$A = 4xy = 4r^2\sin\theta\cos\theta.$

This area is a function of θ and we need to maximize it.

$$\frac{dA}{d\theta} = 4r^2(\cos^2\theta - \sin^2\theta)$$
$$= 4r^2\cos 2\theta$$

For extreme values, $\frac{dA}{d\theta} = 0$ implies

$\cos 2\theta = 0$ or $\theta = \frac{\pi}{4}$

$$\frac{d^2A}{d\theta^2} = -8r^2\sin 2\theta$$
$$\frac{d^2A}{d\theta^2} = -8r^2 < 0 \text{ if } \theta = \frac{\pi}{4}$$

Thus, $\theta = \frac{\pi}{4}$ gives the rectangle of maximum area. The dimensions of the required rectangle are

$$2x = 2r\cos\frac{\pi}{4} = \sqrt{2}\,r$$

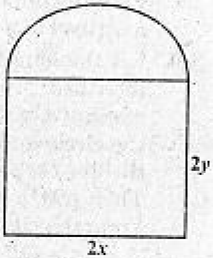
$$2y = 2r \sin\frac{\pi}{4} = \sqrt{2}\, r$$

Thus, the inscribed rectangle of maximum area is a square of side $\sqrt{2}\, r$.

24. A window has the shape of a rectangle surmounted by a semi-circle. Find the dimensions that maximize the area of the window if its perimeter is m metres.

Sol. Let the dimensions of the window be as shown in the figure. Area enclosed is

$$A = 4xy + \frac{1}{2}\pi x^2$$



The perimeter m of the window is

$$m = 4x + 4y + \pi x$$

or $\quad y = \frac{1}{4}(m - 4x - \pi x)$

Therefore, $A = 4x \cdot \frac{1}{4}(m - 4x - \pi x) + \frac{1}{2}\pi x^2$

$$= mx - 4x^2 - \frac{1}{2}\pi x^2$$

$$\frac{dA}{dx} = m - 8x - \pi x$$

For extreme values, $\frac{dA}{dx} = 0$ gives $m - 8x - \pi x = 0$ *i.e.*, $x = \frac{m}{\pi + 8}$

$$\frac{d^2A}{dx^2} = -8 - \pi = -(\pi + 8) < 0$$

Thus, the area is maximum when $x = \frac{m}{\pi + 8}$

$$y = \frac{1}{4}(m - 4x - \pi x) = \frac{m}{\pi + 8}$$

The required dimensions are

Breadth $= \frac{2m}{\pi + 8}$ meters. Height $= \frac{2m}{\pi + 8}$ meters.

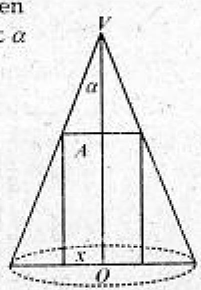
25. Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.

Sol. Let h be the height and α the vertical angle of the cone. Also, let x be the radius of the base of the cylinder.

Let V be the vertex of the cone and O, the centre of the base. Join V to O, cutting the top surface of the cylinder in A. Then height of the cylinder

$$= OA = OV - VA = h - x\cot\alpha$$

Let S be the curved surface of the cylinder. Then



$$S = 2\pi x\,(h - x\cot\alpha) = 2\pi h\, x - 2\pi x^2 \cot\alpha$$

$$\frac{dS}{dx} = 2\pi h - 4\pi x \cot$$

$$\frac{dS}{dx} = 0 \text{ gives } 2\pi h - 4\pi x\cot\alpha = 0$$

or $\quad h = 2x\cot\alpha \quad$ or $\quad x = \frac{h}{2}\tan\alpha$

$$\frac{d^2S}{dx^2} = -4\pi\cot\alpha < 0$$

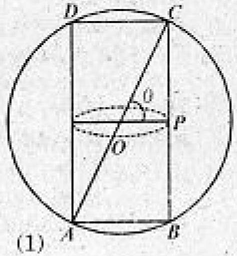
Hence S is maximum when $x = \frac{h}{2}\tan\alpha$

But, radius of the base of the cone $= h\tan\alpha$

Hence the required result.

26. Find the surface of the right cylinder of greatest surface which can be inscribed in a sphere of radius r.

Sol. Let O be the centre of the sphere of radius r and $ABCD$ be any cylinder inscribed in the sphere.



(1)

Let $\angle POC = \theta$.

Height of the cylinder $= |BC| = 2r\sin\theta$

Radius of its base $= |OP| = r\cos\theta$

Total curved surface S of the cylinder is

$$S = 2\pi(r\cos\theta)^2 + 2\pi(r\cos\theta)(2r\sin\theta)$$

$$S = 2\pi(r\cos\theta)^2 + 2\pi(r\cos\theta)(2r\sin\theta)$$

$$= 2\pi r^2(\cos^2\theta + 2\sin\theta\cos\theta)$$

$$= 2\pi r^2(\cos^2\theta + \sin 2\theta)$$

$$\frac{dS}{d\theta} = 2\pi r^2[-2\sin\theta\cos\theta + 2\cos 2\theta]$$

$$= 2\pi r^2[2\cos 2\theta - \sin 2\theta]$$

$\frac{dS}{d\theta} = 0$ gives $\quad \sin 2\theta = 2\cos 2\theta$

$\Rightarrow \quad \tan 2\theta = 2 \quad \Rightarrow \quad 2\theta =$ the acute angle tanarc 2

$$\frac{d^2S}{d\theta^2} = 2\pi r^2[-4\sin 2\theta - 2\cos 2\theta]$$

$$= 2\pi r^2(-2)[2\sin 2\theta - 2\cos 2\theta]$$

$$= -4\pi a^2(2\sin 2\theta + \cos 2\theta)$$

$$= -4\pi a^2\left[2\cdot\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}\right] = -4\pi a^2\left(\frac{5}{\sqrt{5}}\right)$$

$$= -4\pi\sqrt{5}\, a^2 < 0$$

Thus S is maximum when $\tan 2\theta = 2$

i.e., $\sin 2\theta = \dfrac{2}{\sqrt{5}}$ and $\cos 2\theta = \dfrac{1}{\sqrt{5}}$

The maximum value of S

$$= 2\pi r^2\left[\frac{1+\cos 2\theta}{2} + \sin 2\theta\right], \text{ from (1)}$$

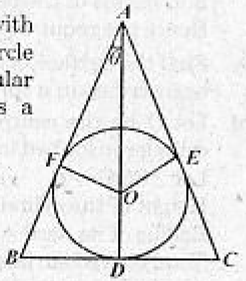
$$= 2\pi r^2 \frac{1}{2}[1 + \cos 2\theta + 2 \text{ in } 2\theta]$$

$$= 2\pi r^2 \frac{1}{2}\left[1 + \frac{1}{\sqrt{5}} + 2 \cdot \frac{2}{\sqrt{5}}\right], \left(\begin{array}{l}\text{Putting the values}\\ \text{of } \cos 2\theta \text{ and } \sin 2\theta\end{array}\right)$$

$$= \pi r^2 [1 + \sqrt{5}].$$

27. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.

Sol. Let ABC be an isosceles triangle with $AB = AC$, O the centre of the inscribed circle and OD, OE, OF respectively perpendicular to BC, CA and AB. Evidently, AOD is a straight line.



Let $\angle OAF = \theta$, then

$AF = AE = r \cot \theta$

$OA = r \csc \theta$

Hence $AD = r + r \csc \theta$

so that $BD = DC = AD \tan \theta$

$P = (r + r \csc \theta) \tan \theta$

$= r (\tan \theta + \sec \theta) = BF = CE$

Perimeter of $\Delta ABC = 2AF + 4BD$

$$= 2r \cot \theta + 4r (\tan \theta + \sec \theta) \quad (1)$$

$$\frac{dP}{d\theta} = 2r[-\csc^2 \theta] + 4r[\sec^2 \theta + \sec \theta \tan \theta]$$

$$= \frac{-2r}{\sin^2 \theta} + 4r\left[\frac{1}{\cos^2 \theta} + \frac{\sin \theta}{\cos^2 \theta}\right]$$

$$= -\frac{2r}{\sin^2 \theta} + \frac{4r(1 + \sin \theta)}{\cos^2 \theta}$$

$$= \frac{-2r \cos^2 \theta + 4r(\sin^2 \theta + \sin^3 \theta)}{\sin^2 \theta \cos^2 \theta}$$

$$= \frac{2r[2\sin^3 \theta - \cos^2 \theta + 2\sin^2 \theta]}{\sin^2 \theta \cos^2 \theta}$$

$$= \frac{2r[2\sin^3 \theta + 3\sin^2 \theta - 1]}{\sin^3 \theta \cos^2 \theta}$$

$$= \frac{2r}{\sin^2 \theta \cos^2 \theta}(2\sin \theta - 1)(\sin \theta + 1)^2$$

$\dfrac{dP}{d\theta} = 0$ gives $\sin \theta = \dfrac{1}{2}, -1$

But $\sin \theta = -1$ is inadmissible. So $\theta = \dfrac{\pi}{6}$

$\dfrac{d^2P}{d\theta^2} = \dfrac{2r}{\sin^2 \theta \cos^2 \theta}(2 \cos \theta)(\sin \theta + 1)^2$ + terms involving the factor $(2 \sin \theta - 1)$ which vanishes when $\theta = \dfrac{\pi}{6}$

When $\theta = \dfrac{\pi}{6}, \dfrac{d^2P}{d\theta^2} > 0$

Thus P is minimum when $\theta = \dfrac{\pi}{6}$

Minimum value of $P = 2r \cot \dfrac{\pi}{6} + 4r\left(\tan \dfrac{\pi}{6} + \sec \dfrac{\pi}{6}\right)$, from (1)

$$= 2r\sqrt{2} + 4r\left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)$$

$$= 2\sqrt{3}\,r + 4r(\sqrt{3}) = 6\sqrt{3}\,r.$$

28. A cone is circumscribed to a sphere of radius r. Show that when the volume of the cone is minimum, its altitude is $4r$ and its semi-vertical angle is $\arcsin\left(\dfrac{1}{3}\right)$.

Sol. Please refer to the figure of Problem 27 which represents the section of the sphere and the cone through the axis of the cone. Let $BD = x$ and $AD = y$. Then the volume of the cone is given by

$$V = \frac{1}{3}\pi x^2 y \quad (1)$$

We now find a relation between x and y. We have

$\csc \theta = \dfrac{AO}{OF} = \dfrac{y - r}{r}$ and $\cot \theta = \dfrac{AD}{BD} = \dfrac{y}{x}$

Now $\csc^2 \theta - \cot^2 \theta = 1 \Rightarrow \dfrac{(y-r)^2}{r^2} - \dfrac{y^2}{x^2} = 1$

$\Rightarrow x^2 (y - r)^2 - r^2 y^2 = r^2 x^2$

$\Rightarrow x^2 (y^2 + r^2 - 2ry) - r^2 y^2 = r^2 x^2$

$\Rightarrow x^2 y^2 + x^2 r^2 - 2rx^2 y - r^2 y^2 = r^2 x^2$

$\Rightarrow y(x^2 y - 2rx^2 - r^2 y) = 0$

$\Rightarrow (x^2 - r^2) y = 2rx^2$

$$\Rightarrow y = \frac{2r x^2}{x^2 - r^2} \quad (2)$$

Substituting the value of y from (2) into (1), we get

$$V = \frac{1}{3}\pi x^2 \frac{2rx^2}{x^2 - r^2} = \frac{2}{3}\pi r\left(\frac{x^4}{x^2 - r^2}\right)$$

$$\frac{dV}{dx} = \frac{2}{3}\pi r \frac{(x^2 - r^2).4x^3 - x^4.2x}{(x^2 - r^2)^2} = \frac{2}{3}\pi r \frac{x^3(x^2 - 2r^2)}{(x^2 - r^2)^2}$$

$$\frac{dV}{dx} = 0 \quad \text{gives} \quad x^2 - 2r^2 = 0$$

$\Rightarrow$ $x = \sqrt{2}\,r$, other value of x being inadmissible.

$$\frac{d^2V}{dx^2} = \frac{2\pi r}{3}\frac{x}{(x^2 - r^2)^2}(2x) + \text{terms involving the factor } (x^2 - 2x^2)$$

When $x = \sqrt{2}\,r$, $\frac{d^2V}{dx^2} > 0$. Therefore,

V is minimum when $x = \sqrt{2}\,r$

From (2), we have, $y = \frac{2r \,.\, 2r^2}{2r^2 - r^2} = \frac{4r^3}{r^2} = 4r.$

Also $\sin\theta = \frac{OF}{OA} = \frac{r}{y - r} = \frac{r}{4r - r} = \frac{1}{3}$

or $\theta = \arcsin\frac{1}{3}$ as required.

29. A farmer has 1000 metres of barbed wire with which he is to fences off three sides of a rectangular field, the fourth side being bounded by a straight canal. How can the farmer enclose the largest field?

Sol. Suppose the length of the side parallel to the canal is x m and that of the one perpendicular to it is y m. Then the area

$A = xy$ is to be maximized subject to

$$x + 2y = 1000 \tag{1}$$

or $x = 1000 - 2y$

Thus $A = 1000y - 2y^2$

$$\frac{dA}{dy} = 1000 - 4y$$

For extreme values, $\frac{dA}{dy} = 0$ gives $y = 250$. It is easy to see that A is maximum for this value of y. From (1), when $y = 250$, $x = 500$. Thus the dimensions of the largest field are 500m by 250 m.

30. A topless rectangular box with a square base is to have volume of 1926 cubic cm. The material for the base costs Rs. 3 per square cm and the material for the sides costs Rs. 2 per square cm. What dimensions should the box have to minimize its cost?

Sol. Let the length of the base be x cm and that of each side be y cm. Then $x^2y = 1296$. (1)

Cost of material for the base $= 3x^2$ Rs.

Cost of material for the four sides $= 8xy$ Rs.

Total cost

$$C = 3x^2 + 8xy \tag{2}$$

This is to be minimized subject to (1). Eliminating y between (1) and (2), we have

$$C = 3x^2 + 8x\,.\,\frac{1296}{x^2} = 3x^2 + \frac{10368}{x}$$

$$\frac{dC}{dx} = x - \frac{10368}{x^2}$$

$$\frac{dC}{dx} = 0 \quad \text{gives} \quad 6x^2 - 10368 = 0$$

or $x^3 = 1728$, $\Rightarrow$ $x = 12$.

C is minimum for this value of x. Putting $x = 12$ in (1), we get $y = 9$. Thus the required dimensions of the box are

12 cm, 12 cm, 9 cm.

31. An open rectangular box is to be made from a sheet of cardboard 8 dm by 5 dm by cutting equal squares from each corner and turning up sides. Find the edge of the square which makes the volume a maximum.

Sol. Let the side of the square cut from each corner be x dm. Then the edges of the box formed by bending the sides are $8 - 2x$, $5 - 2x$ and x decimeters.

Volume V of the box is

$$V = x(8 - 2x)(5 - 2x) : 0 < x < \frac{5}{2}$$

$$= 4x^3 - 26x^2 + 40x$$

$$\frac{dV}{dx} = 12x^2 - 52x + 40$$

For extreme values, $\frac{dV}{dx} = 0$

Therefore, $12x^2 - 52x + 40 = 0$

or $3x^2 - 13x + 10 = 0$ *i.e.*, $x = 1, \frac{10}{3}$

But $x = \frac{10}{3}$ is inadmissible.

$$\frac{d^2V}{dx^2} = 6x - 13 < 0, \text{ for } x = 1$$

Thus V is maximum when $x = 1$ dm.

32. A local train has 1200 passengers at a fare of Rs. 2 each. For every paisa the fare is reduced, 10 more passengers ride the train. What fare should be charged to maximize the revenue?

Sol. Suppose the fare is lowered by x paisa. Then $10x$ more passengers ride the train.
Total number of passengers at the new fare $= 1200 + 10x$.

Total revenue R earned at new fare of Rs.$\left(2 - \frac{x}{100}\right)$ per passenger is

$$R = (1200 + 10x)\left(2 - \frac{x}{100}\right) = 2400 + 8x - \frac{x^2}{100}$$

$$\frac{dR}{dx} = 8 - \frac{x}{5}$$

For extreme values, $\frac{dR}{dx} = 0$ gives $8 - \frac{x}{5} = 0$

or $x = 40$.

R is maximum for this value of x since $\frac{d^2R}{dx^2} < 0$. Thus the fare should be lowered by 40 paisa to earn a maximum revenue.

33. A merchant has 200 quintals of cattle that he can sell at a profit of Rs. 500 per quintal. If the cattle gain 5 quintals per week, but the profit falls by Rs. 10 per quintal per week, when should the cattle be sold to obtain maximum profit?

Sol. Suppose the cattle are sold after x weeks to get maximum profit.
Weight grained by the cattle in x weeks $= 5x$ quintals.
Total weight of cattle after x weeks $= (200 + 5x)$ quintals.
Rate of profit after x weeks $= (500 - 10x)$ Rs. per quintal.
The profit P when the cattle are sold is

$$P = (200 + 5x)(500 - 10x)$$
$$= 100{,}000 + 500x - 50x^2$$

$$\frac{dP}{dx} = 500 - 100x$$

For extreme values $\frac{dP}{dx} = 0$ gives $500 - 100x = 0$ or $x = 5$.

P is maximum for this value of x, since $\frac{d^2P}{dx^2}$ is negative.

Thus the cattle should be sold after 5 weeks to obtain maximum profit.

Exercise Set 7.3 (Page 301)

Determine the nature of the singular point (0, 0) (Problems 1 – 4):

1. $(x^2 + y^2)^2 = 4a^2xy$

Sol. Tangents at the origin are

$xy = 0$

i.e., $x = 0$ and $y = 0$

Since they are real and distinct, the origin is a node

2. $y^2(a^2 - x^2) = x^2(b - x)^2$

Sol. The given equation can be written as

$$a^2y^2 - x^2y^2 = x^2(b^2 - 2bx + x^2)$$

or $x^4 + x^2y^2 - 2bx^3 + b^2x^2 - a^2y^2 = 0$

Equating the lowest degree terms to zero, we get

$b^2x^2 - a^2y^2 = 0$ or $b^2x^2 = a^2y^2$

or $\pm bx = ay$ or $y = \pm\frac{b}{a}x$

Since these tangents at the origin are real and distinct, the origin is a node.

3. $(x^2 + y^2)(2a - x) = b^2x$

Sol. $(x^2 + y^2)x - 2a(x^2 + y^2) + b^2x = 0$

The lowest degree terms equated to zero give

$x = 0$

Therefore, the origin is a cusp.

4. $a^2(x^2 - y^2) = x^2y^2$

Sol. Tangents ate the origin are

$a^2(x^2 - y^2) = 0$

$\Rightarrow (x^2 - y^2) = 0 \Rightarrow (x - y)(x + y) = 0$

Since these tangents are real and distinct, the origin is a node.

Find the position and nature of the multiple points on the given curves (Problems 5 – 10):

5. $x^2(x - y) + y^2 = 0$

Sol. Let $f(x, y) = x^3 - x^2y + y^2$ (1)

$f_x = 3x^2 - 2xy$ (2)

$f_y = -x^2 + 2y$ (3)

$f_{xx} = 6x - 2y$ (4)

$f_{xy} = -2x$ (5)

$f_{yy} = 2$ (6)

Putting each of the equations (2) and (3) equal to zero, we get

$3x^2 - 2xy = 0$ (7)

and $-x^2 + 2y = 0$ (8)

From (8), we have $y = \dfrac{x^2}{2}$

Putting it into (7), we have

$3x^2 - x^3 = 0$

$x^2(3 - x) = 0$

or $x = 0$ and $x = 3$

When $x = 0$, $y = 0$

Thus (0, 0) may be a multiple point

When $x = 3$, $y = \dfrac{9}{2}$

Therefore, $\left(3, \dfrac{9}{2}\right)$ may be the another such point

But (0, 0) satisfies (1) and $\left(3, \dfrac{9}{2}\right)$ does not lie on (1).

Hence (0, 0) is the only multiple point.

At this point

$f_{xx} = 0, \quad f_{yy} = 2, \quad f_{xy} = 0$

Now $(f_{xy})^2 - f_{xx} f_{yy} = 0 - 0 = 0$

Hence (0, 0) is a cusp.

6. $y^3 = x^3 + ax^2$

Sol. $f(x, y) = x^3 - y^3 + ax^2 = 0$

$f_x = 3x^2 + 2ax \,;\, f_{xx} = 6x + 2a$

$f_y = -3y^2 \,;\, f_{yy} = -6y$

$f_{xy} = 0$

$f_x = 0$ gives $x(3x + 2a) = 0$ i.e., $x = 0, -\dfrac{2a}{3}$

$f_y = 0$ gives $3y^2 = 0$ i.e., $y = 0$

Hence the possible multiple points are (0, 0), $\left(-\dfrac{2a}{3}, 0\right)$

Of these only (0, 0) satisfies the given equation.

Hence (0, 0) is the only multiple point.

At (0, 0);

$f_{xy} = 0, \quad f_{xx} = 2a, \quad f_{yy} = 0$

$(f_{xy})^2 - f_{xx} f_{yy} = 0$

Hence (0, 0) is a cusp.

7. $x^4 + y^3 - 2x^3 + 3y^2 = 0$

Sol. Let $f(x, y) = x^4 + y^3 - 2x^2 + 3y^2 = 0$

$f_x = 4x^3 - 6x^2$

$f_y = 3y^2 + 6y$

$f_{xy} = 0$

$f_{xx} = 12x^2 - 12x$

$f_{yy} = 6y + 6$

$f_x = 0$ gives $2x^2(2x - 3) = 0$ i.e., $x = 0, \dfrac{3}{2}$

$f_y = 0$ gives $3y(y + 2) = 0$ i.e., $y = 0, -2$

The possible multiple points are

(0, 0) (0, –2), $\left(\dfrac{3}{2}, 0\right), \left(\dfrac{3}{2}, -2\right)$

Of these only (0, 0) satisfies the given equation of the curve and so (0, 0) is a multiple point.

At (0, 0)

$f_{xy} = 0, \quad f_{xx} = 0, \quad f_{yy} = 6$

$(f_{xy})^2 - f_{xx} f_{yy} = 0 - 0 = 0$

Thus (0, 0) is a cusp.

8. $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$

Sol. Let $f(x, y) = x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$

$f_x = 3x^2 + 4x + 2y + 5$

$f_y = 2x - 2y - 2$

$f_{xy} = 2$

$f_{xx} = 6x + 4$

$f_{yy} = -2$

$f_x = 0$ gives $3x^2 + 4x + 2y + 5 = 0$ (1)

$f_y = 0$ gives $2x - 2y - 2 = 0$ or $x - y - 1 = 0$

or $y = x - 1$ (2)

Putting from (2) into (1), we have

$3x^2 + 4x + 2(x - 1) + 5 + 0$ or $3x^2 + 6x + 3 = 0$

or $x^2 + 2x + 1 = 0$ or $x = -1$

When $x = -1$, $y = -2$ from (1)

The possible multiple point is (–1, –2)

It satisfies the given equation, therefore,, it is multiple point.

At (–1, –2)

$f_{xy} = 2, \quad f_{xx} = 2, \quad f_{yy} = -2$

$(f_{xy})^2 - f_{xx} f_{yy} = 4 - 4(-2) = 0$

Hence $(-1, -2)$ is a cusp.

9. $(2y + x + 1)^2 - 4(1 - x)^5 = 0$

Sol. Let $f(x, y) = (2y + x + 1)^2 - 4(1 - x)^5 = 0$

$f_x = 2(2y + x + 1) + 20(1 - x)^4$

$f_y = 4(2y + x + 1)$

$f_x = 0$ gives $2(2y + x + 1) = -(x - 1)^4$

or $2y + x + 1 = -10(x - 1)^4$

$f_y = 0$ gives $2y + x + 1 = 0$

Also $f_{xy} = 4$, $f_{xx} = 2 - 80(1 - x)^3$, $f_{yy} = 8$

Solving $f_x = 0$, $f_y = 0$ simultaneously, we get

$x = 1$ and $y = -1$

Therefore, $(1, -1)$ may be a multiple point.

Since it satisfies the given equation, it is a multiple point.

At $(1, -1)$

$(f_{xy})^2 - f_{xx} f_{yy} = 16 - (2)(8) = 0$

Thus $(1, -1)$ is a cusp.

10. $(y^2 - a^2)^3 + x^4(2x + 3a)^2 = 0$

Sol. $f(x, y) = (y^2 - a^2)^3 + x^4(2x + 3a)^2$

$f_x = 4x^3(2x + 3a)^2 + 4x^4(2x + 3a)$

$= 4x^3(2x + 3a)(2x + 3a + x) = 12x^3(2x + 3a)(x + a)$

$f_x = 0$ gives $x = 0, -\frac{3}{2}a, -a$

$f_y = 3 \,.\, 2y(y^2 - a^2)^2 = 6y(y^2 - a^2)^2$

$f_y = 0$ gives $y = 0$, $y = a$ $y = -a$

Thus the possible multiple points are

$(0, 0)$, $(0, a)$, $(0, -a)$

$\left(-\frac{3}{2}a, 0\right)$, $\left(-\frac{3}{2}a, a\right)$, $\left(-\frac{3}{2}a, -a\right)$

$(-a, 0)$, $(-a, a)$, $(-a, -a)$

Of these nine points the following satisfy the given equation

$(0, \pm a)$, $\left(-\frac{3}{2}a, \pm a\right)$, $(-a, 0)$

These are the multiple points.

$f_{xx} = 36x^2(2x + 3a)(x + a) + 24x^3(x + a) + 12x^2(2x + 3a)$

$f_{yy} = 6(y^2 - a^2)^2 + 24y^2(y^2 - a^2)$

$= 6(y^2 - a^2)[y^2 - a^2 + 4y^2]$

$= 6(y^2 - a^2)(5y^2 - a^2)$

$f_{xy} = 0$

At $(0, \pm a)$, $f_{xx} = 0$, $f_{yy} = 0$

Therefore, $(0, \pm a)$ are multiple points of order higher than two. It can be easily seem that $(-a, 0)$ is a node and $\left(\frac{-3}{2}a, \pm a\right)$ are cusps.

11. Show that the origin is a node, a cusp or an isolated point on the curve $y^2 = ax^2 + ax^3$ according as a is positive, zero, or negative respectively.

Sol. The lowest degree terms equated to zero give

$y^2 - ax^2 = 0$

When a is positive, we get

$y^2 = ax^2$ *i.e.*, $y = \pm\sqrt{a}x$

which are real and distinct. Therefore, the origin is a node.

(i) When $a = 0$, tangents at the origin are $y^2 = 0$ which are real and coincident. Hence the origin is a cusp.

(ii) When a is negative, tangents at the origin are $y = \pm\sqrt{a}\,x$.

Since a is negative, $\sqrt{a}$ is imaginary.

Hence the tangent at the origin are imaginary and the origin is a conjugate point.

Find equations of the tangent at the multiple points of the given curves (Problems 12 – 13):

12. $x^4 - 4ax^3 - 2ay^3 + 4a^2x^2 + 3a^2y^2 - a^4 = 0$

Sol. $f(x, y) \equiv x^4 - 4ax^3 - 2ay^3 + 4a^2x^2 + 3a^2y^2 - a^4$

$f_x = 4x^3 - 12ax^2 + 8a^2x$

$f_y = -6ay^2 + 6a^2y$

$f_x = 0$ gives

$4x^3 - 12ax^2 + 8a^2x = 0 \Rightarrow 4x(x^2 - 3ax + 2a^2) = 0$

$\Rightarrow 4x(x - 2a)(x - a) = 0 \Rightarrow x = 0, a, 2a$

$f_y = 0$ gives

$-6ay^2 + 6a^2y = 0$ or $-6ay(y - a) = 0$

$\Rightarrow y = 0$, $y = a$

The possible multiple points are

$(0, a)$, (a, a), $(2a, a)$

$(0, 0)$, $(a, 0)$, $(2a, 0)$

Of these $(0, a)$, $(a, 0)$, $(2a, a)$ satisfy the given equation of the curve.

Thus $(0, a)$, $(a, 0)$, $(2a, a)$ are the multiple points.

(i) Shifting the origin to $(0, a)$, we get the new equation as

$$x^4 - 4ax^3 + 4a^2x^2 - 2a(y+a)^3 + 3a^2(y+a)^2 - a^4 = 0$$

$$\Rightarrow x^4 - 4ax^3 + 4a^2x^2 - 2a(y^3 + 3y^2a + 3ya^2 + a^3) + 3a^2(y^2 + 2ay + a^2) - a^4 = 0$$

$$\Rightarrow x^4 - 4ax^3 - 2ay^3 + 4a^2x^2 - 6a^2y^2 + 3a^2y^2 + (-6a^3y + 6a^3y) = 0$$

$$\Rightarrow x^4 - 4ax^3 - 2ay^3 + a^2(4x^2 - 3y^2) = 0$$

Tangents at the new origin are $4x^2 - 3y^2 = 0$

or $3y^2 = 4x^2$ or $y = \pm\dfrac{2}{\sqrt{3}}x$

Hence tangents at $(0, a)$ are $y - a = \pm\dfrac{2}{\sqrt{3}}x$.

Shifting the origin to $(a, 0)$, we get the equation of the curve as

$$(x+a)^4 - 4a(x+a)^3 - 2ay^3 + 4a^2(x+a)^2 + 3a^2y^2 - a^4 = 0$$

$$\Rightarrow x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4 - 4a(x^3+3x^2a+3xa^2+a^3) - 2ay^3 + 4a^2(x^2 + 2xa + a^2) + 3a^2y^2 - a^4 = 0$$

$$\Rightarrow x^4 - 2a^2x^2 - 2ay^3 + 3a^2y^2 = 0$$

$$\Rightarrow x^4 - 2ay^3 - 2a^2x^2 + 3a^2y^2 = 0$$

$$\Rightarrow x^4 - 2ay^3 - a^2(2x^2 - 3y^2) = 0$$

Tangents at the new origin are

$$2x^2 - 3y^2 = 0$$

or $3y^2 = 2x^2$ or $y = \pm\dfrac{\sqrt{2}}{\sqrt{3}}x$

Hence equations of the tangents at the multiple point $(a, 0)$ are

$$y = \pm\sqrt{\frac{2}{3}}(x - a)$$

Shifting the origin to $(2a, a)$, we get the equation of the curve as

$$(x+2a)^4 - 4a(x+2a)^3 - 2a(y+a)^3 + 4a^2(x+2a)^2 + 3a^2(y+a)^2 - a^4 = 0$$

$$\Rightarrow x^4 + 4x^3(2a) + 6x^2(2a)^2 + 4x(2a)^3 + (2a)^4 - 4a[x^3 + 3x^2(2a)^2 + 3x(2a)^2 + (2a)^3] - 2a[y^3 + 3y^2a + 3ya^2 + a^3] + 4a^2[x^2 + 2x(2a) + 4a^2] + 3a^2[y^2 + 2ya + a^2] - a^4 = 0$$

$$\Rightarrow x^4 + 8ax^3 + 24a^2x^2 + 32a^3x + 16x^4 - 4ax^3 - 24a^2x^2 - 48a^3x - 32a^4 - 2ay^3 - 6a^2y^2 - 6a^3y - 2a^4 + 4a^2x^2 + 16a^2x + 16a^4 + 3a^2y^2 + 6a^2y + 3a^4 - a^4 = 0$$

$$\Rightarrow x^4 + a(4x^3 - 2y^3) + a^2(4x^2 - 3y^2) = 0$$

Equating to zero the lowest degree terms, we get

$4x^2 - 3y^2 = 0$ as tangents at the new origin.

Equations of these tangents are

$$3y^2 = 4x^2,\ y = \pm\frac{2}{\sqrt{3}}x$$

So equations of the tangents at $(2a, a)$ are

$$y - a = \pm\frac{2}{\sqrt{3}}(x - 2a).$$

13. $(y-2)^2 = x(x-1)^2$

Sol. $f(x, y) = (y-2)^2 - x(x-1)^2$

$$f_x = -(x-1)^2 - 2x(x-1) = -[(x-1)^2 + 2x(x-1)]$$

$$= -(x-1)[x - 1 + 2x] = -(x-1)(3x-1)$$

$f_x = 0$ gives $x = 1, \dfrac{1}{3}$

$f_y = 2(y-2)$

$f_y = 0$ gives $y = 2$

The possible multiple points are $(1, 2)$, $\left(\dfrac{1}{3}, 2\right)$

Of these only $(1, 2)$ satisfies the equation of the curve.

Shifting the origin to $(1, 2)$, we get the new equation as

$$y^2 - (x+1)x^2 = 0$$

$$y^2 - x^2 - x^3 = 0$$

$$x^3 + x^2 - y^2 = 0$$

Tangents at the new origin are

$$x^2 - y^2 = 0$$

or $y^2 = x^2$ or $y = \pm x$

Hence equations of the tangents at $(1, 2)$ are

$y - 2 = \pm(x-1)$	$y - 2 = -(x-1)$
$y - 2 = x - 1$	$= -x + 1$
or $x - y + 1 = 0$	or $x + y = 3$

Tangents at the multiple point $(1, 2)$ are

$x - y + 1 = 0$ and $x + y = 3$

Find the nature of the cusps on the given curves (Problems 14 – 17):

14. $x^2(x-y) + y^2 = 0$

Sol. The curve has coincident tangents $y^2 = 0$ at the origin. Hence the origin is a cusp and the branches of the curve through it are real. Equation of the curve can be written as

$$y^2 - x^2y + x^3 = 0$$

$$y = \frac{x^2 \pm \sqrt{x^4 - 4x^3}}{2} = \frac{x^2 \pm x\sqrt{x^4 - 4x}}{2} = \frac{x^2 \pm x\sqrt{x(x-4)}}{2}$$

The values of y are real only for negative values of x near origin. Hence the origin is cusp.
Also for any particular negative value of x, y has opposite signs *i.e.*, the curve exists on both sides of the x-axis, the cuspidal tangent.
The cusp is of the first species.
Hence origin is a single cusp of the first species.

15. $x^3 + y^3 - 2ay^2 = 0$

Sol. $y^3 = x^3 + ax^2$ (1)

Tangent at the origin are $x^2 = 0$
i.e., the curve has two coincident tangent $x^2 = 0$ at the origin.
From (1) $ax^2 = y^3$ (Neglecting x^3)

or $x^2 = \dfrac{y^3}{a}$ or $x = \pm y\sqrt{\dfrac{y}{a}}$

The values of x are real only for positive values of y.
Hence origin is a single cusp.
Also, for any particular positive value of y, x has opposite signs. *i.e.*, the curve exists on both sides of the y-axis, the cuspidal tangent.
The cusp is of the first species.
Hence the origin is a single cusp of the first species.

16. $x^6 - ayx^4 - a^3x^2y + a^4y^2 = 0$

Sol. $x^6 - ayx^4 - a^3x^2y + a^4y^2 = 0$ (1)

Tangents at the origin are $y^2 = 0$.
Hence origin is a cusp.
(1) can be written as

$$a^4y^2 - a(x^4 + a^2x^2)y + x^6 = 0$$

$$y = \frac{a(x^4 + a^2x^2) \pm \sqrt{a^2(x^4 + a^2x^2)^2 - 4a^4x^6}}{2a^4}$$

$$= \frac{ax^2(x^2 + a^2) \pm \sqrt{a^2\{a^8 + 2a^2x^6 + a^4x^4 - 4a^2x^6\}}}{2a^4}$$

$$= \frac{a^2x^2(x^2 + a^2) \pm \sqrt{a^2(x^4 - a^2x^2)^2}}{2a^4}$$

$$= \frac{ax^2(x^2 + a^2) \pm a(x^4 - a^2x^2)}{2a^4}$$

$$= \frac{ax^2(x^2 + a^2) \pm ax^2(x^2 - a^2)}{2a^4}$$

$$= \frac{x^2[(x^2 + a^2) \pm (x^2 - a^2)]}{2a^3} = \frac{x^4}{a^3}\,;\ \frac{x^2}{a}$$

The values of y are real and positive for both positive and negative values of x.
The curve exists on one side of the x-axis. This shows that (0, 0) is double cusp of the second species.

17. $y^3 = (x - a)^2(2x - a) = 0$

Sol. $y^3 = (x - a)^2(2x - a) = 0$ (1)

Shifting the origin to the point $(a, 0)$, equation (1) becomes

$$y^3 = x^2(2x + a) \quad (2)$$

Tangents to (2) at the new origin are $x^2 = 0$.
Since the tangents are coincident, the new origin is cusp
From (2), neglecting $2x^3$, we get

$ax^2 = y^3$ or $x = \pm\sqrt{\dfrac{y^3}{a}}$

The value of x are real only for positive values of y. Hence the new origin is a single cusp.
Also for any particular positive value of y, x has opposite signs *i.e.*, the curve exists on both sides of the new y-axis, the cuspidal tangent.
The cusp is of the first species.
Hence the point $(a, 0)$ is a single cusp of the first species.

Exercise Set 7.4 (Page 310)

Discuss and sketch each of the following curves:

1. $3ay^2 = x^2(x - a)$

Sol.

I. Since only even power of y occurs in the given equation, the curve is symmetric about the x-axis.

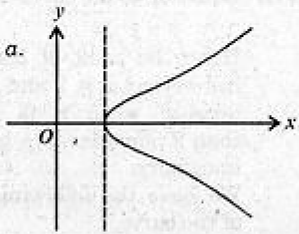
II. It passes through the origin and tangents at the origin are
$x^2 + 3y^2 = 0$ which are imaginary.

III. It has no asymptotes.

IV. It can be written as

$$y^2 = \frac{x^2(x - a)}{3a}$$

$= \pm\dfrac{x\sqrt{x-a}}{\sqrt{3a}}$, y is real only when $x \geq a$.



V. It cuts the x-axis at $(a, 0)$ and $x = a$ is a tangent to the curve at $(a, 0)$.

VI. As x increases beyond a, y also increases.

Hence following is the shape of the curve.

2. $ay^2 = x(a^2 - x^2)$

Sol.

I. It is symmetric about the x-axis.

II. It passes through the origin.
Tangent at the origin is $x = 0$

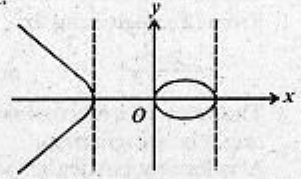
III. It has no asymptotes.

IV. It cuts the x-axis at $x = 0, \pm a$.
$x = \pm a$ are tangents to the curve at $(\pm a, 0)$.
It cuts the y-axis only at the origin.

V. Equation of the curve can be written as

$$y = \pm \frac{\sqrt{x}}{\sqrt{a}} \sqrt{a^2 - x^2}.\ x$$

cannot take any value larger than a.



However, when x is negative, it cannot take any value numerically les than a. The curve is as shown.

3. $y^2 = x^2(4 - x^2)$

Sol.

I. It is symmetric about the x-axis.

II. It is symmetric about the y-axis.

III. On changing x into $-x$ and y into $-y$, the equation of the curve remains unchanged. Therefore, if is symmetric in the opposite quadrants.

IV. The curve is passing through the origin and tangents at the origin are $\ ^2 - 4x^2 = 0$ or $y = \pm 2x$
which are real and distinct.
So the origin is a node.

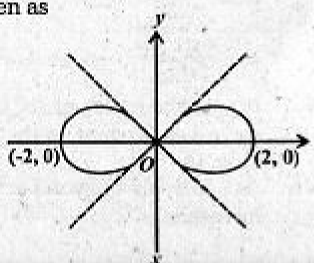
V. It has no asymptotes – oblique or parallel to the coordinate axes.

VI. It cuts the x-axis at $x = 0, x = \pm 2$
It cuts the y-axis at the origin only.

VII. Equation of the curve can be written as

$$y = x\sqrt{4 - x^2}$$

Hence no point of the curve lies beyond $x = 2$ and $x = -2$ because when x is greater than 2 numerically, y becomes imaginary.
We have the following shape of the curve.



4. $x(x^2 + y^2) = a(x^2 - y^2)$

Sol.

I. One changing y into $-y$, equation of the curve remains unchanged.
Hence it is symmetric about the x-axis.

II. It is passing through the origin and tangents at the origin are

$$x^2 - y^2 = 0$$

i.e., $(x - y)(x + y) = 0$ *i.e.,* $y = \pm x$ which are real and distinct.
The origin is a node.

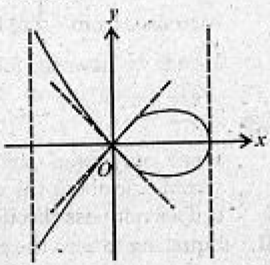
III. Coefficient of y^2 is $x + a$. Therefore,
$x + a = 0$ is an asymptotes parallel to the y-axis.

IV. It cuts the x-axis at $x = a$.
$x - a = 0$ is tangent to the curve at $(a, 0)$.
It cuts the y-axis only at the origin.

V. Equation of t ie curve can be written as

$$y^2 = \frac{x^2(a - x)}{a + x}$$

which shows that no point of the curve lies beyond $x = a$ since y becomes imaginary for $x > a$.
The sketch of the curve is as shown.



5. $y(a^2 + x^2) = a^2x$

Sol.

I. On changing x into $-x$ and y into $-y$, equation of the curve remains unchanged. Therefore, it is symmetric in the opposite quadrants.

II. It passing through the origin. Tangent at the origin is

$$y - x = 0$$

III. Equating to zero the coefficient of the highest power of x, we get
$y = 0$ as an asymptote to the curve

IV. It cuts the coordinate axes only at the origin.

The equation can be written as $y = \dfrac{ax}{a^2 + x^2}$

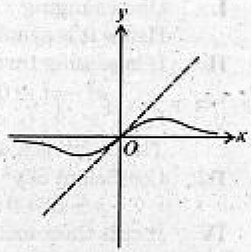
When x is positive, y is positive. When x is negative, y is negative.

$$\frac{dy}{dx} = \frac{(a^2 + x^2)a - ax(2x)}{(a^2 = x^2)^2}$$

$$= \frac{a(a^2 - x^2)}{(a^2 + x^2)^2} \quad \begin{aligned} &> 0 \text{ if } |x| < a \\ &< 0 \text{ if } |x| > a \\ &= 0 \text{ if } |x| = a \end{aligned}$$

Thus the curve has tangents parallel to the x-axis at $\left(a, \frac{1}{2}\right)$ and $\left(-a, -\frac{1}{2}\right)$.

As x increases from 0 to a, y increases from 0 to $\frac{1}{2}$ and as x decreases from 0 to $-a$, y decreases from 0 to $-\frac{1}{2}$. As x increases from a to ∞, y decreases from $\frac{1}{2}$ to 0. As x decreases from $-a$ to $-\infty$, y increases from $-\frac{1}{2}$ to 0.



Hence we have the following sketch of the curve.

6. $y^2x = a(x^2 - a^2)$ (1)

Sol.

I. Since only even powers of y occur in the equation, the curve symmetric about the x-axis.

II It does not pass through the origin.

III. Equating to zero the coefficient of y^3, we get

$x = 0$ as an asymptote.

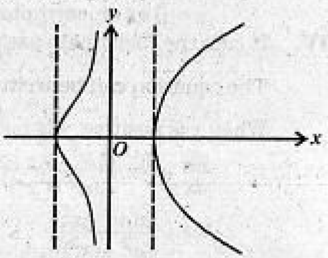
Putting $y = 0$, we get $x = \pm a$. Therefore, the curve meets the x-axis at $x = a$ and $x = -a$.

(1) can be written as $y^2 = \dfrac{a(x^2 - a^2)}{x}$

If x is positive and less than a then y is imaginary. Hence no portion of the curve lies between (0, 0) and $(a, 0)$.

If x is negative and $x^2 < a^2$ then y^2 is positive. If x is negative and $x^2 > a^2$ then y^2 is negative so that y is imaginary. Hence no portion of the curve lies byond $(-a, 0)$.

As x increases from a to ∞, y increases numerically from 0 to ∞.



As x decreases from 0 to $-a$, y increases numerically from 0 to ∞.

$x = \pm a$ are tangents to the curve at $(\pm a, 0)$.

Hence we have sketch of the curve as shown.

7. $x = (y-1)(y-2)(y-3)$

Sol.

I. The curve has no symmetry.

II. It does not pass through the origin.

III. It has no asymptotes.

IV. It cuts the y-axis at $y = 1, y = 2, y = 3$ and the x-axis at $x = -6$.

Equation of the curve may be written as

$$x = y^3 - 6y^2 + 11y - 6$$

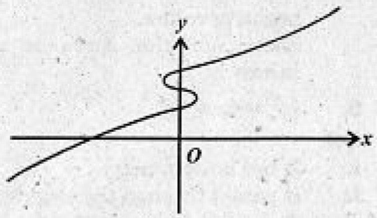
$$\frac{dx}{dy} = 3y^2 - 12y + 11$$

$$\frac{dx}{dy} = 0 \Rightarrow 3y^2 - 12y + 11 = 0$$

$$y = \frac{12 \pm \sqrt{144 - 132}}{6} = \frac{6 \pm \sqrt{3}}{3} = 1.427, 2.577$$

Thus tangents to the curve are parallel to the y-axis at the points where $y = 1.427, 2.577$. As y increase from 1 to 2, x is positive. This portion of the curve lies in the first quadrant. When y increases from 2 to 3, x is negative and this portion of the curve lies in the second quadrant. As y increases from 0 to 1, x remains negative and it increases from -6 to 0. This portion of the curve lies in the second quadrant.

As y increases beyond 3. x increases and remains positive. This portion of the curve lies in the first quadrant. When y is negative, x is also negative and as y increases numerically, x also increases numerically, This portion of the curve lies in the third quadrant.



Hence we have the following sketch.

8. $y^2(2x - 1) = x(x - 1)$ (1)

Sol.

I It is symmetric about the x-axis.

II. It is passing through the origin.

The tangent at the origin is $x = 0$ (y-axis)

III. An asymptote parallel to y-axis is

$$2x - 1 = 0 \qquad i.e., \qquad x = \frac{1}{2}$$

IV. Putting $y = 0$, we get $x = 0, x = 1$

Thus the curve cuts the x-axis at $x = 0, x = 1$.
It does not meet the y-axis.
To shift the origin to (1, 0), we put

$$x - 1 = x' \qquad \text{or} \qquad x = x' + 1$$

Therefore, (1) becomes

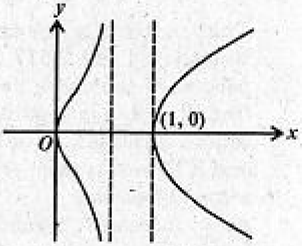
$$y^2 (2x' + 1) = (x' + 1) x'$$

Tangent at the new origin is $x' = 0$
i.e., $x - 1 = 0$ is tangent to the curve at (1, 0).

V. Equation of the curve can be written as

$$y^2 = \frac{x(x-1)}{2x-1}$$

When $0 < x < \frac{1}{2}$, y is real and takes both positive and negative values.



When $\frac{1}{2} < x < 1$, y becomes imaginary and so no part of the curve lies between $x = \frac{1}{2}$ and $x = 1$
Whew $x > 1$, y is real and assumes both positive and negative values.
Sketch of the curve is as shown.

9. $xy^2 = (x + y)^2$

Sol.

I. It has no symmetry

II. It passes through the origin.
Tangents at the origin are $(x + y)^2 = 0$
i.e., There are two coincident tangents
$y = -x$ at the origin.

III. Coefficient of y^2 is $x - 1$. Thus an asymptote parallel to y = axis is $x = 1$

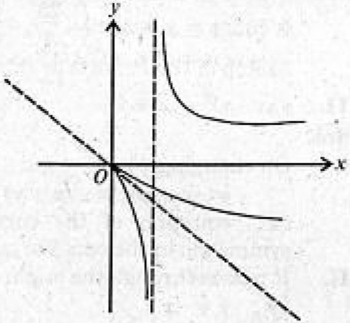
IV. It cuts the coordinate axes only at the origin.
It cuts the asymptote $x = 1$ at $\left(1, -\frac{1}{2}\right)$

V. The equation of the curve can be written as

$$y^2 (x - 1) - 2xy - x^2 = 0$$

Therefore, $y = \dfrac{2x \pm \sqrt{4x^2 + 4x^2(x-1)}}{(2x - 2)}$

$$= \frac{x(1 \pm \sqrt{x})}{x - 1}$$



Thus x cannot take negative values.
When x is positive but < 1, y is negative
When $x > 1$, y decreases as x increases
Sketch of the curve is as shown.

10. $x^2 (y + 1) = x(x - 1)$

Sol.

I. It has no symmetry.

II. It passes through the origin.
Tangents at the origin are

$$x^2 + 4y^2 = 0, \text{ which are imaginary.}$$

Therefore, the origin is an isolated point.

III. Asymptotes parallel to the x-axis is

$$y + 1 = 0 \qquad i.e., \qquad y = -1$$

Asymptotes parallel to y-axis is

$$x - 4 = 0, \qquad x = 4$$

For oblique asymptotes, the equation of the curve can be written as

$$x^2 y - xy^2 + x^2 + 4y^2 = 0$$

Now, $\phi_3(m) = m - m^2 = 0$ gives $m = 0, 1$

$$\phi'_3(m) = 1 - 2m, \qquad \phi_2(m) = 1 + 4m^2$$

To find c, we apply the formula

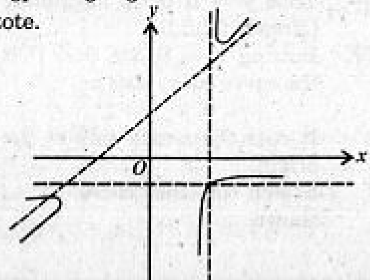
$$c\phi'_3(m) + \phi_2(m) = 0$$

i.e., $c(1 - 2m) + 1 + 4m^2 = 0$ (1)

When $m = 0$, $c = 1$. Thus $y = -1$ is the asymptote already found above.
When $m = 1$, from (1), we have

$$c(1 - 2) + 1 + 4 = 0 \qquad \text{or} \qquad c = 5$$

Hence $y = x + 5$, is an asymptote.



IV. It cuts the coordinate axes at (0, 0) which is an isolated point.
It cuts $y = -1$ at $x = 4$. Thus the curve cuts the asymptote $y = -1$ at (4, –1).
It cuts the asymptote $x = 4$ at $y = -1$, *i.e.*, at (4, –1)

It cuts $y = x + 5$ at $\left(-\frac{25}{4}, -\frac{5}{4}\right)$

Sketch of the curve is as shown.

11. $y(x-y)^2 = x + y$

Sol.

I. On changing x into $-x$ and y into $-y$, equation of curve becomes

$$y(-x+y)^2 = -(x+y) \quad \text{or} \quad y(x-y)^2 = (x+y)$$

i.e., equation of the curve remains unchanged and so it is symmetric in the opposite quadrants.

II. It passes through the origin and tangent there at is $x + y = 0$ *i.e.,* $y = -x$.

III. Coefficient of x^2 is y. Hence $y = 0$ is an asymptote.

For oblique asymptotes, equation of the curve can be written as

$$y(y^2 - 2xy + x^2) - (x + y) = 0$$

Here, $\phi_3(m) = m(m^2 - 2m + 1)$. $\phi_3(m) = 0$ gives $m(m-1)^2 = 0$

$$\phi'_3(m) = 3m^2 - 4m, \; \phi''_3(m) = 6m - 4$$

$$\phi_2(m) = 0, \; \phi'_2(m) = 0$$

$$\phi_1(m) = -1 - m.$$

To find c when $m = 1$, we apply the formula

$$\frac{c^2}{2!}\phi''_3(m) + c\phi'_2(m) + \phi_1(m) = 0$$

i.e., $\frac{c^2}{2}(6m - 4) - (m + 1) = 0$

or $c^2(3m - 2) = m + 1.$ (1)

Putting $m = 1$ in (1), we get $c^2 = 2$ or $c = \pm\sqrt{2}$.

Hence equations of two asymptotes are

$$y = x \pm \sqrt{2}$$

To find c, when $m = 0$, we apply $c\phi'_3(m) + \phi_2(m) = 0$

i.e., $c(3m^2 - 4m) = 0$

or $c = 0$

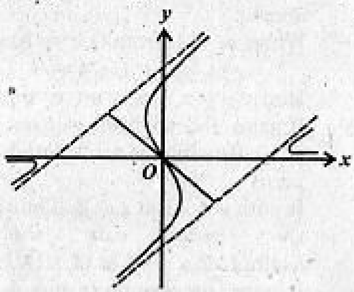
Thus $y = 0$ is an asymptote (already found)

IV. Putting $x = 0$, we find that the curve cuts y-axis at

$$y = 0, \quad y = 1, \quad y = -1$$

It cuts the x-axis only at the origin.

Sketch of the curve is as shown.



12. $y^4 - x^4 + xy = 0$ (1)

Sol.

I. It is symmetric in the opposite quadrants as on changing x into $-x$ and y into $-y$, equation of the curve remains unchanged.

II. It passes through the origin.

Tangents at the origin are $x = 0$, $y = 0$

Thus the origin is a node.

III. There are no asymptotes parallel to the coordinate axes.

For oblique asymptotes, we have

$$\phi_4(m) = m^4 - 1, \; \phi_3(m) = 0$$

$$\phi_4(m) = 0 \quad \text{give} \quad m = \pm 1$$

$$\phi'_4(m) = 4m^3$$

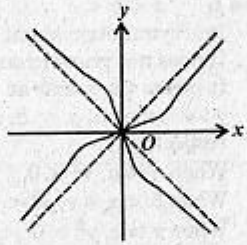
$$c\phi'_4(m) + \phi_3(m) = 0$$

gives $c = 0$.

Hence $y = \pm x$ are oblique asymptotes of (1).

IV. It cuts the coordinate axes at the origin only.

Sketch of the curve is as shown.



13. $x^3 + y^3 = 3ax^2, (a > 0)$

Sol.

I. There is no symmetry about any axis.

II. The curve passes through the origin and tangents at the origin are $x^2 = 0$. Thus the origin is a cusp.

III. There is no asymptote parallel to the axes.

For oblique asymptotes, we have

$$\phi_3(m) = m^3 + 1 = 0 \text{ gives } m = -1$$

$$\phi'_3(m) = 3m^2, \; \phi_2(m) = -3a$$

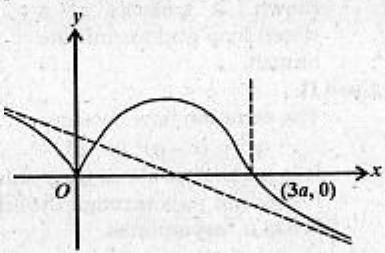
$$c\phi'_3(m) + \phi_2(m) = 0 \quad \text{gives} \quad c(3m^2) - 3a = 0$$

When $m = -1$, $3c = 3a$ or $c = a$

Thus $x + y = a$ is an asymptote of curve.

IV. It cuts the x-axis at $(0, 0)$ and $(3a, 0)$

x and y cannot both be negative, for this would make L.H.S. negative and R.H.S. positive.



Hence no part of the curve exists in the third quadrant.
Sketch of the curve is as shown.

14. $y^2 = (x-a)(x-b)(x-c)$

Sol. We suppose that a, b, c, are positive and consider the following possibilities.

Case I. $a < b < c$
Case II. $a = b < c$
Case III. $a < b = c$
Case IV. $a = b = c$ (I)

Case I. $a < b < c$

It is symmetric about the x-axis.
It does not pass through the origin.
It meets the x-axis at $(a, 0)$, $(b, 0)$ and $(c, 0)$ but it does not meet the y-axis. $x = a$, $x = b$, $x = c$ are tangents at $(a, 0)$, $(b, 0)$ and $(c, 0)$ respectively.

When $x < a, y^2 < 0$, *i.e.*, y is not real when $a < x < b$
When $b < x < c, y^2 < 0$, *i.e.*, y is not real.
When $x > c, y^2 > 0$

Hence no part of the curve lies to the left of the line $x = a$ and between the lines $x = b, x = c$.
As x increases beyond c, y^2 also increases. We have

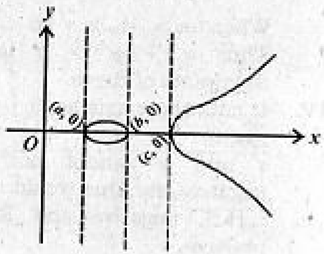
$$2y\frac{dy}{dx} = 3x^2 - 2(a+b+c)x + (ab+bc+ca)$$

$$\text{or}\quad \frac{dy}{dx} = \frac{3x^2 - 2(a+b+c)x + (ab+bc+ca)}{2\sqrt{(x-a)(x-b)(x-c)}}$$

$$= \frac{x^{1/2}\left[3 - 2(a+b+c)\frac{1}{x} + (ab+bc+ca)\left(\frac{1}{x^2}\right)\right]}{\sqrt{\left(1-\frac{a}{x}\right)\left(1-\frac{b}{x}\right)\left(1-\frac{c}{x}\right)}} \longrightarrow \infty$$

as $x \longrightarrow \infty$.

We have the curve as shown. It consists of a closed loop and an infinite branch.



Case II. $a = b < c$

The equation now becomes

$$y^2 = (x-a)^2(x-c)$$

It is symmetric about the y-axis.
It does not pass through the origin.
It has no asymptotes.

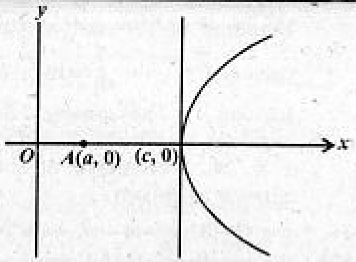
Shifting the origin to $(a, 0)$, equation of the curve becomes

$$y'^2 = x'^2(x'-c)$$

Tangents at the new origin are

$$y'^2 = -cx'^2$$

which are imaginary so that $(a, 0)$ is an isolated point.
The sketch is as shown.



Case III. $a < b = c$.

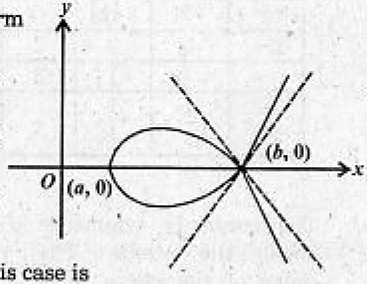
The equation now takes the form

$$y^2 = (x-a)(x-b)^2.$$

It is easily seen that $(b, 0)$ is a node and

$$y = \pm\sqrt{(b-a)}\,(x-b)$$

are the nodal tangents.
The curve may now be easily drawn as shown.



Case IV. $a = b = c$.

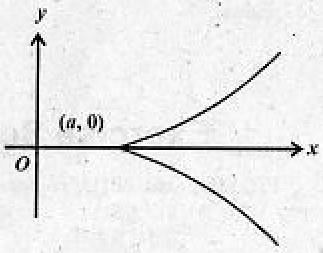
The equation of the curve in this case is

$$y^2 = (x-a)^3$$

Shifting the origin to $(a, 0)$, it takes the from

$$y^2 = x'^3.$$

Tangents at the new origin are $y'^2 = 0$ which are coincident so that $(a, 0)$ is cusp.
The graph of the curve is as shown.



15. $x = t^2 - t + 2, y = t + 3, -\infty < t < \infty$

Sol. Here parametric equations of the curve can be easily transformed into rectangular equation. Substituting $t = y - 3$ from the second equation into the first, we have

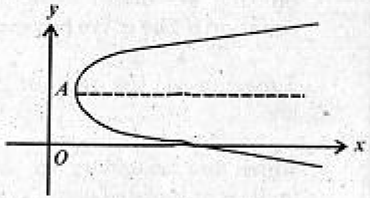
$$x = (y-3)^2 - (y-3) + 2$$

$$\text{or}\quad x = y^2 - 7y + 14 = \left(y - \frac{7}{2}\right)^2 + \frac{7}{2}$$

$$\textit{i.e.,}\quad x - \frac{7}{4} = \left(y - \frac{7}{2}\right)^2$$

which is an equation of a parabola with vertex at $\left(\frac{7}{4}, \frac{7}{2}\right)$.

Equation of the axis of the parabola is $y = \frac{7}{2}$ which is parallel to the x-axis. The parabola meets the x-axis at $x = 14$. The shape of the curve is as shown.

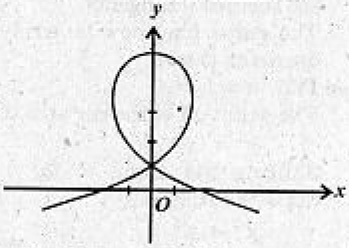


16. $x = t^3 - 3t,\ y = 4 - t^2, -\infty < t < \infty$

Sol. We construct the following table of values

$t =$	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
$x =$	-2	0	2	0	-2	0	2
$y =$	0	1	3	4	3	1	0
$\frac{dy}{dx}$	$\frac{4}{9}$	$\frac{1}{\sqrt{3}}$	∞	0	∞	$\frac{-1}{\sqrt{3}}$	$\frac{-4}{9}$

The curve is symmetric about the y-axis. The shape of the curve is as shown.



Exercise Set 7.5 (Page 316)

1. Find the area of the region bounded by the graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (1)$$

Sol. Here $x = a\cos\theta$ and $y = b\sin\theta$ are parametric equations of the ellipse.

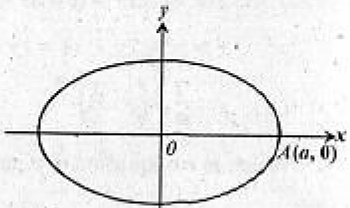
$dx = -a\sin\theta\, d\theta.$

Let A be the required area of the region bounded by the graph of (1).

Then

$$A = 4\int_0^a y\,dx$$

$$= 4\int_{\pi/2}^{0} (b\sin\theta)(-a\sin\theta)\,d\theta$$



$$= 4ab\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 4ab\left(\frac{1}{2}\cdot\frac{\pi}{2}\right), \text{ (by Wallis Formula)}$$

$$= \pi ab.$$

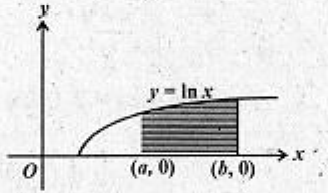
2. Find the area of the region bounded by $y = \ln x$, x-axis, $x = a$, $x = b$

Sol. Area $= \int_a^b y\,dx = \int_a^b \ln x\,dx$

$$= |x\ln x|_a^b - \int_a^b \frac{x}{x}\,dx$$

$$= b\ln b - a\ln a - \int_a^b 1\,.\,dx$$

$$= b\ln b - a\ln a - b + a$$



3. Find the area of the region bounded by $xy = c^2$, x-axis, $x = a$, $x = b$.

Sol. Required area $= \int_a^b y\,dx = \int_a^b \frac{c^2}{x}\,dx = c^2\int_a^b \frac{dx}{x}$

$$= c^2\,|\ln x|_a^b = c^2\,[\ln b - \ln a] = c^2 \ln\frac{b}{a}$$

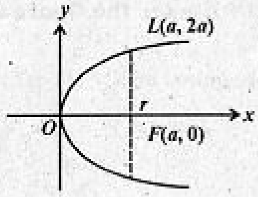
4. Find the area of the region bounded by the graph of the parabola $y^2 = 4ax$ and its latusrectum.

Sol. Required area $= 2\int_0^a y\,dx$

$$= 2\int_0^a \sqrt{4\,ax}\,dx$$

$$= 4\sqrt{a}\int_0^a x^{1/2}\,dx$$

$$= 4\sqrt{a}\left|\frac{x^{3/2}}{3/2}\right|_0^a = 4\sqrt{a}\,.\,\frac{2}{3}\,|x^{3/2}|_0^a = \frac{8}{3}\sqrt{a}\,.\,a^{3/2} = \frac{8a^2}{3}$$



5. Prove that the area of the region bounded by $y = c\cosh\left(\frac{x}{c}\right)$, x-axis, $x = a$, $x = b$ is $c^2\left[\sinh\left(\frac{b}{c}\right) - \sinh\left(\frac{a}{c}\right)\right]$.

Sol. Required area $= \int_a^b y\,dx = \int_a^b c\cosh\frac{x}{c}$

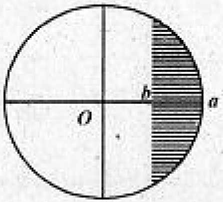
$$= c.c\left|\sinh\frac{x}{c}\right|_a^b = c^2\left[\sinh\left(\frac{b}{c}\right)-\sinh\left(\frac{a}{c}\right)\right]$$

6. Find the area of the smaller segment cut from a circular disc of radius a by a chord at a distance b from the centre, $(a > b)$.

Sol. Let the circular disc be

$$x^2 + y^2 = a^2$$

or $\quad y = \sqrt{a^2 - x^2}$



Required area $= 2\int_b^a y\,dx$

$$= 2\int_b^a \sqrt{a^2 - x^2}\,dx$$

$$= 2\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arcsin\frac{x}{a}\right]_b^a$$

$$= 2\left[\frac{\pi a^2}{4} - \left\{\frac{b\sqrt{a^2-b^2}}{2} + \frac{a^2}{2}\arcsin\frac{b}{a}\right\}\right]$$

$$= \frac{\pi a^2}{4} - \frac{b\sqrt{a^2-b^2}}{2} - a^2\arcsin\frac{b}{a}$$

7. Find the area of the region bounded by the loop of the curve

$$3ay^2 = x\,(x-a)^2$$

Please see the figure in Problem 1 of Exercise Set 6.4

Sol. Required area $= 2\int_0^a y dx = 2\int_0^a \frac{\sqrt{x\,(x-a)^2}}{\sqrt{3a}}\,dx = \frac{2}{\sqrt{3a}}\int_0^a x^{1/2}\,(x-a)\,dx$

$$= \frac{2}{\sqrt{3a}}\int_0^a (x^{3/2} - ax^{1/2})\,dx = \frac{2}{\sqrt{3a}}\left|\frac{2}{5}x^{5/2} - a\,\frac{2}{3}x^{3/2}\right|_0^a$$

$$= \frac{2}{\sqrt{3a}}\left[\frac{2}{5}a^{5/2} - \frac{3}{2}a^{5/2}\right] = \frac{2}{\sqrt{3}\sqrt{a}}\left[\frac{2}{5} - \frac{3}{2}\right]a^{5/2}$$

$$= \frac{2}{\sqrt{3}\sqrt{a}}\cdot\left(-\frac{4}{15}\right)a^{5/2}$$

$$= -\frac{8a^2}{\sqrt{3}\,(15)} = \frac{8a^2}{15\sqrt{3}} \text{ (in absolute value).}$$

8. Find the area of the region lying between the curve

$$x^2\,(x^2 + y^2) = a^2\,(y^2 - x^2) \qquad (1)$$

and its asymptotes.

Sol. The highest power of y in (1) is y^2. Its coefficient is $x^2 - a^2 = 0$. Hence asymptotes parallel to the y-axis are $x = \pm a$.

Equation of the curve is

$$(x^2 - a^2)\,y^2 = -a^2x^2 - x^4$$

or $\quad (a^2 - x^2)\,y^2 = x^2\,(x^2 + a^2) \quad$ or $\quad y^2 = \dfrac{x^2\,(x^2 + a^2)}{a^2 - x^2}$

or $\quad y = \pm\dfrac{x\sqrt{x^2+a^2}}{\sqrt{a^2-x^2}} = \dfrac{x\sqrt{x^2+a^2}}{\sqrt{a^2-x^2}}$, (taking +ve sign)

The curve is symmetric about both the axes. We find the area bounded by the x-axis, $x = 0$, $x = a$ and the curve and multiply it by 4.

$$\text{Area} = 4\int_0^a y\,dx = 4\int_0^a \frac{x\sqrt{x^2+a^2}}{\sqrt{a^2-x^2}}\,dx$$

$$= 4\int_0^a \frac{x\sqrt{a^2+x^2}\,.\sqrt{a^2+x^2}}{\sqrt{a^4-x^4}} = 4\int_0^a \frac{x\,(a^2+x^2)}{\sqrt{a^4-x^4}}\,dx$$

$$= 4\int_0^a \frac{a^2x}{\sqrt{a^4-x^4}}\,dx + 4\int_0^a \frac{x^3}{\sqrt{a^4-x^4}}\,dx$$

$$= 4I_1 + 4I_2 \qquad (2)$$

$I_1 = a^2\displaystyle\int_0^a \frac{x\,dx}{\sqrt{a^4-x^4}}$ | Put $x^2 = a^2\sin\theta$ or $2x\,dx = a^2\cos\theta\,d\theta$

$$= \frac{a^2}{2}\int_0^{\pi/2}\frac{a^2\cos\theta\,d\theta}{a^2\cos\theta} = \frac{a^2}{2}\int_0^{\pi/2} 1\,.\,d\theta = \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4} \qquad (3)$$

$I_2 = \displaystyle\int_0^a x^3\,(a^4 - x^4)^{-1/2}\,dx$ | Put $a^4 - x^4 = t$ or $-4x^3\,dx = dt$

$$= \int_{a^4}^{0} -\frac{1}{4} t^{-1/2}\,dt = \frac{1}{4}\int_{0}^{a^4} t^{-1/2}\,dt$$

$$= \frac{1}{4}\cdot\frac{2}{1}\,|t^{1/2}|_0^{a^4} = \frac{1}{2}\sqrt{a^4} = \frac{1}{2}a^2 \qquad (4)$$

Putting the values from (3) and (4) into (2), we have:

Required area $= 4\left(\frac{\pi a^2}{4}\right) + 4\left(\frac{a^2}{2}\right) = \pi a^2 + 2a^2 = a(\pi^2 + 2).$

9. Find the area of the region bounded by the curve $xy^2 = 4(2 - x)$ and the y-axis.

Sol. Here $y^2 = \frac{4(2-x)}{x}$ or $y = \frac{\pm 2\sqrt{2-x}}{\sqrt{x}}$. Thus the curve is symmetric about the x-axis.

Required area $= 2\int_0^2 y\,dx = 2\int_0^2 \frac{\sqrt{2-x}}{\sqrt{x}}\,dx$

Put $x = 1 + \cos\theta$ or $dx = -\sin\theta\,d\theta$

Now limits of integration are from $-\pi$ to 0

$$\text{Area} = 2\int_{-\pi}^{0} \frac{2.\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}} \times -\sin\theta\,d\theta = 4\int_{0}^{-\pi} \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\sin\theta\,d\theta$$

$$= 4\int_{0}^{-\pi} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$$

$$= 8\int_{0}^{-\pi} \sin^2\frac{\theta}{2}\,d\theta \qquad \left|\; \text{Put } \frac{\theta}{2} = -t \text{ or } d\theta = -2\,dt\right.$$

$$= 8\int_{0}^{\pi/2} \sin^2 t \times (-2\,dt) = -16\int_{0}^{\pi/2} \sin^2 t\,dt$$

$$= -16\,\frac{1}{2}\cdot\frac{\pi}{2} = -4\pi = 4\pi \text{ (in absolute units).}$$

10. Find the area of the region between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Sol. We have $y^2(x^2 - a^2) = a^2x^2$ or $y^2 = \frac{a^2x^2}{a^2 - x^2}$

Asymptotes parallel to y-axis are

$x^2 - a^2 = 0$ i.e., $x = \pm a$.

Since the curve is symmetric about both axis, required area.

$$= 4\int_0^a y\,dx = 4\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\,dx$$

$$= 4\int_0^{\pi/2} \frac{a.\,a\sin\theta.\,a\cos\theta\,d\theta}{a\cos\theta}, \text{ putting } x = a\sin\theta,\ dx = a\cos\theta$$

$$= 4a^2\int_0^{\pi/2} \sin\theta\,d\theta = 4a^2\,|-\cos\theta|_0^{\pi/2} = 4a^2.$$

11. Find the area of the region bounded by the loop of the curve

$ay^2 = x^2(a - x)$

Sol. The curve is symmetric about the x-axis.

$$\text{Area of the loop} = 2\int_0^a y\,dx = 2\int_0^a \frac{x\sqrt{a-x}}{\sqrt{a}}\,dx$$

$$= \int_0^{\pi/2} \frac{a\sin^2\theta.\sqrt{a}\cos\theta.\,2a\sin\theta\cos\theta\,d\theta}{\sqrt{a}},$$

on putting $x = a\sin^2\theta$ and $dx = 2a\sin\theta\cos\theta\,d\theta$

$$= 4a^2\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$$

$$= 4a^2\,\frac{2\,.\,1}{5\,.\,3}, \text{ (by Wallis Formula)} = \frac{8a^2}{15}$$

12. Prove that area of the region bounded by the curve $a^2x^2 = y^3(2a - y)$ is equal to that of the circular disc of radius a.

Sol. The curve is symmetric about the y-axis.

$$\text{Required area} = 2\int_0^{2a} x\,dy = 2\int_0^{2a} \frac{y^{3/2}\sqrt{2a-y}}{a}\,dy.$$

Putting $y = 2a\sin^2\theta$, we obtain $dy = 4a\sin\theta\cos\theta\,d\theta$

and limits of integration are from 0 to $\frac{\pi}{2}$.

Hence area of the region.

$$= 2\int_0^{\pi/2} \frac{(2a)^{3/2}\sin^3\theta\,.\,(2a)^{1/2}\cos\theta\,.\,4a\sin\theta\cos\theta\,d\theta}{a}$$

$$= \frac{2}{a}\int_0^{\pi/2}(2a)^2\,(4a)\sin^4\theta\cos^2\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$$

$$= 32a^2\,\frac{3\,.\,1\,.\,1}{6\,.\,4\,.\,2}\frac{\pi}{2} = \pi a^2 \quad \text{(by Wallis Formula)}$$

= area of the circular disc of radius a.

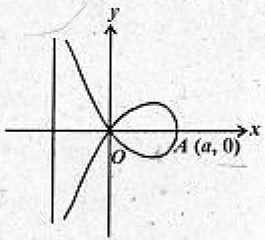
13. Find the area of the region bounded by the loop of the curve $y^2(a + x) = x^2(a - x)$. Also find the area of the region bounded by the curve and its asymptotes.

Sol. The curve $y^2 = x^2\,\dfrac{a - x}{a + x}$ (1)

is symmetric about the x-axis. $a + x = 0$ is its vertical asymptote. x-intercept of the curve is a

From (1), we have $y = \pm x\sqrt{\dfrac{a - x}{a + x}}$

Area of the region bounded by the loop



$$= 2\int_0^a y\,dx = 2\int_0^a x\sqrt{\frac{a - x}{a + x}}$$

Put $x = a\cos\theta$

or $dx = -a\sin\theta\,d\theta$. Then

$$2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx = 2\int_{\pi/2}^0 a\cos\theta\sqrt{\frac{a - a\cos\theta}{a + a\cos\theta}}\,(-a\sin\theta)\,d\theta$$

$$= 2a^2\int_0^{\pi/2}\cos\theta\,\frac{1-\cos\theta}{\sin\theta}\sin\theta\,d\theta$$

$$= 2a^2\int_0^{\pi/2}\left(\cos\theta - \frac{1 + \cos 2\theta}{2}\right)d\theta$$

$$= 2a^2\left[\sin\theta - \frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 2a^2\left(1 - \frac{\pi}{2}\right) = \frac{a^2\,(4 - \pi)}{2}$$

Area of the region bounded by the curve and its asymptote

$$= 2\int_{-a}^0 x\sqrt{\frac{a - x}{a + x}}\,dx\text{, [since } x \text{ is –ive and } y \text{ is +ive]}$$

Substituting $x = a\cos\theta$ as before, the area

$$= 2a^2\int_\pi^{\pi/2}\left(\cos\theta - \frac{1 + \cos 2\theta}{2}\right)d\theta$$

$$= -2a^2\left[\sin\theta - \frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$$

$$= -2a^2\left[-\frac{\pi}{2} - 1 + \frac{\pi}{4}\right] = \frac{a^2\,(4 + \pi)}{2}.$$

14. Find the area of region bounded by one arch of the cycloid $x = a\,(\theta - \sin\theta), y = a\,(1 - \cos\theta)$ and its base.

Sol. Here $x = a\,(\theta - \sin\theta)$, $dx = a\,(1 - \cos\theta)\,d\theta$

Limits of integration are from 0 to 2π.

Required area $\int y\,dx$

$$= \int_0^{2\pi} a\,(1 - \cos\theta)\,.\,a\,(1 - \cos\theta)\,d\theta$$

$$= a^2\int_0^{2\pi}(1 - \cos\theta)^2\,d\theta = \int_0^{2\pi}\left(2\sin^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2\int_0^{2\pi}\sin^4\frac{\theta}{2}\,d\theta$$

$$= 8a^2\int_0^{2\pi}\sin^4 t\,dt \qquad \left(\text{putting } \frac{\theta}{2} = t,\, d\theta = 2dt\right)$$

$$= 16a^2 \int_0^{\pi/2} \sin^4 t\, dt = 16a^2 \frac{3.1}{4.2} \frac{\pi}{2} = 3\pi a^2$$

15. Find the are of the region bounded by the loop of the curve

$$x^4 + y^4 = 2a^2 xy.$$

Sol. When changed to polar coordinates, equation of the curve is

$$r^2 = \frac{2a^2 \sin\theta \cos\theta}{\cos^4\theta + \sin^4\theta}$$

Limits of integration for θ in the fist quadrant are 0 to $\frac{\pi}{2}$.

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{2a^2 \sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta} d\theta$$

$$= a^2 \int_0^{\pi/2} \frac{\sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta} d\theta = a^2 \int_0^{\pi/2} \frac{\tan\theta\sec^2\theta\, d\theta}{1+\tan^4\theta}$$

$$= \frac{a^2}{2}\int_0^{\infty} \frac{dt}{1+t^2} \left(\begin{array}{l}\text{putting } \tan^2\theta = t \text{ so that } 2\tan\theta\sec^2\theta\, d\theta \\ = dt \text{ and limits of } t \text{ are } 0 \text{ to } \infty.\end{array}\right)$$

$$= \frac{a^2}{2} \left|\arctan t\right|_0^{\infty} = \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$$

16. Find the area of the region bounded by the loop of the curve

$$(x+y)^2 (x^2+y^2) = 2a^2 xy.$$

Sol. Changing into polar coordinates, equation of the curve is

$$r^2 (\cos\theta + \sin\theta)^2 = 2a^2 \cos\theta \sin\theta$$

or $$r = \frac{2a^2 \sin\theta\cos\theta}{(\cos\theta+\sin\theta)^2} = \frac{a^2 \sin 2\theta}{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{a^2 \sin 2\theta}{1+\sin 2\theta}$$

$$\text{Area of the loop} = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{a^2 \sin 2\theta}{1+\sin 2\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{(1+\sin 2\theta) - 1}{1+\sin 2\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} 1 \,.\, d\theta - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{1+\sin 2\theta}$$

$$= \frac{a^2}{2} \left|\theta\right|_0^{\pi/2} - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{1+\cos\left(\frac{\pi}{2} - 2\theta\right)}$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2} - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{2\cos^2\left(\frac{\pi}{4} - \theta\right)}$$

$$= \frac{\pi a^2}{4} - \frac{a^2}{4}\int_0^{\pi/2} \sec^2\left(\frac{\pi}{4} - \theta\right) d\theta$$

$$= \frac{\pi a^2}{4} + \frac{a^2}{4}\left|\tan\left(\frac{\pi}{4} - \theta\right)\right|_0^{\pi/2}$$

$$= \frac{\pi a^2}{4} + \frac{a^2}{4}\left[\tan\left(-\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi a^2}{4} + \frac{a^2}{4}[-1-1] = \frac{\pi a^2}{4} - \frac{a^2}{2} = a^2\left(\frac{\pi}{2} - \frac{1}{2}\right)$$

17. Show that the area of the region bounded by the circle $r = a$ and the curve $r = a\cos 5\theta$ is equal to three-fourths of the area of the circle.

Sol. Area of the circle $= \pi a^2$

The curve $r = a\cos 5\theta$ has five loops. It is symmetric about the initial line

It has 10 portions and the limits of integration of one portion are from 0 to $\frac{\pi}{10}$. Area of one portion

$$= \frac{1}{2}\int_0^{\pi/10} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/10} a^2\cos^2 5\theta\, d\theta = \frac{a^2}{2}\int_0^{\pi/10} a^2 \cos^2 5\theta\, d\theta$$

$$= \frac{a^2}{10}\int_0^{\pi/10} \cos^2 t\, dt \qquad \left(\text{putting } 5\theta = t \text{ or } d\theta = \frac{1}{5} dt\right)$$

$$= \frac{a^2}{10} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{40}$$

Hence area of the 10 portions *i.e.*, of all the five loops

$$= 10\left(\frac{\pi a^2}{10}\right) = \frac{\pi a^2}{4}$$

$$\text{Required area} = \pi a^2 - \frac{\pi a^2}{4} = \frac{3}{4}\pi a^2$$

18. Find the area of the region bounded by the cardioid $r = a(1 + \cos\theta)$.

Sol. Required area $= 2 \cdot \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \int_0^{\pi} r^2\, d\theta$

$$= \int_0^{\pi} a^2 (1 + \cos\theta)^2\, d\theta = \int_0^{\pi} a^2 \left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2 \int_0^{\pi} \cos^4\frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2} = t$ or $d\theta = 2dt$ and the limits of integration become 0 and $\frac{\pi}{2}$

$$\text{Required area} = 4a^2 \int_0^{\pi/2} \cos^4 t \cdot 2dt$$

$$= \int_0^{\pi/2} \cos^4 t\, dt = 8a^2\, \frac{3 \cdot 1}{4 \cdot 2}\, \frac{\pi}{2} = \frac{3}{2}\pi a^2$$

19. Find the area of the region bounded by $r^2 = a^2 \cos 2\theta$.

Sol. $r^2 = a^2 \cos 2\theta$ (1)

(1) is symmetric about the initial line.

It is symmetric about the pole.

It is symmetric about the line. $\theta = \frac{\pi}{2}$.

Putting $r = 0$, we have $\cos 2\theta = 0$ and so

$$2\theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{\pi}{4}$$

This curve has four symmetric portions and the limits of integration of one such portion are 0 to $\frac{\pi}{4}$. Hence area of the region

$$= 4 \cdot \frac{1}{2}\int_0^{\pi/4} r^2\, d\theta = 2\int_0^{\pi/4} a^2 \cos 2\theta\, d\theta$$

$$= 2\int_0^{\pi/2} a^2 \cos t \cdot \frac{dt}{2} \qquad \left|\ \begin{array}{l} \text{Put } 2\theta = t \\ \text{or } d\theta = \frac{dt}{2} \end{array}\right.$$

$$= a^2 \int_0^{\pi/2} \cos t\, dt = a^2\, |\sin t|_0^{\pi/2} = a^2$$

20. Find the area of the region bounded by the loop of the folium $x^3 + y^3 = 3axy$.

Sol. The curve is symmetric about the line $y = x$

It meets $y = x$ at $(0, 0)$ and $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

Converting the given equation into polar coordinates, we get

$$r^3(\cos^3\theta + \sin^3\theta) = 3ar^2 \sin\theta\cos\theta$$

or $$r = \frac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$$

Putting $r = 0$, we get $\theta = 0$ and $\theta = \frac{\pi}{2}$ as limits of integration.

$$\text{Area of the loop} = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{9a^2 \sin^2\theta\cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2}\, d\theta$$

$$= \frac{9}{2}a^2 \int_0^{\pi/2} \frac{\tan^2\theta\sec^2\theta}{(1 + \tan^3\theta)^2}\, d\theta$$

Put $1 + \tan^3\theta = z$ so that $3\tan^2\theta\sec^2\theta\, d\theta = dz$ and the limits of integration become $z = 1$ and $z = \infty$

$$\text{Required area} = \frac{3a^2}{2}\int_1^{\infty} \frac{dz}{z^2} = -\frac{3a^2}{2}\left|\frac{1}{z}\right|_1^{\infty} = \frac{3}{2}a^2$$

21. Find the area enclosed by $r = \dfrac{6}{2 - \cos\theta}$

Sol. Here we have

$$2r - r\cos\theta = 6 \tag{1}$$

Transforming into rectangular coordinates (1) becomes

$$2\sqrt{x^2 + y^2} - x = 6$$

or $2^2(x^2 + y^2) = (x + 6)^2$

or $4x^2 + 4y^2 = x^2 + 12x + 36$

i.e., $3(x^2 - 4x + 4) + 4y^2 = 48$

or $3(x - 2)^2 + 4y^2 = 48$

or $\dfrac{(x-2)^2}{16}+\dfrac{y^2}{12}=1$

which is an ellipse with semi-major axis 4 and semi-minor axis $\sqrt{12}$. Hence the area enclosed by the ellipse is

$\pi \cdot 4\sqrt{12} = 8\sqrt{3}\,\pi$. **[See Problem 1]**

22. Find the area of the region that lies outside the cardioid $r = 1 + \cos\theta$ and inside the circle $r = 3\cos\theta$.

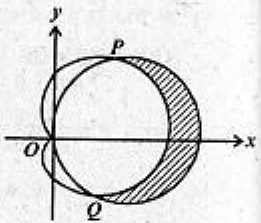
Sol. The required area is the difference of the area enclosed by the circle $r = 3\cos\theta = f(\theta)$ and portion of the area enclosed by $r = 1 + \cos\theta = g(\theta)$. Solving the two equations, we have

$1 + \cos\theta = 3\cos\theta$

or $\cos\theta = \dfrac{1}{2}$ so that $\theta = \pm\dfrac{\pi}{3}$.

Coordinates of the points of intersection are

$P\left(\dfrac{3}{2},\dfrac{\pi}{3}\right)$ and $Q\left(\dfrac{3}{2},\dfrac{-\pi}{3}\right)$



$$\text{Required area} = 2\int_0^{\pi/3}\frac{1}{2}[(f(\theta))^2 - g(\theta)^2]\,d\theta$$

$$= \int_0^{\pi/3}[(3\cos\theta)^2 - (1+\cos\theta)^2]\,d\theta$$

$$= \int_0^{\pi/3}(8\cos^2\theta - 2\cos\theta - 1)\,d\theta$$

$$= \int_0^{\pi/3}[4(1+\cos 2\theta) - 2\cos\theta - 1]\,d\theta$$

$$= [4\theta + 2\sin 2\theta - 2\sin\theta - \theta]\,d\theta = \pi$$

23. Find the area of the region inside the graph of $r^2 = 2a^2\cos 2\theta$ and outside the circle $r = a$.

Sol. $r^2 = 2a^2\cos 2\theta$ and $r = a$

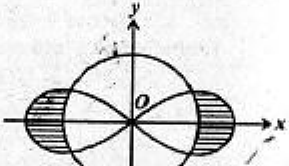
Solving these equations, we have

$1 = 2\cos 2\theta$,

or $\cos 2\theta = \dfrac{1}{2}$

or $2\theta = \dfrac{\pi}{3}, \dfrac{-\pi}{3}$

or $\theta = \dfrac{\pi}{6}, \dfrac{-\pi}{6}$



$$\text{Required area} = \frac{4}{2}\int_0^{\pi/6}(2a^2\cos 2\theta - a^2)\,d\theta$$

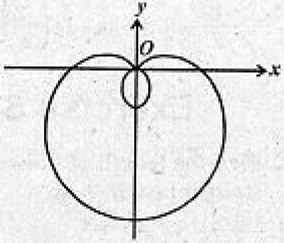
$$= 2\,[a^2\sin 2\theta - a^2\theta]_0^{\pi/6} = 2a^2\left[\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right]$$

$$= a^2\left(\sqrt{3}-\frac{\pi}{3}\right) = \frac{a^2}{3}(3\sqrt{3}-\pi)$$

24. Find the area of the region bounded the smaller loop of the limacon $r = 2 - 3\sin\theta$.

Sol. The left half of the smaller loop of the limacon $r = 2 - 3\sin\theta$ is determined by $\theta = \arcsin\left(\dfrac{2}{3}\right)$ and $\theta = \dfrac{\pi}{2}$

Area enclosed by the smaller loop



$$= 2\int_{\arcsin 2/3}^{\pi/2}\frac{1}{2}(2-3\sin\theta)^2\,d\theta$$

$$= \int_{\arcsin 2/3}^{\pi/2}(4 - 12\sin\theta + 9\sin^2\theta)\,d\theta$$

$$= \left[4\theta + 12\cos\theta + \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta\right]_{\arcsin(2/3)}^{\pi/2}$$

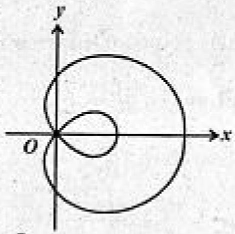
$$= \left[\frac{17}{2}\cdot\frac{\pi}{2} - \frac{17}{2}\arcsin\left(\frac{2}{3}\right) + 12\cos\arcsin\frac{2}{3} - \frac{9}{4}\sin\left(2\arcsin\frac{2}{3}\right)\right]$$

$$= \frac{17\pi}{4} - \frac{17}{2}\arcsin\left(\frac{2}{3}\right) + 12\cdot\frac{\sqrt{5}}{3} - \frac{9}{4}\cdot 2\cdot\frac{2}{3}\cdot\frac{\sqrt{5}}{3}$$

$$= \frac{17\pi}{4} - \frac{17}{2}\arcsin\left(\frac{2}{3}\right) + 3\sqrt{5}.$$

25. Find the area of the region between the two loops of the limacon $r = 2 + 3\cos\theta$.

Sol. The curve is symmetric about the polar axis. The upper half of the outer loop is determined by $\theta = 0$ to $\theta = \arccos\left(-\dfrac{2}{3}\right)$, where r is positive and lower half of the smaller loop is traced by $\theta = \arccos\left(-\dfrac{2}{3}\right)$ to $\theta = \pi$, where r is negative.



$$\text{Required area} = \int_0^{\arccos(-2/3)=u}(2+3\cos\theta)^2\,d\theta - \int_u^{\pi}(2+3\cos\theta)^2\,d\theta$$

$$= \left[\frac{17}{2}\theta + \frac{9}{4}\sin 2\theta + 12\sin\theta\right]_0^u - \left[\frac{17}{2}\theta + \frac{9}{4}\sin 2\theta + 12\sin\theta\right]_u^\pi$$

$$= \frac{17}{2}u + \frac{9}{4}\sin 2u + 12\sin u - \left[\frac{17}{2}\pi - \frac{17}{2}u - \frac{9}{4}\sin 2u - 12\sin u\right]$$

$$= \frac{-17}{2}\pi + 17u + \frac{9}{2}\sin 2u + 24\sin u$$

$$= \frac{-17}{2}\pi + 17u + 9\sin u\cos u + 24\sin u$$

$$\left(\text{where } \cos u = -\frac{2}{3} \text{ and } \sin u = \frac{\sqrt{5}}{3}\right)$$

$$= -\frac{17}{2}\pi + 17\arccos\left(\frac{-2}{3}\right) + 9\left(\frac{-2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) + 24\frac{\sqrt{5}}{3}$$

$$= \frac{-17}{2}\pi + 17\arccos\left(\frac{-2}{3}\right) + 6\sqrt{5}.$$

Exercise Set 7.6 (Page 328)

1. Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the straight line $3y = 8x$.

Sol. We have $y^2 = 4ax$ (1)

and $3y = 8x$ (2)

Putting $y = \frac{8x}{3}$ from (2) into (1), we have

$$\left(\frac{8x}{3}\right)^2 = 4ax \quad\text{or}\quad 64x^2 = 36ax \quad\text{or}\quad x(16x - 9a) = 0$$

i.e., $x = 0, \quad x = \frac{9a}{16}$

So corresponding values of y are

$$y = 0, \quad y = \frac{3a}{2}$$

The points of intersection of (1) and (2) area (0, 0) and $\left(\frac{9a}{16}, \frac{3a}{2}\right)$

From (1), $\frac{dx}{dy} = \frac{y}{2a}$

Therefore, $\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4a^2} = \frac{4a^2 + y^2}{4a^2}$

or $\quad \frac{ds}{dy} = \frac{\sqrt{4a^2 + y^2}}{2a}$

Required length of the arc $= \frac{1}{2a}\int_0^{3a/2}\sqrt{4a^2 + y^2}\,dy$

$$= \frac{1}{2a}\left[\frac{y\sqrt{4a^2 + y^2}}{2} + \frac{4a^2}{2}\ln\frac{y + \sqrt{4a^2 + y^2}}{2a}\right]_0^{3a/2}$$

$$= \frac{1}{4a}\left[\frac{3a}{2}\sqrt{4a^2 + \frac{9a^2}{4}} + 4a^2\ln\frac{\left(\frac{3a}{2} + \sqrt{4a^2 + \frac{9a^2}{4}}\right)}{2a} - 4a^2\ln 1\right]$$

$$= \frac{1}{4a}\left[\frac{3a}{2}\cdot\frac{5a}{2} + 4a^2\ln\frac{\frac{3a}{2} + \frac{5a}{2}}{2a}\right] = \frac{1}{4a}\left[\frac{15a^2}{16} + 4a^2\ln 2\right]$$

$$= \frac{a}{4}\left[\frac{15}{4} + 4\ln 2\right] = a\left[\frac{15}{16} + \ln 2\right] = a\left[\ln 2 + \frac{15}{16}\right]$$

2. Show that in the catenary $y = \cosh\left(\frac{x}{c}\right)$, the length s of the arc from the vertex (where $x = 0$) to any point is given by $s = c\sinh\left(\frac{x}{c}\right)$.

Sol. Here $y = c\cosh\frac{x}{c}$, so $\frac{dy}{dx} = \sinh\frac{x}{c}$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2\frac{x}{c} = \cosh^2\frac{x}{c}$$

or $\quad \frac{ds}{dx} = \cosh\frac{x}{c} \quad$ or $\quad ds = \cosh\frac{x}{c}\,dx$

Required length $= \int_0^x \cosh\frac{x}{c}\,dx = c\left|\sinh\frac{x}{c}\right|_0^x$

$$= c\left(\sinh\frac{x}{c} - 0\right) = c\sinh\frac{x}{c}$$

3. Find the length of the loop of the curve $3ay^2 = x(a - x)^2$.

Sol. The curve is symmetric about the x-axis and meets the x-axis at $x = 0, x = a$. For the loop above the x-axis, we have

$$y = \frac{1}{\sqrt{3a}}\sqrt{x}\,(a - x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{3a}}\left[\frac{a}{2\sqrt{x}} - \frac{3}{2}\sqrt{x}\right] = \frac{1}{2\sqrt{3a}}\left(\frac{a - 3x}{\sqrt{x}}\right)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{12ax}(a^2 - 6ax + 9x^2)$$

$$= \frac{12ax + a^2 - 6ax + 9x^2}{12ax} = \frac{(a+3x)^2}{12ax}$$

Length of the loop above the x-axis

$$= \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^a \frac{a + 3x}{2\sqrt{3a}\,.\sqrt{x}}\, dx$$

$$= \frac{1}{2\sqrt{3a}}\left[\int_0^a \left(\frac{a}{\sqrt{x}} + 3\sqrt{x}\right)\right] dx = \frac{1}{2\sqrt{3a}}\,[2ax\sqrt{x} + 2x^{3/2}]_0^a$$

$$= \frac{1}{2\sqrt{3a}}\,(2a^{3/2} + 2a^{3/2}) = \frac{2a}{\sqrt{3}}$$

Length of the complete loop $= 2, \frac{2a}{\sqrt{3}} = \frac{4a}{\sqrt{3}}$

4. Find the length of the arc of the curve $x = e^\theta \sin\theta$, $y = e^\theta \cos\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

Sol. $x = e^\theta \sin\theta$

$$\frac{dx}{d\theta} = e^\theta \sin\theta + e^\theta\cos\theta = e^\theta(\sin\theta + \cos\theta)$$

$y = e^\theta \cos\theta$

$$\frac{dy}{d\theta} = e^\theta\cos\theta - e^\theta\sin\theta = e^\theta(\cos\theta - \sin\theta)$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= e^{2\theta}[(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2] = 2e^{2\theta}$$

$$\frac{ds}{d\theta} = \sqrt{2}\, e^\theta$$

$$s = \sqrt{2}\int_0^{\pi/2} e^\theta\, d\theta = \sqrt{2}\,|e^\theta|_0^{\pi/2}$$

$$= \sqrt{2}\,|e^{\pi/2} - e^0| = \sqrt{2}\,[e^{\pi/2} - 1]$$

5. Show that the length of the arc if the circle $x^2 + y^2 = a^2$ interpreted between the points where $x = a\cos\alpha$ and $x = a\cos\beta$ is $a(\beta - \alpha)$.

Sol. Parametric equations of the given circle arc

$$x = a\cos\theta, \qquad y = a\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = -a\cos\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\sin^2\theta + a^2\cos^2\theta = a^2$$

or $\frac{ds}{d\theta} = a$ i.e., $ds = a\,d\theta$

$$s = \int_\alpha^\beta a\,d\theta = a\,|\theta|_\beta^\alpha = a(\beta - \alpha).$$

6. Show that the length of the arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ between the points for which $\theta = 0$ and $\theta = 2\alpha$ is $s = 4a\sin\alpha$.

Sol. $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= a^2(1 + \cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$= 2a^2(1 + \cos\theta) = 4a^2\cos^2\frac{\theta}{2}$$

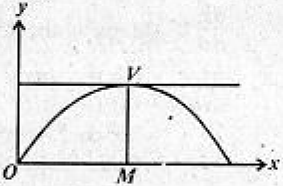
$$\left(\frac{ds}{d\theta}\right)^2 = 4a^2\cos^2\frac{\theta}{2} \text{ or } \frac{ds}{d\theta} = 2a\cos\frac{\theta}{2} \text{ or } ds = 2a\cos\frac{\theta}{2}\,d\theta$$

Hence $s = 2a\int_0^{2a}\cos\frac{\theta}{2}\,d\theta = 2a\left[2\sin\frac{\theta}{2}\right]_0^{2a}$

$$= 4a\left[\sin\frac{\theta}{2}\right]_0^{2a} = 4a\sin a.$$

7. Find the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

Sol. $x = a(\theta - \sin\theta)$



$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

$y = a(1 - \cos\theta)$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = a^2[(1 - \cos\theta)^2 + \sin^2\theta]$$

$$= a^2[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$= a^2\,[2 - 2\cos\theta] = 4a^2 \sin^2\frac{\theta}{2}$$

or $\dfrac{ds}{d\theta} = 2a \sin\dfrac{\theta}{2}$ or $ds = 2a\sin\dfrac{\theta}{2}\,d\theta$

Length of half of one arch

$$= 2a\int_0^{\pi} \sin\frac{\theta}{2}\,d\theta = 2a\left[-2\cos\frac{\theta}{2}\right]_0^{\pi} = 2a\,[-2\,(0) + 2] = 4a$$

Length of the complete arch $= 2\,(4a) = 8a$.

8. Show that the length in one quadrant of the curve $x^{2/3}+y^{2/3} = a^{2/3}$, is equal to $\dfrac{3a}{2}$ and find the length in one quadrant of the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1. \qquad (1)$$

Sol. Parametric equations of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ are

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta, \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$= 9a^2\sin^2\theta\cos^2\theta\,(\cos^2\theta + \sin^2\theta)$$

$$= 9a^2\sin^2\theta\cos^2\theta$$

or $\dfrac{ds}{d\theta} = 3a\sin\theta\cos\theta$ or $ds = 3a\sin\theta\cos\theta\,d\theta$

$$\text{Required length} = s = 3a\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{3a}{2}$$

Parametric equations of the curve (1) are

$$x = a\cos^3\theta, \qquad y = b\sin^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta, \frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= 9a^2\cos^4\theta\sin^2\theta + 9b\sin^4\theta\cos^2\theta$$

$$= 9\sin^2\theta\cos^2\theta\,(a^2\cos^2\theta + b^2\sin^2\theta)$$

$$\frac{ds}{d\theta} = 3\sin\theta\cos\theta\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$$

Limits of integration in the first quadrant are from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

Hence length of the curve in one quadrant is

$$3\int_0^{\pi/2}\sin\theta\cos\theta\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta$$

Put $a^2\cos^2\theta + b^2\sin^2\theta = t^2$

or $(-2a^2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta)\,d\theta = 2t\,dt$

or $(b^2 - a^2)\sin\theta\cos\theta\,d\theta = t\,dt$

or $\sin\theta\cos\theta\,d\theta = \dfrac{t\,dt}{b^2 - a^2}$

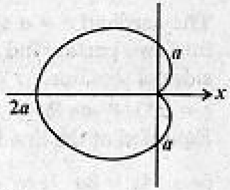
$$\text{Required length} = 3\int_a^b \frac{t\,dt}{b^2 - a^2}\,t = \frac{3}{b^2 - a^2}\int_a^b t^2\,dt$$

$$= \frac{3}{b^2 - a^2}\left[\frac{t^3}{3}\right]_a^b = \frac{b^3 - a^3}{b^2 - a^2}$$

$$= \frac{(b - a)\,(b^2 + ba + a^2)}{(b - a)\,(b + a)} = \frac{a^2 + ab + b^2}{a + b}$$

9. Find the entire length of the cardioid $r = a\,(1 - \cos\theta)$ and show that the arc of the upper half of the curve is bisected by $\theta = \dfrac{2\pi}{3}$.

Sol. $r = a\,(1 - \cos\theta)$



$$\frac{dr}{d\theta} = a\sin\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$= a^2\,(1 - \cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2\,[(1 - \cos\theta)^2 + \sin^2\theta]$$

$$= a^2\,[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$= a^2\,[2 - 2\cos\theta] = 2a^2\,(1 - \cos\theta)$$

$$= 4a^2\sin^2\frac{\theta}{2}$$

$$\frac{ds}{d\theta} = 2a\sin\frac{\theta}{2}$$

Limits of θ for the upper half of the curve are 0 to π

$$\text{Perimeter of the cardioid} = 2\int_0^{\pi} 2a\sin\frac{\theta}{2}\,d\theta = 4a\int_0^{\pi}\sin\frac{\theta}{2}\,d\theta$$

$$= 8a\left[-\cos\frac{\theta}{2}\right]_0^{\pi} = 8a$$

Length of the arc from $\theta = 0$ to $\theta = \frac{2\pi}{3}$

$$= \int_0^{2\pi/3} \frac{ds}{d\theta} d\theta = \int_0^{2\pi/3} 2a \sin\frac{\theta}{2} d\theta$$

$$= -4a\left[\cos\frac{\pi}{3} - 1\right] = -4a\left[\frac{1}{2} - 1\right] = 2a$$

= half of the length of the upper portion which is $4a$.

10. Find the length of the curve $r = \sin^2\left(\frac{\theta}{2}\right)$ from $(0, 0)$ to $(1, \pi)$.

Sol. $r = \sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{2}$

$$\frac{dr}{d\theta} = \frac{\sin\theta}{2}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\left(\frac{1 - \cos\theta}{2}\right)^2 + \frac{\sin^2\theta}{4}} = \sin\left(\frac{\theta}{2}\right)$$

Required length $= \int_0^{\pi} ds = \int_0^{\pi} \sin\left(\frac{\theta}{2}\right) d\theta = \left[2\cos\frac{\theta}{2}\right]_2^{\pi} = 2$

11. The cardioid $r = a\,(1 + \cos\theta)$ is divided by the line $4r\cos\theta = 3a$ into two parts. Find the ratio of the lengths of the arcs on the two sides of this line.

Sol. $r = a\,(1 + \cos\theta)$ (1)

Equation of the line is $4\,r\cos\theta = 3a$ (2)

i.e., $4x = 3a$ *i.e.,* $x = \frac{3a}{4}$

Solving (1) and (2), we have

$$a\,(1 + \cos\theta) = \frac{3a}{4\cos\theta}$$

or $4\cos\theta\,(1 + \cos\theta) = 3$

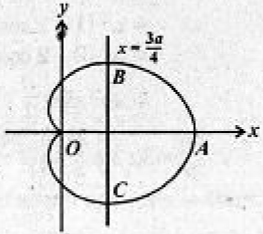
$\Rightarrow$ $4\cos^2\theta + 4\cos\theta - 3 = 0$

or $(2\cos\theta + 3)(2\cos\theta - 1) = 0$

i.e., $\cos\theta = \frac{1}{2}$ or $\cos\theta = -\frac{3}{2}$

$\cos\theta = -\frac{3}{2}$ is inadmissible

$\cos\theta = \frac{1}{2}$ gives $\theta = \frac{\pi}{3}, -\frac{\pi}{3}$



Thus the line cuts the curve at the points B and C, where $\theta = \frac{\pi}{3}$ and $\theta = \frac{-\pi}{3}$.

From $r = a\,(1 + \cos\theta)$, we have

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = a\,(1 + \cos\theta)^2 + a^2\sin^2\theta$$

$$= 2a^2\,(1 + \cos\theta) = 4a^2\cos^2\frac{\theta}{2}$$

or $\frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}$,

Length of the curve above the polar axis

$$= \int_0^{\pi} 2a\cos\frac{\theta}{2} d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 4a.$$

Length of the arc AB

$$= \int_0^{\pi/3} 2a\cos\frac{\theta}{2} d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi/3} = 2a.$$

Thus the line $r\cos\theta = 3a$ bisects the upper half of the arc of the curve. Since the curve is symmetric about the initial line, the gives line divides the arc of the cardioid into two equal parts.

12. For the epicycloid $x = (a + b)\cos\theta - b\cos\left(\frac{(a + b)}{b}\right)\theta$,

$$y = (a + b)\sin\theta - b\sin\left(\frac{a + b}{b}\right)\theta,$$

show that $s = \frac{4b\,(a + b)}{a}\cos\left(\frac{a\theta}{2b}\right)$, where s is measured from the point at which $\theta = \frac{\pi b}{a}$.

Sol. $x = (a + b)\cos\theta - b\cos\left(\frac{a + b}{b}\right)\theta$

$$\frac{dx}{d\theta} = -(a + b)\sin\theta + \frac{b\,(a + b)}{b}\sin\left(\frac{a + b}{b}\right)\theta$$

$$= -(a + b)\left[\sin\theta - \sin\left(\frac{a + b}{b}\right)\theta\right]$$

$$y = (a + b)\sin\theta - b\sin\left(\frac{a + b}{b}\right)\theta$$

$$\frac{dy}{d\theta} = (a+b)\cos\theta - (a+b)\cos\left(\frac{a+b}{b}\right)\theta$$

$$= (a+b)\left[\cos\theta - \cos\left(\frac{a+b}{b}\right)\theta\right]$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= (a+b)^2\left[2 - 2\left(\sin\theta\sin\left(\frac{a+b}{b}\right)\theta + \cos\theta\cos\left(\frac{a+b}{b}\right)\theta\right)\right]$$

$$= (a+b)^2\left[2 - 2\cos\left(\frac{a+b}{b} - 1\right)\theta\right]$$

$$= (a+b)^2\left[2 - 2\cos\frac{a}{b}\theta\right] = 2(a+b)^2\left[1 - \cos\frac{a\theta}{b}\right]$$

$$= 4(a+b)^2\sin^2\frac{a\theta}{2b} \quad \text{or} \quad \frac{ds}{d\theta} = 2(a+b)\sin\frac{a\theta}{2b}$$

Length of the arc measured from $\theta = \frac{b\pi}{a}$ to any value of θ

$$= \int_{b\pi/a}^{\theta} 2(a+b)\sin\frac{a\theta}{2b}\,d\theta = 2(a+b)\int_{b\pi/a}^{\theta}\sin\frac{a\theta}{2b}\,d\theta$$

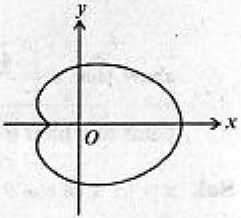
$$= 2(a+b)\frac{2b}{a}\left[-\cos\frac{a\theta}{2b}\right]_{b\pi/a}^{\theta} = -\frac{4b(a+b)}{a}\cos\frac{a\theta}{2b}$$

$$= \frac{4b(a+b)}{a}\cos\frac{a\theta}{2b} \text{ (in absolute value)}$$

13. Show that the perimeter of the limacon $r = a + b\cos\theta$, if $\frac{b}{a}$ is small, is approximately $2\pi a\left(1 + \frac{b^2}{4a^2}\right)$.

Sol. Equation of the curve is

$$r = a + b\cos\theta,\ a > b$$



$$\frac{dr}{d\theta} = -b\sin\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$= (a + b\cos\theta)^2 + b^2\sin^2\theta$$

$$= a^2 + b^2 + 2ab\cos\theta$$

$$= a^2\left[1 + \frac{b^2}{a^2} + \frac{2b}{a}\cos\theta\right]$$

$$= a^2(1 + 2k\cos\theta + k^2), \text{ where } k = \frac{b}{a}$$

$$\frac{ds}{d\theta} = a\{1 + (2k\cos\theta + k^2)\}^{1/2}$$

$$= a\left\{1 + \frac{1}{2}(2k\cos\theta + k^2) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2}(2k\cos\theta + k^2)^2 + \ldots\right\}$$

$$= a\left\{1 + k\cos\theta + \frac{1}{2}k^2 - \frac{1}{2}k^2\cos^2\theta\right\}, \text{ neglecting higher powers of } k$$

$$s = a\int_0^{\pi}\left(1 + k\cos\theta + \frac{1}{2}k^2\sin^2\theta\right)d\theta$$

$$= a\int_0^{\pi}1\,d\theta + ak\int_0^{\pi}\cos\theta\,d\theta + \frac{ak^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$$

$$= a\pi + 0 + ak^2\int_0^{\pi/2}\sin^2\theta\,d\theta$$

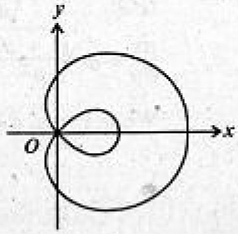
$$= a\pi + ak^2\frac{1}{2}\frac{\pi}{2} \qquad \text{(by Wallis formula)}$$

$$= a\pi\left(1 + \frac{k^2}{4}\right) = a\pi\left(1 + \frac{b^2}{4a^2}\right).$$

14. Prove that the difference between the lengths of the two loops of the limacon $r = a + b\cos\theta$ is $8a$, where $\frac{a}{b}$ is small.

Sol. Since $a < b$, when $\theta = \arccos\left(-\frac{a}{b}\right) = \alpha$ say, $r = 0$ so that the curve passes through the pole for this value of θ. For upper half of the outer loop, θ varies from 0 to $\arccos\left(-\frac{a}{b}\right) = \alpha$, where r is positive and for lower half of the inner loop, θ varies from α to π, where r is negative.

Difference of the length of these two loops is



$$s = \int_0^{\alpha}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta - \int_{\alpha}^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\alpha}[(a + b\cos\theta)^2 + b^2\sin^2\theta]^{1/2}\,d\theta$$

$$- \int_{\alpha}^{\pi}[(a + b\cos\theta)^2 + b^2\sin^2\theta]^{1/2}\,d\theta$$

$$= b\int_0^\alpha \left(1+\frac{2a}{b}\cos\theta\right)^{1/2} d\theta - b\int_\alpha^\pi \left(1+\frac{2a}{b}\cos\theta\right)^{1/2} d\theta$$

$$= b\int_0^\alpha \left(1+\frac{a}{b}\cos\theta\right) d\theta - b\int_\alpha^\pi \left(1+\frac{a}{b}\cos\theta\right) d\theta$$

$$\left(\text{neglecting } \frac{a^2}{b^2} \text{ and higher powers of } \frac{a}{b}\right)$$

$$= b\left[\theta+\frac{a}{b}\sin\theta\right]_0^\alpha - b\left[\theta+\frac{a}{b}\sin\theta\right]_\alpha^\pi$$

$$= b\left[\alpha+\frac{a}{b}\sin\alpha\right] - b\left[\pi-\alpha-\frac{a}{b}\sin\alpha\right]$$

$$= b\left[\alpha-(\pi-\alpha)+\frac{2a}{b}\sin\alpha\right]$$

$$= b\left[2\arccos\left(\frac{-a}{b}\right)-\pi+\frac{2a}{b}\sin\alpha\right]$$

$$= b\left[2\left(\frac{\pi}{2}+\frac{a}{b}\right)-\pi+\frac{2a}{b}\sin\alpha\right], \text{ using Maclaurin's Theorem.}$$

$$= 2a + 2a\left(\sqrt{\frac{b^2-a^2}{b^2}}\right) = 2a + 2a\left(1-\frac{a^2}{b^2}\right)^{1/2} = 2a + 2a = 4a.$$

Total difference of the lengths of the two loops = $8a$.

15. Prove that the length of the arc of the hyperbolic spiral $r\theta = a$ taken from the point $r = a$ to $r = 2a$ is

$$a\left[\sqrt{5}-\sqrt{2}+\ln\frac{2+\sqrt{8}}{1+\sqrt{5}}\right].$$

Sol. $r = \dfrac{a}{\theta}$, $\dfrac{dr}{d\theta} = -\dfrac{a}{\theta^2}$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2}{\theta^2} + \frac{a^2}{\theta^4} = \frac{a^2(\theta^2+1)}{\theta^4}$$

$$\frac{ds}{d\theta} = a\frac{\sqrt{1+\theta^2}}{\theta^2}, \quad ds = \frac{a\sqrt{1+\theta^2}}{\theta^2}\,d\theta.$$

When $r = a$, $\theta = 1$ and when $r = 2a$, $\theta = \dfrac{1}{2}$

$$s = a\int_1^{1/2} \frac{\sqrt{1+\theta^2}}{\theta^2}\,d\theta = a\int_1^{1/2} \frac{1+\theta^2}{\theta^2\sqrt{1+\theta^2}}\,d\theta$$

$$= a\int_1^{1/2} \frac{d\theta}{\theta^2\sqrt{1+\theta^2}} + a\int_1^{1/2} \frac{d\theta}{\sqrt{1+\theta^2}} \qquad (1)$$

Put $\theta = \dfrac{1}{t}$ or $d\theta = -\dfrac{1}{t^2}\,dt$ in the first integral. Then

$$a\int_1^{1/2} \frac{d\theta}{\theta^2\sqrt{1+\theta^2}} = a\int_1^2 \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t^2}\sqrt{1+\frac{1}{t^2}}} = -a\int_1^2 \frac{-t\,dt}{\sqrt{t^2+1}}$$

$$= -\frac{a}{2}\int_1^2 2t\,(t^2+1)^{-1/2}\,dt = -\frac{2a}{2}\left[\sqrt{t^2+1}\right]_1^2$$

$$= -a\left[\sqrt{5}-\sqrt{2}\right] \qquad (2)$$

Now $$\int_1^{1/2} \frac{d\theta}{\sqrt{1+\theta^2}} = \left[\ln\left(\theta+\sqrt{1+\theta^2}\right)\right]_1^{1/2}$$

$$= \ln\left[\frac{1}{2}+\frac{\sqrt{5}}{2}\right] - \ln\left(1+\sqrt{2}\right)$$

$$= \ln\frac{1+\sqrt{5}}{2(1+\sqrt{2})} = \ln\frac{1+\sqrt{5}}{2+\sqrt{8}} \qquad (3)$$

Substituting from (2) and (3) into (1), we have

$$s = -a\left[\sqrt{5}-\sqrt{2}\right] + a\ln\frac{1+\sqrt{5}}{2+\sqrt{8}}$$

$$= -a\left[\sqrt{5}-\sqrt{2}-\ln\frac{1+\sqrt{5}}{2+\sqrt{8}}\right]$$

$$= -a\left[\sqrt{5}-\sqrt{2}+\ln\frac{2+\sqrt{8}}{1+\sqrt{5}}\right]$$

$$= a\left[\sqrt{5}-\sqrt{2}+\ln\frac{2+\sqrt{8}}{1+\sqrt{5}}\right] \text{ in absolute units.}$$

16. Show that the intrinsic equation of the catenary

$$y = c\cosh\left(\frac{x}{c}\right) \text{ is } s = c\tan\alpha.$$

Sol. $y = c\cosh\dfrac{x}{c}$ (1)

$$\frac{dy}{dx} = \sinh\frac{x}{c} \quad (2)$$

$$\left(\frac{ds}{d\theta}\right)^2 = 1 + \left(\frac{dr}{d\theta}\right)^2 = 1 + \sinh^2\frac{x}{c} = \cosh^2\frac{x}{c}$$

$$\frac{ds}{dx} = \cosh\frac{x}{c}$$

Thus $s = \displaystyle\int_0^x \cosh\frac{x}{c}\,dx = c\left[\sinh\frac{x}{c}\right]_0^x = c\sinh\frac{x}{c} = c\left(\frac{dy}{dx}\right)$ from (2)

$= c\tan\alpha$

which is the required intrinsic equation.

17. Show that the intrinsic equation of the parabola $x^2 = 4ay$ is

$$s = a\tan\alpha\sec\alpha + a\ln(\tan\alpha + \sec\alpha).$$

Sol. Equation of the curve is $y = \dfrac{x^2}{4a}$ (1)

$$\frac{dy}{dx} = \frac{x}{2a} \quad (2)$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{4a^2} = \frac{4a^2 + x^2}{4a^2}$$

or $\quad \dfrac{ds}{dx} = \dfrac{\sqrt{4a^2 + x^2}}{2a}$

$$s = \frac{1}{2a}\int_0^x \sqrt{4a^2 + x^2}\,dx$$

$$= \frac{1}{2a}\left[\frac{x\sqrt{4a^2 + x^2}}{2} + 2a^2\sinh^{-1}\frac{x}{2a}\right]_0^x$$

$$= \frac{1}{2a}\left[\frac{x\sqrt{4a^2 + x^2}}{2} + 2a^2\ln\left(\frac{x + \sqrt{x^2 + 4a^2}}{2a}\right)\right] \quad (3)$$

$$\frac{dy}{dx} = \frac{x}{2a} = \tan\alpha \quad \text{or} \quad x = 2a\tan\alpha$$

Putting this values of x into (3), we get

$$s = \frac{1}{2a}\left[\frac{2a\tan\alpha\, 2a\sec\alpha}{2} + 2a^2\ln\frac{2a\tan\alpha + 2a\sec\alpha}{2a}\right]$$

$$= a[\tan\alpha\sec\alpha + \ln(\tan\alpha + \sec\alpha)]$$

which is the required intrinsic equation.

18. Show that the intrinsic equation of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3} \text{ is } s = \frac{3a}{2}\sin^2\alpha.$$

Sol. Parametric equations of the curve are

$$x = a\cos^3\theta \quad \text{and} \quad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad (1)$$

and $\quad \dfrac{dy}{d\theta} = 3a\sin^2\theta\sin\theta \quad (2)$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$= 9a^2\sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta) = 9a^2\sin^2\theta\cos^2\theta$$

or $\quad \dfrac{ds}{d\theta} = 3a\sin\theta\cos\theta$

$$ds = 3a\sin\theta\cos\theta\,d\theta$$

$$s = 3a\int_0^\theta \sin\theta\cos\theta\,d\theta = \frac{3a}{2}\sin 2\theta \quad (3)$$

Dividing (2) by (1), we have

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta} = \tan\theta$$

i.e., $\tan\alpha = \tan\theta$ or $\theta = \alpha$

Putting $\theta = \alpha$ in (3), we get

$s = \dfrac{3a}{2}\sin^2\alpha$ as the required intrinsic equation.

19. Prove that the intrinsic equation of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta) \text{ is } s = 4a\sin\alpha.$$

Sol. Here $x = a(\theta + \sin\theta)$

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \quad (1)$$

$$y = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

Dividing (2) by (1), we get

$$\frac{dy}{dx} = \frac{a \sin \theta}{a(1 + \cos \theta)}$$

$$= \frac{\sin \theta}{1 + \cos \theta} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = \tan \frac{\theta}{2}$$

Hence $\tan \alpha = \tan \frac{\theta}{2}$

or $\alpha = \frac{\theta}{2}$ (3)

Now $\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$

$$= a^2 (1 + \cos \theta)^2 + a^2 \sin^2 \theta$$
$$= a^2 [(1 + \cos \theta)^2 + \sin^2 \theta]$$
$$= a^2 [1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta]$$
$$= a^2 [1 + 2 \cos \theta + 1] = a^2 [2 + 2 \cos \theta]$$
$$= 2a^2 (1 + \cos \theta) = 4a^2 \cos^2 \frac{\theta}{2}$$

$$\frac{ds}{d\theta} = 2a \cos \frac{\theta}{2}$$

$$s = 2a \int_0^\theta \cos \frac{\theta}{2} d\theta = 4a \sin \frac{\theta}{2} = 4a \sin\alpha, \text{ from (3)}$$

is the required intrinsic equation.

20. Show that the intrinsic equation of the cardioid $r = a(1 + \cos \theta)$ is

$s = 4a \sin \frac{\alpha}{3}$ · [Take $\theta = 0$ as the fixed point.].

Sol. Here $\theta = 0$ is the fixed point A on the initial line. The tangent to the curve at A is perpendicular to the initial line. The tangent at P (r, θ) and the tangent at A meet at the point R so that

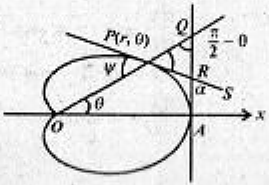
$$\angle ARS = \alpha = \psi + \theta - \frac{\pi}{2} \cdot \quad (1)$$

From the equation of the curve

$r = a(1 + \cos \theta)$, we have

$$\frac{dr}{d\theta} = -a \sin \theta$$

Therefore, $\tan \psi = \frac{r}{dr/d\theta}$



$$= \frac{a(1 + \cos \theta)}{-a \sin \theta}$$

$$= \frac{2 \cos^2 \frac{\theta}{2}}{-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = -\cot \frac{\theta}{2} = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

or $\psi = \frac{\pi}{2} + \frac{\theta}{2}$ (2)

Now $s = \text{Arc } AP = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$

$$= \int_0^\theta [a^2 (1 + 2 \cos \theta + \cos^2 \theta) + a^2 \sin^2 \theta]^{1/2} d\theta$$

$$= \int_0^\theta \sqrt{2a^2 (1 + \cos \theta)} d\theta = 2a \int_0^\theta \cos \frac{\theta}{2} d\theta$$

$$= 4a \sin \frac{\theta}{2} \quad (3)$$

Eliminate θ and ψ from (1), (2) and (3) to get the required intrinsic equation.

Putting the value of ψ into (2) into (1), we have

$$\alpha = \frac{\pi}{2} + \frac{\theta}{2} + \theta - \frac{\pi}{2} = \frac{3\theta}{2} \quad \text{or} \quad \frac{\theta}{2} = \frac{\alpha}{2}$$

Writing this value of $\frac{\theta}{2}$ into (3), the required equation is

$$s = 4a \sin \frac{\alpha}{3} \cdot$$

Exercise Set 7.7 (Page 334)

Find the radius of curvature at any point on each of the given curves (Problems 1 – 8):

1. $y = c \cosh\left(\frac{x}{c}\right)$ (1)

Sol. $\frac{dy}{dx} = \sinh \frac{x}{c}$

$$\frac{d^2y}{dx^2} = \frac{1}{c} \cosh \frac{x}{c}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left(1 + \sinh^2 \frac{x}{c}\right)^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}} = \frac{\cosh^3 \frac{x}{c}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= c\cosh^2\left(\frac{x}{c}\right) = c\frac{y^2}{c^2} = \frac{y^2}{c}, \qquad \text{from (1)}$$

2. $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t),\ a > 0$

Sol. $\dfrac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = a\,t\cos t$

$$\frac{d^2x}{dt^2} = a\cos t - a\,t\sin t$$

$$y = a(\sin t - t\cos t)$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = a\,t\sin t$$

$$\frac{d^2y}{dt^2} = a\sin t + a\,t\cos t$$

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\left|\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right|}$$

$$= \frac{(a^2t^2\cos^2 t + a^2t^2\sin^2 t)^{3/2}}{a\,t\cos t\,(a\sin t + a\,t\cos t) - a\,t\sin t\,(a\cos t - a\,t\sin t)}$$

$$= \frac{a^3t^3}{a^2t^2(\cos^2 t + \sin^2 t)} = \frac{a^3t^3}{a^2t^2} = a\,t.$$

3. $x = a(t - \sin t),\ y = a(1 - \cos t),\ a > 0$

Sol. Differentiating twice the given equations w.r.t. t, we have

$$\frac{dx}{dt} = a(1 - \cos t), \quad \frac{d^2x}{dt^2} = a\sin t$$

$$\frac{dy}{dt} = a\sin t, \quad \frac{d^2y}{dt^2} = a\cos t$$

Now $\rho = \dfrac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\left|\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right|}$

$$= \frac{[a^2 + a^2\cos^2 t - 2a^2\cos t + a^2\sin^2 t]^{3/2}}{|a^2\cos t\,(1 - \cos t) - a^2\sin^2 t|}$$

$$= \frac{[2a^2(1 - \cos t)]^{3/2}}{|a^2\cos t - a^2|}$$

$$= 2a\sqrt{1 - \cos t} = 4a\sin\left(\frac{t}{2}\right)$$

4. $x = a\cos^3\theta,\ y = a\sin^3\theta$

Sol. $x = a\cos^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$\frac{d^2x}{d\theta^2} = -3a\cos^3\theta + 6a\cos\theta\sin^2\theta$$

$y = a\sin^3\theta$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{d^2y}{d\theta^2} = -3a\sin^3\theta + 6a\sin\theta\cos^2\theta$$

$$\rho = \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{3/2}}{\left|\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2}\frac{dy}{d\theta}\right|}$$

$$= \frac{(9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta)^{3/2}}{|-18a^2\sin^2\theta\cos^4\theta + 9a^2\sin^4\theta\cos^2\theta + 9a^2\sin^2\theta\cos^4\theta - 18\,a^2\sin^4\theta\cos^2\theta|}$$

$$= \frac{(9a^2\sin^2\theta\cos^2\theta)^{3/2}}{|9a^2\sin^2\theta\cos^2\theta - 18a^2\sin^2\theta\cos^2\theta|}$$

$$= \frac{27a^3\sin^3\theta\cos^3\theta}{9a^2\sin^2\theta\cos^2\theta} = 3a\sin\theta\cos\theta$$

$$= 3a\left(\frac{y}{a}\right)^{1/3}\left(\frac{x}{a}\right)^{1/3} = 3a\frac{(xy)^{1/3}}{a^{2/3}} = 3\,(axy)^{1/3}$$

5. $r = 2\cos 2\theta$ at $\theta = \dfrac{\pi}{12}$

Sol. $\dfrac{dr}{d\theta} = -4\sin 2\theta,\ \dfrac{d^2r}{d\theta^2} = -8\cos 2\theta$

At $\theta = \dfrac{\pi}{12},\ r = 2\cos\dfrac{\pi}{6} = \sqrt{3}$

$$\frac{dr}{d\theta} = -4\sin\frac{\pi}{6} = -2 \quad \text{and} \quad \frac{d^2r}{d\theta^2} = -8\cos\frac{\pi}{6} = -4\sqrt{3}$$

Radius of curvature at the point $\theta = \dfrac{\pi}{12}$ is

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|} = \frac{[3 + 4]^{3/2}}{3 + 2(-2)^2 - \sqrt{3}(-4\sqrt{3})} = \frac{7\sqrt{7}}{23}$$

6. $r\theta = a$

Sol. $r = \dfrac{a}{\theta}$

$$\frac{dr}{d\theta} = -\frac{a}{\theta^2}, \frac{d^2r}{d\theta^2} = \frac{2a}{\theta^3}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|} = \frac{\left(\frac{a^2}{\theta^2} + \frac{a^2}{\theta^4}\right)^{3/2}}{\frac{a^2}{\theta^2} + \frac{2a^2}{\theta^4} - \frac{2a^2}{\theta^4}} = \frac{\left(\frac{a^2\theta^2 + a^2}{\theta^4}\right)^{3/2}}{\frac{a^2}{\theta^2}}$$

$$= \frac{a^3(1+\theta^3)^{3/2} \cdot \theta^2}{\theta^6 \cdot a^2} = \frac{a\,(1+\theta^2)}{\theta^4}$$

7. $r^n = a^n \sin n\theta$

Sol. Taking ln of both the sides, we get

$$n \ln r = n \ln a + \ln \sin n\theta$$

Differentiating w.r.t. θ, we have

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} \quad \text{or} \quad \frac{dr}{d\theta} = r \cot n\theta$$

$$\frac{d^2r}{d\theta^2} = -rn \csc^2 n\theta + \cot n\theta \frac{dr}{d\theta} = -rn \csc^2 n\theta + r \cot^2 n\theta$$

$$K = \frac{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}$$

$$= \frac{|r^2 + 2r^2 \cot^2 n\theta + r^2 n \csc^2 n\theta - r^2 \cot^2 n\theta|}{(r^2 + r^2 \cot^2 n\theta)^{3/2}}$$

$$= \frac{r^2(1+n)\csc^2 n\theta}{r^3 \csc^3 n\theta} = \frac{(n+1)}{r} \sin n\theta$$

$$= \frac{(n+1)\frac{r^n}{a^n}}{r} = \frac{(n+1)r^{n-1}}{a^n}$$

$$p = \frac{a^n}{(n+1)\,r^{n-1}}$$

8. $r(1 + \cos\theta) = a.$ (1)

Sol. Differentiating (1) w.r.t. θ, we get

$$\frac{dr}{d\theta}(1 + \cos\theta) + r(-\sin\theta) = 0$$

or $$\frac{dr}{d\theta} = \left(\frac{r \sin\theta}{1 + \cos\theta}\right) = r \tan\frac{\theta}{2}$$

$$\frac{d^2r}{d\theta^2} = \frac{dr}{d\theta}\tan\frac{\theta}{2} + r\left(\frac{1}{2}\sec^2\frac{\theta}{2}\right) = r\tan^2\frac{\theta}{2} + \frac{r}{2}\sec^2\frac{\theta}{2}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}$$

$$= \frac{\left[r^2 + r^2\tan^2\frac{\theta}{2}\right]^{3/2}}{r^2 + 2r^2\tan^2\frac{\theta}{2} - r^2\tan^2\frac{\theta}{2} - \frac{r^2}{2}\sec^2\frac{\theta}{2}}$$

$$= \frac{r^3\sec^3\frac{\theta}{2}}{r^2\left(1 + \tan^2\frac{\theta}{2}\right) - \frac{r^2}{2}\sec^2\frac{\theta}{2}}$$

$$= 2r\sec\frac{\theta}{2} = 2r \cdot \sqrt{\frac{2r}{a}}, \text{ from (1)}$$

9. Prove that the radius of curvature at the point $(2a, 2a)$ on the curve $x^2y = a(x^2 + y^2)$ is $2a$.

Sol. Let $f(x, y) = a(x^2 + y^2) - x^2y$

$f_x = 2ax - 2xy$, $f_y = 2ay - x^2$

$f_{xx} = 2a$, $f_{yy} = 2a$, $f_{yx} = -2x = f_{xy}$

At $(2a, 2a)$;

$f_x = 4a^2 - 4a(2a) = 4a^2 - 8a^2 = -4a^2$

$f_y = 2a(2a) - (2a)^2 = 0$

$f_{xx} = 2a$

$f_{yy} = 2a$

$f_{yx} = -2(2a) = -4a$

$$\rho = \frac{[(f_x)^2 + (f_y)^2]^{3/2}}{|(f_y)^2(f_{xx}) - 2f_xf_yf_{xy} + (f_x)^2f_{yy}|} = \frac{(16a^4 + 0)^{3/2}}{(-4a^2)^2(2a)} = \frac{64a^6}{32a^5} = 2a$$

10. Prove that for the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\rho = \dfrac{a^2b^2}{p^3}$, p being length of the perpendicular from the centre of C of the ellipse to the tangent at any point $P(x, y)$ on the ellipse. Prove also that $\rho = \dfrac{|CQ|^3}{ab}$, where CQ is semi-diameter conjugate to CP.

Sol. Parametric equations of the ellipse are

$$x = a\cos\theta, \qquad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a \sin \theta, \qquad \frac{dy}{d\theta} = b \cos \theta$$

$$\frac{d^2x}{d\theta^2} = -a \cos \theta, \qquad \frac{d^2y}{d\theta^2} = -b \sin \theta$$

$$\rho = \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{3/2}}{\left|\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2}\frac{dy}{d\theta}\right|} = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab \sin^2 \theta + ab \cos^2 \theta}$$

$$= \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} \qquad (1)$$

Now $\frac{dy}{dx} = -\frac{b \cos \theta}{a \sin \theta}$

Equation of the tangent at $P(a \cos \theta, b \sin \theta)$ is

$$y - b \sin \theta = -\frac{b \cos \theta}{a \sin \theta}(x - a \cos \theta)$$

or $\quad ya \sin \theta - ab \sin^2 \theta = -b \cos \theta x + ab \cos^2 \theta$

or $\quad (b \cos \theta)x + (a \sin \theta)y - ab = 0$

$$p = \frac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}$$

or $\quad \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} = \frac{ab}{p}$

Putting this value of $\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ into (1), we get

$$\rho = \frac{a^3b^3}{p^3}\frac{1}{ab} = \frac{a^2b^2}{p^3}$$

If CQ is the semi-diameter conjugate to CP, then coordinates of Q are $(-a \sin \theta, b \cos \theta)$.

$$|CQ|^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta \qquad (2)$$

Putting the value of $a^2 \sin^2 \theta + b^2 \cos^2 \theta$ from (2) into (1), we get

$$\rho = \frac{|CQ|^3}{ab}$$

11. Prove that if ρ is the radius of curvature at any point P on the parabola $y^2 = 4ax$ and F is its focus, then ρ^2 varies as $(FP)^3$.

Sol. Here, $\quad f(x, y) = y^2 - 4ax$

$$f_x = -4a, f_y = 2y, f_{xy} = 0, f_{yx} = 0, f_{yy} = 2$$

$$\rho = \frac{[(f_x)^2 + (f_y)^2]^{3/2}}{|-(f_y)^2 f_{xx} + 2f_x f_y f_{xy} - (f_x)^2 f_{yy}|}$$

$$= \frac{(16a^2 + 4y^2)^{3/2}}{16a^2 (2)} = \frac{(16a^2 + 16ax)^{3/2}}{32a^2}$$

$$= \frac{64a^{3/2}(x + a)^{3/2}}{32a^2} = \frac{2(x + a)^{3/2}}{\sqrt{a}}$$

$$\rho^2 = \frac{4(x + a)^3}{a} \qquad (1)$$

Now F is $(a, 0)$ and P is (x, y)

$$FP = \sqrt{(x - a)^2 + y^2} = \sqrt{(x - a) + 4ax}$$

$$= \sqrt{(x + a)^2} = x + a \qquad (2)$$

From (1) and (2), we get

$$\rho^2 = \frac{4(FP)^3}{a} = \frac{4}{a}(FP)^3 \quad i.e., \quad \rho^3 \text{ varies as } (FP)^3$$

12. Prove that for the cardioid $r = a(1 + \cos \theta)$, $\frac{\rho^2}{r}$ is constant.

Sol. Here $\quad r = a(1 + \cos \theta)$

Taking logarithmic of both sides, we have

$$\ln r = \ln a + \ln (1 + \cos \theta)$$

Differentiating both sides w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin \theta}{1 + \cos \theta} = -\tan \frac{\theta}{2}$$

$$\frac{dr}{d\theta} = -r \tan \frac{\theta}{2}$$

$$\frac{d^2r}{d\theta^2} = -\frac{dr}{d\theta}\tan\frac{\theta}{2} - \frac{1}{2} r \sec^2 \frac{\theta}{2}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}$$

$$= \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2r^2 \tan^2 \frac{\theta}{2} - r^2 \tan^2 \frac{\theta}{2} + \frac{r^2}{2}\sec^2\frac{\theta}{2}\right|}$$

$$= \frac{r^3 \sec^3 \frac{\theta}{2}}{\frac{3r^2}{2}\left(1 + \tan^2 \frac{\theta}{2}\right)} = \frac{2}{3} r \sec \frac{\theta}{2} = \frac{2}{3} r \cdot \sqrt{\frac{2a}{r}}$$

or $\quad \frac{\rho^2}{r} = \frac{8a}{9}$, which is constant.

13. Show that for a pedal curve, $\rho = r\frac{dr}{dp}$.

Sol. We know that $p = r \sin \psi \qquad$ **(6.32)**

Differentiating w.r.t. r, we have

$$\frac{dp}{dr} = r\cos\psi\,\frac{d\psi}{dr} + \sin\psi$$

$$= r\frac{dr}{ds}\cdot\frac{d\psi}{dr} + r\frac{d\theta}{ds},\left[\text{since } \tan\psi = r\frac{d\theta}{dr},\ \cos\psi = \frac{dr}{\sqrt{dr^2 + r^3\,d\theta^2}} = \frac{dr}{ds} \text{ and } \sin\psi = r\frac{d\theta}{ds}\right]$$

$$\frac{dp}{dr} = r\frac{d\psi}{ds} + r\frac{d\theta}{ds} = r\frac{d}{ds}(\psi + \theta)$$

$$= r\frac{d\alpha}{ds}, \qquad \text{as } \theta + \psi = \alpha \qquad (6.29)$$

$$= \frac{r}{\rho} \quad \text{or} \quad \rho = r\frac{dr}{dp}.$$

Find the radius of curvature at the point (p, r) of each of the given curves (Problems 14 – 16):

14. $p^2 = ar$

Sol. $p = \sqrt{a}\ r^{1/2}$

$$\frac{dp}{dr} = \sqrt{a}\cdot\frac{1}{2}r^{-1/2} = \frac{\sqrt{a}}{2\sqrt{r}}\ ; \quad \frac{dr}{dp} = \frac{2\sqrt{r}}{\sqrt{a}}$$

$$\rho = r\frac{dr}{dp} = \frac{2r^{3/2}}{\sqrt{a}} = \frac{2}{\sqrt{a}}\left(\frac{p}{\sqrt{a}}\right)^3 = \frac{2p^3}{a^2}.$$

15. $\dfrac{1}{p^2} = \dfrac{A}{r^2} + B$

Sol. Differentiating w.r.t. p, we get

$$-\frac{2}{p^3} = -\frac{2A}{r^3}\frac{dr}{dp} \quad \text{or} \quad \frac{dr}{dp} = \frac{r^3}{Ap^3}$$

$$\rho = r\frac{dr}{dp} = \frac{r^4}{Ap^3}$$

16. $p^2(a^2 + b^2 - r^2) = a^2b^2$

Sol. Differentiating w.r.t. p, we get

$$2p(a^2 + b^2 - r^2) + p^2\left(-2r\frac{dr}{dp}\right) = 0$$

or $\quad a^2 + b^2 - r^2 - pr\dfrac{dr}{dp} = 0$

or $\quad r\dfrac{dr}{dp} = \dfrac{a^2 + b^2 - r^2}{p} = \dfrac{a^2b^2}{p^3}$

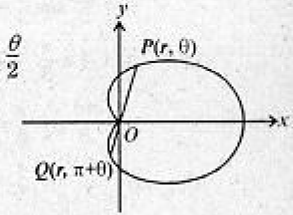
Hence $\quad \rho = \dfrac{a^2b^2}{p^3}$

17. If ρ_1, ρ_2 are the radii of curvature at the extremities of any chord of the cardioid $r = a(1 + \cos\theta)$ which passes through the pole, then prove that

$$\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}.$$

Sol. Here $\quad r = a(1 + \cos\theta)$

From Problem 12, we have



$$\rho = \frac{2}{3}r\sec\frac{\theta}{2} = \frac{2}{3}a(1 + \cos\theta)\sec\frac{\theta}{2}$$

$$= \frac{4a}{3}\cos\frac{\theta}{2}$$

Radius of curvature ρ_1 at $P(r, \theta)$ is

$$\rho_1 = \frac{4a}{3}\cos\frac{\theta}{2}$$

Changing θ into $\theta + \pi$, the radius of curvature ρ_2 at the other extremity of the chord passing through the pole is

$$\rho_2 = \frac{4a}{3}\cos\left(\frac{\theta + \pi}{2}\right) = -\frac{4a}{3}\sin\frac{\theta}{2}$$

Hence $\quad \rho_1^2 + \rho_2^2 = \dfrac{16a^2}{9}\left(\cos^2\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}\right) = \dfrac{16a^2}{9}$

18. Find the radius of curvature of the curve $r = a(1 + \cos\theta)$ at the point where the tangent is parallel to the initial line.

Sol. Here $\quad r = a(1 + \cos\theta)$

Taking ln of both sides, we have

$$\ln r = \ln a + \ln(1 + \cos\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1 + \cos\theta}$$

$$r\frac{d\theta}{dr} = -\frac{1 + \cos\theta}{\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

$$\tan\psi = r\frac{d\theta}{dr} = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

or $\quad \psi = \dfrac{\pi}{2} + \dfrac{\theta}{2}$

Now, $\alpha = \psi + \theta = \dfrac{\pi}{2} + \dfrac{\theta}{2} + \theta = \dfrac{\pi}{2} + \dfrac{3\theta}{2}$

For the tangent to be parallel to the initial line, either

$$\alpha = 0 \quad \text{or} \quad \alpha = \pi$$

$\alpha = 0$ gives $\frac{\pi}{2} + \frac{3\theta}{2} = 0 \Rightarrow \theta = -\frac{\pi}{3}$

$\alpha = \pi$ gives $\frac{\pi}{2} + \frac{3\theta}{2} = \pi \Rightarrow \theta = \frac{\pi}{3}$

From Problem 12, we have

$$\rho = \frac{2r}{3}\sec\frac{\theta}{2} = \frac{2}{3}\cdot 2a\sec^3\frac{\theta}{2}\sec\frac{\theta}{2} = \frac{4a}{3}\cos\frac{\theta}{2}$$

(1) At $\theta = \frac{\pi}{3}$, we hve

$$\rho = \frac{4a}{3}\cos\frac{\pi}{6} = \frac{4a}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{2\sqrt{3}\,a}{3}$$

(2) At $\theta = -\frac{\pi}{3}$, we have

$$\rho = \frac{4a}{3}\cos\left(-\frac{\pi}{6}\right) = \frac{4a}{3}\cos\frac{\pi}{6} = \frac{2\sqrt{3}\,a}{3}$$

Hence in each case, $\rho = \frac{2\sqrt{3}\,a}{3}$

19. Show that for the parabola $y = ax^2 + bx + c$, the radius of curvature ρ is minimum at its vertex.

Sol. $y = ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b$, $\frac{d^2y}{dx^2} = 2a$.

Radius of curvature ρ at any point is

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{[1 + (2ax + b)^2]^{3/2}}{2a}$$

$$\rho = \frac{(1 + b^2 + 4abx + 4a^2x^2)^{3/2}}{2a}$$

$$\frac{d\rho}{dx} = \frac{3}{4a}[1 + b^2 + 4abx + 4a^2x^2]^{1/2}[4ab + 8a^2x]$$

For extreme values of ρ, we have $\frac{d\rho}{dx} = 0$

This gives $4ab + 8a^2x = 0$ *i.e.*, $x = -\frac{b}{2a}$

It is easy to check that ρ is minimum for this value of x.

If $x = -\frac{b}{2a}$, then

$$y = a\left(-\frac{b}{2a}\right) + b\left(-\frac{b}{2a}\right) + c = \frac{-b^2 + 4ac}{4a}$$

Thus ρ is minimum at $A\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$

Equation of the given parabola may be written as

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

or $$y + \frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

Therefore, the vertex of the parabola is $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$ and ρ is minimum at this point.

20. Find the point on the curve $y = \ln x$ where the curvature K is maximum.

Sol. $y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}, \frac{d^2y}{dx^2} = -\frac{1}{x^2}$

Curvature at any point (x, y) of the curve is

$$K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{\left|-\frac{1}{x^2}\right|}{\left(1 + \frac{1}{x^2}\right)^{3/2}} = \frac{x}{(1 + x^2)^{3/2}}$$

$$\frac{dK}{dx} = \frac{(1 = x^2)^{3/2} - 3x^2(1 + x^2)^{1/2}}{(1 + x^2)^3} = \frac{1 - 2x^2}{(1 + x^2)^{5/2}}$$

For extreme values of K, $\frac{dK}{dx} = 0$ gives $x = \pm\frac{1}{\sqrt{2}}$. The negative sign is not admissible and so $x = \frac{1}{\sqrt{2}}$. We see that $\frac{dK}{dx}$ changes sign from positive to negative around $x = \frac{1}{\sqrt{2}}$. Thus, K is maximum at $x = \frac{1}{\sqrt{2}}$.

For $x = \frac{1}{\sqrt{2}}$, $y = \ln\frac{1}{\sqrt{2}} = -\ln\sqrt{2}$. The required point is $\left(\frac{1}{\sqrt{2}}, -\ln\sqrt{2}\right)$.

Exercise Set 7.8 (Page 338)

1. Find an equation of the osculating circle to the curve $y = \ln x$ at the point $(1, 0)$.

Sol. $y = \ln x$

$$y' = \frac{1}{x}, \qquad y'' = \frac{-1}{x^2}$$

$$y'|_{(1,0)} = 1, \qquad y''|_{(1,0)} = -1$$

Radius of curvature ρ at $(1, 0)$ is

$$\rho = \frac{(1+y'^2)^{3/2}}{|y''|} = \frac{\left(1+\frac{1}{x^2}\right)^{3/2}}{\frac{1}{x^2}} = \frac{(1+x^2)^{3/2}}{x} = 2^{3/2}$$

Centre of curvature (h, k) is

$$h = x - \frac{y'(1+y'^2)}{y''} = 1 - 1\,\frac{1+1}{-1} = 1 + 2 = 3$$

$$k = y + \frac{1+y'^2}{y''} = 0 + \frac{1+1}{-1} = -2.$$

Equation of the osculating circle is

$$(x-3)^2 + (y+2)^2 = (2^{3/2})^2 = 8$$

2. Find an equation of the osculating circle to the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1 \text{ at the point } (-2, 0).$$

Sol. Differentiating $\frac{x^2}{4} - \frac{y^2}{9} = 1$, w.r.t. x, we have

$$\frac{x}{2} - \frac{2yy'}{9} = 0 \quad \text{or} \quad 9x - 4yy' = 0 \quad \text{or} \quad y' = \frac{9x}{4y}$$

$$y'' = \frac{9}{4}\left(\frac{y - xy'}{y^2}\right) = \frac{9}{4}\,\frac{\left(y - x\cdot\frac{9x}{4y}\right)}{y^2} = \frac{9}{16}\,\frac{4y^2 - 9x^2}{y^3}$$

$$\rho = \frac{\left(1 + \frac{81x^2}{16y^2}\right)^{3/2}}{\frac{9}{16}\left(\frac{4y^2-9x^2}{y^3}\right)} = \frac{(16y^2 + 81x^2)^{3/2}}{36(4y^2 - 9x^2)}$$

$$\rho \text{ at } (-2, 0) \text{ is } = \left|\frac{(4\times 81)^{3/2}}{36(-36)}\right| = \frac{9}{2}$$

Center of curvature (h, k)

$$h = x - y'\,\frac{(1+y'^2)}{y''} = x - \frac{9x}{4y}\,\frac{(16y^2 + 81x^2)y}{9(4y^2 - 9x^2)}$$

$$= -2 + \frac{9}{2}\,\frac{81\times 4}{9(-9\times 4)} \qquad \text{at } (-2, 0)$$

$$= -2 - \frac{9}{2} = \frac{-13}{2}$$

$$k = y + \frac{1+y'^2}{y''} = 0 \qquad \text{at } (-2, 0)$$

Equation of the osculating circle at $(-2, 0)$ is

$$\left(x + \frac{13}{2}\right)^2 + y^2 = \left(\frac{9}{2}\right)^2.$$

5. Show that the evolute of the ellipse $x = a\cos\theta, y = b\sin\theta$ is

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

Sol. Here $x = a\cos\theta, \quad y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$y' = \frac{dy}{dx} = \frac{-b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$y'' = \frac{d^2y}{dx^2} = \frac{b}{a}\csc^2\theta\,.\frac{d\theta}{dx} = \frac{b}{a}\cdot\frac{\csc^2\theta}{-a\sin\theta} = \frac{-b}{a^2}\csc^3\theta$$

If (X, Y) are the coordinates of the center of curvature at $(a\cos\theta, b\sin\theta)$, we have

$$X = x - \frac{y'(1+y'^2)}{y''}$$

$$= a\cos\theta - \frac{-\frac{b}{a}\cot\theta}{-\frac{b}{a^2}\csc^3\theta}\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)$$

$$= a\cos\theta - a\cos\theta\sin^2\theta\left(1 + \frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$= a\cos\theta - a\cos\theta\sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta - a\cos\theta(1 - \cos^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= \left(\frac{a^2 - b^2}{a}\right)\cos^3\theta \qquad (1)$$

$$Y = y + \frac{1+y'^2}{y''} = b\sin\theta + \frac{1 + \frac{b^2}{a^2}\cot^2\theta}{-\frac{b}{a^2}\csc^3\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta\cos^2\theta$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta(1-\sin^2\theta)$$

$$= -\frac{a^2-b^2}{b}\sin^3\theta \qquad (2)$$

For the evolute we have to eliminate θ between (1) and (2). From (1), we get

$$aX = (a^2-b^2)\cos^3\theta$$

or $\quad (aX)^{2/3} = (a^2-b^3)^{2/3}\cos^2\theta \qquad (3)$

From (2), we have

$$bY = -(a^2-b^2)\sin^3\theta$$

or $\quad (bY)^{2/3} = (a^2-b^2)^{2/3}\sin^2\theta \qquad (4)$

Adding (3) and (4), we obtain

$$(aX)^{2/3} + (bY)^{2/3} = (a-b)^{2/3}$$

Changing (X, Y) into current coordinates (x, y) equation of the evolute is

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{3/2}$$

4. Find the centre of curvature for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Show that its evolute is $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$.

Sol. Parametric equations of the hyperbola are

$$x = a\sec\theta, \qquad y = b\tan\theta$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \qquad \frac{dy}{d\theta} = b\sec^2\theta$$

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{b\csc\theta}{a}$$

$$y'' = \frac{d^2y}{dx^2} = -\frac{b}{a}\csc\theta\cot\theta\,\frac{d\theta}{dx} = -\frac{b}{a^2}\cot^3\theta$$

If (X, Y) is the centre of curvature at $(a\sec\theta, b\tan\theta)$, then

$$X = x - \frac{y'(1+y'^2)}{y''} = \frac{a^2+b^2}{a}\sec^3\theta \qquad (1)$$

(after substitution and simplification)

$$Y = y + \frac{1+y'^2}{y''} = -\frac{a^2+b^2}{b}\tan^3\theta \qquad (2)$$

From (1), we get

$$aX = (a^2+b^2)\sec^3\theta$$

$$(aX)^{2/3} = (a^2+b^2)^{2/3}\sec^2\theta \qquad (3)$$

From (2), we have

$$bY = -(a^2+b^2)\tan^3\theta$$

or $\quad (bY)^{2/3} = (a^2+b^2)^{2/3}\tan^2\theta \qquad (4)$

Subtracting (4) from (3), we get

$$(aX)^{2/3} - (bY)^{2/3} = (a^2+b^2)^{2/3}$$

Changing (X, Y) into current coordinates equation of the evolute is

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}.$$

5. Prove that the evolute of the hyperbola $2xy = a^2$ is

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}.$$

Sol. Here $\quad 2xy = a^2 \quad$ or $\quad y = \frac{a^2}{2x}$

$$y' = \frac{dy}{dx} = -\frac{a^2}{2x^2}, \qquad y'' = \frac{d^2y}{dx^2} = \frac{2a^2}{2x^3} = \frac{a^2}{x^3}$$

If (X, Y) is the centre of curvature at any point (x, y) of the curve, then

$$X = x - \frac{y'(1+y'^2)}{y''} = x + \frac{x}{2}\left(1+\frac{a^4}{4x^4}\right) = \frac{3x}{2} + \frac{a^4}{8x^3}$$

$$Y = y + \frac{(1+y'^2)}{y''} = \frac{a^2}{2x} + \frac{x^3}{a^2}\left(1+\frac{a^4}{4x^4}\right) = \frac{3a^2}{4x} + \frac{x^3}{a^2}$$

$$X + Y = \frac{(a^2+2x^2)^3}{8a^2x^3} \qquad \text{(after simplification)}$$

$$\Rightarrow \quad (X+Y)^{2/3} = \frac{(a^2+2x^2)^2}{4a^{4/3}x^2} \qquad (1)$$

Similarly, $\quad (X-Y)^{2/3} = \frac{(a^2-2x^2)^2}{4a^{4/3}x^2} \qquad (2)$

Subtracting (2) from (1), we have

$$(X+Y)^{2/3} - (X-Y)^{2/3} = \frac{8a^2x^2}{4a^{4/3}x^2} = 2a^{2/3}$$

Changing (X, Y) to current coordinates, the required evolute is

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}.$$

6. Prove that the centres of curvatures at points of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ lie on an equal cycloid.

Sol. $\quad x = a(\theta - \sin\theta), \qquad y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1-\cos\theta), \qquad \frac{dy}{d\theta} = a\sin\theta$$

$$y' = \frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

$$y'' = \frac{d^2y}{dx^2} = -\frac{1}{2}\csc^2\frac{\theta}{2}\,\frac{d\theta}{dx} = -\frac{1}{4a\sin^4\frac{\theta}{2}}$$

If (X, Y) is centre of curvature at any point, then

$$X = x - \frac{y'}{y''}(1+y'^2) = a(\theta + \sin\theta) \qquad (1)$$

$$Y = y + \frac{1+y'^2}{y''} = -a(1-\cos\theta) \qquad (2)$$

Thus parametric equations of evolute of the cycloid are (1) and (2), which represent an equal cycloid.

7. Find the evolute of the four-cusped hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ (or $x = a\cos^3\theta, y = a\sin^3\theta$).

Sol. $x = a\cos^3\theta, \quad y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta, \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$y' = \frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$$y'' = \frac{d^2y}{dx^2} = -\sec^2\theta\frac{d\theta}{dx}$$

$$= -\frac{1}{\cos^2\theta}\left(\frac{1}{-3a\cos^2\theta\sin\theta}\right) = \frac{1}{3a\sin\theta\cos^4\theta}$$

If (X, Y) is the centre of curvature at any point, then

$$X = x - \frac{y'}{y''}(1+y'^2)$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos\theta \qquad (1)$$

(after substitution and simplification)

$$Y = y + \frac{(1+y'^2)}{y''}$$

$$= a\sin^3\theta + 3a\sin\theta\cos^2\theta \qquad (2)$$

(after substitution and simplification)

Adding (1) and (2), we have

$$X + Y = a(\cos^3\theta + \sin^3\theta) + 3a\sin\theta\cos\theta(\sin\theta + \cos\theta)$$

$$= a(\cos\theta + \sin\theta)(\cos^2\theta + \sin^2\theta - \sin\theta\cos\theta) + 3a\sin\theta\cos\theta(\sin\theta+\cos\theta)$$

$$= a(\cos\theta + \sin\theta)[\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta]$$

$$= a(\cos\theta + \sin\theta)^3$$

$$(X+Y)^{2/3} = a^{2/3}(\cos\theta + \sin\theta)^2 \qquad (3)$$

Similarly, subtracting (2) from (1), we get

$$X - Y = a(\cos\theta - \sin\theta)^3$$

$$(X-Y)^{2/3} = a^{2/3}(\cos\theta - \sin\theta)^2 \qquad (4)$$

Adding (3) and (4), we have

$$(X+Y)^{2/3} + (X-Y)^{2/3} = a^{3/2}[(\cos\theta + \sin\theta)^2 + (\cos\theta\sin\theta)^2]$$

$$= a^{2/3}[2(\cos^2\theta + \sin^2\theta)]$$

$$= 2a^{2/3}$$

Changing (X, Y) into current coordinates (x, y), we have equation of the evolute as

$$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$$

8. Show that the evolute of the tractrix

$$x = a\left[\cos t + \ln\tan\left(\frac{t}{2}\right)\right], y = a\sin t$$

is the catenary $y = a\cosh\left(\frac{x}{a}\right)$.

Sol. $x = a\left[\cos t + \ln\tan\left(\frac{t}{2}\right)\right]$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{\frac{1}{2}\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{1-\sin^2 t}{\sin t}\right) = a\frac{\cos^2 t}{\sin t}$$

$$y = a\sin t$$

$$\frac{dy}{dt} = a\cos t$$

$$y' = \frac{dy}{dx} = \frac{a\cos t\,.\,\sin t}{a\cos^2 t} = \tan t$$

$$y'' = \frac{d^2y}{dx^2} = \sec^2 t\frac{dt}{dx} = \frac{1}{\cos^2 t}\cdot\frac{\sin t}{a\cos^2 t} = \frac{\sin t}{a\cos^4 t}$$

If (X, Y) is the centre of curvature at any point, then

$$X = x - \frac{y'}{y''}(1+y'^2)$$

$$= a\left(\cos t + \ln\tan\frac{t}{2}\right) - \frac{\tan t}{\frac{\sin t}{a\cos^4 t}}(1+\tan^2 t)$$

$$= a\left(\cos t + \ln\tan\frac{t}{2}\right) - a\cos^3 t(1+\tan^2 t)$$

$$= a\left(\cos t + \ln\tan\frac{t}{2}\right) - a\cos^3 t\left(\frac{1}{\cos^2 t}\right)$$

$$= a\cos t + a\ln\tan\frac{t}{2} - a\cos t$$

$$= a\ln\tan\frac{t}{2} \quad \text{or} \quad \tan\frac{t}{2} = e^{X/a} \qquad (1)$$

$$Y = y + \frac{1+y'^2}{y''} = a\sin t + \frac{\sec^2 t}{\frac{\sin t}{a\cos^4 t}}$$

$$= a\sin t + \frac{1}{\cos^2 t}\cdot\frac{a\cos^4 t}{\sin t}$$

$$= a\sin t + \frac{a\cos^2 t}{\sin t} = \frac{a(\sin^2 t + \cos^2 t)}{\sin t} = \frac{a}{\sin t} \qquad (2)$$

The evolute of the curve is obtained by eliminating t from (1) and (2).
From (2), we get

$$Y = a\,\frac{1+\tan^2\frac{t}{2}}{2\tan\frac{t}{2}} = a\,\frac{1+e^{\frac{2X}{a}}}{2e^{\frac{X}{a}}}, \text{ [using (1)]}$$

$$= \frac{a}{2}\left(e^{\frac{X}{a}} + e^{\frac{-X}{a}}\right) = a\left(\frac{e^{\frac{X}{a}} + e^{\frac{-X}{a}}}{2}\right) = a\cosh\frac{X}{a}$$

Hence equation of the evolute is

$y = a\cosh\frac{x}{a}$ as required.

9. Show that the centre of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the folium $x^3 + y^3 = 3axy$ is $\left(\frac{21a}{16}, \frac{21a}{16}\right)$.

Sol. Here $f(x, y) = x^3 - y^3 - 3axy$,

$$f_x = 3x^2 - 3ax,\quad f_y = 3y^2 - 3ax$$

$$y' = \frac{-f_x}{f_y} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = \frac{ay - x^2}{y^2 - ax}$$

$$y' \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1$$

Also f_x at $\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{9a^2}{4}$

Again, $f_{xx} = 6a$, $f_{xy} = -3a$, $f_{yy} = 6y$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$,

$$f_{xy} = -3a,\ f_{xx} = 6\left(\frac{3a}{2}\right) = 9a \text{ and } f_{yy} = 6\left(\frac{3a}{2}\right) = 9a$$

$$y''_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{d^2y}{dx^2} = -\frac{(f_y)^2 f_{xx} - 2f_x f_y f_{xy} + (f_x)^2 f_{yy}}{(f_y)^3}$$

$$= -\frac{\frac{81a^4}{16}(9a) - 2\left(\frac{9a^2}{4}\right)\left(\frac{9a^2}{4}\right)(-3a) + \left(\frac{81a^4}{16}\right)(9a)}{\left(\frac{9a^2}{4}\right)^3}$$

$$= -\frac{\frac{729}{16}a^5 + \frac{486a^5}{16} + \frac{729}{16}a^5}{\frac{729}{64}a^6}$$

$$= -\frac{1944a^5}{16} \times \frac{64}{729a^6} = \frac{-32}{3a}$$

If (X, Y) is the centre of curvature, then

$$X = x - \frac{y'}{y''}(1 + y'^2) = \frac{3a}{2} - \frac{-1}{\frac{-32}{3a}}(1+1) = \frac{3a}{2} - \frac{3a}{16} = \frac{21a}{16}$$

$$Y = y + \frac{1+y'^2}{y''} = \frac{3a}{2} + \frac{2}{\frac{-32}{3a}} = \frac{3a}{2} - \frac{3a}{16} = \frac{21a}{16}$$

Hence the center of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is $\left(\frac{21a}{16}, \frac{21a}{16}\right)$.

10. Prove that normals to a curve are tangents to its evolute.

Sol. Let $P(x, y)$ be any point on a curve whose equation is $y = f(x)$. By **Theorem 7.33**, coordinates of the centre of curvature $C(h, k)$ corresponding to the point P are

$$h = x - \rho\sin\alpha \qquad (1)$$

$$k = y + \rho\cos\alpha \qquad (2)$$

Differentiating (1) w.r.t. x, we have

$$\frac{dh}{dx} = 1 - \rho\cos\alpha\,\frac{d\alpha}{dx} - \sin\alpha\,\frac{d\rho}{dx}$$

$$= 1 - \frac{ds}{d\alpha}\cdot\frac{dx}{ds}\cdot\frac{d\alpha}{dx} - \sin\alpha\,\frac{d\rho}{dx}\left(\text{since } \rho = \frac{ds}{d\alpha},\ \cos\alpha = \frac{dx}{ds}\right)$$

$$= -\sin\alpha\,\frac{d\rho}{dx} \qquad (3)$$

From (2), we obtain,

$$\frac{dk}{dx} = \frac{dy}{dx} - \rho\sin\alpha\,\frac{d\alpha}{dx} + \frac{d\rho}{dx}\cos\alpha$$

$$= \frac{dy}{dx} - \rho\sin\alpha\,\frac{d\alpha}{ds}\cdot\frac{ds}{dx} + \cos\alpha\,\frac{d\rho}{dx}$$

$$= \frac{dy}{dx} - \rho\,\frac{\sin\alpha}{\cos\alpha}\cdot\frac{1}{\rho} + \cos\alpha\,\frac{d\rho}{dx}$$

$$= \cos\alpha\,\frac{d\rho}{dx} \qquad (4)$$

From (3) and (4), we get $\frac{dk}{dh} = \frac{\frac{dk}{dx}}{\frac{dh}{dx}} = -\cot\alpha$

Now $\frac{dk}{dh}$ is slope of the tangent to the evolute at $C(h, k)$ and it equals the slope of the normal PC to the curve $y = f(x)$ at $P(x, y)$. Hence the result.

Exercise Set 7.9 (Page 341)

1. Find the envelope of family of lines $y = mx + \frac{a}{m}$, m being the parameter.

Sol. $f(x, y, m) = y - mx - \frac{a}{m} = 0$ (1)

$f_m(x, y, m) = -x + \frac{a}{m^2} = 0$ (2)

From (2), we have

$$m^2 = \frac{a}{x}, \quad \text{or} \quad m = \sqrt{\frac{a}{x}}$$

Substituting this value of m into (1), we get

$$y = x\sqrt{\frac{a}{x}} + a\sqrt{\frac{x}{a}} = 2\sqrt{ax}$$

or $y^2 = 4ax$ is the required envelope.

2. Show that the envelope of the family

(a) $f(x, y, t) = At^2 + Bt + C = 0$ is the discriminant $B^2 - 4AC = 0$.

(b) $f(x, y, t) = At^3 + 3Bt^2 + 3Ct + D = 0$ is $(BC - AD)^2 = 4(BD - C^2)(AC - B^2)$.

Sol.

(a) $f(x, y, t) = At^2 + Bt + C = 0$ (1)

$f_t(x, y, t) = 2At + B = 0$ gives $t = -\frac{B}{2A}$

Substitution of this value of t into (1) yields

$$A\left(\frac{B^2}{4A^2}\right) - \frac{B^2}{2A} + C = 0 \quad \text{or} \quad B^2 - 4AC = 0.$$

(b) $f(x, y, t) = At^3 + 3Bt^2 + 3Ct + D = 0$

If we write $\beta = At + B$, $H = AC - \beta^2$, $G = A^2D - 3ABC + 2B^3$ the given equation becomes

$g(x, y, \beta) = \beta^3 + 3H\beta + G = 0$ (1)

The envelopes of $g(x, y, \beta) = 0$ and $f(x, y, t) = 0$ are identical.

Now, $g_\beta(x, y, \beta) = 3\beta^2 + 3H = 0$ (2)

We eliminate β between (1) and (2) to get the envelope.

From (2), $\beta = \sqrt{-H}$. Substituting this value of β into (1), we have

$G^2 + 4H^3 = 0$

or $(A^2D - 3ABC + 2B^3)^2 + 4(AC - B^2)^3 = 0$

or $A^4D^2 + 9A^2B^2C^2 + 4B^6 - 6A^3BCD + 4A^2B^3D - 12AB^4C + 4A^3C^3 - 12A^2B^2C^2 + 12ACB^4 - 4B^6 = 0$

or $A^4D^2 - 3A^2B^2C^2 - 6A^3BCD + 4A^2B^3D + 4A^3C^3 = 0$

or $A^2D^2 + B^2C^2 - 2ABCD = 4B^2C^2 + 4ABCD - 4B^3D - 4AC^3$

or $(BC - AD) = 4[-B^2(BD - C) + AC(BD - C^2)]$

$= 4(AC - B^2)(BD - C^2)$

3. Find the envelope of the family $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$, θ being the parameter.

Sol. $f(x, y, \theta) = 2yu^2 - 2u^2x\tan\theta + gx^2\sec^2\theta = 0$ (1)

$f_\theta(x, y, \theta) = -2u^2x\sec^2\theta + 2gx^2\sec^2\theta\tan\theta = 0$ (2)

From (2), $\tan\theta = \frac{u^2}{gx}$. Substituting into (1), we obtain

$$2yu^2 - 2u^2x\cdot\frac{u^2}{gx} + gx^2\left(1 + \frac{u^4}{g^2x^2}\right) = 0$$

or $2yu^2g - 2u^4 + g^2x^2 + u^4 = 0$

i.e., $g^2x^2 + 2u^2gy - u^4 = 0$ is the required envelope.

4. Find the envelope of the family $y = mx + \sqrt{a^2m^2 + b^2}$, m being the parameter.

Sol. $f(x, y, m) = (y - mx)^2 - a^2m^2 - b^2 = 0$

or $f(x, y, m) = y^2 - 2mxy + m^2(x^2 - a^2) - b^2 = 0$ (1)

$f_m(x, y, m) = -2xy + 2m(x^2 - a^2) = 0$

This equation gives $m = \frac{xy}{x^2 - a^2}$

Putting this value of m into (1), we get

$$y^2 - \frac{2x^2y^2}{x^2 - a^2} + \frac{x^2y^2}{(x^3 - a^2)^2}(x^2 - a^2) - b^2 = 0$$

or $y^2(x^2 - a^2) - 2x^2y^2 + x^2y^2 - b^2(x^2 - a^2) = 0$

i.e., $a^2y^2 + b^2x^2 = a^2b^2$

or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an equation of the envelope.

5. Find the envelope of the family of straight lines joining the extremities of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol. Let C be the centre of the ellipse and CP and CD be its conjugate semi-diameters. If the point P has coordinates $(a\cos\theta, b\sin\theta)$, then D has coordinates $(-a\sin\theta, b\cos\theta)$. Equation of PD is

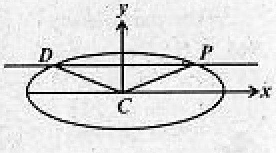
$$y-b\sin\theta=\frac{b(\sin\theta-\cos\theta)}{a(\cos\theta+\sin\theta)}(x-a\cos\theta)$$

or $$\frac{x}{a}(\sin\theta-\cos\theta)-\frac{y}{b}(\sin\theta+\cos\theta)+1=0 \qquad (1)$$

Differentiating (1) w.r.t. θ, we have

$$\frac{x}{a}(\cos\theta+\sin\theta)-\frac{y}{b}(\cos\theta-\sin\theta)=0 \qquad (2)$$

We eliminate θ between (1) and (2) to obtain the envelope.



From (1), we get

$$\frac{x}{a}(\sin\theta-\cos\theta)-\frac{y}{b}(\sin\theta+\cos\theta)=-1 \qquad (3)$$

Squaring (2) and (3) and adding the results, we have

$$\frac{x^2}{a^2}[(\cos\theta+\sin\theta)^2+(\sin\theta-\cos\theta)^2]$$
$$+\frac{y^2}{b^2}[(\cos\theta-\sin\theta)^2+(\sin\theta+\cos\theta)^2]=1$$

or $\frac{2x^2}{a^2}+\frac{2y^2}{b^2}=1$ is an equation of the envelope.

6. Prove that the envelope of an ellipse having its axes the coordinate axes and the sum of these axes constant and equal to $2a$, is the astroid $x^{2/3}+y^{2/3}=a^{2/3}$.

Sol. Equation of an ellipse with its axes as the coordinate axes is

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1, \text{ where } \quad \alpha+\beta=a$$

Here α, β are parameters. We may eliminate one of the parameters and then proceed as before to find the envelope. Alternatively, we use differentials.

$$f(x,y,\alpha,\beta)=\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-1=0 \qquad (1)$$

$$g(\alpha,\beta)=\alpha+\beta-a=0 \qquad (2)$$

Using differentials, we have from (1) and (2)

$$\frac{2x^2}{\alpha^3}d\alpha+\frac{2y^2}{\beta^3}d\beta=0$$

and $$d\alpha+d\beta=0$$

Therefore, $$\frac{2x^2}{\alpha^3}=\frac{2y^2}{\beta^3}=\eta \quad \text{(say)} \qquad (3)$$

Elimination of α, β, η from (1), (2) and (3) will give the envelope.

From (3), we have

$$\frac{2x^2}{\alpha^2}=\eta\alpha, \qquad \frac{2y^2}{\beta^2}=\eta\beta$$

or $$2\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}\right)=\eta(\alpha+\beta) \text{ or } 2=\eta a \text{ or } \eta=\frac{2}{a} \qquad (4)$$

From (3) and (4), we have $\frac{2x^2}{\alpha^3}=\eta=\frac{2}{a}$ or $\alpha^3=ax^2$

or $\alpha=a^{1/3}x^{2/3}$

Similarly, $\beta=a^{1/3}y^{2/3}$.

$$\alpha+\beta=a=a^{1/3}(x^{2/3}+y^{2/3})$$

i.e., $x^{2/3}+y^{2/3}=a^{2/3}$ is the required envelope.

7. A straight line of given length slides with its extremities on two fixed straight lines at right angle. Find the envelope of the circle drawn on the sliding line as diameter.

Sol. Let the fixed straight lines at right angles be the coordinates axes. Let $P(a, 0)$ and $Q(0, b)$ be the coordinates of extremies of the sliding line. Centre of PQ is $\left(\frac{a}{2},\frac{b}{2}\right)$ and $|PQ|=c$. Equation of the circle with PQ as a diameter is

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\frac{c^2}{4^2}$$

i.e., $x^2+y^2-ax-by=0$ (1)

where $a^2+b^2=c^2$ (2)

From (1) and (2), we have

$$x\,da+y\,db=0 \quad \text{and} \quad a\,da+b\,db=0$$

Therefore, $$\frac{x}{a}=\frac{y}{b}=\eta \quad \text{(say)} \qquad (3)$$

To get the envelope, we eliminate a, b, η from (1), (2) and (3).

From (3), we have

$$a=\frac{x}{\eta}, \qquad b=\frac{y}{\eta} \qquad (4)$$

Thus, $$\frac{x^2}{\eta^2}+\frac{y^2}{\eta^2}=a^2+b^2=c^2$$

or $\eta = \dfrac{\sqrt{x^2 + y^2}}{c}$ (5)

From (1) and (4), we get

$$x^2 + y^2 - \frac{x^2 + y^2}{\eta} = 0 \quad \text{or} \quad 1 - \frac{1}{\eta} = 0 \tag{6}$$

Writing the value of η from (5) into (6), we have

$$1 - \frac{c}{\sqrt{x^2 + y^2}} = 0 \text{ or } x^2 + y^2 = c^2$$

as an equation of the envelope.

8. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are connected by the relation

(*i*) $a + b = c$ (*ii*) $a^n + b^n = c^n$

c being a constant.

Sol.

(*i*) $a + b = c$ gives $b = c - a$

$$f(x, y, a) = \frac{x}{a} + \frac{y}{c - a} - 1 = 0 \tag{1}$$

$$f_a(x, y, a) = \frac{-x}{a^2} + \frac{y}{(c - a)^2} = 0 \tag{2}$$

We eliminate a between (1) and (2) to obtain the envelope.

From (2), we have

$$\sqrt{\frac{x}{y}} = \frac{a}{c - a} \quad \text{which yields} \quad \frac{\sqrt{\frac{x}{y}}}{1 + \sqrt{\frac{x}{y}}} = \frac{a}{c}$$

or $$a = \frac{c\sqrt{\frac{x}{y}}}{1 + \sqrt{\frac{x}{y}}} = \frac{c\sqrt{x}}{\sqrt{x} + \sqrt{y}}$$

Substituting this value of a into (1), we get

$$\frac{x(\sqrt{x} + \sqrt{y})}{c\sqrt{x}} + \frac{y(\sqrt{x} + \sqrt{y})}{c\sqrt{y}} - 1 = 0$$

or $\sqrt{x}(\sqrt{x} + \sqrt{y}) + \sqrt{y}(\sqrt{x} + \sqrt{y}) = c$

i.e., $(\sqrt{x} + \sqrt{y})^2 = c$

or $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is the required envelope.'

(*ii*) $$\frac{x}{a} + \frac{y}{b} = 1 \tag{1}$$

where $a^n + b^n = c^n$ (2)

Using differentials, from (1) and (2), we have

$$\frac{x}{a^2}da + \frac{y}{b^2}db = 0 \quad \text{and} \quad na^{n-1}da + nb^{n-1}db = 0$$

Therefore, $$\frac{x/a^2}{na^{n-1}} = \frac{y/b^2}{nb^{n-1}} = \eta \text{ (say)} \tag{3}$$

Elimination of a, b, η from (1), (2) and (3) gives the envelope.

From (3), we obtain

$$\frac{x}{a} = n\eta a^n, \; \frac{y}{b} = n\eta b^n$$

Adding, $\frac{x}{a} + \frac{y}{b} = n\eta(a^n + b^n)$

i.e., $1 = n\eta c^n$ or $\eta = \dfrac{1}{nc^n}$ (4)

Again, from (3), we have

$$a^{n+1} = \frac{x}{n\eta} = xc^n, \text{ using (4)}$$

or $a = x^{\frac{1}{n+1}} c^{\frac{n}{n+1}}$ *i.e.*, $a^n = x^{\frac{n}{n+1}} c^{\frac{n^2}{n+1}}$

Similarly, $b^n = y^{\frac{n}{n+1}} c^{\frac{n^2}{n+1}}$

Adding the last two equations, we get

$$a^n + b^n = c^{\frac{n^2}{n+1}}\left[x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}}\right]$$

or $$\frac{c^n}{c^{\frac{n^2}{n+1}}} = x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}}$$

i.e., $x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}$ is an equation of the envelope.

9. Find an equation of the normal at any point of the curve with parametric equations $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$. Hence deduce that an equation of the evolute of the curve is $x^2 + y^2 = a^2$.

Sol. $y = a(\sin t - t\cos t)$

$x = a(\cos t - t\sin t)$

$$\frac{dy}{dt} = a(\cos t - \cos t) + t\sin t = a\,t\sin t$$

$$\frac{dx}{dt} = a(-\sin t + \sin t + \cos t) = a\,t\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{\sin t}$$

Slope of the normal at any point $= -\dfrac{\cos t}{\sin t}$

Equation of the normal is

$$y - a(\sin t - t\cos t) = -\frac{\cos t}{\sin t}[x - a(\cos t + t\sin t)]$$

or $\quad y\sin t - a(\sin^2 t - t\sin t\cos t)$

$$= -x\cos t + a(\cos^2 t + t\sin t\cos t)$$

or $\quad x\cos t + y\sin t = a \qquad (1)$

Envelope of (1) is evolute of the given curve

$$f(x, y, t) = x\cos t + y\sin t - a = 0$$

$$f_t(x, y, t) = -x\sin t + y\cos t = 0 \qquad (2)$$

Squaring (1) and (2) and adding the results, we have

$$x^2 + y^2 = a^2$$

as the required evolute of the given curve.

10. Prove that an equation of the normal to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ may be written in the form $y\cos\theta - x\sin\theta = a\cos 2\theta$. Hence show that the evolute of the curve is $(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$.

Sol. Parametric equations of the astroid are

$$x = a\sin^3\theta, \qquad y = a\cos^3\theta$$

$$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta, \qquad \frac{dy}{d\theta} = -3a\cos^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\cot\theta.$$

Slope of the normal at any point is $\tan\theta$. Equation of the normal is

$$y - a\cos^3\theta = \frac{\sin\theta}{\cos\theta}(x - a\sin^3\theta)$$

or $\quad y\cos\theta - a\cos^4\theta = x\sin\theta - a\sin^4\theta$

or $\quad x\sin\theta - y\cos\theta + a(\cos^4\theta - \sin^4\theta) = 0$

i.e., $\quad x\sin\theta - y\cos\theta + a\cos 2\theta = 0$

is an equation of the normal in the required form.

To find evolute of the curve, we shall find envelope of the normal.

$$f(x, y, \theta) = x\sin\theta - y\cos\theta + a\cos 2\theta = 0 \qquad (1)$$

$$f_\theta(x, y, \theta) = x\cos\theta + y\sin\theta - 2a\sin 2\theta = 0 \qquad (2)$$

We have to eliminate θ between (1) and (2) to obtain the envelope of the normal (which is evolute of the astroid).

From (1) and (2), we have

$$\frac{x}{2a\sin 2\theta\cos\theta - a\sin\theta\cos 2\theta}$$

$$= \frac{y}{a\cos\theta\cos 2\theta + 2a\sin 2\theta\sin\theta} = \frac{1}{\sin^2\theta + \cos^2\theta} = 1$$

Therefore,

$$x = 2a\sin 2\theta\cos\theta - a\sin\theta(1 - 2\sin^2\theta)$$

$$= 4a\sin\theta(1 - \sin^2\theta) - a\sin\theta(1 - 2\sin^2\theta)$$

$$= 3a\sin\theta - 2a\sin^3\theta$$

$$y = a\cos\theta\cos 2\theta + 2a\sin 2\theta\sin\theta$$

$$= a\cos\theta(2\cos^2\theta - 1) + 4a\cos\theta(1 - \cos^2\theta)$$

$$= 3a\cos\theta - 2a\cos^3\theta$$

$$x + y = 3a(\sin\theta + \cos\theta) - 2a(\sin^3\theta + \cos^3\theta)$$

$$= a(\sin\theta + \cos\theta)(1 + \sin 2\theta) \qquad (3)$$

$$x - y = 3a(\sin\theta - \cos\theta) - 2a(\sin^3\theta - \cos^3\theta)$$

$$= a(\sin\theta - \cos\theta)(1 - \sin 2\theta) \qquad (4)$$

From (3) and (4), on squaring, we get

$$(x + y)^2 = a^2(1 + \sin 2\theta)(1 + \sin 2\theta)^2$$

$$= a^2(1 + \sin 2\theta)^3 \qquad (5)$$

$$(x - y)^2 = a^2(1 - \sin 2\theta)(1 - \sin 2\theta)^2$$

$$= a^2(1 - \sin 2\theta)^3 \qquad (6)$$

From (5) and (6), on the taking cube roots, we have

$$(x + y)^{2/3} = a^{2/3}(1 + \sin 2\theta)$$

$$(x - y)^{2/3} = a^{2/3}(1 - \sin 2\theta)$$

Adding the last two equations, we obtain

$$(x + y)^{2/3} + (x - y)^{2/3} = a^{2/3}(1 + \sin 2\theta + 1 - \sin 2\theta) = 2a^{2/3}$$

as evolute of the astroid.

Chapter 8 ANALYTICAL GEOMETRY OF THREE DIMENSIONS

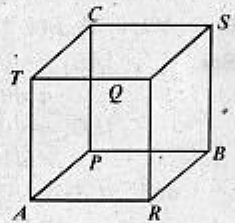
Exercise Set 8.1 (Page 347)

P and Q are the opposite vertices of a parallelepiped having its faces parallel to the coordinate planes. Find the coordinates of the other vertices and sketch the parallelepiped (Problems 1 – 3):

1. $P(-1, 1, 2)$, $Q(2, 3, 5)$

Sol. Complete the parallelepiped with faces parallel to the coordinates planes and PQ as a diagonal. Coordinates of the other vertices are (as in Definition 7.1)

$A(2, 1, 2), B(-1, 3, 2)$
$C(-1, 1, 5), R(2, 3, 2)$
$S(-1, 3, 5), T(2, 1, 5)$



2. $P(2, -1, -3)$, $Q(4, 0, -1)$

Sol. Please refer to the figure in **Problem 1**.
Coordinates of the other vertices are

$A(4, -1, -3), B(2, 0, -3), C(2, -1, -1)$
$R(4, 0, -3), S(2, 0, -1), T(4, -1, -1)$

3. $P(2, 5, -3)$, $Q(-4, 2, 1)$

Sol. Referring to the figure in **Problem 1**, coordinates of the remaining vertices are

$A(-4, 5, -3), B(2, 2, -3), C(2, 5, 1)$
$R(-4, 2, -3), S(2, 2, 1), T(-4, 5, 10).$

Show that the three given points are the vertices of a right triangle, or the vertices of an isosceles triangle, or both. (Problems 4 – 7):

4. $A(1, 5, 0)$, $B(6, 6, 4)$, $C(0, 9, 5)$

Sol.
$$|AB| = \sqrt{(6-1)^2 + (6-5)^2 + (4-0)^2}$$
$$= \sqrt{25 + 1 + 16} = \sqrt{42}$$
$$|BC| = \sqrt{(0-6)^2 + (9-6)^2 + (5-4)^2}$$

$= \sqrt{36+9+1} = \sqrt{46}$

and $|AC| = \sqrt{(1-0)^2+(5-9)^2+(0-5)^2}$

$= \sqrt{1+16+25} = \sqrt{42}$

Since $|AB| = |AC|$, therefore the triangle is isosceles.

5. $A(4, 9, 4)$, $B(0, 11, 2)$, $C(1, 0, 1)$

Sol. $|AB| = \sqrt{(0-4)^2+(11-9)^2+(2-4)^2}$

$= \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$

$|BC| = \sqrt{(1-0)^2+(0-11)^2+(1-2)^2}$

$= \sqrt{1+121+1} = \sqrt{123}$

and $|CA| = \sqrt{(1-4)^2+(0-9)^2+(1-4)^2}$

$= \sqrt{9+81+9} = \sqrt{99} = 3\sqrt{11}$

Clearly $|AB|^2 + |CA|^2 = 24 + 99 = 123 = |BC|^2$.

Thus ABC is a right triangle with right angle at A.

6. $A(1, 0, 2)$, $B(4, 3, 2)$, $C(0, 7, 6)$

Sol. $|AB| = \sqrt{(4-1)^2+(3-0)^2+(2-2)^2}$

$= \sqrt{9+9+0} = \sqrt{18} = 3\sqrt{2}$

$|BC| = \sqrt{(0-4)^2+(7-3)^2+(6-2)^2}$

$= \sqrt{16+16+16} = \sqrt{48} = 4\sqrt{3}$

and $|CA| = \sqrt{(1-0)^2+(0-7)^2+(2-6)^2}$

$= \sqrt{1+49+16} = \sqrt{66}$

Clearly $|AB|^2 + |BC|^2 = |AC|^2$

Hence, ABC is a right triangle with right angle at B.

7. $A(2, 3, 4)$, $B(8, -1, 2)$, $C(-4, 1, 0)$

Sol. $|AB| = \sqrt{(8-2)^2+(-1-3)^2+(2-4)^2}$

$= \sqrt{36+16+4} = \sqrt{56}$

$|BC| = \sqrt{(-4-8)^2+(1+1)^2+(0-2)^2}$

$= \sqrt{144+4+4} = \sqrt{152}$

and $|CA| = \sqrt{(2+4)^2+(3-1)^2+(4-0)^2}$

$= \sqrt{36+4+16} = \sqrt{56}$

Since $|AB| = |CA|$, the triangle ABC is an isosceles one.

8. Show that the points (1, 6, 1), (1, 3, 4), (4, 3, 1) and (0, 2, 0) are the vertices of regular tetrahedron.

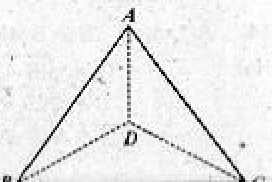
Sol. Let $A\,(1, 6, 1)$, $B(1, 3, 4)$

$C\,(4, 3, 1)$, $D(0, 2, 0)$

They will form the vertices of a regular tetrahedron provided

$|AB| = |AC| = |BC|$

$= |AD| = |CD| = |BD|$



Now, $|AB| = \sqrt{(1-1)^2+(3-6)^2+(4-1)^2}$

$= \sqrt{0+9+9} = \sqrt{18} = 3\sqrt{2}$

$|AC| = \sqrt{(4-1)^2+(3-6)^2+(1-1)^2}$

$= \sqrt{9+9+0} = 3\sqrt{2}$

$|BC| = \sqrt{(4-1)^2+(3-3)^2+(1-4)^2}$

$= \sqrt{9+0+9} = 3\sqrt{2}$

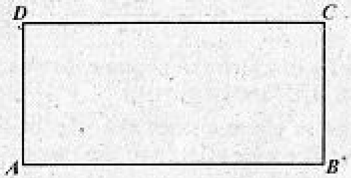
Similarly, $|AD| = |CD| = |BD| = 3\sqrt{2}$

Hence the given points are the vertices of a regular tetrahedron.

9. Show that the points (3, –1, 3), (1, –1, 2), (2, 1, 0) and (4, 1, 1) are the vertices of a rectangle.

Sol. Let $A = (3, -1, 3)$, $B = (1, -1, 2)$, $C = (2, 1, 0)$ and $D = (4, 1, 1)$

They will form a rectangle if $|AB| = |CD|$ and $|AC| = |BD|$ with $\angle A = 90^\circ$.



Now, $|AB| = \sqrt{(1-3)^2+(-1+1)^2+(2-3)^2}$

$= \sqrt{4+0+1} = \sqrt{5}$

$|CD| = \sqrt{(4-2)^2+(1-1)^2+(1-0)^2}$

$= \sqrt{4+0+1} = \sqrt{5}$

Thus $|AB| = |CD|$.

Again $|AC| = \sqrt{(2-3)^2+(1+1)^2+(0-3)^2}$

$= \sqrt{1+4+9} = \sqrt{14}$

$|BD| = \sqrt{(4-1)^2+(1+1)^2+(1-2)^2}$

$= \sqrt{9+4+1} = \sqrt{14}$

Therefore, $|AC| = |BD|$

To prove that angle A is a right angle, we note that

$|AD|^2 + |AB|^2 = [(4-3)^2+(1+1)^2+(1-3)^2] + 5$

$= 1+4+4+5$

$= 14 = |BD|^2$

Hence $\angle A$ is a right angle and so $ABCD$ is a rectangle.

10. Under what conditions on x, y and z is the point $P(x, y, z)$ equidistant from the points (3, –1, 4) and (–1, 5, 0)?

Sol. Let the given points be A and B respectively. Then since $|PA| = |PB|$, we have $(PA)^2 = (PB)^2$

or $(x-3)^2 + (y+1)^2 + (z-4)^2 = (x+1)^2 + (y-5)^2 + (z-0)^2$

or $x^2 - 6x + 9 + y^2 + 2y + 1 + z^2 + 16 - 8z$

$$= (x^2 + 2x + 1) + y^2 - 10y + 25 + z^2$$

or $-8x + 12y - 8z = 0$ or $2x - 3y + 2z = 0$

which is the required condition.

11. Find the coordinates of the point dividing the join of $A(-3, 1, 4)$ and $B(5, -1, 6)$ in the ration $3:5$.

Sol. Coordinates of the point dividing the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $m:n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

Hence the required point dividing the join of given points in the ratio $3:5$ is

$$= \left(\frac{3.5 + 5(-3)}{3+5}, \frac{3(-1) + 5(1)}{3+5}, \frac{3(6) + 5(4)}{3+5}\right) = \left(0, 1, \frac{19}{4}\right)$$

12. Find the ratio in which the yz-plane divides the segment joining the points $A(-2, 4, 7)$ and $B(3, -5, 9)$.

Sol. Suppose the yz-plane divides the join of the given points in the ratio $m:n$. The x-coordinate of the point which divides the join of the given points is

$$x = \frac{m(3) + n(-2)}{m+n}$$

It must be zero as it lies on the yz-plane.

Therefore, $3m - 2n = 0$

or $\dfrac{m}{n} = \dfrac{2}{3}$ *i.e.,* $m:n = 2:3$

13. Show that the centroid of the triangle whose vertices are (x_i, y_i, z_i), $i = 1, 2, 3$; is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

Sol. Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of the triangle.

Mid point of D of BC is $\left(\dfrac{x_1 + x_2}{3}, \dfrac{y_1 + y_2}{3}, \dfrac{z_1 + z_2}{3}\right)$

Coordinates of the point dividing AD in the ratio $2:1$ are

$$\left(\frac{x_1 + 2\left(\frac{x_2 + x_3}{2}\right)}{1+2}, \frac{y_1 + 2\left(\frac{y_2 + y_3}{2}\right)}{1+2}, \frac{z_1 + 2\left(\frac{z_2 + z_3}{2}\right)}{1+2}\right)$$

i.e., $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}, \dfrac{z_1 + z_2 + z_3}{3}\right)$

By symmetry, this point also lies on the two other medians.

Hence coordinates of the centroid of the triangle are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

14. Find the centroid of the tetrahedron whose vertices are (x_i, y_i, z_i), $i = 1, 2, 3, 4$.

Sol. Let $A = (x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ be the vertices of the tetrahedron.

The centroid E of the face BCD is

$$E = \left(\frac{x_2 + x_3 + x_4}{4}, \frac{y_2 + y_3 + y_4}{4}, \frac{z_2 + z_3 + z_4}{4}\right)$$

Coordinates of the point dividing AE in the ratio of $3:1$ are

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

which also lie on lines joining vertices to the centroids of the opposite faces. Thus the coordinates of the centroid of the tetrahedron are

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right).$$

Exercise Set 8.2 (Page 354)

In each of Problems 1 - 4, find parametric equations, direction ratios, direction cosines and measures of the direction angles of the straight line through P and Q:

1. $P(1, -2, 0)$, $Q(5, -10, 1)$

Sol. Here $P = (1, -2, 0)$, $Q = (5, -10, 1)$,. The line is parallel to the vector $\boldsymbol{d} = [5-1, -10+2, 1-0] = [4, -8, 1]$

Parametric equations of the line are

$$x = 1 + 4t, y = -2 + 8t, z = 0 + t$$

Direction ratios of the line PQ are 4, –8, 1.

Also $|PQ| = \sqrt{(4)^2 + (-8)^2 + (1)^2} = \sqrt{16 + 64 + 1} = 9$

Direction cosines of PQ are $\frac{4}{9}, \frac{-8}{9}, \frac{1}{9}$ and the direction angles are

$$\arccos\frac{4}{9}, \arccos\frac{-8}{9} \text{ and } \arccos\frac{1}{9}$$

i.e., 63°37′, 152°44′, 83°37′.

2. $P(6, 5, -3)$, $Q(4, 1, 1)$

Sol. A direction vector of PQ is

$$\boldsymbol{d} = [6-4, 5-1, -3-1] = [2, 4, -4] = 2[1, 2, -2]$$

Equations of the line through $P(6, 5, -3)$ and parallel to $\boldsymbol{d}$ are

$$x = 6 + t, y = 5 + 2t, z = -3 - 2t$$

Direction ratios of PQ are 1, 2, −2 and its direction cosines are

$$\frac{1}{\sqrt{9}}, \frac{2}{\sqrt{9}}, \frac{-2}{\sqrt{9}} \quad i.e., \quad \frac{1}{3}, \frac{2}{3}, \frac{-2}{3}$$

Direction angles of PQ are

$$\alpha = \arccos\frac{1}{3} = 70°32' \quad \beta = \arccos\frac{2}{3} = 48°11'$$

$$\gamma = \arccos\frac{-2}{3} = 131°49'$$

3. $P(1, -5, 1)$, $Q(4, -5, 4)$

Sol. Here direction vector of the line is parallel to

$$\boldsymbol{d} = [3, 0, 3] = 3[1, 0, 1]$$

Equations of PQ are

$$x = 1 + t, y = -5 + 0t, z = 1 + t$$

Direction ratios of PQ are 1, 0, 1

Its direction cosines are $= \frac{1}{\sqrt{1+1}}, \frac{0}{\sqrt{2}}, \frac{3}{\sqrt{2}} = \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

Direction angles of the line are

$$\arccos\frac{1}{\sqrt{2}}, \arccos 0, \arccos\frac{1}{\sqrt{2}} \quad i.e., \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}.$$

4. $P(3, 5, 7)$, $Q(6, -8, 10)$

Sol. Here $\boldsymbol{d} = [6-3, -8-5, 10-7] = [3, -13, 3]$

$$|\boldsymbol{d}| = \sqrt{9 + 169 + 9} = \sqrt{187}$$

Parametric equation of the line through the point P having direction vector $\boldsymbol{d}$ are

$$x = 3 + 3t, y = 5 - 13t, z = 7 + 3t$$

Direction ratios of the line PQ are 3, −13, 3

Direction cosines of PQ are $\frac{3}{\sqrt{187}}, \frac{-13}{\sqrt{187}}, \frac{3}{\sqrt{187}}$

Measures of the direction angles are

$$\alpha = \arccos\left(\frac{3}{\sqrt{187}}\right) = 77°19'38''$$

$$\beta = \arccos\left(\frac{-13}{\sqrt{187}}\right) = 159°19'02''$$

$$\gamma = \arccos\left(\frac{3}{\sqrt{187}}\right) = 77°19'38''$$

5. Find the direction cosines of the coordinates axes.

Sol. Any two points on the x-axis are (0, 0, 0) and $(a, 0, 0)$. Direction ratios of the x-axis are $a - 0, 0, 0 = a, 0, 0$ and hence the direction cosines are

$$= \frac{a}{\sqrt{a^2+0,0}}, \frac{0}{\sqrt{a^2+0,0}}, \frac{0}{\sqrt{a^2+0,0}} = 1, 0, 0$$

Proceeding as above, we find that the direction cosines of the y-axis are 0, 1, 0 and the direction cosines of the z-axis are 0, 0, 1.

6. Prove that if measures of the direction angles of a straight line are α, β and γ, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Sol. We know that if α, β, γ are the direction angles of a line, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

i.e., $(1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

i.e., $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

7. If measures of two of the direction angles of a straight line are 45° and 60°, find measure of the third direction angle.

Sol. Here $\alpha = 45°, \beta = 60°$. We need to find γ

Now $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or $\cos^2 45 + \cos^2 60 + \cos^2\gamma = 1$

or $\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$

or $\cos^2\gamma = 1 - \frac{1}{2} = \frac{1}{4}$

i.e., $\cos\gamma = \pm\frac{1}{2}$ or $\gamma = 60°$

8. The direction cosines l, m, n of two straight lines are given by the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$. Find measure of the angle between them.

Sol. We know that if l, m, n are the direction cosines of a line then

$$l^2 + m^2 + n^2 = 1 \tag{1}$$

Also $l^2 + m^2 - n^2 = 0$ (2)

Subtracting (2) form (1), we get

$$n^2 = \frac{1}{2} \quad \text{or} \quad n = \frac{1}{\sqrt{2}} \quad \text{or} \quad -\frac{1}{\sqrt{2}} \tag{3}$$

Again, adding (1) and (2), we have

$$2(l^2 + m^2) = 1 \quad \text{or} \quad l^2 + m^2 = \frac{1}{2} \tag{4}$$

Now $l + m + n = 0$ (given)

or $l = -\left(m - \frac{1}{\sqrt{2}}\right)$

Putting this values of l into (4), we have

$$\left[-m - \frac{1}{\sqrt{2}}\right]^2 + m^2 = \frac{1}{2}$$

or $m^2 + \frac{2}{\sqrt{2}}m + \frac{1}{2} + m^2 = \frac{1}{2}$

i.e., $2m\left(m + \frac{1}{\sqrt{2}}\right) = 0 \Rightarrow m = 0$ or $\frac{-1}{\sqrt{2}}$

Thus $l = -\left(m + \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$ or 0

Values of l, m, n are $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$ or $0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Similarly, other values of l, m, n can be found by taking $n = -\frac{1}{\sqrt{2}}$.

Thus direction cosines of the two lines are

$$-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \text{ and } 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \theta = \frac{\pi}{3}$$

is the angle between the lines.

9. The direction cosines l, m, n of two straight lines are given by the equations $l + m + n = 0$ and $2lm + 2ln - mn = 0$. Find measure of the angle between them.

Sol.
$$l + m + n = 0 \quad (1)$$
$$2lm + 2ln - mn = 0 \quad (2)$$

From (1), $n = -(l + m)$

Substituting into (2), we have

$$2lm - 2l(l + m) + m(l + m) = 0$$

or $2lm - 2l^2 - 2lm + lm + m^2 = 0$

or $2l^2 - lm - m^2 = 0$

or $(l - m)(2l + m) = 0$

$\Rightarrow l = m$ or $2l = -m$

Solving

$l + m + n = 0$,
and $l - m = 0$, we get
$$\frac{l}{1} = \frac{m}{1} = \frac{n}{-1-1}$$

$l + m + n = 0$
$2l + m = 0$, we have
$$\frac{l}{-1} = \frac{m}{2} = \frac{n}{1-2}$$

Direction cosines of the lines are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$ and $\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$

If θ is measure of the angle between the two lines, then

$$\cos\theta = \frac{1(-1) + 1 \times 2 + (-2)(-1)}{\sqrt{6}.\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

Thus $\theta = \frac{\pi}{3}$

Find equations of the straight lines L and M in symmetric forms. Determine whether the pairs of lines intersect. Find the point of intersection if it exists. (Problems 10 – 12):

10. L : through $A(2, 1, 3)$, $B = (-1, 2, 4)$
M : through $P(5, 1, -2)$, $Q(0, 4, 3)$

Sol. Parametric equations of the lines are

$$L: \begin{cases} x = 2 - 3t \\ y = 1 + t \\ z = 3 + t \end{cases} \quad \text{and} \quad M: \begin{cases} x = 5 - 5s \\ y = 1 + 3s \\ z = -2 + 5s \end{cases}$$

The lines intersect if the system of equations

$$2 - 3t = 5 - 5s \quad (1)$$
$$1 + t = 1 + 3s \quad (2)$$
$$3 + t = -2 + 5s \quad (3)$$

has a solution.

Solving (1) and (2) we obtain $t = \frac{-9}{14}$ and $s = \frac{3}{14}$. But these values of t and s do not satisfy (3). Hence the lines do not intersect. The lines are also non-parallel.

11. L : $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + t(6\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$
M : $\mathbf{r} = (5\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) + ss(14\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

Sol. Equations of the lines are

$$L: \begin{cases} x = 3 + 6t \\ y = 2 \cdot 4t \\ z = -1 - 3t \end{cases} \quad \text{and} \quad M: \begin{cases} x = 5 + 14s \\ y = 4 - 6s \\ z = 7 + 2s \end{cases}$$

The lines intersect if the system of equations

$$3 + 6t = 5 + 14s \quad (1)$$
$$2 - 4t = 4 - 6s \quad (2)$$
$$-1 - 3t = 7 + 2s \quad (3)$$

has a solution.

Solving (1) and (2) we obtain $t = -2$, $s = -1$. These values of t and s satisfy (3). Hence the lines intersect. Substituting $t = -2$ into the equation for L, the point of intersection is $(-9, 10, 5)$.

12. L : through $A(2, -1, 0)$ and parallel to $\mathbf{b} = [4, 3, -2]$

M : through $P(-1, 3, 5)$ and parallel to $\boldsymbol{c} = [1, 7, 3]$.

Sol. Equations of the lines are

$$L:\begin{bmatrix} x = 2 + 4t \\ y = -1 + 3t \\ z = 0 - 2t \end{bmatrix} \quad \text{and} \quad M:\begin{bmatrix} x = -1 + s \\ y = 3 + 7s \\ z = 5 + 3s \end{bmatrix}$$

The lines intersect if the system of equations

$$2 + 4t = -1 + s \qquad (1)$$
$$-1 + 3t = 3 + 7s \qquad (2)$$
$$-2t = 5 + 3s \qquad (3)$$

has a solution.

Solving (1) and (3), we get

$$t = -1 \quad \text{and} \quad s = -1$$

These values of t and s satisfy (2). Hence the lines intersect in a point. Substituting $s = -1$ in the equations for M, the point of intersection is $(-2, -4, 2)$.

Find the distance of the given point P from the given line L. (Problems 13 – 14):

13. $P = (3, -2, 1)$, L : $\begin{cases} x = 1 + t \\ y = 3 - 2t \\ z = -2 + 2t \end{cases}$

Sol. A point on L is $A = (1, 3, -2)$

$$\overrightarrow{AP} = [2, -5, 3]$$

Direction vector of L is $\boldsymbol{b} = [1, -2, 2]$

Required distance

$$d = \frac{|\overrightarrow{AP} \times \boldsymbol{b}|}{|\boldsymbol{b}|} = \frac{1}{3}\left|\text{Det}\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -5 & 3 \\ 1 & -2 & 2 \end{bmatrix}\right|$$

$$= \frac{1}{3}|-4\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}| = \frac{\sqrt{18}}{3} = \sqrt{2}.$$

14. $P = (0, -2, 1)$, L : $\dfrac{x-1}{4} = \dfrac{y+3}{-2} = \dfrac{z+1}{5}$.

Sol. A point on L is $A = (1, -3, -1)$

$$\overrightarrow{AP} = [-1, 1, 2,]$$

Direction vector of L is $\boldsymbol{b} = [4, -2, 5]$

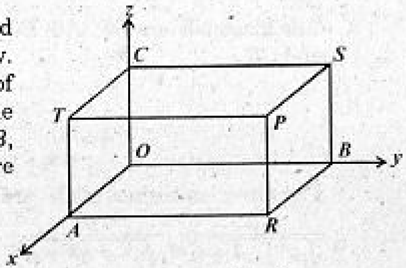
$$d = \frac{|\overrightarrow{AP} \times \boldsymbol{b}|}{|\boldsymbol{b}|} = \frac{1}{45}\left|\text{Det}\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix}\right|$$

$$= \frac{1}{\sqrt{45}}|9\boldsymbol{i} + 13\boldsymbol{j} - \boldsymbol{k}| = \sqrt{\frac{254}{45}}$$

15. If the edges of a rectangular parallelepiped are a, b, c; show that angles between the four diagonals are given by

$$\arccos\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right).$$

Sol. Lengths the edges OA, OB and OC are a, b, c respectively. Therefore, the coordinates of the vertices of the parallelopiped (with OA, OB, OC as coordinate axes) are



$O = (0, 0, 0)$.
$A = (a, 0, 0)$, $B = (0, b, 0)$
$C = (0, 0, c)$, $R = (a, b, 0)$
$S = (0, b, c)$, $T = (a, 0, c)$
and $P = (a, b, c)$

The four diagonals are OP, AS, CR and BT.

Direction ratios of OP are a, b, c

direction cosines of OP are

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Direction ratios of AS are $-a$, b, c

direction cosines of AS are

$$\frac{-a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Therefore, if α is the angle between OP and AS, then

$$\cos\alpha = \frac{-a^2 + b^2 + c^2}{a^2 + b^2 + c^2}, \text{[using } \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2]$$

i.e., $\alpha = \arccos\left(\dfrac{-a^2 + b^2 + c^2}{a^2 + b^2 + c^2}\right).$

Similarly, the angles between the remaining diagonals are found to be one of the angles

$$\arccos\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right).$$

16. A straight line makes angles of measures α, β, γ, δ with the four diagonals of a cube. Prove that

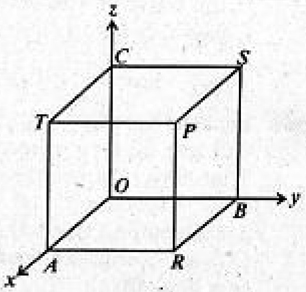
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

Sol. Suppose the edge of the cube is a.

Then as in the previous question the vertices of the cube are $O = (0, 0, 0)$

$A = (a, 0, 0), B = (0, a, 0)$
$C = (0, 0, a), R = (a, a, 0)$
$S = (0, a, a), T = (a, 0, a),$
and $P = (a, a, a)$

The diagonals are OP, AS, BT and CR



The direction cosines of OP are

$$\frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{b}{\sqrt{a^2+a^2+a^2}}, \frac{c}{\sqrt{a^2+a^2+a^2}} = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \qquad (1)$$

Similarly, direction cosines of AS are

$$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \qquad (2)$$

direction cosines of BT are

$$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \qquad (3)$$

and of CR are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ (4)

Let the direction cosines of the line which makes angles α, β, γ and δ with OP, AS,, BT and CR respectively be l, m, n. Then

$$\cos\alpha = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \qquad (5)$$

$$\cos\beta = \frac{-l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \qquad (6)$$

$$\cos\gamma = \frac{l}{\sqrt{3}} + \frac{-m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \qquad (7)$$

and $$\cos\delta = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{-n}{\sqrt{3}} \qquad (8)$$

Squaring (5), (6), (7) and (8) and adding the results, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}(l^2 + m^2 + n^2)$$

$$= \frac{4}{3}, \text{ since } l, m, n, \text{ are the direction cosines.}$$

17. Find equation of the straight line passing through the point $P(0, -3, 2)$ and parallel to the straight line joining the points $A(3, 4, 7)$ and $B(2, 7, 5)$.

Sol. The line joining the points $A(3, 4, 7)$ and $B(2, 7, 5)$ has direction ratios

$$2-3, 7-4, 5-7 \quad \text{or} \quad -1, 3, -2$$

As the required line is parallel to AB, it must have the direction ratios $-1, 3, -2$

Therefore, the line through the point $(0, -3, 2)$ with the direction ratios $-1, 3, -1$ is

$$\frac{x-0}{-1} = \frac{y+3}{3} = \frac{z+1}{-2} \quad \text{or} \quad \frac{x}{1} = -y + \frac{3}{3} = \frac{z+2}{2}$$

18. Find equations of the straight line passing through the point $P(2, 0, -2)$ and perpendicular to each of the straight lines

$$\frac{x-3}{2} = \frac{y}{2} = \frac{z+1}{2} \quad \text{and} \quad \frac{x}{3} = \frac{y+1}{-1} = \frac{z+2}{2}.$$

Sol. Direction ratios of the given lines are 2, 2, 2 and 3 – 1, 2. If the required line has direction ratios c_1, c_2, c_3 then by the condition of perpendicularity, we have

$$2c_1 + 2c_2 + 2c_3 = 0 \quad \text{and} \quad 3c_1 - c_2 + 2c_3 = 0$$

Therefore, $$\frac{c_1}{4+2} = \frac{c_2}{6-4} = \frac{c_3}{-2-6}$$

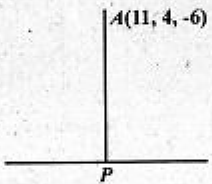
i.e., $\frac{c_1}{6} = \frac{c_2}{2} = \frac{c_3}{-8}$ *i.e.*, $c_1 : c_2 : c_3 = 3 : 1 : -4$

The required line through the point $P(2, 0, -2)$ with d.r.'s c_1, c_2, c_3 is $\frac{x-2}{3} = \frac{y}{1} = \frac{z+2}{-4}$

Find equations of the straight line through the given point *A* and intersecting at right angle the given straight line (Problems 19 – 20):

19. $A = (11, 4, -6)$ and $x = 4 - t, y = 7 + 2t, z = -1 + t$

Sol. Suppose the perpendicular from A meets the given line in P. The given line, in the symmetic form, has equations.



$$\frac{x-4}{-1} = \frac{y-7}{2} = \frac{z+1}{1} = t$$

We shall find t such that $(x, y, z) = (4 - t, 7 + 2t, -1 + t)$ are the coordinates of P.

Direction ratios of AP are

$4 - t - 11, 7 + 2t - 4, -1 + t + 6$

i.e., $-t - 7, 2t + 3, t + 5$ (1)

Direction ratios of the given line are $-1, 2, 1$

Since AP is perpendicular to the given line, so

$$(-1)(-t-7) + 2(2t+3) + 1(t+5) = 0$$

i.e., $t + 4t + t + 7 + 6 + 5 = 0$ or $t = -3$

Hence direction ratios of AP are $-4, -3, 2$

Equations of the required line AP are

$$\frac{x-11}{-4} = \frac{y-4}{-3} = \frac{z+6}{2}$$

20. $A = (5, -4, 4)$ and $\dfrac{x}{-1} = \dfrac{y-1}{1} = \dfrac{z}{-2} = t$ (1)

Sol. Let $P\ (x, y, z)$ be the point on the given line such that AP is perpendicular to (1). Direction ratios of the line AP are

$$x-5, y+4, z-4.$$

Since AP is perpendicular to (1), we have

$$-(x-5) + (y+4) - 2(z-4) = 0$$

or $\quad -(-t-5) + (t+1+4) - 2(-2t-4) = 0$ giving $t = -3$

Thus $P = (3, -2, 6)$

Direction ratios of AP are $1, -1, -1$.

Equations of the required line AP are

$$x = 5 + t, y = -4 - t, z = 4 - t$$

21. Find the length of the perpendicular from the point $P(x_1, y_1, z_1)$ to the straight line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}, \text{ where } l^2 + m^2 + n^2 = 1.$$

Sol. Let $P = (x_1, y_1, z_1)$. The point $A = (\alpha, \beta, \gamma)$ lies on the given line and its direction vector is $\boldsymbol{b} = [l, m, n]$. The required perpendicular distance from P to the given line is

$$d = \frac{|\overrightarrow{AP} \times \boldsymbol{b}|}{|\boldsymbol{b}|} = |[x_1-\alpha, y_1-\beta, z_1-\gamma] \times [l, m, n]|$$

$$= \left|\operatorname{Det}\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1-\alpha & y_1-\beta & z_1-\gamma \\ l & m & n \end{pmatrix}\right| = \left(\Sigma\begin{vmatrix} x_1-\alpha & y_1-\beta \\ l & m \end{vmatrix}^2\right)^{1/2}$$

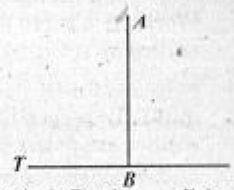
22. Find equations of the perpendicular from the point $P(1, 6, 3)$ to the straight line

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}.$$

Also obtain its length and coordinates of the foot of the perpendicular.

Sol. Let $A = (1, 6, 3)$ and P be the foot of the perpendicular on

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = t$$



If $\quad x = t, y = 1 + 2t, z = 3t + 2$ are coordinates of P, then direction ratios of AP are

$$t-1, 1+2t-6, 3t+2-3$$

or $\quad t-1,\ 3t-5\ ,\ 3t-5,\ 3t-1$

Also direction ratios of the given line are 1, 2, 3. By the condition of perpendicularity, we have

$$(t-1)(1) + (2t-5)^2 + 3(3t-1) = 0$$

or $\quad 14t = 14$ *i.e.,* $t = 1$

Hence coordinates of the foot P of the perpendicular are (1, 3, 5)

Length of the perpendicular $= |AP|$

$$= \sqrt{(1-1)^2 + (3-6)^2 + (5-3)^2}$$

$$= \sqrt{0+9+4} = \sqrt{13}$$

Equation of the perpendicular AP are

$$\frac{x-1}{1-1} = \frac{y-6}{3-6} = \frac{z-3}{5-3} \text{ or } \frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$$

23. Find a necessary and sufficient condition that the points $P(x_1, y_1, y_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ are collinear.

Sol. Equations of a line through $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ are

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$$

Thus $x = x_1 + (x_2-x_1)t, y = y_1 + (y_2-y_1)t\ ;\ z = z_1 + (z_2-z_1)t$

If $\quad (x_3, y_3, z_3)$ lies on this line; then

$$x_3 = x_1 + (x_2-x_1)t = (1-t)x_1 + t x_2$$

$$y_3 = y_1 + (y_2-y_1)t = (1-t)y_1 + t y_2$$

$$z_3 = z_1 + (z_2-z_1)t = (1-t)z_1 + z_2$$

Thus we have

$$(1-t)x_1 + t x_2 - x_3 = 0$$

$$(1-t)y_1 + t y_2 - y_3 = 0$$

$$(1-t)z_1 + t z_2 - z_3 = 0$$

Eliminating t from the last three equations, we get

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x_1 & y_2 & z_3 \\ x_1 & y_2 & z_3 \\ x_1 & y_2 & z_3 \end{vmatrix} = 0$$

which is a necessary condition for the three points to be collinear. Working algebra backward, we find that this condition is also sufficient.

24. If l_1, m_1, n_1; l_2, m_2, n_2, and l_3, m_3, n_3 are direction cosines of three mutually perpendicular lines, prove that the line whose direction-cosines are proportional to $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$ makes congruent angles with them.

Sol. We are given that

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad (1)$$
$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \quad (2)$$
$$l_3 l_1 + m_3 m_1 + n_3 n_1 = 0 \quad (3)$$

Also $l_i^2 + m_i^2 + n_i^2 = 1$, $i = 1, 2, 3$ (4)

Suppose θ_1 is measure of the angle between the lines with d.c.'s l_1, m_1, n_1 and $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$. Then

$$\cos\theta_1 = \frac{l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)}{\sqrt{(l_i^2+m_i^2+n_i^2)}\sqrt{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2}}$$

$$= \frac{l_1^2+m_1^2+n_1^2+(l_1l_2+m_1m_2+n_1n_2)+(l_1l_3+m_1m_3+n_1n_3)}{\Sigma(l_1+l_2+l_3)^2}$$

$$= \frac{1}{\sqrt{\Sigma(l_1 + l_2 + l_3)}}$$

Similarly, measure of the angle θ_2, between lines with d.c.'s l_2, m_2, n_2 and $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$ is given by

$$\cos\theta_2 = \frac{1}{\sqrt{\Sigma(l_1 + l_2 + l_3)}}$$

In a similar manner, $\cos\theta_3 = \dfrac{1}{\sqrt{\Sigma(l_1 + l_2 + l_3)}}$

Thus $\theta_1 = \theta_2 = \theta_3$

25. A variable line in two adjacent positions has direction cosines l, m, n and $l + l\delta, m + m\delta, n + \delta n$. Show that measure of the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Sol. Since l, m, n and $l + \delta l, m + \delta m, n + \delta n$ are direction cosines of a line in two adjacent positions, we have

$$l^2 + m^2 + n^2 = 1 \quad (1)$$

and $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$ (2)

Using (1), we get from (2)

$$2(\delta l + m\,\delta m + n\,\delta n) + (\delta l)^2 + (\delta m)^2 + (\delta m)^2 = 0 \quad (3)$$

The small angle $\delta\theta$ between the two adjacent positions of the line is given by

$$\cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$
$$= 1 + l\,\delta l + m\,\delta m + n\,\delta n$$

or $\quad -1 + \cos\delta\theta = -2\sin^2\left(\frac{\delta\theta}{2}\right) = l\,\delta l + m\,\delta m + n\,\delta n$

i.e., $\quad 4\sin^2\left(\frac{\delta\theta}{2}\right) = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$, (using (3)) (4)

Since $\delta\theta$ is small, so is $\frac{\delta\theta}{2}$

Therefore, $\dfrac{\sin\left(\frac{\delta\theta}{2}\right)}{\delta\theta/2} \longrightarrow 1$

i.e., $\quad \sin\left(\frac{\delta\theta}{2}\right) \longrightarrow \frac{\delta\theta}{2}$

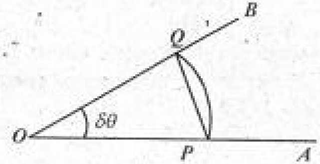
or $\quad \sin^2\left(\frac{\delta\theta}{2}\right) \longrightarrow \frac{(\delta\theta)^2}{4}$ (5)

Hence from (4) and (5), we have

$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

Alternative Method:

Let OA, OB be the two adjacent positions of the line. Let PQ be the arc of the circle with centre O and radius 1. Then coordinates of P and Q are (l, m, n) and $(l + \delta l, m + \delta m, n + \delta n)$ respectively.



Now $\delta\theta$ = arc $PQ \longrightarrow$ chord PQ, since $\delta\theta$ is small
$= [(\delta l)^2 + (\delta m)^2 + (\delta n)^2]^{1/2}$

Therefore, $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Exercise Set 8.3 (Page 363)

Find an equation of the plane through the three given points (Problems 1 – 3):

1. (2, 1, 1), (6, 3, 1), (–2, 1, 2).

Sol. ***1st Method:***

Suppose equation of the required plane is

$$ax + by + cz + 1 = 0 \qquad (1)$$

Since it passes through the points (2, 1, 1); (6, 3, 1); (–2, 1, 2) they satisfy (1). Therefore, we have

$$2a + b + c + 1 = 0$$
$$6a + 3b + c + 1 = 0$$
$$-2a + b + 2c + 1 = 0$$

Solving these equations, we get.

$$a = -\frac{1}{4}, b = \frac{2}{4}, c = -1$$

Equation of the required plane is

$$-\frac{1}{4}x + \frac{2}{4}y - z + 1 = 0 \quad \text{or} \quad x - 2y + 4z - 4 = 0$$

2nd Method:

Let $P = (2, 1, 1), Q = (6, 3, 1), R = (-2, 1, 2)$

$$\overrightarrow{PQ} = [4, 2, 0], \overrightarrow{PR} = [-4, 0, 1]$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 4 & 2 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2i - 4j + 8k \qquad (1)$$

(1) is a normal vector to the required plane. Equation of the plane through $R(-2, 1, 2)$ having (1) as a normal vector is

$$2(x + 2) - 4(y - 1) + 8(z - 2) = 0$$

or $x - 2y + 4z - 4 = 0$

2. (1, –1, 2), (–3, –2, 6), (6, 0, 1).

Sol. Let an equation of the plane be $ax + by + cz = 1$

Since it passes through the points (1, –1, 2), (–3, –2, 6), (6, 0, 1), we have

$$a - b + 2c = 1$$
$$-3a - 2b + 6c = 1$$
$$6a + 0b + c = 1$$

Solving these equations, we have

$$a = \frac{3}{17}, b = \frac{-16}{17}, c = \frac{-1}{17}$$

Hence an equation of the plane is $3x - 16y - z = 17$

3. (–1, 1, 1), (5, –8, –2), (4, 1, 0).

Sol. Let an equation of the plane be $ax + by + cz = 1$

Since it passes through the points (–1, 1, 1), (5, –8, –2) and (4, 1, 0), we have

$$-a + b + c = 1$$
$$5a - 8b - 2c = 1$$
$$4a + b = 1$$

Solving these equation, we obtain $a = \frac{1}{3}, b = \frac{1}{3}, c = \frac{5}{3}$

Hence an equation of the plane is $x - y + 5z = 3$

4. Find equations of the planes bisecting the angles between the planes

$$3x + 2y - 6z + 1 = 0 \text{ and } 2x + y - 5 = 0.$$

Sol. Points on the planes bisecting the angles between the given planes are equidistant from them. Let (x, y, z) be a point on the planes bisecting the angles between the given planes. Equations of the required planes are

$$\frac{2x + y + 2z - 5}{3} = \pm \frac{3x + 2y - 6z + 1}{7}$$

or $14x + 7y + 14z - 35 = \pm (9x + 6y - 18z + 3)$

i.e., $5x + y + 32z - 38 = 0$ and $23x + 13y - 4z - 32 = 0$

5. Transform the equations of the planes $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$ to normal forms and hence find measure of the angle between them.

Sol. $3x - 4y + 5z = 0 \qquad (1)$

Its normal form is

$$\frac{3}{5\sqrt{2}}x - \frac{4}{5\sqrt{2}}y + \frac{5}{5\sqrt{2}}z = 0$$

The plane $2x - y - 2z = 5$ has normal form as

$$\frac{2}{3}x - \frac{1}{3}y - \frac{2}{3}z = \frac{5}{3} \qquad (2)$$

Measure of the angle between the planes is given by

$$\cos\theta = \frac{3}{5\sqrt{5}} \times \frac{2}{3} + \left(-\frac{4}{5\sqrt{2}}\right)\left(-\frac{1}{3}\right) + \left(\frac{5}{5\sqrt{2}}\right)\left(-\frac{2}{3}\right)$$
$$= \frac{2}{5\sqrt{2}} + \frac{4}{15\sqrt{2}} - \frac{2}{3\sqrt{2}} = \frac{6 + 4 - 10}{15\sqrt{2}} = 0$$

Hence $\theta = \frac{\pi}{2}$

6. Find equations of the planes through the points (4, –5, 3) and (2, 3, 1) and parallel to the coordinate exes.

Sol. Let a plane parallel to the x-axis be $ax + by + cz = 1$ where a, b, c are direction ratios of a normal to the plane. The direction cosines of the x-axis are (1, 0, 0). Normal to the plane is perpendicular to the x-axis.

Hence $a.1 + b.0 + c.0 = 0$ or $a = 0$.

Therefore, an equation of the plane parallel to the x-axis is

$$by + cz = 1.$$

Since the points (4, –5, 3) and (2, 3, 1) lie on this plane, we have

$$\left.\begin{aligned}-5b + 3c &= 1\\ \text{and}\quad 3b + c &= 1\end{aligned}\right\}$$

These equations give $b = \frac{1}{7}$, $c = \frac{4}{7}$

The plane parallel to the x-axis and passing through the given points is $\frac{1}{7}y + \frac{4}{7}z = 1$ or $y + 4z = 7$

Similarly, a plane parallel to the y-axis is $ax + cz = 1$. On substituting the given points into this equations, we have

$4a + 3c = 1$ and $2a + c = 1$ which give $a = 1, c = -1$.

The plane parallel to the y-axis and passing through the given points is $x - z = 1$.

The plane parallel to the z-axis can be found similarly as $4x + y - 11 = 0$.

7. Find an equation of the plane through the points (1, 0, 1) and (2, 2, 1) and perpendicular to the plane $x - y - z + 4 = 0$.

Sol. Let the required plane be $ax + by + cz = 1$. As this plane is to be perpendicular to $x - y - z + 4 = 0$, we have

$$a.1 + b(-1) + c(-1) = 0$$

$$\text{i.e.,}\ a - b - c = 0 \qquad (1)$$

Also the points (1, 0, 1) and (2, 2, 1) lie on the required plane

Therefore, $a + c = 1$ (2)

and $2a + 2b + c = 1$ (3)

Solving (1), (2) and (3) simultaneously, we have

$$a = \frac{2}{5}, b = \frac{-1}{5}, c = \frac{3}{5}$$

Hence the required plane is

$$\frac{2}{5}x - \frac{1}{5}y + \frac{3}{5}z = 1 \text{ or } 2x - y + 3z = 5$$

8. Find an equation of the plane which is perpendicular bisector of the line segment joining the points (3, 4, –1) and (5, 2, 7).

Sol. Direction ratios of the line joining the given points are

$$5 - 3, 2 - 4, 7 - (-1) \quad \text{i.e.,} \quad 2, -2, 8$$

These are direction ratios of a normal to the required plane.

Also the mid-point of the line segment joining the given points is

$$\left(\frac{3+5}{2}, \frac{2+4}{2}, \frac{7-1}{2}\right) = (4, 3, 3).$$

Hence equation of the required plane is

$$2(x - 4) + (-2)(y - 3) + 8(z - 3) = 0$$

or $2x - 2y = 8z = 26$ or $x - y + 4z = 13$

9. Show that the join of (0, –1, 0) and (2, 4, –1) intersects the joins of (1, 1, 1) and (3, 3, 9).

Sol. We first show that the two joins are coplanar.

Equation of the plane through (0, –1, 0), (2, 4, –1) and (1, 1, 1) is

$$\begin{vmatrix} x & y & z & 1\\ 0 & -1 & 0 & 1\\ 2 & 4 & -1 & 1\\ -1 & 1 & 1 & 1\end{vmatrix} = 0 \text{ or } 7x - 3y - z - 3 = 0$$

The fourth point (3, 3, 9) also satisfies this equation. Hence the two joins are coplanar.

Direction ratios of the join of

(0, –1, 0) and (2, 4, –1) are 2, 5, –1 (1)

and the direction ratios of the join of

(1, 1, 1) and (3, 3, 9) are 2, 2, 8 (2)

Since (1) and (2) are not proportional, the two joins are not parallel and hence being coplanar, they intersect.

10. The vertices of a tetrahedron are (0, 0, 0), (3, 0, 0), (0, –4, 0) and (0, 0, 5). Find equations of the planes of its faces.

Sol. Let the vertices be denoted by $A = (0, 0, 0)$, $B = (3, 0, 0)$, $C = (0, -4, 0)$ and $D = (0, 0, 5)$.

Equation of any plane through the point A is $ax + by + cz = 0$. If it passes through B and C, then $a = 0$ and $b = 0$. Thus the plane through A, B, C is $cz = 0$ or $z = 0$.

Similarly, plane through A, B and D is $y = 0$ and plane through A, C and D is $x = 0$.

Now we find the plane through B, C and D. This can be written in the intercept form as

$$\frac{x}{3} + \frac{y}{-4} + \frac{z}{5} = 1.$$

Equations of the planes of the four faces are

$$x = 0, y = 0, z = 0, \frac{x}{3} - \frac{y}{4} + \frac{z}{5} = 1$$

11. Find an equation of the plane through $(5, -1, 4)$ and perpendicular to each of the planes $x + y - 2z - 3 = 0$ and $2x - 3y + z = 0$

Sol. A plane through $(5, -1, 4)$ is $a(x-5) + b(y+1) + c(z-4) = 0$

Since it is perpendicular to each of the given planes, we have

$$a + b - 2c = 0 \quad \text{and} \quad 2a - 3b + c = 0$$

or $$\frac{a}{1-6} = \frac{b}{-4-1} = \frac{c}{-3-2} \text{ or } \frac{a}{1} = \frac{b}{1} = \frac{c}{1} \text{ i.e., } a:b:c = 1:1:1$$

Hence equation of the required plane is

$$x - 5 + y + 1 + z - 4 = 0 \text{ or } \quad x + y + z - 8 = 0$$

12. Find an equation of the plane each of whose point is equidistant from the points $A(2, -1, 1)$ and $B(3, 1, 5)$.

Sol. $\overrightarrow{AB} = [1, 2, 4]$

This vector is a normal of the required plane since each points of the plane is equidistant from A, B. Equation of a plane with $\overrightarrow{AB}$ as a normal vector is $x + 2y + 4z + d = 0$.

The mid-point $\left(\frac{5}{2}, 0, 3\right)$ of the line segment $\overrightarrow{AB}$ also lies on this plane. Hence $\frac{5}{2} + 12 + d = 0$ or $d = -\frac{29}{2}$

Equation of the required plane is

$$x + 2y + 4z - \frac{29}{2} = 0 \text{ i.e., } 2x + 4y + 8z - 29 = 0$$

13. Find an equation of the plane through the point $(3, -2, 5)$ and perpendicular to the line $x = 2 + 3t, y = 1 - 6t, z = -2 + 2t$.

Sol. Direction ratios of the line are $3, -6, 2$

Since the plane is perpendicular to the given line, direction ratios of the line are direction ratios of a normal to the plane.

Equation of such a plane is $3x - 6y + 2z + d = 0$

Since it passes through $(3, -2, 5)$, we have

$$9 + 12 + 10 + d = 0 \text{ or } d = -31.$$

Equation of the required plane is $3x - 6y + 2z - 31 = 0$

14. Find parametric equations of the line containing the point $(2, 4, -$ and perpendicular to the plane $3x + 3y - 7z = 9$.

Sol. Since the line is to be perpendicular to the given plane, direction ratios of the line are $3, 3, -7$

Equations of the line through $(2, 4, -3)$ with these direction ratios are

$$\frac{x-2}{3} = \frac{y-4}{3} = \frac{z+3}{-7} = t$$

i.e., $x = 2 + 3t, y = 4 + 3t, z = -3 - 7t$

are the required parametric equations of the line.

15. Write equation of the family of all planes whose distance from the origin is 7. Find those members of the family which are parallel to the plane $x + y + z + 5 = 0$.

Sol. An equation of the family of all required planes in normal form is

$$lx = my + nz = 7$$

where l, m, n are direction cosines of normals to the planes. Equation of the plane $x + y + z + 5 = 0$ in the normal from is

$$\frac{x}{-\sqrt{3}} + \frac{y}{-\sqrt{3}} + \frac{z}{-\sqrt{3}} - \frac{5}{-\sqrt{3}} = 0 \qquad (1)$$

A plane parallel to (1) has a normal vector with direction cosines

$$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \text{ or } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Thus, there are two members of the family parallel to (1). They are

$$-\frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z - 7 = 0$$

and $$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z - 7 = 0$$

16. Fin an equation of the family of the plane which passes through the point $(3, 4, 5)$, has an x-intercept equal to -5 and is perpendicular to the plane $2x + 3y - z = 8$.

Sol. Equation of a plane with intercepts -5, b and c on the x-axis, y-axis and z-axis respectively is

$$\frac{x}{-5} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (1)$$

As this plane is perpendicular to $2x + 3y - z = 8$, we have

$$2\left(-\frac{1}{5}\right) + 3\left(\frac{1}{b}\right) - \frac{1}{c} = 0 \text{ or } \frac{3}{b} - \frac{1}{c} = \frac{2}{5} \qquad (2)$$

Also the plane (1) passes through $(3, 4, 5)$, Therefore

$$\frac{3}{-5} + \frac{4}{b} + \frac{5}{c} = 1 \text{ or } \frac{4}{5} + \frac{5}{c} = 1 + \frac{3}{5} = \frac{8}{5} \qquad (3)$$

From (2) and (3), we have $b = \frac{95}{18}$, $c = \frac{95}{16}$

Equation of the required plane is

$$\frac{x}{-5} + \frac{18y}{95} + \frac{16z}{95} = 1 \text{ or } 19x - 18y - 16z + 95 = 0$$

17. Show that the distance of the point $P(3, -4, 5)$ from the plane $2x + 5y - 6z = 16$ measured parallel to the line

$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$ is $\frac{60}{7}$.

Sol. Equations of the line through $P(3, -4, 5)$ and parallel to the given line are

$$\frac{x-3}{2} = \frac{x+4}{1} = \frac{z-5}{-2} \qquad (1)$$

Any point on this line is $Q(2t + 3, t - 4, 2t + 5)$

If Q also lies on the given plane, then

$$2(2t + 3) + 5(t - 4) - 6(-2t + 5) = 16 \text{ or } t = \frac{20}{7}$$

Thus $Q\left(\frac{61}{7}, \frac{-8}{7}, \frac{-5}{7}\right)$ is the point where the line (1) meets the given plane.

Required distance $= |PQ| = \left[\left(\frac{40}{7}\right)^2 + \left(\frac{20}{7}\right)^2 + \left(\frac{40}{7}\right)^2\right]^{1/2} = \frac{60}{7}$

18. Show that the lines

$L : x = 3 + 2t, \quad y = 2 + t, \quad z = -2 - 3t$

$M : x = -3 + 4s, \quad y = 5 - 4s, \quad z = 6 - 5s$

intersect. Find an equation of the plane containing these lines.

Sol. The lines intersect if the equations

$3 + 2t = -3 + 4s$

$2 + t = 5 - 4s$

$-2 - 3t = 6 - 5s$

have simultaneous solution. Solving the first two equations, we find $t = -1$ and $s = 1$. Last equation is satisfied by these values of t and s. Hence the lines intersect.

Equations of the lines in symmetric forms are

$$L: \frac{x-3}{2} = \frac{y-2}{1} = \frac{z+2}{-3} \qquad (1)$$

$$M: \frac{x+3}{4} = \frac{y-5}{-4} = \frac{z-6}{-5} \qquad (2)$$

A point on the line (1) is $(3, 2, -2)$.

Since the required plane is to contain both the lines, it will contain every point of both lines.

Any plane through $(3, 2, -2)$ is $a(x - 3) + b(y - 2)(+ c(z + 2) = 0$

If this plane contains both (1) and (2), then normal vector of the plane is perpendicular to each of the two lines. Therefore,

$2a + b - 3c = 0$

$4a - 4b - 5c = 0$

$\frac{a}{-17} = \frac{b}{-2} = \frac{c}{-12}$

Hence equation of the desired plane is

$$17(x - 3) + 2(y - 2) + 12(z + 2) = 0$$

i.e., $17x + 2y + 12z - 31 = 0$.

19. If a, b, c are the intercepts of a plane on the coordinate axes and r is the distance of the origin from the plane, prove that

$$\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Sol. Equation of the plane, with a, b, c as intercepts on the x-axis, y-axis and z-axis respectively, is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Distance r of the plane from $(0, 0, 0)$ is

$$r = \frac{|-1|}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)} \quad i.e., \quad \frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

20. Find equations of two planes whose distances from the origin are 3 units each and which are perpendicular to the line through the points $A(7, 3, 1)$ and $B(6, 4, -1)$.

Sol. The line AB is normal to both the required planes. Direction ratios of AB are $1, -1, 2$. Direction cosines of the line AB are

$$\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$$

Equations of the planes at distance 3 unit each from the origin are

$$lx + my + nz = \pm 3$$

where l, m, n are direction cosines of normals to the two planes.

Since AB is normal to the two planes, we have

$$\frac{1}{\sqrt{6}}x - \frac{1}{\sqrt{6}}y + \frac{2}{\sqrt{6}}z = \pm 3$$

or $x - y + 2z = \pm 3\sqrt{6}$ are equations of the desired planes.

Exercise Set 8.4 (Page 366)

1. Prove that the planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ intersect and equations of their line of intersection can be written in the form

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}.$$

Sol. Direction ratios of normals to the two planes are $4, 4, -5$ and $8, 12, -13$ which are not proportional. Thus the planes are not parallel and so they intersect.

To find equations of their line of intersection in a symmetric form, we choose $z = 0$. Then the two equations are

$$4x + 4y = 12 \quad \text{and} \quad 8x + 12y = 32$$

which give $x = 1, y = 2$. Hence $(1, 2, 0)$ is a point on the line.
Similarly, putting $x = 0$, we find

$$4y - 5z = 12 \quad \text{and} \quad 12y - 13z = 32$$

which give $z = -2\,, y = \frac{1}{2}$. Therefore another point on the line is

$$\left(0, \frac{1}{2}, -2\right)$$

The required line passes through $(1, 2, 0)$ and $\left(0, \frac{1}{2}, -2\right)$.

Its equations are

$$\frac{x-1}{0-1} = \frac{y-2}{\frac{1}{2}-2} = \frac{z-0}{-2-0} \quad \text{or} \quad \frac{x-2}{2} = \frac{y-2}{3} = \frac{z}{4}$$

2. Find a symmetric form for the line $x + y + z + 1 = 0 = 4x + y - 2z + 2$.

Sol. Suppose $z = 0$. Then the given equations are

$$x + y + 1 = 0 \quad \text{and} \quad 4x + y + 2 = 0$$

The equations give $x = -\frac{1}{3}, y = \frac{-2}{3}$.

Hence $A\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$ is a point lying on the line.

Again, we let $x = 0$ and have $-y + z + 1 = 0$ and $y - 2z + z = 0$

which give $z = \frac{1}{3}$ and $y = \frac{-4}{3}$

Therefore, another point on the given line is $B\left(0, \frac{-3}{4}, \frac{1}{3}\right)$

Equations of the line through A and B in a symmetric form are

$$\frac{x+\frac{1}{3}}{0+\frac{1}{3}} = \frac{y+\frac{2}{3}}{\frac{-4}{3}+\frac{2}{3}} = \frac{z-0}{\frac{1}{3}-0} \quad \text{or} \quad \frac{x+\frac{1}{3}}{1} = \frac{y+\frac{2}{3}}{-2} = \frac{z}{1}$$

3. Show that the lines

L : $x + 2y - z - 7 = 0 = y + z - 2x - 6$

M : $3x + 6y - 3z - 8 = 0 = 2x - y - z$ are parallel.

Sol. If l_1, m_1, n_1 are direction ratios of the line L then since normals of the two planes are also normal to the line we have

$$1\,.\,l_1 + 2m_1 - n_1 = 0 \quad \text{and} \quad -2l_1 + m_1 + n_1 = 0$$

Therefore, $\frac{l_1}{3} = \frac{m_1}{1} = \frac{n_1}{5}$ (1)

Again if $l_2\,, m_2, n_2$ are direction ratios of the line M we have, as before

$$3l_2 + 6m_2 - 3n_2 = 0 \quad \text{and} \quad 2l_2 - m_2 - n_2 = 0$$

Thus $\frac{l_2}{-9} = \frac{m_2}{-3} = \frac{n_2}{-15}$ (2)

From (1) and (2), we find that

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{-1}{3}$$

Therefore, the two given lines are parallel.

4. Show that the lines

L : $x + 2y - 1 = 0 = 2y - z - 1$

M : $x - y - 1 = 0 = x - 2y - z$

are perpendicular.

Sol. Let direction ratios of the line L be l_1, m_1, n_1. Then we have

$$1\,.\,l_1 + 2m_1 + 0n_1 = 0 \quad \text{and} \quad 0\,l_1 + 2m_1 - n_1 = 0$$

Therefore, $\frac{l_1}{-2} = \frac{m_1}{1} = \frac{n_1}{2} = k_1$ (say)

or $l_1 = -2k_1\,, m_1 = k_1\,, n_1 = 2k_1$ (1)

Now, if l_2, m_2, n_2 are direction ratios of the line M, then we have

$$1\,.\,l_2 - 1\,.\,m_2 + 0n_1 = 0 \quad \text{and} \quad 1\,.\,l_2 - 0m_2 - 2n_2 = 0$$

Hence $\frac{l_2}{2} = \frac{m_2}{2} = \frac{n_2}{1} = k_2$ (say)

or $l_2 = 2k_2, m_2 = 2k_2, n_2 = k_2$ (2)

From (1) and (2), we have

$$l_1\,l_2 + m_1m_2 + n_1n_2 = -4k_1k_2 + 2k_1k_2 + 2k_1k_2 = 0$$

Hence the given lines are perpendicular to each other.

5. Find equations of the straight line through the point $(1, 2, 3)$ and parallel to the line $x - y + 2z - 5 = 0 = 3x + y + z + 6$. (1)

Sol. Suppose direction ratios of the required line are l, m, n. Since the required line is parallel to the line (1), it is perpendicular to the normals of each of the planes constituting the line (1). Therefore,

$$1\,.\,l - 1\,.\,m + 2\,n = 0 \quad \text{and} \quad 3l + m + n = 0$$

From these equations, we get

$$\frac{l}{-3} = \frac{m}{5} = \frac{n}{4}$$

Equations of the required line are

$$\frac{x-1}{-3}=\frac{y-2}{5}=\frac{z-3}{4}$$

6. Find equations of the planes containing the line $x-y-z=0=2x-y+3z-5$ and perpendicular to the coordinate planes.

Sol. Any plane through the given line is

$$x+y-z+k(2x-y+3z-5)=0$$

or $\quad (1+2k)x+(1-k)y+(3k-1)z-5k=0 \qquad (1)$

A normal vector to this plane is

$$[1+2k, 1-k, 3k-1]$$

The plane (1) is perpendiculer to the yz-plane whose equation is $x=0$. Hence $1(2+2k)=0$ which gives $k=-\frac{1}{2}$

Putting this value of k into (1), we have

$$\frac{3}{2}y-\frac{5}{2}z+\frac{5}{2}=0 \qquad \text{or} \qquad 3y-5z+5=0$$

as an equation of the plane perpendicular the to the yz-plane.

Equations of planes perpendicular to the zx and xy-planes can be found in a similar manner.

7. Fin an equation of the plane containing the line $x=2t, y=3t, z=4t$ and intersection of the planes $x+y+z=0$ and $2y-z=0$.

Sol. A plane through the intersection of $x+y+z=0$ and $2y-z=0$ is

$$x+y+z+k(2y-z)=0$$

i.e., $\quad x+(1+2k)y+(1-k)z=0 \qquad (1)$

Since this plane contains the line $x=2t, y=3t, z=4t$, a normal of the plane is perpendicular to the line. Therefore,

$2+3(1+2k)+4(1-k)=0$, (since 2, 3, 4, are direction ratios of the line)

or $\quad 2+3+6k+4-4k=0$ giving $k=-\frac{9}{2}$.

Substituting for k into (1), equation of the required plane is

$$x+(1-9)y+\left(1+\frac{9}{2}\right)z=0 \text{ or } 2x-16y+11z=0.$$

8. Write an equation of the family of planes having x-intercept 5, y-intercept 2 and a nonzero z-intercept. Find the member of the family which is perpendicular to the plane

$3x-2y+z-4=0. \qquad (1)$

Sol. Let nonzero z-intercept be c.

Equation of the required family of planes is

$$\frac{x}{5}+\frac{y}{2}+\frac{z}{c}=1, \quad \text{where } c \text{ is a parameter.}$$

If a member of this family is perpendicular to (1), then we have

$$\frac{3}{5}-\frac{2}{2}+\frac{1}{c}=0, \quad i.e., \quad c=\frac{5}{2}$$

The required plane is

$$\frac{x}{5}+\frac{2}{2}+\frac{z}{5/2}=1 \quad i.e., \quad \frac{x}{5}+\frac{y}{2}+\frac{2z}{5}=1.$$

9. Find an equation of the plane passing through the point (2, –3, 1) and containing the line $x-3=2y=3z-1$.

Sol. The given line can be written as

$$x-2y-3=0=2y-3z+1$$

Therefore, any plane through this line is

$(x-2y-3)+k(2y-3z+1)=0 \qquad (1)$

If it passes through (2, –3, 1), then

$$(2+6-3)+k(-6-3+1)=0 \text{ or } k=\frac{5}{8}$$

Putting this values of k into (1), we have

$$(x-2y-3)+\frac{5}{8}(2y-3z+1)=0 \text{ or } 8x-6y-15z=19$$

is an equation of the required plane.

10. Find an equation of the plane passing through the line of intersection of the planes $2x-y+2z=0$ and $x+2y-2z-3=0$ and at unit distance from the origin.

Sol. Any plane through the intersection of the given planes is

$$2x-y+3z+k(x+2y-2z-3)=0$$

or $\quad (2+k)x+(2k-1)y+(3-2k)z=3k \qquad (1)$

Now the perpendicular distance of this plane from the origin is 1. Therefore,

$$\frac{|-3k|}{\sqrt{(2+k)^2+(2k-1)^2+(3-2k)^2}}=1$$

or $\quad 9k^2=9k^2-12k+14$ or $k=\frac{7}{6}$

Putting this value of k into (1), we get

$$19x+8y+4z=21$$

as an equation of the required plane.

11. Find equations of the perpendicular from the origin to the line $x+2y+3z+4=0=2x+3y+4z+5$. Also find the coordinates of the foot of the perpendicular.

Sol. We first convert equations of the given line in a symmetric form. Let a, b, c be direction ratios of the given line. Since the line is perpendicular to the normal of each of the planes constituting the line, we have

$$a + 2b + 3c = 0$$
and $\quad 2a + 3b + 4c = 0$

or $\quad \dfrac{a}{-1} = \dfrac{b}{2} = \dfrac{c}{-1}$

Therefore, 1, −2, 1 are direction ratios of the line.

Now, we find a point on the given line. Let a point be such that $z = 0$

Then $x + 2y + 4 = 0$ and $2x + 3y + 5 = 0$

Solving these equations, we have $x = 2, y = -3$. Thus a point on the line is (2, −3, 0).

A symmetric form of the given line is

$$\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{1} = t \quad \text{(say)} \qquad (1)$$

Then, $A(2 + t, -3 - 2t, t)$ is any point on the line. Let A be the foot of the perpendicular from $O(0, 0, 0)$ to the line (1). Direction ratios of OA are

$$2 + t, -3 - 2t, t$$

Since OA is perpendicular to (1), we have

$$1(2 + t) - 2(-3 - 2t) + t = 0$$

i.e., $\quad 6t + 8 = 0 \quad$ giving $\quad t = -\dfrac{4}{3}$

Thus foot of the perpendicular is

$$A\left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$$

$$\text{Perpendicular distance} = |OA| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{16}{9}} = \sqrt{\frac{7}{3}}$$

Equations of the perpendicular OA are

$$\frac{x-0}{\frac{2}{3}} = \frac{y-0}{-\frac{1}{3}} = \frac{z-0}{-\frac{4}{3}} \text{ or } \frac{x}{2} = \frac{y}{-1} = \frac{z}{-4}$$

12. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Through A, B, C planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is given by $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Sol. Let an equation of the variable plane be

$$lx + my + nz = p \text{ where } l^2 + m^2 + n^2 = 1 \qquad (1)$$

or $\quad \dfrac{x}{p/l} + \dfrac{y}{p/m} + \dfrac{z}{p/n} = 1$

Thus coordinates of A, B, C are respectively

$$\left(\frac{p}{l}, 0, 0\right), \left(0, \frac{p}{m}, 0\right), \left(0, 2, \frac{p}{n}\right)$$

Equation of the plane through $A\left(\dfrac{p}{l}, 0, 0\right)$ and parallel to yz-plane is $x = \dfrac{p}{l}$

Similarly, equation of the plane through $B\left(0, \dfrac{p}{m}, 0\right)$ and parallel to xz-plane is $y = \dfrac{p}{m}$ and equation of the plane through $C\left(0, 0, \dfrac{p}{n}\right)$ and parallel to xy-plane is $z = \dfrac{p}{n}$.

Thus $\quad l = \dfrac{p}{x}, m = \dfrac{p}{y}, n = \dfrac{p}{z}$.

Since $\quad l^2 + m^2 + n^2 = 1$, we have

$$\frac{p^2}{x^2} + \frac{p^2}{y^2} + \frac{p^2}{z^2} = 1$$

or $\quad x^{-2} + y^{-2} + z^{-2} = p^{-2}$ as the required locus.

13. Let A, B C be the points as in **Problems 12**. Prove that the locus of the centroid of the tetrahedron $OABC$ is

$$x^{-2} + y^{-2} + z^{-2} = 16p^{-2}, O \text{ being the origin.}$$

Sol. From **Problem 12,** we have

$$A\left(\frac{p}{l}, 0, 0\right), B\left(0, \frac{p}{m}, 0\right), C\left(0, 2, \frac{p}{n}\right), O(0, 0, 0)$$

Coordinates of the centroid of the tetrahedron $OABC$ are

$$\left(\frac{p}{4l}, \frac{p}{4m}, \frac{p}{4n}\right).$$

i.e., $\quad x = \dfrac{p}{4l}, y = \dfrac{p}{4m}, z = \dfrac{p}{4n}$

We eliminate l, m, n from these equation to get the required locus.

We have $\quad l = \dfrac{p}{4x}, m = \dfrac{p}{4y}, n = \dfrac{p}{4z}$

Squaring these equations and adding the results, we have

$$1 = l^2 + m^2 + n^2 = \frac{p^2}{16x^2} + \frac{p^2}{16y^2} + \frac{p^2}{16z^2}$$

or $\quad 16 = p^2(x^{-2} + y^{-2} + z^{-2})$ *i.e.*, $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

Exercise Set 8.5 (Page 370)

1. Show that the straight line $\frac{x+3}{2} = \frac{y-4}{-7} = \frac{z}{3}$ is parallel to the plane $4x + 2y + 2z = 9$.

Sol. Direction ratios of a normal to the given line are [2, –7, 3] and direction ratios of a normal to the given plane are [4, 2, –2]. Since $(2)(4) + (-7)(2) + (3)(-2) = 0$, normal to the plane is perpendicular to the given line. Thus the line is parallel to the given plane.

2. Show that the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is perpendicular to the plane $4x + 8y + 12z + 19 = 0$.

Sol. Direction ratios of given line are 1, 2, 3,. And direction ratios of a normal to the plane are 4, 8, 12.

Since $\frac{1}{4} = \frac{2}{8} = \frac{3}{12}$, the given line is parallel to a normal to the given plane

i.e., the given line is perpendicular to the plane.

3. Find the conditions that the straight line $x = mz + a$, $y = nz + b$ may lie in the plane $Ax + By + Cz + D = 0$.

Sol. The given line in a symmetric form is

$$\frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1} = t \qquad \text{(say)}$$

Therefore, any point on the line is (x, y, z) where

$$x = a + mt, \qquad y = b + nt \qquad \text{and} \qquad z = 1$$

This point lies on the given plane if

$$A(a + mt) + B(b + nt) + Ct + D = 0$$

or $\quad (Am + Bn + C)t + Aa + Bb + D = 0$

which must be satisfied for every value of t. This implies that

$$Aa + Bb + D = 0 \qquad \text{and} \qquad Am + Bn + C = 0$$

which are required conditions for the given line to lie in the given plane.

4. Determine the point, if any, common to the straight line $x = 1 + t$, $y = t$, $z = -1 + t$ and the plane $x + y + z = 3$.

Sol. We find t such that $x = 1 + t$, $y = t$ and $z = -1 + t$ lie on the plane $x + y + z = 3$. This requires

$$(1 + t) + t + (-1 + t) = 3 \qquad \text{or} \qquad t = 1$$

Hence the common point is (2, 1, 0).

5. Find an equation of the plane through the point (x_1, y_1, z_1) and through the straight line $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$.

Sol. Any plane through the given line is

$$A(x-a) + B(y-b) + C(z-c) = 0 \qquad (1)$$

where $\quad Al + Bm + Cn = 0 \qquad (2)$

Since (x_1, y_1, z_1) lies on (1), we have

$$A(x_1-a) + B(y_1-b) + C(z_1-c) = 0 \qquad (3)$$

Eliminating A, B, C from (1), (2) and (3), we get

$$\begin{vmatrix} x-a & y-b & z-c \\ l & m & n \\ x_1-a & y_1-b & z_1-c \end{vmatrix} = 0$$

or $\quad \Sigma(x-a)\,[m(z_1-c) - n(y_1-b)] = 0$

which is an equation of the required plane.

6. Find an equation of the plane passing through the straight line $x + 2z = 4$, $y - z = 8$ and parallel to the straight line

$$\frac{x-3}{2} = \frac{y+4}{3} = \frac{z-7}{4}. \qquad (1)$$

Sol. The straight line $x + 2z = 4$, $y - z = 8$ in a symmetric form is

$$\frac{x-4}{-2} = \frac{y-8}{1} = \frac{z}{1}$$

Any plane through this line is

$$a(x-4) + b(y-8) + cz = 0 \qquad (2)$$

where $\quad a(-2) + b(1) + c(1) = 0 \qquad (3)$

As the plane (2) is parallel to the line (1), we have

$$a(2) + b(3) + c(4) = 0 \qquad (4)$$

Eliminating a, b, c from (2), (3) and (4), we obtain

$$\begin{vmatrix} x-4 & y-8 & z \\ -2 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

or $\quad (x-4) + 10(y-8) + (-8)(z) = 0$ *i.e.*, $x + 10y - 8z = 84$

is an equation of the required plane.

7. Find an equation of the plane passing through the point (α, β, γ) and parallel to each of the straight lines

$$\frac{x-a}{l_1} = \frac{y-b}{m_1} = \frac{z-c}{n_1} \text{ and } \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}.$$

Sol. Suppose that the plane through the point (α, β, γ) is

$$a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0 \qquad (1)$$

If (1) is parallel to the given straight lines, we have

$al_1 + bm_1 + cn_1 = 0$ (2)

and $al_2 + bm_2 + cn_2 = 0$ (3)

Eliminating a, b, c from (1), (2) and (3), we get

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

i.e., $\Sigma(x-\alpha)(m_1 n_2 - m_2 n_1) = 0$ which is the required plane.

8. Find an equation of the plane through the straight line

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$$

and parallel to the straight line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ (1)

Sol. Any plane through the given line is

$$ax + by + cz + d + k(a'x + b'y + c'z + d') = 0$$

i.e., $(a + ka')x + (b + kb')y + (c + kc')d + ka' = 0$ (2)

Since this plane is parallel to the line (1), we have

$$(a + ka')l + (b + kb')m + (c + kc')n = 0$$

This gives $k = -\dfrac{al + bm + cn}{a'l + b'm + c'n}$

Putting this value of k into (2), we have equation of the required plane as

$$ax + by + cz + d = -\frac{al + bm + cn}{a'l + b'm + c'n} \times (a'x + b'y + c'z + d') = 0$$

or $(a'l + b'm + c'n)(ax + by + cz + d)$

$= (al + bm + cn)(a'x + b'y + c'z + d')$.

9. Prove that the straight lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-3}{3} = \frac{y-3}{4} = \frac{z-4}{5} \text{ are coplanar.}$$

Sol. We know that if the straight lines

$$\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \text{ and } \frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$$

are coplanar, then

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

So the given straight lines are coplanar if

$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

which is, of course, zero since the subtraction of the second row from the third row makes the first and second rows identical. Hence the given straight lines are coplanar.

10. Prove that the straight lines

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} \text{ and } \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$$

intersect. Also find the point of intersection and the plane through them.

Sol. Let $\dfrac{x-1}{2} = \dfrac{y+1}{-3} = \dfrac{z+10}{8} = t$

This gives $x = 1 + 2t, y = -1 - 3t, z = 8t - 10$ (1)

Similarly, from the second line, we have

$x = 4 + s, y = -3 - 4s, z = -1 + 7s$ (2)

The lines intersect if the equations

$1 + 2t = 4 - s, -1 - 3t = -3 - 4s$ and $8t - 10 = -1 + 7s$

have a simultaneous solution.

Solving the first two equations simultaneously, we have

$t = 2$ and $s = 1$.

The third of these equations is also satisfied by these values of t and s

Putting either $t = 2$ in (1) or $s = 1$ in (2), we have

$x = 5, y = -7, z = 6$

which is the required point of intersection of the given lines.

A plane through these lines must contain the point of intersection $(5, -7, 6)$. Suppose the plane is

$a(x-5) + b(y+7) + c(z-6) = 0$ (3)

Since this plane is to contain both the given lines, we have

$a(2) + b(-3) + c(8) = 0$ (4)

and $a(1) + b(-4) + c(7) = 0$ (5)

Eliminating a, b, c from (3), (4) and (5), we have

$$\begin{vmatrix} x-5 & y+7 & z-6 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

or $(x-5)(11) + (y+7)(-6) + (z-6)(-5) = 0$

i.e., $11x - 6y - 5z = 67$ is an equation of the required plane.

Exercise Set 8.6 (Page 375)

1. Show that the shortest between the lines $x + a = 2y = -12z$ and $x = y + 2a = 6(z - a)$ is $2a$.

Sol. The given lines are

$$\frac{x+a}{1} = \frac{y}{1/2} = \frac{z}{-1/12} \qquad (1)$$

and $x = y + 2a, x = 6(z - a)$ (2)

Any plane through (2) in $(x - y - 2a) + k(x - 6z + 6a) = 0$

or $(1 + k)x - y - 6kz - 2a + 6ka = 0$ (3)

This is parallel to (1) if

$$(1 + k) - \frac{1}{2} - 6k\left(-\frac{1}{12}\right) = 0$$

or $1 + k - \frac{1}{2} + \frac{k}{2} = 0$ or $\frac{3k}{2} = -\frac{1}{2}$ *i.e.*, $k = -\frac{1}{3}$

Substituting this value of k into (3), we get

$$\frac{2}{3}x - y + 2z - 4a = 0 \qquad (4)$$

A point on (1) is $(-a, 0, 0)$. Perpendicular distance of this point from (4) is

$$\frac{\left|-\frac{2}{3}a - 4a\right|}{\sqrt{\frac{4}{9} + 1 + 4}} = \frac{\frac{14a}{3}}{\frac{7}{3}} = 2a$$

which is the shortest distance between the given lines.

2. Find the shortest distance between the x-axis and the straight line $ax + by + cz + d = 0 = a'x + b'y + c'y + c'z + d'$.

Sol. Equations of the x-axis are

$$y = 0 = z \quad \text{or} \quad \frac{x}{1} = \frac{y}{0} = \frac{z}{0} \qquad (1)$$

The other line is

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \qquad (2)$$

Any plane containing the line (2) is

$$(ax + by + cz + d) + k(a'x + b'y + c'z + d') = 0 \qquad (3)$$

or $(a + ka')x + (b + kb')y + (c + kc')z + d + kd' = 0$

This will be parallel to the x-axis if

$$a + ka' = 0 \quad \text{or} \quad k = -\frac{a}{a'}$$

Putting this value of k into (3), we get the plane parallel to (1) as

$$(ax + by + cz + d) - \frac{a}{a'}(a'x + b'y + c'z + d') = 0$$

or $(a'b - ab')y + (a'c - ac')z + (a'd - ad') = 0$ (4)

Shortest distance between (1) and (2)

= Perpendicular distance of the plane (4) from any point, say, (1, 0, 0) on the x-axis.

$$= \frac{a'd - ad'}{\sqrt{(a'b - ab')^2 + (a'c - ac')^2}}$$

3. Show that the shortest distance between the straight lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{and} \quad \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

is $\frac{1}{\sqrt{6}}$ and equations of the straight line perpendicular to both are

$$11x + 2y - 7z + 6 = 0 = 7x + y - 5z + 7.$$

Sol. If l, m, n are direction cosines of the line of shortest distance then, it being perpendicular to the given line, we have

$$2l + 3m + 4n = 0 \quad \text{and} \quad 3l + 4m + 5n = 0$$

Thus $\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9}$ or $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$

i.e., $l = \frac{-1}{\sqrt{6}}, m = \frac{2}{\sqrt{6}}, n = \frac{-1}{\sqrt{6}}$

Length of the shortest distance

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

$$= \frac{-1}{\sqrt{6}}(2 - 1) + \frac{2}{\sqrt{6}}(4 - 2) + \left(\frac{-1}{\sqrt{6}}\right)(5 - 3) = \frac{1}{\sqrt{6}}$$

Equations of the shortest distance are, by the standard formula

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix}$$

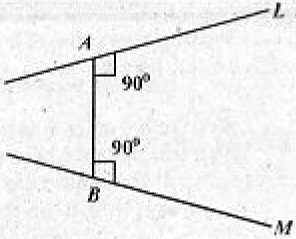
i.e., $11x + 2y - 7z + 6 = 0 = 7x + y - 5z + 7$ as required.

4. Find the shortest between the lines

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \text{ and } \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

Find equations of the straight line perpendicular to both the given straight lines and also its points of intersections with the given straight lines.

Sol. Let the shortest distance AB have direction cosines l, m, n. Then AB being perpendicular to both the given lines, we have



$$1 \cdot l - 2m + n = 0$$

and $7l - 6m + n = 0$

i.e., $\frac{l}{4} = \frac{m}{6} = \frac{n}{8}$ or $\frac{l}{2} = \frac{m}{3} = \frac{n}{4}$

or $l = \frac{2}{\sqrt{29}}, m = \frac{3}{\sqrt{29}}, n = \frac{4}{\sqrt{29}}$

Shortest distance $= |(l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$

$$= \left|\frac{2}{\sqrt{29}}(-1-3) + \frac{3}{29}(-1-5) + \frac{4}{29}(-1-7)\right|$$

$$= \left|-\frac{(8+18+32)}{\sqrt{29}}\right| = \left|\frac{-58}{\sqrt{29}}\right|$$

$$= |-2\sqrt{29}| = 2\sqrt{29}$$

In order to find the coordinates of the points A and B, we note that A lies on

$$L : \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = t \qquad \text{(say)}$$

Then $x = t + 3, y = 5 - 2t, z = 7 - t$ are coordinates of A.

Similarly, $x = 7s - 1, y = -6s - 1, z = -1 + s$ are the coordinates of B.

Therefore, the direction ratios of AB are

$$7s - 1 - t - 3, -6s - 1 - 5 + 2t, -1 + s - 7 + t$$

or $7s - t - 4, -6s + 2t - 6, s + t - 8.$

Since AB is perpendicular to both the lines, we have

$$(7s - t - 4)1 + (-6 + 2t - 6)(-2) + (s + t - 8)(1) = 0$$

and $(7s - t - 4)(7) + (-6s + 2t - 6)(-6) + (s + t - 8)(1) = 0$

i.e., $20s - 4t = 0$ (1)

and $86s - 12t = 0$ (2)

(1) and (2) give $t = 0 = s$

Hence $A = (3, 5, 7)$ and $B = (-1, -1, -1)$

Also direction ratios of AB are $-4, -6, -8$ *i.e.*, $2, 3, 4$.

Equation of the shortest distance AB are $\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$

5. Find the coordinates of the point on the join of $(-3, 7, -13)$ and $(-6, 1, -10)$ which is nearest to the intersection of the planes

$$2x - y - 3z + 32 = 0 \text{ and } 3x + 2y - 15z - 8 = 0.$$

Sol. Equations of the line through the points $(-3, 7, -13)$ and $(-6, 1, -10)$ are

$$\frac{x+3}{3} = \frac{y-7}{-6} = \frac{z+13}{3} \text{ or } \frac{x+3}{1} = \frac{y-7}{2} = \frac{z+13}{-1} = t \qquad (1)$$

A point on (1) is $P(-3 + t, 7 + 2t, -13 - t)$.

We transform the equations

$$\left.\begin{aligned} 2x - y - 3z + 32 &= 0 \\ \text{and} \quad 3x + 2y - 15z - 0 &= 8 \end{aligned}\right\} \qquad (2)$$

of the second straight line into symmetric form. Let $[a, b, c]$ be a direction vector of (2). Then

$$2a - b - 3c = 0$$

and $3a + 2b - 15c = 0$

Therefore, $\frac{a}{21} = \frac{b}{21} = \frac{c}{7}$ or $[a, b, c] = [3, 3, 1]$

Taking $z = 0$, equations (2) become

$$2x - y + 32 = 0 \quad \text{and} \quad 3x + 2y - 8 = 0$$

or $\frac{x}{56} = \frac{y}{112} = \frac{1}{7}$ or $x = -8, y = 16, z = 0$ is a point on (2).

Hence a symmetric form of equations (2) is

$$\frac{x+8}{3} = \frac{y-16}{3} = \frac{z}{1} = s \qquad (3)$$

A point on (3) is

$$Q(-8 + 3s, 16 + 3s, s)$$

$$\overrightarrow{PQ} = [3s - t - 5, 3s - 2t + 9, s + t + 13]$$

Let $\overrightarrow{PQ}$ be normal to both (1) and (3). Then

$$3s - t - 5 + 6s - 4t + 18 - s - t - 13 = 0$$

and $9s - 3t - 15 + 9s - 6t + 27 + s + t + 13 = 0$

i.e., $8s - 6t = 0$ and $19s - 8t + 25 = 0$

Solving the last equations for s and t, we have

$$s = -3, t = -4$$

Substituting $t = -4$ in the coordinates for P, we get

$P(-7, -1, -9)$ as the required point.

6. Find the length and equations of the common perpendicular of the lines

L : $6x + 8y + 3z - 13 = 0, \quad x + 2y + z - 3 = 0$

M : $3x - 9y + 5z = 0, \quad x + y - z = 0$

Sol. We first transform the equations of L and M into symmetric forms. Putting $z = 0$ in the equations for L, we have

$$6x + 8y - 13 = 0 \quad \text{and} \quad x + 2y - 3 = 0$$

Therefore, $\frac{x}{2} = \frac{y}{2} = \frac{1}{4}$ or $x = \frac{1}{2}, y = \frac{5}{4}, z = 0$ is a point on L.

Let $[a, b, c]$ be a direction vector of L. Since L is perpendicular to normal of each plane constituting it, we have

$$6a + 8b + 3c = 0 \quad \text{and} \quad a + 2b + c = 0$$

so $\quad \dfrac{a}{2} = \dfrac{b}{-3} = \dfrac{c}{4}$ or $[a, b, c] = [2, -3, 4]$

A symmetric form of equations of L is

$$\frac{x - \frac{1}{2}}{2} = \frac{y - \frac{5}{4}}{-3} = \frac{z}{4} \qquad (1)$$

Similarly, we can write M as

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \qquad (3)$$

Let $\boldsymbol{n} = [A, B, C]$ be the normal vector of the common perpendicular of both (1) and (2). Then

$$2A - 3B + 4C \quad \text{and} \quad A + 2B + 3C = 0$$

or $\quad \dfrac{A}{-17} = \dfrac{B}{-2} = \dfrac{C}{7}$

Hence $\boldsymbol{n} = [A, B, C] = [-17, -2, 7]$

The points $P\left(\frac{1}{2}, \frac{5}{4}, 0\right)$ and $Q(0, 0, 0)$ lie on (1) and (2) respectively

$$\overrightarrow{PQ} = \left[\frac{1}{2}, \frac{5}{4}, 0\right].$$

Length of the common perpendicular to the two lines is the orthogonal projection of $\overrightarrow{PQ}$ on $\boldsymbol{n}$. Therefore, length of the common perpendicular

$$= \frac{|\overrightarrow{PQ} \cdot \boldsymbol{n}|}{|\boldsymbol{n}|} = \frac{\frac{17}{2} + \frac{5}{2}}{\sqrt{342}} = \frac{11}{\sqrt{342}}$$

To find equations of the common perpendicular, we proceed as follows:

A plane through L is $\quad 6x + 8y + 3z - 13 + k(x + 2y + z - 3) = 0$

i.e., $\quad (6 + k)x + (8 + 2k)y + (3 + k)z - 3k - 13 = 0$

This plane contains the common perpendicular if

$$-17(6 + k) - 2(8 + 2k) + 7(3 + k) = 0$$

or $\quad -102 - 17k - 16 - 4k + 21 + 7k = 0$

or $\quad -14k - 97 = 0 \quad$ or $\quad k = \dfrac{-97}{14}$

Hence the plane is $13x + 82y + 55z - 109 = 0 \qquad (3)$

Similarly, a plane containing M is

$$3x - 9y + 5z + m(x + y - z) = 0$$

or $\quad (3 + m)x + (-9 + m)y + (5 - m)z = 0$

It contains the common perpendicular if

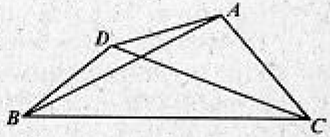
$$-17(3 + m) - 2(-9 + m) + 7(5 - m) = 0 \text{ or } -26m + 2 = 0$$

or $\quad m = \dfrac{1}{13}$. The plane is $\quad 10x - 29y + 16z = 0 \qquad (4)$

(3) and (4) are the required equations of the common perpendicular.

7. Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y + z = 0, z + x = 0, x + y = 0, x + y + z = a$ is $\dfrac{2a}{\sqrt{6}}$ and that the three straight lines of the shortest distances intersect at the point $(-a, -a, -a)$.

Sol. Let the planes $y + z = 0, z + x = 0, x + y = 0$ and $x + y + z$ be ABC, ACD, ADB and BCD respectively.



Equations of line AC (being the intersection of ABC and ACD) are $y + z = 0 = z + x$.

or $\quad \dfrac{x}{1} = \dfrac{y}{1} = \dfrac{z}{-1} \qquad (1)$

The edge opposite to AC is BD which is intersection of the planes ADB and BCD. Its equations are

$$x + y = 0 \quad \text{and} \quad x + y + z = a$$

Any point on this line is $(0, 0, a)$. If l, m, n are direction ratios of this line then

$$1 \cdot l + 1 \cdot m + 0 \cdot n = 0 \quad \text{and} \quad 1 \cdot l + 1 \cdot m + 1 \cdot n = 0$$

or $\quad \dfrac{l}{1} = \dfrac{m}{-1} = \dfrac{n}{0}$

Equation of BD in a symmetric form are $\dfrac{x - 0}{1} = \dfrac{y - 0}{-1} = \dfrac{z - a}{0} \qquad (2)$

Now, if L, M, N are direction cosines of the shortest distance between (1) and (2) then

$$L \cdot 1 + M \cdot 1 + N(-1) = 0$$

and $\quad L \cdot 1 + M(-1) + N(0) = 0$

or $\quad \dfrac{L}{-1} = \dfrac{M}{-1} = \dfrac{N}{-2} \quad$ which give

i.e., $\quad L = \dfrac{1}{\sqrt{6}}, M = \dfrac{1}{\sqrt{6}}, N = \dfrac{2}{\sqrt{6}}$

Shortest distance between the opposite edge AC and BD

$$= |L(x_2 - x_1) + M(y_2 - y_1) + N(z_2 - z_1)|$$

$$= \left|\frac{1}{\sqrt{6}}(0) + \frac{1}{\sqrt{6}}(0) + \frac{2}{\sqrt{6}}(a - 0)\right| = \frac{2a}{\sqrt{6}}$$

Similarly, the distances between the opposite edges AB, CD and BC, AD can be shown each equal to $\frac{2a}{\sqrt{6}}$.

Also equations of the line of shortest distance between AC and BD are

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x & y & z-a \\ 1 & -1 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0$$

i.e., $x - y = 0$ (1)

and $-x + y + z + a = 0$ (2)

Now $(-a, -a, -a)$ satisfies both (1) and (2). Thus this point lies on the line of shortest distance between AC and BD. Similarly, $(-a, -a, -a)$ lies on the other two lines of shortest distances. Hence it lies on the intersection of all three lines of shortest distances.

8. Find the shortest distance between the straight line joining the points $A(3, 2, -4)$ and $B(1, 6, -6)$ and the straight line joining the points $C(-1, 1, -2)$ and $D(-3, 1, -6)$. Also find equations of the shortest distance and coordinates of the feet of the common perpendicular.

Sol. Equations of the line through $A(3, 2, -4)$ and $B(1, 6, -6)$ are

$$\frac{x-3}{1-3} = \frac{y-2}{6-2} = \frac{z+4}{-6+a} \text{ or } \frac{x-3}{-2} = \frac{y-2}{4} = \frac{z+4}{-2} = t \quad (1)$$

Equations of the lines through $C(-1, 1, -2)$ and $D(-3, 1, -6)$ are

$$\frac{x+1}{-3+1} = \frac{y-1}{1-1} = \frac{z+2}{-6+2} = s \text{ or } \frac{x+1}{-2} = \frac{y-1}{0} = \frac{z+2}{-4} = s \quad (2)$$

Suppose P and Q are the feet of the common perpendicular.

Coordinates of P are $(3 - 2t, 2 + 4t, -4 - 2t)$.

Coordinates of Q are $(-1 - 2s, 1, -2 - 4s)$.

$$\overrightarrow{PQ} = [-1 - 2s - 3 + 2t, 1 - 2 - 4t, -2 - 4s + 4 + 2t]$$
$$= [-2s + 2t - 4, -4t - 1, -4s + 2t + 2]$$

Since $\overrightarrow{PQ}$ is perpendicular to (1), we have

$$-2(-2s + 2t - 4) + 4(-4t - 1) - 2(-4s + 2t + 2) = 0$$

$\Rightarrow$ $12s - 24t = 0$ *i.e.*, $s = 2t$ (3)

Also $\overrightarrow{PQ}$ is perpendicular to (2). Therefore,

$$-2(-2s + 2t - 4) - 4(-4s + 2t + 2) = 0$$

or $4s - 4t + 8 + 16s - 8t - 8 = 0$

or $20s - 12t = 0$

Thus $5s - 3t = 0$ *i.e.*, $s = \frac{3}{5}t$ (4)

From (3) and (4), we get $s = 0, t = 0$

Thus $P = (3, 2, -4)$ and $Q = (-1, 1, -2)$

$$|PQ| = \sqrt{(-1-3)^2 + (1-2)^2 + (-2+4)^2}$$
$$= \sqrt{16 + 1 + 4} = \sqrt{21}$$

Equations of PQ are $\frac{x-3}{-4} = \frac{y-2}{-1} = \frac{z+4}{2}$.

Exercise Set 8.7 (Page 379)

1. Find the cylindrical coordinates of the point whose rectangular coordinates are:

(*a*) $(2\sqrt{3}, 2, -2)$ (*b*) $\left(\frac{16}{5}, \frac{12}{5}, 3\right)$

Sol.

(*a*) $(2\sqrt{3}, 2, -2)$

We have, $2\sqrt{3} = r\cos\theta$ and $2 = r\sin\theta$

Squaring these equations and adding the results, we have

$16 = r^2$ or $r = 4$

and $\tan\theta = \frac{1}{\sqrt{3}}$ giving $\theta = \frac{\pi}{6}$

Cylindrical coordinates are $\left(4, \frac{\pi}{6}, -2\right)$

(*b*) Here, $\frac{16}{5} = r\cos\theta$ and $\frac{12}{5} = r\sin\theta$

$r^2 = \frac{400}{25} = 16$ or $r = 4$

$\tan\theta = \frac{3}{4}$ or $\theta = \arctan\left(\frac{3}{4}\right)$

Cylindrical coordinates $\left[4, \arctan\left(\frac{3}{4}\right), 3\right]$

2. Change the following from cylindrical coordinates to rectangular coordinates.

(*a*) $\left(5, \frac{\pi}{6}, 3\right)$ (*b*) $\left(6, \frac{\pi}{3}, -5\right)$

Sol.

(*a*) $\left(5, \frac{\pi}{6}, 3\right)$

$r = 5, \theta = \frac{\pi}{6}$; $x = r\cos\theta = 5\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{2}$

$y = r\sin\theta = 5\sin\frac{\pi}{6} = \frac{5}{\sqrt{2}}$

Rectangular coordinates are $\left(\frac{5\sqrt{3}}{2}, \frac{5}{\sqrt{2}}, 3\right)$

(b) Here $x = 6\cos\frac{\pi}{3} = 3$, $y = 6\sin\frac{\pi}{3} = 3\sqrt{3}$

Rectangular coordinates are $(3, 3\sqrt{3}, -5)$.

3. Find the spherical coordinates of the point whose rectangular coordinates are

(a) $(1, 1, \sqrt{6})$ (b) $(-2, 2\sqrt{3}, 4)$

(c) $(-\sqrt{3}, 1, -2)$ (d) $(4, -4\sqrt{3}, 6)$

Sol.

(a) $\rho = \sqrt{1+1+6} = \sqrt{8}$

$\tan\theta = \frac{y}{x} = 1$ or $\theta = \frac{\pi}{4}$

$z = \rho\cos\phi$ gives $\sqrt{6} = \sqrt{8}\cos\phi$,

$\cos\phi = \frac{\sqrt{3}}{2}$ or $\phi = \frac{\pi}{6}$

Spherical coordinates are $\left(\sqrt{8}, \frac{\pi}{4}, \frac{\pi}{6}\right)$.

(b) $\rho = \sqrt{4+12+16} = 4\sqrt{2}$

$\rho\cos\phi = z \Rightarrow 4\sqrt{2}\cos\phi = 4$

or $\cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}$

$\tan\theta = \frac{2\sqrt{3}}{-2} = -\sqrt{3} \Rightarrow \theta = \frac{2\pi}{3}$

Spherical coordinates are $\left(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$.

(c) $\rho = \sqrt{3+1+4} = 2\sqrt{2}$

$x = -\sqrt{3} = r\cos\theta$; $y = 1 = r\sin\theta$

$r = 2$ and $\tan\theta = \frac{-1}{\sqrt{3}}$ or $\theta = \frac{5\pi}{6}$

$z = \rho\cos\phi$ gives $-2 = 2\sqrt{2}\cos\phi$

or $\cos\phi = \frac{-1}{\sqrt{2}}$ and so $\phi = \frac{3\pi}{4}$

Required coordinates $= \left(2\sqrt{2}, \frac{5\pi}{6}, \frac{3\pi}{4}\right)$

(d) $\rho = \sqrt{16+48+36} = 10$

$4 = r\cos\theta$ and $-4\sqrt{3} = r\sin\theta$

Therefore, $r = \sqrt{16+48} = 8$

$\tan\theta = -\sqrt{3}$ and $\theta = \frac{5\pi}{3}$.

$6 = \rho\cos\phi = 10\cos\phi$

or $\cos\phi = \frac{3}{5}$ or $\phi = \arccos\frac{3}{5}$

Required coordinates $= \left[10, \frac{5\pi}{3}, \text{arcco}\left(\frac{3}{5}\right)\right]$

4. Find the rectangular coordinates of the point whose spherical coordinates are

(a) $\left(5, \frac{\pi}{2}, \frac{\pi}{2}\right)$ (b) $\left(4, \frac{\pi}{3}, \frac{2\pi}{3}\right)$

(c) $\left(0, \frac{\pi}{11}, \frac{\pi}{5}\right)$ (d) $\left(2, \frac{5\pi}{3}, \frac{3\pi}{4}\right)$

Sol.

(a) If (ρ, θ, ϕ) are the spherical polar coordinates of a point then the rectangular coordinates (x, y, z) are given by

$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$

Therefore, $x = 5 \,.\, \sin\frac{\pi}{2}\cos\frac{\pi}{2} = 0$

$y = 5 \,.\, \sin\frac{\pi}{2} \,.\, \sin\frac{\pi}{2} = 5$

and $z = 5 \,.\, \cos\frac{\pi}{2} = 0 = 0$

Hence $(x, y, z) = (0, 5, 0)$

(b) $x = 4 \,.\, \sin\frac{2\pi}{3} \,.\, \cos\frac{\pi}{3} = 4 \,.\, \frac{\sqrt{3}}{2} \,.\, \frac{1}{2} = \sqrt{3}$

$y = 4 \,.\, \sin\frac{2\pi}{3} \,.\, \sin\frac{\pi}{3} = 4 \,.\, \frac{\sqrt{3}}{2} \,.\, \frac{\sqrt{3}}{2} = 3$

$z = 4 \,.\, \cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$

Hence $(x, y, z) = (\sqrt{3}, 3, -2)$

(c) $x = \rho\sin\phi\cos\theta = 0 \,.\, \sin\frac{\pi}{5}\cos\frac{\pi}{11} = 0$

$y = \rho\sin\phi\sin\theta = 0$; $z = \rho\cos\phi = 0$

Rectangular coordinates are $(0, 0, 0)$.

(d) $\rho = 2, \theta = \frac{5\pi}{3}, \phi = \frac{3\pi}{4}$

$x = \rho\sin\phi\cos\theta = 2\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right) = 2 \,.\, \frac{1}{\sqrt{2}} \,.\, \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$$y = \rho \sin \phi \sin \theta = 2 \cdot \frac{1}{\sqrt{2}}\left(\frac{-\sqrt{3}}{2}\right) = -\sqrt{\frac{3}{2}}$$

$$z = \rho \cos \phi = 2 \cos\left(\frac{3\pi}{4}\right) = 2 \cdot \left(\frac{-1}{\sqrt{2}}\right) = -\sqrt{2}.$$

Rectangular coordinates $\left(\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}, -\sqrt{2}\right)$.

In (Problems 5 - 10), express the given equation in rectangular coordinates:

5. $\rho \cos \phi = 2$

Sol. Since $z = \rho \cos \phi$, the given equation in rectangular coordinates is $z = 2$.

6. $\rho = 2 \cos \theta \sin \phi$

Sol. $\rho^2 = 2\rho \cos \theta \sin \phi$ or $x^2 + y^2 + z^2 = 2x$

7. $\rho = 7 \sin \theta \sin \phi$

Sol. $\rho^2 = 7\rho \sin \phi \sin \theta$ or $x^2 + y^2 + z^2 = 7y$

8. $\rho^2 \cos 2\theta = a^2$

Sol. $\rho^2(2 \cos^2 \theta - 1) = a^2$

or $(x^2 + y^2 + z^2)\left(2 \cdot \frac{x^2}{x^2 + y^2} - 1\right) = a^2$

or $(x^2 + y^2 + z^2)(x^2 - y^2) = a^2(x^2 + y^2)$

9. $z = r^2 \cos 2\theta$

Sol. $z = r^2(\cos^2 \theta - \sin^2 \theta) = x^2 - y^2$

10. $z = 1 + \sin \theta$

Sol. $z - 1 = \sin \theta$

or $r(z - 1) = r \sin \theta$

or $r^2(z - 1)^2 = r^2 \sin^2 \theta$

or $(x^2 + y^2)(z - 1)^2 = y^2$

In (Problems 11 - 14) express the given equation in cylindrical and spherical coordinates:

11. $(x + y)^2 - z^2 + 4 = 0$

Sol. For spherical coordinates, we set

$$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$$

in the given equation. It becomes

$$(\rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta)^2 - (\rho \cos \phi)^2 + 4 = 0$$

or $\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cdot \cos \theta) - \rho^2 \cos^2 \phi + 4 = 0$

i.e., $\rho^2 \sin^2 \phi (1 + \sin 2\theta) - \rho^2 \cos^2 \phi + 4 = 0$

or $\rho^2(\sin^2 \phi - \cos^2 \phi) + \rho^2 \sin^2 \phi \cdot \sin 2\theta + 4 = 0$

i.e. $\rho^2(-\cos 2\phi) + \rho^2 \sin^2 \phi \cdot \sin 2\theta + 4 = 0$

i.e., $\rho^2[\sin^2 \phi \cdot \sin 2\theta - \cos 2\phi] + 4 = 0$

For cylindrical coordinates, we put

$x = r \cos \theta, y = r \sin \theta, z = z$, and the given equation becomes

$$(r \cos \theta + r \sin \theta)^2 - z^2 + 4 = 0$$

or $r^2(1 + \sin^2 \theta) - z^2 + 4 = 0$

12. $x^2 + y^2 + 2z = 6$

Sol. In spherical coordinates, the given equation is

$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \theta \sin^2 \phi + 2\rho \cos \phi = 6$$

or $\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + 2\rho \cos \phi = 6$

or $\rho^2 \sin^2 \phi + 2\rho \cos \phi = 6$

In cylinderical coordinates, the given equation is

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta + 2z = 6$$

or $r^2 + 2z = 6$

13. $x^2 - y^2 - z^2 = 1$

Sol. For cylindrical coordinates, we have $x = r \cos \theta$ and $y = r \sin \theta$. The given equation becomes

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta - z^2 = 1$$

or $r^2 \cos 2\theta - z^2 = 1$

For spherical coordinates, we have

$$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$$

and $x^2 - y^2 - z^2 = 1$ becomes

$$\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta - \rho^2 \cos^2 \phi = 1$$

or $\rho^2(\sin^2 \phi \cos 2\theta - \cos^2 \phi) = 1$

14. $3x + y - 4z = 12$

Sol. In cylindrical coordinates, the given equation is

$$3r \cos \theta + r \sin \theta - 4z = 12$$

In spherical coordinates, the equation becomes

$$3\rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta - 4\rho \cos \phi = 12$$

or $\rho(3 \sin \phi \cos \theta + \sin \phi \sin \theta - 4 \cos \phi) = 12$

15. Write the equation of the surface defined by $\frac{(z-1)^2}{4} - \frac{(y+2)^2}{1} = 4(x - 4)$ relative to a new set of parallel axes with origin at $(4, -2, 1)$

Sol. Let the new origin be $O'(4, -2, 1)$. If any point P has coordinates (x, y, z) referred to the original axes and P has coordinates (x', y', z') referred to the new set of axes through O', then

$$x = x' + 4, y = y' - 2, z = z' + 1$$

Putting these values of x, y, z in the given equation, we get

$$\frac{(z'+1-1)^2}{4} - \frac{(y-2+2)^2}{1} = 4(x'+4-4)$$

or $\dfrac{z'^2}{4} - \dfrac{y'^2}{2} = 4x'$ as the required equation.

16. Write the equation $x^2 - 9y^2 - 4z^2 - 6x + 18y + 16z + 20 = 0$ referred to new set of parallel axes with the origin at (3, 1, 2).

Sol. Put $x = x' + 3, y = y' + 1, z = z' + 2$ in the given equation. Therefore, the equation becomes

$(x'+3)^2 - 9(y'+1)^2 - 4(z+2)^2 - 6(x'+3) + 18(y'+1) + 16(z+2) + 20 = 0$

or $x'^2 - 9y'^2 - 4z'^2 + 36 = 0$ or $9y'^2 + 4z'^2 - x'^2 = 36$

or $\dfrac{y'^2}{4} + \dfrac{z'^2}{9} - \dfrac{x'^2}{36} = 1$ is the required equation.

Exercise Set 8.8 (Page 382)

Find an equation of the trace of the given surface in the specified coordinate plane. Identify the trace (Problem 1–4):

1. $x^2 + y^2 + z^2 - 2xz + 5z - 4 = 0$; xy-plane

Sol. For the trace in xy-plane we put $z = 0$ in the equation and get

$$x^2 + y^2 - 4 = 0 \quad \text{or} \quad x^2 + y^2 = 4$$

which is a circle with radius 2 units and centre at the origin.

2. $xy + yz + zx + zx = 1$; xz-plane

Sol. For the trace in xz-plane, put $y = 0$ in the given equation. Therefore, $zx = 1$ in the required trace which is *a* hyperbola.

3. $x^2 + 4y^2 + z^2 + 4xy - 2xz - 2x - 4y + z + 1 = 0$; xy-plane

Sol. The trace in the xy-plane is found by putting $z = 0$ in the given equation. Required trace is

$$x^2 + 4y^2 + 4xy - 2x - 4y + 1 = 0$$

which is a pair of coincident lines.

4. $x^2 + xy - 3xz - 2 = 0$; yz-plane

Sol. Here the trace in yz-plane is found by putting $x = 0$ in the given equation. This implies that $-2 = 0$ which is absurd. Hence there is no trace in the yz-plane.

Find the intercepts of the given surface on the coordinates axes (Problems 5 – 6):

5. $x^2 + 4y^2 + 5xz - 2x + y - 3 = 0$

Sol. The intercepts on the x-axis, y-axis and z-axis are found by putting $y = z = 0$, $z = x = 0$ and $x = y = 0$ respectively in the given equation.

Therefore, $x^2 - 2x - 3 = 0$ gives $x = 3, -1$

Intercepts on the x-axis are 3 and -1.

Intercepts on the y-axis are given by

$$4y^2 + y - 3 = 0$$

or $y = \dfrac{-1 \pm \sqrt{1+48}}{8} = \dfrac{-1 \pm 7}{8} = -1, \dfrac{3}{4}$

Intercepts on the y-axis are -1 and $\dfrac{3}{4}$

There are no intercepts on the z-axis.

6. $2x^2 - z^2 - xy - 8yz + y - z - 2 = 0$ (1)

Sol. Here, we find intercepts on the x-axis by putting $y = 0 = z$ in (1).

i.e., $2x^2 - 2 = 0$ or $x = -1, 1$

Therefore, $-1, 1$ are the intercepts on the x-axis

Putting $x = 0 = z$ in (1), we have

$y - 2 = 0$ *i.e.,* 2 is the intercept on y-axis.

Again, setting $x = 0 = y$ in (1), we get $z^2 + z + 2 = 0$, which gives imaginary roots.

Hence z-intercepts are imaginary.

Exercise Set 8.9 (Page 383)

Write an equation of the surface obtained by revolving the given curve about the specified coordinate axis (Problems 1 – 5):

1. $x^2 + 2y^2 = 8, z = 0$; (*a*) x-axis (*b*) y-axis

Sol.

(*a*) Since the curve is in xy-plane, for surface of revolution about x-axis we replace y^2 by $y^2 + z^2$. Therefore, the required surface is $x^2 + 2(y^2 + z^2) = 8$

(*b*) For surface of revolution about the y-axis, we replace x^2 by $x^2 + z^2$ in the given equation. Hence the required surface is

$$x^2 + z^2 + 2y^2 = 8$$

2. $4x^2 - 9z^2 = 5, y = 0$; (*a*) y-axis (*b*) z-axis

Sol.

(*a*) The curve is in the xz-plane. Therefore, for surface of revolution we replace z^2 by $y^2 + z^2$ and get $4x^2 - 9(y^2 + z^2) = 5$.

(*b*) Here we replace x^2 by $x^2 + y^2$ and have $4(x^2 + y^2) - 9z^2 = 5$ as the required surface of the revolution about z-axis.

3. $6y^2 + 6z^2 = 7, y = 0$; (*a*) y-axis (*b*) z-axis

Sol.

(a) This curve in the yz-plane. For surface of revolution about y-axis, we replace z^2 by $x^2 + z^2$. The required surface is

$$6y^2 + 6(x^2 + z^2) = 7$$

(b) For surface of revolution about z-axis, replace y^2 by $x^2 + y^2$ and get $6(x^2 + y^2) + 6z^2 = 7$ as the required surface.

4. $2x + 3y = 6, z = 0$; **(a)** x-axis **(b)** y-axis

Sol.

(a) We replace y^2 by $y^2 + z^2$ *i.e.*, y by $\sqrt{y^2 + z^2}$

Hence the surface of revolution about x-axis is

$$2x + 3\sqrt{y^2 + z^2} = 6$$

or $\quad 9(y^2 + z^2) = (6 - 2x)^2$

i.e., $\quad 4x^2 - 19(y^2 + z^2) - 24x + 36 = 0$

(b) Similarly, the surface of revolution about y-axis is

$$2\sqrt{x^2 + z^2} + 3y = 6$$

or $\quad 4(x^2 + z^2) = (6 - 3y)^2$

or $\quad 4x^2 - 9y^2 + 4z^2 + 36y - 36 = 0$

5. $y = 2$, **(a)** y-axis **(b)** z-axis

Sol.

(a) The curve is in yz-plane. The surface of revolution about y-axis is $y = 2$ (Since the curve does not contain terms in z).

(b) The surface of revolution about z-axis is

$$\sqrt{x^2 + y^2} = 2 \quad \text{or} \quad x^2 + y^2 = 4$$

State which coordinate axis is the axis of revolution for the given surface and write equations for a generatix in the specified coordinate plane (Problems 6 – 9):

6. $x^2 + y^2 + z = 2$; $\quad xz$-plane

Sol. The given surface contains $x^2 + y^2$ which has been replaced for x^2 in the xz-plane. Therefore, the axis of revolution is the z-axis and hence curve is

$$x^2 + z = 2, y = 0 \quad \text{which is a parabola.}$$

7. $x^2 - 4y^2 - 4z^2 = 8$; xy-plane

Sol. We note that $y^2 + z^2$ has been substituted for y^2. Therefore, the axis of revolution is x-axis and the required curve is $x^2 - 4y^2 = 8, z = 0$ which is a hyperbola.

8. $x^2 - 4y^2 - 4z^2 = 0$; xz-plane

Sol. Here the curve being in the xz-plane, z^2 has been replaced by $y^2 + z^2$. Therefore, the axis of revolution is the x-axis and the curve is

$$x^2 - 4z^2 = 0, y = 0.$$

9. $x^2y^2 + y^2z^2 = 1$; $\quad xz$-plane

Sol. The equation may be written as

$$y^2(x^2 + z^2) = 1 \quad \text{or} \quad x^2 + z^2 = \left(\frac{1}{y}\right)^2$$

This is of the form $x^2 + z^2 = [f(y)]^2$. Hence, y-axis is the axis of revolution and the curve is

$$z^2 = \frac{1}{y^2} \quad \text{or} \quad z = \frac{1}{y}, x = 0.$$

10. Find an equation of the **torus** obtained by revolving about y-axis the circle in the xy-plane with centre at $(a, 0, 0)$ and radius b, where $0 < b < a$.

Sol. Equation of the given circle is $(x - a)^2 + y^2 = b^2$. For the required surface, the axis of revolution being y-axis, we replace x^2 by $x^2 + z^2$ or x by $\sqrt{x^2 + z^2}$.

Hence equation of the torus is

$$\left(\sqrt{x^2 + z^2} - a\right)^2 + y^2 = b^2$$

or $\quad x^2 + z^2 + a^2 - 2a\sqrt{x^2 + z^2} + y^2 - b^2 = 0$

or $\quad (x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + z^2).$

Exercise Set 8.10 (Page 387)

Derive an equation of the cylinder from definition with given directrix and elements parallel to the given vector (Problems 1 – 3):

1. $x^2 + y^2 = 9$, $\quad \boldsymbol{n} = [1, -2, 1]$.

Sol. The line L through $(x_1, y_1, 0)$ and parallel $\boldsymbol{n}$ is

$$L: \begin{cases} x = x_1 + t \\ y = y_1 - 2t \\ z = t \end{cases}$$

Thus, $x_1 = x - z$, $y_1 = y + 2z$. Equation of the directrix is $f(x, y) = x^2 + y^2 - 9 = 0$ and (x_1, y_1) lies on it. Hence $x_1^2 + y_1^2 - 9 = 0$. An equation of the cylinder is

$$(x - z)^2 + (y + 2z)^2 - 9 = 0$$

i.e., $\quad x^2 + y^2 + 5z^2 - 2zx + 4yz - 9 = 0.$

2. $x + y = 4$, $\quad \boldsymbol{n} = [0, 2, -1]$.

Sol. The line l through $(x_1, 0, z_1)$ and parallel to $\boldsymbol{n}$ is

$$L:\begin{cases} x = x_1 \\ y = 2t \\ z = z_1 - t \end{cases}$$

Here $t = \frac{y}{2}$. Hence $x_1 = x$, $z_1 = z + \frac{y}{2}$. Since $(x_1, 0, z_1)$ lies on $x + z = 4$, $x_1 + z_1 = 4$. Therefore, equation of the cylinder is

$$x + z + \frac{y}{2} = 4 \quad \text{or} \quad 2x + y + 2z = 8$$

3. $\frac{z^2}{4} + \frac{y^2}{9} = 1, \quad \boldsymbol{n} = [1, 1, 1]$

Sol. A line parallel to $\boldsymbol{n}$ and through the point $(0, y_1, z_1)$ is

$$L:\begin{cases} x = t \\ y = y_1 + t \\ z = z_1 + t \end{cases}$$

Therefore, $y_1 = y - x$, $z_1 = z - x$. The point $(0, y_1, z_1)$ lies on the directrix. Hence $\frac{z_1^2}{4} + \frac{y_1^2}{9} = 1$.

Equation of the cylinder is $\frac{(z-x)^2}{4} + \frac{(y-x)^2}{9} = 1$.

i.e., $13x^2 + 4y^2 + 9z^2 - 8xy - 18zx - 36 = 0$.

Discuss the given surfaces (Problems 4 – 11):

4. $\frac{x^2}{2} + \frac{y^2}{4} = 1$

Sol. The equation represents an ellipse in the xy-plane. The surface is a right elliptic cylinder parallel to the z-axis.

5. $yz = 2$ (1)

Sol. (1) represents a hyperbola in the yz-plane. Hence it is a right hyperbolic cylinder parallel to the x-axis.

6. $z = k$ (1)

Sol. (1) represent a plane parallel to the xy-plane. Thus it is a right cylinder made up of planes parallel to the xy-plane.

7. $9x^2 + 4z^2 - 18x - 16z - 11 = 0$ (1)

Sol. $(9x^2 - 18x) + (4z^2 - 16z) - 11 = 0$

or $(9x^2 - 18x + 9) + (4z^2 - 16z + 16) - 9 - 16 - 11 = 0$

or $9(x^2 - 2x + 1) + 4(z^2 - 4z + 4) - 36 = 0$

i.e, $9(x-1)^2 + 4(z-2)^2 - 36 = 0$

or $\frac{(x-1)^2}{4} + \frac{(z-2)^2}{9} = 1$

It is an ellipse in the xz-plane with centre at (1, 0, 2). Major axis: 6; minor axis; 4. Hence (1) represents an elliptic cylinder parallel to the y-axis.

8. $z = \sqrt{y-1}$ (1)

Sol. (1) can be written as

$z^2 = y - 1$ which is a parabola in the yz-plane. Hence it represents a right parabolic cylinder parallel to the x-axis.

9. $x^2 + y^2 - 4x + 6y + 11 = 0$ (1)

Sol. (1) may be written as

$(x^2 - 4x + 4) + (y^2 + 6y + 9) - 4 - 9 + 11 = 0$

or $(x-2)^2 + (y+3)^2 = 2$

which is a circle with centre at (2, –3, 0). Thus (1) is represents a right circular cylinder parallel to the z-axis.

10. $z = \sin x$ (1)

Sol. (1) represent the sinusoidal cylinder parallel to the y-axis extending above and below the zx-plane.

11. $r = r_0$ (cylindrical coordinates)

Sol. $r = r_0$ in the rectangular coordinates is $x^2 + y^2 = r_0^2$ which represents a right circular cylinder parallel to the z-axis.

12. Write an equation of the right circular cylinder with radius 2 and centre at (3, 0, 5).

Sol. The right circular cylinder is parallel to the y-axis. Equation of the circle in xz-plane with radius 2 and centre at (3, 0, 5) is

$(x-3)^2 + (z-5)^2 = 4$

or $x^2 + z^2 - 6x - 10z + 30 = 0$

This is the required equation of the right circular cylinder.

13. Write an equation of the right elliptic cylinder whose directrix is in the yz-plane with foci (0, ±3, 0) and major axis 8.

Sol. Semi-major axis a of the ellipse = 4. Also $c = 3$

Therefore, $c^2 = a^2 - b^2$ *i.e.*, $9 = 16 - b^2$

or $b = \sqrt{7}$, which is semi-minor axis of the ellipse.

Equation of the directrix of the cylinder is

$$\frac{y^2}{16} + \frac{z^2}{7} = 1$$

or $7y^2 + 16z^2 - 112 = 0$.

This is the required equation of the right elliptic cylinder.

Find an equation of the cone whose directrix and vertex are given (Problems 14 – 16):

14. Directrix: $y^2 + z^2 = 1, x = 2$; Vertex $A = (0, 0, 0)$

Sol. Let $P(x, y, z)$ be a point on the cone and let the element AP meet the directrix at $Q(x_1, y_1, z_1)$. Then $y_1^2 + z_1^2 = 1, x_1 = 2$ (1)

Equations of the line AQ are $\dfrac{x}{x_1} = \dfrac{y}{y_1} = \dfrac{z}{z_1}$

Since $x_1 = 2$, we have $y_1 = \dfrac{2y}{x}, z_1 = \dfrac{2z}{x}$

Substituting these values of y_1, z_1 into the first equation of (1), we have

$$\left(\frac{2y}{x}\right)^2 + \left(\frac{2z^2}{x}\right)^2 = 1$$

or $4y^2 + 4z^2 - x^2 = 0$, is the required equation of the cone.

15. Directrix: $4x^2 + (y-2)^2 = 4, z = 3$; Vertex $A = (0, 0, 0)$

Sol. Let $P(x, y, z)$ be a point on the cone and let $Q = (x_1, y_1, z_1)$ be the point on the directrix where the element AP meets the cone. Then

$$4x_1^2 + (y^1 - 2)^2 = 4, z_1 = 3$$

Equations of the line PQ are $\dfrac{x}{x_1} = \dfrac{y}{y_1} = \dfrac{z}{z_1}$

or $x_1 = \dfrac{xz_1}{z} = \dfrac{3x}{z}$, since $z_1 = 3$

$y_1 = \dfrac{yz_1}{z} = \dfrac{3y}{z}$

Equation of the cone is $4\left(\dfrac{3x}{z}\right)^2 + \left(\dfrac{3y}{z} - 2\right)^2 = 4$

i.e., $\dfrac{36x^2}{z^2} + \dfrac{9y^2 + 4z^2 - 12yz}{z^2} = 4$

or $36x^2 + 9y^2 + 4z^2 - 12yz = 4z^2$

or $12x^2 + 3y^2 - 4yz = 0$.

16. Directrix: $x^2 + 4y^2 - 2x + 8y - 4 = 0, z = 3$; Vertex $A = (-1, 2, 1)$

Sol. Let $P(x, y, z)$ be a point on the cone and suppose that the element AP meets the directrix at $Q(x_1, y_1, z_1)$. Then

$$x_1^2 + y_1^2 - 2x_1 + 8y_1 - 4 = 0, z_1 = 3 \quad (1)$$

Equation of the line PQ are

$$\frac{x+1}{x_1+1} = \frac{y-2}{y_1-2} = \frac{z-1}{z_1-1} = \frac{z-1}{3-1} = \frac{z-1}{2}, \text{ since } z_1 = 3$$

Therefore, $x_1 + 1 = \dfrac{2(x+1)}{z-1}$ and $y_1 - 2 = \dfrac{2(y-2)}{z-1}$

or $x_1 = \dfrac{2(x+1)}{z-1} - 1 = \dfrac{2x - z + 3}{z-1}$,

$$y_1 = \frac{2(y-2)}{z-1} + 2 = \frac{2y + 2z - 6}{z-1}$$

Substituting these values of x_1, y_1 into the first equation of (1), we have

$$\left(\frac{2x-z+3}{z-1}\right)^2 + 4\left(\frac{2y+2z-6}{z-1}\right)^2 - 2\left(\frac{2x-z+3}{z-1}\right) + 8\left(\frac{2y+2z-6}{z-1}\right) - 4 = 0$$

or $(2x - z + 3)^2 + 16(y + z - 3)^2 - 2(2x - z + 3)(z-1)$
$+ 16(y + z - 3)(z - 1) - 4(z-1)^2 = 0$

or $4x^2 + z^2 + 9 - 4xz + 12x - 6z + 16(y^2 + z^2 + 9 + 2yz - 6y - 6z)$
$-2(2xz - 2x - z^2 + 4z - 3) + 16(yz - y + z^2 - 4z + 3)$
$-4(z^2 - 2z + 1) = 0$

or $4x^2 + 16y^2 + 31z^2 - 8xz + 48yz + 16x - 112y - 166z + 203 = 0$

is the required equation of the cone.

17. Show that an equation of the cone with vertex at (3, 1, 2) and directrix $2x^2 + 3y^2 = 1, z = 0$ is $2x^2 + 3y^2 + 5z^2 - 3yz - 6zx + z - 1 = 0$.

Sol. Any line through (3, 1, 2) is

$$\frac{x-3}{l} = \frac{y-1}{m} = \frac{z-2}{n} = t \quad (1)$$

A point on this line is $(3 + lt, 1 + mt, 2 + nt)$

This lies on $2x^2 + 3y^2 = 1, z = 0$ if

$2(3 + lt)^2 + 3(1 + mt)^2 = 1, 2 + nt = 0$

or $2(9 + l^2t^2 + 6lt) + 3(1 + m^2t^2 + 2mt) = 1, \quad t = -\dfrac{2}{n}$

or $18 + 2l^2t^2 + 12lt + 3 + 3m^2t^2 + 6mt = 1, \quad t = -\dfrac{2}{n}$

or $18 + 2l^2 \times \dfrac{4}{n^2} + 12l\left(-\dfrac{2}{n}\right) + 3 + 3m^2\left(\dfrac{4}{n^2}\right) + 6m\left(-\dfrac{2}{n}\right) = 1$

or $20 + \dfrac{8l^2}{n^2} - \dfrac{24l}{n} + \dfrac{12m^2}{n^2} - \dfrac{12m}{n} = 0$

i.e., $20n^2 + 8l^2 - 24ln + 12m^2 - 12mn = 0$ (2)

Eliminating l, m, n from (1) and (2), we get

$20(z-2)^2 + 8(x-3)^2 - 24(x-3)(z-2) + 12(y-1)^2 - 12(y-1)(z-2) = 0$

or $5(z-2)^2 + 2(x-3)^2 - 6(x-3)(z-2) + 3(y-1)^2 - 3(y-1)(z-2) = 0$

or $2x^2 + 3y^2 + 5z^2 - 3yz - 6zx + z - 1 = 0$.

18. Show that an equation of the cylinder whose generators intersect the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$ and are parallel to $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ is

$a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2 + 2gn(nx - lz) + 2nf(ny - mz) + n^2c = 0$.

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder.
Equation of the generator through P are
$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad (1)$$
Coordinates of any point on (1) are
$$(x_1 + lt, y_1 + mt, z_1 + nt)$$
This lies on the curve if
$$z_1 + nt = 0 \qquad (2)$$
and $a(x_1 + lt)^2 + 2h(x_1 + lt)(y^1 + mt) + b(y_1 + mt)^2$
$$+2g(x_1 + lt) + 2f(y_1 + mt) + c = 0 \qquad (3)$$
Eliminating t from (2) and (3), we get
$$a\left(x_1 - \frac{lz_1}{n}\right)^2 + 2h\left(x_1 - \frac{lz_1}{n}\right)\left(y_1 - \frac{mz_1}{n}\right) + b\left(y_1 - \frac{mz_1}{n}\right)2$$
$$+2g\left(x_1 - \frac{lz_1}{n}\right) + 2f\left(y1 - \frac{mz_1}{n}\right) + c = 0$$
On simplification, the locus of (x_1, y_1, z_1) is
$$a(nx - nl)^2 + 2h(nx - lz)(ny - mz) + b(ny - nz)^2$$
$$+ 2gn(nx - lz) + 2nf(ny - mz) + n^2c = 0$$
which is the required equation of the cylinder.

19. Show that an equation of the cylinder whose generators are parallel to the z-axis and which passes through the curve $x^2 + y^2 + z^2 = 1, x + y + z = 1$ is $x^2 + y^2 + xy - x - y = 0$

Sol. Let (x_1, y_1, z_1) be any point on the cylinder. Direction cosines of the z-axis are 0, 0, 1.
Equations of the generator through $P(x_1, y_1, z_1)$ are
$$\frac{x - x_1}{0} = \frac{y - y_1}{0} = \frac{z - z_1}{1} \qquad (1)$$
Any point on (1) is $(x_1, y_1, z_1 + t)$.
If this point lies on the curve, then
$$x_1^2 + y_1^2 + (z_1 + t)^2 = 1, \qquad (2)$$
$$x_1 + y_1 + z_1 + t = 1 \qquad (3)$$
Eliminating t from (2) and (3), we get
$$x_1^2 + y_1^2 + (1 - x_1 - y_1)^2 = 0$$
or $\quad 2x_1^2 + 2y_1^2 + 2x_1y_1 - 2x_1 - 2y_1 = 0$
The locus of (x_1, y_1, z_1) is $x^2 + y^2 + xy - x - y = 0$
which is an equation of the required cylinder.

20. Show that an equation of the right circular cylinder of radius 3 and whose axis is the line $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$ is
$$5x^2 + 5y^2 + 8z^2 - 8xy + 4xy + 4yz - 6x - 42y - 96z + 225 = 0.$$

Sol. Let $P(x, y, z)$ be any point on the cylinder. $A(1, 3, 5)$ is a point on its axis. Let L be the foot of the perpendicular from P on its axis
Then AL = Projection of AP on its axis
$$= \frac{2(x-1) + 2(y-3) - 1(z-5)}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{2x + 2y - z - 3}{3}$$
Now $AP^2 = AL^2 + LP^2$, where $LP = 3$
Thus $(x-1)^2 + (y-3)^2 + (z-5)^2 = \left[\frac{2x + 2y - z - 3}{3}\right]^2 + 3^2$
or $\quad 5x^2 + 5y^2 + 8z^2 - 8xy + 4yz + 4yz - 6x - 96z + 225 = 0$
is the required equation.

21. Show that an equation to the right circular cone with vertex at O, axis Oz and semi-vertical angle α is $x^2 + y^2 = z^2 \tan^2 \alpha$.

Sol. Let $P(x, y, z)$ be any point on the cone. Direction cosines of OP are proportional to x, y, z. Direction cosines of Oz are 0, 0, 1.
Hence $\quad \cos\alpha = \frac{x(0) + y(0) + z(1)}{\sqrt{x^2 + y^2 + z^2}}$
i.e., $\quad x^2 + y^2 + z^2 = z^2 \sec^2 \alpha$
$$x^2 + y^2 + z^2 = z^2(\tan^2\alpha + 1) = z^2\tan^2\alpha + z^2$$
or $\quad x^2 + y^2 = z^2 \tan^2\alpha$ is the required equation.

22. Show that the general equation to a cone of second degree which passes through the coordinates axes is $fyz + gzx + hxy = 0$.**Sol.**
General equation to a cone of the second degree is
$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$
This is satisfied by direction cosines of the coordinates axes
i.e., by $\quad$ 1, 0, 0; 0, 1, 0; 0, 0, 1
Thus $\quad a = 0,\ b = 0,\ c = 0$
Hence an equation of the cone is
$$2fxz + 2gzx + 2hxy = 0$$
i.e., $\quad fyz + gzx + hxy = 0$

23. Prove that an equation to the cone whose vertex is at the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$ is
$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)^2.$$

Sol. Any line through the origin is

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C} = t \qquad (1)$$

A point on (1) is (At, Bt, Ct)

This lies on the curve if

$$lAt + mBt + nCt = p \qquad (2)$$

and $\quad a(A^2t^2) + b(B^2t^2) + c(C^2t^2) = 1 \qquad (3)$

From (2), $t = \dfrac{p}{Al + Bm + Cn}$

Substituting this value of t into (3), we get

$$(aA^2 + bB^2 + cC^2)\frac{p^2}{(Al + Bm + Cn)^2} = 1$$

or $\quad aA^2 + bB^2 + cC^2 = \left(\dfrac{Al + Bm + Cn}{p}\right)^2 \qquad (4)$

Eliminating A, B, C from (1) and (4), we get

$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)$$

as the required equation.

24. The plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{c}{z} = 1$ meets the coordinates axes in A, B, C.

Prove that an equation to the cone generated by lines drawn from the origin to meet the circle ABC is

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + zy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$

Sol. Coordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. Equations of the circle through A, B, C are given by

$$x^2 + y^2 + z^2 - ax - by - cz = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (1)$$

Any line through the origin is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = t \qquad (2)$$

A point on this line is (lt, mt, nt).

This lies on (1) if

$$\frac{lt}{a} + \frac{mt}{b} + \frac{nt}{c} = 1 \qquad (3)$$

and $\quad (lt)^2 + (mt)^2 + (nt)^2 - a(lt) - b(mt) - c(nt) = 0$

i.e, $\quad (l^2 + m^2 + n^2)t^2 - t(al + bm + cn) = 0 \qquad (4)$

From (3), $t = \dfrac{1}{\dfrac{l}{a} + \dfrac{m}{b} + \dfrac{n}{c}}$

Setting this values of t in (4), we have

$$(l^2 + m^2 + n^2)\cdot\frac{1}{\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right)^2} - \frac{al + bm + cn}{\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right)} = 0$$

$$\Rightarrow \quad (l^2 + m^2 + n^2) - (al + bm + cn)\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right) = 0 \qquad (5)$$

Eliminating l, m, n from (2) and (5), we get

$$(x^2 + y^2 + z^2) - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

or $\quad x^2 + y^2 + z^2 - \left(x^2 + y^2 + z^2 + \dfrac{a}{b}xy + \dfrac{a}{c}xz + \dfrac{b}{a}xy + \dfrac{b}{c}yz + \dfrac{c}{a}xz + \dfrac{c}{b}yz\right) = 0$

or $\quad yz\left(\dfrac{b}{c} + \dfrac{c}{b}\right) + zx\left(\dfrac{c}{a} + \dfrac{a}{c}\right) + xy\left(\dfrac{a}{b} + \dfrac{b}{a}\right) = 0$

as the required equation.

25. Find an equation to the cone whose vertex is the origin and directrix is the circle $x = a$, $y^2 + z^2 = b^2$. Show that the trace of the cone in a plane parallel to the xy-plane is a hyperbola.

Sol. Any line through the origin is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = t \qquad (1)$$

A point on (1) is (lt, mt, nt)

This lies on $x = a$, $y^2 + z^2 = b^2$ if

$$lt = a, \; m^2t^2 + n^2t^2 = b^2$$

Eliminating t from the last two equations, we get

$$m^2\frac{a^2}{l^2} + n^2\frac{a^2}{l^2} = b^2$$

or $\quad a^2(m^2 + n^2) = b^2l^2 \qquad (2)$

Eliminating l, m, n from (1) and (2), we have

$$a^2(y^2 + z^2) = b^2x^2$$

which is the required equation of the cone.

A plane parallel to the xy-plane is $z = k$. Trace of the cone in this plane is $a^2(y^2 + k^2) = b^2x^2$

or $\quad b^2x^2 - a^2y^2 = a^2k^2$, which is a hyperbola.

Exercise Set 8.11 (Page 391)

1. Show that $\rho = c$ is an equation of a sphere **(in spherical coordinates)** of radius c and centre at $(0, 0, 0)$.

Sol. $\rho = c$ implies $\sqrt{x^2 + y^2 + z^2} = c$
or $\quad x^2 + y^2 + z^2 = c^2$ which is an equation of a sphere with centre at the origin and radius c.

2. Find an equation of the sphere whose centre is on the y-axis and which passes through the points $(0, 2, 2)$ and $(4, 0, 0)$.

Sol. Let the centre of the sphere be $(0, b, 0)$ and its radius be r. Then equation of the sphere is $x^2 + (y - b)^2 + z^2 = r^2$.
Since the given points lie on it, we have

$$(0)^2 + (2 - b)^2 + (2)^2 = r^2 \qquad (1)$$

and $\quad (4)^2 + b^2 + (0)^2 = r^2 \qquad (2)$

Subtracting (1) from (2), we get

$$-16 - 4b + 8 = 0 \qquad i.e., \qquad b = -2$$

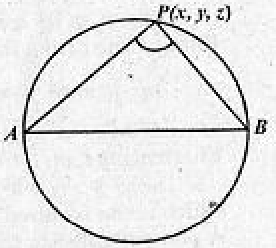
Putting this value of b into (1), we have $r^2 = 20$
Hence equation of the required sphere is

$$(x - 0)^2 + (y + 2)^2 + z^2 = 20$$

or $\quad x^2 + y^2 + z^2 + 4y - 16 = 0$

3. Show that an equation of the sphere having the straight line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.

Sol. Let the given points forming a diameter of the sphere be A and B. If P is any point with coordinates (x, y, z) on the sphere, then APB is a right angle. This implies that AP is perpendicular to BP. Now the direction ratios of AP are $x - x_1$, $y - y_1$, $z - z_1$ and those of BP are $x - x_2, y - y_2, z - z_2$.



Hence $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$
which is the required equation of the sphere.

4. Find an equation of the sphere which passes through the points $A(-3, 6, 0)$, $B(-2, -5, -1)$ and $C(1, 4, 2)$ and whose centre lies on the hypotenuse of the right-angled triangle ABC.

Sol. $A(-3, 6, 0)$, $B(-2, -5, -1)$ and $C(1, 4, 2)$ are the given points.
Now $AB^2 = (-2 + 3)^2 = (-5 - 6)^2 + (-1 - 0)^2$

$$= 1 + 121 + 1 = 123$$

$$BC^2 = (1 + 2)^2 + (4 + 5)^2 + (2 + 1)^2$$
$$= 9 + 81 + 9 = 99$$
$$CA^2 = (1 + 3)^2 + (4 - 6)^2 + (2 - 0)^2$$
$$= 16 + 4 + 4 = 24$$

Thus $AB^2 = BC^2 + CA^2$ and so AB is the hypotenuse of the right triangle ABC.
Since centre of the sphere lies on AB, AB is a diameter for the sphere. Therefore, equation of the sphere is

$$(x + 3)(x + 2) + (y - 6)(y + 5) + (z - 0)(z + 1) = 0$$

or $\quad x^2 + 5x + 6 + y^2 - y - 30 + z^2 + z = 0$
or $\quad x^2 + y^2 + z^2 + 5x - y + z - 24 = 0$
The point $(1, 4, 2)$ lies on this equation.

5. Prove that each of the following equation represents a sphere. Find the centre and radius of each:
(*a*) $x^2 + y^2 + z^2 - 6x + 4z = 0$
(*b*) $x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$
(*c*) $4x^2 + 4y^2 + 4z^2 - 4x + 8y + 24z + 1 = 0 \qquad (1)$

Sol.
(*a*) The given equation written in the standard form is

$$(x - 3)^2 + y^2 + (z + 2)^2 = 13.$$

It is a sphere with centre at $(3, 0, -2)$ and radius $\sqrt{13}$.

(*b*) The equation may be written as

$$(x + 1)^2 + (y - 2) + (z - 3)^2 = -5 + 1 + 4 + 9 = 9,$$

which represents a sphere with centre at $(-1, 2, 3)$ and radius 3.

(*c*) Dividing through by 4 and re-arranging the terms in (1), we have

$$x^2 - x + y^2 + 2y + z^2 + 6z = -\frac{1}{4}$$

or $\quad \left(x - \frac{1}{2}\right)^2 + (y + 1)^2 + (z + 3)^2 = -\frac{1}{4} + \frac{1}{4} + 1 + 9 = 10$

which is a sphere with centre at $\left(\frac{1}{2}, -1, -3\right)$ and radius $\sqrt{10}$.

6. Find an equation of the sphere through the points $(0, 0, 0)$, $(0, 1, -1)$, $(-1, 2, 0)$ and $(1, 2, 3)$. Also find its centre and radius.

Sol. Let an equation of the sphere be

$$x^2 + y^2 + z^2 + 2fx + 2gy + 2hz + c = 0 \qquad (1)$$

Since the given point lie on the sphere, they must satisfy this equation. Therefore, substituting these points into (1), we get

$$c = 0,$$
$$1 + 1 + 2g - 2h = 0, \quad \text{or} \quad g - h + 1 = 0$$
$$1 + 4 - 2f + 4g = 0 \quad \text{or} \quad 4g - 2f + 5 = 0$$

$$1 + 4 + 9 + 2f + 4g + 6h = 0 \quad \text{or} \quad f + 2g + 3h + 7 = 0$$

Solving these equations simultaneously, we have

$$g = \frac{-25}{14}, h = \frac{-11}{14} \quad \text{and} \quad f = \frac{-15}{14}$$

Hence an equation of the sphere is

$$x^2 + y^2 + z^2 + 2\left(\frac{-15}{14}\right)x + 2\left(\frac{-25}{14}\right)y + 2\left(\frac{-11}{14}\right)z = 0$$

or $\quad x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0$

or $\quad 7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0$

Centre of the sphere is $= \left(\frac{15}{14}, \frac{25}{14}, \frac{11}{14}\right)$ and radius

$$= \sqrt{\left(\frac{15}{14}\right)^2 + \left(\frac{25}{14}\right)^2 + \left(\frac{11}{14}\right)^2} = \frac{\sqrt{971}}{14}$$

7. Find an equation of the sphere passing through the point $(0, -2, -4,)$, $(2, -1, -1)$ and having its centre on the straight line $2x - 3y = 0 = 5y + 2z$.

Sol. Let an equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hx + c = 0$$

Its centre is $(-g, -f, -h)$. Since it lies on the given line, we have

$$-2g + 3f = 0 \qquad (1)$$

and $\quad -5f - 2h = 0 \qquad (2)$

Also the given points lie on the sphere. Therefore,

$$4 + 16 - 4f - 8h + c = 0 \qquad (3)$$

and $\quad 4 + 1 + 1 + 4g - 2f - 2h + c = 0 \qquad (4)$

Solving equations (1) to (4) simultaneously, we get

$$f = -2, g = -3, h = 5 \quad \text{and} \quad c = 12$$

Hence the required equation of the sphere is

$$x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$$

8. Find an equation of the sphere which passes through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the point $(1, 2, 3)$.

Sol. A sphere through the given circle is

$$x^2 + y^2 + z^2 - 9 + k(2x + 3y + 4z - 5) = 0 \qquad (1)$$

It passes through $(1, 2, 3)$, then

$$1 + 4 + 9 - 9 + k(2 + 6 + 12 - 5) = 0 \quad i.e., \quad k = -\frac{1}{3}$$

Putting this value of k into (1), we have

$$x^2 + y^2 + z^2 - 9 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

or $\quad 3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$

is the required equation.

9. Find an equation of the sphere through the circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z - 6 = 0$ and touching the plane $z = 0$.

Sol. Any sphere through the given circle is

$$x^2 + y^2 + z^2 - 1 + k(2x + 4y + 5z - 6) = 0$$

or $\quad x^2 + y^2 + z^2 + 2kx + 4ky + 5kz - 1 - 6k = 0 \qquad (1)$

Its centre is $\left(-k, -2k, -\frac{5k}{2}\right)$

and radius $= \sqrt{k^2 + 4k^2 + \frac{25k^2}{4} + 1 + 6k}$

As $z = 0$ is a tangent plane to the sphere, its distance from the centre of the sphere equals the radius of the sphere.

Thus $\quad \frac{\left|-\frac{5k}{2}\right|}{1} = \sqrt{\frac{45k^2}{4} + 6k + 1}$

or $\quad \frac{25k^2}{4} = \frac{45k^2}{4} + 6k + 1 \quad \text{or} \quad 5k^2 + 6k + 1 = 0$

i.e., $\quad k = \frac{-6 \pm \sqrt{36 - 20}}{10} = -1, \frac{-1}{5}$

Putting these values of k into (1), equations of the required spheres are $x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0$

and $\quad x^2 + y^2 = z^2 - \frac{2}{5}x - \frac{4}{5}y - z + \frac{1}{5} = 0.$

10. Show that the two circles $x^2 + y^2 + z^2 = 9$, $x - 2y + 4z - 13 = 0$ and $x^2 + y^2 + z^2 + 6y - 6z + 21 = 0$ $x + y + z + 2 = 0$ lie on the same sphere. Also find its equation.

Sol. Any sphere through the first circle is

$$x^2 + y^2 + z^2 + k(x - 2y + 4z - 13) = 0$$

and a sphere through the second circle is

$$x^2 + y^2 + z^2 + 6y - 6z + 21 + h(x + y + z + 2) = 0 \qquad (2)$$

If the given circles lie on the same sphere, then (1) and (2) must be identical. This requires, (by equating the coefficients of x, y, z and constant terms).

$$k = h, -2k = 6 + h, 4k = -6 + h \quad \text{and} \quad -9 - 13k = 21 + 2h$$

All these equations yield $k = h = -2$. Putting the value of k into (1) or of h into (2), we have

$$x^2 + y^2 + z^2 - 2x + 4y - 8z + 17 = 0$$

as an equation of the sphere on which the two circles lie.

11. Find an equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z - 8 = 0$ is a great circle.

Sol. A sphere through the given circle is

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + k(2x + 3y + 4z - 8) = 0 \qquad (1)$$

Its centre is

$$\left(-k, -\frac{7+3k}{2}, 1-2k\right).$$

If the given circle is a great circle of (1) then the centre of the sphere must lie on the plane $2x + 3y + 4z - 8 = 0$.

Therefore, $-2k - \dfrac{3(7+3k)}{2} + 4(1-2k) - 8 = 0$

or $\quad -29k = 29 \quad$ or $\quad k = -1$

Putting $k = -1$ into (1), we have

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$$

as an equation of the required sphere.

12. Find an equation of the sphere with centre $(2, -1, -1)$ and tangent to the plane $x - 2y + z + 7 = 0$.

Sol. Here radius r of the required sphere is

$$r = \frac{|2+2-1+7|}{\sqrt{1+4+1}} = \frac{10}{\sqrt{6}}$$

Therefore, equation of the sphere is

$$(x-2)^2 + (y-1)^2 + (z+1)^2 = \frac{100}{6} = \frac{50}{3}$$

or $\quad 3x^2 + 3y^2 + 3z^2 - 12x + 6y + 6z + 6z - 32 = 0.$

13. Fin an equation of the plane tangent to the sphere

$$3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0 \text{ at the point } (1, 2, 3).$$

Sol. Centre of the given sphere is

$$M = \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right), P = (1, 2, 3) \ ; \ \overrightarrow{PM} = \left[\frac{-2}{3}, \frac{-3}{2}, \frac{-7}{3}\right]$$

It is a normal vector of the required plane through (1, 2, 3,).

Equation of the plane tangent to the sphere is

$$-\frac{2}{3}(x-1) - \frac{3}{2}(y-2) - \frac{7}{3}(z-3) = 0$$

i.e., $\quad 4x + 9y + 14z - 64 = 0.$

14. Find an equation of the sphere with centre at the point $P(-2, 4, -6)$ and tangent to the

(*a*) xy-plane $\quad$ (*b*) yz-plane $\quad$ (*c*) zx-plane

Sol.

(*a*) Magnitude of the perpendicular from $(-2, 4, -6)$ to the plane $z = 0$ is 6. This is the radius of the required sphere. Equation of the sphere with radius 6 and centre at $P(-2, 4, -6)$ is

$$(x+2)^2 + (y-4)^2 + (z+6)^2 = 36$$

i.e., $\quad x^2 + y^2 + z^2 + 4x - 8y + 12z + 20 = 0$

(*b*) Length of the perpendicular from $P(-2, 4, -6)$ to the plane $x = 0$ is 2 which is radius of the sphere. Equation of the sphere is

$$(x+2)^2 + (y-4)^2 + (z+6)^2 + 2^2$$

or $\quad x^2 + y^2 + z^2 + 4x - 8y + 12z + 52 = 0$

(*c*) Length of the perpendicular from $P(-2, 4, -6)$ to the plane $y = 0$ is 4. Equation of the sphere with centre at $T(-2, 4, -6)$ and radius is

$$(x+2)^2 + (y-4)^2 + (z+6)^2 = 4^2$$

or $\quad x^2 + y^2 + z^2 + 4x - 8y + 12z + 40 = 0.$

15. Find an equation of the surface whose points are equidistant from $P(7, 8, 2)$ and $Q(5, 2, -6)$.

Sol. Let $R(x, y, z)$ be a point on the surface such that

$$|RP| = |RQ|$$

i.e., $\quad (x-7)^2 + (y-8)^2 + (z-2)^2 = (x-5)^2 + (y-2)^2 + (z+6)^2$

or $\quad -14x - 16y - 4z + 117 = -10x - 4y + 12z + 65$

or $\quad 4x + 12y - 16z - 52 = 0$

i.e., $\quad x + 3y - 4z - 13 = 0$

is an equation of the required surface.

16. A point P moves such that the square of its distance from the origin is proportional to its distance from a fixed plane. Show that P always lies on a sphere.

Sol. Let the fixed plane be

$$lx + my + nz = p \quad \text{where } l^2 + m^2 + n^2 = 1$$

$P(x, y, z)$ be a point on the required locus. Therefore, from the hypothesis, we have

$OP^2 = k$(distance of P from the plane), where O is the origin (0, 0, 0).

or $\quad x^2 + y^2 + z^2 = k(lx + my + nz - p)$

i.e., $\quad x^2 + y^2 + z^2 - klx - kmy - knz - kp = 0$

Thus the locus of P is a sphere.

17. A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$.

Sol. Any sphere passing through the origin is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$$

It meets the axes at $(-2u, 0, 0,)$, $(0, -2v, 0)$ and $(0, 0, -2w)$.

Therefore,

$$A = (-2u, 0, 0), B = (0, -2v, 0), C = (0, 0, -2w).$$

Also the radius of this sphere is

$$\sqrt{u^2 + v^2 + w^2} = k$$

or $\quad u^2 + v^2 + w^2 = k \qquad (1)$

Now the centroid of the triangle ABC is

$$\left(\frac{-2u+0+0}{3}, \frac{0-2v+0}{3}, \frac{0+0-3w}{3}\right)$$
$$= \left(\frac{-2u}{3}, \frac{-2v}{3}, \frac{-2w}{3}\right) = (x_1, y_1, z_1) \text{ (say)}$$

Therefore, $x_1 = \frac{-2u}{3}, y_1 = \frac{-2v}{3}, z_1 = \frac{-3w}{3}$

or $u = \frac{-3}{2}x_1, y = \frac{-3y_1}{2}, w = \frac{-3z_1}{2}$

Substituting these values into (1), we get

$$\frac{9}{4}x_1^2 + \frac{9}{4}y_1^2 + \frac{9}{4}z_1^2 = k^2$$

or $9(x_1^2 + y_1^2 + z_1^2) = 4k^2$

Hence the centroid of the $\Delta\, ABC$ lies on the sphere

$$9(x^2 + y^2 + z^2) = 4k^2.$$

18. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in A, B, C. Find an equation of the circumcircle of the triangle ABC. Also find the coordinates of the centre of the circle.

Sol. The circle is the intersection of the given plane by a sphere through A, B, C. For convenience, we take the origin as a forth point on the sphere.

Coordinates of A, B, C, are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

The sphere through O, A, B, C is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 0 & 0 & 0 & 0 & 1 \\ a^2 & a & 0 & 0 & 1 \\ b^2 & 0 & b & 0 & 1 \\ c^2 & 0 & 0 & c & 1 \end{vmatrix} = 0$$

or $\quad x^2 + y^2 + z^2 - ax - by - cz = 0, \qquad (1)$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (2)$$

are equations of the circumcircle of the triangle ABC.

Now the centre P of this circle is the foot of the perpendicular from the centre of the sphere (1) to the plane (2). Coordinates of G, the centre of the sphere, are $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

Direction ratios of PG, which is perpendicular to the plane (2), are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

Therefore, equations of PG are

$$\frac{x-\frac{a}{2}}{\frac{1}{a}} = \frac{y-\frac{b}{2}}{\frac{1}{b}} = \frac{z-\frac{c}{2}}{\frac{1}{c}} = t \text{ (say)}$$

or $\quad x = \frac{a}{2} + \frac{1}{a}t,\ y = \frac{b}{2} + \frac{1}{b}t,\ z = \frac{c}{2} + \frac{1}{c}t$

are coordinates of P. Since P lies on (2), we have

$$\frac{1}{2} + \frac{t}{a^2} + \frac{1}{2} + \frac{t}{b^2} + \frac{1}{2} + \frac{t}{c^2} = 1$$

or $\quad t = -\frac{1}{2}\,\frac{a^2b^2c^2}{b^2c^2 + c^2a^2 + a^2b^2}.$

Foot T of the perpendicular has coordinates $\left(\frac{a}{2} + \frac{t}{a}, \frac{b}{2} + \frac{t}{b}, \frac{c}{2} + \frac{t}{c}\right)$,

where $t = -\frac{1}{2} - \frac{a^2b^2c^2}{b^2c^2 + c^2a^2 + a^2b^2}$.

19. A plane passes through a fixed point (a, b, c) and cuts the axes of coordinates in A, B, C. Find the locus of the centre of the sphere $OABC$ for different positions of the plane, O being the origin.

Sol. Let an equation of the plane be

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1 \qquad (1)$$

As it passes through (a, b, c), we have

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1$$

Coordinates of A, B, C are

$(\alpha, 0, 0)$, $(0, \beta, 0)$, $(0, 0, \gamma)$ respectively.

Suppose equation of the sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (2)$$

Since it passes through $(0, 0, 0)$, $(\alpha, 0, 0)$, $(0, \beta, 0)$, $(0, 0, \gamma)$, we have

$d = 0$

$\alpha^2 + 2u\alpha = 0 \Rightarrow \alpha = -2u$

$\beta^2 + 2v\beta = 0 \Rightarrow \beta = -2v$

$\gamma^2 + 2w\gamma = 0 \Rightarrow \gamma = -2w$

Therefore, from(1), we have $\frac{a}{-2u} + \frac{b}{-2v} + \frac{c}{-2w} = 1$

or $\quad \frac{a}{-u} + \frac{b}{-v} + \frac{c}{-w} = 2$

Locus of the centre $(-u, -v, -w)$ of the sphere (2) is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

or $\quad ax^{-1} + by^{-1} + cz^{-1} = 2.$

20. Find an equation of the sphere circumscribing the tetrahedron whose faces are $x = 0, y = 0, z = 0$ and $lx + my + nz + p = 0$.

Sol. Vertices of the tetrahedron are

$$\left(-\frac{p}{l}, 0, 0\right), \left(0, -\frac{p}{m}, 0\right), \left(0, 0, -\frac{p}{n}\right), (0, 0, 0)$$

Let an equation of the sphere circumscribing the tetrahedron be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (1)$$

Since (1) passes through the vertices of the tetrahedron, we have

$$d = 0$$

$$\frac{p^2}{l^2} - \frac{2p}{l}u = 0 \quad \text{or} \quad u = \frac{p}{2l}$$

Similarly, $v = \frac{p}{2m}$ and $w = \frac{p}{2n}$

Equation of the sphere (1) becomes

$$x^2 + y^2 + z^2 + \frac{p}{l}x + \frac{p}{m}y + \frac{p}{n}z = 0.$$

Exercise Set 8.12 (Page 399)

Discuss and sketch the surface defined by each of the following equations:

1. $x^2 + y^2 + z^2 - 4x + 2y = 11$

Sol. The equation may be written as

$$(x-2)^2 + (y+1)^2 + z^2 = 16$$

This represents a sphere whose centre is at $(2, -1, 0)$ and radius 4.

2. $4x^2 + 4y^2 + 4z^2 - 4x + 16y + 12z + 1 = 0$

Sol. Dividing the given equation by 4, we get

$$x^2 + y^2 + z^2 - x + 4y + 3z + \frac{1}{4} = 0$$

i.e., $$\left(x - \frac{1}{2}\right)^2 + (y+2)^2 + \left(z + \frac{3}{2}\right)^2 = \frac{25}{4}$$

which is an equation of a sphere with centre at $\left(\frac{1}{2}, -2, \frac{-3}{2}\right)$ and radius $\frac{5}{2}$.

3. $4x^2 + 9y^2 + 36z^2 = 36$

Sol. Dividing both sides of the given equation by 36, we have

$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1$$

This is an equation of an ellipsoid.

4. $x^2 + y^2 - 4z^2 = 2$

Sol. The given equation is

$$x^2 + y^2 = 4z^2$$

which is an equation of a right circular cone.

5. $x^2 + y^2 - z^2 - 4 = 0$

Sol. On division by 4, we get

$$\frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{4} = 1$$

which is an equation of a hyperboloid of one sheet.

6. $x^2 + 9z^2 = 36 - 9z^2$

Sol. The given equation is

$$\frac{x^2}{36} + \frac{y^2}{4} + \frac{z^2}{4} = 1$$

which represents an ellipsoid of revolution.

7. $y^2 + z^2 = 4x$

Sol. The given equation is

$$y^2 + z^2 = 4x$$

which is an equation of (an elliptic) paraboloid of revolution.

8. $x^2 + 4y^2 = z^2 - 4$

Sol. The given equation can be written as

$$\frac{x^2}{4} + \frac{y^2}{1} - \frac{z^2}{4} = -1$$

and this represent a hyperboloid of two sheets.

9. $9x^2 - 4y = 9z^2$

Sol. The equation is

$$x^2 - z^2 = \frac{4}{9}y$$

which represents a hyperbolic paraboloid.

10. $x - y^2 - z^2 = 0$

Sol. The can be written as

$$x^2 + y^2 = x$$

which is a paraboloid of revolution.

11. $x^2 + y^2 = 2z - z^2$

Sol. The given equation is

$$(x-1)^2 + y^2 + z^2 = 1$$

which is a sphere with centre $(1, 0, 0)$ and radius 1.

12. $x^2 + 4y^2 = 4 - z$

Sol. The given equation is

$$\frac{x^2}{4}+\frac{y^2}{1}=\frac{-1}{4}(z-1)$$

which is an elliptic paraboloid.

13. $x^2+4y^2=4x-4z^2$

Sol. This equation can be written as

$$\frac{(x-2)^2}{4}+\frac{y^2}{1}+\frac{z^2}{1}=1$$

This represents an ellipsoid with two of its semi-axes equal. It is ellipsoid of revolution with centre at (2, 0, 0).

14. $100x^2+25y^2+100=4z^2$

Sol. On division by 100, the equation is

$$\frac{x^2}{1}+\frac{y^2}{4}-\frac{z^2}{25}=-1$$

which is an equation of a hyperboloid of two sheets.

Exercise Set 8.13 (Page 406)

1. Prove that in a spherical triangle ABC

(*a*) $\sin\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}$

(*b*) $\cos\frac{A}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}$

(*c*) $\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-c)}}$, where $2s=a+b+c$

State and prove similar results for $\frac{B}{2}$ and $\frac{C}{2}$.

Sol.

(*a*) $2\sin^2\frac{A}{2}=1-\cos A$

$$=1-\frac{\cos a-\cos b\cos c}{\sin b\sin c}=\frac{\sin b\sin c-\cos a+\cos b\cos c}{\sin b\sin c}$$

$$=\frac{\cos b\cos c+\sin b\sin c-\cos a}{\sin b\sin c}=\frac{\cos(b-c)-\cos a}{\sin b\sin c}$$

$$=\frac{2\sin\frac{a+b-c}{2}\sin\frac{a-b+c}{2}}{\sin b\sin c}=\frac{2\sin(s-c)\sin(s-b)}{\sin b\sin c}$$

or $\sin^2\frac{A}{2}=\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}$

or $\sin\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}$

(*b*) $2\cos^2\frac{A}{2}=1+\cos A$

$$=1+\frac{\cos a-\cos b\cos c}{\sin b\sin c}=\frac{\sin b\sin c-\cos b\cos c+\cos a}{\sin b\sin c}$$

$$=\frac{\cos a-\cos(b+c)}{\sin b\sin a}=\frac{2\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}}{\sin b\sin c}$$

$$=\frac{2\sin s\sin(s-a)}{\sin b\sin c}\text{ or }\cos\frac{A}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}$$

(*c*) $\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\sqrt{\frac{\sin(s-b)\sin(s-a)}{\sin s\sin(s-a)}}$, by Parts **(*a*)** and **(*b*)**.

From **1(*a*)** and **1(*b*)**, by symmetry, we have

$$\sin\frac{B}{2}=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin a\sin c}};\ \cos\frac{B}{2}=\sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}}$$

$$\sin\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}};\ \cos\frac{C}{2}=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}$$

$$\tan\frac{B}{2}=\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-c)}}$$

2. Prove that in a spherical triangle ABC,

$$\frac{\sin\frac{A+B}{2}}{\cos\frac{C}{2}}=\frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}$$

Sol. From **Problem 1 (*c*)**, we have

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}$$

$$=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s\sin^2(s-a)}}=\frac{r}{\sin(s-a)},$$

where $r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$

Similarly, $\tan\frac{B}{2}=\frac{r}{\sin(s-b)}$ and $\tan\frac{C}{2}=\frac{r}{\sin(s-c)}$

Now $\dfrac{\tan\frac{A}{2}}{\tan\frac{B}{2}} = \dfrac{\sin(s-b)}{\sin(s-a)}$

or $$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}} = \frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)} \tag{1}$$

Also, $$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}} = \frac{\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}-\sin\frac{B}{2}\cos\frac{A}{2}}$$

$$= \frac{\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}} \tag{2}$$

$$\frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)} = \frac{2\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}{2\cos\frac{1}{2}(2s-a-b)\sin\frac{1}{2}(a-b)}$$

$$= \frac{\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c\sin\frac{1}{2}(a-b)} \tag{3}$$

Therefore, from (2) and (3), we have

$$\frac{\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}} = \frac{\sin\frac{1}{2}c\cos\frac{a-b}{2}}{\cos\frac{1}{2}c\sin\frac{a-b}{2}} \quad \text{or} \quad \frac{\left(\sin\frac{A+B}{2}\right)}{\dfrac{\sin\frac{A-B}{2}}{\sin\frac{a-b}{2}}\sin\frac{c}{2}} = \frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}$$

or $$\frac{\sin\frac{A+B}{2}}{\cos\frac{C}{2}} = \frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}$$

$$\left(\text{since } \sin\frac{c}{2}\sin\frac{A-B}{2} = \cos\frac{C}{2}\sin\frac{a-b}{2}\right).$$

3. In any spherical triangle ABC, show that

$$2\cos\frac{a+b}{2}\cos\frac{a-b}{2}\tan\frac{c}{2} = \sin b\cos A + \sin a\cos B.$$

Sol. L.H.S. $= 2\cos\dfrac{a+b}{2}\cos\dfrac{a-b}{2}\tan\dfrac{c}{2}$

$= (\cos a + \cos b)\tan\dfrac{c}{2}$

R.H.S. $= \sin b\cos A + \sin a\cos B$

$= \sin b \times \dfrac{\cos a - \cos b\cos c}{\sin b\sin c} + \sin a \times \dfrac{\cos b - \cos c\cos a}{\sin c\sin a}$

$= \dfrac{\cos a - \cos b\cos c}{\sin c} + \dfrac{\cos b - \cos c\cos a}{\sin c}$

$= \dfrac{\cos a - \cos b\cos c + \cos b - \cos c\cos a}{\sin c}$

$= \dfrac{(\cos a + \cos b) - \cos c(\cos b + \cos a)}{\sin c}$

$= \dfrac{(\cos a + \cos b)(1 - \cos c)}{\sin c} = \dfrac{(\cos a + \cos b)\,.\,2\sin^2\frac{c}{2}}{2\sin\frac{c}{2}\cos\frac{c}{2}}$

$= (\cos a + \cos b)\dfrac{\sin\frac{c}{2}}{\cos\frac{c}{2}} = (\cos a + \cos b)\tan\dfrac{c}{2}.$

4. In an equilateral triangle, show that

(*a*) $\sec A = 1 + \sec a$

(*b*) $\tan^2\dfrac{a}{2} = 1 - 2\cos A$

Sol.

(*a*) $\cos A = \dfrac{\cos a - \cos b\cos c}{\sin b\sin c}$

Since triangle is equilateral, $a = b = c$

$\cos A = \dfrac{\cos a - \cos^2 a}{\sin^2 a} = \dfrac{\cos a(1-\cos a)}{1-\cos^2 a}$

$= \dfrac{\cos a}{1+\cos a} = \dfrac{1}{\sec a + 1}$

or $1 + \sec a = \sec A.$

(*b*) $\cos A = \dfrac{\cos a - \cos b\cos c}{\sin b\sin c}$

As $a = b = c$

$\cos A = \dfrac{\cos a - \cos^2 a}{\sin^2 a} = \dfrac{\cos a(1-\cos a)}{1-\cos^2 a} = \dfrac{\cos a}{1+\cos a}$

$$1-2\cos A = 1-\frac{2\cos a}{1+\cos a} = \frac{1-\cos a}{1+\cos a} = \frac{2\sin^2\frac{a}{2}}{2\cos^2\frac{a}{2}} = \tan^2\frac{a}{2}$$

5. Find the direction of Qibla of each of the given places:

	Place	Latitude ϕ	Longitude λ
(a)	Islamabad	33° 40′ N	73° 8′ E
(b)	Karachi	24° 51.5′ N	67° 2′ E
(c)	Quetta	30° 15′ N	67° 0′ E
(d)	Peshawar	34° 1′ N	71° 40′ E
(e)	New York	40° 49′ N	74° 0′ W
(f)	Canberra	35° 17′ S	149° 8′ E

Sol.

(a) The classical longitude

$l = 73°8'E - 39°49.2'E = 33°18.8'$ (CE)

Now $p = \dfrac{\sin\phi}{\tan l} = \dfrac{\sin 33°40'}{\tan 33°18.8'}$

$$q = \frac{\cos 33°40' \tan 21°25.2'}{\sin 33°18.8'}$$

$\log p = \log\sin 33°40' - \log\tan 33°18.8'$

$= -0.25620 + 0.182295 = -0.0739$

$p = 0.8435$

$\log q = \log\cos 33°40' + \log\tan 21°25.2' - \log\sin 33°18.8'$

$= -0.079732 - 0.4064 - 0.26026 = -0.746392$

or $q = 0.1793$

Now, $\tan i = p - q = 0.8435 - 0.1793$

$= 0.6642$

or $i = 33°35.5'$ south of west.

(b) Here $l = 67°2'E - 39°49'.2E = 66°62' - 39°49.2$

$= 27°12.8'$ (CE)

$p = \dfrac{\sin 24°51.5'}{\tan 27°12.8'}$; $q = \dfrac{\cos 24°51.5' \tan 21°25.2'}{\sin 27°12.8'}$

$\log p = \log\sin 24°51.5' - \log\tan 27°12.8'$

$= -0.37636 + 0.28885 = -0.08751$

or $p = 0.8175$

$\log q = \log\cos 24°51.5' + \log\tan 21°25.2' - \log\sin 27°12.8'$

$= -0.04222 - 0.4064 + 0.33979$

$= -0.44862 + 0.33979 = -0.10833$

or $q = 0.77834$

$\tan i = p - q = 0.8175 - 0.77834 = 0.03916$

$i = 2°14.55'$ south of west.

(c) $l = 67°E - 39°49.2'E = 66°60' - 39°49.2' = 27°10.8'$

$p = \dfrac{\sin 30°15'}{\tan 27°10.8'}$, $q = \dfrac{\cos 30°15' \tan 21°15.5'}{\sin 27°10.8'}$

$\log p = \sin 30°15' - \log\tan 27°10.8'$

$= -0.29776 + 0.28946 = -0.0083$

$p = 0.9811$

$\log q = \log\cos 30°15' + \log\tan 21°25.2' - \log\sin 27°10.8'$

$= 0.06356 - 0.4064 + 0.34028 = -0.46996 + 0.34028$

$= -0.12968$

or $q = 0.74185$

$p - q = 0.9811 - 0.74185 = 0.23925$

$\tan i = 0.23925$

or $i = 13°27.3'$ south of west.

(d) $l = 71°40'E - 39°49.2'E = 70°100' - 39°49.2'$

$= 31°50.8'$

$p = \dfrac{\sin 34°1'}{\tan 31°50.8'}$, $q = \dfrac{\cos 34°1' \tan 21°25.2'}{\sin 31°50.8'}$

$\log p = \log\sin 34°1' - \log\tan 21°25.8'$

$= -0.25225 + 0.20680 = -0.04545$

or $p = 0.90063$

$\log q = \log\cos 34°1' + \log\tan 21°25.2' - \log\sin 31°50.8'$

$= -0.08151 - 0.4064 + 0.27765$

$= -0.48791 + 0.27765 = -0.21026$

or $q = 0.61622$

$p - q = 0.2844 = \tan i$

or $i = 15°52.55'$ south of west.

(e) $\phi_o = 21°25.2'N$

$\lambda_o = 39°49.2'N$

$l = 74° + 39°49.2'E = 113°49.2'$ CW

$p = \dfrac{\sin\phi}{\tan l} = \dfrac{\sin 40°49'}{\tan 113°49.2'} = -0.288562$

$q = \dfrac{\cos\phi \tan\phi_o}{\sin l} = \dfrac{\cos 40°49' \tan 21°25.2'}{\sin 113°49.2'} = 0.324537$

$\tan i = p - q = -0.6131$

$i = -31.51°$

Since i is negative, the direction of Qibla is $31°30.6'$ north of east.

(f) $l = \lambda - \lambda_o$ CE $= 149°8' - 39°.49.2'$

$= 109°18.8'$ CE

$$p = \frac{\sin 35°17'}{\tan 109°18.8'} = -0.196905109$$

$$q = \frac{\cos 35°17' \tan 21°25.2'}{\sin 109°18.8'} = 0.3393320$$

$$\tan i = p - q = -0.5362371$$

$$i = -28° \ 12.1'$$

The direction of Qibla is 28°12.1′ north of west.

6. Prove that for a place on the equator the direction of Qibla is inclined $\arctan(\tan \phi_o \csc l)$ north of west or north of east according as its classical longitude l is east or west.

Sol. Here $p = \dfrac{\sin 0}{\tan l} = 0$, $q = \dfrac{\cos 0 \tan \phi_o}{\sin l} = \dfrac{\tan \phi_0}{\sin l}$

Hence $\tan i = \tan \phi_o \csc l$

i.e., $i = \arctan(\tan \phi_o \csc l)$.

7. Prove that for a place on the same parallel of latitude as the Khana-e-Ka'aba the direction of Qibla is inclined at $\arctan\left(\sin \phi_o \tan \dfrac{l}{2}\right)$ north of west or north of east according as its classical longitude l is east or west.

Sol. Here $p = \dfrac{\sin \phi_o}{\tan l}$, since $\phi = \phi_o$

$$q = \frac{\cos \phi_o \tan \phi_o}{\sin l}$$

$$\tan i = p - q = \frac{\sin \phi_o}{\tan l} - \frac{\cos \phi_o \tan \phi_o}{\sin l}$$

$$= \frac{\sin \phi_o \cos l}{\sin l} - \frac{\cos \phi_o \sin \phi_o}{\sin l \cos \phi_o}$$

$$= \frac{\sin \phi_o \cos l}{\sin l} - \frac{\sin \phi_o}{\sin l}$$

$$= \frac{\sin \phi_o}{\sin l}(\cos l - 1) = \frac{\sin \phi_o}{\sin l} 2 \sin^2 \frac{l}{2}$$

$$= \frac{\sin \phi_o}{2 \sin \frac{l}{2} \cos \frac{l}{2}} 2 \sin^2 \frac{l}{2} = \sin \phi_o \tan \frac{l}{2}$$

or $\quad i = \arctan\left(\sin \phi_o \tan \dfrac{l}{2}\right)$.

Chapter 9 FUNCTIONS OF SEVERAL VARIABLES

Exercise Set 9.1 (Page 411)

1. Verify Euler's Theorem for

(*a*) $u = \arcsin\left(\dfrac{x}{y}\right) + \arctan\left(\dfrac{y}{x}\right)$

(*b*) $u = x^n \ln\left(\dfrac{y}{x}\right)$ (*c*) $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Sol.

(*a*) Here u is a homogeneous function of zero degree. Therefore, by Euler's theorem we must have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \qquad (1)$$

Now $u = \arcsin\left(\dfrac{x}{y}\right) + \text{arctna}\left(\dfrac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{-x}{y^2} + \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{x} = \frac{-x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} - \frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} = 0$$

(*b*) Here u is a homogeneous function of degree n. Therefore, by Euler's theorem we must have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad (1)$$

To verify this, we have

$$\frac{\partial u}{\partial x} = nx^{n-1} \ln\frac{y}{x} + x^n \cdot \frac{1}{y/x} \cdot \frac{-y}{x^2} = nx^{n-1} \ln\frac{y}{x} - x^{n-1}$$

and $$\frac{\partial u}{\partial y} = x^n \cdot \frac{1}{y/x} \cdot \frac{1}{x} = \frac{x^n}{y}.$$

Hence $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \ln\frac{y}{x} - x^n + y\frac{x^n}{y}$$

$$= nx^n \ln\frac{y}{x} - x^n + x^n = nx^n \ln\frac{y}{x} = nu$$

(c) Here $u = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}} = \dfrac{x^{1/4}\left[1+\left(\frac{y}{x}\right)^{1/4}\right]}{x^{1/5}\left[1+\left(\frac{y}{x}\right)^{1/5}\right]}$

$$= x^{\frac{1}{4}-\frac{1}{5}}\left[\frac{1+\left(\frac{y}{x}\right)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}\right] = x^{\frac{1}{20}}\ \frac{1+\left(\frac{y}{x}\right)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}$$

Thus u is a homogeneous function of degree $\frac{1}{20}$ and hence by Euler's theorem we must have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}u = \frac{1}{20}u$$

To verify this, we have

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\cdot\left(\frac{1}{4}x^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{-4/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

and $$\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\cdot\left(\frac{1}{4}y^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{-4/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}+\frac{1}{4}y^{1/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{1/5}+\frac{1}{5}x^{1/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})\left(\frac{1}{20}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{1}{20}\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}} = \frac{1}{20}\cdot u$$

2. If $u = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Sol. We have $\dfrac{\partial u}{\partial y} = \dfrac{-y}{x^2}f'\left(\frac{y}{x}\right)$

and $\dfrac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right)\dfrac{1}{x} = \dfrac{1}{x}f'\left(\frac{y}{x}\right)$

Thus $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{-y}{x}f'\left(\frac{y}{x}\right) + \dfrac{y}{x}f'\left(\frac{y}{x}\right) = 0.$

1. If $u = xyf\left(\frac{x}{y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u.$

Sol. We have $\dfrac{\partial u}{\partial x} = y\left[f\left(\frac{x}{y}\right) + xf'\left(\frac{x}{y}\right)\cdot\frac{1}{y}\right], \dfrac{\partial u}{\partial y} = x\left[f\left(\frac{x}{y}\right) - yf'\left(\frac{x}{y}\right)\cdot\frac{x}{y^2}\right]$

Hence $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = xy\left[f\left(\frac{x}{y}\right) + \frac{x}{y}f'\left(\frac{x}{y}\right) + f\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right)\right]$

$$= 2xyf\left(\frac{y}{x}\right) = 2u.$$

4. If $z = \arctan\left(\frac{y}{x}\right)$, verify that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 2u$

Sol. Differentiating partially w.r.t. x, we have

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2} = (-y)(x^2+y^2)^{-1}$$

$$\frac{\partial^2 z}{\partial x^2} = (-1)\,(y)\,(x^2+y^2)^{-2}\,(-2x) = \frac{2xy}{(x^2+y^2)^2} \qquad (1)$$

Again, $$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$$

$$\frac{\partial^2 z}{\partial y^2} = (-1)\,(x)\,(x^2+y^2)^{-2}\,(2y) = \frac{-2xy}{(x^2+y^2)^2} \qquad (2)$$

Adding (1) and (2), we get $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

5. If $u = \arcsin\left(\dfrac{x^2+y^2}{x+y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$

Sol. Writing tx, ty for x, y in the R.H.S. of the given equation, we have

$$\arcsin\frac{t(x^2+y^2)}{x+y} \neq t\arcsin\frac{x^2+y^2}{x+y}$$

Hence u is not a homogeneous function.

Let $z = \sin u = \dfrac{x^2+y^2}{x+y}$.

Then z is a homogeneous function of degree 1.

Therefore, by Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1.z \quad\text{or}\quad x\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = z$$

i.e., $x\cdot\cos u\cdot\dfrac{\partial u}{\partial x} + y\cdot\cos u\cdot\dfrac{\partial u}{\partial y} = z = \sin u$

Dividing by $\cos u$, we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{\sin u}{\cos u} = \tan u.$

6. If $u = \arcsin\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{y}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Sol. Proceeding as in **Q. 5**, it is easy of see that u is not a homogeneous function.

Let $z = \sin u = \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

This shows that z is a homogeneous function of zero degree. Therefore, by Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0\,.\,z = 0 \quad \text{or} \quad x\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = 0$$

or $\quad x\cos u\dfrac{\partial u}{\partial x} + y\cos u\dfrac{\partial u}{\partial y} = 0.$

i.e., $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

7. If $u = \ln\left(\dfrac{x^2+y^2}{x+y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

Sol. Here u is a homogeneous function (verify !)

Let $z = e^u = \dfrac{x^2+y^2}{x+y}$. Then z is a homogeneous function of degree 1. By Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1\,.\,z$$

or $\quad x\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x} + y\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y} = z = e^u$

or $\quad x\,.\,e^u\,.\dfrac{\partial u}{\partial x} + y\,.\,e^u.\dfrac{\partial u}{\partial y} = e^u$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.$

8. If $u = f(x, y)$ is a homogeneous function of degree n, prove that

$$x^2 f_{xx} + 2xy\, f_{xy} + y^2 f_{yy} = n\,(n-1)\,f.$$

Sol. Since f is a homogeneous function of degree n, we have

$$xf_x + yf_y = nf \tag{1}$$

Differentiating (1) w.r.t. x and y respectively, we get

$$xf_{xx} + f_x + yf_{yx} = nf_x \tag{2}$$

and $\quad xf_{xy} + yf_{yx} + f_y = nf_y \qquad (3)$

Assuming $f_{xy} = f_{yx}$ and multiplying (2) by x and (3) by y and adding the results, we have

$$x^2f_{xx} + 2xyf_{xy} + y^2f_{yy} + xf_x + yf_y = n(xf_x + yf_y)$$

or $\quad x^2f_{xx} + 2xyf_{xy} + y^2f_{yy} = (n-1)\,(xf_x + yf_y) = n(n-1)f$, using (1).

9. If $u = f(r)$, where $r = \sqrt{x^2+y^2}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).$$

Sol. We have $\dfrac{\partial u}{\partial x} = f'(r)\,.\,\dfrac{\partial r}{\partial x} = f'(r)\,.\,\dfrac{1}{2}(x^2+y^2)^{-1/2}\,2x = f'(r)\,.\,\dfrac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\,.\,\frac{\partial r}{\partial x}\cdot\frac{x}{\sqrt{x^2+y^2}} + f'(r)\,\frac{\sqrt{x^2+y^2}\,.\,1 - x\,.\,\frac{1}{2}(x^2+y^2)^{-1/2}\,2x}{(\sqrt{x^2+y^2})^2}$$

$$= f''(r)\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + f'(r)\,\frac{y^2}{(x^2+y^2)^{3/2}}$$

$$= f''(r)\,.\,\frac{x^2}{x^2+y^2} + f'(r)\,\frac{y^2}{(x^2+y^2)^{3/2}} \tag{1}$$

By symmetry, we have

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\,\frac{y^2}{x^2+y^2} + f'(r)\,.\,\frac{x^2}{(x^2+y^2)^{3/2}} \tag{2}$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2+y^2}{x^2+y^2}\right] + f'(r)\left[\frac{x^2+y^2}{(x^2+y^2)^{3/2}}\right]$$

$$= f''(r) + f'(r)\,.\,\frac{r^2}{r^3} = f''(r) + \frac{1}{r}f'(r).$$

10. If $V = \rho^m$, where $\rho^2 = x^2 + y^2 + z^2$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)\,\rho^{m-2}$$

Sol. We have $\dfrac{\partial V}{\partial x} = m\rho^{m-1}\,.\,\dfrac{\partial \rho}{\partial x} \qquad (1)$

Now, $\quad \rho^2 = x^2 + y^2 + z^2$

Differentiating both the sides w.r.t. x, we have

$$2\,.\,\rho\,.\,\frac{\partial \rho}{\partial x} = 2x \quad \text{or} \quad \frac{\partial \rho}{dx} = \frac{x}{\rho}$$

Putting this value of $\dfrac{\partial \rho}{\partial x}$ into (1),we get

$$\frac{\partial V}{\partial x} = m\rho^{m-1}\,.\,\frac{x}{\rho} = m\rho^{m-2}\,.\,x$$

Differentiating w.r.t. x, we obtain

$$\frac{\partial^2 V}{\partial x^2} = m\left[\rho^{m-2}\,.\,1 + (m-2)\,\rho^{m-3}\,.\,\frac{\partial \rho}{\partial x}\,.\,x\right]$$

$$= m\left[\rho^{m-2} + (m-2)\,\rho^{m-3}\left(\frac{x}{\rho}\right)x\right]$$

$$= m\,[\rho^{m-2} + (m-2)\rho^{m-4}\,.\,x^2] \qquad (2)$$

By symmetry, we get

$$\frac{\partial^2 V}{\partial y^2} = m\,[\rho^{m-2} + (m-2)\,\rho^{m-4}\,.\,y^2] \qquad (3)$$

and $$\frac{\partial^2 V}{\partial z^2} = m\,[\rho^{m-2} + (m-2)\,\rho^{m-4}\,.\,z^2] \qquad (4)$$

Adding (2), (3) and (4), we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m\,[3\rho^{m-2} + (m-2)\,(\rho^{m-4})\,(x^2+y^2+z^2)]$$

$$= m\,[3\rho^{m-2} + (m-2)\,\rho^{m-4}\,.\,\rho^2]$$

$$= m\,[3\rho^{m-2} + (m-2)\,\rho^{m-2}]$$

$$= m\,(m+1)\,\rho^{m-2}, \text{ as required.}$$

Exercise Set 9.2 (Page 414)

1. Approximate $\sqrt{299^2 + 399^2}$ by means of differentials.

Sol. Let $f(x,y) = \sqrt{x^2+y^2}$

$$f(300,400) = \sqrt{300^2 + 400^2} = 500$$

$$\Delta f \approx df = f_x\,dx + f_y\,dy$$

$$= \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy \qquad (1)$$

With $x = 300, y = 400, dx = \Delta x = -1$ and $dy = \Delta y = -1$, we have from (1)

$$df = \frac{300}{500}(-1) + \frac{400}{500}(-1) = -\frac{7}{5}$$

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta f$$

$$\approx f(x,y) + df$$

or $$f(299,399) = f(300,400) + df = 500 - \frac{7}{5} = 498\frac{3}{5} = 498.6$$

Hence $\sqrt{299^2 + 399^2} = 498.6$

2. If $\theta = \arctan\left(\frac{y}{x}\right)$, use differentials to find an approximate value of θ when $x = 0.95$ and $y = 1.05$.

Sol. $\theta(x,y) = \arctan\left(\frac{y}{x}\right) \Rightarrow \theta(1,1) = \frac{\pi}{4}$

With $x = 1, y = 1, dx = -0.05$ and $dy = 0.05$, we need to find

$$\theta(x+\Delta x, y+\Delta y) = \theta(0.95, 1.05) = \arctan\left(\frac{1.05}{0.95}\right)$$

Now, $\theta(x+\Delta x, y+\Delta y) = \theta(x,y) + \Delta\theta \approx \theta(x,y) + d\theta$

But $$d\theta = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \left(-\frac{1}{2}\right)(-0.05) + \frac{1}{2}(0.05) = 0.05$$

Therefore, $$\arctan\left(\frac{1.05}{0.95}\right) \approx \theta(1,1) + d\theta = \arctan\frac{1}{1} + d\theta$$

$$= \frac{\pi}{4} + 0.05 = 0.8354$$

3. If $u = \sqrt{x+2y}$ and x changes from 3 to 2.98 while y changes from 0.5 to 0.51, find an approximate value for the change in u.

Sol. We know that $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad (1)$

Here $u = \sqrt{x+2y}$, $x = 3, y = 0.5$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x+2y}} = \frac{1}{2\sqrt{3+2(0.5)}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{x+2y}}.2 = \frac{1}{\sqrt{x+2y}} = \frac{1}{\sqrt{3+2(0.5)}} = \frac{1}{2}$$

Also $dx = 2.98 - 3 = -0.02$ and $dy = 0.51 - 0.5 = 0.01$

Substituting these values into (1), we get

$$du = \frac{1}{4}(-0.02) + \frac{1}{2}(0.01) = 0.$$

Hence there is no change in u.

4. If $u = x^2 + y^2 + z^2 + xy^2z^3$ and x changes from 2 to 2.01, y changes from 1 to 1.02 and z changes from -1 to -0.99, find an approximate value for the change in u.

Sol. Here $dx = 2.01 - 2 = 0.01$, $dy = 1.02 - 1 = 0.02$ and $dz = -0.99 - (-1) = 0.01$

Also $$\frac{\partial u}{\partial x} = 2x + y^2z^3 = 3$$

$$\frac{\partial u}{\partial y} = 2y + 2xyz^3 = 2 + 2(2)(1)(-1)^3 = -2$$

and $$\frac{\partial u}{\partial z} = 2z + 3xy^2z^2 = -2 + 6 = 4 \qquad \text{at } x = 2, y = 1, z = -1.$$

Change in $u = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$$= 3(0.01) - 2(0.02) + 4(0.01) = 0.03.$$

5. A rectangular plate expands in such a way that its length changes from 10 to 10.03 and its breadth changes from 8 to 8.02. Find an approximate value for the change in its area

Sol. Suppose that the length and breadth are x and y respectively.
Then $dx = 10.03 - 10 = 0.03$, $dy = 8.02 - 8 = 0.02$

Area $A = x \cdot y$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy$$
$$= y\,dx + x\,dy = 8(0.03) + 10\,(0.02)$$
$$= 0.24 + 0.2 = 0.44 \text{ is the change in area}$$

6. The lateral surface of a cone is computed from the formula $S = \pi r\sqrt{r^2 + h^2}$, where r is the radius of the base and h is the height. If r is calculated as 6 with an accuracy of 1 % and h as 8 with an accuracy of 0.25%, with what accuracy will be the area S?

Sol. Here $r = 6$, $h = 8$

$$dr = 6 \times \frac{1}{100} = 0.06,\ dh = 8 \times \frac{25}{100 \times 100} = 0.02$$

Now as $S = \pi r\sqrt{r^2 + h^2}$,

$$dS = \frac{\partial S}{\partial r}dr + \frac{\partial S}{\partial h}dh$$
$$= \pi\left[\sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}}\right]dr + \pi r\frac{h}{\sqrt{r^2 + h^2}}dh$$
$$= \pi\left[10 + \frac{36}{10}\right][0.06] + \pi.\frac{6\times 8}{10}0.02$$
$$= \pi[0.6 + 0.216] + \pi(0.096) = \pi[0.912] \qquad (1)$$

But $S = \pi r\sqrt{r^2 + h^2} = \pi \cdot 6\sqrt{36 + 64} = 60\pi$

The change (1) is for 60π. Therefore % change

$$= \frac{\pi(0.912)}{60\pi} \times 100 = 1.5\%.$$

7. The volume V of a rectangular parallelepiped having sides x, y and z is given by the formula $V = xyz$. If this solid is compressed from above so that z is decreased by 2% while x and y each is increased by 0.75% approximately, what percentage change will be in V?

Sol. We have $V = xyz$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\,dx + zx\,dy + xy\,dz$$

Now $dx = \frac{75}{100 \times 100}x$, $dy = \frac{75}{100 \times 100}y$, $dz = \frac{-2}{100}z$

Change in volume

$$= dV = yz\left(\frac{3}{400}x\right) + zx\left(\frac{3}{400}y\right) + xy\left(\frac{-2}{100}z\right) = \frac{-xyz}{200}$$

Percentage change in volume

$$= \frac{dV}{V} \times 100 = \frac{-xyz}{200xyz} \times 100$$
$$= -\frac{1}{2} = -0.5\%, \text{ which is decrease in volume.}$$

8. A formula for the area Δ of a triangle is $\Delta = \frac{1}{2}ab\sin C$, where a, b are two adjacent sides and C is the angle included. Approximately what error is made in computing Δ if a is taken to be 9.1 instead of 9, b is taken to be 4.08 instead of 4 and C is taken to be $30°\ 3'$ instead of $30°$?

Sol. $\Delta = \frac{1}{2}ab\sin C$

$$d\Delta = \frac{\partial \Delta}{\partial a}da + \frac{\partial \Delta}{\partial b}db + \frac{\partial \Delta}{\partial C}dC$$

Here $da = 9.1 - 9 = 0.1$
$db = 4.08 - 4 = 0.08$

$$dC = 30°3' - 30° = 3' = \left(\frac{3}{60}\right)^{\circ} = \frac{3}{60} \times \frac{\pi}{180} \text{ radians}$$

Hence $d\Delta = \frac{1}{2}(b\sin C)\,da + \frac{1}{2}a\sin C.\,db + \frac{1}{2}ab\cos C.\,dC$

$$= \frac{1}{2}(4)\left(\frac{1}{2}\right)(0.1) + \frac{1}{2}(9)\left(\frac{1}{2}\right)(0.08) + \frac{1}{2}(4)(9)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{3\pi}{60\times 180}\right)$$
$$= 0.1 + 0.18 + 0.015 = 0.293$$

$$\%\text{ change in area} = \frac{0.293}{9} \times 100 = 3.25\%.$$

9. The dimensions of a box are measured to be 10 in., 12 in. and 15 in. and the measurements are correct to 0.02 in. Find the maximum error if the volume is calculated from the given measurements. Also find the percentage error.

Sol. The volume V of the box with dimension x, y, z (in inches) is

$$V = xyz$$

We shall approximate the maximum error by means of differentials.

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\,dx + zx\,dy + xy\,dz.$$

The maximum error in volume is obtained by taking

$dx = \Delta x = 0.02$, $dy = \Delta y = 0.02$ and $dz = \Delta z = 0.02$ with
$x = 10$, $y = 12$ and $z = 15$.

Therefore,

$$dV = (12)(15)(0.02) + (10)(15)(0.02) + (10)(12)(0.02) = 9$$

is the maximum error.

$$V = 10 \times 12 \times 15 = 1800 \text{ cu. in.}$$

Relative maximum error $= \dfrac{dV}{V} = \dfrac{9}{1800} = \dfrac{1}{200} = 0.005$

Percentage error $= 0.005 \times 100 = 0.5\%$.

10. Evaluate sine $29^o \cos 28^o \tan 44^o$ by using differentials.

Sol. Let $f(x, y, z) = \sin x \cos y \tan z$

Take $x = \dfrac{\pi}{6}$, $y = \dfrac{\pi}{6}$; $z = \dfrac{\pi}{4}$

$$dx = \frac{-\pi}{180},\ dy = \frac{-\pi}{90},\ dz = \frac{-\pi}{180}$$

With these values, we shall have

$$\sin 29^0 \cos 28^0 \tan 44^0 = f(x + \Delta x, y + \Delta y, z + \Delta z)$$
$$= f(x, y, z) + df \qquad (1)$$

Now

$$df = \cos x \cos y \tan z\, dx - \sin x \sin y \tan z\, dy + \sin x \cos y \sec^2 z\, dz$$

$$= \left(\cos\frac{\pi}{6}\cos\frac{\pi}{6}\tan\frac{\pi}{4}\right)\left(-\frac{\pi}{180}\right) - \left(\sin\frac{\pi}{6}\sin\frac{\pi}{6}\tan\frac{\pi}{4}\right)\left(-\frac{\pi}{90}\right) + \left(\sin\frac{\pi}{6}\cos\frac{\pi}{6}\sec^2\frac{\pi}{4}\right)\left(-\frac{\pi}{180}\right)$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1\left(\frac{-\pi}{180}\right) - \frac{1}{2}\cdot\frac{1}{2}\cdot 1\left(\frac{-\pi}{90}\right) + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 2\left(-\frac{\pi}{180}\right)$$

$$= \frac{\pi}{180}\left[-\frac{3}{4} - \frac{1}{2} - \frac{\sqrt{3}}{2}\right] = 0.0175\,(-1.1160) = -0.01953.$$

$$f\left(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{4}\right) = \sin\frac{\pi}{6}\cos\frac{\pi}{6}\tan\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1 = \frac{\sqrt{3}}{4} = 0.4330.$$

Therefore, $f(29^o, 28^o, 44^o) = \sin 29^o \cos 28^o \tan 44^o$

$$= f\left(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{4}\right) + df$$
$$= 0.4330 - 0.0195 = 0.4135$$

By actual calculation, $\sin 29^o \cos 28^o \tan 44^o = 0.4132$.

Exercise Set 9.3 (Page 418)

1. If $u = x - y^2, x = 2r - 3s + 4, y = -r + 8s - 5$, find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial s}$

Sol. We know that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = (1)(2) + (-2y)(-1) = 2(1 + y)$$

Again, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial s} = (1)(-3) + (-2y)(8) = -(3 + 16y).$

2. If $z = \dfrac{\cos y}{x}, x = u^2 - v, y = e^v$, find $\dfrac{\partial z}{\partial u}$, $\dfrac{\partial z}{\partial v}$

Sol. We have $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial u}$

$$= \frac{-\cos y}{x^2}(2u) + \frac{-\sin y}{x}\cdot 0 = \frac{-2u.\cos y}{x^2}$$

and $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial v}$

$$= \frac{-\cos y}{x^2}.(-1) + \frac{-\sin y}{x}.e^v = \frac{1}{x^2}[\cos y - x e^v \sin y]$$

$$= \frac{1}{x^2}[\cos y - xy \sin y],\ \text{since } y = e^v.$$

Find $\dfrac{dy}{dx}$ (Problems 3 – 6):

3. $\sin xy - e^{xy} - x^2y = 0$

Sol. Here $f(x, y) = \sin xy - e^{xy} - x^2y = 0$

$$f_x = y\cos xy - ye^{yx} - 2xy$$
$$f_y = x\cos xy - xe^{xy} - x^2$$
$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{(y\cos xy - ye^{xy} - 2xy)}{x\cos xy - xe^{xy} - x^2} = \frac{y(\cos xy - e^{xy} - 2x)}{x(x + e^{xy} - \cos xy)}$$

4. $3(x^2 + y^2)^2 = 25(x^2 - y^2)$

Sol. $f(x, y) = 3(x^2 + y^2)^2 - 25(x^2 - y^2) = 0$

$$f_x = 6(x^2 + y^2).2x - 50x$$
$$f_x = 6(x^2 + y^2).2y + 50y$$
$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{12x(x^2 + y^2) - 50x}{12y(x^2 + y^2) + 50y} = \frac{25x - 6x(x^2 + y^2)}{25y + 6y(x^2 + y^2)}$$

5. $f(x, y) = x^y - y^x = 0$

Sol. $f(x, y) = x^y - y^x$

$$f_x = yx^{y-1} - y^x \ln y$$

$$= yx^{y-1} - x^y \ln y = x^{y-1}(y - x \ln y)$$

and $f_y = x^y \ln x - xy^{x-1} = x^y \ln x - \frac{x}{y} . y^x = x^y \ln x - \frac{x}{y} x^y$

$$= x^y \frac{(y \ln x - x)}{y}$$

Hence $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x^{y-1}(y - x \ln y)}{x^y (y \ln x - x)} \cdot y$

$$= \frac{y(y - x \ln y)}{x(x - y \ln x)}.$$

6. $(\tan x)^y + y^{\cot x} = a$

Sol. We have $f(x, y) = (\tan x)^y + y^{\cot x} - a = 0$

$$f_x = y(\tan x)^{y-1} . \sec^2 x - y^{\cot x} . \ln y (\csc^2 x)$$

and $f_y = (\tan x)^y . \ln \tan x + (\cot x) y^{\cot x - 1}$

Hence $\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{y \sec^2 x (\tan x)^{y-1} - \csc^2 x\, y^{\cot x} \ln y}{(\tan x)^y \ln \tan x + (\cot x) y^{\cot x - 1}}$

7. If $F(x,y,z) = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Sol. We know that if $f(x, y) = 0$, then

$$\frac{dy}{dx} = \frac{-f_x}{f_y} \qquad (1)$$

Now in $F(x, y, z) = 0$ we may regard z as a function of x and y. In order to find $\frac{\partial z}{\partial x}$, we treat y as constant and use (1)

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad (2)$$

Similarly, we get $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

Here $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are partial derivatives because z is a function of two variables x and y.

8. If $f(x, y, z) = 0$ and $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y} \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

Sol. From $f(x, y) = 0$, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

From $\phi(y, z) = 0$, we get

$$\frac{dz}{dy} = \frac{-\phi_y}{\phi_z} \qquad (2)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = \frac{\frac{\partial \phi}{\partial y} \cdot \frac{\partial f}{\partial x}}{\frac{\partial \phi}{\partial z} \cdot \frac{\partial f}{\partial y}}$$

Cross multiplying, we obtain

$$\frac{\partial \phi}{\partial z} \cdot \frac{\partial f}{\partial y} \cdot \frac{dz}{dx} = \frac{\partial \phi}{\partial y} \frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

9. If $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a$, show that $\frac{d^2y}{dx^2} = \frac{-a}{(1 - x^2)^{3/2}}$.

Sol. We have $f(x, y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2} - a$

$$f_x = \sqrt{1 - y^2} - \frac{1 . y . 2x}{2\sqrt{1 - x^2}} = \frac{\sqrt{1 - y^2}\sqrt{1 - x^2} - xy}{\sqrt{1 - x^2}}$$

$$f_y = \frac{-x}{2\sqrt{1 - y^2}} . 2y + \sqrt{1 - x^2} = \frac{-xy + \sqrt{1 - x^2}\sqrt{1 - y^2}}{\sqrt{1 - y^2}}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}} \qquad (1)$$

Differentiating (1) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{1 - x^2}\left(\frac{-1}{2\sqrt{1 - y^2}} 2y\right)\frac{dy}{dx} - \sqrt{1 - y^2} \times \left(-\frac{1}{2}\frac{2x}{\sqrt{1 - x^2}}\right)}{1 - x^2}$$

$$= \frac{y\frac{\sqrt{1 - x^2}}{\sqrt{1 - y^2}} \frac{dy}{dx} - \frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}} . x}{(1 - x^2)} = \frac{-y\frac{\sqrt{1 - x^2}}{\sqrt{1 - y^2}}\left(\frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}}\right) - \frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}} . x}{(1 - x^2)}$$

$$= -\frac{(y\sqrt{1 - x^2} + x\sqrt{1 - y^2})}{(1 - x^2)^{3/2}} = \frac{-a}{(1 - x^2)^{3/2}}.$$

10. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^2}$$

Sol. Here $f_x = 2ax + 2hy + 2g$

and $f_y = 2hx + 2by + 2f$

$$\frac{dy}{dx} = -\frac{ax+hy+g}{hx+by+f} \qquad (1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{(hx+by+f)\left(a+h\frac{dy}{dx}\right)-(ax+hy+g)\left(h+b\frac{dy}{dx}\right)}{(hx+by+f)^2}$$

$$= -\frac{dy}{dx}(h^2x+hby+hf-abx-hby-gb)/(hx+by+f)^2$$

$$\frac{-(ahx+aby+af-ahx-h^2y-hg)}{(hx+by+f)^2}$$

$$= \frac{\frac{ax+hy+g}{hx+by+f}(h^2x+hf-abx-gb)-(aby+af-h^2y-hg)}{(hx+by+f)^2}$$

$$= \frac{(ax+hy+g)(h^2x+hf-abx-gb)-(hx+by+f)(aby+af-h^2y-hg)}{(hx+by+f)^3}$$

$$= \frac{h^2(ax^2+2hxy+by^2+2gx+2fy)-ab(ax^2+2hxy+by^2+2gx+2fy-af^2-bg^2+2fgh)}{(hx+by+f)^3}$$

$$= \frac{h^2(-c)-ab(-c)-af^2-bg^2+2fgh}{(hx+by+f)^3} = \frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^2}$$

11. Find $\frac{d^2y}{dx^2}$ if $x^3+y^3=3axy$

Sol. Here $f(x, y) = x^3+y^3-3axy = 0$

$f_x = 3x^2-3ay$ and $f_y = 3y^2-3ax$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{ay-x^2}{y^2-ax} \qquad (1)$$

Differentiating (1) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\left(a\frac{dy}{dx}-2x\right)-(ay-x^2)\left(2y\frac{dy}{dx}-a\right)}{(y^2-ax)^2}$$

$$= \frac{\frac{dy}{dx}(ay^2-a^2x-2ay^2+2x^2y)-(2xy^2-2ax^2-a^2y+ax^2)}{(y^2-ax)^2}$$

$$= \frac{\frac{ay-x^2}{y^2-ax}(2x^2y-ay^2-a^2x)-(2xy^2-ax^2-a^2y)}{(y^2-ax)^2}$$

$$= 2ax^2y^2-a^2y^3-a^3xy-2x^4y+ax^2y^2+a^2x^3/(y^2-ax)^3$$

$$+\frac{-(2xy^4-ax^2y^2-a^2y^3-2ax^2y^2+a^2x^3+a^3xy)}{(y^2-ax)^3}$$

$$= \frac{-(2xy^4-ax^2y^2-a^2y^3-2ax^2y^2+a^2x^3+a^3xy)}{(y^2-ax)^3}$$

$$+\frac{3a^2x^2-a^2y^3-a^3xy-2x^4y+a^2x^3}{(y^2-ax)^3}$$

$$= \frac{6ax^2y^2-2a^3xy-2xy(x^3+y^3)}{(y^2-ax)^3}$$

$$= \frac{6ax^2y^2-2a^3xy-2xy(3axy)}{(y^2-ax)^3}$$

$$= \frac{-2a^3xy}{(y^2-ax)^3} = \frac{2a^3xy}{(ax-y^2)^3}.$$

Exercise Set 9.4 (Page 420)

In Problems 1 – 3, find the rate of change of u at the given point and in the given direction.

1. $u = 2xy - \frac{y}{x}$; (1, 2): [2, – 3, 0]

Sol. We have

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2y + \frac{y}{x^2} = 6\\ \frac{\partial u}{\partial y} &= 2x - \frac{1}{x} = 1\\ \frac{\partial u}{\partial z} &= 0\end{aligned}\right\} \text{ at } (1, 2)$$

grad $u = [6, 1, 0]$

Also $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} = [2, -3, 0]$

$$\frac{du}{ds} = \text{Rate of change of } u$$

$$= \frac{\mathbf{v}\cdot \textbf{grad}\, u}{|\mathbf{v}|} = \frac{(2)(6)+(-3)(1)+0}{\sqrt{4+9}} = \frac{9}{\sqrt{13}}.$$

2. $u = ye^{-x}(x^2+y^2+z^2+1)$; (0; 0; 0); [2, 1, 2]

Sol. Here

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= ye^{-x}(2x-x^2+y^2+z^2+1) = 0\\ \frac{\partial u}{\partial y} &= e^{-x}(x^2+z^2+1+3y^2) = 1\\ \frac{\partial u}{\partial z} &= ye^{-x}(2z) = 0\end{aligned}\right\} \text{ at } (0, 0, 0)$$

grad $u = [0, 1, 0]$

Also $\mathbf{v} = [2, 1, 2]$

$$\frac{du}{ds} = \frac{\mathbf{v}.\,\mathbf{grad}\,u}{|\mathbf{v}|} = \frac{2.0 + 1.1 + 2.0}{\sqrt{4+1+4}} = \frac{1}{\sqrt{9}} = \frac{1}{3}.$$

3. $u = \sinh(x+y) + \cosh z$; $(1; 0; 1)$, $[-2, 2, -1]$

Sol.

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \cosh(x+y) = \cosh 1\\ \frac{\partial u}{\partial y} &= \cosh(x+y) = \cosh 1\\ \frac{\partial u}{\partial z} &= \sinh z = \sinh 1\end{aligned}\right\} \text{ at } \quad (1, 0, 1)$$

$\mathbf{grad}\,u = [\cosh 1, \cosh 1, \sinh 1]$

$\mathbf{v} = [-2, 2, -1]$

$$\frac{du}{ds} = \frac{\mathbf{v}.\mathbf{grad}\,u}{|\mathbf{v}|} = \frac{-2\cosh 1 + 2\cosh 1 - \sinh 1}{3}$$

$$= \frac{-\left(\dfrac{e-e^{-1}}{2}\right)}{3} = \frac{-(e^2-1)}{6e} = \frac{1-e^2}{6e}.$$

4. Let $u = x^2 + y^2$. Find the direction of the greatest rate of change of u at (a, b) and the magnitude of this greatest rate of change. Find the direction of no change at (a, b).

Sol. We know that the directional derivative is greatest in the direction of the gradient itself and the magnitude of this greatest directional derivative is the magnitude of the gradient vector. Therefore,

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2x = 2a\\ \frac{\partial u}{\partial y} &= 2y = 2b\end{aligned}\right\} \text{ at } (a, b)$$

$$\frac{\partial u}{\partial z} = 0$$

Hence $\mathbf{grad}\,u = [2a, 2b, 0] = 2a\mathbf{i} + 2b\mathbf{j}$ is the direction of greatest rate of change.

Also its magnitude $= \sqrt{4a^2 + 4b^2} = 2\sqrt{a^2+b^2}$.

Again, the directions of no change are the directions perpendicular to $\mathbf{grad}\,u$.

If $[s, t]$ is perpendicular to $\mathbf{grad}\,u$, the $2as + 2bt = 0$

$$\Rightarrow \quad \frac{s}{t} = -\frac{b}{a} \quad \text{or} \quad s = -kb,\ t = ka,$$

$[s, t] = [-kb, ka]$ or $[-b, a]$ is the direction of no change at (a, b), which can also be written as $-b\mathbf{i} + a\mathbf{j}$.

5. If $u = \arctan\left(\dfrac{y}{x}\right)$, find the direction of the greatest rate of change of u at (a, b) and the magnitude of the greatest rate of change. Find also the direction of no change at (a, b).

Sol. We have

$$\frac{\partial u}{\partial x} = \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\left(\frac{-y}{x^2}\right) = -\frac{y}{x^2+y^2} = \frac{-b}{a^2+b^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{a}{a^2+b^2}$$

$\mathbf{grad}\,u$ = Direction of greatest rate of change

$$= \left[\frac{-b}{a^2+b^2}, \frac{a}{a^2+b^2}, 0\right] = \frac{-b}{a^2+b^2}\mathbf{i} + \frac{a}{a^2+b^2}\mathbf{j}$$

Magnitude of greatest rate of change

$$= |\mathbf{grad}\,u| = \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}} = \frac{1}{\sqrt{a^2+b^2}}.$$

Direction of no change is just a vector perpendicular to $\mathbf{grad}\,u$ which is

$\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, since $\mathbf{v}$ is perpendicular to $\mathbf{grad}\,u$

6. The temperature distribution for the semi-circular plate $x^2 + y^2 \le 1$, $y \ge 0$, is given by the formula $T = 3x^2y - y^3 + 27$ under certain conditions. Find $\dfrac{dT}{ds}$ at $A\left(0, \dfrac{1}{2}\right)$ in the direction of y-axis. Also find (i) $\dfrac{dT}{ds}$ at A in the direction of $[1, -2]$, (ii) the direction of greatest rate of change at A, (iii) the magnitude of the greatest rate of change (iv) the direction of the isothermal through A (the direction of zero rate of change at A).

Sol.

$$\left.\begin{aligned}\frac{dT}{dx} &= 6xy = 0\\ \frac{\partial T}{\partial y} &= 3x^2 - 3y^2 = \frac{-3}{4}\end{aligned}\right\} \text{ at } \left(0, \frac{1}{2}\right)$$

$$\mathbf{grad}\,T = \left[0, \frac{-3}{4}\right]$$

Also unit vector in the direction of y-axis is $[0, 1]$.

$$\frac{dT}{ds} = \left[0, \frac{-3}{4}\right].[0, 1] = \frac{-3}{4}$$

Again $\mathbf{v} = [1, -2]$, $|\mathbf{v}| = \sqrt{5}$

Hence (i) $\frac{dT}{ds}$ in the direction of **v**

$$= \frac{\textbf{grad } T.\textbf{v}}{|\textbf{v}|} = \frac{\left[0, \frac{-3}{4}\right][1,-2]}{\sqrt{5}} = \frac{3}{2\sqrt{5}}.$$

(ii) The direction of greatest rate of change

$$= \textbf{grad } \; T = \left[0, \frac{-3}{4}\right].$$

(iii) Magnitude of greatest rate of change

$$= |\textbf{grad } T| = \sqrt{(0)^2 + \left(\frac{-3}{4}\right)^2} = \frac{3}{4}.$$

(iv) Direction of the isothermal through A

= Direction perpendicular to **grad** T

$$= [1,0] = i, \text{ since } = [1,0].\left[0, \frac{-3}{4}\right] = 0.$$

7. Repeat **Problem 6** with $T = 4x^3y - 4xy^3 + 27z$ and $A\left(\frac{1}{2}, \frac{1}{2}\right)$

Sol. Here $\frac{\partial T}{\partial x} = 12x^2y - 4y^3 = 12\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^3$

$$= \frac{3}{2} - \frac{1}{2} = 1 \quad \text{at } A\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\frac{\partial T}{\partial y} = 4x^3 - 12xy^2 = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{3}{2} = -1.$$

grad $T = [1, -1]$.

As the unit vector in the direction of y – axis is [0, 1],we have

$$\frac{dT}{ds} = [1,-1].[0,1] = -1$$

Now $\textbf{v} = [1,-2]$, $|\textbf{v}| = \sqrt{5}$

(i) $\frac{dT}{ds}$ in the direction of **v**

$$= \frac{\textbf{grad } T.\textbf{v}}{\textbf{v}} = \frac{[1,-1].[1,-2]}{\sqrt{5}} = \frac{1+2}{\sqrt{5}} = \frac{3}{\sqrt{5}}.$$

(ii) The direction of greatest rate of change

= **grad** $T = [1,-1] = \textbf{i} - \textbf{j}$.

(iii) Magnitude of greatest rate of change

$$= |\textbf{grad } T| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}.$$

(iv) Direction of no rate of change at A

= Direction perpendicular to **grad** T

$= [1, 1] = \textbf{i} + \textbf{j}$, since $(\textbf{i} - \textbf{j}).(\textbf{i} + \textbf{j}) = 0$.

Exercise Set 9.5 (Page 424)

In Problems 1 – 10 find equations for the tangent plane and the normal line to the given surface at the indicated point P:

1. $4x^2 - y^2 + 3z^2 = 10$, $P(2, -3, 1)$

Sol. $f(x, y, z) = 4x^2 - y^2 + 3z^2 - 10$

grad $f = 8x\textbf{i} - 2y\textbf{j} + 6z\textbf{k}$

grad $f\,|_{P(2,-3,1)} = 16\textbf{i} + 6\textbf{j} + 6\textbf{k}$ is a normal vector at P

which is a normal vector of the tangent plane to the surface at $P(2, -3, 1)$. Equation of the tangent plane is

$16(x-2) + 6(y+3) + 6(z-1) = 0$

i.e., $8x + 3y + 3z - 10 = 0$

Equations of the normal line to the surface through $P(2,-3,1)$ are

$x = 2 + 16t, y = -3 + 6t, z = 1 + 6t$

or $\frac{x-2}{8} = \frac{y+3}{3} = \frac{z-1}{3}$.

2. $x^2 + y^2 + z^2 = 14$, $P(1, -2, 3)$

Sol. $f(x, y, z) = x^2 + y^2 + z^2 - 14$

grad $f = 2x\textbf{i} + 2y\textbf{j} + 2x\textbf{k}$

grad $f\,|_P = 2\textbf{i} - 4\textbf{j} + 6\textbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at $P(1, -2, 3)$ is

$2(x-1) - 4(y+2) + 6(z-3) = 0$

i.e., $x - 2y + 3z - 14 = 0$.

Equations of the normal line to the surface through P are

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3}.$$

3. $9x^2 + 4y^2 - z^2 = 36$, $P(2, 3, 6)$

Sol. $f(x, y, z,) = 9x^2 + 4y^2 - z^2 - 36$

grad $f = 18x\textbf{i} + 8y\textbf{j} - 2z\textbf{k}$

grad $f\,|_P = 36\textbf{i} + 24\textbf{j} - 12\textbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$36(x-2) + 24(y-3) - 12(z-6) = 0$

or $3x + 2y - z - 6 = 0$

Equations of the normal line to the surface through P are

$$\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-6}{-1}.$$

4. $x^2 - 2y^2 - z^2 = 4$, $P(-6, 2, \sqrt{24})$

Sol $f(x, y, z) = x^2 - 2y^2 - z^2 - 4$

grad $f = 2x\mathbf{i} - 4y\mathbf{j} - 2z\mathbf{k}$

grad $f|_P = -12\mathbf{i} - 8\mathbf{j} - 2\sqrt{24}\,\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$-12(x+6) - 8(y-2) - 2\sqrt{24}(z - \sqrt{24})$$

or $\quad 3x + 2y + \sqrt{6}\,z + 2 = 0$

Equations of the normal line to the surface through $P(-6, 2, \sqrt{24})$ are

$$\frac{x+6}{3} = \frac{y-2}{2} = \frac{z-\sqrt{24}}{\sqrt{6}}.$$

5. $z = x^2 + y^2, P(-2, 1, 5)$

Sol. $f(x, y, z) = x^2 + y^2 - z$

grad $f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$

grad $f|_P = -4\mathbf{j} + 2\mathbf{i} - \mathbf{k}$ is normal vector at P.

Equation of the tangent plane to the surface at P is

$$-4(x+2) + 2(y-1) - (z-5) = 0$$

or $\quad 4x - 2y + z - 5 = 0$

Equations of the normal line to the surface through P are

$$\frac{x+2}{4} = \frac{y-1}{-2} = \frac{z-5}{1}.$$

6. $xz = 4, P(-2, 2, -2)$

Sol. $f(x, y, z) = xz - 4$

grad $f = z\mathbf{i} + 0\mathbf{j} + x\mathbf{k}$

grad $f|_P = -2\mathbf{i} + 0\mathbf{j} - 2\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$-2(x+2) + 0(y-2) - 2(z+2) = 0$$

or $\quad x + z + 4 = 0$

Equations of the normal line to the surface through P are

$$\frac{x+2}{1} = \frac{y-2}{0} = \frac{z+2}{1}.$$

7. $x^2 + z^2 = \frac{a^2}{h^2}y^2, P\left(\frac{a}{\sqrt{2}}, h, \frac{a}{\sqrt{2}}\right)$

Sol. $f(x, y, z) = x^2 + z^2 - \frac{a^2}{h^2}y^2$

grad $f = 2x\mathbf{i} - 2\frac{a^2}{h^2}y\mathbf{j} + 2z\mathbf{k}$

grad $f|_P = \sqrt{2}\,a\mathbf{i} - 2\frac{a^2}{h}\mathbf{j} + \sqrt{2}\,a\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$\sqrt{2}\,a\left(x - \frac{a}{\sqrt{2}}\right) - \frac{2a^2}{h}(y-h) + \sqrt{2}\,a\left(z - \frac{a}{\sqrt{2}}\right) = 0$$

or $\quad \sqrt{2}\,ax - \frac{2a^2}{h}y + \sqrt{2}\,az = 0$

or $\quad \sqrt{2}\,h(x+z) - 2ay = 0$

Equations of the normal line to the surface through P are

$$\frac{x - \frac{a}{\sqrt{2}}}{\sqrt{2}\,h} = \frac{y-h}{2a} = \frac{z - \frac{a}{\sqrt{2}}}{\sqrt{2}\,h}.$$

8. $z = e^x \cos y, P\left(0, \frac{\pi}{2}, 0\right)$

Sol. $f(x, y, z) = e^x \cos y - z$

grad $f = e^x\cos y\,\mathbf{i} - e^x \sin y\,\mathbf{j} - \mathbf{k}$

grad $f|_P = 0\mathbf{i} - \mathbf{j} - \mathbf{k}$ is a normal vector at P

Equation of the tangent plane to the surface at P is

$$-1\left(y - \frac{\pi}{2}\right) - z = 0 \quad \text{or} \quad y + z - \frac{\pi}{2} = 0$$

Equations of the normal line to the surface through P are

$$\frac{x}{0} = \frac{y - \frac{\pi}{2}}{1} = \frac{z}{1}.$$

9. $x = \ln\left(\frac{y}{2z}\right), P(0, 2, 1)$

Sol. $f(x, y, z) = x - \text{In}\left(\frac{y}{2z}\right) = x - \text{In}\, y + \text{In}\, z + \text{In}\, 2$

grad $f = \mathbf{i} - \frac{1}{y}\mathbf{j} + \frac{1}{z}\mathbf{k}$

grad $f|_P = \mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$x - \frac{1}{2}(y-2) + z - 1 = 0$$

or $\quad 2x - y + 2z = 0$

Equations of the normal line to the surface through P are

$$\frac{x}{2} = \frac{y-2}{-1} = \frac{z-1}{2}.$$

10. $x^{2/3} + y^{2/3} + z^{2/3} = 9, \quad P(1, 8, -8)$

Sol. $f(x, y, z) = x^{2/3} + y^{2/3} + z^{2/3} - 9$

$\text{grad } f = \frac{2}{3}x^{-1/3}\mathbf{i} + \frac{2}{3}y^{-1/3}\mathbf{j} + \frac{2}{3}z^{-1/3}\mathbf{k}$

$\text{grad } f\,|_P = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$\frac{2}{3}(x-1) + \frac{1}{3}(y-8) - \frac{1}{3}(z+8) = 0$$

or $\quad 2x + y - z - 18 = 0$

Equations of the normal line to the surface through P are

$$\frac{x-1}{2} = \frac{y-8}{1} = \frac{z+8}{-1}.$$

11. Find the point on $x^2 - 2y^2 - 4z^2 = 16$ at which the tangent plan parallel to the plane $4x - 2y + 4z = 5$

Sol. Let $P(x_1, y_1, z_1)$ be the required point on the given surface

$f(x, y, z) = x^2 - 2y^2 - 4z^2 - 16 = 0$

$\text{grad } f\,|_P = 2x_1\,\mathbf{i} - 4y_1\,\mathbf{j} - 8z_1\,\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane at P is

$$2x_1(x - x_1) - 4y_1(y - y_1) - 8z_1(z - z_1) = 0$$

or $\quad xx_1 - 2yy_1 - 4zz_1 - 16 = 0$

This plane is to be parallel to

$$4x - 2y + 4z - 5 = 0$$

Hence $\frac{x_1}{4} = \frac{-2y_1}{-2} = \frac{-4z_1}{4} = k$ (say)

Therefore, $x_1 = 4k, y_1 = k, z_1 = -k$

Since (x_1, y_1, z_1) lies on the surface, we have

$$16k^2 - 2k^2 - 4k^2 = 16$$

or $\quad k = \pm\frac{4}{\sqrt{10}} = \pm\frac{2\sqrt{2}}{\sqrt{5}}$

Thus $P\left(\frac{8\sqrt{2}}{\sqrt{5}}, \frac{2\sqrt{2}}{\sqrt{5}}, \frac{-2\sqrt{2}}{\sqrt{5}}\right)$ and $Q\left(\frac{-8\sqrt{2}}{\sqrt{5}}, \frac{-2\sqrt{2}}{\sqrt{5}}, \frac{2\sqrt{2}}{\sqrt{5}}\right)$ are the required points.

12. Two surfaces are said to be **tangent** at a common point P if each has the same tangent plane at P Show that the surfaces $x^2 + z^2 + 4y = 0$ and $x^2 + y^2 + z^2 - 6z + 7 = 0$ are tangent at $P(0, -1, 2)$.

Sol. $f(x, y, z) = x^2 + z^2 + 4y$

$\text{grad } f = 2x\mathbf{i} + 2z\mathbf{k} + 4\mathbf{j}$

$\text{grad } f\,|_P = 0\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ is a normal vector at P.

Tangent plane to $f(x, y, z) = 0$ at $P(0, -1, 2)$ is

$$4(y+1) + 4(z-2) = 0$$

or $\quad y + z - 1 = 0 \qquad (1)$

$g(x, y, z) = x^2 + y^2 + z^2 - 6z + 7$

$\text{grad } g = 2x\mathbf{i} + 2y\mathbf{j} + (2z - 6)\mathbf{k}$

$\text{grad } g\,|_P = -2\mathbf{j} - 2\mathbf{k}$ is a normal vector at P.

Tangent plane to $g(x, y, z) = 0$ at P is $-2(y+1) - 2(z-2) = 0$ or $y + z - 1 = 0$ which is the same as (1). Hence the two surfaces are tangent at P

13. Show that the sphere $x^2 + y^2 + z^2 = 18$ and the cone $x^2 + z^2 = (y-6)^2$ are tangent along their intersection.

Sol. We find the points of intersection of the two surfaces

$$x^2 + y^2 + z^2 = 18 \quad \text{and} \quad x^2 + z^2 = (y-6)^2$$

Subtracting, we get

$$y^2 = 18 - (y-6)^2$$

This gives $y = 3$

If $\quad f(x, y, z) = x^2 + y^2 + z^2 - 18$

then $\text{grad } f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ which shows that $2x_1, 2y_1, 2z_1$ are the direction ratios of the normal to $f(x, y, z) = 0$ at (x_1, y_1, z_1).

But $y_1 = 3$. Therefore direction ratios are $2x_1, 6, 2z_1$.

Again, if $g(x, y, z) = x^2 - (y-6)^2 + z^2 = 0$

then $\text{grad } g = 2x\mathbf{i} - 2(y-6)\mathbf{j} + 2z\mathbf{k}$

$\text{grad } g(x_1, y_1, z_1) = 2x_1\mathbf{i} - 2(y_1 - 6)\mathbf{j} + 2z_1\mathbf{k}$ is a normal vector at P.

$2x_1, -2(y_1 - 6), 2z_1$ are the direction ratios of the normal to $g(x, y, z)$ at (x_1, y_1, z_1) But $y_1 = 3$. Therefore, $2x_1, 6, 2z_1$ are the direction ratios of the normal in both the cases which shows that the given sphere and the cone are tangent along their intersection.

14. Show that the surfaces $z = 16 - x^2 - y^2$ and $63z = x^2 + y^2$ intersect orthogonally.

Sol. $f(x, y, z) = x^2 + y^2 + z - 16 = 0 \qquad (1)$

$f_x = 2x, f_y = 2y, f_z = 1$

$g(x, y, z) = x^2 + y^2 - 63z = 0 \qquad (2)$

$g_x = 2x, g_y = 2y, g_z = -63$

Subtracting (2) from (1), we get $z = \frac{1}{4}$.

Eliminating z from (1) and (2), we have $x^2 + y^2 = \frac{63}{4}$

The two surfaces intersect along the curve

$$z = \frac{1}{4},\ x^2 + y^2 = \frac{63}{4}$$

Now, $f_x g_x + f_y g_y + f_z g_z = 4(x^2 + y^2) - 63 = 4\left(\frac{63}{4}\right) - 63 = 0$

Hence the surfaces intersect orthogonally.

15. Prove that all normal lines of the sphere $x^2 + y^2 + z^2 = a^2$ pass through the centre of the sphere.

Sol. Let $P(x_1, y_1, z_1)$ be any point on the sphere

$$f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$$

grad $f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

grad $f\,|_P = 2x_1\mathbf{i} + 2y_1\mathbf{j} + 2z_1\mathbf{k}$ is a normal vector at P.

Equations of the normal line through P are

$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{z_1} \quad \text{or} \quad xy_1 - yx_1 = 0 = yz_1 - zy_1$$

which pass through (0, 0, 0) – the centre of the sphere.

Since (x_1, y_1, z_1) is any point on the sphere, all normal lines pass through the centre of the sphere.

16. Show that the ellipsoid $\frac{x^2}{12} + \frac{y^2}{16} + \frac{z^2}{12} = 1$ and the hyperbolid $\frac{y^2}{3} - x^2 - z^2 = 1$ intersect orthogonally.

Sol. Let $f(x, y, z) = \frac{y^2}{16} + \frac{x^2}{12} + \frac{z^2}{12} - 1 = 0$ (1)

and $g(x, y, z) = \frac{y^2}{3} - (x^2 + z^2) - 1 = 0$ (2)

$$f_x = \frac{x}{6},\ f_y = \frac{y}{8},\ f_z = \frac{z}{6}$$

$$g_x = -2x,\ g_y = \frac{2y}{3},\ g_z = -2z$$

$$f_x g_x + f_y g_y + f_z g_z = -\frac{x^2}{3} + \frac{y^2}{12} - \frac{z^2}{3} = \frac{y^2}{12} - \frac{1}{3}(x^2 + z^2).$$

The two surfaces intersect at the points where (from (1) and (2))

$$\frac{y^2}{16} + \frac{1}{12}\left(\frac{y^2}{3} - 1\right) - 1 = 0$$

or $13y^2 = 156$ or $y = \pm 2\sqrt{3}$

At $y = \pm 2\sqrt{3}$, $x^2 + z^2 = 3$, from (2)

Hence at all common points of the two surfaces, we have

$$f_x g_x + f_y g_y + f_z g_z = \frac{y^2}{12} - \frac{1}{3}(x^2 + z^2) = 1 - 1 = 0$$

Hence the two surfaces intersect orthogonally.

17. Find the point on $z = 4x^2 + 9y^2$ at which the normal line is parallel to the line through $A(-2, 4, 3)$ and $B(5, -1, 2)$

Sol. Let $P(x_1, y_1, z_1)$ be the required point.

grad $f\,|\,P = 8x_1\mathbf{i} + 18y_1\mathbf{j} - \mathbf{k}$ is a normal vector at P

Direction ratios of the normal are $8x_1$, $18y_1$, -1.

Direction ratios of the line AB are 7, – 5, – 1.

Since the normal line is parallel to AB, we have

$$\frac{8x_1}{7} = \frac{18y_1}{-5} = \frac{-1}{-1}$$

Therefore, $x_1 = \frac{7}{8}$, $y_1 = \frac{-5}{18}$ and $z_1 = \frac{37}{18}$

The required point is $\left(\frac{7}{8}, \frac{-5}{18}, \frac{37}{18}\right)$.

18. Where and at what angle do the cone $x^2 + y^2 = \frac{1}{2}z^2$ and the cylinder $x^2 + y^2 = 4$ intersect?

Sol. For the point of intersection we solve $x^2 + y^2 = \frac{1}{2}z^2$ and $x^2 + y^2 = 4$ simultaneously. We get $z^2 = 8$ or $z = \pm 2\sqrt{2}$. Therefore, intersection of the cone and the cylinder is the circle $x^2 + y^2 = 4$ in the planes $z = 2\sqrt{2}$ and $z = -2\sqrt{2}$.

Now for the cone, we have $f(x, y, z) = x^2 + y^2 - \frac{1}{2}z^2 = 0$

grad $f = 2x\mathbf{i} + 2y\mathbf{j} - z\mathbf{k}$ is a normal vector at P

This shows that $2x$, $2y$, $-z$ are the direction ratios of the normal to the tangent plane of the cone at (x, y, z).

For the cylinder, we have

$$g(x, y, z) = x^2 + y^2 - 4 = 0$$

grad $g = 2x\mathbf{i} + 2y\mathbf{j} + 0\mathbf{k}$ is a normal vector at P.

Angle between the normals at the common point is the required angle. Therefore,

$$\cos\theta = \frac{(2x)(2x) + (2y)(2y) + 0}{\sqrt{4x^2 + 4y^2 + z^2}\sqrt{4x^2 + 4y^2 + 0}}$$

$$= \frac{4(x^2 + y^2)}{\sqrt{4(x^2 + y^2) + z^2}\sqrt{4(x^2 + y^2)}}$$

$$= \frac{4 \times 4}{\sqrt{4 \times 4 + 8}} \frac{1}{\sqrt{4 \times 4}}, \text{ as } x^2 + y^2 = 4 \text{ and } z^2 = 8$$

$$= \frac{16}{4\sqrt{24}} = \frac{4}{2\sqrt{6}} = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$

or $\theta = \arccos \sqrt{\frac{2}{3}}$ is the required angle at which the cone and the cylinder intersect.

19. For the surface defined by the parametric equations $x = 2 \cosh u \cos v$, $y = 3 \cosh u \sin v$, $z = 6 \sinh u$, find a vector normal to the surface at the point for which $u = 1, v = \frac{\pi}{3}$

Sol. We have

$r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$= (2 \cosh u \cos v)\mathbf{i} + (3 \cosh u \sin v)\mathbf{j} + (6 \sinh u)\mathbf{k}$

$$\frac{\partial r}{\partial u} = (2 \sinh u \cos v)\,\mathbf{i} + (3 \sinh u \sin v)\,\mathbf{j} + (6 \cosh u)\,\mathbf{k}$$

$$= \left(\frac{1}{2} \times 2 \sinh u\right)\mathbf{i} + \left(\frac{\sqrt{3}}{2} 3 \sinh u\right)\mathbf{j} + (6 \cosh u)\,\mathbf{k} \text{ at } v = \frac{\pi}{3}$$

$$= \sinh u\, \boldsymbol{i} + \frac{3\sqrt{3}}{2} \sinh u\, \mathbf{j} + 6 \cosh u\, \mathbf{k}$$

$$\frac{\partial r}{\partial v} = (-2 \cosh u.\, \sin v)\,\mathbf{i} + (3 \cosh u \cos v)\,\mathbf{j} + 0\,\mathbf{k}$$

$$= \left(\frac{\sqrt{3}}{2} \times -2 \cosh u\right) + \left(\frac{1}{2} \cdot 3 \cosh u\right)\mathbf{j} + 0\,\mathbf{k} \text{ at } v = \frac{\pi}{3}$$

$$= -\sqrt{3} \cosh u\, \boldsymbol{i} + \frac{3}{2} \cosh u\, \mathbf{j} + 0\mathbf{k}$$

The required vector (normal to the surface) is

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh u & \frac{3\sqrt{3}}{2} \sinh u & 6 \cosh u \\ -\sqrt{3} \cosh u & \frac{3}{2} \cosh u & 0 \end{vmatrix}$$

$$= -9 \cosh^2 u\, \mathbf{i} - 6\sqrt{3} \cosh^2 u\mathbf{j} + \left(\frac{3}{2} + \frac{9}{2}\right) \sinh u \cosh u\, \mathbf{k}$$

$$= -9 \cosh^2 u\, \mathbf{i} - 6\sqrt{3} \cosh^2 u\, \mathbf{j} + 3 \sinh 2u\, \mathbf{k},\ u = 1.$$

$$= [-9 \cosh^2 u\mathbf{i},\ -6\sqrt{3} \cosh^2 u\mathbf{j},\ 3 \sinh 2u\ \mathbf{k}],\ u = 1.$$

20. For the surface defined by $x = (3 + \cos \phi) \cos \theta$, $y = (3 + \cos \phi) \sin \theta$, $z = \sin \phi$, $0 \le \theta < 2\pi$, $-\pi < \phi \le \pi$, show that the parametric curves for which ϕ is constant and θ varies are circles in planes parallel to the xy-plane. Also show that the parametric curves for which θ is constant and ϕ varies lie in planes through the z-axis. Find a vector normal to this surface at the point for which $\theta = \frac{\pi}{4}, \phi = \frac{2\pi}{3}$.

Sol. First we consider the parametric curves for which ϕ is consta[nt]. Since $x = (3 + \cos\phi)\cos \theta$, $y = (3 + \cos \phi) \sin \theta$, squaring [an]d adding we get

$$x^2 + y^2 = (3 + \cos \phi)^2$$

which represents a circle in the plane z = constant

i.e., a plane parallel to the xy-plane. Again if θ is constant then we have $\frac{x}{\cos\theta} + \frac{-y}{\sin\theta} = 0$ which represents a plane containing the z-axis.

Now if $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$= (3 + \cos \phi) \cos \theta\mathbf{i} + (3 + \cos \phi) \sin \theta\mathbf{j} + \sin \phi\,\mathbf{k},$$

then $\frac{\partial r}{\partial \theta} = -(3 + \cos \phi) \sin \theta\mathbf{i} + (3 + \cos \phi) \cos \theta\mathbf{j} + 0\,\mathbf{k}$

$$= -\left(3 - \frac{1}{2}\right)\frac{1}{\sqrt{2}}\mathbf{i} + \left(3 - \frac{1}{2}\right)\frac{1}{\sqrt{2}}\mathbf{j} + 0\mathbf{k}$$

$$= -\frac{5}{2\sqrt{2}}\mathbf{i} + \frac{5}{2\sqrt{2}}\mathbf{j} + 0\mathbf{k}, \text{ at } \theta = \frac{\pi}{4}, \phi = \frac{2\pi}{3}$$

$$\frac{\partial r}{\partial \phi} = -\sin \phi \cos\theta\mathbf{i} - \sin\phi \sin \theta\mathbf{j} + \cos\phi\,\mathbf{k}$$

$$= -\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}\mathbf{i} - \frac{\sqrt{3}}{2\sqrt{2}}\mathbf{j} - \frac{1}{2}\mathbf{k}, \text{ at } \theta = \frac{\pi}{4}, \phi = \frac{2\pi}{3}$$

Therefore, any vector normal to the surface is

$$\frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{-\sqrt{3}}{2\sqrt{2}} & \frac{-\sqrt{3}}{2\sqrt{2}} & \frac{-1}{2} \\ \frac{-5}{2\sqrt{2}} & \frac{5}{2\sqrt{2}} & 0 \end{vmatrix} = \frac{5}{4\sqrt{2}}\mathbf{i} + \frac{5}{4\sqrt{2}}\mathbf{j} - \frac{5\sqrt{3}}{4}\mathbf{k}$$

$$= \left[\frac{5}{4\sqrt{2}}, \frac{5}{4\sqrt{2}}, -\frac{5\sqrt{3}}{4}\right]$$

Exercise Set 9.6 (Page 433)

Find the extrema of each of the following (Problems 1–14):

1. $f(x,y) = x^2 - xy + y^2 + 6x$

Sol.
$$f(x, y) = x^2 - xy + y^2 + 6x$$
$$f_x(x, y) = 2x - y + 6$$
$$f_{xx}(x, y) = 2$$
$$f_y(x, y) = -x + 2y$$
$$f_{yy}(x, y) = 2$$
$$f_{xy}(x, y) = f_{yx}(x, y) = -1$$

For critical points, we have

$$2x - y + 6 = 0 \qquad (1)$$
$$-x + 2y = 0 \qquad (2)$$

Adding two times of equation (2) to (1), we have

$3y + 6 = 0$ or $y = -2$

Setting $y = -2$ into (2), we get $x = -4$.

The only critical point is $(-4, -2)$

Now $f_{xx}(x, y).f_{yy}(x, y) - [f_{xy}(x, y)]^2 = 4 - 1 > 0$

and $f_{xx}(x, y), f_{yy}(x, y)$ are both positive. $f(x, y)$ has a local minimum at $(-4, -2)$. The minimum value is

$$16 - 8 + 4 - 24 = -12.$$

2. $f(x,y) = \frac{1}{x} + xy - \frac{8}{y}$

Sol. $f(x,y) = \frac{1}{x} + xy - \frac{8}{y}$

$$f_x(x, y) = -\frac{1}{x^2} + y \qquad f_y(x, y) = x + \frac{8}{y^2}$$
$$f_{xx}(x, y) = \frac{2}{x^3} \qquad f_{yy}(x, y) = -\frac{16}{y^3}$$
$$f_{xy}(x, y) = 1 = f_{yx}(x, y)$$

For critical points, we have

$$-\frac{1}{x^2} + y = 0 \qquad (1)$$
$$x + \frac{8}{y^2} = 0 \qquad (2)$$

Substituting $y = \frac{1}{x^2}$ into (2), we get

$$x + 8x^4 = 0$$
or $$x(1 + 8x^3) = 0$$

Thus $x = 0$ or $x = -\frac{1}{2}$

$f(x, y)$ is not defined at $x = 0$ so this value is not admissible.

Putting $x = -\frac{1}{2}$ into (1), we find $y = 4$

A critical point of $f(x, y)$ is $\left(-\frac{1}{2}, 4\right)$

$$f_{xx}\left(-\frac{1}{2}, 4\right) = -16,\ f_{yy}\left(-\frac{1}{2}, 4\right) = -\frac{1}{4}$$
$$f_{xx}\left(-\frac{1}{2}, 4\right).f_{yy}\left(-\frac{1}{2}, 4\right) - \left[f_{xy}\left(-\frac{1}{2}, 4\right)\right]^2 = (-16)\left(-\frac{1}{4}\right) - 1 = 3 > 0$$

But $f_{xx}\left(-\frac{1}{2}, 4\right)$ and $f_{yy}\left(-\frac{1}{2}, 4\right)$ are both negative.

Therefore $\left(-\frac{1}{2}, 4\right)$ is a point of relative maxima.

3. $f(x,y) = 2x^2 + xy^2 - 4x - 1$

Sol.
$$f_x = 4x + y^2 - 4, \quad f_{xx} = 4$$
$$f_y = 2xy, \quad f_{yy} = 2x$$
$$f_{yx} = 2y = f_{xy}$$

For critical points,

$$f_x = y^2 + 4x - 4 = 0 \qquad (1)$$
and $$f_y = 2xy = 0 \qquad (2)$$

From (2), we have $x = 0$ or $y = 0$

If $x = 0$, then (1) gives $y = \pm 2$

If $y = 0$, then from (1), we get $x = 1$

Thus critical points, are

$(1, 0), (0, -2), (0, 2)$

$f_{yy}(1, 0) = 2$, and $f_{yx}(1, 0) = 0$

Now, $f_{xx}(1, 0) f_{yy}(1, 0) - [f_{yx}(1, 0)]^2 = 8 > 0$ and $f_{xx}(1, 0) > 0$

Therefore local minima at $(1, 0)$.

$f_{xx}(0, \pm 2) = 4, f_{yy}(0, \pm 2) = 0$ and $f_{yx}(0, \pm 2) = \pm 4$

Therefore,

$$f_{xx}(0, \pm 2).f_{yy}(0, \pm 2) - [f_{yx}(0, \pm 2)]^2 = -16 < 0$$

Thus $(0, 2)$ and $(0, -2)$ are saddle points.

4. $f(x,y) = x^2 + 6xy + 2y^2 - 6x + 10y - 5$

Sol. $f(x, y) = x^2 + 6xy + 2y^2 - 6x + 10y - 5$

$f_x(x, y) = 2x + 6y - 6$ | $f_y(x,y) = 6x + 4y + 10$

$f_{xx}(x, y) = 2$ | $f_{yy}(x, y) = 4$

$$f_{yx}(x,y) = 6 = f_{xy}(x, y)$$

For critical points, we have

$$2x + 6y - 6 = 0$$
$$6x + 4y + 10 = 0$$

i.e., $x + 3y - 3 = 0$ (1)

$3x + 2y + 5 = 0$ (2)

Subtracting three times of (1) from (2), we get

$-7y + 14 = 0$ or $y = 2$

Putting this value of y into (1), we find $x = -3$

Thus $(-3, 2)$ is the only critical point.

$$f_{xx}(-3, 2).f_{yy}(-3, 2) - [f_{yx}(-3, 2)]^2 = 2 \times 4 - 6^2 < 0$$

Thus $(-3, 2)$ is a saddle point.

5. $f(x,y) = 6x^3y^2 - x^4y^2 - x^3y^3$

Sol. $f_x(x, y) = 18x^2y^2 - 4x^3y^2 - 3x^2y^3$

$f_y(x, y) = 12x^3y - 2x^4y - 3x^3y^2$

For extreme value, we have

$$f_x = x^2y^2(18 - 4x - 3y) = 0$$

and $f_y = x^3y(12 - 2x - 3y) = 0$

i.e., $18 - 4x - 3y = 0$ (1)

and $12 - 2x - 3y = 0$ (2)

Subtracting (1) from (2), we get

$-6 + 2x = 0$ or $x = 3$

Substituting $x = 3$ into (2), we have

$12 - 6 - 3y = 0$ or $y = 2$

Thus critical point is $(3, 2)$.

$$f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3$$
$$f_{yy} = 12x^3 - 2x^4 - 6x^3y$$
$$f_{yx} = 36x^2y - 8x^3y - 9x^2y^2$$
$$f_{xx}(3, 2) = -144 < 0, f\ \ (3, 2) = -162 < 0$$
$$f_{yx}(3, 2) = -108$$
$$f_{xx}(3, 2).f_{yy}(3, 2) - [f_{xy}(3, 2)]^2 = (-144)(-162) - (-108)^2 > 0$$

Therefore, f has a local maxima at $(3, 2)$.

6. $f(x,y) = 2x^4 + y^2 - x^2 - 2y$

Sol. $f_x = 8x^3 - 2x$, $f_{xx} = 24x^2 - 2$

$f_y = 2y - 2$, $f_{yy} = 2$

$f_{xy} = 0 = f_{yx}$

For critical points, we have

$$f_x = 2x(4x^2 - 1) = 0 \quad \text{and} \quad f_y = 2(y - 1) = 0$$

Thus $y = 1$ and $x = 0, \frac{1}{2}, -\frac{1}{2}$.

Critical points are

$$(0, 1)\left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, 1\right)$$

$$f_{xx}(0, 1) = -2, f_{xx}\left(-\frac{1}{2}, 1\right) = 4 = f_{xx}\left(\frac{1}{2}, 1\right)$$

$$f_{yy}(0, 1) = f_{yy}\left(-\frac{1}{2}, 1\right) = f_{yy}\left(\frac{1}{2}, 1\right) = 2$$

Thus $f_{xx}(0, 1) f_{yy}(0, 1) - [f_{xy}(0, 1)]^2 = -4 - 0 < 0$

Thus $(0, 1)$ is a saddle point

$$f_{xx}\left(\pm\frac{1}{2}, 1\right) f_{yy}\left(\pm\frac{1}{2}, 1\right) - \left[f_{xy}\left(\pm\frac{1}{2}, 1\right)\right]^2 = 8 - 0 > 0$$

Thus local minima at $\left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, 1\right)$.

7. $f(x,y) = 18x^2 - 32y^2 - 36x - 128y$

Sol. $f_x = 36x - 36$, $f_{xx} = 36$,

$f_y = -64y - 128, f_{yy} = -64, f_{xy} = 0 = f_{yx}$

For critical points,

$$f_x = 36(x - 1) = 0$$

and $f_y = -64(y + 2) = 0$

Thus $(1, -2)$ is the only critical point

$$f_{xx}(1, -2).f_{yy}(1, -2) - [f_{xy}(1, 2)]^2$$
$$= 36(-64) - 0 < 0$$

Thus $(1, -2)$ is a saddle point.

8. $f(x,y) = e^{-(x^2+y^2+2x)}$

Sol. $f_x = (-2x - 2)e^{-(x^2+y^2+2x)}$

$$f_{xx} = (-2x - 2)^2 e^{-(x^2+y^2+2x)} - 2e^{-(x^2+y^2+2x)}$$
$$f_y = -2y\, e^{-(x^2+y^2+2x)}$$
$$f_{yy} = 4y^2 e^{-(x^2+y^2+2x)} - 2e^{-(x^2+y^2+2x)}$$
$$f_{xy} = -2y(-2x + 2)e^{-(x^2+y^2+2x)} = f_{yx}.$$

For critical points, we have

$$f_x = (-2x-2)\,e^{-(x^2+y^2+2x)} = 0$$

and $f_y = -2y\,e^{-(x^2+y^2+2x)} = 0$

Therefore, $x = -1, y = 0$ and a critical point is $(-1, 0)$

Now $f_{xx}(-1,0)\,f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 = (-2e)(-2e) > 0$

and $f_{xx}(-1, 0) < 0$

Thus $(-1, 0)$ is a point of local maxima.

9. $f(x,y) = 2x^3 + y^2 - 9x^2 - 4y + 12x - 2$

Sol. $f_x = 6x^2 - 18x + 12,\quad f_{xx} = 12x - 18$

$f_y = 2y - 4,\quad f_{yy} = 2, f_{xy} = 0 = f_{yx}$

For critical points, we have

$$f_x = 6(x^2 - 3x + 2) = 0$$

and $f_y = 2(y-2) = 0$

i.e., $x = 1, 2$ and $y = 2$

Critical points are $(1, 2)$ and $(2, 2)$

Now, $f_{xx}(1,2)\,f_{yy}(1,2) - [f_{xy}(1,2)]^2 = -12 < 0$

Therefore $(1, 2)$ is a saddle point.

$f_{xx}(2,2)\,f_{yy}(2,2) - [f_{xy}(2,2)]^2 = 12 > 0$ and $f_{xx}(2,2) > 0$

Hence $(2, 2)$ is a point of local minima.

10. $f(x,y) = x^2 - e^{y^2}$

Sol. $f(x, y) = x^2 - e^{y^2}$

$f_x = 2x,\quad f_{xx} = 2$

$f_y = -2y\,e^{y^2},\quad f_{yy} = -4y^2 e^{y^2} - 2e^{y^2}, f_{yx} = 0 = f_{xy}$

For critical points,

$f_x = 2x = 0$ and $f_y = -2y\,e^{y^2} = 0$

Therefore $(0, 0)$ is the only critical point.

$f_{xx}(0,0)\,f_{yy}(0,0) - [f_{xy}(0,0)]^2 = -4 < 0$

Thus $(0, 0)$ is a saddle point.

11. $f(x,y) = \sin x + \sin y$

Sol. $f_x = \cos x,\quad f_{xx} = -\sin x$

$f_y = \cos y,\quad f_{yy} = -\sin y, f_{yx} = 0 = f_{xy}$

For critical points

$f_x = \cos x = 0$ and $f_y = \cos y = 0$

Therefore $\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$ is a critical point. (m, n are integers).

$$f_{xx}\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right) = -\sin\left(m\pi + \frac{\pi}{2}\right) = \begin{cases} -1 & \text{if } m \text{ even} \\ 1 & \text{if } m \text{ odd} \end{cases}$$

Similarly, $f_{yy}\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right) = -\sin\left(n\pi + \frac{\pi}{2}\right) = \begin{cases} -1 \text{ if } n \text{ even} \\ 1 \text{ if } n \text{ odd} \end{cases}$

Now, $f_{xx}\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right) \cdot f_{yy}\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right) - [f_{yx}]^2$

$$\left.\begin{array}{l} > 0 \\ > 0 \\ < 0 \end{array}\right\} \begin{array}{l} \text{if both } m,n \text{ even} \\ \text{if both } m,n \text{ odd} \\ \text{if one of } m,n \text{ odd and other even.} \end{array}$$

Thus (i) maximum for both m, n even

(ii) minimum for both m, n odd

(iii) saddle point otherwise.

12. $f(x, y) = y^2 - 6y\cos x + 6$

Sol. $f_x = 6y\sin x,\quad f_{xx} = 6y\cos x$

$f_y = 2y - 6\cos x,\ f_{yy} = 2$

$f_{xy} = 6\sin x = f_{yx}$

For critical points, we have

$f_x = 6y\sin x = 0$ (1)

and $f_y = 2y - 6\cos x = 0$ (2)

From (1), $y = 0$ or $x = 2n\pi$

Setting $x = 2n\pi$ into (2) we get $y = 3$

$(2n\pi, 3)$ is a critical point.

$f_{xx}(2n\pi, 3)\,f_{yy}(2n\pi, 3) - [f_{xy}(2n\pi, 3)]^2 = 36 - 0 > 0$

$f_{xx}(2n\pi, 3) = 18 > 0$

Thus f is minimum at $(2n\pi, 3)$.

Setting $y = 0$ into (2), we have

$$x = (2n+1)\frac{\pi}{2}$$

$$f_{xx}f_{yy} - \left[f_{xy}\left(\frac{2n+1}{2}\pi, 0\right)\right]^2 = 0 - 36 < 0$$

Thus $\left(\frac{2n+1}{2}\pi, 0\right)$ is a saddle point.

13. $f(x, y) = \cos x + \cos y + \cos(x+y)$

Sol. $f_x = -\sin x - \sin(x+y), f_{xx} = -\cos x - \cos(x+y)$

$f_y = -\sin y - \sin(x+y), f_{yy} = -\cos y - \cos(x+y)$

$f_{yx} = f_{xy} = -\cos(x+y)$

For critical points,

$$-\sin x - \sin(x+y) = 0 \quad (1)$$

and $\quad -\sin y - \sin(x+y) = 0$

Therefore, $\sin x = \sin y$

Setting, $\sin x = \sin y$ into (1), we get

$$\sin x + \sin x \cos x + \cos x \sin x = 0$$

or $\quad \sin x(1 + 2\cos x)$ i.e., $x = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

$x = 0, \quad y = 0$

$x = \pi, \quad y = \pi$

$x = \frac{2\pi}{3}, \quad y = \frac{2\pi}{3}$

Thus $(0,0)$ (π,π), $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, \frac{4\pi}{3}\right)$ are the critical points.

$$f_{xx}(0,0) f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 4 - 1 > 0$$

$$f_{xx}(0,0) = -2 < 0$$

Therefore maximum at $(0,0)$.

$$f_{xx}\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) f_{yy}\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) - \left[f_{xy}\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)\right]^2 = 1 - \frac{1}{4} > 0$$

Thus minimum at $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$. Similarly, minimum at $\left(\frac{4\pi}{3}, \frac{4\pi}{3}\right)$

$$f_{xx}(\pi,\pi) f_{yy}(\pi,\pi) - [f_{xy}(\pi,\pi)]^2 = 0 - 1 < 0.$$

Hence (π,π) is a saddle point.

14. $f(x,y) = (x+1)(y+1)(x+y+1)$

Sol. $f_x = (y+1)[(x+y+1) + (x+1)]$

$= (y+1)(2x+y+2)$

$f_{xx} = 2(y+1)$

$f_y = (x+1)[(x+y+1) + (y+1)]$

$= (x+1)(x+2y+2)$

$f_{yy} = 2(x+1)$

$f_{xy} = (y+1) + (2x+y+2)$

$= 2x + 2y + 3 = f_{yx}$

For critical points, we have

$$f_x = (y+1)(2x+y+2) = 0 \quad (1)$$

and $\quad f_y = (x+1)(x+2y+2) = 0 \quad (2)$

From (1), $y + 1 = 0$ or $2x + y + 2 = 0$

i.e., $\quad y = -1$ or $\quad y = -2x - 2$

Setting $y = -1$ into (2), we have

$(x+1)x = 0. \quad$ i.e., $x = 0, -1$

Critical points are $(0,-1), (-1,-1)$

Setting $y = -2x - 2$ into (2), we get $(x+1)(-3x-2) = 0$

i.e., $x = -1$ or $x = -\frac{2}{3}$.

If $\quad x = -1, y = 0, \quad$ if $x = -\frac{2}{3}, y = -\frac{2}{3}$

Critical points are $(-1, 0), \left(-\frac{2}{3}, -\frac{2}{3}\right)$

$$f_{xx}(0,-1) f_{yy}(0,-1) - [f_{xy}(0,-1)]^2 = 0 - 1 < 0$$

$(0,-1)$ is saddle point.

$$f_{xx}(-1,-1) f_{yy}(-1,-1) - [f_{xy}(-1,-1)]^2 = 0 - 1 < 0$$

$(-1,-1)$ is a saddle point.

$$f_{xx}(-1,0) f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 = 0 - 1 < 0$$

$(-1, 0)$ is a saddle point.

$$f_{xx}\left(-\frac{2}{3}, -\frac{2}{3}\right) f_{yy}\left(-\frac{2}{3}, -\frac{2}{3}\right) - \left[f_{xy}\left(-\frac{2}{3}, -\frac{2}{3}\right)\right]^2 = \frac{2}{3} \cdot \frac{2}{3} - \frac{1}{9} > 0$$

$\left(-\frac{2}{3}, -\frac{2}{3}\right)$ is a point of relative minimum.

15. Find the point of the sphere $x^2 + y^2 + x^2 + = 49$ that is nearest to the point $(2, 1, 3)$.

Sol. Let $Q(x, y, z)$ be the point on the sphere that is nearest to $(2, 1, 3)$

$$d^2 = (x-2)^2 + (y-1)^2 + (z-3)^2.$$

d is to be minimized subject to $x^2 + y^2 + z^2 = 49$

or $\quad z^2 = 49 - (x^2 + y^2)$

$$d^2 = (x-2)^2 + (y-1)^2 + (\sqrt{49 - x^2 - y^2} - 3)^2 = f(x,y)$$

$$= x^2 - 4x + 4 + y^2 - 2y + 1 + 49 - x^2 - y^2 + 9 - 6\sqrt{49 - x^2 - y^2}$$

$$= -4x - 2y + 63 - 6\sqrt{49 - x^2 - y^2}$$

$$f_x = -4 + \frac{6x}{\sqrt{49 - x^2 - y^2}}; \; f_y = -2 + \frac{6y}{\sqrt{49 - x^2 - y^2}}$$

For critical points

$$f_x = 0 = -4\sqrt{49 - x^2 - y^2} + 6x$$

and $\quad f_y = 0 = -2\sqrt{49 - x^2 - y^2} + 6y$

i.e., $\quad 4(49 - x^2 - y^2) = 9x^2$

and $\quad 49 - x^2 - y^2 = 9y^2 \quad (1)$

Therefore, $\frac{9x^2}{4} = 9y^2$ or $y = \pm\frac{x}{2}$

Putting this value of y into (1), we get

$$49 - x^2 - \frac{x^2}{4} = \frac{9x^2}{4} \quad \text{or} \quad 7x^2 = 98$$

$$x = \pm\sqrt{14} = \pm\frac{14}{\sqrt{14}} \; ; \; y = \pm\frac{\sqrt{14}}{2} = \pm\frac{7}{\sqrt{14}}$$

$$z^2 = 49 - (x^2 + y^2) = 49 - 14 - \frac{14}{4}$$

or $\quad z = \pm\frac{21}{\sqrt{14}}$

Required point is $\left(\frac{14}{\sqrt{14}}, \frac{7}{\sqrt{14}}, \frac{21}{\sqrt{14}}\right)$.

16. The sum of the length and girth (perimeter of a cross-section) of the packages accepted by post office is 270 centimeters. Find the dimensions of the rectangular package of greatest volume that can be sent by post.

Sol. Let x, y, z in centimeters be the length, breadth and height respectively of the package. By the given condition, we have

$$x + 2y + 2z = 270 \qquad (1)$$

Volume V of the package is

$$V = xyz = 2yz(135 - y - z), \text{ using (1)}$$

$$\frac{\partial V}{\partial y} = 2z(135 - y - z) - 2yz = 2z(135 - 2y - z)$$

$$\frac{\partial V}{\partial z} = 2y(135 - y - z) - 2yz = 2z(135 - y - 2z0$$

For extrema, we have

$$\frac{\partial V}{\partial y} = 0 = 2z(135 - 2y - z)$$

$$\frac{\partial V}{\partial z} = 0 = 2y(135 - y - 2z)$$

Solving these equations, we find

$$y = 45, z = 45$$

$$\frac{\partial^2 V}{\partial y^2} = -4z, \frac{\partial^2 V}{\partial z^2} = -4y, \frac{\partial^2 V}{\partial y\, \partial z} = 270 - 4y - 4z^2$$

$$\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 V}{\partial z^2} - \left(\frac{\partial^2 V}{\partial y\, \partial z}\right)^2 = 16yz - (270 - 4y - 4z^2) < 0 \text{ for } y = z = 45$$

Thus V is maximum at $y = z = 45$.

From (1), we have

$$x = 270 - 2(y + z) = 270 - 180 = 90$$

Dimensions of the package of greatest volume are 90 cm × 45 cm × 45 cm.

17. Show that the volume of the largest parallelepiped (with faces parallel to the coordinate planes) that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$.

Sol. Let the coordinates of one vertex of the parallelepiped be (x, y, z)

The volume V of the parallelepiped is

$$V = 8xyz$$

or $\quad \frac{V^2}{64a^2b^2c^2} = \frac{x^2}{a^2}\cdot\frac{y^2}{b^2}\cdot\frac{z^2}{c^2}$

Let $\quad t = \frac{x^2}{a^2},\ u = \frac{y^2}{b^2},\ w = \frac{z^2}{c^2}$

Then we have to maximize

$$f(t, u, w) = tuw \text{ where } t + u + w = 1$$

or $\quad f(t, u) = tu(1 - t - u)$

$$f_t = u(1 - t - u) - tu = u - u^2 - 2ut, f_{tt} = -2u$$

$$f_u = t(1 - t - u) - tu = t - t^2 - 2ut, f_{uu} = -2t,$$

$$f_{ut} = 1 - 2t - 2u = f_{tu}.$$

For critical points, we have

$$f_t = u\,(1 - u - 2t) = 0$$

and $\quad f_u = t\,(1 - t - 2u) = 0$

Since $u \neq 0$, $t \neq 0$, therefore

$$1 - u - 2t = 0 \qquad (1)$$

and $\quad 1 - t - 2u = 0 \qquad (2)$

Multiply (1) by 2 and subtract the result from (2). We have

$$-1 + 3t = 0 \quad \text{or} \quad t = \frac{1}{3}$$

Substituting $t = \frac{1}{3}$ into (1), we get $u = \frac{1}{3}$.

Finally, we have $\mathbf{w} = \frac{1}{3}$

Hence, $\quad \frac{V^2}{64a^2b^2c^2} = tuw = \frac{1}{27}$

or $\quad V = \frac{8abc}{3\sqrt{3}}$

18. Find the minimum distance between the lines $x = t, y = 3 - 2t, z = 1 + 2t \qquad (1)$

and $\quad x = -1 - s, y = s, z = 4 - 3s. \qquad (2)$

Sol. $P(t, 3-2t, 1+2t)$ and $Q(-1-s, s, 4-3s)$ are points on the lines (1) and (2) respectively.

$$|PQ|^2 = (t+s+1)^2 + (3-2t-s)^2 + (2t+3s-3)^2 \qquad (3)$$
$$= t^2 + s^2 + 1 + 2ts + 2s + 2t + 9 + 4t^2 + s^2 - 12t - 6s + 4ts + 4t^2 + 9s^2 + 9 - 12t - 18s + 12ts$$
$$= 9t^2 + 11s^2 + 18ts - 22t - 22s + 19 = f(t, s)$$
$$f_t = 18t + 18s - 22, \qquad f_{tt} = 18$$
$$f_s = 22s + 18t - 22, \qquad f_{ss} = 22, \qquad f_{ts} = 12$$

For critical points, we have

$$f_t = 18t + 18s - 22 = 0 \quad \text{or} \quad 9t + 6s - 11 = 0$$

and $f_s = 18t + 22s - 22 = 0$ or $6t + 11s - 11 = 0$

Solving these equations, we have $s = 0,\ t = \dfrac{11}{9}$.

$\left(\dfrac{11}{9}, 0\right)$ is critical point of (3).

Now, $f_{tt} \cdot f_{ss} - [f_{ts}]^2 = 18 \times 22 - 122 > 0$

Thus (3) has relative minimum at $\left(\dfrac{11}{9}, 0\right)$

The points P and Q have coordinates

$$P\left(\frac{11}{9}, \frac{5}{9}, \frac{31}{9}\right) \text{ and } Q(-1, 0, 4)$$

$$|PQ| = \sqrt{\left(\frac{20}{9}\right)^2 + \left(\frac{5}{9}\right)^2 + \left(\frac{-5}{9}\right)^2} = \sqrt{\frac{400}{81} + \frac{25}{81} + \frac{25}{81}} = \frac{\sqrt{50}}{3}$$

is the required minimum distance.

19. Find the dimensions of the largest rectangular box with three of its faces in the coordinate planes and the vertex opposite the origin in the first octant and on the plane $2x + y + 3z = 6$. (1)

Sol. Let x, y, z be the dimensions of the box . Then volume V of the box is

$$V = xyz$$

or $V = f(x, z) = xz(6 - 2x - 3z)$, using (1)

$$f_x = 6z - 4xz - 3z^2, \qquad f_{xx} = -4z$$
$$f_z = 6x - 2x^2 - 6xz, \qquad f_{zz} = -6x$$
$$f_{xz} = 6 - 4x - 6z$$

For critical points, we have

$$f_x = 6z - 4xz - 3z^2 = 0$$

and $f_z = 6x - 2x^2 - 6xz = 0$

or $z(6 - 4x - 3z) = 0$ (1)

and $x(6 - 4x - 3z) = 0$ (2)

$(x \neq 0, \quad z \neq 0)$

Multiply $6 - 2x - 6z = 0$ by 2 and subtract from $6 - 4x - 3z = 0$ to get

$$-6 + 9z = 0 \quad \text{or} \quad z = \frac{2}{3}$$

Setting $z = \dfrac{2}{3}$ into $6 - 2x - 6z = 0$, we find $x = 1$.

$$y = 6 - 2x - 3z = 6 - 2 - 2 = 2$$

The dimensions of the box are

$$x = 1, \quad y = 2, \quad z = \frac{2}{3}.$$

20. A closed rectangular box with volume 16 ft^3 is to be made of three different materials. The cost of the material for the top and the bottom is Rs. 9 per sq. ft., the cost of material for the front and the back is Rs. 8 per sq ft. and the cost of the material for the other two sides is Rs. 6 per sq. ft Find the dimensions of the box so that the cost of the material is a minimum

Sol. Let the dimensions of the top of the box be x ft by y ft and let its height be z ft. Then $xyz = 16$ (1)

Let C denote the total rupee cost of the material used for constructing the box.

$$C = 18xy + 16xz + 12yz$$
$$= 18xy + 16x \cdot \frac{16}{yx} + 12y \cdot \frac{16}{yx}, \text{ using (1)}$$
$$= 18xy + \frac{256}{y} + \frac{192}{x}$$
$$C_x = 18y - \frac{192}{x^2}, \quad C_{xx} = \frac{384}{x^2}$$
$$C_y = 18x - \frac{256}{y^2}, \quad C_{yy} = \frac{512}{y^3}, C_{xy} = 18 = C_{yx}$$

For critical points, we have

$$C_x = 18y - \frac{192}{x^2} = 0 \text{ and } C_y = 18x - \frac{256}{y^2} = 0$$

or $18x^2y - 192 = 0$ (2)

and $18xy^2 - 256 = 0$ (3)

From (2), $y = \dfrac{32}{3x^2}$ Substituting into (3), we get

$$18x\left(\frac{32}{3x^2}\right)^2 - 256 = 0$$

or $64(32 - 4x^3) = 0$ or $x = 2$

Therefore, $y = \frac{32}{12} = \frac{8}{3}$.

Critical point is $\left(2, \frac{8}{3}\right)$

$$C_{xx}\left(2, \frac{8}{3}\right) C_{yy}\left(2, \frac{8}{3}\right) - \left[C_{xy}\left(2, \frac{8}{3}\right)\right]^2 = 48 \times 27 - 18^2 > 0$$

Thus $\left(2, \frac{8}{3}\right)$ is a point of local minimum

When $x = 2, y = \frac{8}{3}$, then from (1), $z = 3$

The desired dimensions of the box are 2ft by $\frac{8}{3}$ ft by 3 ft.

Exercise Set 9.7 (Page 438)

1. Find the point of the plane $x + 2y - z = 3$ nearest to the origin.

Sol. The distance of a point (x, y, z) on the plane to the origin is

$$d = \sqrt{x^2 + y^2 + z^2}$$

We shall minimize $d^2 = x^2 + y^2 + z^2$ subject to $x + 2y - z = 3$.
Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(x + 2y - z - 3)$$
$$F_x = 2x + \lambda = 0 \;; \qquad \lambda = -2x$$
$$F_y = 2y + 2\lambda = 0 \;; \qquad \lambda = -y$$
$$F_z = 2z - \lambda = 0 \;; \qquad \lambda = -2z$$
$$F_\lambda = x + 2y - z - 3 = 0$$

Therefore,

$$-2x = -y = 2z$$

or $\quad y = 2x \quad$ and $\quad z = -x$

Substituting into $x + 2y - z = 3$, we get $6x = 3$ or $x = \frac{1}{2}$.

The desired point is $\left(\frac{1}{2}, 1, \frac{-1}{2}\right)$.

2. Find the extrema of $f(x, y, z) = xy + z$ subject to $x^2 + y^2 + z^2 = 1$.

Sol. Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = xy + z + \lambda(x^2 + y^2 + z^2 - 1)$$
$$F_x = y + 2\lambda x = 0 \;; \qquad \lambda = \frac{-y}{2x}$$
$$F_y = x + 2\lambda y = 0 \;; \qquad \lambda = \frac{-x}{2y}$$

$$F_z = 1 + 2\lambda z = 0 \;; \qquad \lambda = -\frac{1}{2z}$$
$$F_\lambda = x^2 + y^2 + z^2 - 1 = 0 \qquad \text{(A)}$$

Therefore,

$$-\frac{1}{2z} = \frac{-x}{2y} = \frac{-y}{2x}$$

and from $\frac{1}{z} = \frac{x}{y}$, we get $y = xz$ (1)

From $-\frac{1}{2z} = \frac{-y}{2x}$, we have $x = yz$ (2)

From (1) and (2), we find $z = \pm 1$.
When $z = 1, y = x$ and when $z = -1, y = -x$.
Substituting into (A), we obtain

$$x^2 + x^2 + 1 - 1 = 0 \qquad i.e., \quad x = 0.$$

Thus the critical points are

$$(0, 0, 1) \quad \text{and} \quad (0, 0, -1)$$

Clearly, the maximum and minimum of $f(x, y, z)$ are attained at $(0, 0, 1)$ and $(0, 0, -1)$ respectively.

3. Find the maximum value of $f(x, y, z) = x^4 + y^4 + z^4$ subject to $x + y + z = 1$.

Sol. Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = x^4 + y^4 + z^4 + \lambda(x + y + z - 1)$$
$$F_x = 4x^3 + \lambda = 0 \;; \qquad \lambda = -4x^3$$
$$F_y = 4y^3 + \lambda = 0 \;; \qquad \lambda = -4y^3$$
$$F_z = 4z^3 + \lambda = 0 \;; \qquad \lambda = -4z^3$$
$$F_\lambda = x + y + z - 1 = 0$$

Therefore, we get

$$x^3 = y^3 = z^3 \qquad \text{or} \qquad x = y = z$$

Substituting into $x + y + z = 1$, we obtain

$$3x = 1 \quad \text{or} \quad x = \frac{1}{3}$$

Thus the critical point is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Desired value $= \frac{1}{81} + \frac{1}{81} + \frac{1}{81} = \frac{1}{27}$.

4. Find the extrema of $f(x, y, z) = 4x - 3y + 2z$ subject to $x^2 + y^2 - 6z = 0$.

Sol. Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = 4x - 3y + 2z + \lambda(x^2 + y^2 - 6z)$$

$$F_x = 4 + 2\lambda x = 0; \qquad \lambda = \frac{-2}{x}$$

$$F_y = -3 + 2\lambda y = 0; \qquad \lambda = \frac{3}{2y}$$

$$F_z = 2 - 6\lambda = 0; \qquad \lambda = \frac{1}{3}$$

$$F_\lambda = x^2 + y^2 - 6z = 0$$

Thus $\frac{-2}{x} = \frac{1}{3} = \frac{3}{2y}$ and so $x = -6, \quad y = \frac{9}{2}$

Substituting into $x^2 + y^2 = 6z$, we have

$$36 + \frac{81}{4} = 6z \qquad \text{or} \qquad z = \frac{225}{24}$$

The critical point is $\left(-6, \frac{9}{2}, \frac{225}{24}\right)$ and the extreme value is

$$-24 - \frac{27}{2} + \frac{225}{12} = \frac{-225}{12}$$

which is the minimum value.

5. Find the points on $x^2 + y^2 + z^2 = 1$ closest to and farthest from the point (1, 2, 3)

Sol. We have to find the extrema of

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$$

subject to $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = (x-1)^2 + (y-2)^2 + (z-3)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

$$F_x = 2(x-1) + 2\lambda x = 0; -\lambda = \frac{x-1}{x} \qquad (1)$$

$$F_y = 2(y-2) + 2\lambda y = 0; -\lambda = \frac{y-2}{y} \qquad (2)$$

$$F_z = 2(z-3) + 2\lambda z = 0; -\lambda = \frac{z-3}{z} \qquad (3)$$

$$F_\lambda = x^2 + y^2 + z^2 - 1 = 0 \qquad (A)$$

From (1) and (2), we have $\frac{x-1}{x} = \frac{y-2}{y}$ *i.e.*, $y = 2x$

From (1) and (3), we get $\frac{x-1}{x} = \frac{z-3}{z}$ *i.e*, $z = 3x$

The point $(x, y, z) = (x, 2x, 3x)$ lies on (A). Substituting into (A), we find

$$x = \pm\frac{1}{\sqrt{14}}, y = \pm\frac{2}{\sqrt{14}}, z = \pm\frac{3}{\sqrt{14}}$$

When $x = \frac{1}{\sqrt{14}}$ then $y = \frac{2}{\sqrt{14}}, z = \frac{3}{\sqrt{14}}$

When $x = \frac{-1}{\sqrt{14}}$ then $y = \frac{-2}{\sqrt{14}}, z = \frac{-3}{\sqrt{14}}$

The critical points are

$P\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$ and $Q\left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$

It is easy to see that P is closest to and Q is farthest from (1,2,3).

6. Find the points on the paraboloid $z = 2 - x^2 - y^2$ that is closest to the point (1, 1, 2)

Sol. We have to find the point (x, y, z) on the paraboloid such that $(x-1)^2 + (y-1)^2 + (z-2)^2$ is minimum.

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = (x-1)^2 + (y-1)^2 + (z-2)^2 + \lambda(x^2 + y^2 + z - 2)$$

$$F_x = 2(x-1) + 2\lambda x = 0 \quad ; \quad -\lambda = \frac{x-1}{x}$$

$$F_y = 2(y-1) + 2\lambda y = 0 \quad ; \quad -\lambda = \frac{y-1}{y}$$

$$F_z = 2(z-2) + \lambda = 0 \quad ; \quad -\lambda = 2(z-2)$$

$$F_\lambda = x^2 + y^2 + z - 2 = 0$$

Thus $\frac{x-1}{x} = \frac{y-1}{y} = 2(z-2)$ and so we find $x = y$ and $z = \frac{x-1}{2x} + 2$

Substituting into $z = 2 - x^2 - y^2$, we get

$$\frac{x-1}{2x} + 2 = 2 - 2x^2$$

or $\quad 4x^3 + x - 1 = 0$

By inspection, $x = \frac{1}{2}$ and so $y = \frac{1}{2}$ and $z = \frac{1}{2} - 1 + 2 = \frac{3}{2}$

The required point is $\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$.

7. Find the dimensions of a topless box so that the volume is a maximum when the surface area is 24 square metres.

Sol. Let the dimension of the box be x, y and z metres.

Surface area $= xy + 2xz + 2yz$ sq. m. The volume xyz of the box is to be maximized subject to

$$xy + 2xz + 2yz = 24.$$

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = xyz + \lambda(xy + 2xz + 2yz - 24)$$

$$F_x = yz + \lambda y + 2\lambda z \qquad = 0 \qquad (1)$$

$$F_y = xz + \lambda x + 2\lambda z = 0 \quad (2)$$
$$F_z = xy + 2\lambda x + 2\lambda y = 0 \quad (3)$$
$$F_\lambda = xy + 2xz + 2yz - 24 = 0 \quad (A)$$

Multiplying (1) by x, (2) by y and (3) by z respectively, we get

$$xyz + \lambda xy + 2\lambda zx = 0 \quad (4)$$
$$xyz + \lambda xy + 2\lambda yz = 0 \quad (5)$$
$$xyz + 2\lambda xz + 2\lambda yz = 0 \quad (6)$$

From (4) and (5), we find $xz = yz$ or $x = y$. From (4) and (6), we get

$$xy = 2yz \text{ so that } x = 2z.$$

Substituting into (A), we have

$$4z^2 + 4z^2 + 4z^2 = 24 \quad \text{or} \quad z = \sqrt{2}$$

Thus dimensions of the box are $2\sqrt{2}, 2\sqrt{2}, \sqrt{2}$ metres.

8. Find the dimensions of the box with maximum volume that can be enclosed by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ if the sides of the box are parallel to the coordinate planes.

Sol. Let the corner of the box in the first octant be (x, y, z).
Then the dimensions of the box are $2x$, $2y$ and $2z$.
We have to find the maximum of $f(x, y, z) = 8xyz$ subject to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$F_x = 8yz + \frac{2\lambda x}{a^2} = 0 \quad \text{or} \quad 4xyz + \frac{\lambda x^2}{a^2} = 0$$

$$F_y = 8zx + \frac{2\lambda y}{b^2} = 0 \quad \text{or} \quad 4xyz + \frac{\lambda y^2}{b^2} = 0$$

$$F_z = 8xy + \frac{2\lambda z}{c^2} = 0 \quad \text{or} \quad 4xyz + \frac{\lambda z^2}{c^2} = 0$$

$$F_\lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

Therefore, we get $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$.

Since $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, we find that

$$3x^2 = a^2, \quad 3y^2 = b^2, \quad 3z^2 = c^2$$

The corner of the box in first octant has positive coordinates, so this point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$. The box with maximum volume has sides $\frac{2a}{\sqrt{3}}, \frac{2b}{\sqrt{3}}, \frac{2c}{\sqrt{3}}$ and the maximum volume is $\frac{8abc}{3\sqrt{3}}$ cu units.

9. Find the minimum volume of a tetrahedron bounded by the coordinate planes and a plane tangent to the sphere $x^2+y^2+z^2 = 1$.

Sol. Let $P(u, v, w)$ be a point on the sphere. An equation of the tangent plane to the sphere at P is $xu + yv + zw - 1 = 0$.
The tetrahedron is bounded by the planes $x = 0, y = 0, z = 0$ and $ux + vy + wz - 1 = 0$.
The vertices of the tetrahedron are

$$(0, 0, 0), \left(\frac{1}{u}, 0, 0\right), \left(0, \frac{1}{v}, 0\right), \left(0, 0, \frac{1}{w}\right)$$

Volume of the tetrahedron

$$= \frac{1}{6}\begin{vmatrix} \frac{1}{u} & 0 & 0 & 1 \\ 0 & \frac{1}{v} & 0 & 1 \\ 0 & 0 & \frac{1}{w} & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{6uvw}$$

We have to minimize $f(u, v, w) = \frac{1}{6uvw}$ subject to $u^2 + v^2 + w^2 = 1$.

Using Lagrange multipliers, we have

$$F(u, v, w, \lambda) = \frac{1}{6uvw} + \lambda(u^2 + v^2 + w^2 - 1)$$

$$F_u = \frac{-vw}{6(uvw)^2} + 2\lambda u = 0$$

$$F_v = \frac{-uw}{6(uvw)^2} + 2\lambda v = 0$$

$$F_w = \frac{-uv}{6(uvw)^2} + 2\lambda w = 0$$

$$F_\lambda = u^2 + v^2 + w^2 - 1 = 0 \quad (1)$$

Thus $u^2 = \frac{1}{12\lambda uvw} = v^2 = w^2$

Substituting into (1), we have $3u^2 = 1$ or $u = \frac{1}{\sqrt{3}} = v = w$

Minimum volume of the tetrahedron is

$$\frac{1}{6\left(\frac{1}{\sqrt{3}}\right)^3} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \text{ cu. units.}$$

10. Find the point on the line $x + y + 2z - 12 = 0 = x - 3y - 2z + 16$ nearest to the origin.

Sol. Let $P(x, y, z)$ be a point on the line. Then we have to minimize $x^2 + y^2 + z^2$ subject to the given constraints.

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda_1, \lambda_2) = x^2+y^2+z^2+\lambda_1(x+y+2z-12)+\lambda_2(x-3y-2z+16)$$

$$F_x = 2x + \lambda_1 + \lambda_2 = 0 \quad (1)$$
$$F_y = 2y + \lambda_1 - 3\lambda_2 = 0 \quad (2)$$
$$F_z = 2z + 2\lambda_1 - 2\lambda_2 = 0 \quad (3)$$
$$F_{\lambda_1} = x + y + 2z - 12 = 0$$
$$F_{\lambda_2} = x - 3y - 2z + 16 = 0$$

From (1) and (2), we get $x - y + 2\lambda_2 = 0$

From (2) and (3), we find $2y - z - 2\lambda_2 = 0$

Adding the last two equations, we have $x + y - z = 0$

Setting $z = x + y$ into the equations of the line, we obtain

$$x + y - 4 = 0 \quad \text{and} \quad -x - 5y + 16 = 0$$

Therefore $y = 3, x = 1$ and $z = 4$

The required point is (1, 3, 4).

11. Find the points that are on both the ellipsoid $x^2 + y^2 + 9z^2 = 25$ and the plane $x + 3y - 2z = 0$ which are closest to and farthest from the origin.

Sol. Let $P(x, y, z)$ be a point on the intersection of the ellipsoid and the plane. Then we have to find the extrema of $d = \sqrt{x^2 + y^2 + z^2}$ or of $d^2 = x^2 + y^2 + z^2$.

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda_1, \lambda_2) = x^2+y^2+z^2+\lambda_1(x^2+y^2+9z^2-25) + \lambda_2(x+3y-2z)$$

$$F_x = 2x + 2\lambda_1 x + \lambda_2 = 0$$
$$F_y = 2y + 2\lambda_1 y + 3\lambda_2 = 0$$
$$F_z = 2z + 18\lambda_1 z - 2\lambda_2 = 0$$
$$F_{\lambda_1} = x^2 + y^2 + 9z^2 - 25 = 0 \quad (A)$$
$$F_{\lambda_2} = x + 3y - 2z = 0 \quad (B)$$

or

$$2xyz + 2\lambda_1 xyz + \lambda_2 yz = 0 \quad (1)$$
$$2xyz + 2\lambda_1 xyz + 3\lambda_2 xz = 0 \quad (2)$$

$$2xyz + 18\lambda_1 xyz - 2\lambda_2 xy = 0$$

From (1) and (2), we have

$$3\lambda_2 xz - \lambda_2 yz = 0$$

or $\quad 3xz - yz = 0$

i.e., $\quad z = 0 \quad$ or $\quad y = 3x$

Using $z = 0$ and the two constraints, we get

$$x = -3y \text{ and so } \quad 9y^2 + y^2 = 25$$

i.e., $y = \pm\frac{5}{\sqrt{10}}$. Therefore, $x = \frac{-15}{\sqrt{10}}$ if $y = \frac{5}{\sqrt{10}}$

and $x = \frac{15}{\sqrt{10}}$ if $y = \frac{-5}{\sqrt{10}}$.

The two points are $A\left(\frac{-15}{\sqrt{10}}, \frac{5}{\sqrt{10}}, 0\right)$, $B\left(\frac{15}{\sqrt{10}}, \frac{-5}{\sqrt{10}}, 0\right)$

If $y = 3x$, then substituting into (B), we find $z = \frac{x + 3y}{2}$

Putting into (A), we get

$$x^2 + 9x^2 + 9\left(\frac{x + 9x}{2}\right) = 25$$

or $$x = \pm\frac{5}{\sqrt{235}}$$

If $\quad x = \frac{5}{\sqrt{235}}$, then $y = \frac{15}{\sqrt{235}}$ and so $z = \frac{25}{\sqrt{235}}$

If $\quad x = \frac{-5}{\sqrt{235}}$ then $y = \frac{-15}{\sqrt{235}}$ and $z = \frac{-25}{\sqrt{235}}$

The two points are

$$C\left(\frac{5}{\sqrt{235}}, \frac{15}{\sqrt{235}}, \frac{25}{\sqrt{235}}\right), D\left(\frac{-5}{\sqrt{235}}, \frac{-15}{\sqrt{235}}, \frac{-25}{\sqrt{235}}\right)$$

The distance of each of the four points A, B, C and D from (0, 0, 0) is $5, 5, \sqrt{\frac{175}{47}}, \sqrt{\frac{175}{47}}$

The points A and B are farthest from the origin while the points C and D are closest to the origin.

Exercise Set 9.8 (Page 446)

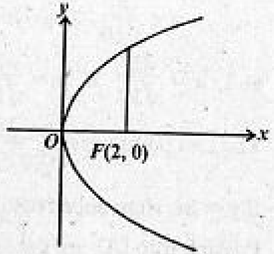
1. Find the volume generated by revolving the area in the first quadrant bounded by the parabola $y^2 = 8x$ and its latus-rectum about the x-axis.

Sol. Here the limits for x are from $x = 0$ to $x = 2$. Therefore, the required volume is

$$= \int_0^2 \pi y^2\, dx$$

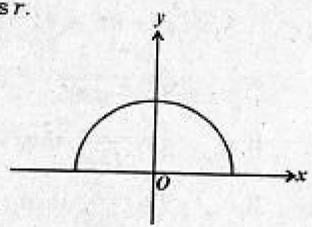
$$= \int_0^2 \pi(8x)\, dx = 8\pi \int_0^2 x\, dx$$

$$= 8\pi \left|\frac{x^2}{2}\right|_0^2 = 16\pi$$



2. Find the volume of a sphere of radius r.

Sol. Volume of the sphere is the volume generated by the circle $x^2 + y^2 = r^2$ when it revolves about the x-axis. Therefore, volume of the sphere



$$= \int_{-r}^{r} \pi y^2\, dx = 2\int_0^r \pi(r^2 - x^2)\, dx$$

$$= 2\pi\left|r^2x - \frac{x^3}{3}\right|_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

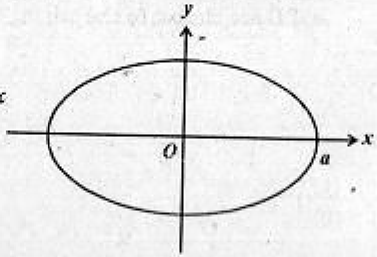
3. Find the volume of the spheroid formed by the revolution of the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the (i) major axis (ii) minor axis.

Sol.

(i) The required volume is

$$= \int_{-a}^{a} \pi y^2\, dx = 2\int_0^a \pi y^2\, dx$$

$$= 2\int_0^a \pi\left(1 - \frac{x^2}{a^2}\right) b^2\, dx$$



$$= 2\frac{b^2}{a^2}\pi\left|a^2x - \frac{x^3}{3}\right|_0^a = 2\frac{\pi b^2}{a^2}\left(a^3 - \frac{a^3}{3}\right) = \frac{4}{3}\pi ab^2$$

(ii) Here the required volume is

$$= \int_{-b}^{b} \pi x^2\, dy = 2\int_0^b \pi\left(1 - \frac{y^2}{b^2}\right) a^2\, dy$$

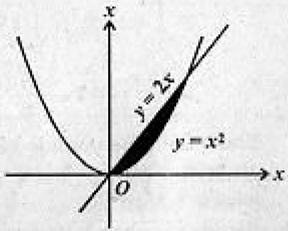
$$= \frac{2\pi a^2}{b^2}\int_0^b (b^2 - y^2)\, dy = \frac{2\pi a^2}{b^2}\left|b^2y - \frac{y^3}{3}\right|_0^b = \frac{4}{3}\pi ba^2$$

4. Find the volume of the solid generated by revolving the area enclosed by $y = 2x$ and $y = x^2$ about the y-axis.

Sol. The points of intersection of the two equations are given by

$2x = x^2$, or $x = 0,2$

Thus, (0, 0,), (2, 4) are the two points of intersection. The desired volume



$$= \int_0^4 \left[\pi(\sqrt{y})^2 - \pi\left(\frac{y}{2}\right)^2\right] dy$$

$$= \pi\int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8\pi}{3}$$

5. Find the volume of a right circular cone having base radius r and height h.

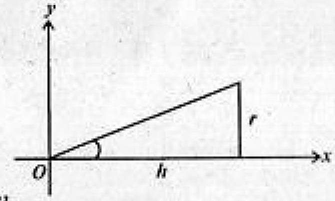
Sol. The cone is generated by revolving about the x-axis the area enclosed by

$y = \frac{r}{h}x$, $x = h$ and the x-axis

$$\text{Desired volume} = \int_0^h \pi y^2\, dx$$

$$= \pi\int_0^h \frac{r^2}{h^2}x^2\, dx$$

$$= \pi\frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2 h}{3}.$$



[Solve also by Cross Sectional Method.]

6. The area in the first quadrant bounded by $x = 2y^3 - y^4$ and the y-axis is revolved abut the x-axis. Find the volume of the resulting solid.

Sol. By the shell method, desired volume

$$= \int_0^2 2\pi y(2y^3 - y^4)dy = 2\pi\left[\frac{2y^5}{5} - \frac{y^6}{6}\right]_0^2 = \frac{64\pi}{15}.$$

7. A basin is formed by the revolution of the area bounded by the curve $x^3 = 64y$, $(y > 0)$ about the axis of y. If the depth of the basin is 8 cm, how many cubic cm. of water would it hold?

Sol. The required volume is

$$= \int_0^8 \pi x^2\, dy = \int_0^8 \pi\,(64y)^{2/3}\, dy = 16\pi \int_0^8 y^{2/3}\, dy = 16\,\pi\left[3.\frac{y^{5/3}}{5}\right]_0^8$$

$$= \frac{48\pi}{5}(8)^{5/3} = \frac{48\pi}{5} \times 32 = \frac{1536}{5}\,\pi \text{ cubic cm.}$$

8. Show that the volume generated by revolving the area bounded by an arch of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ about its base is $5\,\pi^2 a^3$.

Sol. Required volume

$$= \pi \int_0^{2\pi} y^2\, dx = \pi \int_0^{2\pi} a^2\,(1-\cos\theta)^2\,(a)\,(1-\cos\theta)\, d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1-\cos\theta)^3\, d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1 + 3\cos^2\theta)\, d\theta \text{ as } \int_0^{2\pi} \cos\theta\, d\theta = 0 \text{ and} \int_0^{2\pi} \cos^3\theta\, d\theta = 0$$

$$= \pi a^3 \int_0^{2\pi} \left[1 + \frac{3}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= \pi a^3 \left|\frac{5\theta}{2} + \frac{3}{4}\sin 2\theta\right|_0^{2\pi} = \pi a^3\left[\frac{5}{2}2\pi + 0\right] = 5\pi^2 a^3.$$

9. Find the volume of a right pyramid whose height is h and has a square base with each side of length a.

Sol. Let the axis of the pyramid be along the y-axis. A typical cross section is a square with side s, where s is a function of y.

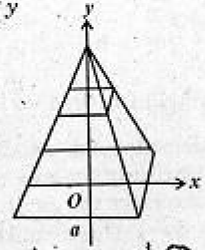
Now $\frac{s}{a} = \frac{y}{h}$ or $s = \frac{a}{h}y$

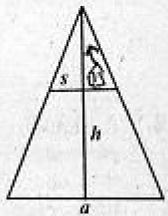
Area of the cross section

$$= \left(\frac{a}{h}\right)^2 y^2$$

$$\text{Volume} = \int_0^h \left(\frac{a}{h}\right)^2 y^2 dy$$

$$= \frac{a^2}{h^2}\int_0^h y^2\, dy = \frac{1}{3}a^3 h$$





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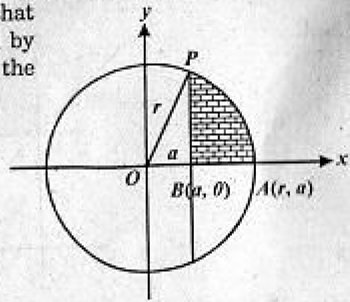
10. Find the volume of the solid that remains after boring a hole of radius a through the centre of a solid sphere of radius $r > a$.

Sol. Suppose the sphere is generated by the right half of the circular disc $x^2 + y^2 \le r^2$ revolved about the y-axis. Let the hole be vertical with its centre line coincident with the y-axis.

The upper half of the solid that remains after boring is generated by revolving the shaded area about the y-axis. The required volume



$$= 2\int_a^r 2\pi x\sqrt{r^2 - x^2}\, dx$$

$$= 4\pi\left[-\frac{1}{3}(r^2 - x^2)^{3/2}\right]_a^r$$

$$= \frac{4\pi}{3}(r^2 - a^2)^{3/2}.$$

11. Find the volume formed by the revolution of the area enclosed by the loop of the curve $y^2 = x^2\frac{a-x}{a+x}$ about the x-axis.

Sol. The required volume is

$$V = \int_0^a \pi y^2 dx = \pi\int_0^a x^2\frac{a-x}{a+x}dx$$

$$= \pi \int_0^a \left(-x^2 + 2ax - 2a^2 + \frac{2a^3}{x+a}\right) dx$$

$$= \pi \left| -\frac{x^3}{3} + ax^2 - 2a^2x + 2a^3 \ln (x+a) \right|_0^a$$

$$= \pi\left(-\frac{4}{3}a^3 + 2a^3 \ln 2a - 2a^3 \ln a\right)$$

$$= 2\pi a^3\left(\ln 2a - \ln a - \frac{2}{3}\right) = 2\pi a^3\left(\ln \frac{2a}{a} - \frac{2}{3}\right).$$

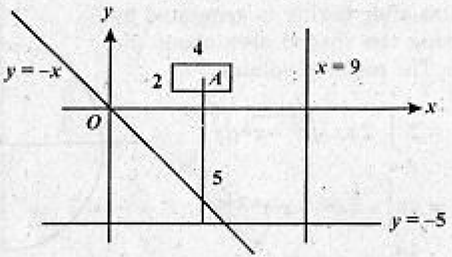
12. A doughnut-shaped solid, called **torus** (or **anchor ring**), is generated by revolving an area enclosed by a circle about a line that does not intersect the circle. Find the volume of the torus if the circle is $(x-b)^2 + y^2 = a^2$ and the line is the y-axis, $(0 < a < b)$.

Sol. Centre of gravity of the circle $(x-b)^2 + y^2 = a^2$ is its centre $(b, 0)$

Area of the circle is revolved about the y-axis. **By Theorem 9.26,** volume of the torus $= \pi a^2 (2\pi b) = 2\pi^2 a^2 b$.

13. A rectangular area whose length and breadth are 4 and 2 units respectively and whose centroid is at (4, 3) is revolved about (i) the straight line $x = 9$, (ii) the straight line $y = -5$ and (iii) the straight line $y = -x$. Find the volume generated in each case.

Sol.



(*i*) If $A = (4, 3)$ be the centroid of the rectangular area then its distance from the straight line $x = 9$ is $9 - 4 = 5$ units. Therefore, the distance convered by the point A about the line

$x = 9$ in one revolution is $2.\ \pi.\ 5 = 10\pi$

Also area of the rectangle $= 4 \times 2 = 8$ square units.

Required volume $= 8 \times 10\pi$ cubic units.

(*ii*) Distance of the centroid A from the line $y = -5$ is

$3 - (-5) = 8$ units.

Distance covered by A is one revolution about $y = -5$ is

$2\pi.\ 8 = 16\pi$ units.

Volume of revolution in this case

$= 8 \times 16\pi = 128\pi$ cubic units

(*iii*) Perpendicular distance of the centroid $A = (4, 3)$ from the line

$y = -x$ or $x + y = 0$ is $\frac{4+3}{\sqrt{2}} = \frac{7}{\sqrt{2}}$.

Distance covered by the centroid in one revolution about the line

$$2\pi \frac{7}{\sqrt{2}} = 7\sqrt{2}\,\pi.$$

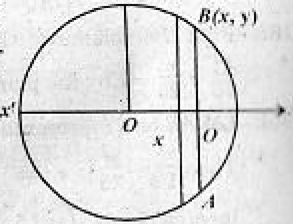
Required volume $= 8 \times 7\sqrt{2}\,\pi = 56\sqrt{2}\,\pi$ cubic units.

14. A solid has a circular base of radius 4 units. Find the volume of the solid if every plane section perpendicular to a fixed diameter is an equilateral triangle

Sol. Since the base is circular, we take the fixed diameter $x'Ox$, *i.e.,* the x-axis with centre at the origin O. If AB represents one section with breadth δx then AB also forms one side of the equilateral triangle. Equation of the circle is

$x^2 + y^2 = 16$.

$|AB| = 2y$.



Now area of the equilateral triangle with each side $2y$

$$= \frac{1}{2} 2y.\ 2y \sin 60^\circ = \sqrt{3}\,y^2$$

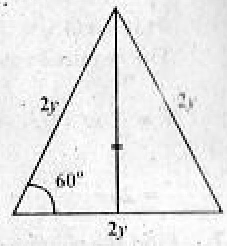
Volume of one triangular strip $= \sqrt{3}\,y^2\,\delta x$

Hence the required volume

$$= \int_{-4}^{4} (\sqrt{3}\,y^2)\,dx = 2\int_0^4 \sqrt{3}\,(16 - x^2)dx$$

$$= 2\sqrt{3}\left|16x - \frac{x^3}{3}\right|_0^4 = 2\sqrt{3}\left(64 - \frac{64}{3}\right)$$

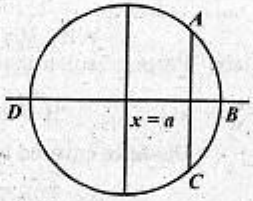
$$= \frac{256\sqrt{3}}{3} \text{ cubic units.}$$



15. Find the volumes of the two portions in which a sphere of radius r is divided by the plane $x = a$, $(a < r)$.

Sol. Sphere, being the solid generated by the circle $x^2 + y^2 = r^2$, when cut by the plane $x = a$, is divided into two portion ABC and ADC.

Volume of the portion ABC



$$= \int_a^r \pi y^2\,dx = \pi\int_a^r (r^2 - x^2)\,dx$$

$$= \pi\left|r^2x - \frac{x^3}{3}\right|_a^r$$

$$= \pi\left(r^3 - \frac{r^3}{3} - r^2a + \frac{a^3}{3}\right)$$

$$= \pi\left(\frac{2}{3}r^3 - r^2a + \frac{a^3}{3}\right) = \frac{2.\,\pi}{3}r^3 - \frac{a\pi}{3}(3r^2 - a^2).$$

But the volume of the sphere $= \frac{4}{3}\pi r^3$.

Hence volume of the portion ACD

$$= \frac{4}{3}\pi r^3 - \text{volume of the portion } ABC$$

$$= \frac{4}{3}\pi r^3 - \left[\frac{2\pi}{3}r^3 - \frac{a\pi}{3}(3r^2 - a^2)\right] = \frac{2\pi}{3}r^3 + \frac{a\pi}{3}(3r^2 - a^2).$$

16. Find the volume of the solid cut off from the elliptic paraboloid $\frac{x^2}{16} + \frac{y^2}{25} = z$ by the plane $z = 10$.

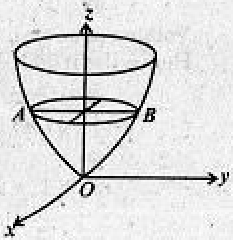
Sol. Let AB be a representative ellipse on the paraboloid.

Then $\frac{x^2}{16} + \frac{y^2}{25} = z \Rightarrow \frac{x^2}{16z} + \frac{y^2}{25z} = 1$ is an equation of this ellipse

Its area is $\pi(4\sqrt{z})(5\sqrt{z})$,

$\left(\text{using } \pi ab \text{ as the area of the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$.

∴ Volume of the elliptic dise with breadth δz is $(\pi(4\sqrt{z})(5\sqrt{z})\,\delta z)$.



The required volume is

$$V = \int_0^{10} \pi(4\sqrt{z})(5\sqrt{z})\,dz = 20\pi\int_0^{10} z\,dz$$

$$= 20\pi\left|\frac{z^2}{2}\right|_0^{10} = 1000\pi.$$

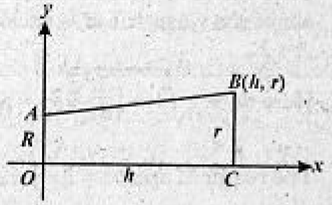
17. Find the volume of a frustrum of a right circular cone of altitude h, lower base radius R and upper base radius r.

Sol. The frustrum of the cone is generated by revolving about x-axis the area enclosed by the line AB, $x = 0$, $x = h$ and the x-axis.

Equation of the line AB is

$$y - R = \frac{r - R}{h}x$$

i.e., $y = R + \frac{r - R}{h}x$



$$\text{Desired volume} = \int_0^h \pi y^2 dx$$

$$= \pi\int_0^h \left(R + \frac{r - R}{h}x\right)^2 dx$$

$$= \frac{\pi h}{3(r - R)}\left[\left(R + \frac{r - R}{h}x\right)^3\right]_0^h$$

$$= \frac{\pi h}{3(r - R)}(r^3 - R^3) = \frac{1}{3}\pi h(r^2 + rR + R^2).$$

If $R = 0$, the resulting solid is the complete right cirucular cone and the volume reduces to the familiar formula $\frac{1}{3}\pi r^2 h$.

Exercise Set 9.9 (Page 452)

1. Find the area of the surface of revolution generated by revolving about the x-axis the area bounded by an arc of the parabola $y^2 = 12x$ from $x = 0$ to $x = 3$.

Sol. Required area $A = 2\pi\int y\,ds$

Here $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{6}{y}\right)^2}\,dx$

$$= \sqrt{1 + \frac{36}{12x}}\,dx = \sqrt{\frac{x + 3}{x}}\,dx$$

Hence required area

$$= 2\pi\int_0^3 \sqrt{12x}\sqrt{\frac{x + 3}{x}}\,dx = 2\pi(2\sqrt{3})\int_0^3 \sqrt{x + 3}\,dx$$

$$= 2\pi(2\sqrt{3})\left|\frac{(x + 3)^{3/2}}{3/2}\right|_0^3 = 2\pi\frac{4}{\sqrt{3}}\left[(6)^{3/2} - (3)^{3/2}\right]$$

$$= 2\pi\cdot 4\,\frac{3\sqrt{3}}{\sqrt{3}}\left[2\sqrt{2} - 1\right] = 12(2\sqrt{2} - 1).\,2\pi$$

$= 24\,(2\sqrt{2}-1)\,\pi$ square units.

2. Find the area of the surface of revolution generated by revolving about the y-axis the area enclosed by the arc of $x = y^3$ from $y = 0$ to $y = 1$.

Sol. Here $ds = \sqrt{1+\left(\dfrac{dx}{dy}\right)^2}\,dy = \sqrt{1+9y^4}\,dy$

The required area $A = 2\pi \displaystyle\int_0^1 x\,ds = 2\pi \int_0^1 y^3\sqrt{1+9y^4}\,dy$

Put $1 + 9y^4 = t^2$ or $36y^3 dy = 2t\,dt$

or $dy = \dfrac{1}{18y^3}\,t\,dt$

Also when $y = 0$, $t = 1$ and when $y = 1$, $t = \sqrt{10}$.

Therefore, $A = 2\pi \displaystyle\int_1^{\sqrt{10}} \frac{t}{18}\,t.dt = \frac{\pi}{9}\left|\frac{t^3}{3}\right|_1^{\sqrt{10}} = \frac{\pi}{27}(10\sqrt{10}-1)$ sq. units

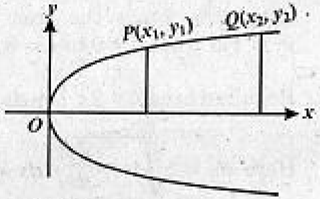
3. Find the surface area of a belt of the paraboloid formed by revolving the area bounded by the curve $y^2 = 4ax$ about the x-axis.

Sol. Let the belt be formed by revolving about x-axis the area enclosed by the curve, the x-axis, $x = x_1$ and $x = x_2$.

Here $ds = \sqrt{1+\left(\dfrac{2a}{y}\right)^2}\,dx$

$= \sqrt{1+\dfrac{4a^2}{4ax}}\,dx$

$= \sqrt{\dfrac{x+a}{x}}\,dx$



Required area

$$A = 2\pi\int y\,ds = 2\pi\int_{x_1}^{x_2}\sqrt{4ax}\sqrt{\frac{x+a}{x}}\,dx$$

$$= 4\pi\sqrt{a}\int_{x_1}^{x_2}(x+a)^{1/2}\,dx$$

$$= 4\pi\sqrt{a}\left|\frac{(x+a)^{3/2}}{3/2}\right|_{x_1}^{x_2} = \frac{8\pi\sqrt{a}}{3}\left((x_2+a)^{3/2}-(x_1+a)^{3/2}\right).$$

4. Find the surface area of a sphere of radius r.

Sol. The required surface area is generated by revolving the circle $x^2 + y^2 = r^3$ about the x-axis.

Here $ds = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1+\dfrac{x^2}{y^2}}\,dx$

$= \sqrt{\dfrac{y^2+x^2}{y^2}}\,dx = \sqrt{\dfrac{r^2}{r^2-x^2}}\,dx$

Required area

$$A = 2\pi\int y\,ds = 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}\,dx$$

$$= 4\pi\int_0^r r\,dx = 4\pi[rx]_0^r = 4\pi r^2.$$

5. Find the area on a sphere of radius r included between two parallel planes at distances r_1 and r_2 from the centre, $(r_1 < r_2 < r)$.

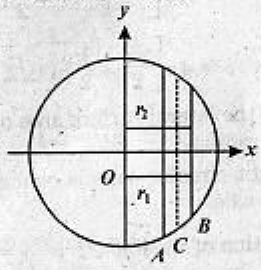
Sol. Let C be any plane intersecting the sphere. If y be the radius of this circle, then we have $y^2 = r^2 - x^2$ where x is the distance of the plane from the origin.

Area of small circular strip at C whose width is δx is

$2\pi y\,\delta s = 2\pi\sqrt{r^2-x^2}\,\delta s$

Required surface area

$$S = \int_{r_1}^{r_2} 2\pi\sqrt{r^2-x^2}\,\frac{ds}{dx}\,dx$$



But $\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

$= \sqrt{1+\dfrac{x^2}{y^2}} = \sqrt{\dfrac{x^2+y^2}{y^2}}$

$= \sqrt{\dfrac{r^2}{y^2}} = \dfrac{r}{y} = \dfrac{r}{\sqrt{r^2-x^2}}$

Hence $S = \displaystyle\int_{r_1}^{r_2} 2\pi\sqrt{r^2-x^2}\,.\,\frac{r}{\sqrt{r^2-x^2}}\,dx$

$= 2\pi\,[x]_{r_1}^{r_2} = 2\pi(r_2 - r_1)$

6. Show that the surface of the solid obtained by revolving the area bounded by the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ about the x-axis is $2\pi\left[\sqrt{2} + \ln(\sqrt{2}+1)\right]$.

Sol. Since $y = \sin x$, so $\dfrac{dy}{dx} = \cos x$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\cos^2 x}\,dx$$

Required area is

$$A = 2\pi\int_0^{\pi} y\,ds = 2\pi\int_0^{\pi}\sin x\,(1+\cos^2 x)^{1/2}\,dx$$

$$= 4\pi\int_0^{\pi/2}\sin x\,(1+\cos^2 x)^{1/2}\,dx$$

Put $\cos x = t$ or $-\sin x\,dx = dt$ in the above integral.

Then $A = 4\pi\displaystyle\int_1^0 -(1+t^2)^{1/2}\,dt = 4\pi\int_0^1 (1+t^2)^{1/2}\,dx$

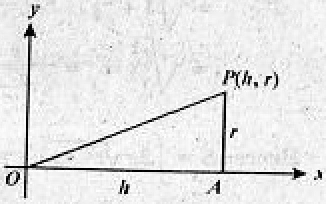
$$= 4\pi\left[\frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\ln\frac{t+\sqrt{t^2+1}}{1}\right]_0^1$$

$$= 4\pi\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)\right] = 2\pi[\sqrt{2} + \ln(\sqrt{2}+1)].$$

7. Find the lateral surface area of a right circular cone of height h and base radius r.

Sol. A right circular cone is generated by revolving the line OP about the x-axis.

Equation of the line OP is $y = \dfrac{r}{h}x$



Required area

$$= \int_0^h 2\pi y\,ds$$

$$= 2\pi\int_0^h\left(\frac{r}{h}x\right)\sqrt{1+\frac{r^2}{h^2}}\,dx$$

$$= 2\pi\,\frac{r}{h^2}\sqrt{r^2+h^2}\left[\frac{x^2}{2}\right]_0^h = \pi r\sqrt{r^2+h^2}$$

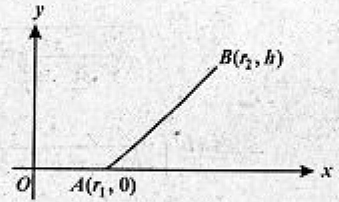
$= \pi r l$, where $l = |OP|$ = slant height of the cone.

8. Find the surface area generated by revolving the line segment between $(r_1, 0)$ and (r_2, h) about the y-axis.

Sol. Equation of the line AB is

$$y = \frac{h}{r_2 - r_1}(x - r_1)$$

or $\quad x = r_1 + \dfrac{r_2 - r_1}{h}y$



Area of the surface generated by revolving AB about the y-axis

$$= 2\pi\int_0^h x\,ds = 2\pi\int_0^h x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= 2\pi\int_0^h x\sqrt{1+\left(\frac{r_2-r_1}{h}\right)^2}\,dy$$

$$= 2\pi\int_0^h\left(r_1 + \frac{r_2-r_1}{h}y\right)\sqrt{1+\left(\frac{r_2-r_1}{h}\right)^2}\,dy$$

$$= \frac{2\pi}{h}\sqrt{h^2+(r_2-r_1)^2}\left[r_1 y + \frac{r_2-r_1}{h}\cdot\frac{y^2}{2}\right]_0^h$$

$$= 2\pi\frac{|AB|}{h}\left[r_1 h + \frac{r_2-r_1}{2}h\right] = 2\pi|AB|\frac{r_1+r_2}{2}.$$

9. Prove that the surface area of the prolate spheroid formed by the revolution of an area enclosed by an ellipse of eccentricity e about its major axis is

$$2\,(\text{Area of the ellipse}) \times \left(\sqrt{1-e^2} + \frac{\arcsin e}{e}\right).$$

Sol. Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\frac{b^4x^2}{a^4y^2}}\,dx = \frac{\sqrt{a^4y^2+b^4x^2}}{a^2y}\,dx$$

Required area

$$A = 2\pi \int y\, ds = 2\pi \int_{-a}^{a} \frac{y\sqrt{a^4y^2 + b^4x^2}}{a^2y}\, dx$$

$$= \frac{4\pi}{a^2} \int_0^a \sqrt{a^4\left(\frac{a^2 - x^2}{a^2}\right) b^2 + b^4x^2}\ dx$$

$$= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - x^2\,(a^2 - b^2)}\, dx$$

But $b^2 = a^2\,(1 - e^2)$ or $a^2 - b^2 = a^2e^2$

Hence $A = \dfrac{4\pi b}{a^2} \displaystyle\int_0^a \sqrt{a^4 - a^2e^2x^2}\, dx$

$$= \frac{4\pi b}{a} \int_0^a \sqrt{a^2 - e^2x^2}\, dx = \frac{4\pi b}{a} e \int_0^a \sqrt{\frac{a^2}{e^2} - x^2}\, dx$$

$$= \frac{4\pi b}{a} e \left[\frac{x\sqrt{\dfrac{a^2}{e^2} - x^2}}{2} + \frac{a^2}{2e^2} \arcsin \frac{xe}{a}\right]_0^a$$

$$= \frac{4\pi b}{a} e \left[\frac{a\sqrt{a^2 - e^2a^2}}{2e} + \frac{a^2}{2e^2} \arcsin e\right]$$

$$= 2\pi ab\left[\sqrt{1 - e^2} + \frac{1}{e} \arcsin e\right]$$

$$= 2(\text{Area of the ellipse})\left[\sqrt{1 - e^2} + \frac{1}{e} \arcsin e\right].$$

10. Prove that the surface area of the ellipsoid formed by the revolution of the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about its major axis is $2\pi\left(a^2 + \dfrac{b^2}{e} \ln\sqrt{\dfrac{1+e}{1-e}}\right)$, e being the eccentricity of the ellipse.

Sol. Here, as in Problem 9,

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \frac{a^4y^2}{b^4x^2}}\, dy = \frac{\sqrt{b^4x^2 + a^4y^2}}{b^2x}\, dy$$

Required area

$$A = 2\pi \int_{-b}^{b} \frac{\sqrt{b^4x^2 + a^4y^2}}{b^2}\, dy$$

$$= \frac{4\pi}{b^2} \int_0^b \sqrt{b^2\,(a^2b^2 - a^2y^2) + a^4y^2}\, dy$$

$$= \frac{4\pi}{b^2} a \int_0^b \sqrt{b^4 + (a^2 - b^2)\,y^2}\, dy$$

$$= \frac{4\pi a}{b^2} \int_0^b \sqrt{b^4 + a^2e^2y^2}\, dy = \frac{4\pi a^2e}{b^4} \int_0^b \sqrt{\frac{b^4}{a^2e^2} + y^2}\, dy$$

$$= \frac{4\pi a^2e}{b^2} \left[\frac{y\sqrt{\dfrac{b^4}{a^2e^2} + y^2}}{2} + \frac{b^4}{2a^2e^2} \ln \frac{y + \sqrt{y^2 + \dfrac{b^4}{a^2e^2}}}{b^2/ae}\right]_0^b$$

$$= \frac{2\pi a^2e}{b^2}\left[\frac{b\sqrt{b^4 + a^2e^2b^2}}{ae} + \frac{b^4}{a^2e^2} \ln \frac{bae + \sqrt{b^2a^2e^2 + b^4}}{b^2}\right]$$

$$= \frac{2\pi a^2e}{b^2}\left[\frac{b^2\sqrt{b^2 + a^2e^2}}{ae} + \frac{b^4}{a^2e^2} \ln \frac{ae + \sqrt{a^2e^2 + b^2}}{b}\right]$$

$$= 2\pi a^2 e\left[\frac{a}{ae} + \frac{b^2}{a^2e^2} \ln \frac{ae + a}{b}\right] = 2\pi\left[a^2 + \frac{b^2}{e} \ln \frac{a(1 + e)}{a\sqrt{1 - e^2}}\right]$$

$$= 2\pi\left[a^2 + \frac{b^2}{e} \ln\sqrt{\frac{1 + e}{1 - e}}\right], \text{ using } b^2 = a^2\,(1 - e^2).$$

11. Find the area of the surface generated by revolving the curve $x = e^\theta \sin\theta$, $y = e^\theta \cos\theta$, from $\theta = 0$ to $\theta = \pi$ about the y-axis.

Sol. $\dfrac{dx}{d\theta} = e^\theta\,(\sin\theta + \cos\theta)$, $\dfrac{dy}{d\theta} = e^\theta\,(\cos\theta - \sin\theta)$

$$\frac{ds}{d\theta} = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (2e^{2\theta})^{1/2} = \sqrt{2}\, e^\theta$$

Desired area $= 2\pi \displaystyle\int_0^\pi x\, ds = 2\pi \int_0^\pi e^\theta \sin\theta . \sqrt{2}\, e^\theta\, d\theta$

$$= 2\sqrt{2}\,\pi \int_0^{\pi} e^{2\theta} \sin\theta \, d\theta$$

$$= 2\sqrt{2}\,\pi \frac{1}{5}\left[(2e^{2\theta}\sin\theta - \cos\theta)\right]_0^{\pi} = \frac{2\sqrt{2}\,\pi\,[2e^{2\pi}+1]}{5}$$

12. Find the area of the surface generated by revolving the area enclosed by the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ about $y = 0$.

Sol. Here $\dfrac{dx}{d\theta} = a(1-\cos\theta)$, $\quad \dfrac{dy}{d\theta} = a\sin\theta$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{(1-\cos\theta)^2 + \sin^2\theta} = a\sqrt{2 - 2\cos\theta} = 2a\sin\frac{\theta}{2}$$

Required surface area is

$$A = 2\int_0^{\pi} 2\pi y \frac{ds}{d\theta}\,d\theta = 4\pi\int_0^{\pi} a(1-\cos\theta)\cdot 2a\sin\frac{\theta}{2}\,d\theta$$

$$= 8\pi a^2 \int_0^{\pi} 2\sin^2\frac{\theta}{2}\cdot\sin\frac{\theta}{2}\,d\theta = 16\pi a^2\int_0^{\pi}\sin^3\frac{\theta}{2}\,d\theta$$

Put $\dfrac{\theta}{2} = z$ or $d\theta = 2dz$

Then $A = 16\pi a^2 \displaystyle\int_0^{\pi/2} \sin^3 z\cdot(2dz)$

$$= 32\pi a^2\cdot\frac{2}{3} \quad \text{(by Wallis formula)}$$

$$= \frac{64\pi a^2}{3}.$$

13. Show that the surface area formed by revolving the area enclosed by the loop of the curve $3ay^2 = x\,(x-a)^2$ about the x-axis is $\pi a^2/3$.

Sol. Here we have $6ay\dfrac{dy}{dx} = (x-a)^2 + 2x\,(x-a)$

$$= (x-a)\,(3x-a)$$

or $\quad \dfrac{dy}{dx} = \dfrac{(x-a)\,(3x-a)}{6ay}$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\frac{(x-a)^2\,(3x-a)^2}{36a^2y^2}}\,dx$$

$$= \sqrt{1+\frac{(3x-a)^2}{12a\cdot x}}\,dx = \sqrt{\frac{12ax+(3x-a)^2}{12ax}}\,dx$$

$$= \sqrt{\frac{(3x+a)^2}{12ax}}\,dx = \frac{3x+a}{\sqrt{12ax}}\,dx$$

Hence the required area $= 2\pi\displaystyle\int_0^{a}\sqrt{\frac{x}{3a}}\,(x-a)\left(\frac{3x+a}{\sqrt{12ax}}\right)dx$

$$= \frac{2\pi}{6a}\int_0^{a}(3x^2 - 2ax - a^2)\,dx$$

$$= \frac{\pi}{3a}\left[x^3 - ax^2 - a^2x\right]_0^a = \frac{\pi}{3a}\cdot a^3 = \frac{\pi}{3}a^2.$$

14. Find the surface area of the **torus** formed by revolving a disc of radius a about a straight line in its plane at a distance b from the centre ($b > a$).

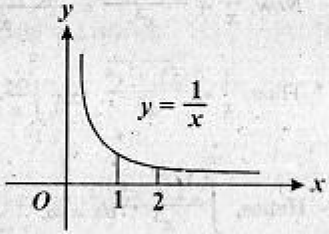
Sol. The circumference of the disc $= 2\pi a$. Also the c.g. describes a circle of radius b. Hence the required area

$$= (2\pi a)\,(2\pi b) = 4\pi^2 ab.$$

15. The curve $y = \dfrac{1}{x}$, $1 \le x \le 2$, is rotated about the x-axis. Find the area of the resulting surface. If $1 \le x < \infty$, show that the volume of solid generated is finite but its surface area is infinite.

Sol. Required area $= 2\pi\displaystyle\int_1^2 y\,ds = 2\pi\int_1^2 \frac{1}{x}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 2\pi\int_1^2 \frac{\sqrt{1+x^4}}{x^3}\,dx$$



Put $x^2 = \tan\theta$, $2x\,dx = \sec^2\theta\,d\theta$.

Then $I = \displaystyle\int \frac{\sqrt{1+x^4}}{x^3}\,dx$

$$= \int\frac{\sec\theta\cdot\sec^2\theta\,d\theta}{2\tan^2\theta} = \int\frac{d\theta}{2\sin^2\theta\cos\theta}$$

Again put $\sin\theta = u$, $\cos\theta\, d\theta = du$, so that

$$I = \int \frac{du}{2u^2(1-u^2)} = \int\left(\frac{1}{u^2} + \frac{1}{1-u^2}\right) du = -\frac{1}{u} + \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|$$

$$= \frac{-1}{\sin\theta} + \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|$$

$$= -\left(\frac{\sqrt{1+x^4}}{x^2}\right) + \frac{1}{2}\ln\left|\frac{1+\frac{x^2}{\sqrt{1+x^4}}}{1-\frac{x^2}{\sqrt{1+x^4}}}\right|$$

$$\text{Area} = 2\pi\left[-\frac{\sqrt{1+x^4}}{x^2} + \frac{1}{2}\ln\frac{\sqrt{1+x^4}+x^2}{\sqrt{1+x^4}-x^2}\right]_1^2$$

$$= 2\pi\left[-\frac{\sqrt{17}}{4} + \frac{1}{2}\ln\frac{\sqrt{17}+4}{\sqrt{17}-4} + \sqrt{2} - \frac{1}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

If $1 \le x < \infty$, then

$$\text{Volume} = \int_1^\infty \pi y^2\, dx = \pi\int_1^\infty \frac{dx}{x^2} = \pi\left[-\frac{1}{x}\right]_\infty^1 = \pi.$$

$$\text{And area} = 2\pi\int_1^\infty \frac{\sqrt{1-x^4}}{x^3}\, dx$$

We know that if $f(x) \le g(x)$ and if

$$\int_1^\infty f(x)\, dx = \infty, \text{ then } \int_1^\infty g(x)\, dx = \infty.$$

$$\text{Now, } \frac{-1}{x} + \frac{\sqrt{1+x^4}}{x^3} = \frac{-x^2+\sqrt{1+x^4}}{x^3} \ge 0 \text{ for all } x \ge 1.$$

$$\text{Thus, } \frac{1}{x} < \frac{\sqrt{1+x^4}}{x^3}. \text{ But } \int_1^\infty \frac{dx}{x} = \infty.$$

$$\text{Hence, } \int_1^\infty \frac{\sqrt{1+x^4}}{x^3}\, dx = \infty.$$

Exercise Set 9.10 (Page 456)

1. Find the area of solid generated by revolving the circle $r = a$, $0 \le \theta \le \frac{\pi}{4}$, about the polar axis.

Sol. $x = a\cos\theta, y = a\sin\theta$ are parametric equations of the circle $r = a$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2$$

$$\text{Desired area} = 2\pi\int_0^{\pi/4} y\, ds = 2\pi\int_0^{\pi/4} a\sin\theta\cdot a\, d\theta$$

$$= 2\pi a^2\,[-\cos\theta]_0^{\pi/4} = 2\pi a^2\left(1-\frac{1}{\sqrt{2}}\right) = 2\pi a^2\left(1-\frac{1}{\sqrt{2}}\right).$$

2. Find the area of the surface generated by revolving $r = 2a\sin\theta$ about the polar axis.

Sol. $r = 2a\sin\theta = f(\theta)$

$$f'(\theta) = \frac{dr}{d\theta} = 2a\cos\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2 = 4a^2\sin^2\theta + 4a^2\cos^2\theta = 4a^2$$

$$ds = 2a\, d\theta.$$

$$\text{Required area} = 2\,.\,2\pi\int_0^{\pi/2} f(\theta)\sin\theta\, ds$$

$$= 4\pi\int_0^{\pi/2} 2a\sin^2\theta\,.\,2a\, d\theta = 16\pi a^2\frac{1}{2}\frac{\pi}{2} = 4\pi^2 a^2.$$

3. The arc of the spiral $r = e^\theta$ from $(1, 0)$ to $(e, 1)$ is revolved about the line $\theta = \frac{\pi}{2}$. Find the area of the resulting surface.

Sol. $r = f(\theta) = e^\theta$

$$\left(\frac{ds}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2 = e^{2\theta} + e^{2\theta} = 2e^{2\theta}$$

$$\text{Required area} = 2\pi\int_0^1 x\, ds = 2\pi\int_0^1 e^\theta\cos\theta\sqrt{2}\, e^\theta\, d\theta$$

$$= 2\sqrt{2}\,\pi \int_0^1 e^{2\theta} \cos\theta\, d\theta$$

$$= 2\sqrt{2}\,\pi \left[\frac{e^{2\theta}}{5}(2\cos\theta + \sin\theta)\right]_0^1$$

$$= \frac{2\sqrt{2}\,\pi}{5}\left[e^2(2\cos 1 + \sin 1) - 2\right].$$

4. Find the surface area of the solid generated by revolving the curve $r = e^{\theta/2}$ about the polar axis, $0 \le \theta \le \pi$.

Sol. $r = f(\theta) = e^{\theta/2}$

$$f'(\theta) = \frac{1}{2}e^{\theta/2}$$

$$\left(\frac{ds}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2 = e^{\theta} + \frac{1}{4}e^{\theta} = \frac{5}{4}e^{\theta}.$$

$$\text{Required area} = 2\pi \int_0^{\pi} e^{\theta/2} \sin\theta \frac{\sqrt{5}}{2} e^{\theta} d\theta$$

$$= \sqrt{5}\,\pi \int_0^{\pi} e^{\theta} \sin\theta\, d\theta = \frac{\sqrt{5}\pi}{2}\left[e^{\theta}(\sin\theta - \cos\theta)\right]_0^{\pi}$$

$$= \frac{\sqrt{5}\pi}{2}(e^{\pi} + 1).$$

5. Prove that the volume of the solid generated by the revolution of the area enclosed by the limacon $r = a + b\cos\theta$, $(a > b)$ is $\frac{4}{3}\pi a(a^2 + b^2)$.

Sol. Required volume

$$= \frac{2}{3}\pi \int_0^{\pi} r^3 \sin\theta\, d\theta = \frac{2}{3}\pi \int_0^{\pi} (a + b\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{2}{3}\pi\left[-\frac{(a + b\cos\theta)^4}{4b}\right]_0^{\pi} = \frac{-2}{3}\pi\left[\frac{(a-b)^4 - (a+b)^4}{4b}\right]$$

$$= -\frac{2}{3}\frac{\pi}{4b}\left[a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 - a^4 - 4a^3b - 6a^2b^2 - 4ab^3 - b^4\right]$$

$$= \frac{\pi}{6b}\left[8a^3b + 8ab^3\right] = \frac{4}{3}\pi a\left[a^2 + b^2\right]$$

6. Find the volume and area of the surface of the solid generated by the revolution of the area enclosed by the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line.

Sol. Required volume V is given by

$$V = \int_0^{\pi/4} \frac{2\pi}{3} r^3 \sin\theta\, d\theta = \frac{4\pi}{3}\int_0^{\pi/4} a^3(\cos 2\theta)^{3/2}\sin\theta\, d\theta$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/4}(2\cos^2\theta - 1)^{3/2}\sin\theta\, d\theta$$

Put $\sqrt{2}\cos\theta = z$ or $-\sqrt{2}\sin\theta\, d\theta = dz$

$$V = \frac{4\pi a^3}{3}\int_{\sqrt{2}}^{1}(z^2 - 1)^{3/2}\left(-\frac{dz}{\sqrt{2}}\right) = \frac{4\pi a^3}{3\sqrt{2}}\int_1^{\sqrt{2}}(z^2 - 1)^{3/2} dz$$

Now let $I = \int (z^2-1)^{3/2} dz = z(z^2-1)^{3/2} - \int z \cdot \frac{3}{2}(z^2-1)^{1/2}\, 2z\, dz$

$$= z(z^2-1)^{3/2} - 3\int z^2(z^2 - a)^{1/2} dz$$

$$= z(z^2-1)^{3/2} - 3\int (z^2 - 1 + 1)(z^2-1)^{1/2} dz$$

$$= z(z^2-1)^{3/2} - 3\int (z^2-1)^{3/2} dz - 3\int (z^2-1)^{1/2} dz$$

Thus $4I = z(z^2-1)^{3/2} - 3\int (z^2-1)^{1/2} dz$

$$= z(z^2-1)^{3/2} - 3\left[\frac{z(z^2-1)^{1/2}}{2} - \frac{1}{2}\ln\left(z + (z^2-1)^{1/2}\right)\right]$$

or $$I = \frac{z(z^2-1)^{3/2}}{4} - \frac{3}{8}\left[z(z^2-1)^{1/2} - \ln\left(z + \sqrt{z^2-1}\right)\right]$$

Hence, $$V = \frac{4\pi a^3}{3\sqrt{2}}\left[\left|\frac{z(z^2-1)^{3/2}}{4} - \frac{3}{8}\left[z(z^2-1)^{1/2} - \ln\left(z + \sqrt{z^2-1}\right)\right]\right|\right]_1^{\sqrt{2}}$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\left[\frac{\sqrt{2}}{4} - \frac{3}{8}(\sqrt{2}) - \ln(\sqrt{2}+1)\right]$$

$$= \frac{\pi a^3}{3\sqrt{2}}\left[\sqrt{2} - \frac{3\sqrt{2}}{2} + \frac{3}{2}\ln(\sqrt{2}+1)\right]$$

$$= \frac{\pi a^3}{3\sqrt{2}}\left[\frac{-\sqrt{2}}{2} + \frac{3}{2}\ln(\sqrt{2}+1)\right]$$

$$= \frac{\pi a^3}{2}\left[\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1) - \frac{1}{3}\right].$$

Surface S is given by

$$S = 2\int_0^{\pi/4} 2\pi(r\sin\theta)\,ds = 4\pi\int_0^{\pi/4} r\sin\theta\,\frac{ds}{d\theta}\,d\theta,$$

where $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}}$

$$= a\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}}$$

$$S = 4\pi\int_0^{\pi/4} a.\sqrt{\cos 2\theta}.\sin\theta.\frac{a}{\sqrt{\cos 2\theta}}\,d\theta$$

$$= 4\pi a^2\int_0^{\pi/4}\sin\theta\,d\theta = -4\pi a^2\,|\cos\theta|_0^{\pi/4}$$

$$= -4\pi a^2\left[\frac{1}{\sqrt{2}} - 1\right] = (4 - 2\sqrt{2})\,\pi a^2.$$

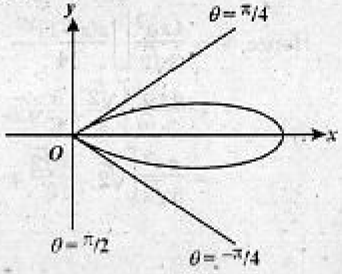
7. Show that the volume of the solid formed by revolving the area bounded by one loop of the curve $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$ is $\frac{\pi^2 a^3}{4\sqrt{2}}$.

Sol. Here the limits of integration are from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

Therefore, the required volume is

$$V = \int_{-\pi/4}^{\pi/4}\frac{2}{3}\pi r^3\cos\theta\,d\theta$$

$$= \frac{4}{3}\pi.\int_0^{\pi/4} r^3\cos\theta\,d\theta$$



$$= \frac{4}{3}\pi a^3\int_0^{\pi/4}(\cos 2\theta)^{3/2}\cos\theta\,d\theta$$

Put $\sqrt{2}\sin\theta = \sin t$ or $\sqrt{2}\cos\theta\,d\theta = \cos t\,dt$ and we have

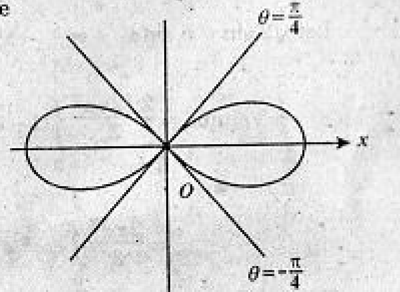
$$V = \frac{4\pi a^3}{3}\int_0^{\pi/4}(1 - \sin^2 t)^{3/2}\frac{\cos t\,dt}{\sqrt{2}}$$

or $$V = \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/4}\cos^3 t\cos t\,dt = \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/4}\cos^4 t\,dt$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\cdot\frac{3\,.\,1}{4\,.\,2}\cdot\frac{\pi}{2},\text{ (by Wallis formula)} = \frac{\pi^2 a^3}{4\sqrt{2}}.$$

8. The area enclosed by the lemniscate $r^2 = a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the volume and area of the surface of the solid generated are respectively $\frac{1}{4}\pi^2 a^3$ and $4\pi a^2$.

Sol. Tangent to the curve at the pole are $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{\pi}{4}$. For one loop (on the right hand side) θ varies from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$. Therefore, the volume generated by one loop is



$$= \int_{-\pi/4}^{\pi/4}\frac{2}{3}\pi r^3\sin\left(\theta + \frac{\pi}{4}\right)d\theta$$

$$= \frac{2}{3}\pi\int_{-\pi/4}^{\pi/4}a^3(\cos 2\theta)^{3/2}\left[\frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)\right]d\theta$$

$$= \frac{\sqrt{2}}{3}\pi a^3\left[\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2\theta)^{3/2}\cos\theta\, d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2\theta)^{3/2}\sin\theta\, d\theta\right]$$

$$= \frac{\sqrt{2}}{3}\pi a^3\left[\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(2\cos^2\theta-1)^{3/2}\sin\theta\, d\theta\right]$$

The first integral is even (for if we replace θ by $-\theta$ we get the same integral), therefore we can replace it by

$$2\int_0^{\frac{\pi}{4}}(1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta$$

The second integral is odd, it vanishes and so the volume is

$$= \frac{\sqrt{2}\pi a^3}{3}.2\int_0^{\frac{\pi}{4}}(1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta$$

Let $\sqrt{2}\sin\theta = \sin\phi$ or $\sqrt{2}\cos\theta\, d\theta = \cos\phi\, d\phi$.

$$\text{Volume} = \frac{2\sqrt{2}\pi a^3}{3}\int_0^{\frac{\pi}{4}}(1-\sin^2\phi)^{3/2}\frac{1}{\sqrt{2}}\cos\phi\, d\phi$$

$$= \frac{2\pi a^3}{3}\int_0^{\frac{\pi}{4}}\cos^4\phi\, d\phi = \frac{2\pi a^3}{3}\cdot\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} = \frac{\pi^2 a^3}{8}.$$

The volume generated by both the loops is

$$2.\frac{\pi^2 a^3}{8} = \frac{\pi^2 a^3}{4}.$$

Now we find the surface area generated by the given curve. Surface area generated by one loop is

$$S = 2\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r\sin\left(\theta+\frac{\pi}{4}\right)\frac{ds}{d\theta}d\theta$$

Here $$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}}$$

$$= a\sqrt{\frac{\cos^2 2\theta+\sin^2 2\theta}{\cos 2\theta}} = \frac{a^2}{r}.$$

$$\text{Hence } S = 2\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r\left[\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)\right]\frac{a^2}{r}d\theta$$

$$= \frac{2\pi a^2}{\sqrt{2}}\left|\sin\theta-\cos\theta\right|_{-\pi/4}^{\pi/4} = \frac{2\pi a^2}{\sqrt{2}}\left[\frac{2}{\sqrt{2}}\right] = 2\pi a^2$$

The surface generated by both the loops is $2.2\pi a^2 = 4\pi a^2$.

9. Find the area of the surface generated by revolving the area enclosed by the upper half of the cardioid $r = a(1-\cos\theta)$ about the initial line.

Sol. The required surface S is

$$= \int_0^{\pi} 2\pi r\sin\theta\frac{ds}{d\theta}d\theta$$

Here $$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}$$

$$= a\sqrt{2-2\cos\theta}$$

$$= a\cdot\sqrt{2}.\sqrt{2}\sin\frac{\theta}{2}. = 2a\sin\frac{\theta}{2}$$

Hence $$S = \int_0^{\pi}2\pi\left(a(1-\cos\theta).\sin\theta.2a\sin\frac{\theta}{2}\right)d\theta$$

$$= 4\pi a^2\int_0^{\pi}\left(2\sin^2\frac{\theta}{2}\right)2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.\sin\frac{\theta}{2}d\theta$$

$$= 16\pi a^2\int_0^{\pi}\sin^4\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$= 16\pi a^2 2.\left[\frac{\sin^5\theta/2}{5}\right]_0^{\pi} = \frac{32}{5}\pi a^2.$$

10. The upper half of the area inside the cardioid $r = 2a(1 + \cos\theta)$ and outside the parabola $\frac{2a}{r} = 1 + \cos\theta$ is revolved about the initial line. Show that the volume of the solid generated is $18\pi a^3$.

Sol. The curves intersect at the points where $2a(1 + \cos\theta) = \frac{2a}{1 + \cos\theta}$

or $(1 + \cos\theta)^2 = 1$

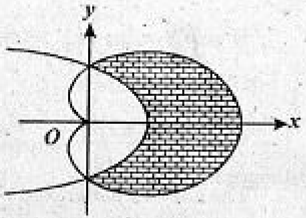
or $\cos\theta(\cos\theta + 2) = 0$

Therefore, $\cos\theta = 0$

or $\cos\theta = -2$

But $\cos\theta = -2$ is not possible. So

$\cos\theta = 0$ i.e, $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$



Required volume

$$= \frac{2}{3}\pi\int_0^{\pi/2}\left\{(2a(1 + \cos\theta))^3 - \left(\frac{2a}{1 + \cos\theta}\right)^3\right\}\sin\theta\, d\theta$$

$$= \frac{-16\pi a^3}{3}\int_0^{\pi/2}(1+\cos\theta)^3 d(-\cos\theta) + \frac{-16\pi a^3}{3}\int_0^{\pi/2}(1+\cos\theta)^{-3} d(-\cos\theta)$$

$$= \frac{-4\pi a^3}{3}[(1 + \cos\theta)^4]_0^{\pi/2} - \frac{8\pi a^3}{3}[(1 + \cos\theta)^{-2}]_0^{\pi/2}$$

$$= \frac{-4\pi a^3}{3}(1 - 16) - \frac{8\pi a^3}{3}\left(1 - \frac{1}{4}\right) = 20\pi a^3 - 2\pi a^3 = 18\pi a^3.$$

Chapter 10 MULTIPLE INTEGRALS

Exercise Set 10.1 (Page 463)

Evaluate (Problems 1 – 19):

1. $\int_0^1\int_1^2 dx\, dy$

Sol. $\int_0^1\int_1^2 dx\, dy = \int_0^1 [|x|_1^2 dy = \int_0^1 [2 - 1]\, dy$

$= \int_0^1 dy = |y|_0^1 = |1 - 0| = 1.$

2. $\int_1^2\int_0^3 (x + y)\, dx\, dy$

Sol. $\int_1^2\int_0^3 (x + y)\, dx\, dy = \int_1^2\left(\left[\frac{x^2}{2}\right]_0^3 + [xy]_0^3\right) dy$

$= \int_1^2\left[\left(\frac{9}{2} - \frac{0}{2}\right) + (3y - 0y)\right] dy$

$= \int_1^2\left[\frac{9}{2} + 3y\right] dy = \frac{9}{2}[y]_1^2 + 3\left[\frac{y^2}{2}\right]_1^2$

$= \frac{9}{2}(2 - 1) + \frac{3}{2}(4 - 1) = 9.$

3. $\int_2^4\int_1^2 (x^2 + y^2)\, dy\, dx$

Sol. $$\int_2^4\int_1^2 (x^2+y^2)\,dy\,dx = \int_2^4\left[x^2y+\frac{y^3}{3}\right]_1^2 dx$$

$$= \left[\int_2^4 x^2(2-1)+\frac{1}{3}(8-1)\right]dx$$

$$= \int_2^4\left[x^2+\frac{7}{3}\right]dx = \left[\frac{x^3}{3}+\frac{7}{3}x\right]_2^4$$

$$= \frac{1}{3}(64-8)+\frac{7}{3}(4-2) = \frac{70}{3}.$$

4. $$\int_0^1\int_{x^2}^{x} xy^2\,dy\,dx$$

Sol. $$\int_0^1\int_{x^2}^{x} xy^2\,dy\,dx = \int_0^1\left(x\left[\frac{y^3}{3}\right]_{x^2}^{x}\right)dx = \int_0^1\left[x\left(\frac{x^3}{3}-\frac{x^6}{3}\right)\right]dx$$

$$= \int_0^1\left(\frac{x^4}{3}-\frac{x^7}{3}\right)dx = \left[\frac{x^5}{15}-\frac{x^8}{24}\right]_0^1$$

$$= \left(\frac{1}{15}-\frac{0}{15}\right)-\left(\frac{1}{24}-\frac{0}{24}\right) = \frac{1}{40}.$$

5. $$\int_1^2\int_0^{y^{3/2}} \frac{x}{y^2}\,dx\,dy$$

Sol. $$\int_1^2\int_0^{y^{3/2}} \frac{x}{y^2}\,dx\,dy = \int_1^2\frac{1}{y^2}\left(\left[\frac{x^2}{2}\right]_0^{y^{3/2}}\right)dy = \int_1^2\frac{1}{y^2}\left(\frac{y^3}{2}-\frac{0}{2}\right)dy$$

$$= \frac{1}{2}\int_1^2 y\,dy = \frac{1}{2}\left|\frac{y^2}{2}\right|_1^2 = \frac{1}{4}\left(4-\frac{1}{2}\right) = \frac{3}{4}.$$

6. $$\int_0^1\int_x^{\sqrt{x}} (y+y^3)\,dy\,dx$$

Sol. $$\int_0^1\int_x^{\sqrt{x}} (y+y^3)\,dy\,dx = \int_0^1\left(\left[\frac{y^2}{2}+\frac{y^4}{4}\right]_x^{\sqrt{x}}\right)dx$$

$$= \frac{1}{2}\int_0^1\left[x-x^2+\frac{x^2}{2}-\frac{x^4}{2}\right]dx$$

$$= \frac{1}{2}\int_0^1\left[x-\frac{x^2}{2}-\frac{x^4}{2}\right]dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^5}{10}\right]_0^1 = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{6}-\frac{1}{10}\right) = \frac{7}{60}.$$

7. $$\int_0^1\int_0^{x^2} xe^y\,dy\,dx$$

Sol. $$\int_0^1\int_0^{x^2} xe^y\,dy\,dx = \int_0^1\left(x\,[e^y]_0^{x^2}\right)dx = \int_0^1 [x(e^{x^2}-e^0)]\,dx$$

$$= \int_0^1 (xe^{x^2}-x)\,dx = \left[\frac{1}{2}e^{x^2}-\frac{x^2}{2}\right]_0^1 = \frac{1}{2}e-\frac{1}{2}-\frac{1}{2}e^0$$

$$= \frac{e}{2}-1.$$

8. $$\int_2^4\int_y^{8-y} y\,dx\,dy$$

Sol. $$\int_2^4\int_y^{8-y} y\,dx\,dy = \int_2^1 [yx]_y^{8-y}\,dy = \int_2^4 [y(8-y-y)]\,dy$$

$$= \int_2^4 (8y-2y^2)\,dy = \left[4y^2-\frac{2}{3}y^3\right]_2^4$$

$$= 4\times 16-\frac{2}{3}\times 64-4\times 4+\frac{2}{3}\times 8 = 48-\frac{112}{3} = \frac{32}{3}.$$

9. $\int_0^4\int_{y/2}^2 e^{x^2}\,dx\,dy$

Sol. The region of integration is bounded by $0 \le y \le 4$, $x = \frac{y}{2}$ and $x = 2$.

This region is also enclosed by $0 \le x \le 2$, $y = 0$ and $y = 2x$. The given integral is

$$= \int_0^2\int_0^{2x} e^{x^2}\,dy\,dx = \int_0^2 e^{x^2}\,[y]_0^{2x}\,dx$$

$$= \int_0^2 2xe^{x^2}\,dx = [e^{x^2}]_0^2 = e^4 - 1.$$

10. $\int_0^2\int_{y^2}^4 y\cos x^2\,dx\,dy$

Sol. We change the order of integration. The region of integration is $0 \le y \le 2, y^2 \le x \le 4$. This is equivalent to $0 \le x \le 4, 0 \le y \le \sqrt{x}$. The given integral equals

$$\int_0^4\int_0^{\sqrt{x}} y\cos x^2\,dy\,dx = \frac{1}{2}\int_0^4 [y^2]_0^{\sqrt{x}}\cos x^2\,dx = \frac{1}{2}\int_0^4 x\cos x^2\,dx$$

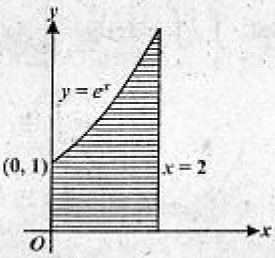
$$= \frac{1}{4}[\sin x^2]_0^4 = \frac{1}{4}\sin 16.$$

11. $\iint_D dy\,dx$ and $\iint_D dx\,dy$, where D is the region bounded by the y-axis, the line $x = 2$ and the curve $y = e^x$.

Sol. $\iint_D dy\,dx = \int_0^2\int_1^{e^x} dy\,dx = \int_0^2 (e^x - 1)\,dx$

$$= [(e^x - x)]_0^2 = e^2 - 3$$

$$\iint_D dy\,dx = \int_1^{e^2}\int_{\ln y}^2 dx\,dy$$



$$= \int_1^{e^2} (2 - \ln y)dy = [2y - y\ln y + y]_1^{e^2} = e^2 - 3.$$

12. $\int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta$

Sol. $\int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\left[\frac{r^3}{3}\right]_0^2 d\theta = \int_0^{\pi/2}\frac{8}{3}\cos\theta\,d\theta$

$$= \frac{8}{3}[\sin\theta]_0^{\pi/2} = \frac{8}{3}(1 - 0) = \frac{8}{3}.$$

13. $\int_0^{2\pi}\int_0^{1-\cos\theta} r^3\cos^2\theta\,dr\,d\theta$

Sol. $\int_0^{2\pi}\int_0^{1-\cos\theta} r^3\cos^2\theta\,dr\,d\theta = \int_0^{2\pi}\cos^2\theta\left[\frac{r^4}{4}\right]_0^{1-\cos\theta} d\theta$

$$= \int_0^{2\pi}\frac{1}{4}[\cos^2\theta(1 - \cos\theta)^4]d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\cos^2\theta[1 - 4\cos\theta + 6\cos^2\theta - 4\cos^3\theta + \cos^4\theta]\,d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}[\cos^2\theta - 4\cos^3\theta + 6\cos^4\theta - 4\cos^5\theta + \cos^6\theta]\,d\theta$$

$$= \frac{1}{4}\cdot 2\int_0^{\pi}[\cos^2\theta - 4\cos^3\theta + 6\cos^4\theta - 4\cos^5\theta + \cos^6\theta]\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\cos^2\theta\,d\theta - 2\int_0^{\pi}\cos^3\theta\,d\theta + 3\int_0^{\pi}\cos^4\theta\,d\theta - 2\int_0^{\pi}\cos^5\theta\,d\theta + \frac{1}{2}\int_0^{\pi}\cos^6\theta\,d\theta$$

$$= \frac{1}{2}.2\int_0^{\pi/2}\cos^2\theta\,d\theta - 2(0) + 3.2\int_0^{\pi/2}\cos^4\theta\,d\theta - 2\,(0) + \frac{1}{2}.2\int_0^{\pi/2}\cos^6\theta\,d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\,d\theta + 6\int_0^{\pi/2}\cos^4\theta\,d\theta + \int_0^{\pi/2}\cos^6\theta\,d\theta$$

$$= \frac{1}{2}\frac{\pi}{2} + 6\,\frac{3\,.\,1}{4\,.\,2}\frac{\pi}{2} + \frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}\frac{\pi}{2}, \text{ by Wallis formula}$$

$$= \left(\frac{1}{2}+\frac{9}{4}+\frac{15}{48}\right)\frac{\pi}{2} = \left(\frac{24+108+15}{48}\right)\frac{\pi}{2} = \frac{49}{32}\pi.$$

14. $\iint_D e^{-(x^2+y^2)}\,dx\,dy$, where D is the region in the first quadrant bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Sol. Changing into polar coordinates, the given integral is

$$= \int_0^{\pi/2}\int_1^2 e^{-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_1^2 d\theta$$

$$= \frac{1}{2}(e^{-1}-e^{-4})\left(\frac{\pi}{2}\right) = \frac{\pi}{4}\left(e^{-1}-e^{-4}\right).$$

15. $\iint_D \frac{dx\,dy}{1+x^2+y^2}$, where D is the closed disc of radius a with centre at the origin.

Sol. Changing into polar coordinates, the given integral is

$$= \int_0^{2\pi}\int_0^a \frac{r\,dr\,d\theta}{1+r^2} = \int_0^{2\pi}\left[\frac{1}{2}\ln(1+r^2)\right]_0^a d\theta$$

$$= \frac{1}{2}\ln(1+a^2)\int_0^{2\pi} d\theta = \pi\ln(1+a^2).$$

16. $\iint_D \frac{x^2}{(x^2+y^2)^2}\,dA$, where D is the region in the first quadrant bounded by the circles $x^2+y^2=a^2$, $x^2+y^2=b^2$, $0<a<b$.

Sol. Changing into polar coordinates, the given integral is

$$= \int_0^{\pi/2}\int_a^b \frac{r^2\cos^2\theta}{r^4}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\cos^2\theta\,[\ln r]_a^b\,d\theta = \ln\frac{b}{a}\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{\pi}{4}\ln\frac{b}{a}.$$

17. $\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\,dy\,dx.$

Sol. Changing into polar coordinates, the given integral is

$$= \int_0^{\pi}\int_0^a r^3.\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^5}{5}\right]_0^a d\theta = \frac{a^5}{5}\int_0^{\pi} d\theta = \frac{\pi a^5}{5}.$$

18. $\int_0^1\int_0^{\sqrt{1-y^2}} \sin(x^2+y^2)\,dx\,dy$

Sol. Changing into polar coordinates, we have the integral

$$= \int_0^{\pi/2}\int_0^1 (\sin r^2)\,r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{1}{2}\cos r^2\right]_0^1 d\theta$$

$$= \frac{1}{2}(1-\cos 1)\int_0^{\pi/2} d\theta = \frac{\pi}{4}(1-\cos 1).$$

19. $\int_0^4\int_0^{\sqrt{4y-y^2}} (x^2+y^2)\,dx\,dy$

Sol. The region of integration is bounded by

$0 \le x \le \sqrt{4y-y^2}$ and $0 \le y \le 4$

Now $x=\sqrt{4y-y^2}$ is the circle $x^2+y^2-4y=0$

or $x^2+y^2=4y$. In polar coordinates this takes the form

$r^2 = 4r\sin\theta$, or $r = 4\sin\theta$

On changing into polar coordinates, the given integral is

$$= \int_0^{\pi/2}\int_0^{4\sin\theta} r^2.\,r\,dr\,d\theta = \int_0^{\pi/2} 64\sin^4\theta\,d\theta$$

$$= 64\cdot\frac{3\,.\,1}{4\,.\,2}\cdot\frac{\pi}{2}, \text{ (using Wallis formula) } = 12\pi.$$

20. (*a*). Let D_a be the region bounded by the circle $x^2 + y^2 = a^2$.

$$\text{Define} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy.$$

Evaluate this improper integral.

Sol.(*a*). Changing into polar coordinates, the given integral

$$I = \lim_{a\to\infty}\int_0^{2\pi}\int_0^{a} e^{-r^2}\,r\,dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\left[-\frac{1}{2}e^{-r^2}\right]_0^{a} d\theta$$

$$= \lim_{a\to\infty}\int_0^{2\pi}\frac{1}{2}(1-e^{a^2})\,d\theta = \lim_{a\to\infty}\left[\frac{1}{2}\theta - \frac{1}{2}e^{-a^2}\theta\right]_0^{2\pi}$$

$$= \pi - \lim_{a\to\infty}\frac{\pi}{e^{a^2}} = \pi.$$

20. (*b*). Use part (*a*) to prove that $\int_0^{\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$

Sol. $$\int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dy\,dx = \int_{-a}^{a} e^{-y^2}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)dy$$

$$= \left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right) = \left(\int_{-a}^{a} e^{-x^2}\,dx\right)^2 = 4\left(\int_{0}^{a} e^{-x^2}\,dx\right)^2$$

Letting $a \to \infty$, we have

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= 4\left(\int_0^{\infty} e^{-x^2}\,dx\right)^2 = \pi \text{ from Part } (a) \text{ above.}$$

Therefore, $\int_0^{\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$

Exercise Set 10.2 (Page 468)

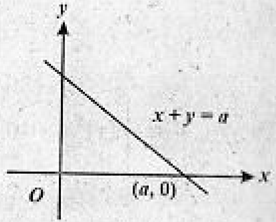
1. By means of double integration, find the area of the region bounded by

(a). the coordinate axes and the straight line $x + y = a$

Sol.(a). The required area

$$A = \int_0^{a}\int_{a-x}^{0} dy\,dx = \int_0^{a}[y]_{a-x}^{0}\,dx$$

$$= \int_0^{a}(x-a)\,dx = \left[\frac{x^2}{2}\right]_0^{a} = \frac{a^2}{2}.$$



1(b). the y-axis, the straight line $y = 2x$, and the straight line $y = 4$

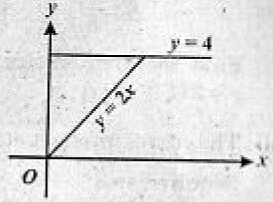
Sol.

(b). Required area

$$A = \int_0^{2}\int_{2x}^{4} dy\,dx = \int_0^{2}[y]_{2x}^{4}\,dx$$

$$= \int_0^{2}(4-2x)\,dx$$

$$= [4x - x^2]_0^2 = 8 - 4 = 4.$$



2. Find the area bounded by the parabola $y = x^2$ and the straight line $y = 2x + 3$

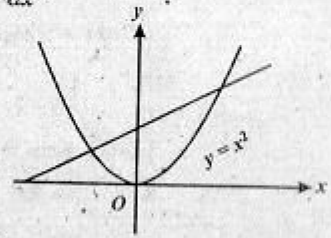
Sol. Solving $y = x^2$ and $y = 2x + 3$ simultaneously, we get the limits of integration for x as $-1, 3$. The required area is

$$= \int_{-1}^{3}\int_{x^2}^{2x+3} dy\,dx = \int_{-1}^{3}[y]_{x^2}^{2x+3}\,dx$$

$$= \int_{-1}^{3}[2x + 3 - x^2]\,dx$$

$$= \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3}$$

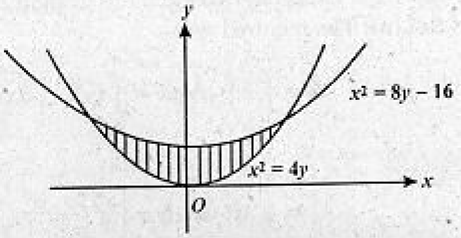
$$= 9 + 9 - 9 - \left(1 - 3 + \frac{1}{3}\right) = \frac{32}{3}.$$



3. Find the area bounded by $x^2 = 4y$ and $8y = x^2 + 16$

Sol. Solving the equation $x^2 = 4y$ and $8y = x^2 + 16$, we get $x = \pm 4$. The required area is

$$= \int_{-4}^{4} \int_{\frac{x^2}{4}}^{\frac{x^2}{8}+2} dy\, dx$$



$$= \int_{-4}^{4} [y]_{\frac{x^2}{4}}^{\frac{x^2}{8}+2} dx$$

$$= \int_{-4}^{4} \left[\frac{x^2}{8} + 2 - \frac{x^2}{4}\right] dx = \int_{-4}^{4} \left[2 - \frac{x^2}{8}\right] dx$$

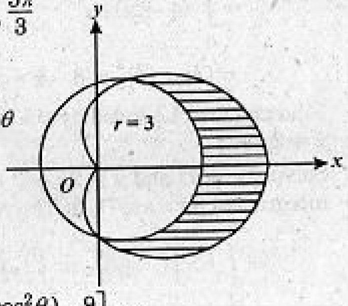
$$= \left[2x - \frac{1}{8}\frac{x^3}{3}\right]_{-4}^{4} = 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}.$$

4. Find the area outside the circle $r = 3$ and inside the cardioid $r = 2(1 + \cos\theta)$.

Sol. The curves intersect at $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Required area

$$= 2\int_{0}^{\pi/3} \int_{3}^{2(1+\cos\theta)} r\, dr\, d\theta$$



$$= 2\int_{0}^{\pi/3} \left[\frac{r^2}{2}\right]_{3}^{2(1+\cos\theta)} d\theta$$

$$= 2\int_{0}^{\pi/3} \left[\frac{4(1 + 2\cos\theta + \cos^2\theta)}{2} - \frac{9}{2}\right] d\theta$$

$$= \int_{0}^{\pi/3} (-5 + 8\cos\theta + 4\cos^2\theta) d\theta$$

$$= \int_{0}^{\pi/3} (-5 + 8\cos\theta + 2(1 + \cos 2\theta)\, d\theta$$

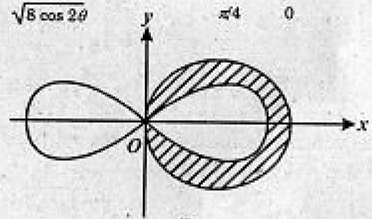
$$= [-3\theta + 8\sin\theta + \sin 2\theta]_{0}^{\pi/3}$$

$$= -3 \cdot \frac{\pi}{3} + 8 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2} - \pi.$$

5. Find the area inside the circle $r = 4\sin\theta$ and outside the leminscate $r^2 = 8\cos 2\theta$.

Sol. Since the two curves intersect at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ and $r^2 = 8\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, the desired area

$$= 2\int_{\pi/6}^{\pi/4} \int_{\sqrt{8\cos 2\theta}}^{4\sin\theta} r\, dr\, d\theta + 2\int_{\pi/4}^{\pi/2} \int_{0}^{4\sin\theta} r\, dr\, d\theta$$



$$= 2\int_{\pi/6}^{\pi/4} \left[\frac{r^2}{2}\right]_{\sqrt{8\cos 2\theta}}^{4\sin\theta} d\theta + 2\int_{\pi/4}^{\pi/2} \left[\frac{r^2}{2}\right]_{0}^{4\sin\theta}$$

$$= 2\int_{\pi/6}^{\pi/4} (8\sin^2\theta - 4\cos 2\theta)\, d\theta + 2\int_{\pi/4}^{\pi/2} 8\sin^2\theta\, d\theta$$

$$= 8\int_{\pi/6}^{\pi/4} (1 - 2\cos 2\theta)\, d\theta + 8\int_{\pi/4}^{\pi/2} (1 - \cos 2\theta)\, d\theta$$

$$= 8\left[\theta - \sin 2\theta\right]_{\pi/6}^{\pi/4} + 8\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= 8\left[\frac{\pi}{4} - 1 - \frac{\pi}{6} + \frac{\sqrt{3}}{2}\right] + 8\left[\frac{\pi}{2} - \frac{\pi}{4} + \frac{1}{2}\right]$$

$$= 8\left[\frac{3\pi - 12 - 2\pi + 6\sqrt{3}}{12}\right] + 8\left[\frac{\pi}{4} + \frac{1}{2}\right]$$

$$= \frac{2\pi}{3} - 4\sqrt{3} - 8 + 2\pi + 4 = \frac{8\pi}{3} + 4\sqrt{3} - 4.$$

6. Find the volume in the first octant between the planes $z = 0$, $z = x + y + 2$ and inside the cylinder $x^2 + y^2 = 16$

Sol. $$V = \int_0^4 \int_0^{\sqrt{16-x^2}} (x + y + 2)\, dy\, dx$$

$$= \int_0^4 \left[xy + \frac{y^2}{2} + 2y\right]_0^{\sqrt{16-x^2}} dx$$

$$= \int_0^4 \left[x\sqrt{16-x^2} + \frac{16-x^2}{2} + 2\sqrt{16-x^2}\right] dx$$

$$= -\frac{1}{3}\left[(16-x^2)^{3/2}\right]_0^4 + \left[8x - \frac{x^3}{6}\right]_0^4 + 2\left[\frac{x\sqrt{16-x^2}}{2} + 8\arcsin\frac{x}{4}\right]_0^4$$

$$= \frac{64}{3} + \left(32 - \frac{64}{6}\right) + 2\left[8\left(\frac{\pi}{2}\right)\right] = \frac{64}{3} + \frac{64}{3} + 8\pi = \frac{128}{3} + 8\pi.$$

7. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$

Sol. $$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx = 2\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} 4\, dy\, dx$$

$$= 8\int_0^2 \sqrt{4-x^2}\, dx = 16\int_0^2 \sqrt{4-x^2}\, dx$$

$$= 16\left[x\frac{\sqrt{4-x^2}}{2} + 2\arcsin\frac{x}{2}\right]_0^2 = 16\left(2 \cdot \frac{\pi}{2}\right) = 16\pi.$$

8. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, a, b and c being positive.

Sol. From $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, we have $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$.

Volume of the solid

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$$

$$= c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{x}{a}b\left(1 - \frac{x}{a}\right) - \frac{1}{2b}\cdot b^2\left(1 - \frac{x}{a}\right)^2\right] dx$$

$$= c\int_0^a \left[b - \frac{bx}{a} - \frac{bx}{a} + b\frac{x^2}{a^2} - \frac{b}{2}\left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right)\right] dx$$

$$= c\int_0^a \left(\frac{b}{2} - \frac{bx}{a} + \frac{bx^2}{2a^2}\right) dx = c\left[\frac{bx}{2} - \frac{b}{a}\frac{x^2}{2} + \frac{b}{2}\frac{x^3}{3a^2}\right]_0^a$$

$$= c\left[\frac{ba}{2} - \frac{ba}{2} + \frac{ba}{6}\right] = \frac{abc}{6}.$$

9. Find the volume of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy – plane.

Sol. The region D in the xy – plane is bounded by $x^2 + y^2 = 4$.

$$\text{Volume} = \iint_D (4 - x^2 - y^2)\, dx\, dy = \int_0^{2\pi}\int_0^2 (4 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta = 4\int_0^{2\pi} d\theta = 8\pi.$$

10. Find the volume of the solid bounded by the graphs of $x^2 + y^2 = 4$, $z = \sqrt{16 - x^2 - y^2}$, $z = 0$.

Sol. Required volume

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{16 - x^2 - y^2}\, dy\, dx$$

We change the integral into polar coordinates. From $x = r\cos\theta$, $y = r\sin\theta$, we have $x^2 + y^2 = r^2$, so that

$z = \sqrt{16 - r^2}$. The bounds become $0 \le r \le 2$ and $0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{2} \sqrt{16 - r^2}\, r\, dr\, d\theta = \frac{1}{3}\int_0^{2\pi} [(16 - r^2)^{3/2}]_0^2\, d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} [(12)^{3/2} - (16)^{3/2}]\, d\theta = -\frac{1}{3}\int_0^{2\pi} (24\sqrt{3} - 64)\, d\theta$$

$$= -\frac{1}{3}(24\sqrt{3} - 64)\, \theta \Big|_0^{2\pi} = -\frac{1}{3}(24\sqrt{3} - 64)\, 2\pi$$

$$= \frac{-16\pi}{3}(3\sqrt{3} - 8) = \frac{16\pi}{3}(8 - 3\sqrt{3}).$$

11. Find the centre of gravity of the plane area of uniform density bounded by

$x^2 = 4y$ and $8y = x^2 + 16$ in the first quadrant.

Sol. The parbolas intersect at the points where

$$\frac{x^2}{4} = \frac{x^2 + 16}{8}, \qquad \text{or} \qquad x = \pm 4$$

Points if intersection are (4, 4), (–4, 4). Let $(\bar{x}, \bar{y})$ be coordinates of the centre of gravity of the area lying in the first quadrant. Then

$$\bar{x} = \frac{\iint x\, dA}{\iint dA} \quad ; \quad \bar{y} = \frac{\iint y\, dA}{\iint dA}$$

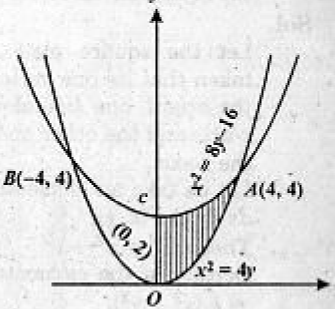
$$\text{Now, } \iint dA = \int_0^4 \int_{x^2/4}^{(x^2+16)/8} dy\, dx = \int_0^4 \left(\frac{x^2 + 16}{8} - \frac{x^2}{4}\right) dx$$

$$= \frac{1}{8}\int_0^4 (16 - x^2)\, dx = \frac{1}{8}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{16}{3}.$$

$$\iint x\, dA = \int_0^4 \int_{x^2/4}^{(x^2+16)/8} x\, dy\, dx = \frac{1}{8}\int_0^4 (16x - x^3)\, dx$$

$$= \frac{1}{8}\left[8x^2 - \frac{x^4}{4}\right]_0^4 = 8$$

Hence $\bar{x} = \dfrac{8}{\frac{16}{3}} = \dfrac{3}{2}$



$$\iint y\, dA = \int_0^4 \int_{x^2/4}^{(x^2+16)/8} y\, dy\, dx$$

$$= \frac{1}{2}\int_0^4 \left[\left(\frac{x^2 + 16}{8}\right)^2 - \left(\frac{x^2}{4}\right)^2\right] dx$$

$$= \frac{1}{2}\cdot\frac{1}{64}\int_0^4 (256 + 32x^2 - 3x^4)\, dx$$

$$= \frac{1}{128}\left[256x + \frac{32x^3}{3} - \frac{3x^5}{5}\right]_0^4$$

$$= \frac{1}{128}\cdot 256 \times 4\left[1 + \frac{2}{3} - \frac{3}{5}\right] = \frac{8 \times 16}{15}.$$

$$\bar{y} = \frac{128/15}{16/3} = \frac{8}{5}$$

Centre of gravity $(\bar{x}, \bar{y}) = \left(\dfrac{3}{2}, \dfrac{8}{5}\right)$.

12. Find the mass of semicircular wire whose density varies as the distance from the diameter joining the ends.

Sol.

Let O be the centre of the semi-circular wire of radius r. Let PQ be a small strip of the wire so that $m\ \angle POQ = d\theta$. Then

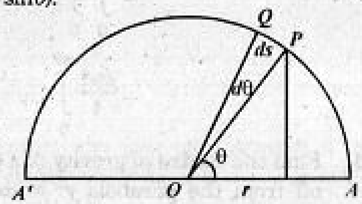
$$\frac{ds}{r} = d\theta, \; ds = r\, d\theta$$

Height of PQ above the diameter $= r\sin\theta$.

Density of the strip $PQ = k\,(r\sin\theta)$.

Required mass

$$= \int_0^{\pi} r\, d\theta\, k r \sin\theta$$

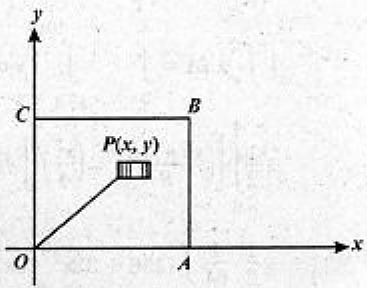


$$= kr^2 \int_0^{\pi} \sin\theta\, d\theta$$

$$= kr^2\,[1 - \cos\theta]_0^{\pi} = 2kr^2.$$

13. Find the mass of the square plate of side a if the density varies as the square of the distance from a vertex.

Sol.
Let the square plate be so taken that its one vertex is at the origin, one side along the x-axis and the other one along the y-axis.
Let us take an elemental area $dx\,dy$ at $P(x, y)$.
Then $OP^2 = x^2 + y^2$
Density of the elemental area $= k\,(x^2 + y^2)$.
Thus the required mass



$$= \int_0^a \int_0^a k\,(x^2 + y^2)\,dx\,dy = \int_0^a k\left[\frac{x^3}{3} + xy^2\right]_0^a dy$$

$$= k\int_0^a \left(\frac{a^3}{3} + ay^2\right) dy = k\left[\frac{a^3}{3}y + \frac{ay^3}{3}\right]_0^a = k\left[\frac{a^4}{3} + \frac{a^4}{3}\right] = k\,\frac{2a^4}{3}.$$

14. Find the mass of a circular plate of radius a if the density varies as the square of the distance from a point on the circumference to the centre of the circle.

Sol. Let an equation of the circular plate be $x^2 + y^2 = a^2$.
Density $= k(x^2 + y^2)$.

Mass of the plate $= \int\int k\,(x^2 + y^2)\,dA$

$$= \int_0^{\pi/2} \int_0^a kr^2\,r\,dr\,d\theta = k\int_0^{\pi/2} \left|\frac{r^4}{4}\right|_0^a d\theta$$

$$= \frac{ka^4}{4}\int_0^{\pi/2} d\theta = \frac{ka^4}{4}\,.\,2\pi = \frac{k\pi a^4}{2}.$$

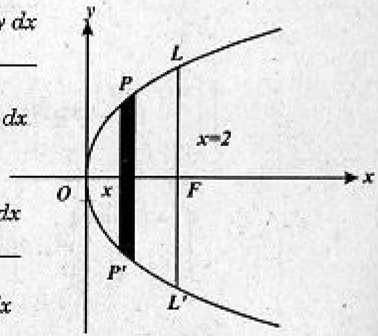
15. Find the centre of gravity of a plate in the form of the segment cut off from the parabola $y^2 = 8x$ by its latus rectum $x = 2$, if the density varies as the distance from the latus rectum.

Sol. Cuzt off a small strip PP' of breadth dx at a distance of x from the vertex and a distance of $2 - x$ from the latus rectum.

The density of the strip $= k\,(2 - x)$, where k is any constant.

Let $(\bar{x}\ ,\ \bar{y})$ be the centre gravity of the centre plate. Then by symmetry $\bar{y} = 0$

$$\bar{x} = \frac{\displaystyle\int_0^2 \int_0^{\sqrt{8x}} k\,(2 - x)\,x\,dy\,dx}{\displaystyle\int_0^2 \int_0^{\sqrt{8x}} k\,(2 - x)\,dy\,dx}$$



$$= \frac{\displaystyle\int_0^2 x\,(2 - x)\,2\sqrt{2}\,x^{1/2}\,dx}{\displaystyle\int_0^2 (2 - x)\,2\sqrt{2}\,x^{1/2}\,dx}$$

$$= \frac{\displaystyle\int_0^2 (2x^{3/2} - x^{5/2})dx}{\displaystyle\int_0^2 (2x^{1/2} - x^{3/2})dx} = \frac{2.\frac{2}{5}\,[x^{5/2}]_0^2 - \frac{2}{7}\,[x^{7/2}]_0^2}{2.\frac{2}{3}\,[x^{3/2}]_0^2 - \frac{2}{5}\,[x^{5/2}]_0^2}$$

$$= \frac{\frac{4}{5}.\,4\sqrt{2} - \frac{2}{7}.\,8\sqrt{2}}{\frac{4}{3}.\,2\sqrt{2} - \frac{2}{5}.\,4\sqrt{2}} = \frac{\frac{16\sqrt{2}}{5} - \frac{16\sqrt{2}}{7}}{\frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}} = \frac{\frac{2}{5} - \frac{2}{7}}{\frac{1}{3} - \frac{1}{5}}$$

$$= \frac{4}{35} \times \frac{15}{2} = \frac{6}{7}$$

Hence the centre of gravity is $\left(\frac{6}{7}, 0\right)$.

16. Find the centre of gravity of a plate in the form of the upper half of the cardioid $r = a\ (1 + \cos\theta\,)$ if the density varies as the distance from the pole.

Sol. Here $r = a\ (1 + \cos\theta\,)$. Let us take any strip at a point P of the plate so that its area is $r\,d\theta\,dr$. Its density $= kr$.

If $(\bar{x}\ ,\bar{y})$ is the centre of gravity of the upper half then

$$\bar{x} = \frac{\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} (r\,d\theta\,dr)\,kr.r\cos\theta}{\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} (r\,d\theta\,dr)\,kr}$$

$$= \frac{\displaystyle\int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r^3\,dr\right]\cos\theta\,d\theta}{\displaystyle\int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r^2\,d\theta\right]d\theta}$$

$$= \frac{\displaystyle\int_0^{\pi}\frac{a^4(1+\cos\theta)^4}{4}.\cos\theta\,d\theta}{\displaystyle\int_0^{\pi}\frac{a^3(1+\cos\theta)^3}{3}\,d\theta}$$

$$= \frac{3a}{4}\,\frac{\displaystyle\int_0^{\pi}(1+\cos\theta)^4\cos\theta\,d\theta}{\displaystyle\int_0^{\pi}(1+\cos\theta)^3\,d\theta}$$

$$= \frac{3a}{4}\,\frac{\displaystyle\int_0^{\pi}16\cos^8\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta}{\displaystyle\int_0^{\pi}8\cos^6\frac{\theta}{2}\,d\theta}$$

$$= \frac{3a}{2}\,\frac{\displaystyle\int_0^{\pi}\left(2\cos^{10}\frac{\theta}{2}-\cos^8\frac{\theta}{2}\right)d\theta}{\displaystyle\int_0^{\pi}\cos^6\frac{\theta}{2}\,d\theta}. \qquad \text{Put } t=\frac{\theta}{2} \text{ or } d\theta=\frac{1}{2}dt$$

$$= \frac{3a}{2}\,\frac{\displaystyle\int_0^{\pi/2}(2\cos^{10}t-\cos^8 t)\,dt}{\displaystyle\int_0^{\pi/2}\cos^6 t\,dt}$$

$$= \frac{3a}{2}\,\frac{\left(2\,\dfrac{9.7.5.3.1}{10.8.6.4.2}-\dfrac{7.5.3.1}{8.6.4.2}\right)\dfrac{\pi}{2}}{\left(\dfrac{5.3.1}{6.4.2}\right)\dfrac{\pi}{2}}$$

$$= \frac{3a}{2}\,\frac{\dfrac{63}{128}-\dfrac{35}{128}}{\dfrac{5}{16}} = \frac{3a}{2}\left(\frac{28}{128}\times\frac{16}{5}\right) = \frac{21a}{20}.$$

$$\bar{y} = \frac{\displaystyle\int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r^3 dr\right]\sin\theta\,d\theta}{\displaystyle\int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r^2 dr\right]d\theta}$$

$$= \frac{3a\displaystyle\int_0^{\pi}(1+\cos\theta)^4\sin\theta\,d\theta}{4\displaystyle\int_0^{\pi}(1+\cos\theta)^3\,d\theta}$$

Put $1+\cos\theta = t$ or $-\sin\theta\,d\theta = dt$

$$\int_0^{\pi} (1+\cos\theta)^4 \sin\theta\, d\theta = -\int_2^0 t^4\, dt = \int_0^2 t^4\, dt = \left|\frac{t^5}{5}\right|_0^2 = \frac{32}{5}$$

$$\int_0^{\pi} (1+\cos\theta)^3\, d\theta = \int_0^{\pi} (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\, d\theta$$

$$= \int_0^{\pi} 1.\, d\theta + 3\int_0^{\pi} \cos^2\theta\, d\theta$$

$$\left[\text{Applying} \int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx + \int_0^{a} f(2a-x)\, dx\right]$$

$$= \pi + 3.2\int_0^{\pi/2} \cos^2\theta\, d\theta = \pi + 6.\frac{1}{2}.\frac{\pi}{2} = \pi + \frac{3\pi}{2} = \frac{5\pi}{2}$$

Hence $\bar{y} = \dfrac{3a}{4}.\dfrac{\frac{32}{5}}{\frac{5\pi}{2}} = \dfrac{3a}{4}\dfrac{64}{25\pi} = \dfrac{48a}{25\pi}$

Centre of gravity $(\bar{x}, \bar{y}) = \left(\dfrac{21a}{20}, \dfrac{48a}{25\pi}\right)$.

17. Find I_x, I_y, I_0 for the area enclosed by the loop of $y^2 = x^2(2-x)$.

Sol. Let A be the area of the loop. Then

$$A = 2\int_0^2 y\, dx = 2\int_0^2 x\sqrt{2-x}\, dx$$

$$= 2\int_0^{\pi/2} (2\sin^2\theta)\sqrt{2}\cos\theta.\, 4\sin\theta\cos\theta\, d\theta$$

(Putting $x = 2\sin^2\theta$ or $dx = 4\sin\theta\cos\theta\, d\theta$)

$$A = 16\sqrt{2}\int_0^{\pi/2} \sin^3\theta\cos^2\theta\, d\theta$$

$$= 16\sqrt{2}\,\frac{2.1}{5.3.1} = \frac{32\sqrt{2}}{15} \qquad (1)$$

$$I_x = \int_0^2 \int_{-x\sqrt{2-x}}^{x\sqrt{2-x}} y^2\, dy\, dx = 2\int_0^2 \int_0^{x\sqrt{2-x}} y^2\, dy\, dx$$

$$= \frac{2}{3}\int_0^2 [y^3]_0^{x\sqrt{2-x}}\, dx = \frac{2}{3}\int_0^2 x^3(2-x)^{3/2}\, dx$$

Put $x = 2\sin^2\theta \Rightarrow dx = 4\sin\theta\cos\theta\, d\theta$. Then

$$I_x = \frac{2}{3}\int_0^{\pi/2} (2\sin 2\theta)^3\, 2^{3/2}\cos^3\theta.\, 4\sin\theta\cos\theta\, d\theta$$

$$= \frac{128\sqrt{2}}{3}\int_0^{\pi/2} \sin^7\theta\cos^4\theta\, d\theta$$

$$= \frac{128\sqrt{2}}{3}\,\frac{6.4.2.3}{11.9.7.5.3.1}$$

$$= 60A\left(\frac{16}{11.9.7.5}\right) \qquad \text{from (1)}$$

$$= \frac{64}{231}A$$

$$I_y = \int_0^2 \int_{-x\sqrt{2-x}}^{x\sqrt{2-x}} x^2\, dy\, dx = 2\int_0^2 \left[\int_0^{x\sqrt{2-x}} dy\right] x^2\, dx$$

$$= 2\int_0^2 x\sqrt{2-x}\, x^2\, dx = 2\int_0^2 x^3\sqrt{2-x}\, x^2\, dx$$

$$= 2\int_0^{\pi/2} 8\sin^6\theta\sqrt{2}\cos\theta.\, 4\sin\theta\cos\theta\, d\theta$$

(on putting $x = 2\sin^2\theta$ or $dx = 4\sin\theta\cos\theta\, d\theta$)

$$I_y = 64\sqrt{2}\int_0^{\pi/2} \sin^7\theta\cos^2\theta\, d\theta$$

$$= 64\sqrt{2}\,\frac{6.4.2.1}{9.7.5.3.1} = (30A)\left(\frac{16}{9.7.5}\right) = \frac{32}{21}A.$$

$$I_0 = \int_0^2 \int_{-x\sqrt{2-x}}^{x\sqrt{2-x}} (x^2+y^2)\, dy\, dx = 2\int_0^2 \left[\int_0^{x\sqrt{2-x}} (x^2+y^2)\, dy\right] dx$$

$$= 2\int_0^2 \left[x^2y + \frac{y^3}{3}\right]_{-x\sqrt{2-x}}^{x\sqrt{2-x}} dx$$

$$= 2\int_0^2 x^2 .\, x\sqrt{2-x}\, dx + \frac{2}{3}\int_0^2 x^3(2-x)^{3/2}\, dx$$

$$= 2\int_0^2 x^3\sqrt{2-x}\, dx + \frac{2}{3}\int_0^2 x^3(2-x)^{3/2}\, dx$$

$$= 2\int_0^2 x^3\sqrt{2-x}\left(1+\frac{2-x}{3}\right) dx = 2\int_0^2 x^3\sqrt{2-x}\left(\frac{5-x}{3}\right) dx$$

$$= \frac{10}{3}\int_0^2 x^3\sqrt{2-x}\, dx - \frac{2}{3}\int_0^2 x^4\sqrt{2-x}\, dx$$

$$= \frac{10}{3}\int_0^{\pi/2} 8\sin^6\theta.\ \sqrt{2}\cos\theta.\ 4\sin\theta\cos\theta\, d\theta$$

$$-\frac{2}{3}\int_0^{\pi/2} 16\sin^8\theta.\ \sqrt{2}\cos\theta.\ 4\sin\theta\ \cos\theta\, d\theta$$

$$\left(\begin{array}{l}\text{Putting } x = 2\sin^2\theta \\ \text{or } dx = 4\sin\theta\cos\theta\, d\theta\end{array}\right)$$

$$= \frac{320\sqrt{2}}{3}\int_0^{\pi/2} \sin^7\theta\cos^2\theta\, d\theta - \frac{128\sqrt{2}}{3}\int_0^{\pi/2} \sin^9\theta\cos^2\theta\, d\theta$$

$$= \frac{320\sqrt{2}}{3}\ \frac{6.4.2.1.}{9.7.5.3.1} - \frac{128\sqrt{2}}{3}\ \frac{8.6.4.2.1}{11.9.7.5.3.1}$$

$$= \frac{30A}{3}.\frac{16}{63} - \frac{60A}{3}\ \frac{8\times 16}{99+35} = \frac{160A}{63} - \frac{512A}{693}$$

$$= \left(\frac{1760-512}{693}\right)A = \frac{1248}{693}A = \frac{416}{231}A.$$

18. Find I_x and I_y of the area of

(a) the circle $r = 2(\sin\theta + \cos\theta)$

(b) one loop of $r^2 = \cos 2\theta$.

Sol.

(a) The given equation can be written as

$$r^2 = 2(r\sin\theta + r\cos\theta)$$

$$\Rightarrow\quad x^2 + y^2 = 2(y + x) \quad\Rightarrow\quad (x-1)^2 + (y-1)^2 = 2$$

Total area of the circle $= \pi(\sqrt{2})^2 = 2\pi$

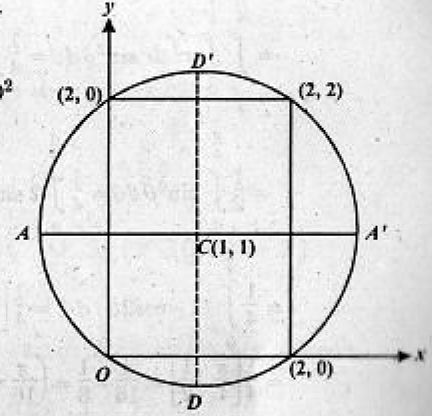
M.I. about $AA' = \dfrac{2\pi.2}{4} = \pi$

M.I. about $Ox = I_x$

$$= \pi + 2\pi(1)^2$$
$$= \pi + 2\pi$$
$$= 3\pi$$

Similarly, $I_y = 3\pi$

Hence $I_x = I_y = 3\pi$



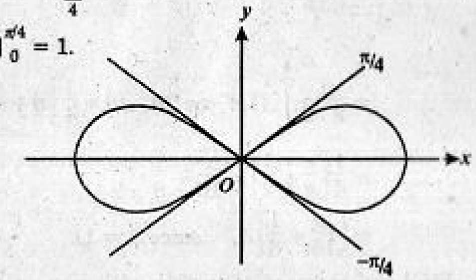
But $A = 2\pi$, where A is the area of the circle. Therefore,

$$I_x = I_y = \frac{3A}{2}.$$

(b) Let A be the area of the loop. Then

$$A = 2.\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta\, d\theta = 2\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta$$

$$= [\sin 2\theta]_0^{\pi/4} = 1.$$



$$I_x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r^2 \sin^2\theta\, r\, d\theta\, dr$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r^3\, dr \sin^2\theta\, d\theta = \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{4}} 2\sin^2\theta\, d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\, d\theta = \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{16} - \frac{1}{8} = \left(\frac{\pi}{16} - \frac{1}{8}\right)A, \text{as } A = 1$$

$$I_y = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r^2 \cos^2\theta\, d\theta\, dr = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r^3\, dr \cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta$$

$$= \frac{1}{2\,.\,2}\int_0^{\frac{\pi}{4}} (1+\cos 2\theta)\, d\theta = \frac{1}{4}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + \frac{1}{2}\right] = \frac{\pi}{16} + \frac{1}{8}$$

$$= \left(\frac{\pi}{16} + \frac{1}{8}\right)A, \quad (\text{since } A = 1).$$

19. Find the moment of inertia with respect to the x-axis of a plate having for its edges one arch of the curve $y = \sin x$ and the x-axis if its density varies as the distance from the x-axis.

Sol. Here $A = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} ky\, dy\, dx = \frac{k}{2}\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{k}{2}\frac{1}{2}\cdot\frac{\pi}{2} = \frac{k\pi}{8}$

or $\quad k = \frac{8A}{\pi}.$

$$I_x = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} y^2 \cdot ky\, dy\, dx = \frac{k}{4}\int_0^{\frac{\pi}{2}} \sin^4 x\, dx$$

$$= \frac{k}{4}\frac{3\,.\,1}{4\,.\,2}\frac{\pi}{2} = \frac{3\pi k}{64} = \frac{3\pi}{64}\left(\frac{8A}{\pi}\right) = \frac{3}{8}A.$$

Exercise Set 10.3 (Page 477)

1. Evaluate in SIX different ways

$I = \iiint_S (x + 2y + 4z)\, dx\, dy\, dz$, where S is defined by

$1 \le x \le 2,\ -1 \le y \le 0,\ 0 \le z \le 3$

Sol. $I = \int_1^2 \int_{-1}^0 \int_0^3 (x + 2y + 4z)\, dz\, dy\, dx$

$$= \int_1^2 \int_{-1}^0 [(x + 2y)z + 2z^2]_0^3\, dy\, dx$$

$$= \int_1^2 \int_{-1}^0 (18 + 3x + 6y)\, dy\, dx = \int_1^2 [18y + 3xy + 3y^2]_{-1}^0\, dx$$

$$= \int_1^2 (15 + 3x)\, dx = \left[15x + \frac{3x^2}{2}\right]_1^2 = \frac{39}{2}.$$

The other five orders of integration are similar.

2. Evaluate $I = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} dz\, dy\, dx$. Also change the order of integration so that z-integration is performed last and find its value.

Sol. $I = \int_0^4 \int_0^{4-x} (4-x-y)\,dy\,dx = \int_0^4 \left[4y - xy - \frac{y^2}{2}\right]_0^{4-x} dx$

$$= \int_0^4 \left[4(4-x) - x(4-x) - \frac{(4-x)^2}{2}\right] dx$$

$$= \int_0^4 \left(8 - 4x + \frac{x^2}{2}\right) dx = \left[8x - 2x^2 + \frac{x^3}{6}\right]_0^4 = \frac{32}{3}.$$

If z – integration is performed last, then

$$I = \int_0^4 \int_0^{4-z} \int_0^{4-y-z} dx\,dy\,dz \quad \text{and} \quad I = \int_0^4 \int_0^{4-z} \int_0^{4-x-z} dy\,dx\,dz$$

$$\int_0^4 \int_0^{4-z} \int_0^{4-y-z} dx\,dy\,dz = \int_0^4 \int_0^{4-z} (4-y-z)\,dy\,dz$$

$$= \int_0^4 4y - \frac{y^2}{2} - yz\Big|_0^{4-z} dz = \int_0^4 \left[4(4-z) - \frac{1}{2}(4-z)^2 - z(4-z)\right] dz$$

$$= \int_0^4 \left(8 - 4z + \frac{z^2}{2}\right) dz = 8z - 4\frac{z^2}{2} + \frac{z^3}{6}\Big]_0^4 = 32 - 32 + \frac{64}{6} = \frac{32}{3}$$

Similarly,

$$\int_0^4 \int_0^{4-z} \int_0^{4-x-z} dy\,dx\,dz = \frac{32}{3}.$$

Evaluate (Problems 3 – 10):

3. $\int_0^2 \int_0^1 \int_0^1 xyz\sqrt{2 - x^2 - y^2}\,dx\,dy\,dz$

Sol. $I = \int_0^2 \int_0^1 yz\left[-\frac{1}{3}(2 - x^2 - y^2)^{3/2}\right]_0^1 dy\,dz$

$$= \int_0^2 \int_0^1 yz\left[-\frac{1}{3}(1 - y^2)^{3/2} + \frac{1}{3}(2 - y^2)^{3/2}\right] dy\,dz$$

$$= \int_0^2 z\left[\frac{1}{15}(1 - y^2)^{5/2} - \frac{1}{15}(2 - y^2)^{5/2}\right]_0^1 dz$$

$$= \int_0^2 z\left(\frac{4\sqrt{2}}{15} - \frac{2}{15}\right) dz = \left[\left(\frac{4\sqrt{2}}{15} - \frac{2}{15}\right)\frac{z^2}{2}\right]_0^2$$

$$= 2\left(\frac{4\sqrt{2}}{15} - \frac{2}{15}\right) = \frac{4}{15}(2\sqrt{2} - 1).$$

4. $\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} x\,dz\,dx\,dy$

Sol. $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} x\,[z]_0^{\sqrt{a^2-x^2-y^2}}\,dx\,dy$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} x\sqrt{a^2 - x^2 - y^2}\,dx\,dy$$

$$= \int_0^a \left[-\frac{1}{3}(a^2 - x^2 - y^2)^{3/2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a -\frac{1}{3}[0 - (a^2 - y^2)^{3/2}]\,dy = \frac{1}{3}\int_0^a (a^2 - y^2)^{3/2}\,dy$$

Put $y = a\sin\theta$ so that $dy = a\cos\theta\,d\theta$ and

$$I = \frac{1}{3}\int_0^{\pi/2} a^4\cos^4\theta\,d\theta = \frac{a^4}{3}\left[\frac{3}{4.\,2}\cdot\frac{\pi}{2}\right] = \frac{\pi a^4}{16}.$$

5. $\int_0^2 \int_0^{\sqrt{4-z^2}} \int_{y^2+z^2-4}^{4-y^2-z^2} dx\,dy\,dz$

Sol. $I = \int_0^2 \int_0^{\sqrt{4-z^2}} [x]_{y^2+z^2-4}^{4-y^2-z^2}\,dy\,dz$

$$= 2\int_0^2\int_0^{\sqrt{4-z^2}} (4-z^2-y^2)\,dy\,dz = 2\int_0^2\left[(4-z^2)\,y-\frac{y^3}{3}\right]_0^{\sqrt{4-z^2}} dz$$

$$= \frac{4}{3}\int_0^2 (4-z^2)^{3/2}\,dz$$

Put $z = 2\sin\theta$ so that $dz = 2\cos\theta\,d\theta$ and

$$= I = \frac{4}{3}\int_0^{\pi/2} 8.2\cos^4\theta\,d\theta = \frac{64}{3}\cdot\frac{3}{4.2}\cdot\frac{\pi}{2}\;4\pi.$$

6. $\iiint_S z\,dx\,dy\,dz$, S bounded by

$z = \sqrt{x^2+y^2}$, $z = 0, x = \pm 1, y = \pm 1.$

Sol. $I = \int_{-1}^{1}\int_{-1}^{1}\int_0^{\sqrt{x^2+y^2}} z\,dz\,dy\,dx = \int_{-1}^{1}\int_{-1}^{1}\left[\frac{z^2}{2}\right]_0^{\sqrt{x^2+y^2}} dy\,dx$

$$= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} (x^2+y^2)\,dy\,dx = \frac{1}{2}\int_{-1}^{1}\left[x^2y+\frac{y^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\int_{-1}^{1}\left(2x^2+\frac{2}{3}\right)dx = \frac{1}{2}\left[2\frac{x^3}{3}+\frac{2}{3}x\right]_{-1}^{1} = \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}.$$

7. $\iiint_S 15x^2z^2\,dx\,dy\,dz$, S bounded by

$x^2+y^2 = 1, x^2+z^2 = 1.$

Sol. $I = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 15z^2x^2\,dz\,dy\,dx$

$$= 15\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left[x^2\frac{z^3}{3}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx$$

$$= 15\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{2}{3}x^2(1-x^2)^{3/2}\,dy\,dx$$

$$= 10\int_{-1}^{1}\left[x^2(1-x)^{3/2}y\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 10\int_{-1}^{1} 2x^2(1-x^2)^{3/2}\sqrt{1-x^2}\,dx$$

$$= 20\int_{-1}^{1} x^2(1-x^2)^2\,dx = 20\int_{-1}^{1}(x^2-2x^4+x^6)\,dx$$

$$= 20\left[\frac{x^3}{3}-\frac{2x^5}{5}+\frac{x^7}{7}\right]_{-1}^{1}$$

$$= 20\left[\frac{1}{3}-\frac{2}{5}+\frac{1}{7}-\left(-\frac{1}{3}+\frac{2}{5}-\frac{1}{7}\right)\right] = 20\times 2\left[\frac{8}{3\times 5\times 7}\right] = \frac{64}{21}.$$

8. $\iiint_S x^2y^2z\,dx\,dy\,dz$, S defined by.

$0 \le z \le x^2-y^2, 0 \le x \le 1, 0 \le y \le 1$

Sol. $I = \int_0^1\int_0^1\int_0^{x^2-y^2} x^2y^2z\,dz\,dy\,dx$

$$= \int_0^1\int_0^1\left[x^2y^2\frac{z^2}{2}\right]_0^{x^2-y^2} dy\,dx = \frac{1}{2}\int_0^1\int_0^1 x^2y^2(x^4-2x^2y^2+y^4)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\left[x^6\frac{y^3}{3}-2x^4\frac{y^5}{5}+x^2\frac{y^7}{7}\right]_0^1 dx = \frac{1}{2}\int_0^1\left(\frac{x^6}{3}-\frac{2x^4}{5}+\frac{x^2}{7}\right)dx$$

$$= \frac{1}{2}\left[\frac{x^7}{21}-\frac{2x^5}{25}+\frac{x^3}{21}\right]_0^1 = \frac{1}{2}\left[\frac{1}{21}-\frac{2}{25}+\frac{1}{21}\right] = \frac{4}{525}.$$

9. $\iiint_S (x+1)\,dx\,dy\,dz$, S defined by

$y = 0, y = x$ for $0 \le x \le 1$ and $-y^2 \le z \le x^2$.

Sol. $I = \int_0^1\int_0^x\int_{-y^2}^{x^2} (x+1)\,dz\,dy\,dx = \int_0^1\int_0^x [(x+1)z]_{-y^2}^{x^2}\,dy\,dx$

$$= \int_0^1\int_0^x (x+1)(x^2+y^2)\,dy\,dx = \int_0^1 \left[(x+1)\,x^2y + (x+1)\frac{y^3}{3}\right]_0^x dx$$

$$= \int_0^1 \left[(x+1)x^3 + (x+1)\frac{x^3}{3}\right]_0^x dx = \frac{4}{3}\int_0^1 (x^4+x^3)\,dx$$

$$= \frac{4}{3}\left[\frac{x^5}{5}+\frac{x^4}{4}\right]_0^1 = \frac{4}{3}\left[\frac{1}{5}+\frac{1}{4}\right] = \frac{3}{5}.$$

10. $\iiint_S yz\,dx\,dy\,dz$: S in the first octant bounded above by $z = 1$ and below by $z = \sqrt{x^2+y^2}$

Sol. Since S is in the first octant, the region D in the xy-plane is also in the first quadrant. The surfaces $z = 1$ and $z = \sqrt{x^2+y^2}$ intersect in the curve $1 = \sqrt{x^2+y^2}$ in the xy-plane.

$$I = \int_0^1\int_0^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^1 yz\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{\sqrt{1-y^2}} \left[y\frac{z^2}{2}\right]_{\sqrt{x^2+y^2}}^1 dx\,dy$$

$$= \int_0^1\int_0^{\sqrt{1-y^2}} \frac{y}{2}(1-x^2-y^2)\,dx\,dy$$

$$= \int_0^1 \frac{y}{2}\left[x-\frac{x^3}{3}-y^2x\right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{y}{2}\left[\sqrt{1-y^2}\,(1-y^2)-\frac{(1-y^2)^{3/2}}{3}\right]dy$$

$$= \frac{1}{3}\int_0^1 y(1-y^2)^{3/2}\,dy = \frac{1}{3}\left[\frac{(1-y^2)^{3/2}}{-5}\right]_0^1 \frac{1}{3}\cdot\frac{1}{5} = \frac{1}{15}.$$

Find the volume of the given solid (Problems 11 – 13):

11. Bounded by the coordinate planes and $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}} = 1.$

Sol. Required volume

$$= \int_0^{a}\int_0^{b(1-\sqrt{x/a})^2}\int_0^{c(1-\sqrt{x/a}-\sqrt{x/b})^2} dz\,dy\,dx$$

$$= \int_0^a\int_0^{b(1-\sqrt{x/a})^2} c\left(1-\sqrt{\frac{x}{a}}-\sqrt{\frac{y}{b}}\right)^2 dy\,dx$$

$$= \int_0^a\int_0^{b(1-\sqrt{x/a})^2} c\left[\left(1-\sqrt{\frac{x}{a}}\right)^2-2\left(1-\sqrt{\frac{x}{a}}\right)\sqrt{\frac{y}{b}}+\frac{y}{b}\right]dy\,dx$$

$$= \int_0^a c\left[\left(1-\sqrt{\frac{x}{a}}\right)^2 y-2\left(1-\sqrt{\frac{x}{a}}\right)\frac{2}{3}\cdot\frac{y^{3/2}}{\sqrt{b}}+\frac{y^2}{2b}\right]_0^{b(1-\sqrt{x/a})^2} dx$$

$$= c\int_0^a\left[b\left(1-\sqrt{\frac{x}{a}}\right)^4-\frac{4b^{3/2}}{3\sqrt{b}}\left(1-\sqrt{\frac{x}{a}}\right)^4+\frac{b^2}{2b}\left(1-\sqrt{\frac{x}{a}}\right)^4\right]dx$$

$$= c\int_0^a\left(1-\sqrt{\frac{x}{a}}\right)^4\left(b-\frac{4}{3}b+\frac{b}{2}\right)dx$$

$$= \frac{1}{6}bc\int_0^a\left[1-4\sqrt{\frac{x}{a}}+6\frac{x}{a}-4\left(\frac{x}{a}\right)^{3/2}+\left(\frac{x}{a}\right)^2\right]dx$$

$$= \frac{1}{6}bc\left[x-\frac{4}{\sqrt{a}}\frac{x^{3/2}}{3/2}+6\frac{x^2}{2a}-\frac{4}{a^{3/2}}\cdot\frac{x^{5/2}}{5/2}+\frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{1}{6}bc\left[a-\frac{8}{3}a+3a-\frac{8}{5}a+\frac{1}{3}a\right] = \frac{1}{6}abc\left[4-\frac{8}{3}-\frac{8}{5}+\frac{8}{5}+\frac{1}{3}\right] = \frac{abc}{90}$$

12. Bounded above by $z = 4-x^2-y^2$ and below by $z = 4-2x$.

Sol. The region D in the xy-plane is the curve of intersection of the surfaces $z = 4-x^2-y^2$ and $z = 4-2x$

i.e., $4 - 2x = 4 - x^2 - y^2$ or $x^2 + y^2 - 2x = 0$.

Thus $-\sqrt{2x - x^2} \le y \le \sqrt{2x - x^2}$; $0 \le x \le 2$.

Required volume

$$V = \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_{4-2x}^{4-x^2-y^2} dz\,dy\,dx$$

$$= \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (4 - x^2 - y^2 - 4 + 2x)\,dy\,dx$$

$$= \int_0^2 \left[(2x - x^2)\,y - \frac{y^3}{3}\right]_{-\sqrt{2x-x^2}}^{\sqrt{2x-x}} dx$$

$$= \int_0^2 \left[2(2x - x^2)^{3/2} - \frac{2}{3}(2x - x^2)^{3/2}\right] dx = \frac{4}{3}\int_0^2 (2x - x^2)^{3/2}\,dx$$

Put $x - 1 = X$ so that $dx = dX$

$$V = \frac{4}{3}\int_{-1}^{1} (1 - X^2)^{3/2}\,dX$$

Now put $X = \sin\theta$; $dX = \cos\theta\,d\theta$ so that

$$V = \frac{4}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{8}{3}\cdot\frac{3}{4\,.\,2}\cdot\frac{\pi}{2} = \frac{\pi}{2}.$$

13. Bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol. Required volume

$$= 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx$$

$$= 8c\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\;dy\,dx$$

$$= 8c\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} \left[\frac{b^2}{a^2}\left(\frac{a^2 - x^2}{b^2}\right) - \frac{y^2}{b^2}\right]^{1/2} dy\,dx$$

$$= 8c\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} \frac{1}{b}\sqrt{\frac{b^2}{a^2}(a^2 - x^2) - y^2}\;dy\,dx$$

$$= \frac{8c}{b}\int_0^a \left[y\sqrt{\frac{b^2}{a^2}(a^2 - x^2) - y^2} + \frac{b^2(a^2 - x^2)}{2a^2}\arcsin\left(\frac{y}{\frac{b}{a}\sqrt{a^2 - x^2}}\right)\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{8c}{b}\int_0^a \left(0 + \frac{b^2(a^2 - x^2)}{2a^2}\cdot\frac{\pi}{2}\right) dx = \frac{2\pi b\,c}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4\pi abc}{3}.$$

14. $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{1} z(x^2 + y^2)\,dz\,dy\,dx$

by changing to cylindrical coordinates.

Sol. The solid is bounded below by $z = x^2 + y^2$ and above $z = 1$. The region D in the xy-plane is $x^2 + y^2 = 1$. Changing into cylindrical coordinates, we have

$$I = \int_0^{2\pi}\int_0^1\int_{r^2}^1 (zr^2)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[r^3\cdot\frac{z^2}{2}\right]_{r^2}^1 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\left(\frac{1}{2} - \frac{r^2}{2}\right) dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\left[\frac{r^4}{4} - \frac{r^6}{6}\right]_0^1 d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12}$$

15. Use cylindrical coordinates to evaluate

$I = \iiint_S z\sqrt{x^2 + y^2}\,dV$, where S is the hemisphere

$x^2 + y^2 + z^2 \le 4$, $z \ge 0$.

Sol. $I = \int_0^2 \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} (z\,r)\, r\, dz\, d\theta\, dr$

$= \int_0^2 \int_0^{2\pi} \left[r^2 \cdot \frac{z^2}{2} \right]_0^{\sqrt{4-r^2}} d\theta\, dr = \int_0^2 \int_0^{2\pi} \left(2r^2 - \frac{1}{2} r^4 \right) d\theta\, dr$

$= \int_0^2 \left[\left(2r^2 - \frac{1}{2} r^4 \right) \theta \right]_0^{2\pi} dr = \pi \int_0^2 (4r^2 - r^4)\, dr = \pi \left[\frac{4}{3} r^3 - \frac{r^5}{5} \right]_0^2$

$= \pi \left[\frac{32}{3} - \frac{32}{5} \right] = \frac{64}{15}.$

16. Evaluate $I = \int\int_S\int \sqrt{x^2+y^2}\, dV$, where S is bounded above by the plane $y + z = 4$, below by $z = 0$ and on the sides by $x^2 + y^2 = 16$.

Sol. Changing into cylindrical coordinates, we note that

$z = 4 - y = 4 - r\cos\theta;\ 0 \le r \le 4$

$I = \int_0^{2\pi} \int_0^4 \int_0^{4 - r\cos\theta} r.r\, dz\, dr\, d\theta$

$= \int_0^{2\pi} \int_0^4 r^2 (4 - r\cos\theta)\, dr\, d\theta = \int_0^{2\pi} \left[4\frac{r^3}{3} - \frac{r^4}{4} \cos\theta \right]_0^4 d\theta$

$= \int_0^{2\pi} \left[\frac{256}{3} - 64\cos\theta \right] d\theta = \left[\frac{256}{3}\theta - 64\sin\theta \right]_0^{2\pi} = \frac{512}{3}\pi.$

17. Use spherical coordinates to evaluate

$I = \int\int_S\int z\sqrt{x^2+y^2+z^2}\, dx\, dy\, dz$, where S is defined by

$\sqrt{x^2+y^2} \le z \le \sqrt{1 - x^2 - y^2}.$

Sol. S is bounded below by the cone $z^2 = x^2 + y^2$ and above by the hemisphere $z^2 = 1 - x^2 - y^2$. In spherical coordinates equation of the cone becomes

$\rho^2 \cos^2\phi = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin^2\phi$

or $\tan^2\phi = 1$ giving $\phi = \frac{\pi}{4}.$

The hemisphere has the equation $\rho = 1$

$I = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 (\rho\cos\phi)\, \rho\, (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$

$= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{\rho^5}{5} \right]_0^1 \cos\phi \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \cos\phi \sin\phi\, d\phi\, d\theta$

$= \frac{1}{5} \int_0^{2\pi} \left[\frac{\sin^2\phi}{2} \right]_0^{\pi/4} d\theta = \frac{1}{5} \cdot \frac{1}{4} \int_0^{2\pi} d\theta = \frac{\pi}{10}.$

18. Evaluate $I = \int\int_S\int \frac{dx\, dy\, dz}{x^2+y^2+z^2}$, where S is the region above $z = 0$ bounded by the cone $z = \sqrt{3x^2 + 3y^2}$ and the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 25$.

Sol. Equations of the spheres in spherical coordinates are $\rho = 3$ and $\rho = 5$. Equation of the cone is $z^2 = 3(x^2 + y^2)$

i.e., $\rho^2 \cos^2\phi = 3(\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta)$

or $\tan^2\phi = \frac{1}{3}$, *i.e.,* $\phi = \frac{\pi}{6}$

$I = \int_0^{2\pi} \int_0^{\pi/6} \int_3^5 \frac{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}{\rho^2} = 2 \int_0^{2\pi} \int_0^{\pi/6} \sin\phi\, d\phi\, d\theta$

$= 2 \int_0^{2\pi} [-\cos\phi]_0^{\pi/6}\, d\theta = 2 \int_0^{2\pi} \left(1 - \frac{\sqrt{3}}{2} \right) d\theta = 2\pi(2 - \sqrt{3}).$

19. Evaluate $I = \int\int_S\int \sqrt{z}\, dx\, dy\, dz$, where S is in first octant bounded by $x^2 + y^2 + z^2 = 16$ and the planes $z = 0$, $x = \sqrt{3}\,y$, $x = y$.

Sol. Changing into spherical coordinates, we have

$I = \int_0^{\pi/2} \int_{\pi/6}^{\pi/4} \int_0^4 \sqrt{\rho\cos\phi}\, (\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi$

$$= \frac{2}{7}\int_{0}^{\pi/2}\int_{\pi/6}^{\pi/4}[\rho^{7/2}]_0^4\sqrt{\cos\phi}\,\sin\phi\, d\theta\, d\phi$$

$$= \frac{2}{7}(4)^{7/2}\int_{0}^{\pi/2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\sqrt{\cos\phi}\,\sin\phi\, d\phi$$

$$= \frac{2}{7}\,.\,128\,.\,\frac{\pi}{12}\left[-\frac{(\cos\phi)^{3/2}}{3/2}\right]_0^{\pi/2} = \frac{64\pi}{21}\times\frac{2}{3} = \frac{128\pi}{63}$$

Alternative method: Changing into cylinderical coordinates, we get

$$I = \int_{\pi/6}^{\pi/4}\int_{0}^{4}\int_{0}^{\sqrt{16-r^2}}\sqrt{z}\, r\, dz\, dr\, d\theta$$

$$= \int_{\pi/6}^{\pi/4}\int_{0}^{4}\frac{2}{3}(\sqrt{16-r^2}\,)^{3/2}\, r\, dr\, d\theta = \int_{\pi/6}^{\pi/4}\frac{1}{3}\left[-\frac{(16-r^2)^{7/4}}{7/4}\right]_0^4 d\theta$$

$$= \frac{1}{3}\times\frac{4}{7}\,.\,2^7\int_{\pi/6}^{\pi/4}d\theta = \frac{4\times 128}{21}\times\frac{\pi}{12} = \frac{128\pi}{63}.$$

20. Find the volume bounded by the torus $\rho = 3\sin\phi$

Sol. Required volume

$$= \int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{3\sin\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_{0}^{2\pi}\int_{0}^{\pi}\left[\frac{\rho^3}{3}\right]_0^{3\sin\phi}\sin\phi\, d\phi\, d\theta$$

$$= 9\int_{0}^{2\pi}\int_{0}^{\pi}\sin^4\phi\, d\phi\, d\theta = 9\int_{0}^{2\pi}\int_{0}^{\pi}\left(\frac{1-\cos 2\phi}{2}\right)^2 d\phi\, d\theta$$

$$= 9\int_{0}^{2\pi}\int_{0}^{\pi}\left[\frac{3-4\cos 2\phi + \cos 4\phi}{8}\right]d\phi\, d\theta$$

$$= \frac{9}{8}\int_{0}^{2\pi}\left[3\phi - \frac{4\sin 2\phi}{2} + \frac{\sin 4\phi}{4}\right]_0^{\pi}d\theta = \frac{9}{8}\int_{0}^{2\pi}3\pi\, d\theta = \frac{27}{4}\pi.$$

Exercise Set 10.4 (Page 480)

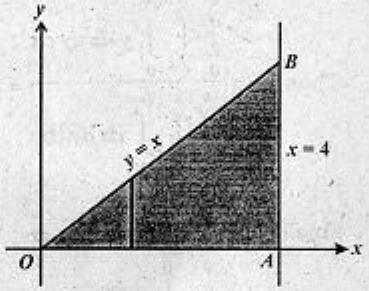
1. Find the centroid of each of the following volumes:

(a) Under $z^2 = xy$ and above the triangle $y = x$, $y = 0$, $x = 4$.

Sol. The base is the triangle OAB. This can be swept if we vary x from 0 to 4 and y from 0 to x (*i.e.*, from x-axis to the line $y = x$).

The variation of z from the triangle OAB to the surface $z^2 = xy$ implies z varies from 0 to $\sqrt{xy}$.

Let $G(\bar{x}, \bar{y}, \bar{z})$ be the required centroid.



$$\int_{0}^{4}\int_{0}^{x}\int_{0}^{\sqrt{xy}}dz\, dy\, dx$$

$$= \int_{0}^{4}\int_{0}^{x}[z]_0^{\sqrt{xy}}\, dy\, dx$$

$$= \int_{0}^{4}\int_{0}^{x}\sqrt{x}\sqrt{y}\, dy\, dx$$

$$= \int_{0}^{4}\sqrt{x}\left[\frac{y^{3/2}}{3/2}\right]_0^x dx = \frac{2}{3}\int_{0}^{4}x^2 dx = \frac{2}{3}\left[\frac{x^3}{3}\right]_0^4 = \frac{128}{9}$$

Now, $$\int_{0}^{4}\int_{0}^{x}\int_{0}^{\sqrt{xy}}x\, dz\, dy\, dx = \int_{0}^{4}\int_{0}^{x}x\,|z|_0^{\sqrt{xy}}\, dy\, dx = \int_{0}^{4}\int_{0}^{x}x^{3/2}y^{1/2}\, dy\, dx$$

$$= \int_{0}^{4}x^{3/2}\left[\frac{y^{3/2}}{3/2}\right]_0^x dx = \frac{2}{3}\int_{0}^{4}x^3\, dx = \frac{2}{3}\left[\frac{x^4}{4}\right]_0^4 = \frac{128}{3}.$$

Hence $$\bar{x} = \frac{\displaystyle\int_{0}^{4}\int_{0}^{x}\int_{0}^{\sqrt{xy}}x\, dx\, dy\, dx}{\displaystyle\int_{0}^{4}\int_{0}^{x}\int_{0}^{\sqrt{xy}}dz\, dy\, dx} = \frac{128/3}{128/9} = 3$$

Again, $\int_0^4\int_0^x\int_0^{\sqrt{xy}} y\,dz\,dy\,dx$

$$= \int_0^4\int_0^x y\,[z]_0^{\sqrt{xy}}\,dy\,dx = \int_0^4\int_0^x \sqrt{x}\,y^{3/2}\,dy\,dx$$

$$= \int_0^4 \sqrt{x}\left[\frac{y^{5/2}}{5/2}\right]_0^x dx = \frac{2}{3}\int_0^4 x^3 dx = \frac{2}{5}\left[\frac{x^4}{4}\right]_0^4 = \frac{128}{5}$$

Thus $\bar{y} = \dfrac{\int_0^4\int_0^x\int_0^{\sqrt{xy}} y\,dz\,dy\,dx}{\int_0^4\int_0^x\int_0^{\sqrt{xy}} dz\,dy\,dx} = \dfrac{128/5}{128/9} = \dfrac{9}{5}$

Finally, $\int_0^4\int_0^x\int_0^{\sqrt{xy}} z\,dz\,dy\,dx$

$$= \int_0^4\int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{xy}} dy\,dx = \frac{1}{2}\int_0^4\int_0^x xy\,dy\,dx$$

$$= \frac{1}{2}\int_0^4 x\left[\frac{y^2}{2}\right]_0^x dx = \frac{1}{2}\int_0^4 x^3 dx = \frac{1}{4}\left[\frac{x^4}{4}\right]_0^4 = 16$$

So, $\bar{z} = \dfrac{\int_0^4\int_0^x\int_0^{\sqrt{xy}} z\,dz\,dy\,dx}{\int_0^4\int_0^x\int_0^{\sqrt{xy}} dz\,dy\,dx} = \dfrac{16}{128/9} = \dfrac{144}{128} = \dfrac{9}{8}$

Thus $G(\bar{x}, \bar{y}, \bar{z}) = G\left(3, \frac{9}{5}, \frac{9}{8}\right)$.

1. **(b)** within the cylinder $r = 2\cos\theta$, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$.

Sol. Equation of the cylinder is given in the cylindrical polar coordinates as $r = 2\cos\theta$. (1)

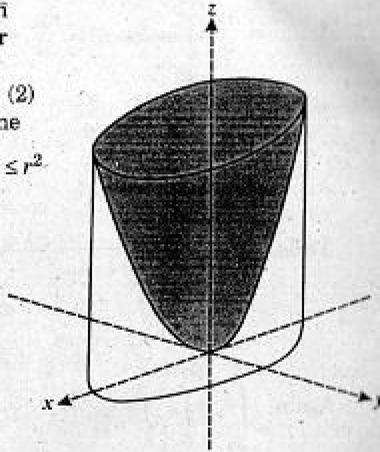
Equation of the given paraboloid in cylindrical polar coordinates is

$$z = r^2 \qquad (2)$$

Clearly, for the required volume

$0 \le r \le 2\cos\theta, \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}, 0 \le z \le r^2$

Let $G(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the volume bounded by the cylinder (1), under the paraboloid (2) and above the plane $z = 0$ (*i.e.*, the *xy*-plane)



Now, $\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r|z|_0^{r^2}\,dr\,d\theta$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left|\frac{r^4}{4}\right|_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = 4.2\int_0^{\pi/2}\cos^4\theta\,d\theta = 8.\frac{3.1\pi}{4.2\,2} = \frac{3\pi}{2}$$

Also, $\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r\cos\theta.\,r\,dz\,dr\,d\theta$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,|z|_0^{r^2}\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^4\cos\theta\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left|\frac{r^5}{5}\right|_0^{2\cos\theta}\cos\theta\,d\theta$$

$$= \frac{32}{5} \int_{-\pi/2}^{\pi/2} \cos^6\theta \, d\theta = \frac{32}{5} \cdot 2 \int_{0}^{\pi/2} \cos^6\theta \, d\theta$$

$$= \frac{64}{5} \cdot \frac{5 \, . \, 3 \, . \, 1}{6 \, . \, 4 \, . \, 2} = \frac{3\pi}{2}$$

Thus, $$\bar{x} = \frac{\displaystyle\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \int_{0}^{r^2} r\cos\theta \, r \, dz \, dr \, d\theta}{\displaystyle\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \int_{0}^{r^2} r \, dz \, dr \, d\theta} = \frac{2\pi}{3\pi/2} = \frac{4}{3}.$$

Again, $$\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \int_{0}^{r^2} r\sin\theta \, r \, dz \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} r^2 \sin\theta \, |z|_0^{r^2} \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} r^4 \sin\theta \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left|\frac{r^5}{5}\right|_0^{2\cos\theta} \sin\theta \, d\theta$$

$$= \frac{32}{5} \int_{-\pi/2}^{\pi/2} \cos^6\theta \, \sin\theta \, d\theta = 0, \text{ as } \cos^5\theta \sin\theta \text{ is odd function of } \theta.$$

Hence $$\bar{y} = \frac{\displaystyle\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \int_{0}^{r^2} r\sin\theta \, r \, dz \, dr \, d\theta}{\displaystyle\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \int_{0}^{r^2} r \, dz \, dr \, d\theta} = 0$$

Finally, $$\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \int_{0}^{r^2} z \, r \, dz \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} r \left|\frac{z^2}{2}\right|_0^{r^2} dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \frac{r^5}{2} \, dr \, d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left|\frac{r^6}{6}\right|_0^{2\cos\theta} d\theta = \frac{32}{3} \int_{0}^{\pi/2} \cos^6\theta \, d\theta$$

$$= \frac{32}{5} \cdot 2 \int_{0}^{\pi/2} \cos^6\theta \, d\theta = \frac{32}{3} \cdot \frac{5 \, . \, 3 \, . \, 1}{6 \, . \, 4 \, . \, 2} \cdot \frac{\pi}{2} = \frac{5\pi}{3}.$$

Thus, $$\bar{z} = \frac{\displaystyle\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \int_{0}^{r^2} z \, r \, dz \, dr \, d\theta}{\displaystyle\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \int_{0}^{r^2} r \, dz \, dr \, d\theta} = \frac{5\pi/3}{3\pi/2} = \frac{10}{9}$$

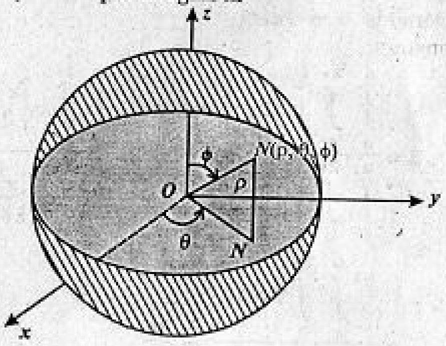
Thus $G\ (\bar{x}, \bar{y}, \bar{z}) = G\left(\frac{4}{3}, 0, \frac{10}{9}\right)$.

2. Find the mass of a sphere of radius r if the density varies inversely as the square of the distance from the centre.

Sol. Equation of a sphere with centre at the origin and radius a in spherical polar coordinates is $\rho = a$

The volume element $\rho^2 \sin\phi \, d\phi \, d\theta \, \partial\rho$ at $P(\rho, \theta, \phi)$ has density $\frac{k}{\rho^2}$, as the distance $|OP|^2 = \rho^2 = x^2 + y^2 + z^2$.

Clearly, $0 \le \rho \le a$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$ The mass M of the sphere is given by the triple integral as



$$M = \int_0^a \int_0^{2\pi} \int_0^{\pi} \frac{k}{\rho^2} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

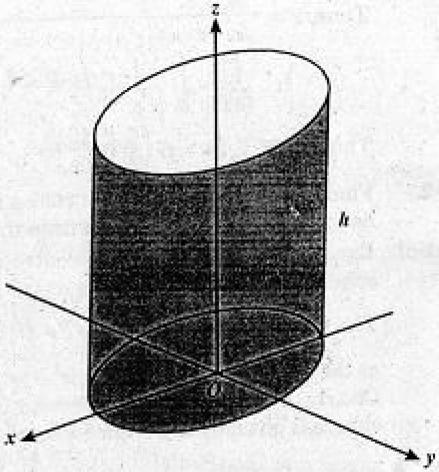
$$= \int_0^a \int_0^{2\pi} |-k\cos\phi|_0^{\pi} \, d\theta \, d\rho = \int_0^a \int_0^{2\pi} 2k \, d\theta \, d\rho$$

$$= \int_0^a 2k\,|\theta|_0^{2\pi}\, d\rho = \int_0^a 4k\pi\, d\rho = 4k\,\pi|\,\rho|_0^a = 4k\,\pi a.$$

3. Find the centre of gravity of a right circular cylinder of radius r and height h if the density varies as the distance from the base.

Sol. Equation of a cylinder in cylindrical polar coordinates is $r = a$ and axis of the cylinder is the z-axis. Now x varies from $-a$ to a and θ varies from 0 to 2π. The variation of z is from 0 to h.

By symmetry the *c. g.* lies on the z-axis and let it be $G(0, 0, \bar{z})$. The density of the mass element $r\, dz\, d\theta\, dr$ at $P(r, \theta, z)$ of height z from the base *i.e.*, the xy-plane) is λz, where λ is constant.



$$\bar{z} = \frac{\int_0^a \int_0^{2\pi} \int_0^h \lambda z^2 r\, dz\, d\theta\, dr}{\int_0^a \int_0^{2\pi} \int_0^h \lambda z r\, dz\, d\theta\, dr} = \frac{\int_0^a \int_0^{2\pi} r \left|\frac{z^3}{3}\right|_0^h d\theta\, dr}{\int_0^a \int_0^{2\pi} r \left|\frac{z^2}{2}\right|_0^h d\theta\, dr}$$

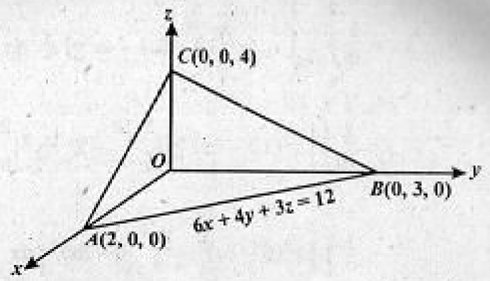
$$= \frac{\frac{h^3}{3}\int_a^0 \int_0^{2\pi} r\, d\theta\, dr}{\frac{h^2}{2}\int_0^a \int_0^{2\pi} r\, d\theta\, dr} = \frac{2h}{3}$$

Hence centre of gravity is $\left(0, 0, \frac{2h}{3}\right)$.

4. Find the moments of inertia I_x, I_y, I_z of the following volumes:

(a) bounded by the coordinate planes and $6x + 4y + 3z = 12$.

Sol. The volume is as shown in the figure



Clearly, $0 \le x \le 2,\ 0 \le y \le \frac{6-3x}{2},\ 0 \le z \le \frac{12 - 6x - 4y}{4}$

V = volume of the tetrahedron $OABC$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} dz\, dy\, dx = \int_0^2 \int_0^{\frac{6-3x}{2}} |z|_0^{\frac{12-6x-4y}{3}} dy\, dx$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} \frac{12 - 6x - 4y}{3}\, dy\, dx$$

$$= \frac{1}{3}\int_0^2 \left[(12 - 6x)\,|y|_0^{\frac{6x-3x}{2}} - 2|\,y^2|_0^{\frac{6x-3x}{2}}\right] dx$$

$$= \frac{1}{3}\int_0^2 \left[(6 - 3x)^2 - \frac{1}{2}(6 - 3x)^2\right] dx$$

$$= \frac{1}{6}\int_0^2 (6 - 3x)^2\, dx = \frac{1}{6}\left|\frac{(6 - 3x)^3}{-9}\right|_0^2 = \frac{1}{6}\left[0 + \frac{63}{9}\right] = 4.$$

$$\int_0^2 \int_0^{(6-3x)/2} \int_0^{(12-6x-4y)/3} x^2\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} x^2\, |z|_0^{\frac{12-6x-4y}{3}} dy\, dx = \int_0^2 \int_0^{\frac{6-3x}{2}} x^2 \left(\frac{12 - 6x - 4y}{3}\right) dy\, dx$$

$$= \frac{1}{3}\int_0^2 \int_0^{\frac{6-3x}{2}} [x^2(12-6x) - 4x^2y]\, dy\, dx$$

$$= \frac{1}{3}\int_0^2 \left[x^2(12-6x)\, |y|_0^{\frac{6x-3x}{2}} - 2x^2|y^2|_0^{\frac{6x-3x}{2}}\right] dx$$

$$= \frac{1}{3}\int_0^2 \left[x^2(6-3x)^2 - \frac{2x^2}{4}(6-3x)^2\right] dx$$

$$= \frac{1}{6}\int_0^2 x^2(6-3x)^2\, dx = \frac{1}{6}\int_0^2 x^2[36 + 9x^2 - 36x]dx$$

$$= \frac{1}{6}\int_0^2 (36x^2 + 9x^4 - 36x^3)\, dx = \frac{1}{6}\left| 12x^3 + \frac{9}{5}x^5 - 9x^4\right|_0^2$$

$$= \frac{1}{6}\left[96 + \frac{288}{5} - 144\right] = \frac{1}{6}\left[\frac{288}{5} - 48\right]$$

$$= \frac{1}{6}\left[\frac{288-240}{5}\right] = \frac{48}{6\times 5} = \frac{8}{5} = \frac{2}{5}V.$$

Again, $\displaystyle\int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} y^2\, dz\, dy\, dx = \int_0^2 \int_0^{\frac{6-3x}{2}} y^2 |\, z\, |_3^{12-6x-4y}\, dy\, dx$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} y^2\left(\frac{12-6x-4y}{3}\right) dy\, dx$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} \left[y^2\frac{(12-6x)}{3} - \frac{4}{3}y^3\right] dy\, dx$$

$$= \int_0^2 \left|\frac{2y^3}{9}(6-3x) - \frac{y^4}{3}\right|_0^{\frac{6-3x}{2}} dx$$

$$= \int_0^2 \left[\frac{2}{9}\left(\frac{(6-3x)^3}{8}(6-3x) - \frac{\left(\frac{(6-3x)^4}{2}\right)}{3}\right)\right] dx$$

$$= \int_0^2 \left[\frac{1}{36}(6-3x)^4 - \frac{(6-3x)^4}{48}\right] dx = \frac{1}{144}\int_0^2 (6-3x)^4\, dx$$

$$= \frac{-1}{432}\int_0^2 (6-3x)^4(-3)\, dx = \frac{-1}{432}\left|\frac{(6-3x)^5}{5}\right|_0^2 = \frac{-1}{432}\left[-\frac{6^5}{5}\right]$$

$$= \frac{6.6.6.6.6}{6.6.6.2.5} = \frac{18}{5} = \frac{9}{10}V$$

Finally, $\displaystyle\int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4x}{3}} z^2\, dz\, dy\, dx = \int_0^2 \int_0^{\frac{6-3x}{2}} \left|\frac{z^3}{3}\right|_0^{\frac{12-6x-4y}{3}} dy\, dx$

$$= \frac{1}{3^4}\int_0^2 \int_0^{\frac{6-3x}{2}} (12-6x-4y)^3\, dy\, dx$$

$$= \frac{1}{3^4}\int_0^2 \left|\frac{(12-6x-4y)^4}{-16}\right|_0^{\frac{6x-3x}{2}} dx$$

$$= -\frac{1}{16.3^4}\int_0^2 [0-(12-6x)^4]\, dx$$

$$= -\frac{1}{16.3^4}\left[\frac{(12-6x)^5}{-30}\right]_0^2$$

$$= -\frac{1}{16.3^4}\left[\frac{(12)^5}{30}\right] = \frac{32}{5} = \frac{8}{5}V$$

$$I_x = \int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} (y^2+z^2)\, dz\, dy\, dx$$

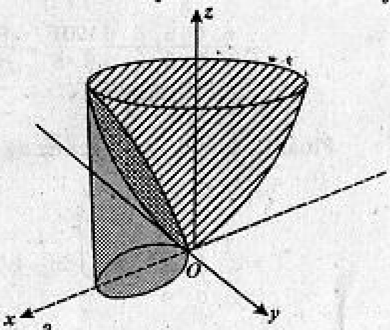
$$= \frac{9}{10}V + \frac{8}{5}V = \frac{125}{10}V = \frac{5}{2}V.$$

$$I_y = \int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} (x^2+y^2)\,dz\,dy\,dx = \frac{2}{5}V + \frac{8}{5}V = 2V$$

$$I_z = \int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} (x^2+y^2)\,dz\,dy\,dx$$

$$= \frac{2}{5}V + \frac{9}{10}V = \frac{13}{10}V.$$

4. (b) inside $x^2 + y^2 = 4x$, bounded above by $z = 0$ and below by $x^2 + y^2 = 4z$.



Sol. Let V be the volume inside the cylinder $x^2 + y^2 = 4x$ and below by the paraboloid $x^2 + y^2 = 4z$ and above the plane $z = 0$. In cylindrical polar coordinates equation of the cylinder becomes $r^2 = 4r\cos\theta$ and that of the paraboloid $r^2 = 4z$.

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 4\cos\theta,\ 0 \le z \le \frac{r^2}{4}$$

$$V = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4} r\,dz\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{4\cos\theta} r\,|z|_0^{\frac{r^2}{4}}\,dr\,d\theta$$

$$= \frac{2}{4}\int_0^{\pi/2}\int_0^{4\cos\theta} r^3\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{4\cos\theta} d\theta$$

$$= \frac{4^4}{8}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{256}{8}\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2} = 6\pi.$$

$$2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4} x^2\,r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4} r^2\cos^2\theta\,r\,dz\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{4\cos\theta} r^3\cos^2\theta\,|z|_0^{\frac{r^2}{4}}\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{4\cos\theta}\frac{r^5}{4}\cos^2\theta\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left|\frac{r^6}{6}\right|_0^{4\cos\theta}\cos^2\theta\,d\theta$$

$$= \frac{4^6}{12}\int_0^{\pi/2}\cos^8\theta\,d\theta$$

$$= \frac{4^6}{12}\cdot\frac{7.5.3.1}{8.6.4.2}\cdot\frac{\pi}{2} = \frac{140V}{18}$$

$$2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4} y^2\,r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4} r^2\sin^2\theta\,r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4} r^3\sin^2\theta\,|z|_0^{\frac{r^2}{4}}\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{4\cos\theta}\frac{r^5}{4}\sin^2\theta\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left|\frac{r^6}{6}\right|_0^{4\cos\theta}\sin^2\theta\,d\theta$$

$$= \frac{4^6}{12}\int_0^{\pi/2}\sin^6\theta\,d\theta$$

$$= \frac{4^6}{12}\cdot\frac{1.5.3.1}{8.6.4.2}\cdot\frac{\pi}{2} = \frac{20V}{18}.$$

$$2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4} z^2 r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4}\frac{r^5}{16}\,dz\,dr\,d\theta$$

$$=\frac{1}{8}\int_0^{\pi/2}\int_0^{4\cos\theta} r^5\,|z|_0^{\frac{r^2}{4}}\,dr\,d\theta$$

$$=\frac{1}{32}\int_0^{\pi/2}\int_0^{4\cos\theta} r^7\,dr\,d\theta$$

$$=\frac{1}{32}\int_0^{\pi/2}\left|\frac{r^8}{8}\right|_0^{4\cos\theta} d\theta$$

$$=\frac{4^8}{32}\times\frac{1}{8}\int_0^{\pi/2}\sin^8\theta\,d\theta$$

$$=\frac{4^8}{32\times 8}\cdot\frac{7.5.3.1}{8.6.4.2}\cdot\frac{\pi}{2}$$

$$=\frac{105V}{32\times 144}=\frac{35V}{18}.$$

Thus, $I_x = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4}(y^2+z^2)\,r\,dz\,dr\,d\theta$

$$=\frac{20V}{18}+\frac{35V}{18}=\frac{55V}{18}.$$

$$I_y = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4}(x^2+z^2)\,r\,dz\,dr\,d\theta$$

$$=\frac{140V}{18}+\frac{35V}{18}=\frac{175V}{18}.$$

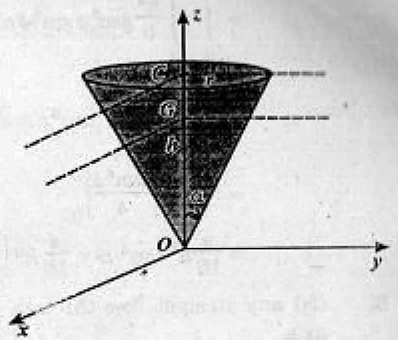
$$I_z = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{r^2/4}(x^2+y^2)\,r\,dz\,dr\,d\theta$$

$$=\frac{140V}{18}+\frac{20V}{18}=\frac{80V}{9}.$$

5. For the right circular cone of radius r and height h, find the moment of inertia with respect to:

Sol. **(a)** its axis

Let us take a right circular cone with vertex at O (0, 0, 0), its axis along the z-axis as shown in figure.



Use spherical polar coordinates ρ, θ, ϕ.

$x = \rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$.

Now the respective variations of coordinates for the points of the cone are

$$0\le\rho\le h\sec\phi,\ 0\le\phi\le\alpha,\ 0\le\theta\le 2\pi$$

The volume V of the cone is given by the following triple integral

$$V=\int_0^{2\pi}\int_0^{\alpha}\int_0^{h\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\alpha}\left|\frac{\rho^3}{3}\right|_0^{h\sec\phi}\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\alpha}\frac{h^3}{3}\sec^3\phi\sin\phi\,d\phi\,d\theta$$

$$=\frac{h^3}{3}\int_0^{2\pi}\int_0^{\alpha}\tan\phi\sec^2\phi\,d\phi\,d\theta=\frac{h^3}{3}\int_0^{2\pi}\left|\frac{\tan^2\phi}{2}\right|_0^{\alpha}d\theta$$

$$=\frac{h^3}{3}\tan^2\alpha\,|\theta|_0^{2\pi}=\frac{2\pi h^3}{6}\tan^2\alpha=\frac{\pi h^3}{3}\cdot\frac{r^2}{h^2}=\frac{1}{3}\pi r^2 h.$$

Now $x^2+y^2=\rho^2\sin^2\phi$. Hence

$$I_z=\int_0^{2\pi}\int_0^{\alpha}\int_0^{h\sin\phi}(x^2+y^2)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\alpha}\int_0^{h\sin\phi}\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\alpha}\left|\frac{\rho^5}{5}\right|_0^{h\sec\phi}\sin^3\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\alpha} \frac{h^5}{5} \sec^5\phi \sin^3\phi \, d\phi \, d\theta$$

$$= \frac{h^5}{5} \int_0^{2\pi} d\theta \int_0^{\alpha} \tan^3\phi \sec^2\phi \, d\theta$$

$$= \frac{2\pi h^5}{5} \left| \frac{\tan^4\phi}{4} \right|_0^{\alpha}$$

$$= \frac{\pi}{10} h^5 \tan^4 \alpha = \frac{\pi}{10} h^5 \left(\frac{r^4}{h^4}\right) = \frac{3}{10} r^2 \left(\frac{\pi r^2 h}{3}\right) = \frac{3r^2}{10} V.$$

5. **(b)** any straight line through its vertex and perpendicular to its axis.

Sol. A line perpendicular to the axis (z-axis) through the vertex can be taken as the x-axis

Now, $\displaystyle\int_0^{2\pi} \int_0^{\alpha} \int_0^{h\sec\phi} y^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= \int_0^{2\pi} \int_0^{\alpha} \int_0^{h\sec\phi} \rho^4 \sin^3\phi \sin^2\theta \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\alpha} \left| \frac{\rho^5}{5} \right|_0^{h\sec\phi} \sin^3\phi \sin^2\theta \, \phi \, d\phi \, d\theta$$

$$= \frac{h^5}{5} \int_0^{2\pi} \int_0^{\alpha} \sec^5\phi \sin^3\phi \cos^2\theta \, d\phi$$

$$= \frac{4h^5}{5} \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\alpha} \sec^2\phi \tan^3\phi \, d\phi$$

$$= \frac{4h^5}{5} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \left| \frac{\tan^4\phi}{4} \right|_0^{\alpha}$$

$$= \frac{\pi h^5}{20} \tan^4 \alpha = \frac{\pi h^5}{20} \cdot \frac{r^4}{h^4} = \frac{\pi h r^4}{20}.$$

Also, $\displaystyle\int_0^{2\pi} \int_0^{\alpha} \int_0^{h\sec\phi} z^2 \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta$

$$= \int_0^{2\pi} \int_0^{\alpha} \int_0^{h\sec\phi} \rho^4 \cos^2\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\alpha} \left| \frac{\rho^5}{5} \right|_0^{h\sec\phi} \cos^2\phi \sin\phi \, d\phi$$

$$= \frac{2\pi h^5}{5} \int_0^{\alpha} \sec^5\phi \cos^2\phi \sin\phi \, d\phi$$

$$= \frac{2\pi h^5}{5} \int_0^{\alpha} \tan\phi \sec^2\phi \, d\phi = \frac{2\pi h^5}{5} \left| \frac{\tan^2\phi}{2} \right|_0^{\alpha}$$

$$= \frac{\pi h^5}{5} . \tan^2 \alpha = \frac{\pi h^5}{5} \cdot \frac{r^2}{h^2} = \frac{\pi h^3 r^2}{5}.$$

Hence, $\displaystyle I_x = \int_0^{2\pi} \int_0^{\alpha} \int_0^{h\sec\phi} (y^2 + z^2) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= \frac{\pi h r^4}{20} + \frac{\pi h^3 r^2}{5} = \frac{\pi h r^2}{3} \cdot \frac{3}{5}\left[h^3 + \frac{r^2}{4}\right] = \frac{3}{5}\left(h^2 + \frac{r^2}{4}\right) V.$$

5. **(c)** any line through its centre of gravity and perpendicular to its axis.

Sol. The symmetry of the cone shows that its *c.g.* lies on the z-axis.

Let $G(0, 0, \bar{z})$ be the *c.g.* of the cone.

$$\int_0^{2\pi} \int_0^{\alpha} \int_0^{h\sec\phi} z \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\alpha} \int_0^{h\sec\phi} \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\alpha} \left| \frac{\rho^4}{4} \right|_0^{h\sec\phi} \sin\phi \cos\phi \, d\phi$$

$$= \frac{2\pi h^4}{4} \int_0^{\alpha} \sec^4\phi \cos\phi \, d\phi$$

$$= \frac{\pi h^4}{2} \int_0^{\alpha} \tan\phi \sin\phi \cos\phi \, d\phi$$

$$= \frac{\pi h^4}{2}\left|\frac{\tan^2\phi}{2}\right|_0^{\alpha}$$

$$= \frac{\pi h^2 r^2}{4}\tan^2\alpha = \frac{\pi h^4}{4}\cdot\frac{r^2}{h^2} = \frac{\pi h^2 r^2}{4}$$

$$\text{Thus, } \bar{z} = \frac{\int_0^{2\pi}\int_0^{\alpha}\int_0^{h\sec\phi} z\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}{\int_0^{2\pi}\int_0^{\alpha}\int_0^{h\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta} = \frac{\pi h^2 r^2/4}{\pi r^2 h/3} = \frac{3h}{4}.$$

Hence $G\left(0, 0, \frac{3h}{4}\right)$ is the *c.g.* of the cone.

By the principle of parallel axes of moments of inertia, we have M.I. about the x-axis = M.I. about an axis through G and parallel to the x-axis + M.I. of mass V placed at G about the x-axis.

or $\frac{3}{5}\left(h^2 + \frac{r^2}{4}\right)V$ = M.I. about an axis through G and parallel to the x-axis + $V\left(\frac{3h}{4}\right)^2$, by **5(b)**

Thus M.I. about an axis through G and parallel to the x-axis

$$= \frac{3}{5}\left(h^2 + \frac{r^2}{4}\right)V - \frac{9h^2}{16}V$$

$$= \frac{3}{80}(16h^2 + 4r^2 - 15h^2)\,V = \frac{3}{80}(h^2 + 4r^2)\,V.$$

5. **(d)** any diameter of its base.

Sol. By the principle of parallel axes of moments inertia, we have:

M.I. about a diameter (parallel to the x-axis) through C

= M.I. about an axis through G (parallel to the x-axis) + M.I. of mass V placed at G about an axis through C (parallel to the x-axis)

$$= \frac{3}{80}(h^2 + 4r^2)\,V + \left(\frac{h}{4}\right)V$$

$$= \frac{V}{80}[3h^2 + 12r^2 + 5h^2] = \frac{V}{80}(8h^2 + 12r^2)$$

$$= \frac{1}{20}(2h^2 + 3r^2)\,V.$$