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Solutions Manual For

MATHEMATICAL METHODS

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The Complex Number System.

The set 'C' $= R \times R = \{(a,b) / a,b \in R\}$ is called the set of complex number if the following conditions are satisfied.

i) $(a,b)+(c,d) = (a+c,\ b+d)$ (Addition)

ii) $(a,b)\cdot(c,d) = (ac-bd,\ ad+bc)$ (Multiplication)

iii) $K(a,b) = (Ka, Kb)$ where $K \in R$ (Scalar Multiple)

iv) $(a,b) = (c,d)$ iff $a=c,\ b=d$ (Equality)

Note. $\boxed{(a,b) = a+bi}$ a = Real Part, b = Imag Part

$i=(0,1)$

$(a,b) = (a,0)+(0,b)$
$= a(1,0)+b(0,1)$
$= a(1)+b(i)$
$(a,b) = a+bi$

$i = (0,1)$
$i \cdot i = (0,1)(0,1)$
$i^2 = (0\cdot0-1\cdot1,\ 0\cdot1+1\cdot0)$
$i^2 = (-1,0)$
$i^2 = -1+0i$
$i^2 = -1$

where $(a,b) \in C$ & $a+bi \in C$
$\because (a,b) = a(1,0)+b(0,1)$
$(a,b) = a\cdot1 + bi = a+bi$

Modulus of $(a,b) \in C$

If $z=(a,b)$ then $|z| = \sqrt{a^2+b^2}$

Conjugate of $(a,b) \in C$

If $z=(a,b)=a+bi$
then $\bar{z} = \overline{a+bi} = a-bi$

Multiplicative Identity in C. $(1,0) = 1 = 1+0i$

Additive Identity in C. $(0,0) = 0 = 0+0i$

Additive Inverse of (a,b) is $(-a,-b)$

Multiplicative Inverse of $(a,b) \in C$

Let $z = (a,b) = a+ib$

$z^{-1} = (a+ib)^{-1} = \dfrac{1}{a+ib}$

$= \dfrac{1}{a+ib} \times \dfrac{(a-ib)}{(a-ib)}$

$z^{-1} = \dfrac{a-ib}{a^2+b^2} = \dfrac{a}{a^2+b^2} - \dfrac{ib}{a^2+b^2}$

$z^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

(To verify $zz^{-1} = (1,0) = 1$)

To Prove $z\cdot\bar{z} = |z|^2$ $= (\text{Re}\,z)^2+(\text{Im}\,z)^2$

LHS $z\cdot\bar{z}$

$= (a+bi)(a-bi)$

$= a^2+b^2 = (\text{Re}\,z)^2+(\text{Im}\,z)^2$

$\boxed{z\bar{z} = |z|^2}$

Also $z^2 = (a+bi)(a+bi)$

$z^2 = a^2-b^2+2abi$

$|z^2| = \sqrt{(a^2-b^2)^2+(2ab)^2}$

$= \sqrt{a^4+b^4-2a^2b^2+4a^2b^2}$

$= \sqrt{a^4+b^4+2a^2b^2}$

$= \sqrt{(a^2+b^2)^2} = a^2+b^2 = |z|^2$

$\therefore \boxed{|z^2| = |z|^2}$

Imp

Th Let z_1, z_2 be complex numbers. Show that

(i) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

(ii) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Sol (i) Let $z_1 = a+bi \quad \bar{z}_1 = a-bi$

$z_2 = c+di, \quad \bar{z}_2 = c-di$

LHS

$$z_1+z_2 = a+bi+c+di$$

$$z_1+z_2 = (a+c)+(b+d)i$$

$$\overline{z_1+z_2} = (a+c)-(b+d)i \quad \text{—— (i)}$$

RHS

$$\bar{z}_1+\bar{z}_2 = a-bi+c-di$$

$$= (a+c)-(b+d)i \quad \text{—— (ii)}$$

(i) = (ii) $\Rightarrow$ $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

(ii)

LHS

$$z_1 z_2 = (a+bi)(c+di)$$

$$= ac+adi+bci+bd(i^2)$$

$$= ac+adi+bci+bd(-1)$$

$$z_1 z_2 = (ac-bd)+(ad+bc)i$$

$$\overline{z_1 z_2} = (ac-bd)-i(ad+bc) \quad \text{—— (i)}$$

RHS

$$\bar{z}_1 \bar{z}_2 = (a-bi)(c-di)$$

$$= ac-adi-bci+bd(i)^2$$

$$\bar{z}_1\bar{z}_2 = (ac-bd)-i(ad+bc) \quad \text{—— (ii)}$$

(1) = (2) $\Rightarrow$ $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(iii) LHS

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}$$

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$

$$= \frac{ac-adi+bci-bd(i^2)}{c^2-(id)^2}$$

$$= \frac{ac-adi+bci-bd(-1)}{c^2+d^2}$$

$$\frac{z_1}{z_2} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(ac+bd)-i(bc-ad)i}{c^2+d^2} \quad \text{—— (i)}$$

Imp

Th Show that the modulus $|z|$ of a complex number $z = a+bi$ is the distance of a point from origin, or length of $\overline{OP}$.

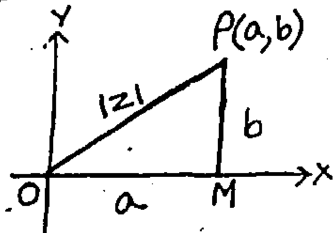
Sol We know to each complex number $z = a+bi$ there corresponds a pt $P(a,b)$, in the cartesian plane and vice versa.

But the point (a,b) in the plane is represented as

$|\overline{OM}| = a$

$|\overline{PM}| = b$

$|\overline{OP}| = |z|$



By pythagoras Th.

$$|\overline{OP}|^2 = |\overline{OM}|^2 + |\overline{PM}|^2$$

$$|OP|^2 = a^2+b^2$$

$$|OP| = \sqrt{a^2+b^2}$$

$$|z| = \sqrt{a^2+b^2}$$ Distance of (a,b) from $(0,0)$

Note 1)

$z = a+ib$

$\bar{z} = a-ib$

$\bar{\bar{z}} = a+ib \quad \therefore \boxed{\bar{\bar{z}} = z}$

Note 2)

$z = a+ib \Rightarrow |z| = \sqrt{a^2+b^2}$

$\bar{z} = a-ib \Rightarrow |\bar{z}| = \sqrt{a^2+b^2}$

$-z = -a-ib \Rightarrow |-z| = \sqrt{a^2+b^2}$

$\therefore |z| = |-z| = |\bar{z}|$

RHS

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{a-bi}{c-di}$$

$$= \frac{a-bi}{c-di} \times \frac{c+di}{c+di}$$

$$= \frac{ac+adi-bci-bd(i^2)}{c^2-(id)^2}$$

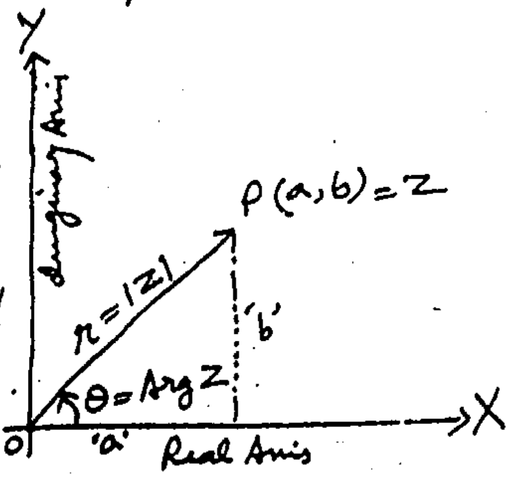
$$= \frac{ac+adi-bci+bd}{c^2+d^2}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{(ac+bd)-i(bc-ad)}{c^2+d^2} \quad \text{—— (ii)}$$

(i) = (ii) $\Rightarrow$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Complex Plane

A complex number $z = a+bi$ corresponds to the pt (a,b) in the XY-plane and vice versa.



The XY-plane in which a complex number z is represented by a vector $\overline{OP}$ is called Complex Plane or Z-Plane, X-axis is called Real Axis and Y-axis is called Imaginary Axis. & figure so obtained is called Argand Diagram.

The inclination 'θ' of a complex vector $\overline{OP}$ with positive direction of x-axis is called Argument or Amplitude of z, written as arg z.

$$\arg z = \theta = \tan^{-1}\frac{b}{a}$$

$\sin\theta = \frac{b}{|z|}$

$\cos\theta = \frac{a}{|z|}$

If value of θ is such that $-\pi < \theta \leq \pi$ then θ is called Principal argument of z i.e Arg z

Arg of 0 is not defined.

Imp Note

i) When $z=(a,b)$ is in Ist Quad then angle is 'θ'
ii) When $z=(a,b)$ is in 2nd Quad then angle is $(\pi-\theta)$
iii) When $z=(a,b)$ is in 3rd Quad then angle is $-(\pi-\theta)$
iv) When $z=(a,b)$ is in 4th Quad then angle is $-$'θ'

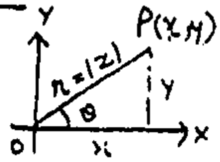
It is because $-\pi<\theta\leq\pi$ i.e value of θ is not greater than π.

Properties

i) $|z| = |-z| = |\bar{z}|$
ii) $|z|^2 = |z^2| = z\bar{z}$
iii) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$
iv) $|z_1 z_2| = |z_1||z_2|$
v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
vi) $|\mathrm{Re}\, z| \leq |z|$
vii) $|\mathrm{Im}\, z| \leq |z|$
viii) $z\bar{z} = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2$
ix) $|z_1 - z_2| = |z_2 - z_1|$
x) $|z_1 - z_2| \geq |z_1| - |z_2|$
xi) $|z_1 + z_2| \leq |z_1| + |z_2|$

Polar Form of Complex Number

From fig



$x = r\cos\theta, \quad y = r\sin\theta$

$$z = x+iy \quad \text{——— ①}$$
$$= r\cos\theta + i\,r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$z = r\,\mathrm{cis}\,\theta \quad \text{——— ⑪}$$

① is Cartesian form of complex number z.
⑪ is Polar form of complex number z.

Imp

Th For all $z_1, z_2 \in C$

$$\boxed{|z_1 z_2| = |z_1||z_2|}$$

Proof Let $z_1 = a+ib$, $z_2 = c+id$, then

$$z_1 z_2 = (a+ib)(c+id)$$
$$z_1 z_2 = ac+iad+ibc+i^2bd$$
$$z_1 z_2 = (ac-bd)+i(ad+bc)$$
$$|z_1 z_2| = \sqrt{(ac-bd)^2+(ad+bc)^2}$$
$$= \sqrt{a^2c^2+b^2d^2-\cancel{2acbd}+a^2d^2+b^2c^2+\cancel{2adbc}}$$
$$= \sqrt{a^2(c^2+d^2)+b^2(d^2+c^2)}$$
$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$
$$= \sqrt{a^2+b^2}\,\sqrt{c^2+d^2}$$
$$|z_1 z_2| = |z_1||z_2| \quad \text{proved}$$

Th Prove that for $z_1, z_2 \in C$

$$\boxed{|z_1|-|z_2| \le |z_1+z_2| \le |z_1|+|z_2|}$$

Proof Let $|z_1+z_2|^2$

$= (z_1+z_2)(\overline{z_1+z_2})$ $\quad \because |z|^2 = z\bar{z}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$ $\quad \because \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$ $\quad \because z_1\bar{z}_2+z_2\bar{z}_1 = 2Re\, z_1\bar{z}_2$

$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2$ $\quad \because |Re\, z| \le |z|$

$\le |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$ $\quad \because |z_1 z_2| = |z_1||z_2|$, $|\bar{z}| = |z|$

$= |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

$|z_1+z_2|^2 \le (|z_1|+|z_2|)^2$

$\boxed{|z_1+z_2| \le |z_1|+|z_2|}$ —— (I)

Now $|z_1| = |z_1+z_2-z_2|$ $\quad (+z_2-z_2)$

$\le |z_1+z_2| + |-z_2|$

$|z_1| = |z_1+z_2| + |z_2|$

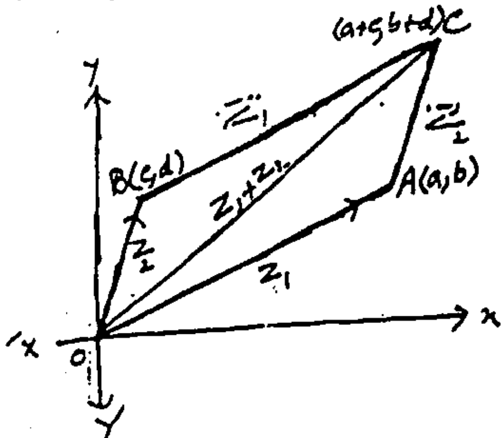
$\boxed{|z_1|-|z_2| \le |z_1+z_2|}$ —— (II)

Combining (I) & (II) $|z_1|-|z_2| \le |z_1+z_2| \le |z_1|+|z_2|$

Imp **2nd Method** (Also see on Page 10)

For any two complex numbers

$$|z_1|-|z_2| \le |z_1+z_2| \le |z_1|+|z_2|$$



Let $|z_1| = |\overline{OA}|$ $\quad |z_1+z_2| = |\overline{OC}|$

$|z_2| = |\overline{OB}|$

In ΔOAC $\quad |\overline{OA}|+|\overline{AC}| > |\overline{OC}|$ —— (I)

$|z_1|+|z_2| > |z_1+z_2|$

For Collinear pts

$|\overline{OA}|+|\overline{AC}| = |\overline{OC}|$

$|z_1|+|z_2| = |z_1+z_2|$ —— (II)

Combining (I) & (II)

$|z_1|+|z_2| \ge |z_1+z_2|$ proved

Now $|z_1| = |z_1+z_2-z_2|$

$|z_1| \le |z_1+z_2| + |-z_2|$

$|z_1| \le |z_1+z_2| + |z_2|$

$|z_1|-|z_2| \le |z_1+z_2|$ —— (III)

Combining (II) + (III) we get the result.

To Prove $|z_1|-|z_2| \le |z_1-z_2|$

Proof Since $|z_1+z_2| \le |z_1|+|z_2|$

Replace z_1 by z_1-z_2

$\therefore |z_1-z_2+z_2| \le |z_1-z_2|+|z_2|$

$\Rightarrow |z_1| \le |z_1-z_2|+|z_2|$

$|z_1|-|z_2| \le |z_1-z_2|$

To Prove $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Proof $\because z\bar{z} = |z|^2$

$\Rightarrow \frac{1}{z\bar{z}} = \frac{1}{|z|^2}$

$\Rightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2}$ proved.

EXERCISE 1.1

EXPRESS each of the following complex numbers in the polar form. (Problem 1—6):

Q.1 Let $z = x + iy = -\sqrt{3} + i \Rightarrow x = -\sqrt{3}, \ y = 1$, $\quad r = |z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$

$\because \cos\theta = \dfrac{x}{r} = \dfrac{-\sqrt{3}}{2} \Rightarrow \theta = \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

& $\sin\theta = \dfrac{y}{r} = \dfrac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\dfrac{1}{2}\right)$ $\Bigg\} \Rightarrow \theta = \dfrac{5\pi}{6}$

(x, is -ive & y, is +ive, So θ lies in 2nd Quad. Since 2nd Quadrant, $\therefore \theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$)

Hence $z = r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$

$= 2\,\text{cis}\,\dfrac{5\pi}{6}$ Ans

$\boxed{\text{2nd Method}}$

$z = x + iy = -\sqrt{3} + i \Rightarrow x = -\sqrt{3}, \ y = 1$

$\tan\theta = \dfrac{y}{x} = \dfrac{1}{-\sqrt{3}}$

$\theta = \tan^{-1}\left(\dfrac{1}{-\sqrt{3}}\right) = \dfrac{5\pi}{6}$

($\because$ x -ive, y +ive, $\therefore$ θ lies in 2nd Quad. So Principal Arg $= \pi - \theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$)

Hence $z = r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$

Q.2 Let $z = x + iy = -i = 0 + (-i) \Rightarrow x = 0, \ y = -1$

$\Rightarrow r = |z| = \sqrt{0^2 + (-1)^2} = 1$

$\because \cos\theta = \dfrac{x}{r} = \dfrac{0}{1} = 0 \Rightarrow \theta = \cos^{-1}(0)$

& $\sin\theta = \dfrac{y}{r} = -\dfrac{1}{1} = -1 \Rightarrow \theta = \sin^{-1}(-1)$ $\Bigg\} \theta = -\dfrac{\pi}{2}$

($\because$ x +ive & y -ive $\therefore$ 4th Quad. So Principal Arg $z = -\theta = -\dfrac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta)$

$= 1\left(\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right)$

or $z = \left(\cos\dfrac{\pi}{2} - i\sin\dfrac{\pi}{2}\right)$

$\boxed{\text{2nd Method}}$

$z = x + iy = -i$

$\Rightarrow x = 0, \ y = -1$

$\tan\theta = \dfrac{y}{x} = \dfrac{-1}{0} = \infty$

$\theta = \tan^{-1}(\infty) = -\dfrac{\pi}{2}$

($\because$ x +ive, y -ive $\therefore$ 4th Quad. So Principal Arg $= -\theta = -\dfrac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta)$

$= 1\left(\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right)$

$z = \cos\dfrac{\pi}{2} - i\sin\dfrac{\pi}{2}$

Q.3 Let $z = x + iy = -1 - \sqrt{3}\,i \Rightarrow x = -1, \ y = -\sqrt{3}$

$\Rightarrow r = |z| = \sqrt{(-1)^2 + (-\sqrt{3})^2}$

$\Rightarrow r = \sqrt{1+3} = \sqrt{4} = 2$

$\because \cos\theta = \dfrac{-1}{2} \Rightarrow \theta = \cos^{-1}\left(-\dfrac{1}{2}\right)$

& $\sin\theta = \dfrac{-\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ $\Bigg\} \theta = -\dfrac{2\pi}{3}$

(3) $z = x + iy = -1 - \sqrt{3}i \Rightarrow x = -1, \ y = -\sqrt{3}$

$\tan\theta = \dfrac{y}{x} = \dfrac{-\sqrt{3}}{-1}$ $\quad r = \sqrt{1+3} = \sqrt{4}$, $r = 2$

$\theta = \tan^{-1}\left(\dfrac{-\sqrt{3}}{-1}\right)$

$\theta = -\dfrac{2\pi}{3}$

$\because$ x -ive, y -ive III Quad

$\therefore -\left(\pi - \dfrac{\pi}{3}\right) = -\dfrac{2\pi}{3}$

$\because$ x is $-$ve $\therefore \theta$ is in 3rd Quad.
y is $-$ve

So Principal $\text{Arg}\, z = -(\pi - \theta)$

Principal $\text{Arg}\, z = -\left(\pi - \frac{\pi}{3}\right)$

$= -\frac{2\pi}{3}$

Hence $z = r(\cos\theta + i\sin\theta)$

$z = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$

$= 2\left(\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right)$

Q.4 Let $z = x + iy = -1 + i \Rightarrow x = -1, \; y = 1$

$\Rightarrow r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\theta = \frac{3\pi}{4} \quad \therefore \theta = \frac{3\pi}{4}$.

$\because$ x is $-$ve, y is $+$ve $\therefore \theta$ is in 2nd Quad

Principal $\text{Arg}\, z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence $z = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= \sqrt{2}\, \text{Cis}\, \frac{3\pi}{4}$ Ans

Q4 2nd Method

$z = x + iy = -1 + i$ $\quad x = -1, \; y = 1$

$r = \sqrt{1+1} = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$

$\theta = \tan^{-1}(-1) = \frac{3\pi}{4}$

x $-$ve, y $+$ve II Quad

$\therefore \pi - \theta$

$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\,S\,\frac{3\pi}{4}\right)$

Q5 Let $z = (-2 + 2i)(1 - i)$

$= -2 + 2i + 2i - 2i^2$

$= \cancel{-2} + 4i + \cancel{2}$

$z = 4i$

$z = 0 + 4i \Rightarrow x = 0, \; y = 4$

$r = |z| = \sqrt{0^2 + 4^2} = 4$

$\cos\theta = \frac{x}{r} = \frac{0}{4} = 0 \Rightarrow \theta = \frac{\pi}{2}$

$\sin\theta = \frac{y}{r} = \frac{4}{4} = 1 \Rightarrow \theta = \frac{\pi}{2}$

(x is $+$ve, y is $+$ve) θ is in Ist Quad So Principal $\text{Arg}\, z = \frac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4\,\text{Cis}\,\frac{\pi}{2}$ Ans.

Q5 $z = (-2 + 2i)(1 - i)$

$= 4i$

$z = x + iy = 4i$

$x = 0, \; y = 4$

$r = \sqrt{0^2 + 4^2}$

$r = 4$

$\tan\theta = \frac{y}{x} = \frac{4}{0} = \infty$

$\theta = \tan^{-1}(\infty)$

$\theta = \frac{\pi}{2}$

$\because$ x $+$ve, y $+$ve I Q.

$\therefore z = 4\left(\cos\frac{\pi}{2} + i\,S\,\frac{\pi}{2}\right)$

Q6 Let $z = -\dfrac{34i}{5-3i}$

$= \dfrac{-34i}{5-3i} \times \dfrac{5+3i}{5+3i}$

$= \dfrac{-34i(5+3i)}{25+9}$

$= -i(5+3i)$

$= -5i+3 \quad \Rightarrow x = 3, \; y = -5$

$r = \sqrt{3^2+(-5)^2} = \sqrt{34}$

$\theta = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\left(\dfrac{-5}{3}\right)$

$\because x$ +ive, y −ive $\therefore \theta$ is in IVth Q $\therefore$ Principal Arg $= -\theta$

So Principal Argz $z = -\tan^{-1}\left(\dfrac{5}{3}\right)$

$= \tan^{-1}\left(-\dfrac{5}{3}\right)$

$z = r(\cos\theta + i\sin\theta)$

$= \sqrt{34}\left(\cos\left(\tan^{-1}\left(-\frac{5}{3}\right)\right) + i\sin\left(\tan^{-1}\left(-\frac{5}{3}\right)\right)\right)$

$= \sqrt{34}\ \text{cis}\left(\tan^{-1}\left(-\frac{5}{3}\right)\right)$

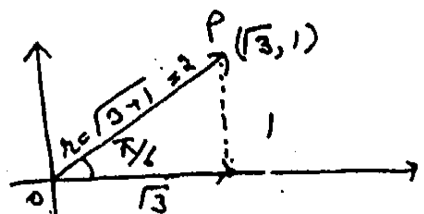
Q7 Express the given Complex number in Cartesian form and in Argand Diagram.

$z = 2\,\text{cis}\left(\frac{\pi}{6}\right)$

$= 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$= \sqrt{3} + i \quad = (\sqrt{3}, 1)$



$\because x$ is +ive & y is +ive so I st Quad.

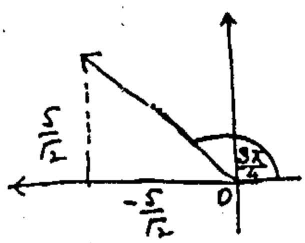
Q8 $z = 5\,\text{cis}\left(\frac{3\pi}{4}\right)$

$= 5\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

$= 5\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$= \frac{-5}{\sqrt{2}} + i\frac{5}{\sqrt{2}}$

$= \left(-\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$



$\frac{3\pi}{4} = 135°$

$\because x$ is −ive, y is +ive so IInd Q

Q9 $z = \sqrt{3}\, Cis\, \frac{7\pi}{6}$

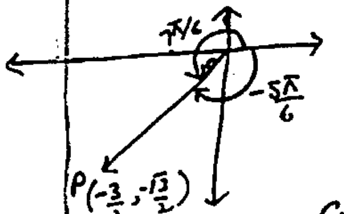
$= \sqrt{3}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= \sqrt{3}\left(\cos\left(2\pi - \frac{5\pi}{6}\right) + i\sin\left(2\pi - \frac{5\pi}{6}\right)\right)$

$= \sqrt{3}\left(\cos\frac{5\pi}{6} - i\sin\frac{\pi}{6}\right)$

$= \sqrt{3}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= -\frac{3}{2} - i\frac{\sqrt{3}}{2} = \left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right)$



Note: both are equal
$\left(\pi + \frac{\pi}{6}\right) = \frac{7\pi}{6} = \left(2\pi - \frac{5\pi}{6}\right)$

$\frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$

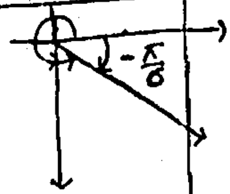
$\frac{5\pi}{6} = 150°$

$\cos\left(2\pi - \frac{5\pi}{6}\right) = \cos\frac{5\pi}{6}$

$\sin\left(2\pi - \frac{5\pi}{6}\right) = -\sin\frac{5\pi}{6}$

x is $-ve$, y is $-ve$ So III rd Quad.
$\cos^{-1}\frac{\sqrt{3}}{2} = \boxed{\frac{\pi}{6}}$, $\sin^{-1}\frac{1}{2} = \boxed{\frac{\pi}{6}}$ So $-(\pi - \theta)$
$-\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$

Q10 $z = \frac{5\, Cis\left(\frac{\pi}{3}\right)}{2\, Cis\left(\frac{\pi}{2}\right)}$



$= \frac{5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}$

$= \frac{5\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}{2(0 + i\cdot 1)}$

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$
$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

x & $\div$ by i $= \frac{5\cdot\left(\frac{i}{2} - \frac{\sqrt{3}}{2}\right)}{-2}$

$= \frac{5}{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{5\sqrt{3}}{4} - \frac{5}{4}i = \left(\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$

2nd Method

$z = \frac{5\, Cis\left(\frac{\pi}{3}\right)}{2\, Cis\left(\frac{\pi}{2}\right)}$

$= \frac{5}{2}\, Cis\left(\frac{\pi}{3} - \frac{\pi}{2}\right)$

$= \frac{5}{2}\, Cis\left(-\frac{\pi}{6}\right)$

$= \frac{5}{2}\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$

$= \frac{5}{2}\left[\frac{\sqrt{3}}{2} - i\frac{1}{2}\right]$

$= \frac{5}{4}\left(\sqrt{3} - i\right)$

x +ve, y -ve So IV th Q.
So $-\theta$
So $-\frac{\pi}{6}$

Q11 (i)

Find $|z|$ where $z = -2i(1+i)(2+4i)(3+i)$

$|z| = |-2i(1+i)(2+4i)(3+i)|$ $\quad \because |z_1 z_2| = |z_1|\cdot|z_2|$

$= |-2i|\,|1+i|\cdot|2+4i|\cdot|3+i|$

$= \sqrt{(-2)^2}\sqrt{1^2+1^2}\sqrt{2^2+4^2}\sqrt{3^2+1^2}$

$= 2\sqrt{2}\sqrt{20}\sqrt{10}$

$= 2\sqrt{2}\; 2\sqrt{5}\; \sqrt{5}\sqrt{2}$

$= 4(2)(5) = 40$

(ii) $z = \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$

$|z| = \left|\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right|$

$= \frac{|(3+4i)(-1+2i)|}{|(-1-i)(3-i)|}$ $\quad \because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$|z| = \frac{|3+4i|\cdot|-1+2i|}{|-1-i|\,|3-i|}$

$= \frac{\sqrt{9+16}\;\sqrt{1+4}}{\sqrt{1+1}\;\sqrt{9+1}}$

$= \frac{5\;\sqrt{5}}{\sqrt{2}\;\sqrt{10}}$

$= \frac{5\sqrt{5}}{2\sqrt{5}}$

$= \frac{5}{2}$ Ans.

Imp **Q12 i)** Show that $z = a+ib$ is real iff $\boxed{z = \bar{z}}$

Let $z = a+ib$ is real $\Rightarrow b=0$ (Imaginary part $b=0$)

$\therefore z = a$ —— ①

$\bar{z} = a$ —— ⑪

from ① & ⑪ $\therefore \boxed{z = \bar{z}}$

Conversely Suppose $z = \bar{z}$

$a+ib = \overline{a+ib}$

$a+ib = a-ib$

$\cancel{a}-\cancel{a}+ib+ib = 0$

$2ib = 0$

$b = 0 \quad \because \left(\begin{matrix} i=\sqrt{-1} \neq 0 \\ 2 \neq 0 \end{matrix}\right.$

Hence $z = a+0i$

$z = a$ which is real

Imp Q12 (ii) Show that $z = a+ib$ is pure imaginary iff $\boxed{z = -\bar{z}}$

(Real part $a=0$)

Let $z = a+ib$ is pure imaginary $\Rightarrow a=0$

So $z = ib$ —— ①

$\bar{z} = -ib$ —— ②

$\bar{z} = -(z)$ using ① in ②

$\boxed{z = -\bar{z}}$

Conversely Let $z = -\bar{z}$

$a+ib = -(a-ib)$

$a+\cancel{ib}+a-\cancel{ib} = 0$

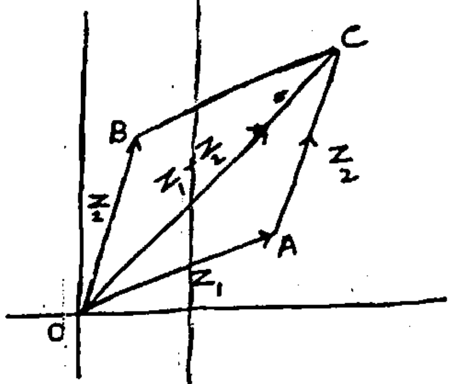
$2a = 0$

$a = 0$ (Real part of z is zero)

So $z = 0+ib$

$z = ib$ which is pure imaginary.

Example 3 Let z_1, z_2 be two complex numbers. Determine the greatest and least values of $|z_1+z_2|$



Sol Let $z_1 = \overline{OA}$

& $z_2 = \overline{OB}$

then $z_1+z_2 = \overline{OC}$

Now $|z_1| = |\overline{OA}|$

$|z_2| = |\overline{OB}| = |\overline{AC}|$

& $|z_1+z_2| = |\overline{OC}|$

In $\triangle OAC$

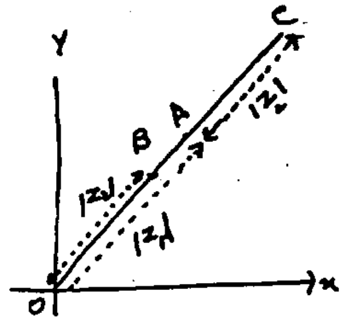
$|\overline{OA}|+|\overline{AC}| > |\overline{OC}|$ ($\because$ Sum of lengths of two sides is greater than the length of third side in any $\triangle$)

$|z_1|+|z_2| > |z_1+z_2|$ —— (I)

when $\operatorname{Arg} z_1 = \operatorname{Arg} z_2$

then OA is // to OB

hence //gm OABC becomes straight line

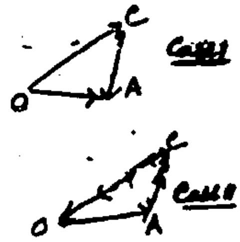


$\therefore$ OA + AC = OC

$|z_1| + |z_2| = |z_1+z_2|$ —— (II)

from (I) & (II) $|z_1|+|z_2| \geqslant |z_1+z_2|$ —— (III)

Thus greatest possible value of $|z_1+z_2|$ is $|z_1|+|z_2|$



In $\triangle OAC$ OC + CA > OA and CO + OA > CA

$|z_1+z_2|+|z_2| > |z_1|$ $\qquad$ $|z_1+z_2|+|z_1| > |z_2|$

$|z_1+z_2| > |z_1|-|z_2|$ —— (IV) $\qquad$ $|z_1+z_2| > |z_2|-|z_1|$

$|z_1+z_2| > -(|z_1|-|z_2|)$

$-|z_1+z_2| < |z_1|-|z_2|$ —— (V)

from (IV) & (V) $-|z_1+z_2| < |z_1|-|z_2| < |z_1+z_2|$ —— (VI)

& (VI) together with the extreme case when O, A, B, C are collinear gives

$$||z_1|-|z_2|| \leqslant |z_1+z_2| \text{ —— (VII)} \qquad \because \begin{pmatrix} -a \leq x \leq a \\ \Rightarrow |x| \leq a \end{pmatrix}$$

Thus least possible value of $|z_1+z_2|$ is $||z_1|-|z_2||$

Combining (III) & (VII)

$$||z_1|-|z_2|| \leqslant |z_1+z_2| \leqslant |z_1|+|z_2|$$

x ———— x

A.P. S2001

Q13 Prove analytically for complex No z_1, z_2

$$||z_1|-|z_2|| \leqslant |z_1-z_2| \leqslant |z_1+z_2| \leqslant |z_1|+|z_2|$$

Sol Let $|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$ — $\because |z|^2 = z\bar{z}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$ — $\because \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2$ — $\because z_1\bar{z}_2 + z_2\bar{z}_1 = 2Re z_1\bar{z}_2$ proved earlier

$\leqslant |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$ — $\because |Re z| \leqslant |z|$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ — $\because |z_1\bar{z}_2| = |z_1||\bar{z}_2|$ & $|\bar{z}_1| = |z_1|$

$|z_1+z_2|^2 \leq (|z_1|+|z_2|)^2$

Taking Square Root $\boxed{|z_1+z_2| \leqslant |z_1|+|z_2|}$ —— (I)

Now $|z_1| = |z_1+z_2-z_2|$ $(+ \,\&\, -z_2)$

$\leqslant |z_1+z_2| + |-z_2|$ — $\because |z_1+z_2| \leqslant |z_1|+|z_2|$

$= |z_1+z_2| + |z_2|$ — $\because |z_2| = |-z_2|$

$\boxed{|z_1|-|z_2| \leqslant |z_1+z_2|}$ —— (II)

Put $z_2 = -z_2$ in (II)

$|z_1|-|-z_2| \leqslant |z_1-z_2|$

$\boxed{|z_1|-|z_2| \leqslant |z_1-z_2|}$ —— (III)

Also $|z_2| = |z_2-z_1+z_1|$ $(+ \,\&\, -z_1)$

$\leqslant |z_2-z_1| + |z_1|$

$|z_2|-|z_1| \leqslant |z_2-z_1|$

$|z_2|-|z_1| \leqslant |z_1-z_2|$ — $\because |z_1-z_2| = |z_2-z_1|$

$-|z_1-z_2| \leqslant |z_1|-|z_2|$ —— (IV)

from (III) & (IV) $\therefore -|z_1-z_2| \leqslant |z_1|-|z_2| \leqslant |z_1-z_2|$

$\therefore ||z_1|-|z_2|| \leqslant |z_1-z_2|$ —— (V) — as if $-a \leqslant x \leqslant a$ then $|x| \leqslant a$

Now Obviously $|z_1-z_2| \leqslant |z_1+z_2|$ —— (VI)

from (I) & (VI) $|z_1-z_2| \leqslant |z_1+z_2| \leqslant |z_1|+|z_2|$ —— (VII)

from (V) & (VII) $||z_1|-|z_2|| \leqslant |z_1-z_2| \leqslant |z_1+z_2| \leqslant |z_1|+|z_2|$ Proved.

$z_1 = 24+7i$, $|z_2| = 6$

$|z_1| = \sqrt{(24)^2+7^2}$

$= \sqrt{576+49}$

$= \sqrt{625} = 25$

(14) We know $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

So greatest value of $|z_1 + z_2|$ is

$$= |z_1| + |z_2| = 25+6 = 31$$

also since $||z_1| - |z_2|| \leq |z_1 + z_2|$

So least value of $|z_1 + z_2|$ is

$$= ||z_1| - |z_2||$$

$$= |25 - 6| = |19| = 19$$

Q.15 If z_1, z_2 are complex numbers, show that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

PROOF L.H.S. $|z_1 + z_2|^2 + |z_1 - z_2|^2$

$$= (z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2}) \quad (\because z\bar{z} = |z|^2)$$

$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - \cancel{z_2\bar{z}_1}$$

$$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2 - 2(z_1\bar{z}_1 + z_2\bar{z}_2)$$

$$= 2(|z_1|^2 + |z_2|^2) = \text{R.H.S.}$$

599 Imp

Q.16 Prove that $\left|\dfrac{az+b}{\bar{b}z+\bar{a}}\right| = 1$ for $|z| = 1$

Sol we have L.H.S. $= \left|\dfrac{az+b}{\bar{b}z+\bar{a}}\right| = \dfrac{|az+b|}{|\bar{b}z+\bar{a}|} \quad \left(\because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right)$

$(\because |z| = |\bar{z}|)$

$$= \frac{|az+b|}{|\overline{\bar{b}z+\bar{a}}|} = \frac{|az+b|}{|\bar{\bar{b}}\bar{z}+\bar{\bar{a}}|}$$

$$= \frac{|az+b|}{|\bar{\bar{b}}\bar{z}+\bar{\bar{a}}|} = \frac{|az+b|}{|b\bar{z}+a|} \quad (\because \bar{\bar{z}} = z)$$

$$= \frac{|z|\,|az+b|}{|z|\,|b\bar{z}+a|} = \frac{|z|\,|az+b|}{|bz\bar{z}+az|} \quad (\because |z_1z_2| = |z_1||z_2|)$$

$$= \frac{|z||az+b|}{|az+b|} \qquad \left(\because \; z\bar{z} = |z|^2 = 1 \text{ as } |z| = 1 \text{ given}\right)$$

$$= |z| = 1 \; \text{R.H.S}$$

Q.17 Find locus of the points in the plane which satisfying each of the following conditions:

Part-(i) $|z-5| = 6$

Sol $|z-5| = 6 \rightarrow$ ① let $z = a+ib$

Then ① will become

$|a+ib-5| = 6$ or $|(a-5)+ib| = 6$

or $\sqrt{(a-5)^2 + b^2} = 6$ or $(a-5)^2 + (b-0)^2 = 36$

which shows that the locus is a ⊙ having centre at point $(5,0)$ and radius $= 6$ unit

Part-(ii) $|z-2i| \geq 1$

Sol we have $|z-2i| \geq 1 \rightarrow$ (i)

let $z = x+iy$. So (i) will be

$|x+iy-2i| \geq 1$ or $|x+i(y-2)| \geq 1$

or $\sqrt{x^2+(y-2)^2} \geq 1$ or $x^2+(y-2)^2 \geq 1$

or $(x-0)^2+(y-2)^2 \geq 1 \longrightarrow$ (ii)

inequality give that required locus is a set of those points that lies on the ⊙ or outside the circle having centre at $(0,2)$ and radius $= 1$

Part-(iii) $Re(z+2) = -1$

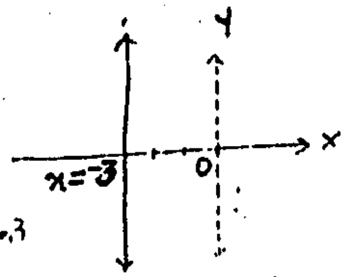
Sol We are given that

$Re(z+2) = -1 \longrightarrow$ (i)

let $z = x+iy$, put in (i)

$\Rightarrow \quad Re\,(x+iy+2) = -1$

or $\quad Re\,\big((x+2)+iy\big) = -1$



$\Rightarrow \quad x+2 = -1 \quad$ or $\quad x = -3$

The locus is the line $\parallel$ to Y-axis on left side of y-axis.

x ——————— x

Part - (iv) $\quad Re(i\bar{z}) = 3$

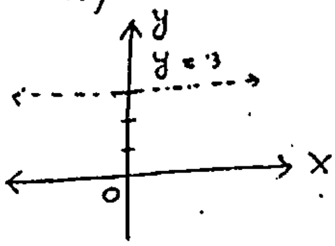
Sol $\quad$ we have $\quad Re\,(i\,\bar{z}) = 3 \longrightarrow$ (i)

let $\quad z = x+iy \quad$ then (i) will be

$Re\left(i\;\overline{x+iy}\right) = 3 \quad$ or $\quad Re\big(i\,(x-iy)\big) = 3$

or $\quad Re\,(y+ix) = 3 \quad \Rightarrow \quad y = 3$

$\Rightarrow$ loucus is the horizontal line $y=3$.



x ——————— x

Part - (v) $\quad |z+i| = |z-i| \longrightarrow$ (i)

Sol $\quad$ Put $\quad z = x+iy$ in (i) we get

$|x+iy+i| = |x+iy-i|$

or $\quad |x+i(y+1)| = |x+i(y-1)|$

$\Rightarrow \quad \sqrt{x^2+(y+1)^2} = \sqrt{x^2+(y-1)^2}$

Sq. we get

$x^2+(y+1)^2 = x^2+(y-1)^2$

or $\quad (y+1)^2 = (y-1)^2$

$\Rightarrow \quad y^2+2y+1 = y^2-2y+1 \quad \Rightarrow \quad 2y = -2y$

or $\quad 4y = 0 \quad \Rightarrow \quad y = 0 \longrightarrow$ (ii)

Eq. (ii) gives that the required locus is set of all those points that lies on x-axis.

Part-(vi) $|z+3| + |z+1| = 4 \rightarrow (i)$

Sol Put $z = x+iy$ in (i) we get

$$|x+yi+3| + |x+iy+1| = 4$$

or $|(x+3)+iy| + |(x+1)+iy| = 4$

or $\sqrt{(x+3)^2+y^2} + \sqrt{(x+1)^2+y^2} = 4$

$\Rightarrow \sqrt{(x+3)^2+y^2} = 4 - \sqrt{(x+1)^2+y^2}$

Sq. we get

$$(x+3)^2+y^2 = 16+(x+1)^2+y^2 - 8\sqrt{(x+1)^2+y^2}$$

$$\cancel{x^2}+\cancel{y^2}+6x+9-\cancel{x^2}-2x-17-\cancel{y^2} = -8\sqrt{(x+1)^2+y^2}$$

$$4x-8 = -8\sqrt{(x+1)^2+y^2}$$

or $x-2 = -2\sqrt{(x+1)^2+y^2}$

Sq. we get

$$x^2-4x+4 = 4(x^2+2x+1+y^2)$$

or $3x^2+12x+4y^2=0$ which is required locus

×——×

Part-(vii) $-1 \leq Re\, z < 1$

Sol Put $x+iy = z$ in (i)

$\Rightarrow -1 \leq Re(x+iy) < 1$

or $-1 \leq x < 1$

$\Rightarrow$ The value of x lies in the interval $[-1, 1[$.

×——×

Part-(viii) $Im\, z < 0 \rightarrow (i)$

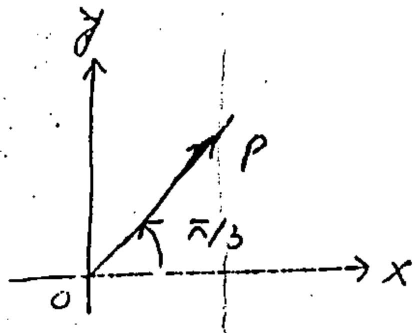
Sol Put $z = x+iy$ in (i)

$\Rightarrow Im(x+iy)<0$ or $y<0$ which is required locus i.e. value of y is $-ive$.

Part - (ix) $\quad \text{Arg}\, z = \frac{\pi}{3}$

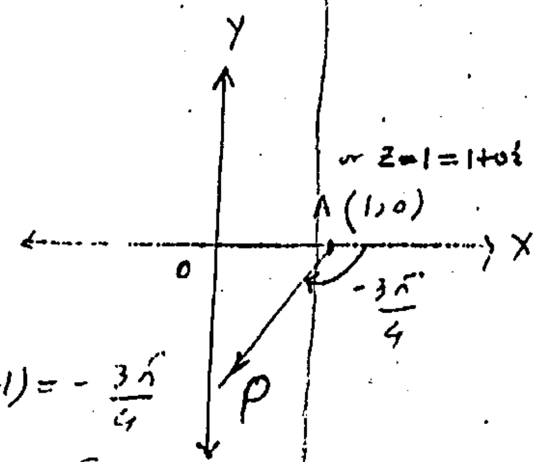
Sol Let $z = \overrightarrow{OP}$. then $\text{Arg}\, z = \text{Arg}\, \overrightarrow{OP} = \frac{\pi}{3}$

So, the required locus is a line $\overrightarrow{OP}$ that makes $\theta = \frac{\pi}{3}$ with +ive x-axis, as shown in fig



× ———— ×

Part -(x) $\quad \text{Arg}(z-1) = -\frac{3\pi}{4}$



Sol put $z = x + iy$

$\Rightarrow \text{Arg}(z-1) = \text{Arg}(x+iy-1) = -\frac{3\pi}{4}$

or $\text{Arg}\big((x-1) + iy\big) = -\frac{3\pi}{4}$

or $\text{Arg}\Big((x-1) + i(y-0)\Big) = -\frac{3\pi}{4}$

The required locus is represented by the line $\overrightarrow{AP}$ which makes an angle of measure $= -\frac{3\pi}{4}$ with +ive x-axis at pt $A(1,0)$ on x-axis

× ———— ×

De Moivre's Theorem

Statement:- If n is any integer, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \rightarrow (1)$$

Proof:- Case-1 Put $n=0$ in (1), then

$L.H.S = (\cos\theta + i\sin\theta)^0 = 1$

$R.H.S = \cos(0\cdot\theta) + i\sin(0\cdot\theta) = \cos 0 + i\sin 0 = 1 + 0i = 1$

$\Rightarrow L.H.S. = R.H.S.$

Ex 1.2

1-2-1

① Write the following expression in the form of $a+ib$.

(i) $(-\sqrt{3}+i)^2$

Let $z = (-\sqrt{3}+i)$ $\Rightarrow x = -\sqrt{3}$, $y = 1$

$r = |z| = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{3+1} = 2$

$\cos\theta = \frac{x}{r} = \frac{-\sqrt{3}}{2}$ $\Rightarrow \theta = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

$\sin\theta = \frac{y}{r} = \frac{1}{2}$ $\Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$

$\theta = \frac{5\pi}{6}$

$\because$ x is $-ve$, y is $+ve$ } So θ lies in 2nd Quad.

$\therefore$ Principal Arg of $z = \pi - \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Hence $z = r(\cos\theta + i\sin\theta)$

$-\sqrt{3}+i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

squaring $(-\sqrt{3}+i)^2 = 2^2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)^2$

$= 4\left(\cos 2\left(\frac{5\pi}{6}\right) + i\sin 2\left(\frac{5\pi}{6}\right)\right)$

$= 4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$= 4\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$

$= 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$= 4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 2 - i2\sqrt{3}$

$\frac{5\pi}{3} - 2\pi = \frac{-\pi}{3}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(ii) $(-3i)^4$

Let $z = -3i = 0 - 3i$ $\Rightarrow x = 0$, $y = -3$

$r = |z| = \sqrt{0^2+(-3)^2} = 3.$

$\cos\theta = \frac{x}{r} = \frac{0}{3} = 0$ $\Rightarrow \theta = \cos^{-1}(0)$

$\sin\theta = \frac{y}{r} = \frac{-3}{3} = -1$ $\Rightarrow \theta = \cos^{-1}(-1)$

$\Rightarrow \theta = \frac{-\pi}{2}$

$\because$ x is $+ve$, y is $-ve$ } So θ lies in 4^{th} Quad.

$\therefore$ Principal Arg of $z = -\theta = -\frac{\pi}{2}$

$z = r(\cos\theta + i\sin\theta)$

$-3i = 3\left(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})\right)$

$(-3i)^4 = 3^4\left[\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})\right]^4$

$= 81\left[\cos(4\cdot\frac{\pi}{2}) + i\sin(-4\cdot\frac{\pi}{2})\right]$

$= 81\left(\cos(2\pi) + i\sin(-2\pi)\right)$

$= 81\left(\cos 2\pi - i\sin 2\pi\right)$

$= 81\ (1-0)$

$(-3i)^4 = 81$ Ans

(iii) $\left(\frac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i}\right)^6$

Let $z = \frac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i}$

$= \frac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i} \times \frac{1-\sqrt{3}\,i}{1-\sqrt{3}\,i}$

$= \frac{(1-\sqrt{3}i)^2}{1+3}$

$= \frac{1^2+(\sqrt{3}i)^2 - 2\cdot 1\cdot\sqrt{3}\,i}{4}$

$= \frac{1-3-2\sqrt{3}\,i}{4}$

$= -\frac{2}{4} - \frac{2\sqrt{3}}{4}i$

$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\left.\begin{array}{l} x = -\frac{1}{2} \\ y = -\frac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow r = |z| = \sqrt{\frac{1}{4}+\frac{3}{4}}$

$\boxed{r = 1}$

$\left.\begin{array}{ll} \cos\theta = \frac{x}{r} = -\frac{1/2}{1} & \Rightarrow \theta = \cos^{-1}(-\frac{1}{2}) \\ \sin\theta = \frac{y}{r} = -\frac{\sqrt{3}}{2} & \Rightarrow \theta = \sin^{-1}(-\frac{\sqrt{3}}{2}) \end{array}\right\} \Rightarrow \theta = -\frac{2\pi}{3}$

$\because$ $\left.\begin{array}{l} x \text{ is -ive} \\ y \text{ is -ive} \end{array}\right\}$ So θ lies in IIIrd Quad
So Principal Arg of $z = -(\pi - \theta)$
$= -(\pi - \frac{\pi}{3}) = \boxed{-\frac{2\pi}{3}}$

$\therefore\ z = r(\cos\theta + i\sin\theta)$

$\therefore\ \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1\left(\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3})\right)$

$$So\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^6 = 1^6\left[\cos 6\left(-\frac{2\pi}{3}\right) + i\sin 6\left(-\frac{2\pi}{3}\right)\right]$$

$$= \cos(-4\pi) + i\sin(-4\pi)$$

$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

$$= \cos 4\pi - i\sin 4\pi$$

$$= 1 - 0 = 1 \quad \text{ans.}$$

Q. No. 2

Part-(i) Simplify $\dfrac{(\cos 2\theta + i\sin 2\theta)^5(\cos 3\theta - i\sin 3\theta)^6}{(\cos 4\theta - i\sin 4\theta)^7(\cos 5\theta + i\sin 5\theta)^8}$

Sol

$$= \frac{(\cos 2\theta + i\sin 2\theta)^5(\cos(-3\theta) + i\sin(-3\theta))^6}{[\cos(-4\theta) + i\sin(-4\theta)]^7[\cos 5\theta + i\sin 5\theta]^8}$$

$$= \frac{[(\cos\theta + i\sin\theta)^2]^5[(\cos\theta + i\sin\theta)^{-3}]^6}{[(\cos\theta + i\sin\theta)^{-4}]^7[(\cos\theta + i\sin\theta)^5]^8}$$

$$= \frac{(\cos\theta + i\sin\theta)^{10}(\cos\theta + i\sin\theta)^{-18}}{(\cos\theta + i\sin\theta)^{-28}(\cos\theta + i\sin\theta)^{40}}$$

$$= (\cos\theta + i\sin\theta)^{10-18+28-40}$$

$$= (\cos\theta + i\sin\theta)^{-20}$$

$$= \cos(-20\theta) + i\sin(-20\theta)$$

$$= \cos 20\theta - i\sin 20\theta \quad \text{ans.}$$

Part-(ii) $\dfrac{(\cos\alpha - i\sin\alpha)^{11}}{(\cos\beta + i\sin\beta)^9}$

Sol

$$= \frac{\{\cos(-\alpha) + i\sin(-\alpha)\}^{11}}{(\cos\beta + i\sin\beta)^9}$$

$\because$ To make $\cos\theta + i\sin\theta$ i.e $+$ive sign.

$$= [(\cos\alpha + i\sin\alpha)^{-1}]^{11}\{\cos\beta + i\sin\beta\}^{-9}$$

$$= (\cos\alpha + i\sin\alpha)^{-11}(\cos\beta + i\sin\beta)^{-9}$$

$$= [\cos(-11\alpha) + i\sin(-11\alpha)][\cos(-9\beta) + i\sin(-9\beta)]$$

$$= (\cos(-11\alpha)\cos(-9\beta) - \sin(-11\alpha)\sin(-9\beta)) + i\begin{pmatrix}\cos(-11\alpha)\sin(-9\beta)\\ +\sin(-11\alpha)\cos(-9\beta)\end{pmatrix}$$

$$= \cos(-11\alpha - 9\beta) + i\sin(-11\alpha - 9\beta)$$

$$= Cis(-11\alpha - 9\beta) = Cis(-(11\alpha + 9\beta)) \quad \text{Ans.}$$

x ———— x

Part-(iii) $\dfrac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)}$

Sol

$$= \frac{(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)}{(\cos\gamma\cos\delta - \sin\gamma\sin\delta) + i(\sin\gamma\cos\delta + \cos\gamma\sin\delta)}$$

$$= \frac{\cos(\alpha+\beta) + i\sin(\alpha+\beta)}{\cos(\gamma+\delta) + i\sin(\gamma+\delta)}$$

$$= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)][\cos(\gamma+\delta) + i\sin(\gamma+\delta)]^{-1}$$

$$= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)][\cos(-\gamma-\delta) + i\sin(-\gamma-\delta)]$$

$$= [\cos(\alpha+\beta)\cos(-\gamma-\delta) - \sin(\alpha+\beta)\sin(-\gamma-\delta)]$$
$$+ i[\sin(\alpha+\beta)\cos(-\gamma-\delta) + \sin(-\gamma-\delta)\cos(\alpha+\beta)]$$

$$= \cos(\alpha+\beta-\gamma-\delta) + i\sin(\alpha+\beta-\gamma-\delta)$$

$$= Cis(\alpha+\beta-\gamma-\delta)$$

x ———— x

Part-(iv) $\left(3\,Cis\frac{\pi}{6}\right)^7 / \left(4\,Cis\frac{\pi}{3}\right)^6$

Sol:

$$= 3^7\left(Cis\frac{\pi}{6}\right)^7 / 4^6\left(Cis\frac{\pi}{3}\right)^6$$

$$= \frac{3^7}{4^6}\cdot\frac{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^7}{\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^6}$$

1.2-5

$$= \frac{3^7}{4^6} \cdot \left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{-6}$$

$$= \frac{3^7}{4^6} \cdot \left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\left(\cos\left(-6\cdot\frac{\pi}{3}\right) + i\sin\left(-6\cdot\frac{\pi}{3}\right)\right)$$

$$= \frac{3^7}{4^6} \left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right]\left[\cos(-2\pi) + i\sin(-2\pi)\right]$$

$$= \frac{3^7}{4^6} \left[\cos\left(\frac{7\pi}{6} - 2\pi\right) + i\sin\left(\frac{7\pi}{6} - 2\pi\right)\right]$$

$\because \cos\frac{7\pi}{6}\cos(-2\pi) - \sin\frac{7\pi}{6}\sin(-2\pi) = \cos\left(\frac{7\pi}{6} + (-2\pi)\right)$

$$= \frac{3^7}{4^6} \operatorname{Cis}\left(-\frac{5\pi}{6}\right) \quad \text{Ans.}$$

x ———— x

Q.3 (i) Prove that $\left[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)\right]^n + \left[(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)\right]^n$

$$= 2^{n+1}\sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\theta+\phi+\pi}{2}\right)$$

Sol: L.H.S. $\left[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)\right]^n + \left[(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)\right]^n$

using formulas

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

and $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$ we get

$$= \left[\left(-2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right) + i\left(2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right)\right]^n$$
$$+ \left[\left(-2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right) - i\left(2\cos\frac{\theta+\phi}{2}\cdot\sin\frac{\theta-\phi}{2}\right)\right]^n$$

$$= 2^n\sin^n\left(\frac{\theta-\phi}{2}\right)\left[-\sin\frac{\theta+\phi}{2} + i\cos\frac{\theta+\phi}{2}\right]^n$$
$$+ 2^n\sin^n\left(\frac{\theta-\phi}{2}\right)\left[-\sin\frac{\theta+\phi}{2} - i\cos\frac{\theta+\phi}{2}\right]^n$$

$\because \cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta$ and $\sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta$

$$= 2^n\sin^n\left(\frac{\theta-\phi}{2}\right)\left[\left\{\cos\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)\right\}^n\right.$$
$$\left. + \left\{\cos\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right) - i\sin\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)\right\}^n\right]$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right)\left[\left\{\cos n\left(\frac{\pi+\theta+\phi}{2}\right)+i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)\right\}\right.$$
$$\left.+\left\{\cos n\left(\frac{\pi+\theta+\phi}{2}\right)-i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)\right\}\right]$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right)\left[\begin{array}{l}\cos n\left(\frac{\pi+\theta+\phi}{2}\right)+\cancel{i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)}\\ \cos n\left(\frac{\pi+\theta+\phi}{2}\right)-\cancel{i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)}\end{array}\right]$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right)\, 2\cos n\left(\frac{\pi+\theta+\phi}{2}\right)$$

$$= 2^{n+1} \sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\pi+\theta+\phi}{2}\right) = RHS$$

x ———————————— x

P. (ii) $\left(\dfrac{1+\sin x+i\cos x}{1+\sin x-i\cos x}\right)^n = \cos n\left(\frac{\pi}{2}-x\right)+i\sin n\left(\frac{\pi}{2}-x\right)$

Sol: L.H.S $\left(\dfrac{1+\sin x+i\cos x}{1+\sin x-i\cos x}\right)^n$

$$= \left(\frac{(\sin^2 x+\cos^2 x)+(\sin x+i\cos x)}{1+\sin x-i\cos x}\right)^n$$

$$= \left(\frac{(\sin x+i\cos x)(\sin x-i\cos x)+(\sin x+i\cos x)}{1+\sin x-i\cos x}\right)^n$$

$$= \left(\frac{(\sin x+i\cos x)\cancel{(\sin x-i\cos x+1)}}{\cancel{(1+\sin x-i\cos x)}}\right)^n$$

$$= (\sin x+i\cos x)^n$$

$\boxed{\because \cos\left(\frac{\pi}{2}-x\right)=\sin x \quad \text{and} \quad \sin\left(\frac{\pi}{2}-x\right)=\cos x}$

$$= \left[\cos\left(\frac{\pi}{2}-x\right)+i\sin\left(\frac{\pi}{2}-x\right)\right]^n$$

applying Demoivre's th.

$$= \cos n\left(\frac{\pi}{2}-x\right)+i\sin n\left(\frac{\pi}{2}-x\right)$$

$$= RHS$$

Q.4 $2\cos\theta = x + \frac{1}{x}$; $2\cos\phi = y + \frac{1}{y}$, $2\cos\psi = z + \frac{1}{z}$,

then prove that

Part-(i) $2\cos(\theta+\phi+\psi) = xyz + \frac{1}{xyz}$

PROOF we have $2\cos\theta = x + \frac{1}{x}$, $\Rightarrow x = \cos\theta + i\sin\theta$

$2\cos\phi = y + \frac{1}{y} \Rightarrow y = \cos\phi + i\sin\phi$

and. $2\cos\psi = z + \frac{1}{z} \Rightarrow z = \cos\psi + i\sin\psi$

Then $x.y.z = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)$

$= [(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \sin\phi\cdot\cos\theta)]$
$[\cos\psi + i\sin\psi]$

$= [\cos(\theta+\phi) + i\sin(\theta+\phi)][\cos\psi + i\sin\psi]$

$= (\cos(\theta+\phi)\cos\psi - \sin(\theta+\phi)\sin\psi)$
$+ i(\sin(\theta+\phi)\cos\psi + \cos(\theta+\phi)\sin\psi)$

$x.y.z = \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi) \longrightarrow ①$

Similarly $\frac{1}{x.y.z} = \frac{1}{(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)}$

$= \frac{1}{\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)}$

$= [\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)]^{-1}$

$\frac{1}{xyz} = \cos(\theta+\phi+\psi) - i\sin(\theta+\phi+\psi \longrightarrow ②$

+' Eqns ① and ②, we get

$xyz + \frac{1}{xyz} = 2\cos(\theta+\phi+\psi)$ Proved.

x———————x

Part-(ii) $2\cos(m\theta + n\phi) = x^m y^n + \frac{1}{x^m y^n}$

Sol: we are given that $2\cos\theta = x + \frac{1}{x}$

it is because if $x = \cos\theta + i\sin\theta$

1.2-8

and $2\cos\phi = y + \frac{1}{y} \Rightarrow y = \cos\phi + i\sin\phi$

So $x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

and $y^n = (\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$

then $x^m y^n = (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$

$= (\cos m\theta\cos n\phi - \sin m\theta\sin n\phi) + i(\sin m\theta\cos n\phi + \sin n\phi\cos m\theta)$

$$x^m y^n = \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) \longrightarrow ①$$

and $\frac{1}{x^m y^n} = \frac{1}{(\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)}$

$= \frac{1}{\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)}$

$= [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1}$

$$\frac{1}{x^m y^n} = \cos(m\theta + n\phi) - i\sin(m\theta + n\phi) \longrightarrow ②$$

Add equations ① and ②, we get

$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

Proved

Q.5(i) Find Three Cube roots of '$8i$'

Sol Let $z^3 = 8i = 8(0 + i)$

or $z^3 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\left\{\begin{array}{l} x=0,\ y=1,\ r=\sqrt{0+1}=1 \\ \theta \text{ lies on} \\ \theta = \sin^{-1} 1 \end{array}\right\} \theta = \frac{\pi}{2}$

lies 1st and $y+ive$

or $z^3 = 2^3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\theta + 2k\pi$

$= 2^3\left(\cos\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{\pi}{2} + 2k\pi\right)\right)$, where $k \in Z$

$$\Rightarrow Z_k = 2\left[\cos\left(2\pi k+\frac{\pi}{2}\right)+i\sin\left(2\pi k+\frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

where $k=0,1,2.$

$$\therefore Z_k = 2\left[\cos\left(\frac{4k\pi+\pi}{6}\right)+i\sin\left(\frac{4k\pi+\pi}{6}\right)\right]$$

1.2-9

So put $k=0,1,2$, then required three roots are given by.

for $k=0$, $Z_0 = 2\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right] = 2\left\{\frac{\sqrt3}{2}+\frac{i\cdot 1}{2}\right\} \Rightarrow \boxed{\sqrt3+i = Z_0}$

for $k=1$, $Z_1 = 2\left\{\cos\left(\frac{4\pi+\pi}{6}\right)+i\sin\left(\frac{4\pi+\pi}{6}\right)\right\} = 2\left\{\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right\}$

$$= 2\left\{-\frac{\sqrt3}{2}+\frac{i}{2}\right\} \Rightarrow \boxed{Z_1 = -\sqrt3+i}$$

and 3rd root is obtained by $k=2$, we get

$$Z_2 = 2\left\{\cos\left(\frac{8\pi+\pi}{6}\right)+i\sin\left(\frac{8\pi+\pi}{6}\right)\right\} = 2\left[\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right]$$

$$= 2\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right] = 2(0-i) \Rightarrow \boxed{Z_2 = -2i}$$

x ———— x

Part-(ii) Find four fourth roots of each of the following complex number.

(a) $-16i$, (b) 64, (c) $-2\sqrt3+2i$

(a) Since we have find four fourth roots of $-16i$

So put $z^4 = -16i = 16(0-i)$

$$= 2^4\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$$

$x=0, y=-1$, $r=\sqrt{0^2+(-1)^2}=1$

$\cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \cos^{-1}0$

$\sin\theta = \frac{y}{r} = \frac{-1}{1} = -1 \Rightarrow \theta = \cos^{-1}(-1)$

$\Rightarrow \theta = -\frac{\pi}{2}$

(angle is in IV Q $\therefore 1-\theta$, Hence $-\frac{\pi}{2}$)

$$z^4 = 2^4\left[\cos\left(2\pi k-\frac{\pi}{2}\right)+i\sin\left(2k\pi-\frac{\pi}{2}\right)\right]$$

So fourth root of $-16i$ is

$$Z_k = 2\left[\cos\left(2\pi k-\frac{\pi}{2}\right)+i\sin\left(2k\pi-\frac{\pi}{2}\right)\right]^{\frac{1}{4}}$$

where $k=0,1,2,3$

$$= 2\left[\cos\frac{1}{4}\left(\frac{4k\pi-\pi}{2}\right)+i\sin\left(\frac{4k\pi-\pi}{2}\right)\frac{1}{4}\right]$$

$$Z_k = 2\left[\cos\left(\frac{4k\pi-\pi}{8}\right)+i\sin\left(\frac{4k\pi-\pi}{8}\right)\right], \quad k=0,1,2,3 \longrightarrow (1)$$

So required four roots can be obtained by putting $K = 0, 1, 2, 3$ in ①, we get $\boxed{1\text{-}2\text{-}10}$

$$Z_0 = 2\left[\cos\left(\frac{0-\pi}{8}\right) + i\sin\left(\frac{0-\pi}{8}\right)\right] = 2\left(\cos\left(\frac{-\pi}{8}\right) + i\sin\left(\frac{-\pi}{8}\right)\right)$$

or $\boxed{Z_0 = 2\,\text{cis}\left(\frac{-\pi}{8}\right)}$

for $K=1$, $Z_1 = 2\left[\cos\left(\frac{4\pi-\pi}{8}\right) + i\sin\left(\frac{4\pi-\pi}{8}\right)\right] = 2\ \text{cis}\left(\frac{3\pi}{8}\right)$

for $K=2$, $Z_2 = 2\left[\cos\left(\frac{8\pi-\pi}{8}\right) + i\sin\left(\frac{8\pi-\pi}{8}\right)\right] = 2\ \text{cis}\left(\frac{7\pi}{8}\right)$

for $K=3$, $Z_3 = 2\left[\cos\left(\frac{12\pi-\pi}{8}\right) + i\sin\left(\frac{12\pi-\pi}{8}\right)\right] = 2\ \text{cis}\left(\frac{11\pi}{8}\right)$ $\because \frac{11\pi}{8} - 2\pi = -\frac{5\pi}{8}$

$= 2\,\text{cis}\left(-\frac{5\pi}{8}\right)$

x———————x

(b) let $Z^4 = 64 = 64(1+0i) = 64(\cos 0 + i\sin 0)$

$$= 64\left(\cos(2\pi K + 0) + i\sin(2\pi K + 0)\right)$$

$$Z^4 = 64\left[\cos 2\pi K + i\sin 2\pi K\right]$$

So four 4th root of 64 are

$$Z_k = (64)^{\frac{1}{4}}\left[\cos 2\pi K + i\sin 2\pi K\right]^{\frac{1}{4}}, \text{ where } k = 0, 1, 2, 3$$

$$= (16\times 4)^{\frac{1}{4}}\left[\cos\frac{2\pi K}{4} + i\sin\frac{2\pi K}{4}\right]$$

$$= (2^4 \cdot 2^2)^{\frac{1}{4}}\left(\cos\frac{\pi K}{2} + i\sin\frac{\pi K}{2}\right]$$

$$Z_k = 2\sqrt{2}\left[\cos\frac{\pi K}{2} + i\sin\frac{\pi K}{2}\right]$$

for first root, put $K = 0$, $\Rightarrow Z_0 = 2\sqrt{2}\left[\cos 0 + i\sin 0\right] = 2\sqrt{2}(1 + i\cdot 0)$

$\boxed{Z_0 = 2\sqrt{2}}$

put $K = 1$, $Z_1 = 2\sqrt{2}\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\} = 2\sqrt{2}\left\{0 + i\right\} \Rightarrow \boxed{2\sqrt{2}\,i = Z_1}$

put $K = 2$, $Z_2 = 2\sqrt{2}\left(\cos\pi + i\sin\pi\right) = 2\sqrt{2}\left[-1 + 0i\right] \Rightarrow \boxed{Z_2 = -2\sqrt{2}}$

put $K = 3$, $Z_3 = 2\sqrt{2}\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right] = 2\sqrt{2}\left(0 - i\right]$

$\Rightarrow \boxed{Z_3 = -2\sqrt{2}\,i}$

(C) Let $Z^4 = -2\sqrt{3} + 2i$ $\quad r = |z| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{12+4} = 4$

$x = -2\sqrt{3}$, $y = 2$, $r = 4 = |z|$

$\cos\theta = \frac{x}{r} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$, $\sin\theta = \frac{y}{r} = \frac{2}{4} = \frac{1}{2}$ $\Rightarrow \theta = \frac{5\pi}{6}$

(x $-$, y $+$: 2nd Q. $\therefore \pi - \theta$. So $\pi - \frac{\pi}{6} = \boxed{\frac{5\pi}{6}}$)

$$= 4\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$$

$$= 4\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$$

$$z^4 = 4\left[\cos\left(\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(\frac{5\pi}{6} + 2k\pi\right)\right]$$

$$\Rightarrow z^4 = (4)\left[\cos\left(2k\pi + \frac{5\pi}{6}\right) + i\sin\left(2k\pi + \frac{5\pi}{6}\right)\right]$$

$$\therefore Z_k = (4)^{\frac{1}{4}}\left[\cos\left(\frac{12k\pi + 5\pi}{6}\right) + i\sin\left(\frac{12k\pi + 5\pi}{6}\right)\right]^{\frac{1}{4}}$$ where $k = 0, 1, 2, 3$

$$\Rightarrow Z_k = (2^2)^{\frac{1}{4}}\left[\cos\left(\frac{12k\pi + 5\pi}{24}\right) + i\sin\left(\frac{12k\pi + 5\pi}{24}\right)\right]$$

$$= 2^{\frac{1}{2}} \operatorname{cis}\left(\frac{k\pi}{2} + \frac{5\pi}{24}\right) \quad \text{where } k = 0, 1, 2, 3$$

put $k = 0$, we get

$$Z_0 = \sqrt{2}\left[\cos\frac{5\pi}{24} + i\sin\frac{5\pi}{24}\right] = \sqrt{2}\operatorname{cis}\frac{5\pi}{24}$$

for $k = 1$, $\quad Z_1 = \sqrt{2}\left[\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right] = \sqrt{2}\operatorname{cis}\frac{17\pi}{24}$

for $k = 2$, $\quad Z_2 = \sqrt{2}\left[\cos\frac{29\pi}{24} + i\sin\frac{29\pi}{24}\right] = \sqrt{2}\operatorname{cis}\left(\frac{-19\pi}{24}\right)$ $\quad \frac{29\pi}{24} - 2\pi = \frac{-19\pi}{24}$

for $k = 3$, $\quad Z_3 = \sqrt{2}\left[\cos\frac{41\pi}{24} + i\sin\frac{41\pi}{24}\right] = \sqrt{2}\operatorname{cis}\left(\frac{-7\pi}{24}\right)$ $\quad \frac{41\pi}{24} - 2\pi = \frac{-7\pi}{24}$

x ——— x

Q.6 Find six 6th roots of (a) -1, (b) $1+i$

Sol (a) Let $Z^6 = -1 = (-1 + 0i)$ $\quad r = |z| = \sqrt{(-1)^2 + 0^2} = +1$

$\cos\theta = \frac{x}{r} = \frac{-1}{1} = -1 \Rightarrow \theta = \cos^{-1}(-1)$, $\sin\theta = \frac{y}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \sin^{-1}(0)$ $\Rightarrow \theta = \pi$

(x $-$, y $+$, So QII $\therefore \pi - \theta$, $\pi - 0 = \boxed{\pi}$)

$$= 1\left[\cos\pi + i\sin\pi\right]$$

$$Z^6 = \left[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\right]$$

So six 6th roots of '-1' are given by

$$Z_k = \left[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\right]^{\frac{1}{6}}$$

or $\quad Z_k = \cos\left(\frac{2k\pi + \pi}{6}\right) + i\sin\left(\frac{2k\pi + \pi}{6}\right)$, where $k = 0, 1, 2, 3, 4, 5$

So for $k = 0$, $\quad Z_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

for $k=1$, $Z_1 = \cos\left(\frac{2\pi+\pi}{6}\right) + i\sin\left(\frac{2\pi+\pi}{6}\right) = \text{Cis}\frac{\pi}{2}$

$\Rightarrow \boxed{Z_1 = 0 + i}$

1.2-12

for $k=2$, $Z_2 = \cos\left(\frac{4\pi+\pi}{6}\right) + i\sin\left(\frac{4\pi+\pi}{6}\right) = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$\Rightarrow \boxed{Z_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}}$ $\left(\because \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2},\ \sin\frac{5\pi}{6} = \frac{1}{2}\right)$

for $k=3$, $Z_3 = \cos\left(\frac{6\pi+\pi}{6}\right) + i\sin\left(\frac{6\pi+\pi}{6}\right) = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$\frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$

$= \cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)$

$= \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} = \boxed{-\frac{\sqrt{3}}{2} - \frac{i}{2}}$

for $k=4$, $Z_4 = \cos\left(\frac{8\pi+\pi}{6}\right) + i\sin\left(\frac{8\pi+\pi}{6}\right)$

$= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = \text{Cis}\left(-\frac{\pi}{2}\right)$ $\quad \frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$

or $\boxed{Z_4 = 0 - i}$

for $k=5$, $Z_5 = \cos\left(\frac{10\pi+\pi}{6}\right) + i\sin\left(\frac{10\pi+\pi}{6}\right) = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

$Z_5 = \cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)$

or $\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$ $\Rightarrow \boxed{Z_5 = \frac{\sqrt{3}}{2} - \frac{i}{2}}$

(b) let $Z^6 = 1 + i$ $\quad \because |Z| = |1+i| = \sqrt{1+1} = \sqrt{2} = r$

$\theta = \tan^{-1}\frac{y}{x} = \frac{1}{1} = 1 \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

OR $\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

x +ive, y +ive so Ist Q. $\therefore \theta$

$= \sqrt{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right] = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$Z^6 = \sqrt{2}\left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]$

So Six 6th roots of $1+i$ are

$Z_k = \left[(2)^{\frac{1}{2}}\right]^{\frac{1}{6}}\left[\cos\left(\frac{8\pi k+\pi}{4}\right) + i\sin\left(\frac{8k\pi+\pi}{4}\right)\right]^{\frac{1}{6}}$

or $Z_k = (2)^{\frac{1}{12}}\left[\cos\left(\frac{8\pi k+\pi}{24}\right) + i\sin\left(\frac{8\pi k+\pi}{24}\right)\right]$

where $k = 0, 1, 2, 3, 4, 5$

for $k=0$, $\quad Z_0 = (2)^{\frac{1}{12}}\left[\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right] = 2^{\frac{1}{12}}\operatorname{cis}\frac{\pi}{24}$

for $k=1$, $\quad Z_1 = (2)^{\frac{1}{12}}\left[\cos\frac{9\pi}{24} + i\sin\frac{9\pi}{24}\right] = 2^{\frac{1}{12}}\operatorname{cis}\frac{3\pi}{8}$

for $k=2$, $\quad Z_2 = (2)^{\frac{1}{12}}\left[\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right] = 2^{\frac{1}{12}}\operatorname{cis}\frac{17\pi}{24}$

for $k=3$, $\quad Z_3 = (2)^{\frac{1}{12}}\left[\cos\frac{25\pi}{24} + i\sin\frac{25\pi}{24}\right]$

$$= (2)^{\frac{1}{12}}\left[\cos\left(\pi+\frac{\pi}{24}\right) + i\sin\left(\pi+\frac{\pi}{24}\right)\right]$$

$$= (2)^{\frac{1}{12}}\left[-\cos\frac{\pi}{24} - i\sin\frac{\pi}{24}\right]$$

$$= -(2)^{\frac{1}{12}}\left[\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right] = -2^{\frac{1}{12}}\operatorname{cis}\frac{\pi}{24}$$

(Also $\frac{25\pi}{24} - 2\pi = -\frac{23\pi}{24}$, $Z_3 = 2^{\frac{1}{12}}\operatorname{cis}\left(-\frac{23\pi}{24}\right)$)

for $k=4$, $\quad Z_4 = (2)^{\frac{1}{12}}\left[\cos\frac{33\pi}{24} + i\sin\frac{33\pi}{24}\right]$

$$= (2)^{\frac{1}{12}}\left[\cos\left(\pi+\frac{9\pi}{24}\right) + i\sin\left(\pi+\frac{9\pi}{24}\right)\right]$$

$$= (2)^{\frac{1}{12}}\left[\cos\left(\pi+\frac{3\pi}{8}\right) + i\sin\left(\pi+\frac{3\pi}{8}\right)\right]$$

$$= (2)^{\frac{1}{12}}\left[-\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right]$$

$$= -(2)^{\frac{1}{12}}\left[\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right]$$

(Also $\frac{33\pi}{24} - 2\pi = -\frac{15\pi}{24}$, $Z_4 = 2^{\frac{1}{12}}\operatorname{cis}\left(-\frac{15\pi}{24}\right)$)

for $k=5$, $\quad Z_5 = (2)^{\frac{1}{12}}\left[\cos\frac{41\pi}{24} + i\sin\frac{41\pi}{24}\right]$

$$= (2)^{\frac{1}{2}}\left[\cos\left(\pi+\frac{17\pi}{24}\right) + i\sin\left(\pi+\frac{17\pi}{24}\right)\right]$$

$$= +(2)^{\frac{1}{2}}\left[-\cos\frac{17\pi}{24} - i\sin\frac{17\pi}{24}\right]$$

$$= -(2)^{\frac{1}{2}}\left[\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right]$$

(Also $\frac{41\pi}{24} - 2\pi = -\frac{7\pi}{24}$, $Z_5 = 2^{\frac{1}{12}}\operatorname{cis}\left(-\frac{7\pi}{24}\right)$)

Q:7 Find the squares of all the fifth roots of $\frac{1}{2}+\frac{\sqrt{3}}{2}i$.

Sol

$r=\sqrt{(\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$

Let $z^5 = \frac{1}{2}+\frac{\sqrt{3}}{2}i = 1\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$

$\cos\theta=\frac{x}{r}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$ or $z^5 = \left[\cos(2k\pi+\frac{\pi}{3})+i\sin(2k\pi+\frac{\pi}{3})\right]$

$\sin\theta=\frac{y}{r}=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{3}$

1st Quad

$\boxed{1\cdot 2-14}$

$$z_k = \left[\cos\left(\frac{6k\pi+\pi}{3}\right)+i\sin\left(\frac{6k\pi+\pi}{3}\right)\right]^{1/5}$$

$$z_k = \left[\cos\left(\frac{6k\pi+\pi}{15}\right)+i\sin\frac{6k\pi+\pi}{15}\right] \text{ where } k=0,1,2,3,4$$

Now square of all the fifth roots are

$$z_k = \left[\cos\left(\frac{6k\pi+\pi}{15}\right)+i\sin\left(\frac{6k\pi+\pi}{15}\right)\right]^2 \text{ where } k=0,1,2,3,4$$

$$\text{or } z_k = \cos\left(\frac{12k\pi+2\pi}{15}\right)+i\sin\left(\frac{12k\pi+2\pi}{15}\right) \text{ where } k=0,1,2,3,4$$

for $k=0$, $z_0 = \cos\frac{2\pi}{15}+i\sin\frac{2\pi}{15} = \text{cis}\,\frac{2\pi}{15}$

for $k=1$, $z_1 = \cos\frac{14\pi}{15}+i\sin\frac{14\pi}{15} = \text{cis}\,\frac{14\pi}{15}$

for $k=2$, $z_2 = \cos\frac{26\pi}{15}+i\sin\frac{26\pi}{15}$

$\frac{26\pi}{15}-2\pi=\frac{-4\pi}{15}$ $\quad = \cos(\frac{-4\pi}{15})+i\sin(\frac{-4\pi}{15}) = \text{cis}(\frac{-4\pi}{15})$

for $k=3$, $z_3 = \cos\frac{38\pi}{15}+i\sin\frac{38\pi}{15}$

$\frac{38\pi}{15}-2\pi=\frac{8\pi}{15}$ $\quad z_3 = \cos\frac{8\pi}{15}+i\sin\frac{8\pi}{15} = \text{cis}(\frac{8\pi}{15})$ $\quad 15\overline{)50}\,3$ $\quad (3+\frac{5}{15})\pi$

$z_4 = \cos\frac{50\pi}{15}+i\sin\frac{50\pi}{15}$

for $k=4$, $z_4 = \cos\frac{20\pi}{15}+i\sin\frac{20\pi}{15}$

$\frac{50\pi}{15}-2\pi=\frac{20\pi}{15}$ $\quad = \cos(\frac{-2\pi}{3})+i\sin(\frac{-2\pi}{3}) = \text{cis}\,(-\frac{2\pi}{3})$

$\frac{20\pi}{15}-2\pi=-\frac{10\pi}{15}=\frac{-2\pi}{3}$

(18)

Q.8 (i) Solve the equation $x^7+1=0$ $\boxed{1.2-15}$

Sol We have $x^7+1=0$

$$\Rightarrow x^7 = -1 = -1+0i = 1(\cos\pi + i\sin\pi)$$

or $x^7 = \cos(\pi+2k\pi) + i\sin(\pi+2k\pi)$

$$\Rightarrow x = \left[\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)\right]^{\frac{1}{7}}$$

So Seven 7th roots of -1 are

$$x_k = \cos\left(\frac{\pi+2k\pi}{7}\right) + i\sin\left(\frac{\pi+2k\pi}{7}\right)$$

Where $k = 0,1,2,3,4,5,6$

$\frac{9\pi}{7} = \left(1+\frac{2}{7}\right)\pi = \pi + \frac{2\pi}{7}$

for $k=0$, $x_0 = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7} = \text{cis}\,\frac{\pi}{7}$

for $k=1$, $x_1 = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7} = \text{cis}\,\frac{3\pi}{7}$

for $k=2$, $x_2 = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7} = \text{cis}\,\frac{5\pi}{7}$

for $k=3$, $x_3 = \cos\pi + i\sin\pi = -1+0i = -1$

for $k=4$, $x_4 = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$

$\frac{9\pi}{7} - 2\pi = -\frac{5\pi}{7}$

$$= \cos\left(-\frac{5\pi}{7}\right) + i\sin\left(-\frac{5\pi}{7}\right)$$

$$x_4 = \text{cis}\left(-\frac{5\pi}{7}\right)$$

for $k=5$ $\quad x_5 = \cos\left(\frac{11\pi}{7}\right) + i\sin\left(\frac{11\pi}{7}\right)$

$\frac{11\pi}{7} - 2\pi = -\frac{3\pi}{7}$

$$= \cos\left(-\frac{3\pi}{7}\right) + i\sin\left(-\frac{3\pi}{7}\right)$$

$$= \text{cis}\left(-\frac{3\pi}{7}\right)$$

for $k=6$, $x_6 = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$

$\frac{13\pi}{7} - 2\pi = -\frac{\pi}{7}$

$$= \cos\left(-\frac{\pi}{7}\right) + i\sin\left(-\frac{\pi}{7}\right) = \text{cis}\left(-\frac{\pi}{7}\right)$$

Note we can also take values of $k = 0, \pm1, \pm2, \pm3$ instead of $k = 0,1,2,3,4,5,6$

1.2-15

Q-(ii) $x^7 + x^4 + x^3 + 1 = 0$

Sol $x^4[x^3+1] + 1[x^3+1] = 0$

$\Rightarrow (x^4+1)(x^3+1) = 0$

$\Rightarrow x^4+1 = 0$ and $x^3+1 = 0$

In case of $x^4+1=0 \Rightarrow x^4 = -1 = -1+0i$

or $x^4 = \cos\pi + i\sin\pi = \cos(2k\pi+\pi) + i\sin(2k\pi+\pi)$

$\Rightarrow x_k = \cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)$, where $k = 0,1,2,3$

for $k=0$, $x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt2} + \frac{i}{\sqrt2}$

for $k=1$, $x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt2} + \frac{i}{\sqrt2}$

for $k=2$, $x_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \cos\frac{-3\pi}{4} + i\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt2} - \frac{i}{\sqrt2}$

$\frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$

for $k=3$, $x_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt2} - \frac{i}{\sqrt2}$

$\frac{7\pi}{4} - 2\pi = -\frac{\pi}{4}$

In case of $x^3+1=0 \Rightarrow x^3 = -1 = -1+0i$

or $x^3 = \cos\pi + i\sin\pi = \cos(2k\pi+\pi) + i\sin(2k\pi+\pi)$

$\Rightarrow x_k = \cos\left(\frac{2k\pi+\pi}{3}\right) + i\sin\left(\frac{2k\pi+\pi}{3}\right)$, $k = 0, 1, -1$

for $k=-1$, $x_{-1} = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt3}{2}i$

for $k=0$, $x_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt3}{2}i$

for $k=1$, $x_1 = \cos\pi + i\sin\pi$

$= -1 + 0 = -1$

1.2-17

Q.8(iii) $x^6+1=\sqrt{3}i$

$r=|z|=\sqrt{(1)^2+(\sqrt{3})^2}=\sqrt{1+3}=\sqrt{4}=2$

$\Rightarrow x^6=-1+\sqrt{3}i$

or $x^6=2\left[-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right]$ (x and $\div$ by 2, we get)

2nd quad

For finding θ, find $\theta = \cos^{-1}(-0.5)$

or $x^6=2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]$

$=2\left[\cos\left(2k\pi+\frac{2\pi}{3}\right)+i\sin\left(2k\pi+\frac{2\pi}{3}\right)\right]$

$\Rightarrow x=(2)^{\frac{1}{6}}\left[\cos\left(2k\pi+\frac{2\pi}{3}\right)+i\sin\left(2k\pi+\frac{2\pi}{3}\right)\right]^{1/6}$

or $x=(2)^{\frac{1}{6}}\left[\cos\left(\frac{6k\pi+2\pi}{18}\right)+i\sin\left(\frac{6k\pi+2\pi}{18}\right)\right]$

where $k=0,1,2,3,4,5$

for $k=0$, $x_0=(2)^{\frac{1}{6}}\left[\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right]$

for $k=1$, $x_1=(2)^{\frac{1}{6}}\left[\cos\frac{8\pi}{18}+i\sin\frac{8\pi}{18}\right]=(2)^{\frac{1}{6}}\operatorname{cis}\left(\frac{4\pi}{9}\right)$

for $k=2$, $x_2=(2)^{\frac{1}{6}}\left[\cos\frac{7\pi}{9}+i\sin\frac{7\pi}{9}\right]=(2)^{\frac{1}{6}}\operatorname{cis}\left(\frac{7\pi}{9}\right)$

for $k=3$, $x_3=(2)^{\frac{1}{6}}\left[\cos\frac{20\pi}{18}+i\sin\frac{20\pi}{18}\right]$

$\frac{20\pi}{18}-2\pi=-\frac{16\pi}{18}$

$=(2)^{\frac{1}{6}}\left[\cos\left(-\frac{16\pi}{18}\right)+i\sin\left(-\frac{16\pi}{18}\right)\right]=(2)^{\frac{1}{6}}\operatorname{cis}\left(-\frac{8\pi}{9}\right)$

for $k=4$, $x_4=(2)^{\frac{1}{6}}\left[\cos\frac{13\pi}{9}+i\sin\frac{13\pi}{9}\right]=(2)^{\frac{1}{6}}\operatorname{cis}\left(\frac{5\pi}{9}\right)$

$\frac{13\pi}{9}-2\pi=-\frac{5\pi}{9}$

for $k=4$, $x_5=(2)^{\frac{1}{6}}\left[\cos\frac{16\pi}{9}+i\sin\frac{16\pi}{9}\right]$

$\frac{16\pi}{9}-2\pi=-\frac{2\pi}{9}$

$=(2)^{\frac{1}{6}}\left[\cos\left(-\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right]=(2)^{\frac{1}{6}}\operatorname{cis}\left(-\frac{2\pi}{9}\right)$

x————x

Q.9 Solve the equation $x^{12}-1=0$ and find which of its roots satisfy the equation $x^4+x^2+1=0$

Sol: $x^{12}-1=0$ gives

$x^{12}=1=1+0i=\cos 0+i\sin 0$

or $x^{12}=\cos(0+2k\pi)+i\sin(0+2k\pi)$

Easy

(21)

12-18

2nd Method: Now $x^4+x^2+1=0$

$x^4+x^2+1=0$
$x^4+x^2+x^2-x^2+1=0$
$x^4+2x^2+1-x^2=0$
$(x^2+1)^2-x^2=0$
$(x^2+1+x)(x^2+1-x)=0$

Either $x^2+1+x=0$
$x=\frac{-1\pm\sqrt{1-4}}{2}$
$x=\frac{-1\pm i\sqrt{3}}{2}$

OR $x^2+1-x=0$
$x=\frac{1\pm\sqrt{1-4}}{2}$
$=\frac{1\pm i\sqrt{3}}{2}$

Put $x^2=y$

$y^2+y+1=0$

$y=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$

x and by 2 to complete square

$x^2=\frac{-2\pm 2\sqrt{-3}}{4}$

$=\frac{-2\pm 2\cdot 1\cdot\sqrt{-3}}{4}$

$=\frac{(1-3)\pm 2\cdot 1\cdot\sqrt{-3}}{4}$

$x^2=\frac{1^2+(\sqrt{-3})^2\pm 2\cdot 1\cdot\sqrt{-3}}{4}$

$x^2=\left(\frac{1\pm\sqrt{-3}}{2}\right)^2$

$x=\pm\left(\frac{1\pm\sqrt{-3}}{2}\right)$

$\Rightarrow\ x=\frac{1\pm\sqrt{3}i}{2},\ \frac{-1\pm\sqrt{3}i}{2}$

$x=\frac{1+\sqrt{3}i}{2},\ \frac{1-\sqrt{3}i}{2},\ \frac{-1+\sqrt{3}i}{2},\ \frac{-1-\sqrt{3}i}{2}$

We see that in twelve 12th roots of $x^{12}=-1$

$x_2,\ x_4,\ x_8$ and x_{10} satisfy the roots of equation $x^4+x^2+1=0$

x————————x

Q.10 Expand the following in series of Sines or Cosines of multiples of θ

$(a+b)^n=a^n+na^{n-1}\cdot b+\frac{n(n-1)}{L2}a^{n-2}b^2+\dots\dots b^n$

(i) $\cos^4\theta$

Sol: Let $x=\cos\theta+i\sin\theta$, then $\frac{1}{x}=\frac{1}{\cos\theta+i\sin\theta}$
$=(\cos\theta+i\sin\theta)^{-1}$
$\frac{1}{x}=\cos\theta-i\sin\theta$

$\Rightarrow x+\frac{1}{x}=2\cos\theta$

So $(2\cos\theta)^4=\left(x+\frac{1}{x}\right)^4$

$2^4\cos^4\theta=x^4+4x^3\left(\frac{1}{x}\right)+\frac{4\cdot 3}{2\cdot 1}x^2\cdot\frac{1}{x^2}+\frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1}x\cdot\frac{1}{x^3}+\frac{1}{x^4}$

$=x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$

$$\text{or}\quad x^{12} = \cos 2\pi k + i \sin 2\pi k$$

$$\Rightarrow x_k = (\cos 2\pi k + i \sin 2\pi k)^{1/12}$$

$$\Rightarrow x_k = \cos \frac{\pi k}{6} + i \sin \frac{\pi k}{6}$$

where $k = 0, 1, 2, \ldots \ldots 11$

For $k=0$, $x_0 = \cos 0 + i \sin 0 = 1$

For $k=1$, $x_1 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} + i \frac{1}{2} = \frac{\sqrt{3}+i}{2}$

For $k=2$, $x_2 = \cos \frac{2\pi}{6} + i \sin \frac{2\pi}{6} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i = \boxed{\frac{1+\sqrt{3}i}{2}}$

For $k=3$, $x_3 = \cos \frac{3\pi}{6} + i \sin \frac{3\pi}{6} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i = i$

For $k=4$, $x_4 = \cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \cos\left(\frac{\pi}{3} + \frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$

$$= \cos \frac{\pi}{3} \cdot \cos \frac{\pi}{3} - \sin \frac{\pi}{3} \sin \frac{\pi}{3} + i \left(2 \sin \frac{\pi}{3} \cos \frac{\pi}{3}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + 2i \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{4} - \frac{(\sqrt{3})^2}{4} + i \frac{\sqrt{3}}{2} = -\frac{2}{4} + \frac{\sqrt{3}i}{2} = -\frac{1}{2} + \frac{\sqrt{3}i}{2} = \boxed{\frac{-1+\sqrt{3}i}{2}}$$

$$x_5 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{i}{2} = \frac{-\sqrt{3}+i}{2}$$

$$x_6 = \cos \pi + i \sin \pi = -1 + 0i = -1$$

$\because \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$ $\quad x_7 = \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} = \cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{i}{2}$

$\frac{8\pi}{6} - 2\pi = -\frac{4\pi}{6}$ $\quad x_8 = \cos \frac{8\pi}{6} + i \sin \frac{8\pi}{6} = \cos\left(-\frac{4\pi}{6}\right) + i \sin\left(-\frac{4\pi}{6}\right) = \operatorname{cis}\left(-\frac{2\pi}{3}\right) = \boxed{\frac{-1-i\sqrt{3}}{2}}$

$\frac{9\pi}{6} - 2\pi = -\frac{3\pi}{6}$ $\quad x_9 = \cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6} = \cos\left(-\frac{3\pi}{2}\right) + i \sin\left(-\frac{3\pi}{2}\right) = 0 - i(1) = -i$

$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$ $\quad x_{10} = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)$

$$= \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} = \boxed{\frac{1-\sqrt{3}i}{2}}$$

$\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$ $\quad x_{11} = \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} = \cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)$

$$= \cos \frac{\pi}{6} - i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{i}{2} = \frac{\sqrt{3}-i}{2}$$

$$2^4\cos^4\theta = \left(x^4+\frac{1}{x^4}\right)+4\left(x^2+\frac{1}{x^2}\right)+6$$

If $x = \cos\theta + i\sin\theta$
then $x^4 = \cos 4\theta + i\sin 4\theta$..
$\frac{1}{x^4} = \frac{1}{\cos 4\theta + i\sin 4\theta}$
$\frac{1}{x^4} = \cos 4\theta - i\sin 4\theta$
$x^4 + \frac{1}{x^4} = 2\cos 4\theta$

$$= 2\cos 4\theta + 4\cdot 2\cos 2\theta + 6$$

$$2^4\cos^4\theta = 2\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

$$\cos^4\theta = \frac{1}{2^3}\left[\cos 4\theta + 4\cos 2\theta + 3\right] = \frac{1}{8}\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

x———————x

(ii) $\sin^4\theta$

Sol if $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

so $x - \frac{1}{x} = 2i\sin\theta$, thus

$$(2i\sin\theta)^4 = \left(x-\frac{1}{x}\right)^4 = x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}$$

(∵ similar to part (i))

$$2^4 i^4 \sin^4\theta = \left(x^4+\frac{1}{x^4}\right) - 4\left(x^2+\frac{1}{x^2}\right) + 6$$ $\boxed{1\cdot 2-20}$

$$= 2\cos 4\theta - 4(2\cos 2\theta) + 6$$

$$16 i^4\sin^4\theta = 2(\cos 4\theta - 4\cos 2\theta + 3)$$

$$\sin^4\theta = \frac{2}{16}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$

$$\Rightarrow \sin^4\theta = \frac{1}{8}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$

x———————x

(iii) $\sin^6\theta$

Sol, Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$x - \frac{1}{x} = 2i\sin\theta$$

$$\Rightarrow (2i\sin\theta)^6 = \left(x-\frac{1}{x}\right)^6$$

$$(2i\sin\theta)^6 = x^6 - 6x^5\cdot\frac{1}{x} + \frac{6\cdot 5}{2\cdot 1}x^4\cdot\frac{1}{x^2} - \frac{6\cdot5\cdot4}{3\cdot2\cdot1}x^3\cdot\frac{1}{x^3} + \frac{6\cdot5\cdot4\cdot3}{4\cdot3\cdot2\cdot1}x^2\cdot\frac{1}{x^4}$$

$$- \frac{6\cdot5\cdot4\cdot3\cdot2}{5\cdot4\cdot3\cdot2\cdot1}\cdot x\cdot\frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

$$2^6\cdot i^6\sin^6\theta = \left(x^6+\frac{1}{x^6}\right) - 6\left(x^4+\frac{1}{x^4}\right) + 15\left(x^2+\frac{1}{x^2}\right) - 20$$

$$-2^6\sin^6\theta = (2\cos 6\theta) - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$-2^6\sin^6\theta = 2\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right]$$

$$\Rightarrow \boxed{\sin^6\theta = -\frac{1}{2^5}\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right]}$$

ans

x —————————— x

1.2-21

(iv) 10 $\cos^7\theta = ?$

Sol: let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\Rightarrow x + \frac{1}{x} = 2\cos\theta$$

So $\left[2\cos\theta\right]^7 = \left(x+\frac{1}{x}\right)^7$

$$2^7\cos^7\theta = x^7 + 7x^6\cdot\frac{1}{x} + \frac{7\cdot6}{2\cdot1}x^5\cdot\frac{1}{x^2} + \frac{7\cdot6\cdot5}{3\cdot2\cdot1}x^4\cdot\frac{1}{x^3} + \frac{7\cdot6\cdot5\cdot4}{4\cdot3\cdot2\cdot1}x^3\cdot\frac{1}{x^4} + \frac{7\cdot6\cdot5\cdot4\cdot3}{5\cdot4\cdot3\cdot2\cdot1}x^2\cdot\frac{1}{x^5}$$

$$+ \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2}{6\cdot5\cdot4\cdot3\cdot2\cdot1}x\cdot\frac{1}{x^6} + \frac{1}{x^7}$$

or $$2^7\cos^7\theta = x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$$

$$2^7\cos^7\theta = \left(x^7+\frac{1}{x^7}\right) + 7\left(x^5+\frac{1}{x^5}\right) + 21\left(x^3+\frac{1}{x^3}\right) + 35\left(x+\frac{1}{x}\right)$$

$$= 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos\theta)$$

$$\Rightarrow 2^7\cos^7\theta = 2\left[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right]$$

$$\Rightarrow \boxed{\cos^7\theta = \frac{1}{2^6}\left[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right]}$$

ans.

(v) Q-10 $\sin^9\theta = ?$ 1-2-22

Sol:- let $x = \cos\theta + i\sin\theta$ then

$\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x - \frac{1}{x} = 2i\sin\theta$

$\Rightarrow (2i\sin\theta)^9 = \left(x - \frac{1}{x}\right)^9$

or $2^9 i^9 \sin^9\theta = x^9 - 9x^8\frac{1}{x} + \frac{9\cdot8\, x^7}{2\cdot1\cdot x^2} - \frac{9\cdot8\cdot7}{3\cdot2\cdot1}\frac{x^6}{x^3} + \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}\frac{x^5}{x^4}$

$- \frac{9\cdot8\cdot7\cdot6\cdot5}{5\cdot4\cdot3\cdot2\cdot1}\cdot\frac{x^4}{x^5} + \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4}{6\cdot5\cdot4\cdot3\cdot2\cdot1}\cdot\frac{x^3}{x^6} - \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}\frac{x^2}{x^7}$

$+ \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}{8!}\frac{x}{x^8} - \frac{1}{x^9}$

$2^9 i^{8+1}\sin^9\theta = x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x}$

$+ \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$

$2^9 i \sin^9\theta = x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x}$

$+ \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$

$= \left(x^9 - \frac{1}{x^9}\right) - 9\left(x^7 - \frac{1}{x^7}\right) + 36\left(x^5 - \frac{1}{x^5}\right) - 84\left(x^3 - \frac{1}{x^3}\right)$

$+ 126\left(x - \frac{1}{x}\right)$

$= 2i\sin9\theta - 9(2i\sin7\theta) + 36(2i\sin5\theta) - 84(2i\sin3\theta)$

$+ 126(2i\sin\theta)$

$2^9 i \sin^9\theta = 2i\left[\sin9\theta - 9\sin7\theta + 36\sin5\theta - 84\sin3\theta + 126\sin\theta\right]$

$$\boxed{\sin^9\theta = \frac{1}{2^8}\left[\sin9\theta - 9\sin7\theta + 36\sin5\theta - 84\sin3\theta + 126\sin\theta\right]}$$

Ans.

(vi) Q-10 $\sin^6\theta\cos^2\theta = ?$

Sol:- let $x = \cos\theta + i\sin\theta$
then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Big\}$ $\therefore x + \frac{1}{x} = 2\cos\theta$
and $x - \frac{1}{x} = 2i\sin\theta$

$\boxed{1.2-23}$

$$2^6 i^6 2^2 \sin^6\theta \cos^2\theta = \left(x-\frac{1}{x}\right)^6\left(x+\frac{1}{x}\right)^2$$

$$2^8 i^6 \sin^6\theta\cos^2\theta = \left[x-\frac{1}{x}\right]^4\left[x-\frac{1}{x}\right]^2\left[x+\frac{1}{x}\right]^2 = \left[x-\frac{1}{x}\right]^4\left[\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\right]^2$$

$$= \left[x-\frac{1}{x}\right]^4\left[x^2-\frac{1}{x^2}\right]^2$$

$$= \left[x^4 - 4x^3\cdot\frac{1}{x} + \frac{4\cdot3}{2\cdot1}x^2\cdot\frac{1}{x^2} - \frac{4\cdot3\cdot2}{3\cdot2\cdot1}\frac{x}{x^3} + \frac{1}{x^4}\right]\left[x^4+\frac{1}{x^4}-2\right]$$

$$= \left[x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}\right]\left[x^4+\frac{1}{x^4}-2\right]$$

$$= x^8 + 1 - 2x^4 - 4x^6 - \frac{4}{x^2} + 8x^2 + 6x^4 + \frac{6}{x^4} - 12 - 4x^2$$
$$- \frac{4}{x^6} + \frac{8}{x^2} + 1 + \frac{1}{x^8} - \frac{2}{x^4}$$

$\boxed{i^6 = -1}$

$$= \left(x^8+\frac{1}{x^8}\right) - 4\left(x^6+\frac{1}{x^6}\right) + \left(-2x^4+6x^4+\frac{6}{x^4}-\frac{2}{x^4}\right)$$
$$+\left(8x^2-4x^2+\frac{8}{x^2}-\frac{4}{x^2}\right) - 10$$

$$-2^8\sin^6\theta\cos^2\theta = \left(x^8+\frac{1}{x^8}\right) - 4\left(x^6+\frac{1}{x^6}\right) + 4\left(x^4+\frac{1}{x^4}\right) + 4\left(x^2+\frac{1}{x^2}\right) - 10$$

$$= 2\cos 8\theta - 4(2\cos 6\theta) + 4(2\cos 4\theta) + 4(2\cos 2\theta) - 10$$

$$-2^8\cdot\sin^6\theta\cos^2\theta = 2\left[\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 5\right]$$

$$\Rightarrow \sin^6\theta\cos^2\theta = -\frac{1}{2^7}\left(\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 5\right) \quad \text{Ans}$$

or

$$\boxed{\sin^6\theta\cos^2\theta = \frac{1}{2^7}\left(-\cos 8\theta + 4\cos 6\theta - 4\cos 4\theta - 4\cos 2\theta + 5\right)}$$

—— x —— ans —— x ——

(vii) Q-10 $\cos^4\theta\,\sin^3\theta = ?$

Sol: Let $x = \cos\theta + i\sin\theta$
then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Bigg\} \Rightarrow x+\frac{1}{x} = 2\cos\theta$ and $x-\frac{1}{x} = 2i\sin\theta$

So $(2\cos\theta)^4(2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3$

$$2^4 \cdot 2^3 i^3 \cos^4\theta \sin^3\theta = \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$$

$$-2^7 i \cos^4\theta \sin^3\theta = \left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^3 \qquad \boxed{1\cdot 2-24}$$

$$= \left(x+\frac{1}{x}\right)\left[x^2-\frac{1}{x^2}\right]^3$$

$$= \left(x+\frac{1}{x}\right)\left[(x^2)^3 - 3(x^2)^2\left(\frac{1}{x^2}\right) + 3x^2\cdot\frac{1}{(x^2)^2} - \frac{1}{(x^2)^3}\right]$$

$$= \left(x+\frac{1}{x}\right)\left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right]$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} + x^5 - 3x + \frac{3}{x^3} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right) + \left(x^5 - \frac{1}{x^5}\right)$$

$$= 2i\sin 7\theta - 6i\sin 3\theta - 6i\sin\theta + 2i\sin 5\theta$$

$$-2^7 i\cos^4\theta\sin^3\theta = 2i\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$$

$$\boxed{\Rightarrow \cos^4\theta\sin^3\theta = -\frac{1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]}$$

—x—— Ans ——x—

VIII Q-10 $\cos^5\theta \sin^7\theta = ?$

Sol:- Let $x = \cos\theta + i\sin\theta$, $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Big\} \Rightarrow x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

So: $(2\cos\theta)^5(2i\sin\theta)^7 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^7$

$$-2^{12} i\cos^5\theta\sin^7\theta = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^2$$

$$= \left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^5\left[x-\frac{1}{x}\right]^2$$

$$= \left[x^2-\frac{1}{x^2}\right]^5\left[x-\frac{1}{x}\right]^2$$

$$= \left[(x^2)^5 - 5(x^2)^4\cdot\frac{1}{x^2} + \frac{5\cdot 4}{2\cdot 1}(x^2)^3\cdot\frac{1}{(x^2)^2} - \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1}(x^2)^2\cdot\frac{1}{(x^2)^3} + \frac{5\cdot 4\cdot 3\cdot 2}{4\cdot 3\cdot 2}x^2\cdot\frac{1}{(x^2)^4} - \frac{1}{(x^2)^5}\right]\left[x^2+\frac{1}{x^2}-2\right]$$

$$-2^{12} i \cos^5\theta \sin^7\theta = \left[x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right]\left(x^2 + \frac{1}{x^2} - 2\right)$$

$$-2^{12} i \cos^5\theta \sin^7\theta = x^{12} - 5x^8 + 10x^4 - 10 + \frac{5}{x^4} - \frac{1}{x^8} + x^8 - 5x^4 + 10 - \frac{10}{x^4} + \frac{5}{x^8} - \frac{1}{x^{12}} - 2x^{10} + 10x^6 - 20x^2 + \frac{20}{x^2} - \frac{10}{x^6} + \frac{2}{x^{10}}$$

1.2-25

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

$$= 2i\sin 12\theta - 2(2i\sin 10\theta) - 4(2i\sin 8\theta) + 10(2i\sin 6\theta) + 5(2i\sin 4\theta) - 20(2i\sin 2\theta)$$

$$-2^{12} i \cos^5\theta \sin^7\theta = 2i\left(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\right)$$

$$\Rightarrow \boxed{\cos^5\theta \sin^7\theta = -\frac{1}{2^{11}}\left(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\right)}$$

—x— Ans —x—

Q.11 Show that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$

Sol:- let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Rightarrow$ $2\cos\theta = x + \frac{1}{x}$, $2i\sin\theta = x - \frac{1}{x}$

So $(2\cos\theta)^4 = \left(x + \frac{1}{x}\right)^4$

$$2^4\cos^4\theta = x^4 + 4\cancel{x^3}\cdot\frac{1}{x} + \frac{4\cdot 3}{2\cdot 1}x^2\frac{1}{x^2} + \frac{4\cdot 3\cdot 2}{\cancel{3\cdot 2\cdot 1}}x\frac{1}{x^3} + \frac{4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1}\frac{1}{x^4} \rightarrow (1)$$

$$(2i\sin\theta)^4 = \left(x - \frac{1}{x}\right)^4 = x^4 - 4\cancel{x^3}\frac{1}{x} + \frac{4\cdot 3}{2\cdot 1}x^2\frac{1}{x^2} - \frac{4\cdot 3\cdot 2}{\cancel{3\cdot 2\cdot 1}}x\frac{1}{x^3} + \frac{1}{x^4} \rightarrow (2)$$

Add (1) and (2), we get

$$2^4\cos^4\theta + 2^4 i^4\sin^4\theta = 2\left\{x^4 + \frac{4\cdot 3}{2\cdot 1} + \frac{1}{x^4}\right\}$$

$$2^4\left[\cos^4\theta+\sin^4\theta\right]=2\left\{x^4+6+\frac{1}{x^4}\right\} \qquad (2^4=1)$$

$$\cos^4\theta+\sin^4\theta=\frac{1}{2^3}\left[\left(x^4+\frac{1}{x^4}\right)+6\right]=\frac{1}{8}\left[2\cos 4\theta+6\right]$$

$$\boxed{\cos^4\theta+\sin^4\theta=\frac{1}{4}\left[\cos 4\theta+3\right]}$$

— x — Ans — x —

$\boxed{1.2-26}$

Q.12 Prove that $64(\cos^8\theta+\sin^8\theta)=\cos 8\theta+28\cos 4\theta+35$

SOL: Let $x=\cos\theta+i\sin\theta$, $\frac{1}{x}=\cos\theta-i\sin\theta$ $\Rightarrow$ $x+\frac{1}{x}=2\cos\theta$ and $x-\frac{1}{x}=2i\sin\theta$

So. $(2\cos\theta)^8+(2i\sin\theta)^8=\left[\left(x+\frac{1}{x}\right)^8+\left(x-\frac{1}{x}\right)^8\right]$

$$2^8\left[\cos^8\theta+\sin^8\theta\right]=\left\{\begin{array}{l} x^8+8x^7\frac{1}{x}+\frac{8\cdot7}{2}x^6\cdot\frac{1}{x^2}+\frac{8\cdot7\cdot6}{3\cdot2}x^5\cdot\frac{1}{x^3}+\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2}x^4\cdot\frac{1}{x^4} \\ +\frac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2}x^3\cdot\frac{1}{x^5}+\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3}{6\cdot5\cdot4\cdot3\cdot2}x^2\cdot\frac{1}{x^6}+\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}{7\cdot6\cdot5\cdot4\cdot3\cdot2}x\cdot\frac{1}{x^7}+\frac{1}{x^8} \\ x^8-8x^7\frac{1}{x}+\frac{8\cdot7}{2}x^6\cdot\frac{1}{x^2}-\frac{8\cdot7\cdot6}{3\cdot2}x^5\cdot\frac{1}{x^3}+\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2}x^4\cdot\frac{1}{x^4} \\ -\frac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2}x^3\cdot\frac{1}{x^5}+\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3}{6\cdot5\cdot4\cdot3\cdot2}x^2\cdot\frac{1}{x^6}-\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}{7\cdot6\cdot5\cdot4\cdot3\cdot2}x\cdot\frac{1}{x^7}+\frac{1}{x^8} \end{array}\right\}$$

$$2^8\left[\cos^8\theta+\sin^8\theta\right]=2\left[x^8+28x^6\cdot\frac{1}{x^2}+70+28\cdot x^2\cdot\frac{1}{x^6}+\frac{1}{x^8}\right]$$

$$=2\left[\left(x^8+\frac{1}{x^8}\right)+28\left(x^4+\frac{1}{x^4}\right)+70\right]$$

$$=2\left[2\cos 8\theta+28(2\cos 4\theta)+70\right]$$

$$2^6(\cos^8\theta+\sin^8\theta)=\cancel{2}\left[\cos 8\theta+28\cos 4\theta+35\right]$$

$$\boxed{\Rightarrow\ 64(\cos^8\theta+\sin^8\theta)=\cos 8\theta+28\cos 4\theta+35}$$

Proved.

Q-13 PROVE THAT:- [1-2-27]

P-(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

PROOF Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x - \frac{1}{x} = 2i\sin\theta$

Thus $(2i\sin\theta)^3 = \left(x - \frac{1}{x}\right)^3 = x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$

$2^3 i^3 \sin^3\theta = \left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$

$-2^3 i \sin^3\theta = 2i\sin 3\theta - 3(2i\sin\theta)$

$-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$\Rightarrow -4\sin^3\theta = \sin 3\theta - 3\sin\theta$

$\Rightarrow$ $\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

Proved

Part-(ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

PROOF:- Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x + \frac{1}{x} = 2\cos\theta$

Thus $(2\cos\theta)^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + 3x^2\cdot\frac{1}{x} + 3x\cdot\frac{1}{x^2} + \frac{1}{x^3}$

$2^3\cos^3\theta = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

$2^3\cos^3\theta = \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$

$\cancel{2^3}\,2\cos^3\theta = 2\cos 3\theta + 3(2\cos\theta)$

$2^2\cos^3\theta = \cos 3\theta + 3\cos\theta$

$\Rightarrow$ $\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$

—x— Proved —x—

ALTERNATE OF PART-(i) AND PART-(ii)

$\because (\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^3$

(By using De-moivre's Th.)

But $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

or $\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$$= \left(\cos^3\theta - 3\cos\theta\sin^2\theta\right) + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$$

$$= \left[\cos^3\theta - 3\cos\theta(1-\cos^2\theta)\right] + i\left[3\sin\theta(1-\sin^2\theta) - \sin^3\theta\right]$$

$$= \left[\cos^3\theta - 3\cos\theta + 3\cos^3\theta\right] + i\left[3\sin\theta - 3\sin^3\theta - \sin^3\theta\right]$$

$\cos 3\theta + i\sin 3\theta = \left[4\cos^3\theta - 3\cos\theta\right] + i\left[3\sin\theta - 4\sin^3\theta\right]$

Equating real and imaginary parts we get

$$\boxed{\begin{aligned} \cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ \text{and}\quad \sin 3\theta &= 3\sin\theta - 4\sin^3\theta \end{aligned}}$$

Proved

$\boxed{1.2\text{-}28}$

<u>PART-(iii) & (iv)</u> $\sin 4\theta = 4\left(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta\right)$

and $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

<u>PROOF:-</u>

$\because (\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta \longrightarrow (1)$

(De-Moivre's Th.)

but $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta + \frac{4\cdot3}{2}i^2\cos^2\theta\sin^2\theta$

$+ \frac{4\cdot3\cdot2}{3\cdot2}i^3\cos\theta\sin^3\theta + i^4\sin^4\theta$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

(Binomial Th.)

or $(\cos\theta + i\sin\theta)^4 = \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right) + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$

using (1)

$\cos 4\theta + i\sin 4\theta = \left[\cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + \sin^4\theta\right] + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$

$\cos 4\theta + i\sin 4\theta = \left[\cos^4\theta + 6\cos^4\theta - 6\cos^2\theta + (1-\cos^2\theta)^2\right] + i\left[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right]$

$= \left[7\cos^4\theta - 6\cos^2\theta + 1 + \cos^4\theta - 2\cos^2\theta\right] + i\left[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right]$

$\cos 4\theta + i\sin 4\theta = \left[8\cos^4\theta - 8\cos^2\theta + 1\right] + i\left[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right]$

Equating real and imaginary parts, we get

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ ---------→ Part (iv)

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ ⟶ Part (iii)

$\boxed{12-29}$

(v) $\dfrac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

1st Method

$x^5 = \cos 5\theta + i\sin 5\theta =$

$\frac{1}{x^5} = \cos 5\theta - i\sin 5\theta$

$x^5 - \frac{1}{x^5} = 2i\sin 5\theta$

$\left(x - \frac{1}{x}\right)^5 = 2^5 i^5 \sin^5\theta$

$32 i\sin^5\theta = x^5 - 5x^4\frac{1}{x} + \frac{5\cdot4}{2}x^3\frac{1}{x^2} - \frac{5\cdot4\cdot3}{1\cdot2\cdot3}x^2\frac{1}{x^3} + \frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}x\frac{1}{x^4} - \frac{1}{x^5}$

$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$

$= (2i\sin 5\theta) - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$32 i\sin^5\theta = 2i(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta.$

$\sin\theta(16\sin^4\theta) + 5(3\sin\theta - 4\sin^3\theta) - 10\sin\theta = \sin 5\theta.$

$\frac{\sin\theta}{\sin\theta}(16\sin^4\theta + 15 - 20\sin^2\theta - 10) = \frac{\sin 5\theta}{\sin\theta}$

$16(1-\cos^2\theta)^2 + 15 - 20(1-\cos^2\theta) - 10$

$16(1 + \cos^4\theta - 2\cos^2\theta) + 15 - 20 + 20\cos^2\theta - 10$

$16 + 16\cos^4\theta - 32\cos^2\theta + 15 - 20 + 20\cos^2\theta - 10$

$1 + 16\cos^4\theta - 12\cos^2\theta. \quad = \frac{\sin 5\theta}{\sin\theta}$

proved.

2nd Method.

Part-(V) $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

[1·2-30]

PROOF According to De moivre's Th.

$$[\cos\theta + i\sin\theta]^5 = \cos 5\theta + i\sin 5\theta \longrightarrow ①$$

but $[\cos\theta+i\sin\theta]^5 = \cos^5\theta + 5i\cos^4\theta\cdot\sin\theta + \frac{5\cdot4}{2\cdot1}\cos^3\theta\, i^2\sin^2\theta - \frac{5\cdot\cancel{4}\cdot\cancel{3}}{\cancel{3}\cdot\cancel{2}\cdot1} i\cos^2\theta\sin^3\theta$
by Binomial Th
$+ \frac{5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1}\cos\theta\sin^4\theta + i\sin^5\theta$

$[\cos\theta+i\sin\theta]^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

using ①, we get

$$\cos 5\theta + i\sin 5\theta = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Equating imaginary parts, we get

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$\Rightarrow$ $\frac{\sin 5\theta}{\sin\theta} = \frac{\cancel{\sin\theta}(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta)}{\cancel{\sin\theta}}$
by $\sin\theta$

$$= 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + \cos^4\theta + 1 - 2\cos^2\theta$$

or $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

x————Proved————x

Q.14 Prove that $\tan 6\theta = 2t\left(\frac{3-10t^2+3t^4}{1-15t^2+15t^4-t^6}\right)$ where $t=\tan\theta$

PROOF According to Demoivre's Th.

$$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta \longrightarrow ①$$

but $(\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6i\cos^5\theta\sin\theta - \frac{6\cdot5}{2}\cos^4\theta\sin^2\theta - \frac{6\cdot5\cdot4}{\cancel{3}\cdot\cancel{2}} i\cos^3\theta\sin^3\theta$
by Binomial Th
$+\frac{6\cdot5\cdot\cancel{4}\cdot\cancel{3}}{\cancel{4}\cdot3\cdot\cancel{2}}\cos^2\theta\sin^4\theta + \frac{6\cdot5\cdot4\cdot3\cdot2}{5\cdot4\cdot3\cdot2} i\cos\theta\sin^5\theta - \sin^6\theta$

$$\cos 6\theta + i\sin 6\theta = \left\{\begin{array}{l}\cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta \\ +15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta\end{array}\right\}$$

$\boxed{1.2-31}$

Equating real and imaginary parts.

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \longrightarrow \text{(ii)}$$

and $\sin 6\theta = 6\cos^5\theta\sin\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta$

$$\frac{\sin 6\theta}{\cos 6\theta} = \left(\frac{6\cos^5\theta\sin\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}\right)$$

$$\tan 6\theta = \frac{\dfrac{6\cos^5\theta\sin\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta}}{\dfrac{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}{\cos^6\theta}}$$

$$= \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

R.H.S.

$$= \frac{2\tan\theta(3 - 10\tan^2\theta + 3\tan^4\theta)}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

$$= \frac{2t(3 - 10t^2 + 3t^4)}{1 - 15t^2 + 15t^4 - t^6}$$

Q.15 Prove that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$ and hence solve the equation $1-3t^2 = 3t - t^3$

Sol: Since $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \to ①$

but $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ by B.Th.

using ①, we get

$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

$= [\cos^3\theta - 3\cos\theta(1-\cos^2\theta)] + i[3(1-\sin^2\theta)\sin\theta - \sin^3\theta]$

$= [\cos^3\theta - 3\cos\theta + 3\cos^3\theta] + i[3\sin\theta - 3\sin^3\theta - \sin^3\theta]$

$\cos 3\theta + i\sin 3\theta = [4\cos^3\theta - 3\cos\theta] + i[3\sin\theta - 4\sin^3\theta]$

Equating real and imaginary parts, we get

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\boxed{1\cdot 2\text{-}32}$

$\Rightarrow \dfrac{\sin 3\theta}{\cos 3\theta} = \dfrac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}$

$= \dfrac{\sin\theta(3-4\sin^2\theta)}{\cos\theta(4\cos^2\theta - 3)} \quad \dfrac{\tan\theta(3-4\sin^2\theta)}{4\cos^2\theta - 3}$

$= \dfrac{\tan\theta[3\cdot 1 - 4\sin^2\theta]}{4\cos^2\theta - 3\cdot 1} = \dfrac{\tan\theta[3(\cos^2\theta + \sin^2\theta) - 4\sin^2\theta]}{4\cos^2\theta - 3(\sin^2\theta + \cos^2\theta)}$

$= \dfrac{\tan\theta[3\cos^2\theta - \sin^2\theta]}{\cos^2\theta - 3\sin^2\theta}$

÷ N & D by $\cos^2\theta$

$\tan 3\theta = \dfrac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta}$ proved

Now Put $\tan\theta = t$

$\Rightarrow \tan 3\theta = \dfrac{t(3-t^2)}{1-3t^2} = \dfrac{3t-t^3}{1-3t^2} \to ②$

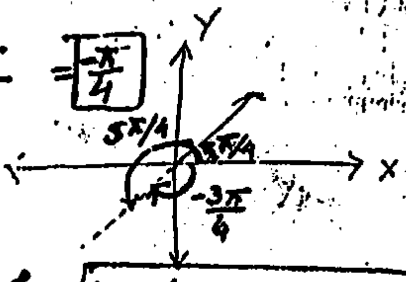
Since we are asked to solve $1-3t^2 = 3t-t^3$

$\Rightarrow 1 = \dfrac{3t-t^3}{1-3t^2} \to ③ \Rightarrow \tan 3\theta = 1$ (from ② and ③)

$3\theta = \tan^{-1}(1) = \pi/4, \frac{5\pi}{4}, -\frac{3\pi}{4}$

$\Rightarrow 3\theta = \frac{\pi}{4}, \quad 3\theta = \frac{5\pi}{4}, \quad$ and $\quad 3\theta = -\frac{3\pi}{4}$

$\Rightarrow \boxed{\theta = \frac{\pi}{12}}, \quad \boxed{\theta = \frac{5\pi}{12}}, \quad \theta = \frac{-3\pi}{(3)4} = \boxed{\frac{-\pi}{4}}$



Since $t = \tan\theta$

So $t = \tan\frac{\pi}{12}$, $\quad t = \tan\frac{5\pi}{12}$ & $\boxed{t = \tan\left(\frac{-\pi}{4}\right) = -1}$

$\because \tan\frac{\pi}{6} = \tan\left(\frac{\pi}{12} + \frac{\pi}{12}\right)$ $\quad \left(\because \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}\right)$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$ $\quad \boxed{\because \sin\frac{\pi}{6} = \frac{1}{2} \text{ \& } \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}}$

$\Rightarrow 1 - \tan^2\frac{\pi}{12} = 2\sqrt{3}\tan\frac{\pi}{12}$

or $\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} = 1$

$\Rightarrow \tan^2\pi/12 + 2\sqrt{3}\tan\pi/12 + 3 = 1+3$ (completing square)

$\Rightarrow \left(\tan\frac{\pi}{12} + \sqrt{3}\right)^2 = 2^2$

$\Rightarrow \tan\frac{\pi}{12} + \sqrt{3} = 2 \quad \Rightarrow \boxed{\tan\frac{\pi}{12} = 2 - \sqrt{3} = t}$

Also $\tan\frac{5\pi}{6} = \tan\left(\frac{5\pi}{12} + \frac{5\pi}{12}\right)$ $\quad \boxed{\sin\frac{5\pi}{6} = \frac{1}{2} \Rightarrow \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \Rightarrow \tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}}$

or $\frac{-1}{\sqrt{3}} = \frac{2\tan\frac{5\pi}{12}}{1-\tan^2\frac{5\pi}{12}}$

$-1 + \tan^2\frac{5\pi}{12} = 2\sqrt{3}\tan\frac{5\pi}{12}$

or $\tan^2\frac{5\pi}{12} - 2\sqrt{3}\tan\frac{5\pi}{12} = 1$

Completing sq. we get

$\tan^2\frac{5\pi}{12} - 2\sqrt{3}\tan\frac{5\pi}{12} + (\sqrt{3})^2 = 1+3$

$\left(\tan\frac{5\pi}{12} - \sqrt{3}\right)^2 = 2^2$

$\Rightarrow \tan\frac{5\pi}{12} - \sqrt{3} = 2$

$\Rightarrow \boxed{t = \tan\frac{5\pi}{12} = 2 + \sqrt{3}}$

Hence the required roots of cubic equation $1 - 3t^2 = 3t - t^3$ are

$-1, \; 2+\sqrt{3}, \; 2-\sqrt{3}$

ans

Q.16 Prove that $\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}=\frac{1}{2}$ [1.2-34]

Sol Consider the seventh roots of unity

i.e let $x^7=1 \Rightarrow x^7=1+0i$

$\Rightarrow x^7=\cos 0+i\sin 0=\cos(0+2k\pi)+i\sin(0+2\pi k)$

or $x^7=\cos 2\pi k+i\sin 2k\pi$

So seven 7th roots of unity are

$$x_k=(\cos 2\pi k+i\sin 2k\pi)^{\frac{1}{7}}$$

where $k=0,\pm1,\pm2,\pm3$

$\Rightarrow x_k=\cos\frac{2\pi k}{7}+i\sin\frac{2k\pi}{7} \rightarrow (1)$, where $k=0,\pm1,\pm2,\pm3$

after putting values of k in (1), we get its seven roots

i.e $1,\ \cos\frac{2\pi}{7}\pm i\sin\frac{2\pi}{7},\ \cos\frac{4\pi}{7}\pm i\sin\frac{4\pi}{7},\ \cos\frac{6\pi}{7}\pm i\sin\frac{6\pi}{7}$

(Since $\sin(-\alpha)=-\sin\alpha$ and $\cos(-\alpha)=\cos\alpha$)

Now from theory of equations, the sum of root of

$x^7-1=0$ is zero

$$\Rightarrow 1+\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)+\left(\cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}\right)+\left(\cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7}\right.$$
$$\left.\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}+\cos\frac{4\pi}{7}-i\sin\frac{4\pi}{7}+\cos\frac{6\pi}{7}-i\sin\frac{6\pi}{7}\right)=0$$

$\Rightarrow 1+2\cos\frac{2\pi}{7}+2\cos\frac{4\pi}{7}+2\cos\frac{6\pi}{7}=0$

$\Rightarrow 2\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)=-1$

$\boxed{\cos(\pi-\alpha)=-\cos\alpha}$

$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$

$\cos\frac{2\pi}{7}+\cos\left(\pi-\frac{3\pi}{7}\right)+\cos\left(\pi-\frac{\pi}{7}\right)=-\frac{1}{2}$

$\pi-\frac{3\pi}{7}=\frac{7\pi-3\pi}{7}=\frac{4\pi}{7}$

$\pi-\frac{\pi}{7}=\frac{7\pi-\pi}{7}=\frac{6\pi}{7}$

$\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{\pi}{7}=-\frac{1}{2}$

$\Rightarrow \boxed{\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}=\frac{1}{2}}$ Proved

Q.17 Prove the following relations ($\forall$ m, n belongs to $\mathbb{Z}$ mean m & n are integers)

$\boxed{1.2\text{-}35}$

(i) $Z^m Z^n = Z^{m+n}$

Proof Let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^m = r^m(\cos\theta + i\sin\theta)^m$

$z^m = r^m(\cos m\theta + i\sin m\theta)$

Similarly $z^n = r^n(\cos n\theta + i\sin n\theta)$

Then L.H.S. $= z^m \cdot z^n$

$= r^m[\cos m\theta + i\sin m\theta]\, r^n[\cos n\theta + i\sin n\theta]$

$= r^m r^n[\cos m\theta + i\sin m\theta][\cos n\theta + i\sin n\theta]$

$= r^{m+n}\left[\begin{array}{l}(\cos m\theta\cos n\theta - \sin m\theta\sin n\theta) \\ \quad + i(\sin m\theta\cos n\theta + \sin n\theta\cos m\theta)\end{array}\right]$

$= r^{m+n}[\cos(m\theta + n\theta) + i\sin(m\theta + n\theta)]$

$= r^{m+n}[\cos(m+n)\theta + i\sin(m+n)\theta]$

$= r^{m+n}(\cos\theta + i\sin\theta)^{m+n}$ by Demoivre's Th.

$= z^{m+n} =$ R.H.S.

×———————×

(ii) $(z^m)^n = z^{mn}$

Proof:- Let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^m = r^m(\cos\theta + i\sin\theta)^m$ (De-moivre's Th.)

$z^m = r^m(\cos m\theta + i\sin m\theta)$

$\Rightarrow (z^m)^n = r^{mn}(\cos m\theta + i\sin m\theta)^n$ (De-moivre's Th.)

$$L.H.S = \left(Z^m\right)^n = r^{mn}\left[\cos mn\theta + i\sin mn\theta\right] = r^{mn}\left[\cos\theta + i\sin\theta\right]^{mn}$$

$$= Z^{mn} = R.H.S.$$

× ——— ×

(iii) $(Z_1 Z_2)^n = Z_1^n Z_2^n$

PROOF Let $Z_1 = r_1[\cos\theta_1 + i\sin\theta_1]$ and $Z_2 = r_2[\cos\theta_2 + i\sin\theta_2]$

Hence $Z_1 \cdot Z_2 = r_1[\cos\theta_1 + i\sin\theta_1] \cdot r_2[\cos\theta_2 + i\sin\theta_2]$

$$= r_1 r_2[\cos\theta_1 + i\sin\theta_1][\cos\theta_2 + i\sin\theta_2]$$

$$= r_1 r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$$

$$Z_1 Z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$L.H.S = (Z_1 Z_2)^n = r_1^n r_2^n\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]^n$$

by Demoivre th.

$\boxed{\begin{aligned} &\text{If } Z = r(\cos\theta + i\sin\theta) \\ &\Rightarrow Z^n = [r(\cos\theta + i\sin\theta)]^n \\ &= r^n(\cos\theta + i\sin\theta)^n \\ &= r^n[\cos n\theta + i\sin n\theta] \end{aligned}}$

$$= r_1^n r_2^n\left[\cos n(\theta_1+\theta_2) + i\sin n(\theta_1+\theta_2)\right]$$

$$= r_1^n r_2^n\left[\cos(n\theta_1+n\theta_2) + i\sin(n\theta_1+n\theta_2)\right]$$

$$= r_1^n r_2^n\left[(\cos n\theta_1\cos n\theta_2 - \sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 + \cos n\theta_1\sin n\theta_2)\right]$$

$$= r_1^n r_2^n\left[(\cos n\theta_1\cos n\theta_2 + i^2\sin n\theta_1\sin n\theta_2) + (i\sin n\theta_1\cos n\theta_2 + i\cos n\theta_1\sin n\theta_2)\right]$$

$$= r_1^n r_2^n\left[(\cos n\theta_1\cos n\theta_2 + i\cos n\theta_1\sin n\theta_2) + (i\sin\theta_1 n\cos n\theta_2 + i^2\sin n\theta_1\sin n\theta_2)\right]$$

$$= r_1^n r_2^n\left[\cos n\theta_1(\cos n\theta_2 + i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 + i\sin n\theta_2)\right]$$

$$= r_1^n r_2^n\left[(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 + i\sin n\theta_2)\right]$$

$$= r_1^n(\cos n\theta_1 + i\sin n\theta_1) \cdot r_2^n(\cos n\theta_2 + i\sin n\theta_2)$$

$$= r_1^n(\cos\theta_1 + i\sin\theta_1)^n \cdot r_2^n(\cos\theta_2 + i\sin\theta_2)^n$$

$$= Z_1^n Z_2^n = R.H.S$$

(iv) $\frac{z^m}{z^n} = z^{m-n}$, $z \neq 0$

Proof let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^m = r^m(\cos\theta + i\sin\theta)^m$ & $z^n = r^n(\cos\theta + i\sin\theta)^n$

$$L.H.S = \frac{z^m}{z^n} = \frac{r^m(\cos\theta + i\sin\theta)^m}{r^n(\cos\theta + i\sin\theta)^n}$$

$$= r^{m-n} \cdot \frac{(\cos m\theta + i\sin m\theta)}{(\cos n\theta + i\sin n\theta)}$$

$$= r^{m-n} \cdot (\cos m\theta + i\sin m\theta)(\cos n\theta + i\sin n\theta)^{-1}$$

$$= r^{m-n}[\cos m\theta + i\sin m\theta][\cos(-n\theta) + i\sin(-n\theta)]$$

$$= r^{m-n}[\cos m\theta + i\sin m\theta][\cos n\theta - i\sin n\theta].$$

$$= r^{m-n}\left[(\cos m\theta\cos n\theta + \sin m\theta\sin n\theta) + i(\sin m\theta\cos n\theta - \cos m\theta\sin n\theta)\right]$$

$$= r^{m-n}[\cos(m\theta - n\theta) + i\sin(m\theta - n\theta)]$$

$$= r^{m-n}[\cos(m-n)\theta + i\sin(m-n)\theta]$$

$$= r^{m-n}[\cos\theta + i\sin\theta]^{m-n}$$ using Demoivre Th.

$$= z^{m-n} = R.H.S.$$

(v) $\left(\frac{z_1}{z_2}\right)^n = \frac{z_1^n}{z_2^n}$, $z_2 \neq 0$

Proof let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\boxed{1.2\text{-}38}$$

$$L.H.S = \left(\frac{z_1}{z_2}\right)^n = \left(\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\right)^n$$

$$\Rightarrow \left(\frac{z_1}{z_2}\right)^n = \frac{r_1^n}{r_2^n}\left(\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\cos n(\theta_1 - \theta_2) + i\sin n(\theta_1 - \theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left[\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 + \sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 - i\sin n\theta_2\cos n\theta_1) + (i\sin n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left(\cos n\theta_1(\cos n\theta_2 - i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 - i\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 - i\sin n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n} \times (\cos\theta_1 + i\sin\theta_1)^n(\cos n\theta_2 + i\sin n\theta_2)^{-1}$$

$$= \frac{r_1^n}{r_2^n}\frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos n\theta_2 + i\sin n\theta_2)} = \frac{r_1^n}{r_2^n} \times \frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos\theta_2 + i\sin\theta_2)^n} = \frac{z_1^n}{z_2^n}$$

$$= R.H.S$$

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$\boxed{1\cdot 3-1}$

Exponential Function

Prove $e^{ix} = \cos x + i\sin x$

We know
$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \frac{x^6}{\lfloor 6} + \frac{x^7}{\lfloor 7} + \cdots$$

Put $x = ix$
$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{\lfloor 2} + \frac{(ix)^3}{\lfloor 3} + \frac{(ix)^4}{\lfloor 4} + \frac{(ix)^5}{\lfloor 5} + \frac{(ix)^6}{\lfloor 6} + \cdots$$
$$= 1 + ix + i^2\frac{x^2}{\lfloor 2} + (i)^3\frac{x^3}{\lfloor 3} + (i)^4\frac{x^4}{\lfloor 4} + (i)^5\frac{x^5}{\lfloor 5} + (i)^6\frac{x^6}{\lfloor 6} + \cdots$$
$$= 1 + ix + i^2\frac{x^2}{\lfloor 2} + i(i)^2\frac{x^3}{\lfloor 3} + (i^2)^2\frac{x^4}{\lfloor 4} + i(i^2)^2\frac{x^5}{\lfloor 5} + i^2(i^2)^2\frac{x^6}{\lfloor 6}$$
$$= 1 + ix - \frac{x^2}{\lfloor 2} - i\frac{x^3}{\lfloor 3} + (1)\frac{x^4}{\lfloor 4} + i(1)\frac{x^5}{\lfloor 5} - (1)\frac{x^6}{\lfloor 6} + \cdots$$
$$= \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots\right) + \left(ix - i\frac{x^3}{\lfloor 3} + i\frac{x^5}{\lfloor 5} - \cdots\right)$$
$$= \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots\right) + i\left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots\right)$$
$$= \cos x + i\sin x$$

Note: if $z = x + iy$ in Cartesian form
$z = r(\cos\theta + i\sin\theta)$ in Polar form
$z = re^{i\theta}$ in Exponential form.

Note: $a^z = e^{z\ln a}$ if $a \in R$, $a > 0$
$\because \ln a^z = \ln e^{z\ln a}$
$z\ln a = z\ln a(\ln e)$
$z\ln a = z\ln a$

Hyperbolic Functions

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\quad$ $\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$

$\cosh x = \frac{e^x + e^{-x}}{2}$ $\quad$ $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\quad$ $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Osborn's Rule

Relation b/w Trig Fns & Hyperbolic fns.

$\sin iz = i\sinh z$	$\sinh iz = i\sin z$
$\cos iz = \cosh z$	$\cosh iz = \cos z$
$\tan iz = i\tanh z$	$\tanh iz = i\tan z$
$\cot iz = -i\coth z$	$\coth iz = -i\cot z$
$\sec iz = \operatorname{sech} z$	$\operatorname{sech} iz = \sec z$
$\operatorname{cosec} iz = -i\operatorname{cosech} z$	$\operatorname{cosech} iz = -i\operatorname{cosec} z$

Note: To Prove Osborn's Rule just Put $iz = z$ & solve.

Prove $\sin iz = i\sinh z$

Proof $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

Put $z = iz$
$$\sin iz = \frac{e^{iiz} - e^{-iiz}}{2i} = \frac{e^{i^2 z} - e^{-i^2 z}}{2i} = \frac{e^{(-1)z} - e^{-(-1)z}}{2i}$$
$$= \frac{e^{-z} - e^{z}}{2i} = i\left(\frac{e^{-z} - e^{z}}{2ii}\right) = i\left(\frac{e^{-z} - e^{z}}{-2}\right)$$
$$= i\left(\frac{e^{z} - e^{-z}}{2}\right) = i\sinh z$$

Prove $\cos iz = \cosh z$

$\cos z = \frac{e^{iz} + e^{-iz}}{2}$

Put $z = iz$
$$\cos iz = \frac{e^{iiz} + e^{-iiz}}{2} = \frac{e^{i^2 z} + e^{-i^2 z}}{2}$$
$$= \frac{e^{-z} + e^{z}}{2} = \cosh z$$

Trigonometric Functions

We know
$e^{ix} = \cos x + i\sin x$
$e^{-ix} = \cos x - i\sin x$
$e^{ix} + e^{-ix} = 2\cos x$
$\frac{e^{ix} + e^{-ix}}{2} = \cos x$
Similarly $e^{ix} - e^{-ix} = 2i\sin x$
$\frac{e^{ix} - e^{-ix}}{2i} = \sin x$

$\sec x = \frac{2}{e^{ix} + e^{-ix}}$

$\operatorname{cosec} x = \frac{2i}{e^{ix} - e^{-ix}}$

$\tan x = \frac{\sin x}{\cos x} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$

$\cot x = \frac{i(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}}$

Q1 $\boxed{\text{Ex } 1\cdot 3}$

Show that e^z is never zero.

Sol $e^z \cdot \frac{1}{e^z} = 1$ Since the multiplicati[ve] inverse of e^z exists so e^z is never zer[o]

(4)

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[1.2-38]

$$L.H.S = \left(\frac{z_1}{z_2}\right)^n = \left(\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\right)^n$$

$$\Rightarrow \left(\frac{z_1}{z_2}\right)^n = \frac{r_1^n}{r_2^n}\left(\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\cos n(\theta_1 - \theta_2) + i\sin n(\theta_1 - \theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left(\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 + \sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 - i\sin n\theta_2\cos n\theta_1) + (i\sin n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left(\cos n\theta_1(\cos n\theta_2 - i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 - i\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 - i\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n} \times (\cos\theta_1 + i\sin\theta_1)^n(\cos n\theta_2 + i\sin n\theta_2)^{-1}$$

$$= \frac{r_1^n}{r_2^n}\frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos n\theta_2 + i\sin n\theta_2)} = \frac{r_1^n}{r_2^n} \times \frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos\theta_2 + i\sin\theta_2)^n} = \frac{z_1^n}{z_2^n}$$

$$= R.H.S$$

(Q) (iii) $|e^{iz}| = 1$

Proof Since $e^{iz} = \cos z + i \sin z$

$$\Rightarrow |e^{iz}| = |\cos z + i \sin z|$$
$$= \sqrt{\cos^2 z + \sin^2 z} = 1 = \text{R.H.S.}$$

×———————×

(iv) $e^{z_1} = e^{z_2} \Leftrightarrow z_1 - z_2 = 2k\pi i$, where k is an integer

Proof Suppose that

$$e^{z_1} = e^{z_2} \Rightarrow e^{z_1} \cdot e^{-z_2} = e^{z_2} \cdot e^{-z_2}$$

$$e^{z_1 - z_2} = e^{z_2 - z_2} = e^0 = 1$$

Put $z = z_1 - z_2$,

$e^z = 1$, if and only if $z = 2k\pi i$

k is any integer, then

$e^{2k\pi i} = \cos 2k\pi + i \sin 2k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$

$= 1 + 0i = 1$

(Since $\cos 2k\pi = 1$, $\sin 2k\pi = 0$)

$$e^{z_1 - z_2} = 1 \Rightarrow e^{z_1 - z_2} = e^0 = e$$

$$e^{z_1} \cdot e^{-z_2} = 1 \Rightarrow e^{z_1} = e^{z_2}$$

2nd Method

$e^{z_1} = e^{z_2}$

$\Rightarrow \dfrac{e^{z_1}}{e^{z_2}} = 1$ or

$\Rightarrow e^{z_1 - z_2} = 1$

which is possible only if $z_1 - z_2$ is an integral multiple of $2\pi i$ (as proved in (iii))

$\Rightarrow z_1 - z_2 = 2\pi i k$ where k is integer

Conversely

$z_1 - z_2 = 2\pi i k$

$e^{z_1 - z_2} = e^{2\pi i k}$

$= \cos 2k\pi + i \sin 2k\pi$

$= 1 + 0$

$= 1$

$e^{z_1 - z_2} = 1$

$e^{z_1} = e^{z_2}$

suppose that

$e^z = 1$, taking $z = x + iy$

$$\Rightarrow e^{x+iy} = e^x(\cos y + i \sin y) = 1$$

$$= e^x \cos y + i e^x \sin y = 1 = 1 + 0i$$

$$\Rightarrow e^x \cos y = 1 \text{ and } e^x \sin y = 0$$

but $e^x \neq 0 \Rightarrow \cos y = 1$ and $\sin y = 0$

(58) $\boxed{1\cdot 3-3}$

Q: Imp

(iii) $e^z = 1 \Leftrightarrow z$ is an integral multiple of $2\pi i$

Proof: Let $z = 2\pi i K$ where K is any integer

i.e. $K = 0, \pm 1, \pm 2, \pm 3, \ldots$

Then $e^z = e^{2\pi i K} = \cos 2\pi K + i \sin 2\pi K$ —— (E)

$= 1 + 0 = 1$

Since $\cos 2\pi K = 1$ and $\sin 2\pi K = 0$ when K is any integer

Conversely Suppose that $e^z = 1$

and if $z = x + iy$

$\Rightarrow e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos y + i \sin y]$

$\therefore 1 = e^x \cos y + i e^x \sin y$ $\quad \because e^z = 1$ supposed

Comparing Real & Imaginary Parts

$\Rightarrow 1 = e^x \cos y$ & $0 = e^x \sin y$

$\Rightarrow$ Since $e^x \neq 0$ $\therefore \cos y \neq 0$ $\qquad \Rightarrow \because e^x \neq 0, \therefore \sin y = 0$

$\sin y = 0 \Rightarrow \boxed{y = n\pi}$ where n is any integer

i.e. $n = 0, \pm 1, \pm 2, \ldots$

Now $\cos y = \cos n\pi = (-1)^n$ where $n = 0, \pm 1, \pm 2, \ldots$

$\therefore e^x \cos y = 1$

becomes $e^x (-1)^n = 1$

$\therefore (-1)^n > 0$ $\qquad$ since $e^x > 0$ and product of two +ives or product of two −ives is positive. So $(-1)^n$ is +ive $\because$ product of e^x & $(-1)^n$ is +ive 1 and $e^x > 0$ so $(-1)^n$ must be +ive

$\Rightarrow n$ must be even i.e. $\boxed{n = 2K}$ where $K = 0, \pm 1, \pm 2, \ldots$

$\therefore \Rightarrow e^x (-1)^{2K} = 1 \Rightarrow e^x \cdot 1 = 1 = e^0 \Rightarrow x = 0$

$\Rightarrow z = 0 + iy = in\pi = i(2K)\pi = 2K\pi i$

where $K = 0, \pm 1, \pm 2, \ldots$

and it only possible when ~~y is an even integer~~

~~i.e~~ $y = 2k\pi$ and $x = 0$

$\Rightarrow z = 0 + iy = 0 + 2k\pi i$

$\Rightarrow z_1 - z_2 = 2k\pi i$

x ———— x

(v) Show that $|e^z| = e^x$, when $z = x + iy$

Sol L.H.S $= |e^z| = |e^{x+iy}|$ Since $z = x+iy$ (given)

$= |e^x \cdot e^{iy}| = |e^x||e^{iy}|$

$= |e^x| \, |\cos y + i\sin y|$

$= |e^x|\sqrt{\cos^2 y + \sin^2 y} = |e^x| \cdot 1 = |e^x| = R.H.S$

x ———— x

(vi) $e^{z_1} \cdot e^{z_2} \cdots\cdots e^{z_n} = e^{z_1 + z_2 + z_3 + \cdots\cdots z_n}$. where $n = 1, 2, 3 \ldots$

Proof we shall prove it by Induction.

Case I Put $n = 2$, then

$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$

Hence c-1 is true for $n=2$

2nd Method

$z_1 = (x_1 + iy_1)$ $z_2 = (x_2 + iy_2)$

LHS $e^{x_1+iy_1} \cdot e^{x_2+iy_2} \cdot e^{x_3+iy_3} \cdots\cdots e^{x_n+iy_n}$

$= e^{x_1+x_2+\cdots x_n}(\cos y_1 + i\sin y_1) \cdots\cdots (\cos y_n + i\sin y_n)$

$= e^{x_1+x_2\cdots x_n}[\cos(y_1+y_2\cdots y_n) + i\sin(y_1+y_2 \cdots y_n)]$

$= e^{x_1+x_2\cdots x_n} e^{i(y_1+y_2 \cdots\cdots y_n)}$

$= e^{(x_1+iy_1)+(x_2+iy_2)+\cdots\cdots(x_n+iy_n)}$

$= e^{z_1+z_2+\cdots\cdots z_n}$

C-2 let it is true for $n = k$.

$e^{z_1} \cdot e^{z_2} \cdot e^{z_3} \cdots\cdots e^{z_k} = e^{z_1+z_2+z_3+\cdots+z_k} \rightarrow (1)$

Now we have to prove it that it is true for $n = k+1$. for 'x' eq (1) by $e^{z_{k+1}}$ both sides, we get

$$\left(e^{z_1} \cdot e^{z_2} \cdot e^{z_3} \cdots e^{z_k}\right) \cdot e^{z_{k+1}} = e^{z_1+z_2+z_3+\cdots z_k} \cdot e^{z_{k+1}}$$

$$\Rightarrow e^{z_1} \cdot e^{z_2} \cdot e^{z_3} \cdots e^{z_{k+1}} = e^{z_1+z_2+z_3+\cdots z_k+z_{k+1}}$$

$$= e^{z_1+z_2+\cdots+z_{k+1}}$$

$\Rightarrow$ given statement is true for $n = k+1$. Thus it is true for all +ive integral values of n.

x ———— x

(vii) $(e^z)^n = e^{nz}$, where n is any integer

Proof let $z = x+iy$, then

$$L.H.S = (e^z)^n = \left(e^{x+iy}\right)^n = \left(e^x \cdot e^{iy}\right)^n = \left[e^x(\cos y + i\sin y)\right]^n$$

$$= e^{nx}\left[\cos ny + i\sin ny\right] \quad \text{(De moivre's Th.)}$$

$$= e^{nx} \cdot e^{iny} = e^{nx+iny} = e^{n(x+iy)}$$

$$= e^{nz} = R.H.S.$$

x ———— x

Q.2 (i) Prove that $\forall\ z_1, z_2, z_3 \in C$
$1+\tan^2 z = \sec^2 z$

L.H.S $\quad 1+\tan^2 z$ $\qquad \left(\because \tan z = \dfrac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\right)$

$$= 1+\left(\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\right)^2$$

$$= 1+\frac{e^{2iz}+e^{-2iz}-2}{i^2(e^{2iz}+e^{-2iz}+2)} = 1+\frac{2-e^{2iz}-e^{-2iz}}{e^{2iz}+e^{-2iz}+2}$$

$(\because i^2=-1)$

1.3 - 6

$$= \frac{\cancel{e^{2iz}} + \cancel{e^{-2iz}} + 2 + 2 - \cancel{e^{2iz}} - \cancel{e^{-2iz}}}{e^{2iz} + e^{-2iz} + 2}$$

$$= \frac{4}{e^{2iz} + e^{-2iz} + 2} = \left(\frac{2}{e^{iz} + e^{-iz}}\right)^2 = \sec^2 z$$
$= R.H.S.$

(ii) $1 + \cot^2 z = \operatorname{cosec}^2 z$

Proof: L.H.S $= 1 + \cot^2 z$

$$= 1 + \left[\frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}\right]^2 = 1 + \frac{(-1)(e^{2iz} + e^{-2iz} + 2)}{e^{2iz} + e^{-2iz} - 2}$$

(using $i^2 = -1$)

$$= \frac{\cancel{e^{2iz}} + \cancel{e^{-2iz}} - 2 - \cancel{e^{2iz}} - \cancel{e^{-2iz}} - 2}{e^{2iz} + e^{-2iz} - 2}$$

$$= \frac{-4}{e^{2iz} + e^{-2iz} - 2} = \frac{4i^2}{(e^{iz} - e^{-iz})^2} = \left(\frac{2i}{e^{iz} - e^{-iz}}\right)^2$$

$= \operatorname{cosec}^2 z = R.H.S$

(iii) $\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$

R.H.S. $\sin z_1 \cos z_2 - \cos z_1 \sin z_2$

$$= \left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2} - e^{-iz_2}}{2i}\right)$$

$$= \frac{e^{iz_1} \cdot e^{iz_2} + e^{iz_1} \cdot e^{-iz_2} - e^{-iz_1} \cdot e^{iz_2} - e^{-iz_1} \cdot e^{-iz_2}}{4i} - \frac{e^{iz_1} \cdot e^{iz_2} - e^{iz_1} \cdot e^{-iz_2} + e^{-iz_1} \cdot e^{iz_2} - e^{-iz_1} \cdot e^{-iz_2}}{4i}$$

$$\frac{\cancel{e^{iz_1+iz_2}} + e^{iz_1-iz_2} - e^{-iz_1+iz_2} - e^{-iz_1-iz_2} - e^{iz_1+iz_2} + e^{iz_1-iz_2} - e^{-iz_1+iz_2} + \cancel{e^{-iz_1-iz_2}}}{4i}$$

$$= \frac{2\left(e^{iz_1-iz_2} - e^{-iz_1+iz_2}\right)}{4i} = \frac{e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}}{2i}$$

$$= \sin(z_1 - z_2) = L.H.S.$$

x ———— x

(iv) $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.

R.H.S $= \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$$= \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)$$

$$= \frac{e^{iz_1}\cdot e^{iz_2} + \cancel{e^{iz_1}e^{-iz_2}} + \cancel{e^{-iz_1}\cdot e^{iz_2}} + e^{-iz_1}\cdot e^{-iz_2}}{4}$$

$$+ \frac{e^{iz_1}e^{iz_2} - \cancel{e^{iz_1}e^{-iz_2}} - \cancel{e^{-iz_1}e^{iz_2}} + e^{-iz_1}e^{-iz_2}}{4}$$

$$= \frac{e^{iz_1}\cdot e^{iz_2} + e^{-iz_1}\cdot e^{-iz_2} + e^{iz_2}e^{iz_1} + e^{-iz_1}\cdot e^{-iz_2}}{4}$$

$$= \frac{2\left(e^{iz_1+iz_2} + e^{-iz_1-iz_2}\right)}{4} = \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2}$$

$$= \cos(z_1+z_2) = R.H.S.$$

x ———— x

(8)

(68) $\boxed{1\cdot 3-8}$

Q·2(vi) $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$

PROOF $\cos^2 z - \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$

$$= \frac{e^{2iz}+e^{-2iz}+2}{4} - \frac{e^{2iz}+e^{-2iz}-2}{4i^2}$$

$$= \cancel{\frac{e^{2iz}+e^{-2iz}+2-e^{2iz}-e^{-2iz}+2}{4}}$$

$$= \frac{e^{2iz}+e^{-2iz}+\cancel{2}}{4} + \frac{e^{2iz}+e^{-2iz}-\cancel{2}}{4}$$

$$= \frac{2\left(e^{2iz}+e^{-2iz}\right)}{4} = \frac{e^{2iz}+e^{-2iz}}{2}$$

$$= \cos 2z = L.H.S$$

$$2\cos^2 z - 1 = 2\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 - 1$$

$$= \cancel{2}\left(\frac{e^{2iz}+e^{-2iz}+2}{4_2}\right) - 1$$

$$= \frac{e^{2iz}+e^{-2iz}+2}{2} - 1 = \frac{e^{2iz}+e^{-2iz}+\cancel{2}-\cancel{2}}{2}$$

$$= \frac{e^{2iz}+e^{-2iz}}{2} = \cos 2z = L.H.S$$

$$1-2\sin^2 z = 1-2\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$$

$$= 1-2\left(\frac{e^{2iz}+e^{-2iz}-2}{4i^2}\right)$$

$$= 1+\frac{e^{2iz}+e^{-2iz}-2}{2} \qquad (\because i^2=-1)$$

$$= \frac{\cancel{2}+e^{2iz}+e^{-2iz}-\cancel{2}}{2} = \frac{e^{2iz}+e^{-2iz}}{2} = \cos 2z = L.H.S$$

2(vii) $\sin 2z = 2\sin z\cos z$

R.H.S. $= 2\sin z\cos z$

$$= \cancel{2}\left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iz}+e^{-iz}}{\cancel{2}}\right)$$

$$= \frac{e^{2iz}-e^{-2iz}}{2i} = \sin 2z = L.H.S.$$

x ——— x

2(viii) $\cos z_1 - \cos z_2 = 2\sin\frac{z_1+z_2}{2}\sin\frac{z_2-z_1}{2}$

L.H.S. $\cos z_1 - \cos z_2$

$$= \frac{e^{iz_1}+e^{-iz_1}}{2} - \frac{e^{iz_2}+e^{-iz_2}}{2}$$

$$\boxed{\sin z = \frac{e^{iz}-e^{-iz}}{2i}}$$

$$= \left(e^{iz_1}+e^{-iz_1}-e^{iz_2}-e^{-iz_2}\right)/2 \longrightarrow (1)$$

R.H.S. $2\sin\frac{z_1+z_2}{2}\sin\frac{z_2-z_1}{2}$

$$= \cancel{2}\left(\frac{e^{i\left(\frac{z_1+z_2}{2}\right)}-e^{-i\left(\frac{z_1+z_2}{2}\right)}}{\cancel{2}i}\right)\left(\frac{e^{i\left(\frac{z_2-z_1}{2}\right)}-e^{-i\left(\frac{z_2-z_1}{2}\right)}}{2i}\right)$$

$$= \frac{1}{2i^2}\left[e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{iz_2-iz_1}{2}} - e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{-iz_2+iz_1}{2}} - e^{\frac{-iz_1-iz_2}{2}}\cdot e^{\frac{iz_2-iz_1}{2}} + e^{\frac{-iz_1-iz_2}{2}}\cdot e^{\frac{-iz_2+iz_1}{2}}\right]$$

$$= -\frac{1}{2}\left[e^{\frac{2iz_2}{2}} - e^{\frac{2iz_1}{2}} - e^{\frac{-2iz_1}{2}} + e^{\frac{-2iz_2}{2}}\right]$$

$$= \frac{e^{iz_1}+e^{-iz_1}-e^{iz_2}-e^{-iz_2}}{2} \longrightarrow (2)$$

from (1) and (2)

L.H.S. = R.H.S.

1·3−10 66

(10)

Q.2 (ix) $\sin z_1 + \sin z_2 = 2\sin\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2}$

$$L.H.S. = \sin z_1 + \sin z_2$$
$$= \frac{e^{iz_1} - e^{-iz_1}}{2i} + \frac{e^{iz_2} - e^{-iz_2}}{2i}$$
$$= \frac{e^{iz_1} - e^{-iz_1} + e^{iz_2} - e^{-iz_2}}{2i} \longrightarrow (1)$$

$$R.H.S. = 2\sin\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2}$$
$$= \cancel{2}\left(\frac{e^{\frac{i(z_1+z_2)}{2}} - e^{-i\left(\frac{z_1+z_2}{2}\right)}}{2i}\right)\left(\frac{e^{i\left(\frac{z_1-z_2}{2}\right)} + e^{-i\left(\frac{z_1-z_2}{2}\right)}}{\cancel{2}}\right)$$
$$= \frac{1}{2i}\left[e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{iz_1-iz_2}{2}} + e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{-iz_1+iz_2}{2}} - e^{\frac{-iz_1-iz_2}{2}}\cdot e^{\frac{iz_1-iz_2}{2}} - e^{\frac{-iz_1-iz_2}{2}}\cdot e^{\frac{-iz_1+iz_2}{2}}\right]$$
$$= \frac{1}{2i}\left[e^{\frac{2iz_1}{2}} + e^{\frac{2iz_2}{2}} - e^{\frac{-2iz_2}{2}} - e^{\frac{-2iz_1}{2}}\right]$$
$$= \frac{1}{2i}\left(e^{iz_1} - e^{-iz_1} + e^{iz_2} - e^{-iz_2}\right) \longrightarrow (2)$$

from (1) and (2) $\Rightarrow L.H.S. = R.H.S.$

× ——— ×

Q.2 (x) $\sin 3z = 3\sin z - 4\sin^3 z$

$$L.H.S. \quad 3\sin z - 4\sin^3 z$$
$$= 3\left(\frac{e^{iz} - e^{-iz}}{2i}\right) - 4\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^3$$
$$= 3\left(\frac{e^{iz} - e^{-iz}}{2i}\right) - 4\left(\frac{e^{3iz} - 3e^{2iz}e^{-iz} + 3e^{iz}e^{-2iz} - e^{-3iz}}{8i^3}\right)$$
$$= \frac{3e^{iz} - 3e^{-iz}}{2i} - 4\left(\frac{e^{3iz} - 3e^{iz} + 3e^{-iz} - e^{-3iz}}{-4(2i)}\right)$$

$$= \frac{3e^{iz} - 3e^{-iz} + e^{3iz} - 3e^{iz} + 3e^{-iz} - e^{-3iz}}{2i}$$

$$= \frac{e^{3iz} - e^{-3iz}}{2i} = \operatorname{Sin} 3z = L.H.S.$$

Q.2(xi) $\tan(z_1+z_2) = \dfrac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$

Sol: If z_1, z_2 are complex numbers, then

$\operatorname{Sin}(z_1+z_2) = \operatorname{Sin} z_1 \operatorname{Cos} z_2 + \operatorname{Cos} z_1 \operatorname{Sin} z_2$

and $\operatorname{Cos}(z_1+z_2) = \operatorname{Cos} z_1 \operatorname{Cos} z_2 - \operatorname{Sin} z_1 \operatorname{Sin} z_2$

$\Rightarrow L.H.S. = \tan(z_1+z_2)$

$= \dfrac{\operatorname{Sin}(z_1+z_2)}{\operatorname{Cos}(z_1+z_2)}$, putting values we get

$= \dfrac{\operatorname{Sin} z_1 \operatorname{Cos} z_2 + \operatorname{Cos} z_1 \operatorname{Sin} z_2}{\operatorname{Cos} z_1 \operatorname{Cos} z_2 - \operatorname{Sin} z_1 \operatorname{Sin} z_2}$

$\div$ each term of numerator and denominator by $\operatorname{Cos} z_1 \cdot \operatorname{Cos} z_2$, we get

$$= \frac{\dfrac{\operatorname{Sin} z_1 \cancel{\operatorname{Cos} z_2}}{\operatorname{Cos} z_1 \cancel{\operatorname{Cos} z_2}} + \dfrac{\cancel{\operatorname{Cos} z_1} \operatorname{Sin} z_2}{\cancel{\operatorname{Cos} z_1} \operatorname{Cos} z_2}}{1 - \dfrac{\operatorname{Sin} z_1}{\operatorname{Cos} z_1} \dfrac{\operatorname{Sin} z_2}{\operatorname{Cos} z_2}}$$

$$= \frac{\dfrac{\operatorname{Sin} z_1}{\operatorname{Cos} z_1} + \dfrac{\operatorname{Sin} z_2}{\operatorname{Cos} z_2}}{1 - \dfrac{\operatorname{Sin} z_1}{\operatorname{Cos} z_1} \cdot \dfrac{\operatorname{Sin} z_2}{\operatorname{Cos} z_2}}$$

$$= \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2} = R.H.S.$$

Q·2(xii) $\tan(z_1-z_2)=\dfrac{\tan z_1-\tan z_2}{1+\tan z_1\tan z_2}$

L.H.S $\tan(z_1-z_2)$

$$=\frac{\sin(z_1-z_2)}{\cos(z_1-z_2)} \longrightarrow (i)$$

we know that if z_1 and z_2 are any complex numbers then

$$\sin(z_1-z_2)=\sin z_1\cos z_2-\cos z_1\sin z_2$$

and $\cos(z_1-z_2)=\cos z_1\cos z_2+\sin z_1\sin z_2$

putting these values in (i) we get

$$\tan(z_1-z_2)=\frac{\sin z_1\cos z_2-\cos z_1\sin z_2}{\cos z_1\cos z_2+\sin z_1\sin z_2}$$

÷ nume- and denome- by $\cos z_1\cos z_2$

we get

$$\tan(z_1-z_2)=\frac{\dfrac{\sin z_1\cos z_2}{\cos z_1\cos z_2}-\dfrac{\cos z_1\sin z_2}{\cos z_1\cos z_2}}{1+\dfrac{\sin z_1}{\cos z_1}\cdot\dfrac{\sin z_2}{\cos z_2}}$$

$$=\frac{\dfrac{\sin z_1}{\cos z_1}-\dfrac{\sin z_2}{\cos z_2}}{1+\dfrac{\sin z_1}{\cos z_1}\cdot\dfrac{\sin z_2}{\cos z_2}}$$

$$=\frac{\tan z_1-\tan z_2}{1+\tan z_1\tan z_2} = R.H.S.$$

X———————————X

1.3 (i) Show that

$$\overline{\sin z}=\sin\bar z$$

L.H.S $\overline{\sin z}$

let $z=x+iy$, then

$$\sin z = \sin(x+iy)$$
$$= \sin x \cos(iy) + \cos x \sin(iy)$$
$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$\because \cos iz = \cosh z$ and $\sin iz = i \sinh z$

$$\Rightarrow \overline{\sin z} = \overline{\sin x \cosh y + i \cos x \sinh y}$$
$$= \sin x \cosh y - i \cos x \sinh y \longrightarrow ①$$

and $\sin \bar{z} = \sin(\overline{x+iy}) = \sin(x-iy)$
$$= \sin x \cos(iy) - \sin(iy) \cos x$$
$$= \sin x \cosh y - i \sinh y \cos x \longrightarrow ②$$

from ① and ②
$$\overline{\sin z} = \sin \bar{z}$$

× ——————— ×

Q.3(ii) $\overline{\cos z} = \cos \bar{z}$

$\because \cos iz = \cosh z$ and $\sin iz = i \sinh z$

L.H.S $= \overline{\cos z}$

let $z = x+iy$. then
$$\cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$
$$= \cos x \cosh y - i \sin x \sinh y$$
$$\Rightarrow \overline{\cos z} = \cos x \cosh y + i \sin x \sinh y \longrightarrow ①$$

and $\cos \bar{z} = \cos(\overline{x+iy}) = \cos(x-iy)$
$$= \cos x \cos iy + \sin x \sin iy$$
$$= \cos x \cosh y + i \sin x \sinh y \longrightarrow ②$$

from ① and ②
$$\overline{\cos z} = \cos \bar{z}$$

× ——————— ×

Q.3(iii) $\overline{\tan z} = \tan \bar{z}$

Sol L.H.S $= \overline{\tan z}$

70 $\boxed{1\cdot 3-14}$

Let $z = x+iy$

then $\tan z = \tan(x+iy)$ $\qquad \because \tan iz = i\tanh z$

$$= \frac{\tan x + \tan iy}{1-\tan x \tan iy}$$

$$\tan z = \frac{\tan x + i\tanh y}{1-i\tan x\tanh y}$$

$$\Rightarrow \overline{\tan z} = \overline{\left(\frac{\tan x + i\tanh y}{1-i\tan x \tanh y}\right)} \qquad \boxed{\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}}$$

$$= \frac{\overline{\tan x + i\tanh y}}{\overline{1-i\tan x\tanh y}}$$

$$= \frac{\tan x - i\tanh y}{1+i\tan x\tanh y} \qquad \cancel{\tan} \tan x$$

$$= \frac{\tan x - \tan iy}{1+\tan x\tan iy} = \tan(x-iy) = \tan\bar{z}$$

$$= R.H.S.$$

x———x

Q.3(iv) $\sin(-z) = -\sin z$

L.H.S $\quad \sin(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{2i}$

$$= \frac{e^{-iz} - e^{iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right)$$

$$= -\sin z = R.H.S$$

x———x

Q.3(v) $\cos(-z) = \cos z$

L.H.S $\quad \cos(-z) = \dfrac{e^{i(-z)} + e^{-i(-z)}}{2}$

$$= \frac{e^{-iz} + e^{+iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos z = R.H.S.$$

Q.3(vi) $\tan(-z) = -\tan z$

L.H.S. $\tan(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{i\left(e^{i(-z)} + e^{-i(-z)}\right)} = \dfrac{e^{-iz} - e^{iz}}{i\left(e^{-iz} + e^{iz}\right)}$

$$= -\left[\frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}\right] = -\tan z = R.H.S.$$

Q.3(vii) $\sinh(-z) = -\sinh z$

L.H.S. $\sinh(-z) = \dfrac{e^{-z} - e^{-(-z)}}{2} = \dfrac{e^{-z} - e^{+z}}{2}$

$$= -\left(\frac{e^{z} - e^{-z}}{2}\right) = -\sinh z = R.H.S.$$

Q.3(viii) $\cosh(-z) = \cosh z$

L.H.S. $= \cosh(-z) = \dfrac{e^{-z} + e^{-(-z)}}{2} = \dfrac{e^{z} + e^{z}}{2} = \dfrac{e^{z} + e^{-z}}{2} = \cosh z$
$= R.H.S$

Q.3(ix) $\tanh(-z) = -\tanh z$

L.H.S $= \tanh(-z) = \dfrac{e^{-z} - e^{-(-z)}}{e^{-z} + e^{-(-z)}} = \dfrac{e^{-z} - e^{z}}{e^{-z} + e^{z}}$

$$= -\left(\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}\right) = -\tanh z = R.H.S.$$

Q.3(x) $\overline{\tanh z} = \tanh \bar{z}$

Sol. We know that $i \tanh z = \tan(iz)$

Ex # $\boxed{1.3-16}$

Q4 (i) $\cosh^2 z - \sinh^2 z = 1$ To Prove

$\cos^2 z + \sin^2 z = 1$ by putting $z = iz$

$\cos^2 iz + \sin^2 iz = 1$ $\because \cos iz = \cosh z$

$\cosh^2 z + (i\sinh z)^2 = 1$ $\because \sin iz = i\sinh z$

$\cosh^2 z - \sinh^2 z = 1$ $\because i^2 = -1$

(ii) To Prove $\operatorname{sech}^2 z = 1 - \tanh^2 z$

We know

$\sec^2 z = 1 + \tan^2 z$ by putting $z = iz$

$\sec^2 iz = 1 + \tan^2 iz$ $\because \sec iz = \operatorname{sech} z$

$\operatorname{sech}^2 z = 1 + (i\tanh z)^2$ $\because \tan iz = i\tanh z$

$\operatorname{sech}^2 z = 1 - \tanh^2 z$ $\because i^2 = -1$

(iii) To Prove $\operatorname{cosech}^2 hz = \coth^2 hz - 1$

We know

$1 + \cot^2 z = \operatorname{cosec}^2 z$ by putting $z = iz$

$1 + \cot^2 iz = \operatorname{cosec}^2 iz$

$1 + (-i\coth z)^2 = (-i\operatorname{cosech} z)^2$ $\left.\begin{array}{l}\because \cot iz = -i\coth z \\ \because \operatorname{cosec} iz = -i\operatorname{cosech} z\end{array}\right\}$

$1 - \coth^2 z = -\operatorname{cosech}^2 z$

$\operatorname{cosech}^2 z = \coth^2 z - 1$

(iv) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$

We know

$\cos 2z = \cos^2 z - \sin^2 z$

$\cos 2iz = (\cos iz)^2 - (\sin iz)^2$

$= \cosh^2 z - (i\sinh z)^2$

$\cosh 2z = \cosh^2 z + \sinh^2 z$ proved

$= \cosh^2 z + (\cosh^2 z - 1)$

$= 2\cosh^2 z - 1$ proved

$= 2(1 + \sinh^2 z) - 1$

$= 2 + 2\sinh^2 z - 1$

$= 1 + 2\sinh^2 z$ proved

OR $\cosh 2z = 2\cosh^2 z - 1$

We know $\cos 2z = 2\cos^2 z - 1$

$\cos 2iz = 2(\cos iz)^2 - 1$

$\cosh 2z = 2\cosh^2 z - 1$ proved

OR $\cosh 2z = 1 + 2\sinh^2 z$

We know

$\cos 2z = 1 - 2\sin^2 z$

$\cos 2(iz) = 1 - 2(\sin iz)^2$

$\cosh 2z = 1 - 2(i\sinh z)^2$

$\cosh 2z = 1 + 2\sinh^2 z.$ proved

Q4(ix) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

Sol We know

$$\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

$$\tan(iz_1 \pm iz_2) = \frac{\tan iz_1 \pm \tan iz_2}{1 \mp \tan iz_1 \tan iz_2}$$ by putting $z_1 = iz_1$, $z_2 = iz_2$

$$\tan i(z_1 + z_2) = \frac{i\tanh z_1 \pm i\tanh z_2}{1 \mp (i)^2 \tanh z_1 \tanh z_2}$$ $(\because \tan i z_1 = i\tanh z_1)$

$$i\tanh(z_1 + z_2) = i\left(\frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}\right)$$

$$\tanh(z_1 + z_2) = \frac{\cancel{i}(\tanh z_1 \pm \tanh z_2)}{\cancel{i}(1 \pm \tanh z_1 \tanh z_2)}$$ Proved

Q4(x) $\tanh 3z = \dfrac{3\tanh z + \tanh^3 z}{1 + 3\tanh^2 z}$

Sol We know

$$\tan 3z = \frac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$$

$$\tan 3iz = \frac{3\tan iz - \tan^3 iz}{1 - 3\tan^2 iz}$$

$$i\tanh 3z = \frac{3i\tanh z - (i\tanh z)^3}{1 - 3ii\tan^2 hz}$$

$$i\tanh 3z = \frac{i(3\tanh z + \tan^3 hz)}{1 + 3\tan^2 hz}$$

$$\tanh 3z = \frac{\cancel{i}(3\tanh z + \tan^3 hz)}{\cancel{i}(1 + 3\tan^2 hz)}$$ Proved

1-3-18

Q4 (v) To Prove $\sinh 2z = 2\sinh z \cosh z$

We know

$$\sin 2z = 2\sin z \cos z$$

$$\sin 2iz = 2\sin iz \cos iz$$

$$i\sinh 2z = 2(i\sinh z)(\cosh z)$$

$$\sinh 2z = 2\sinh z \cosh z \quad \text{proved}$$

(vi) To Prove $\sinh 3z = 3\sinh z + 4\sinh^3 z$

We know

$$\sin 3z = 3\sin z - 4\sin^3 z$$

$$\sin 3iz = 3\sin iz - 4\sin^3 iz$$

$$i\sinh 3z = 3i\sinh z - 4(i\sinh z)^3$$

$$i\sinh 3z = 3i\sinh z + 4i\sinh^3 z$$

$$\sinh 3z = \frac{\cancel{i}}{\cancel{i}}(3\sinh z + 4\sinh^3 z)$$

proved

vii) To Prove $\cosh 3z = 4\cosh^3 z - 3\cosh z$

We know

$$\cos 3z = 4\cos^3 z - 3\cos z$$

$$\cos 3iz = 4\cos^3 iz - 3\cos iz$$

$$\cos 3hz = 4\cos^3 hz - 3\cosh z \quad \text{proved}$$

viii) To Prove $\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2$

We know

$$\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

$$\sin(iz_1 - iz_2) = \sin iz_1 \cos iz_2 - \cos iz_1 \sin iz_2$$

$$\sin i(z_1 - z_2) = i\sinh z_1 \cosh z_2 - \cosh z_1 (i\sinh z_2)$$

$$i\sinh(z_1 - z_2) = i(\sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2)$$

$$\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2$$

proved

$\boxed{1.3-19}$ 85

Q5 (i) If $z = x+iy$, Prove that
$\sin z = \sin x \cosh y + i \cos x \sinh y$

Sol $\sin z = \sin(x+iy)$
$= \sin x \cos iy + \cos x \sin iy$
$= \sin x \cosh y + i \cos x \sinh y$

Proved

ii) Prove that $\tan z = \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$

$\tan z = \tan(x+iy)$

$= \dfrac{\sin(x+iy)}{\cos(x+iy)}$

x & ÷ by $2\cos(x-iy)$

$= \dfrac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$

$= \dfrac{\sin(x+\cancel{iy}+x-\cancel{iy}) + \sin(\cancel{x}+iy-\cancel{x}+iy)}{\cos(x+\cancel{iy}+x-\cancel{iy}) + \cos(\cancel{x}+iy-\cancel{x}+iy)}$

$= \dfrac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy}$

$= \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$ Proved

2nd Method

$\tan z = \tan(x+iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)} = \dfrac{\sin x \cos iy + \cos x \sin iy}{\cos x \cos iy - \sin x \sin iy}$

$= \dfrac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}$

$= \dfrac{(\sin x \cosh y + i \cos x \sinh y)}{(\cos x \cosh y - i \sin x \sinh y)} \times \dfrac{(\cos x \cosh y + i \sin x \sinh y)}{(\cos x \cosh y + i \sin x \sinh y)}$

$= \dfrac{\sin x \cos x \cosh^2 y - \cos x \sin x \sinh^2 y + i(\sin^2 x \cosh y \sinh y + \cos^2 x \sinh y \cosh y)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$

$= \dfrac{\sin x \cos x (\cosh^2 y - \sinh^2 y) + i \cosh y \sinh y (\sin^2 x + \cos^2 x)}{\cos^2 x (1+\sinh^2 y) + \sin^2 x \sinh^2 y}$

(x & ÷ by 2)

$= \dfrac{\sin x \cos x \cdot 1 + i \cosh y \sinh y}{\cos^2 x + \sinh^2 y(\cos^2 x + \sin^2 x)} = \dfrac{2\sin x \cos x + i\, 2\cosh y \sinh y}{2\cos^2 x + 2\sinh^2 y}$

$\because \cos 2x = 2\cos^2 x - 1$
$\because \cosh 2y = 2\sinh^2 y + 1$
$\sinh 2y = 2\cosh y \sinh y$

$= \dfrac{\sin 2x + i \sinh 2y}{(2\cos^2 x - 1) + (2\sinh^2 y + 1)} = \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$ proved

(2+) Ex 1.4 Ex # 1.4 (i)

Q11 Prove that
$\sec(x+iy) = \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cosh 2y}$

Sol $\sec(x+iy)$ x & ÷ by $2\cos(x-iy)$

$= \dfrac{1}{\cos(x+iy)} \times \dfrac{2\cos(x-iy)}{2\cos(x-iy)}$

$= \dfrac{2(\cos x \cos iy + \sin x \sin iy)}{2\cos(x+iy)\cos(x-iy)}$

$= \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos(x+\cancel{iy}+x-\cancel{iy}) + \cos(\cancel{x}+iy-\cancel{x}+iy)}$

$= \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cos 2iy}$

$= \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cos 2hy}$

proved

Q6 If $\sin(A+iB) = x+iy$ then show that,

(i) $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$ (ii) $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$

Sol

$$\sin(A+iB) = x+iy$$

$$\sin A \cos iB + \cos A \sin iB = x+iy$$

$$\Rightarrow \sin A \cosh B + i\cos A \sinh B = x+iy$$

Equating Real & Imaginary Parts

$\sin A \cosh B = x$ —① & $\cos A \sinh B = y$ —②

$\frac{x}{\sin A} = \cosh B$ $\frac{y}{\cos A} = \sinh B$

Squaring & subtracting

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1 \text{ proved}$$

from ① $\frac{x}{\cosh B} = \sin A$

from ② $\frac{y}{\sinh B} = \cos A$

Squaring + adding

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A$$

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

proved

Q7 (i) If $\tan(\alpha+i\beta) = x+iy$, then Show that $x^2+y^2+2x\cot 2\alpha = 1$

Sol $\tan(\alpha+i\beta) = x+iy$

$$\alpha+i\beta = \tan^{-1}(x+iy) \text{ —①}$$

Conjugate

$$\alpha-i\beta = \tan^{-1}(x-iy) \text{ —②}$$

Add ① & ② To eliminate β

$$\alpha+\cancel{i\beta}+\alpha-\cancel{i\beta} = \tan^{-1}(x+iy)+\tan^{-1}(x-iy)$$

$$2\alpha = \tan^{-1}\left(\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}\right)$$

$$2\alpha = \tan^{-1}\left[\frac{2x}{1-(x^2+y^2)}\right]$$

$$\tan 2\alpha = \frac{2x}{1-x^2-y^2}$$

Reciprocal

$$\cot 2\alpha = \frac{1-x^2-y^2}{2x}$$

$$2x\cot 2\alpha = 1-x^2-y^2$$

$$x^2+y^2+2x\cot 2\alpha = 1$$

proved.

(ii) If $\tan(\alpha+i\beta) = x+iy$ then show that $x^2+y^2-2y\coth 2\beta = -1$

Sol $\tan(\alpha+i\beta) = x+iy$

$$\alpha+i\beta = \tan^{-1}(x+iy) \text{ —①}$$

$$\alpha-i\beta = \tan^{-1}(x-iy) \text{ —②}$$

(To eliminate α) Subtracting

$$\alpha+i\beta-(\alpha-i\beta) = \tan^{-1}(x+iy)-\tan^{-1}(x-iy)$$

$$2i\beta = \tan^{-1}\left[\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}\right]$$

$$\tan 2i\beta = \frac{2iy}{1+x^2+y^2}$$

$$i\tanh 2\beta = \frac{2iy}{1+x^2+y^2}$$

$$\tanh 2\beta = \frac{2y}{1+x^2+y^2}$$

$$\coth 2\beta = \frac{1+x^2+y^2}{2y}$$

$$2y\coth 2\beta = 1+x^2+y^2$$

$$-1 = x^2+y^2-2y\coth 2\beta$$

proved

Note $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

1·3–21

Q8 If $\sin(\theta+\phi i) = \cos\alpha + i\sin\alpha$, prove that $\cos^2\theta = \pm\sin\alpha$

Sol $\sin(\theta+\phi i) = \cos\alpha + i\sin\alpha$

$\sin\theta\cos(i\phi)+\cos\theta\sin(i\phi) = \cos\alpha + i\sin\alpha$

$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \cos\alpha + i\sin\alpha$

Equating real & imaginary parts

$\sin\theta\cosh\phi = \cos\alpha$ & $\cos\theta\sinh\phi = \sin\alpha$

$\cosh\phi = \dfrac{\cos\alpha}{\sin\theta}$ $\qquad$ $\sinh\phi = \dfrac{\sin\alpha}{\cos\theta}$

squaring & subtracting

$$\cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$$

$$1 = \frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\sin^2\theta\cos^2\theta = \cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta$$

$$(1-\cos^2\theta)\cos^2\theta = (1-\sin^2\alpha)\cos^2\theta - \sin^2\alpha(1-\cos^2\theta)$$

$$\cos^2\theta - \cos^4\theta = \cos^2\theta - \cancel{\sin^2\alpha\cos^2\theta} - \sin^2\alpha + \cancel{\sin^2\alpha\cos^2\theta}$$

$$\cancel{\cos^2\theta} - \cancel{\cos^2\theta} + \sin^2\alpha = \cos^4\theta$$

$$\pm\sin\alpha = \cos^2\theta \quad \text{Proved.}$$

Q10 Prove that $\sinh\left(\frac{x}{2}\right) = \sqrt{\dfrac{\cosh x - 1}{2}}$ if $x \geq 0$

$= -\sqrt{\dfrac{\cosh x - 1}{2}}$ if $x < 0$

Sol RHS $\sqrt{\dfrac{\cosh x - 1}{2}}$

$$= \sqrt{\frac{\left(\frac{e^x+e^{-x}}{2}\right)-1}{2}} = \sqrt{\frac{e^x+e^{-x}-2}{4}}$$

$$= \sqrt{\frac{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2}{4}} = \pm\left(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{2}\right) = \pm\sinh\left(\frac{x}{2}\right)$$

$\Rightarrow \sqrt{\dfrac{\cosh x - 1}{2}} = \sinh\left(\frac{x}{2}\right)$ for $x \geq 0$

& $\sqrt{\dfrac{\cosh x - 1}{2}} = -\sinh\left(\frac{x}{2}\right)$ for $x < 0$

$\Rightarrow \sinh\left(\frac{x}{2}\right) = \sqrt{\dfrac{\cosh x-1}{2}}$ for $x \geq 0$

$\Rightarrow \sinh\left(\frac{x}{2}\right) = -\sqrt{\dfrac{\cosh x-1}{2}}$ for $x < 0$

proved

2nd Method

$$\cosh x = 1 + 2\sinh^2\frac{x}{2}$$

$$\cosh x - 1 = 2\sinh^2\frac{x}{2}$$

$$\frac{\cosh x - 1}{2} = \sinh^2\frac{x}{2}$$

$$\pm\sqrt{\frac{\cosh x - 1}{2}} = \sinh\frac{x}{2}$$

1·3−22

Q9. (i) If $\tan(\theta+\phi i) = \tan\alpha + i\sec\alpha$,

Prove that $e^{2\phi} = \pm\cot\frac{\alpha}{2}$

Sol: $\tan(\theta+\phi i) = \tan\alpha + i\sec\alpha$

$\theta+\phi i = \tan^{-1}(\tan\alpha + i\sec\alpha)$ — (i)

$\theta-\phi i = \tan^{-1}(\tan\alpha - i\sec\alpha)$ — (ii)

$\tan^{-1}(\tan\alpha+i\sec\alpha) - \tan^{-1}(\tan\alpha - i\sec\alpha)$

$2\phi i = \tan^{-1}\left[\dfrac{(\tan\alpha+i\sec\alpha)-(\tan\alpha-i\sec\alpha)}{1+(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}\right]$

$\tan 2i\phi = \dfrac{2i\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}$

$i\tanh 2\phi = \dfrac{2i\sec\alpha}{\sec^2\alpha+\sec^2\alpha}$

$\tanh 2\phi = \dfrac{\cancel{2}\sec\alpha}{\cancel{2}\sec^2\alpha}$

$= \dfrac{1}{\sec\alpha}$

$\tanh 2\phi = \cos\alpha$

$\dfrac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \dfrac{\cos\alpha}{1}$

Componendo Dividendo $\left(\because \frac{a}{b} = \frac{a+b}{a-b}\right)$

$\dfrac{(e^{2\phi}-e^{-2\phi})+(e^{2\phi}+e^{-2\phi})}{(e^{2\phi}-e^{-2\phi})-(e^{2\phi}+e^{-2\phi})} = \dfrac{\cos\alpha+1}{\cos\alpha-1}$

$\dfrac{\cancel{2}e^{2\phi}}{-\cancel{2}e^{-2\phi}} = \dfrac{\cos\alpha+1}{-(1-\cos\alpha)}$

$e^{4\phi} = \dfrac{2\cos^2\alpha/2}{2\sin^2\frac{\alpha}{2}}$

$e^{4\phi} = \cot^2\frac{\alpha}{2}$

$\boxed{e^{2\phi} = \pm\cot\frac{\alpha}{2}}$

(ii) If $\tan(\theta+\phi i) = \tan\alpha + i\sec\alpha$

Prove that $2\theta = n\pi + \frac{\pi}{2} + \alpha$

Sol: Add (i) & (ii)

$2\theta = \tan^{-1}(\tan\alpha+i\sec\alpha)+\tan^{-1}(\tan\alpha-i\sec\alpha)$

$= \tan^{-1}\left[\dfrac{(\tan\alpha+i\sec\alpha)+(\tan\alpha-i\sec\alpha)}{1-(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}\right]$

$\tan 2\theta = \dfrac{2\tan\alpha}{1-(\tan^2\alpha+\sec^2\alpha)}$

$= \dfrac{2\tan\alpha}{1-\tan^2\alpha-\sec^2\alpha}$

$= \dfrac{2\tan\alpha}{1-\tan^2\alpha-(1+\tan^2\alpha)}$

$= \dfrac{2\tan\alpha}{-2\tan^2\alpha}$

$= -\dfrac{1}{\tan\alpha}$

$\tan 2\theta = -\cot\alpha$

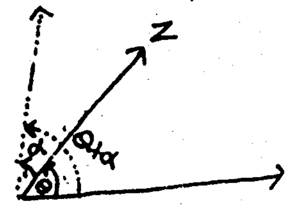
$= \tan\left(\frac{\pi}{2}+\alpha\right)$

$\tan 2\theta = \tan\left(n\pi+\frac{\pi}{2}+\alpha\right)$

$2\theta = n\pi+\frac{\pi}{2}+\alpha$

proved.

Q11 Show that multiplication of a vector z by $e^{i\alpha}$, where α is a real number, rotates the vector z counterclockwise through an angle 'α'.



Sol $z = r(\cos\theta + i\sin\theta)$

$= r\,e^{i\theta}$

$z\,e^{i\alpha} = r\,e^{i\theta}\,e^{i\alpha}$

$= r\,e^{i(\theta+\alpha)}$

$z\,e^{i\alpha} = r\left(\cos(\theta+\alpha) + i\sin(\theta+\alpha)\right)$

Thus vector z is rotated through an angle α counterclockwise.

Q12 Show that $2+i = \sqrt{5}\,e^{i\tan^{-1}\left(\frac{1}{2}\right)}$

$z = 2+i \Rightarrow r = |z| = \sqrt{2^2+1^2} = \sqrt{5}$

$\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{5}}$

$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{5}}$

$\tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\frac{1}{2}$

In polar form $z = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$

$2+i = \sqrt{5}\,e^{i\tan^{-1}\frac{1}{2}}$

(ii) $z = -3-4i \Rightarrow r = |z| = \sqrt{9+16} = \sqrt{25} = 5$

$\cos\theta = \frac{x}{r} = \frac{-3}{5}$

$\sin\theta = \frac{y}{r} = \frac{-4}{5}$

$\tan\theta = \frac{-4}{-3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\theta = \pi + \tan^{-1}\left(\frac{4}{3}\right)$ $\because$ 3rd Quad

In polar form $z = r\,e^{i\theta}$

$= 5\,e^{i\left(\pi + \tan^{-1}\frac{4}{3}\right)}$ Ans.

Logarithmic Function:-

Let $z \in$ where $z \neq 0$
& $w \in$ such that $e^{w} = z \Rightarrow w = \ln z$, '$w$' is Natural logarithm of z.

A particular value of 'w' satisfying the eq $e^{w} = z$ is given by $\ln|z| + i\arg$

$e^{\ln|z| + i\arg z} = e^{\ln|z|}\, e^{i\arg z} = |z|\, e^{i\theta} = r(\cos\theta + i\sin\theta) = z$

This particular value of ω is called Principal logarithm of z denoted by $\operatorname{Log} z$.

$\boxed{\operatorname{Log} z = \ln|z| + i\arg z} = \boxed{\ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} \quad \text{if } z = x+iy}$

The general value of w satisfying the eq $e^{w} = z$ is given by $\ln|z| + i\arg z + 2n\pi i$

$$e^{\ln|z| + i\arg z + 2n\pi i} = e^{\ln|z|}\, e^{i\arg z}\, e^{2n\pi i} = |z|\, e^{i\theta}\, e^{2\pi n i}$$
$$= r e^{i\theta}(\cos 2\pi n + i\sin 2\pi n)$$
$$= r e^{i\theta}(1+0)$$
$$= r(\cos\theta + i\sin\theta)$$
$$= z$$

Ex 1.4

$\operatorname{Log} z = \ln|z| + i\arg z$

when $z = x+iy$, $\operatorname{Log}(x+iy) = \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$

Q1 (i) Prove that $\operatorname{Log} i = \frac{\pi i}{2}$

$$\operatorname{Log} i = \operatorname{Log}(0+1i)$$
$$= \ln\sqrt{0^2+1^2} + i\tan^{-1}\left(\tfrac{1}{0}\right)$$
$$= \ln 1 + i\frac{\pi}{2}$$
$$= 0 + \frac{i\pi}{2}$$
$$= \frac{i\pi}{2} \quad \text{Ans}$$

x+ive, y+ive
So Ist Quad
$\therefore\ \theta$
$\theta = \tan^{-1}(\infty) = \frac{\pi}{2}$

(ii) $\operatorname{Log}(-5) = \ln 5 + i\pi$ To Prove

$$\operatorname{Log}(-5) = \ln\sqrt{(-5)^2+0^2} + i\tan^{-1}\left(\tfrac{0}{-5}\right)$$
$$= \ln\sqrt{25} + i\tan^{-1}(0)$$
$$= \ln 5 + i\pi$$

x-ive y+ive
So IInd Q
$\therefore\ \pi-\theta$
$\theta = \tan^{-1}(0) = 0$
Principal $\arg z = \pi-\theta = \pi - 0$
$= \pi$

1.4-2

Q1 (iii) $\text{Log}(-1+i) = \frac{1}{2}\ln 2 + \frac{3}{4}\pi i$ To Prove

$$\text{Log}(-1+i) = \ln\sqrt{1+1} + i\tan^{-1}\left(\frac{1}{-1}\right)$$
$$= \ln\sqrt{2} + i\tan^{-1}(-1)$$
$$= \ln 2^{\frac{1}{2}} + i\tan^{-1}(-1)$$
$$= \frac{1}{2}\ln 2 + i\,\frac{3\pi}{4}$$

x -ive y +ive
So IInd Quad.
$\therefore \pi - \theta$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$
Principal arg z $= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

---

iv) $\text{Log}(1+i) = \frac{1}{2}\ln 2 + \frac{\pi}{4} i$ To Prove

$$\text{Log}(1+i) = \ln\sqrt{1^2+1^2} + i\tan^{-1}\left(\frac{1}{1}\right)$$
$$= \ln\sqrt{2} + i\tan^{-1} 1$$
$$= \frac{1}{2}\ln 2 + i\,\frac{\pi}{4}$$

x +ive, y +ive
Ist Quad $\therefore \theta$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

---

v) $\text{Log}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -\frac{2}{3}\pi i$ To Prove

$$\text{Log}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \ln\sqrt{\frac{1}{4}+\frac{3}{4}} + i\tan^{-1}\left(\frac{-\sqrt{3}/2}{-1/2}\right)$$
$$= \ln\sqrt{\frac{4}{4}} + i\tan^{-1}(\sqrt{3})$$
$$= \ln 1 + i\left(-\frac{2\pi}{3}\right)$$
$$= 0 - \frac{2}{3}\pi i$$

x -ive, y -ive
3rd Quad.
$\therefore \theta - \pi$
$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$
$\therefore \theta - \pi = \frac{\pi}{3} - \pi$
$= -\frac{2\pi}{3}$

---

vi) $\text{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4} i$ To Prove

$$\text{Log}(1-i) = \ln\sqrt{1^2+(-1)^2} + i\tan^{-1}\left(\frac{-1}{1}\right)$$
$$= \ln\sqrt{2} + i\tan^{-1}(-1)$$
$$= \frac{1}{2}\ln 2 + i\left(-\frac{\pi}{4}\right)$$
$$= \frac{1}{2}\ln 2 - \frac{\pi}{4} i$$

x +ive, y -ive
So 4th Quad
$\therefore -\theta$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$
Principal arg z $= -\theta = -\frac{\pi}{4}$

1·4−3

Q2 i) To Prove $\coth^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$

Let $\coth^{-1} z = \omega \Rightarrow z = \coth\omega$

$$\frac{z+1}{z-1} = \frac{\coth\omega + 1}{\coth\omega - 1}$$

$$= \frac{\frac{e^{\omega}+e^{-\omega}}{e^{\omega}-e^{-\omega}} + 1}{\frac{e^{\omega}+e^{-\omega}}{e^{\omega}-e^{-\omega}} - 1}$$

$$= \frac{e^{\omega} + \cancel{e^{-\omega}} + e^{\omega} - \cancel{e^{-\omega}}}{e^{\omega}+e^{-\omega}-e^{\omega}+e^{-\omega}} \quad \text{LCM}$$

$$= \frac{\cancel{2}e^{\omega}}{\cancel{2}e^{-\omega}}$$

$$= e^{\omega}e^{\omega}$$

$$= e^{2\omega}$$

$$\log\left(\frac{z+1}{z-1}\right) = 2\omega$$

$$\frac{1}{2}\log\left(\frac{z+1}{z-1}\right) = \omega = \coth^{-1} z$$

To Prove $\operatorname{Sech}^{-1} z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Let $\operatorname{Sech}^{-1} z = \omega \Rightarrow z = \operatorname{Sech}\omega$

So, $\frac{1+\sqrt{1-z^2}}{z} = \frac{1+\sqrt{1-\operatorname{Sech}^2\omega}}{\operatorname{Sech}\omega}$

$(\because \operatorname{Sech}^2 z + \operatorname{Tanh}^2 z = 1)$

$$= \frac{1+\tanh\omega}{\operatorname{Sech}\omega}$$

$$= \frac{1}{\operatorname{Sech}\omega} + \left(\frac{\sinh\omega}{\cancel{\cosh\omega}}\right)\cdot\cancel{\cosh\omega}$$

$$= \cosh\omega + \sinh\omega$$

$$= \frac{e^{\omega}+e^{-\omega}}{2} + \frac{e^{\omega}-e^{-\omega}}{2}$$

$$= \frac{\cancel{2}e^{\omega}}{\cancel{2}} \quad \text{LCM}$$

$$\log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = \omega$$

$$\boxed{\log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = \operatorname{Sech}^{-1} z} \text{ proved}$$

To Prove

⑥ $\operatorname{Cosech}^{-1} z = \log\left(\frac{1+\sqrt{1+z^2}}{z}\right)$

Let $\operatorname{Cosech}^{-1} z = \omega \Rightarrow z = \operatorname{Cosech}\omega$

So $\frac{1+\sqrt{1+z^2}}{z} = \frac{1+\sqrt{1+\operatorname{Cosech}^2\omega}}{\operatorname{Cosech}\omega}$

$\because \coth^2 z - \operatorname{Cosech}^2 z = 1$

$$= \frac{1+\sqrt{\coth^2\omega}}{\operatorname{Cosech}\omega}$$

$$= \frac{1+\coth\omega}{\operatorname{Cosech}\omega}$$

$$= \frac{1}{\operatorname{Cosech}\omega} + \frac{\coth\omega}{\operatorname{Cosech}\omega}$$

$$= \sinh\omega + \frac{\cosh\omega}{\cancel{\sinh\omega}}\cdot\frac{\cancel{\sinh\omega}}{1}$$

$$= \frac{e^{\omega}-e^{-\omega}}{2} + \frac{e^{\omega}+e^{-\omega}}{2}$$

$$= \frac{\cancel{2}e^{\omega}}{\cancel{2}}$$

$$\log\left(\frac{1+\sqrt{1+z^2}}{z}\right) = \omega$$

$$\boxed{\log\left(\frac{1+\sqrt{1+z^2}}{z}\right) = \operatorname{Cosech}^{-1} z} \text{ proved}$$

Hyperbolic Identities

$$\begin{cases} \cosh^2 z - \sinh^2 z = 1 \\ \operatorname{Sech}^2 z + \operatorname{Tanh}^2 z = 1 \\ \coth^2 z - \operatorname{Cosech}^2 z = 1 \end{cases}$$

Trig Identities

$$\begin{cases} \cos^2\theta + \sin^2\theta = 1 \\ \sec^2\theta - \tan^2\theta = 1 \\ \operatorname{Cosec}^2\theta - \cot^2\theta = 1 \end{cases}$$

Inverse Hyperbolic Functions :-

To Prove $\sinh^{-1} z = \log(z+\sqrt{1+z^2})$

Let $\sinh^{-1} z = \omega \Rightarrow z = \sinh\omega$

So $z + \sqrt{1+z^2} = \sinh\omega + \sqrt{1+\sinh^2\omega}$

$= \sinh\omega + \sqrt{\cosh^2\omega}$

$= \sinh\omega + \cosh\omega$

$= \frac{e^{\omega}-e^{-\omega}}{2} + \frac{e^{\omega}+e^{-\omega}}{2}$

$= \frac{\cancel{2}e^{\omega}}{\cancel{2}}$ LCM

$\log(z+\sqrt{z^2+1}) = \omega$

$\boxed{\log(z+\sqrt{z^2+1}) = \sinh^{-1} z}$ proved

To Prove $\cosh^{-1} z = \log(z+\sqrt{z^2-1})$

Let $\cosh^{-1} z = \omega \Rightarrow z = \cosh\omega$

So $z+\sqrt{z^2-1} = \cosh\omega + \sqrt{\cosh^2\omega - 1}$

$= \cosh\omega + \sqrt{\sinh^2\omega}$

$= \cosh\omega + \sinh\omega$

$= \frac{e^{\omega}+e^{-\omega}}{2} + \frac{e^{\omega}-e^{-\omega}}{2}$

$= \frac{\cancel{2}e^{\omega}}{\cancel{2}}$ LCM

$\log(z+\sqrt{z^2-1}) = \omega$

$\boxed{\log(z+\sqrt{z^2-1}) = \cosh^{-1} z}$

IHF

1) $\sinh^{-1} z = \log(z+\sqrt{1+z^2})$
2) $\cosh^{-1} z = \log(z+\sqrt{z^2-1})$
3) $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$
4) $\coth^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$
5) $\operatorname{sech}^{-1} z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$
6) $\operatorname{cosech}^{-1} z = \log\left(\frac{1+\sqrt{z^2+1}}{z}\right)$

3) **To Prove** $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

Let $\tanh^{-1} z = \omega \Rightarrow z = \tanh\omega$

So $\frac{1+z}{1-z} = \frac{1+\tanh\omega}{1-\tanh\omega}$

$= \frac{1+\left(\frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}}\right)}{1-\left(\frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}}\right)}$

$= \frac{e^{\omega}+\cancel{e^{-\omega}}+e^{\omega}-\cancel{e^{-\omega}}}{\cancel{e^{\omega}}+e^{-\omega}-\cancel{e^{\omega}}+e^{-\omega}}$ LCM

$= \frac{\cancel{2}e^{\omega}}{\cancel{2}e^{-\omega}}$

$= e^{\omega}\cdot e^{\omega}$

$\frac{1+z}{1-z} = e^{2\omega}$

$\log\left(\frac{1+z}{1-z}\right) = 2\omega$

$\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \omega = \tanh^{-1} z$

$\boxed{1.4-4}$

Inverse Trigonometric Functions:-

1) **To Prove** $\sin^{-1}z = \frac{1}{i}\log(iz+\sqrt{1-z^2})$

Let $\sin^{-1}z = \omega \Rightarrow z = \sin\omega$

So $iz+\sqrt{1-z^2} = i\sin\omega + \sqrt{1-\sin^2\omega}$

$= i\sin\omega + \sqrt{\cos^2\omega}$

$= i\sin\omega + \cos\omega$

$= e^{i\omega}$

$\log(iz+\sqrt{1-z^2}) = i\omega$

$\frac{1}{i}\log(iz+\sqrt{1-z^2}) = \omega$

$$\boxed{\frac{1}{i}\log(iz+\sqrt{1-z^2}) = \sin^{-1}z}$$

Q 3(i) 2) **To Prove** $\cos^{-1}z = \frac{1}{i}\log(z+\sqrt{z^2-1})$

Let $\cos^{-1}z = \omega \Rightarrow z = \cos\omega$

So $z+\sqrt{z^2-1} = \cos\omega + \sqrt{\cos^2\omega - 1}$

$= \cos\omega + \sqrt{-(1-\cos^2\omega)}$

$= \cos\omega + \sqrt{-\sin^2\omega}$

$= \cos\omega + i\sin\omega$

$z+\sqrt{z^2-1} = e^{i\omega}$

$\log(z+\sqrt{z^2-1}) = i\omega$

$\frac{1}{i}\log(z+\sqrt{z^2-1}) = \omega$

$$\boxed{\frac{1}{i}\log(z+\sqrt{z^2-1}) = \cos^{-1}z}$$

$\sin^{-1}z = \frac{1}{i}\log(iz+\sqrt{1-z^2})$

$\cos^{-1}z = \frac{1}{i}\log(z+\sqrt{z^2-1})$

$\tan^{-1}z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$

$\sec^{-1}z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

$\operatorname{cosec}^{-1}z = \frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$

$\cot^{-1}z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$

3) **To Prove** $\tan^{-1}z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$

Let $\tan^{-1}z = \omega \Rightarrow z = \tan\omega$

So $\frac{1+iz}{1-iz} = \frac{1+i\tan\omega}{1-i\tan\omega}$

$= \frac{1+i\frac{\sin\omega}{\cos\omega}}{1-i\frac{\sin\omega}{\cos\omega}}$

LCM $= \frac{\cos\omega + i\sin\omega}{\cos\omega - i\sin\omega}$

$= \frac{e^{i\omega}}{e^{-i\omega}}$

$= e^{i\omega}e^{i\omega}$

$\frac{1+iz}{1-iz} = e^{2i\omega}$

$\log\left(\frac{1+iz}{1-iz}\right) = 2i\omega$

$\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = \omega$

$$\boxed{\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = \tan^{-1}z}$$

Q3

(iii) To Prove $\cot^{-1} z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$

Let $\cot^{-1} z = \omega \Rightarrow z = \cot\omega$

So $\frac{z+i}{z-i} = \frac{\cot\omega + i}{\cot\omega - i}$

$= \frac{\frac{\cos\omega}{\sin\omega}+i}{\frac{\cos\omega}{\sin\omega}-i}$

$= \frac{\cos\omega + i\sin\omega}{\cos\omega - i\sin\omega}$

$= \frac{e^{i\omega}}{e^{-i\omega}} = e^{i\omega}e^{i\omega}$

$= e^{2i\omega}$

$\log\left(\frac{z+i}{z-i}\right) = 2i\omega$

$\frac{1}{2i}\log\left(\frac{z+i}{z-i}\right) = \omega$

$\boxed{\frac{1}{2i}\log\left(\frac{z+i}{z-i}\right) = \cot^{-1} z}$ Proved

(v) To Prove $\operatorname{cosec}^{-1} z = \frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$

Let $\operatorname{cosec}^{-1} z = \omega \Rightarrow z = \operatorname{cosec}\omega$

So $\frac{i+\sqrt{z^2-1}}{z} = \frac{i+\sqrt{\operatorname{cosec}^2\omega - 1}}{\operatorname{cosec}\omega}$

$= \frac{i+\sqrt{\cot^2\omega}}{\operatorname{cosec}\omega}$

$= \frac{i+\cot\omega}{\operatorname{cosec}\omega}$

$= \frac{i+\frac{\cos\omega}{\sin\omega}}{\frac{1}{\sin\omega}}$ LCM

$= \frac{i\sin\omega + \cos\omega}{1}$

$= e^{i\omega}$

$\log\frac{i+\sqrt{z^2-1}}{z} = i\omega$

$\frac{1}{i}\log\left[\frac{i+\sqrt{z^2-1}}{z}\right] = \omega$

$\boxed{\frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right) = \operatorname{cosec}^{-1} z}$ Proved

Q3 (iv) To Prove $\sec^{-1} z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Let $\sec^{-1} z = \omega \Rightarrow z = \sec\omega$

So $\frac{1+\sqrt{1-z^2}}{z} = \frac{1+\sqrt{1-\sec^2\omega}}{\sec\omega}$

$= \frac{1+\sqrt{-\tan^2\omega}}{\sec\omega}$ $\left(\because \sec^2\omega - \tan^2\omega = 1,\ -\tan^2\omega = 1-\sec^2\omega\right)$

$= \frac{1+i\tan\omega}{\sec\omega}$

$= \left(1+i\frac{\sin\omega}{\cos\omega}\right)\cdot\frac{\cos\omega}{1}$

$= \frac{\cos\omega + i\sin\omega}{\cancel{\cos\omega}}\cdot\frac{\cancel{\cos\omega}}{1}$

$\frac{1+\sqrt{1-z^2}}{z} = e^{i\omega}$

$\log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = i\omega \Rightarrow \omega = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)^2$

$\sec^{-1} z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)^2$ proved

Complex Power: If z and w are any two complex numbers s.t $z \neq 0$

then we define

$$\boxed{z^{w} = e^{w\,\mathrm{Log}\,z}}$$

Q1 Prove that $i^{i} = e^{-\pi/2}$

LHS $i^{i} = e^{i\,\mathrm{Log}\,i}$

$= e^{i\{\ln\sqrt{0+1^2} + i\tan^{-1}(\frac{1}{0})\}}$ ∵ Q1

$= e^{i(0+i\frac{\pi}{2})}$

$i^{i} = e^{-\frac{\pi}{2}}$ proved

ii) Prove that $(-1)^{i} = e^{-\pi}$

LHS $(-1)^{i} = e^{i\,\mathrm{Log}(-1)}$

$= e^{i\{\ln\sqrt{(-1)^2+0^2} + i\tan^{-1}(0)\}}$

$= e^{i\{0 + i\pi\}}$ ∵ QII

$= e^{-\pi}$

$(-1)^{i} = e^{-\pi}$ proved

iii) Prove that $(-i)^{-i} = e^{-\frac{\pi}{2}}$

LHS $(-i)^{-i} = e^{-i\,\mathrm{Log}(-i)}$

$= e^{-i\{\ln\sqrt{0^2+(-1)^2} + i\tan^{-1}(\frac{-1}{0})\}}$

$= e^{-i\{0 + i\tan^{-1}(-\infty)\}}$ ∵ QII

$= e^{-i\{0 + i(-\frac{\pi}{2})\}}$

$= e^{-\frac{\pi}{2}}$ proved

iv) Prove that $a^{i} = \cos(\ln a) + i\sin(\ln a)$

LHS $a^{i} = e^{i\,\mathrm{Log}\,a}$

$= e^{i\{\ln\sqrt{a^2+0^2} + i\tan^{-1}(\frac{0}{a})\}}$

$= e^{i\{\ln a + i(0)\}}$ ∵ IVth Q

$= e^{i\ln a}$

$a^{i} = \cos\ln a + i\sin\ln a$ proved

Q5 Prove that $\tanh^{-1} z = \sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right)$

Sol Let $\tanh^{-1} z = w \Rightarrow z = \tanh w$

So $\frac{z}{\sqrt{1-z^2}} = \frac{\tanh w}{\sqrt{1-\tanh^2 w}}$

$= \frac{\tanh w}{\sqrt{\operatorname{sech}^2 w}}$

$= \frac{\tanh w}{\operatorname{sech} w}$

$= \frac{\sinh w}{\cosh w} \cdot \frac{\cosh w}{1}$

$= \sinh w$

$\sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right) = w$

$\sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right) = \tanh^{-1} z$

Q6 Show that if $z = x+iy$ then $\mathrm{Log}\left(\frac{z}{\bar z}\right) = 2i\tan^{-1}\left(\frac{y}{x}\right)$

Sol $z = x+iy$

$\bar z = x-iy$

$\mathrm{Log}\left(\frac{z}{\bar z}\right) = \mathrm{Log}\,z - \mathrm{Log}\,\bar z$

$= \mathrm{Log}(x+iy) - \mathrm{Log}(x-iy)$

$= \left[\ln\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)\right] - \left[\ln\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{-y}{x}\right)\right]$

$= \ln\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2} - \left(-i\tan^{-1}\frac{y}{x}\right)$

$\mathrm{Log}\frac{z}{\bar z} = 2i\tan^{-1}\frac{y}{x}$ proved

Q. If $a^{\alpha+i\beta} = (x+iy)^{p+iq}$ then Prove that

i) $\alpha = \frac{1}{2}\log_a(x^2+y^2) - \tan^{-1}\left(\frac{y}{x}\right)\log_a e$

ii) $\log_a(x^2+y^2) = 2(\alpha \quad \beta q)$
p^2+q^2

Sol

(given) $a^{\alpha+i\beta} = (x+iy)^{p+iq}$

$$e^{(\alpha+i\beta)\operatorname{Log} a} = e^{(p+iq)\operatorname{Log}(x+iy)}$$

$$(\alpha+i\beta)\operatorname{Log} a = (p+iq)\operatorname{Log}(x+iy)$$

$$(\alpha+i\beta)\left[\ln\sqrt{a^2} + i\tan^{-1}\frac{0}{a}\right] = (p+iq)\left[\ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}\right]$$

$$(\alpha+i\beta)\left[\ln a + i\cdot 0\right] = (p+iq)\left[\frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\frac{y}{x}\right]$$

$$\alpha\ln a + i\beta\ln a = \left[\frac{p}{2}\ln(x^2+y^2) - q\tan^{-1}\frac{y}{x}\right] + i\left[\frac{q}{2}\ln(x^2+y^2) + p\tan^{-1}\frac{y}{x}\right] \quad -(1)$$

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Equating Real & Imaginary Parts

$$\alpha\ln a = \frac{p}{2}\ln(x^2+y^2) - q\tan^{-1}\frac{y}{x}$$

$$\beta\ln a = \frac{q}{2}\ln(x^2+y^2) + p\tan^{-1}\frac{y}{x}$$

$$\beta = \frac{q\ln(x^2+y^2)}{} + \frac{p\tan^{-1}\frac{y}{x}}{}$$

Q8 If $\operatorname{Log}\sin(x+iy) = u+iv$ show that

i) $\cosh 2y = \cos 2x + 2e^{2u}$

ii) $e^{2y} = \dfrac{\cos(x-v)}{\cos(x+v)}$

Sol $\operatorname{Log}\sin(x+iy) = u+iv$ given

$$\sin(x+iy) = e^{u+iv}$$

$$\sin x\cos iy + \cos x\sin iy = e^{u}.e^{iv}$$

$$\sin x\cosh y + i\cos x\sinh y = e^{u}(\cos v + i\sin v)$$

Equating Real & Imaginary parts.

$$\sin x\cosh y = e^{u}\cos v \quad \text{——(I)}$$

$$\cos x\sinh y = e^{u}\sin v \quad \text{——(II)}$$

Squaring & adding (I) & (II) (To change into double angle as required)

$\because 2\cosh^2\theta - 1 = \cosh 2\theta$
$1 + 2\sinh^2\theta = \cosh 2\theta$

$$\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = e^{2u}(\cos^2 v + \sin^2 v)$$

$$\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cosh 2y}{2}\right) + \left(\frac{1+\cos 2x}{2}\right)\left(\frac{\cosh 2y - 1}{2}\right) = e^{2u}$$

$$\frac{1-\cos 2x + \cosh 2y - \cos 2x\cosh 2y}{4} + \frac{\cosh 2y + \cos 2x\cosh 2y - 1 - \cos 2x}{4} = e^{2u}$$

$$\frac{\cancel{1}-\cancel{1} - 2\cos 2x + 2\cosh 2y - \cancel{\cos 2x\cosh 2y} + \cancel{\cos 2x\cosh 2y}}{4} = e^{2u}$$

$$2(\cosh 2y - \cos 2x) = 4e^{2u}$$

$$\cosh 2y - \cos 2x = 2e^{2u}$$

$$\boxed{\cosh 2y = \cos 2x + 2e^{2u}} \text{ proved}$$

Dividing eq (I) by (II)

$$\frac{\sin x\cosh y}{\cos x\sinh y} = \frac{\cos v}{\sin v}$$

$$\frac{\cosh y}{\sinh y} = \frac{\cos v\cos x}{\sin v\sin x}$$

$$\frac{e^{y}+e^{-y}}{e^{y}-e^{-y}} = \frac{\cos v\cos x}{\sin v\sin x}$$

Comp & Div. $\left(\frac{a}{b} = \frac{a+b}{a-b}\right)$

Now apply Componendo & Dividendo

$$\frac{e^{y}+\cancel{e^{-y}}+e^{y}-\cancel{e^{-y}}}{\cancel{e^{y}}+e^{-y}-\cancel{e^{y}}+e^{-y}} = \frac{\cos v\cos x + \sin v\sin x}{\cos v\cos x - \sin v\sin x}$$

$$\frac{\cancel{2}e^{y}}{\cancel{2}e^{-y}} = \frac{\cos(x-v)}{\cos(x+v)}$$

$$e^{y}e^{y} = \frac{\cos(x-v)}{\cos(x+v)}$$

$$e^{2y} = \frac{\cos(x-v)}{\cos(x+v)} \text{ proved}$$

(9) Show that $\text{Log}(1+\cos\theta+i\sin\theta) = \ln(2\cos\frac{\theta}{2})+i\frac{\theta}{2}$

LHS $\text{Log}(1+\cos\theta+i\sin\theta)$

$$= \ln\sqrt{(1+\cos\theta)^2+\sin^2\theta} + i\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \ln\sqrt{1+\cos^2\theta+2\cos\theta+\sin^2\theta} + i\tan^{-1}\left(\frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^2\frac{\theta}{2}}\right)$$

$$= \ln\sqrt{1+1+2\cos\theta} + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \ln\sqrt{2(1+\cos\theta)} + i\frac{\theta}{2}$$

$$= \ln\sqrt{2\left(2\cos^2\frac{\theta}{2}\right)} + i\frac{\theta}{2}$$

$$= \ln 2\cos\frac{\theta}{2} + i\frac{\theta}{2} \quad \text{proved}$$

Imp (10) Prove that $\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4}+\frac{i}{2}\ln\left(\frac{x+y}{x-y}\right)$

Let $\alpha+i\beta = \tan^{-1}\left(\frac{x+iy}{x-iy}\right)$ —(1)

$\alpha-i\beta = \tan^{-1}\left(\frac{x-iy}{x+iy}\right)$ —(2)

Add (1)+(2) $2\alpha = \tan^{-1}\left[\dfrac{\frac{x+iy}{x-iy}+\frac{x-iy}{x+iy}}{1-\left(\cancel{\frac{x+iy}{x-iy}}\right)\left(\cancel{\frac{x-iy}{x+iy}}\right)}\right]$

$$2\alpha = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\boxed{\alpha = \frac{\pi}{4}}$$

Subtracting (2) from (1)

$$2i\beta = \tan^{-1}\left[\dfrac{\frac{x+iy}{x-iy}-\frac{x-iy}{x+iy}}{1+\cancel{\frac{x+iy}{x-iy}}\cancel{\frac{x-iy}{x+iy}}}\right]$$

$$\tan 2i\beta = \frac{(x+iy)^2-(x-iy)^2}{2(x^2+y^2)}$$

$$i\tanh 2\beta = \frac{\cancel{x^2}\cancel{-y^2}+2ixy-\cancel{x^2}\cancel{+y^2}+2ixy}{2(x^2+y^2)}$$

$$i\tanh 2\beta = \frac{2ixy}{x^2+y^2}$$

$$\tanh 2\beta = \frac{2xy}{x^2+y^2}$$

$$\frac{e^{2\beta}-e^{-2\beta}}{e^{2\beta}+e^{-2\beta}} = \frac{2xy}{x^2+y^2}$$

By Comp & Dividendo

$$\Rightarrow \frac{e^{2\beta}-\cancel{e^{-2\beta}}+e^{2\beta}+\cancel{e^{-2\beta}}}{\cancel{e^{2\beta}}-e^{-2\beta}-\cancel{e^{2\beta}}-e^{-2\beta}} = \frac{2xy+x^2+y^2}{2xy-x^2-y^2}$$

$$\frac{\cancel{2}e^{2\beta}}{\cancel{-2}e^{-2\beta}} = \frac{(x+y)^2}{-(x-y)^2}$$

$$e^{2\beta}e^{2\beta} = \left(\frac{x+y}{x-y}\right)^2$$

$$e^{4\beta} = \left(\frac{x+y}{x-y}\right)^2$$

Take Square Root $\quad e^{2\beta} = \frac{x+y}{x-y}$

Take log $\quad 2\beta = \ln\left(\frac{x+y}{x-y}\right)$

$$\beta = \frac{1}{2}\ln\left(\frac{x+y}{x-y}\right)$$

Hence

$$\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4}+\frac{i}{2}\ln\left(\frac{x+y}{x-y}\right)$$

proved.

$\boxed{1\cdot 4 - 10}$

See Pag # 85 Ex# 1.4(i) TP

Q11 To Prove $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} + i\ln\left(\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right)$

Sol Let $\alpha + i\beta = \cos^{-1}(\cos\theta + i\sin\theta)$ —— ①

(Now we find values of α & β and put in ① to get required result.)

$\therefore \cos(\alpha + i\beta) = \cos\theta + i\sin\theta$

$\cos\alpha\cos i\beta - \sin\alpha\sin i\beta = \cos\theta + i\sin\theta$

$\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta = \cos\theta + i\sin\theta$

Equating Real & Imaginary parts

$\cos\alpha\cosh\beta = \cos\theta$ & $-\sin\alpha\sinh\beta = \sin\theta$

$\cosh\beta = \dfrac{\cos\theta}{\cos\alpha}$ —— ⑪ & $\sinh\beta = \dfrac{\sin\theta}{-\sin\alpha}$ —— ⑪

Since

$\cosh^2\beta - \sinh^2\beta = 1$ (To eliminate β)

$\left(\dfrac{\cos\theta}{\cos\alpha}\right)^2 - \left(\dfrac{-\sin\theta}{\sin\alpha}\right)^2 = 1$

$\dfrac{\cos^2\theta\sin^2\alpha - \sin^2\theta\cos^2\alpha}{\cos^2\alpha\sin^2\alpha} = 1$

$\cos^2\theta\sin^2\alpha - \sin^2\theta\cos^2\alpha = \cos^2\alpha\sin^2\alpha$

$(1-\sin^2\theta)\sin^2\alpha - \sin^2\theta(1-\sin^2\alpha) = (1-\sin^2\alpha)\sin^2\alpha$

$\sin^2\alpha - \cancel{\sin^2\theta\sin^2\alpha} - \sin^2\theta + \cancel{\sin^2\theta\sin^2\alpha} = \sin^2\alpha - \sin^4\alpha$

$\cancel{-}\sin^2\theta = \cancel{\sin^2\alpha} - \cancel{\sin^2\alpha} \cancel{-} \sin^4\alpha$

$\sin\theta = \sin^2\alpha$

$\sqrt{\sin\theta} = \sin\alpha$

$\boxed{\sin^{-1}\sqrt{\sin\theta} = \alpha}$

Now Since $\cosh^2\beta - \sinh^2\beta = 1$

$\cosh^2\beta = 1 + \sinh^2\beta$

$\cosh\beta = \sqrt{1+\sinh^2\beta}$ —— ⓘⓥ

from ⑪ $\sinh\beta = \dfrac{\sin\theta}{-\sin\alpha} = \dfrac{\sin\theta}{-\sqrt{\sin\theta}} = -\sqrt{\sin\theta}$

∴ ⓘⓥ becomes $\cosh\beta = \sqrt{1+\left(-\sqrt{\sin\theta}\right)^2} = \sqrt{1+\sin\theta}$

$\cosh\beta + \sinh\beta = \sqrt{1+\sin\theta} + \left(-\sqrt{\sin\theta}\right)$

$\dfrac{e^{\beta}+e^{-\beta}}{2} + \dfrac{e^{\beta}-e^{-\beta}}{2} = \sqrt{1+\sin\theta} - \sqrt{\sin\theta}$

$\dfrac{2e^{\beta}}{2} = \sqrt{1+\sin\theta} - \sqrt{\sin\theta}$

$e^{\beta} = \sqrt{1+\sin\theta} - \sqrt{\sin\theta}$

$\beta = \ln\left[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right]$

Put values of α & β in ①

$\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} + i\ln\left[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right]$

proved

1·4–11

Q(ii) Prove that $\tan^{-1}(\cos\theta + i\sin\theta) = \pm\frac{\pi}{4} + \frac{i}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

Sol Let $\alpha + i\beta = \tan^{-1}(\cos\theta + i\sin\theta)$ —(i)

$\alpha - i\beta = \tan^{-1}(\cos\theta - i\sin\theta)$ —(ii)

Adding (i)+(ii)

$\tan^{-1}(\cos\theta + i\sin\theta) + \tan^{-1}(\cos\theta - i\sin\theta) = 2\alpha$

$\Rightarrow \tan^{-1}\left[\frac{\cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta}}{1-(\cos\theta+i\sin\theta)(\cos\theta - i\sin\theta)}\right] = 2\alpha$

$\Rightarrow \tan^{-1}\left[\frac{2\cos\theta}{1-(\cos^2\theta + \sin^2\theta)}\right] = 2\alpha$

$\Rightarrow \tan^{-1}\left[\frac{2\cos\theta}{1-1}\right] = 2\alpha$

$\Rightarrow \tan^{-1}(\pm\infty) = 2\alpha$ as $\cos\theta > 0$ or $\cos\theta < 0$

$\Rightarrow \pm\frac{\pi}{2} = 2\alpha$

$\Rightarrow \boxed{\pm\frac{\pi}{4} = \alpha}$

Again subtracting

$\tan^{-1}(\cos\theta + i\sin\theta) - \tan^{-1}(\cos\theta - i\sin\theta) = 2i\beta$

$\Rightarrow \tan^{-1}\left[\frac{\cancel{\cos\theta} + i\sin\theta - \cancel{\cos\theta} + i\sin\theta}{1+(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}\right] = 2i\beta$

$\Rightarrow \tan^{-1}\left[\frac{2i\sin\theta}{1+\cos^2\theta + \sin^2\theta}\right] = 2i\beta$

$\Rightarrow \tan^{-1}\left(\frac{\cancel{2}i\sin\theta}{\cancel{1+1}}\right) = 2i\beta$

$\Rightarrow i\sin\theta = \tan 2i\beta$

$\Rightarrow \cancel{i}\sin\theta = \cancel{i}\tanh 2\beta$

$\sin\theta = \frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta}}$

Comp & Dividendo

$\frac{\sin\theta + 1}{\sin\theta - 1} = \frac{e^{2\beta} - \cancel{e^{-2\beta}} + e^{2\beta} + \cancel{e^{-2\beta}}}{\cancel{e^{2\beta}} - e^{-2\beta} - \cancel{e^{2\beta}} - e^{-2\beta}}$

$\frac{\sin\theta + 1}{\sin\theta - 1} = \frac{\cancel{2}e^{2\beta}}{-\cancel{2}e^{-2\beta}} = -e^{4\beta}$

$e^{4\beta} = \frac{1+\sin\theta}{1-\sin\theta}$

$4\beta = \ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

$\beta = \frac{1}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

Put values of α & β in (i) we get

$\tan^{-1}(\cos\theta + i\sin\theta) = \pm\frac{\pi}{4} + \frac{i}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

proved.

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1.5-1

Summation of Series

Some Important Formulae.

1) $a + ar + ar^2 + \cdots\cdots ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$ G. Series

2) $a + ar + ar^2 + \cdots\cdots \infty = \dfrac{a}{1-r}$ $|r|<1$ & $n\to\infty$ Infinite G. Series

3) $1 - \frac{1}{2}x + \frac{1.3}{2.4}x^2 - \frac{1.3.5}{2.4.6}x^3 \cdots\cdots \infty = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$ (B. Series, $n=-\frac{1}{2}$)

4) $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots\cdots \infty = \dfrac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$ (B. series, $n=-\frac{1}{2}$)

5) $x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots\cdots \infty = \sin x$

6) $1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \frac{x^8}{\lfloor 8} \cdots\cdots \infty = \cos x$

7) $1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots \infty = e^x$

8) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots\cdots \infty = \ln(1+x)$

9) $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \cdots\cdots \infty = \ln(1-x)$

10) $1 + \frac{x}{2} - \frac{1}{2.4}x^2 + \frac{1.3}{2.4.6}x^3 - \frac{1.3.5}{2.4.6.8}x^4 \cdots\cdots \infty = \sqrt{1+x}$ (B Series, $n=+\frac{1}{2}$)

11) $1 - \frac{x}{2} - \frac{x^2}{2.4} - \frac{1.3}{2.4.6}x^3 - \frac{1.3.5}{2.4.6.8}x^4 \cdots\cdots \infty = \sqrt{1-x}$ (B. Series, $n=+\frac{1}{2}$)

To Prove $1 - e^{iA} = e^{\frac{iA}{2}}\left(-2i\sin\frac{A}{2}\right)$

LHS $1 - e^{iA} = e^{\frac{iA}{2}}\left(e^{\frac{-iA}{2}} - e^{\frac{iA}{2}}\right) = (2i)e^{\frac{iA}{2}}\left(\dfrac{e^{\frac{-iA}{2}} - e^{\frac{iA}{2}}}{2i}\right)$

$\dfrac{e^{\frac{iA}{2}} - e^{\frac{-iA}{2}}}{2i} = \sin\frac{A}{2}$

$\times e$, $\div$ by $2i$; Take $-$ common $= -2i\, e^{\frac{iA}{2}}\left(\dfrac{e^{\frac{iA}{2}} - e^{\frac{-iA}{2}}}{2i}\right)$

$= -2i\, e^{\frac{iA}{2}} \sin\frac{A}{2}$.

To Prove $e^{\theta} = \sinh\theta + \cosh\theta$

RHS $= \dfrac{e^{\theta} - e^{-\theta}}{2} + \dfrac{e^{\theta} + e^{-\theta}}{2} = \dfrac{e^{\theta} - \cancel{e^{-\theta}} + e^{\theta} + \cancel{e^{-\theta}}}{2} = \dfrac{2e^{\theta}}{2} = e^{\theta} = $ LHS.

Ex 1.5

In each of problem 1–5, evaluate the indicated sum.

1. (i) $\sin A + \sin 2A + \sin 3A + \cdots \sin nA$.

(ii) $\cos A + \cos 2A + \cos 3A + \cdots \cos nA$.

Sol: Let $S = \sin A + \sin 2A + \sin 3A + \cdots + \sin nA$

& $C = \cos A + \cos 2A + \cos 3A + \cdots + \cos nA$

$\Rightarrow C + iS = (\cos A + i\sin A) + (\cos 2A + i\sin 2A) + \cdots + (\cos nA + i\sin nA)$

$= e^{iA} + e^{2iA} + e^{3iA} + \cdots e^{niA}$

Geometric Series with $a = e^{iA}$, $n = n$, $r = e^{iA}$

$S_n = a\dfrac{(r^n - 1)}{r - 1}$

$\Rightarrow C + iS = e^{iA}\dfrac{(e^{niA} - 1)}{e^{iA} - 1}$

Taking $e^{\frac{niA}{2}}$ common from N

Taking $e^{\frac{iA}{2}}$ common from D

$= e^{iA}\left\{\dfrac{e^{\frac{niA}{2}}\left(e^{\frac{niA}{2}} - e^{-\frac{niA}{2}}\right)}{e^{\frac{iA}{2}}\left(e^{\frac{iA}{2}} - e^{-\frac{iA}{2}}\right)}\right\}$

$= e^{iA}\cdot e^{\frac{niA}{2}}\cdot e^{-\frac{iA}{2}}\dfrac{\left(e^{\frac{niA}{2}} - e^{-\frac{niA}{2}}\right)}{\left(e^{\frac{iA}{2}} - e^{-\frac{iA}{2}}\right)}$

$= e^{\frac{2iA + niA - iA}{2}}\dfrac{\dfrac{\left(e^{\frac{niA}{2}} - e^{-\frac{niA}{2}}\right)}{2i}}{\dfrac{\left(e^{\frac{iA}{2}} - e^{-\frac{iA}{2}}\right)}{2i}}$ ÷ N & D by $2i$

$= e^{iA\left(\frac{1+n}{2}\right)}\cdot\dfrac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$

$= \left\{\cos\left(\frac{1+n}{2}\right)A + i\sin\left(\frac{1+n}{2}\right)A\right]\dfrac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$

Comparing Real & Imaginary Parts

$C = \cos\left(\frac{1+n}{2}\right)A\cdot\dfrac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$ & $S = \sin\left(\frac{1+n}{2}\right)A\cdot\dfrac{\sin\frac{nA}{2}}{\frac{A}{2}}$

$\cos A + \cos 2A + \cos 3A \cdots \cos nA = \cos\left(\frac{1+n}{2}\right)A\cdot\dfrac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$ & $\sin A + \sin 2A + \cdots \sin nA = \sin\left(\frac{n+1}{2}\right)A\cdot\dfrac{\sin\frac{nA}{2}}{\sin\frac{A}{2}}$

1.5-3

② Let $C = \cos\theta + \cos 3\theta + \cos 5\theta + \dots \cos(2n-1)\theta$

Sol & $S = \sin\theta + \sin 3\theta + \sin 5\theta + \dots \sin(2n-1)\theta$

$\Rightarrow C + iS = (\cos\theta + i\sin\theta) + (\cos 3\theta + i\sin 3\theta) + \dots\dots (\cos(2n-1)\theta + i\sin(2n-1)\theta)$

$= e^{i\theta} + e^{3i\theta} + e^{5i\theta} + \dots e^{(2n-1)i\theta}$

Geometric Series

$a = e^{i\theta},\ n = n,\ r = e^{2i\theta}$

$S_n = \dfrac{a(r^n - 1)}{r - 1}$

$\Rightarrow C + iS = \dfrac{e^{i\theta}(e^{2ni\theta} - 1)}{(e^{2i\theta} - 1)}$

$= \cancel{e^{i\theta}} \left[\dfrac{e^{ni\theta}(e^{ni\theta} - e^{-ni\theta})}{\cancel{e^{i\theta}}(e^{i\theta} - e^{-i\theta})}\right]$

$= \dfrac{e^{ni\theta}\left(\dfrac{e^{ni\theta} - e^{-ni\theta}}{2i}\right)}{\left(\dfrac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$ ÷ N & D by $2i$

$\Rightarrow C + iS = (\cos n\theta + i\sin n\theta)\left(\dfrac{\sin n\theta}{\sin\theta}\right)$

Comparing Real part only

$C = \cos n\theta \cdot \dfrac{\sin n\theta}{\sin\theta}$

$\cos\theta + \cos 3\theta + \cos 5\theta + \dots \cos(2n-1)\theta = \cos n\theta \cdot \dfrac{\sin n\theta}{\sin\theta}$

$= \dfrac{2}{2}\cos n\theta \dfrac{\sin n\theta}{\sin\theta}$

$= \dfrac{\sin 2n\theta}{2\sin\theta}$

(3) $1+x\cos\theta+x^2\cos 2\theta+\dots x^n\cos n\theta$

Sol Let $C=1+x\cos\theta+x^2\cos 2\theta+\dots x^n\cos n\theta$

& $S=\quad x\sin\theta+x^2\sin 2\theta+\dots x^n\sin n\theta$

$C+iS=1+x(\cos\theta+i\sin\theta)+x^2(\cos 2\theta+i\sin 2\theta)+\dots x^n(\cos n\theta+i\sin n\theta)$

$=1+xe^{i\theta}+x^2e^{i2\theta}+x^3e^{i3\theta}+\dots+x^ne^{in\theta}$

G. Series $a=1,\ r=xe^{i\theta}$, $n=n+1$, $S_n=\dfrac{a(r^n-1)}{r-1}$

$$C+iS=1\left[\frac{(xe^{i\theta})^{n+1}-1}{xe^{i\theta}-1}\right]=\frac{x^{n+1}e^{(n+1)i\theta}-1}{xe^{i\theta}-1}$$

$$=\frac{x^{n+1}\{\cos(n+1)\theta+i\sin(n+1)\theta\}-1}{x(\cos\theta+i\sin\theta)-1}$$

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$$=\frac{(x^{n+1}\cos(n+1)\theta-1)+i(x^{n+1}\sin(n+1)\theta)}{(x\cos\theta-1)+i(x\sin\theta)}$$

$$=\frac{(x^{n+1}\cos(n+1)\theta-1)+i(x^{n+1}\sin(n+1)\theta)}{(x\cos\theta-1)+i(x\sin\theta)}\cdot\frac{(x\cos\theta-1)-i(x\sin\theta)}{(x\cos\theta-1)-i(x\sin\theta)}$$

$$C+iS=\frac{\left[(x^{n+1}\cos(n+1)\theta-1)(x\cos\theta-1)+(x^{n+1}\sin(n+1)\theta)(x\sin\theta)\right]+i\left[x^{n+1}\sin(n+1)\theta(x\cos\theta-1)-(x^{n+1}\cos(n+1)\theta-1)(x\sin\theta)\right]}{(x\cos\theta-1)^2+(x\sin\theta)^2}$$

comparing Real part only

$$C=\frac{x^{n+2}\cos(n+1)\theta\cos\theta-x^{n+1}\cos(n+1)\theta-x\cos\theta+1+x^{n+2}\sin(n+1)\theta\sin\theta}{x^2\cos^2\theta+1-2x\cos\theta+x^2\sin^2\theta}$$

$$=\frac{x^{n+2}\left[\cos(n+1)\theta\cos\theta+\sin(n+1)\theta\sin\theta\right]-x^{n+1}\cos(n+1)\theta-x\cos\theta+1}{x^2(\cos^2\theta+\sin^2\theta)-2x\cos\theta+1}$$

$$C=\frac{x^{n+2}\cos\left[(n+1)\theta-\theta\right]-x^{n+1}\cos(n+1)\theta-x\cos\theta+1}{x^2-2x\cos\theta+1}$$

$$C=\frac{x^{n+2}\cos n\theta-x^{n+1}\cos(n+1)\theta-x\cos\theta+1}{x^2-2x\cos\theta+1}$$ Ans

(4) $S = 3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \cdots (2n+1)\sin n\alpha$

Sol Let $S = 3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \cdots (2n+1)\sin n\alpha$

& $C = 3\cos\alpha + 5\cos 2\alpha + 7\sin 3\alpha + \cdots (2n+1)\cos n\alpha$

$C+iS = 3(\cos\alpha + i\sin\alpha) + 5(\cos 2\alpha + i\sin 2\alpha) + \cdots (2n+1)(\cos n\alpha + i\sin n\alpha)$

$C+iS = 3e^{i\alpha} + 5e^{i2\alpha} + 7e^{i3\alpha} + \cdots (2n+1)e^{in\alpha}$ —— ① (Not G. Series)

Multiply eq ① by $e^{i\alpha}$ & subtract

$(C+iS)e^{i\alpha} = \quad + 3e^{i2\alpha} + 5e^{i3\alpha} + 7e^{i4\alpha} \cdots (2n-1)e^{in\alpha} + (2n+1)e^{i(n+1)\alpha}$ —— ②

$(C+iS)(1-e^{i\alpha}) = 3e^{i\alpha} + 2e^{i2\alpha} + 2e^{i3\alpha} + \cdots + 2e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$

$= (e^{i\alpha} + 2e^{i\alpha}) + 2e^{i2\alpha} + 2e^{i3\alpha} + \cdots 2e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$

$= e^{i\alpha} + 2(e^{i\alpha} + e^{i2\alpha} + e^{i3\alpha} + \cdots e^{in\alpha}) - (2n+1)e^{i(n+1)\alpha}$

$= e^{i\alpha} + 2\left[\dfrac{e^{i\alpha}(e^{ni\alpha} - 1)}{(e^{i\alpha} - 1)}\right] - (2n+1)e^{i(n+1)\alpha}$ G. Series $a = e^{i\alpha}$, $r = e^{i\alpha}$, $n = n$

LCM $= \dfrac{e^{i\alpha}(e^{i\alpha}-1) + 2e^{i\alpha}(e^{ni\alpha} - 1) - (2n+1)e^{i(n+1)\alpha}\cdot(e^{i\alpha}-1)}{(e^{i\alpha}-1)}$

$(C+iS)(1-e^{i\alpha})(e^{i\alpha}-1) = e^{i\alpha}\left[e^{i\alpha} - 1 + 2e^{ni\alpha} - 2 - (2n+1)e^{i(n+1)\alpha} + (2n+1)e^{ni\alpha}\right]$ ✓

$-(C+iS)(e^{i\alpha}-1)^2 = e^{i\alpha}\left[e^{i\alpha} - 3 + (2+2n+1)e^{ni\alpha} - (2n+1)e^{i(n+1)\alpha}\right]$

$-(C+iS)\left[-4e^{i\alpha}\sin^2\frac{\alpha}{2}\right] = e^{i\alpha}\left[e^{i\alpha} - 3 + (2n+3)e^{ni\alpha} - (2n+1)e^{i(n+1)\alpha}\right]$

$(C+iS) = \dfrac{\cancel{e^{i\alpha}}\left(e^{i\alpha} - 3 + (2n+3)e^{ni\alpha} - (2n+1)e^{i(n+1)\alpha}\right)}{4\cancel{e^{i\alpha}}\sin^2\frac{\alpha}{2}}$

$= \dfrac{\cos\alpha + i\sin\alpha - 3 + (2n+3)(\cos n\alpha + i\sin n\alpha) - (2n+1)(\cos(n+1)\alpha + i\sin(n+1)\alpha)}{4\left(\frac{1-\cos\alpha}{2}\right)}$

$(e^{i\alpha} - 1)^2$

$= \left[e^{\frac{i\alpha}{2}}\left(e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}}\right)\right]^2$

$= \left[e^{\frac{i\alpha}{2}}\left(\dfrac{e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}}}{2i}\right)2i\right]^2$

$= e^{\left(\frac{i\alpha}{2}\right)^2}\sin^2\frac{\alpha}{2}\,(-4)$

$= -4e^{i\alpha}\sin^2\frac{\alpha}{2}$

Comparing Imaginary Part

$S = \dfrac{\sin\alpha + (2n+3)\sin n\alpha - (2n+1)\sin(n+1)\alpha}{2(1-\cos\alpha)}$

(5) $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + \cdots \cos^2 n\theta$

$$= \frac{1+\cos 2\theta}{2} + \frac{1+\cos 4\theta}{2} + \frac{1+\cos 6\theta}{2} + \cdots \frac{1+\cos 2n\theta}{2}$$

$$= \left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots n\text{ terms}\right) + \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{2} + \frac{\cos 6\theta}{2} \cdots \frac{\cos 2n\theta}{2}$$

$$S_n = \frac{n}{2} + \frac{1}{2}(\cos 2\theta + \cos 4\theta + \cos 6\theta + \cdots \cos 2n\theta) \quad \text{---(1)}$$

Let $C = \cos 2\theta + \cos 4\theta + \cos 6\theta + \cdots \cos 2n\theta$

$S = \sin 2\theta + \sin 4\theta + \sin 6\theta + \cdots \sin 2n\theta$

$$C+iS = e^{2i\theta} + e^{4i\theta} + e^{6i\theta} + \cdots e^{2ni\theta}$$

Geometric Series $a = e^{2i\theta}$, $r = e^{2i\theta}$, $n = n$

$S_n = \frac{a(r^n - 1)}{r-1}$

$$C+iS = \frac{e^{2i\theta}\left(e^{2i\theta n} - 1\right)}{e^{2i\theta} - 1}$$

$$= \frac{e^{2i\theta} \cdot e^{ni\theta}\left(e^{ni\theta} - e^{-ni\theta}\right)}{e^{i\theta}\left(e^{i\theta} - e^{-i\theta}\right)}$$

$$= e^{i\theta} \cdot e^{ni\theta}\left(\frac{e^{ni\theta} - e^{-ni\theta}}{e^{i\theta} - e^{-i\theta}}\right)$$

$$= e^{(n+1)i\theta}\frac{\left(\frac{e^{ni\theta} - e^{-ni\theta}}{2i}\right)}{\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$$

$$= e^{(n+1)i\theta} \cdot \frac{\sin n\theta}{\sin\theta}$$

$$C+iS = (\cos(n+1)\theta + i\sin(n+1)\theta)\frac{\sin n\theta}{\sin\theta}$$

Comparing Real Part

$$C = \cos(n+1)\theta \, \frac{\sin n\theta}{\sin\theta}$$

Hence from (1) $S_n = \frac{n}{2} + \frac{1}{2}\frac{\sin n\theta}{\sin\theta}\cos(n+1)\theta$

⑥ Find sum of Infinite series.

Let $S = \sin\theta + \frac{1}{2}\sin 3\theta + \frac{1\cdot 3}{2\cdot 4}\sin 5\theta + \cdots\cdots \infty$

$C = \cos\theta + \frac{1}{2}\cos 3\theta + \frac{1\cdot 3}{2\cdot 4}\cos 5\theta + \cdots\cdots \infty$

$$C+iS = e^{i\theta} + \frac{1}{2}e^{3i\theta} + \frac{1\cdot 3}{2\cdot 4}e^{5i\theta} + \cdots\cdots \infty$$

$$= e^{i\theta}\left(1 + \frac{1}{2}e^{2i\theta} + \frac{1\cdot 3}{2\cdot 4}e^{4i\theta} + \cdots\cdots \infty\right)$$

$$= e^{i\theta}\left(1 - e^{2i\theta}\right)^{-\frac{1}{2}} \qquad \because (1-x)^{-\frac{1}{2}} = \frac{1}{\sqrt{(1-x)}} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$$

$$= e^{i\theta}\left(e^{2i\theta}\right)^{-\frac{1}{2}}\left(e^{-2i\theta} - 1\right)^{-\frac{1}{2}}$$

$$= e^{i\theta}e^{-i\theta}\left(\cos(-2\theta) + i\sin(-2\theta) - 1\right)^{-\frac{1}{2}}$$

$$= e^{0}\left(\cos 2\theta - i\sin 2\theta - 1\right)^{-\frac{1}{2}}$$

$$= \left[(\cos 2\theta - 1) - i\sin 2\theta\right]^{-\frac{1}{2}} \qquad \cos 2\theta = 1 - 2\sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \left(-2\sin^2\theta - i2\sin\theta\cos\theta\right)^{-\frac{1}{2}}$$

$$= \left\{2\sin\theta(-\sin\theta - i\cos\theta)\right\}^{-\frac{1}{2}}$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left(-\sin\theta - i\cos\theta\right)^{-\frac{1}{2}}$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{2}+\theta\right) - i\sin\left(\frac{\pi}{2}+\theta\right)\right]^{-\frac{1}{2}} \qquad \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta, \quad \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(-\frac{\pi}{4}-\frac{\theta}{2}\right) - i\sin\left(-\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$C+iS = (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

Comparing Imaginary parts

$$S = (2\sin\theta)^{-\frac{1}{2}}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$S = \frac{\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{\sqrt{2\sin\theta}} \quad \text{Ans.}$$

(7) $\sinh\theta + \frac{\sinh 2\theta}{\lfloor 2} + \frac{\sinh 3\theta}{\lfloor 3} + \cdots\infty$

Sol $=\left(\frac{e^{\theta}-e^{-\theta}}{2}\right) + \left(\frac{e^{2\theta}-e^{-2\theta}}{2\lfloor 2}\right) + \left(\frac{e^{3\theta}-e^{-3\theta}}{2\lfloor 3}\right) + \cdots\infty$

$=\frac{1}{2}\left[e^{\theta}-e^{-\theta} + \frac{e^{2\theta}-e^{-2\theta}}{\lfloor 2} + \frac{e^{3\theta}-e^{-3\theta}}{\lfloor 3} + \cdots\infty\right]$

$=\frac{1}{2}\left[e^{\theta} + \frac{e^{2\theta}}{\lfloor 2} + \frac{e^{3\theta}}{\lfloor 3} + \cdots\infty\right] - \frac{1}{2}\left[e^{-\theta} + \frac{e^{-2\theta}}{\lfloor 2} + \frac{e^{-3\theta}}{\lfloor 3} + \cdots\infty\right]$

Add & Subtract 1

$\Rightarrow \frac{1}{2}\left[1+e^{\theta} + \frac{e^{2\theta}}{\lfloor 2} + \frac{e^{3\theta}}{\lfloor 3} + \cdots\infty\right] - \frac{1}{2}\left[1+e^{-\theta} + \frac{e^{-2\theta}}{\lfloor 2} + \frac{e^{-3\theta}}{\lfloor 3} + \cdots\infty\right]$

$\because e^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\infty$

$=\frac{1}{2}\left(e^{e^{\theta}}\right) - \frac{1}{2}\left(e^{e^{-\theta}}\right)$

$=\frac{1}{2}\left[e^{e^{\theta}} - e^{e^{-\theta}}\right]$

* $=\frac{1}{2}\left[e^{\cosh\theta+\sinh\theta} - e^{\cosh\theta-\sinh\theta}\right]$

$=\frac{e^{\cosh\theta}}{2}\left(e^{\sinh\theta} - e^{-\sinh\theta}\right)$

$=e^{\cosh\theta}\left(\frac{e^{\sinh\theta} - e^{-\sinh\theta}}{2}\right)$

$=e^{\cosh\theta}\sinh(\sinh\theta)$ Ans

* $\cosh\theta + \sinh\theta = e^{\theta}$

LHS $= \frac{e^{\theta}+e^{-\theta}}{2} + \frac{e^{\theta}-e^{-\theta}}{2} = \frac{e^{\theta}+\cancel{e^{-\theta}}+e^{\theta}-\cancel{e^{-\theta}}}{2}$

$=\frac{2e^{\theta}}{2} = e^{\theta}$ RHS

$=$

Q13 $C = 1 + c\cos\theta + \frac{c^2}{\lfloor 2}\cos 2\theta + \frac{c^3}{\lfloor 3}\cos 3\theta + \cdots$

$S = c\sin\theta + \frac{c^2}{\lfloor 2}\sin 2\theta + \frac{c^3}{\lfloor 3}\sin 3\theta + \cdots$

$C+iS = 1 + c(\cos\theta + i\sin\theta) + \frac{c^2}{\lfloor 2}(\cos 2\theta + i\sin 2\theta) + \cdots$

$= 1 + ce^{i\theta} + \frac{c^2}{\lfloor 2}e^{i2\theta} + \frac{c^3}{\lfloor 3}e^{3i\theta} + \cdots$

$= 1 + ce^{i\theta} + \frac{(ce^{i\theta})^2}{\lfloor 2} + \frac{(ce^{i\theta})^3}{\lfloor 3} + \cdots$

$= e^{ce^{i\theta}}$ $\because e^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\infty$

$= e^{c(\cos\theta + i\sin\theta)}$

$= e^{c\cos\theta}\cdot e^{ic\sin\theta}$

$C+iS = e^{c\cos\theta}\left(\cos(c\sin\theta) + i\sin(c\sin\theta)\right)$

Comparing Real Part

$C = e^{c\cos\theta}\cos(c\sin\theta)$ Ans.

Q8 $S = \sin\alpha\cdot\sin\alpha + \sin^2\alpha\cdot\sin 2\alpha + \sin^3\alpha\cdot\sin 3\alpha + \cdots\infty$

$C = \sin\alpha\cdot\cos\alpha + \sin^2\alpha\cdot\cos 2\alpha + \sin^3\alpha\cdot\cos 3\alpha + \cdots\infty$

$C+iS = \sin\alpha(\cos\alpha + i\sin\alpha) + \sin^2\alpha(\cos 2\alpha + i\sin 2\alpha) + \cdots\infty$

$= \sin\alpha\, e^{i\alpha} + \sin^2\alpha\, e^{2i\alpha} + \sin^3\alpha\, e^{3i\alpha} + \cdots\infty$ Infinite Geometric Series

$a = \sin\alpha\, e^{i\alpha}$

$r = \sin\alpha\, e^{i\alpha}$

$S_\infty = \dfrac{a}{1-r}$

$$C+iS = \frac{a}{1-r} = \frac{\sin\alpha\, e^{i\alpha}}{1-\sin\alpha\, e^{i\alpha}}$$

$$= \frac{\sin\alpha\, \cancel{e^{i\alpha}}}{\cancel{e^{i\alpha}}(e^{-i\alpha} - \sin\alpha)}$$

$$= \frac{\sin\alpha}{(\cos\alpha - i\sin\alpha) - \sin\alpha}$$

$$= \frac{\sin\alpha}{(\cos\alpha - \sin\alpha) - i\sin\alpha}$$

$$= \frac{\sin\alpha}{(\cos\alpha - \sin\alpha) - i\sin\alpha}\cdot\frac{[(\cos\alpha - \sin\alpha) + i\sin\alpha]}{[(\cos\alpha - \sin\alpha) + i\sin\alpha]}$$

$$= \frac{\sin\alpha\,[(\cos\alpha - \sin\alpha) + i\sin\alpha]}{(\cos\alpha - \sin\alpha)^2 + \sin^2\alpha}$$

$$= \frac{\sin\alpha(\cos\alpha - \sin\alpha) + i\sin^2\alpha}{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha + \sin^2\alpha}$$

$$C+iS = \frac{\sin\alpha(\cos\alpha - \sin\alpha) + i\sin^2\alpha}{1 - \sin 2\alpha + \sin^2\alpha}$$

Comparing imaginary parts

$$S = \frac{\sin^2\alpha}{1-\sin\ [illegible]}$$ Ans.

1.5-10

Q(9) $C = 1 - \frac{1}{2}\cos\theta + \frac{1\cdot 3}{2\cdot 4}\cos 2\theta - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 3\theta + \cdots\cdots \infty$

$S = -\frac{1}{2}\sin\theta + \frac{1\cdot 3}{2\cdot 4}\sin 2\theta - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 3\theta + \cdots\cdots \infty$

$C+iS = 1 - \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1\cdot 3}{2\cdot 4}(\cos 2\theta + i\sin 2\theta) - \cdots\cdots \infty$

$= 1 - \frac{1}{2}e^{i\theta} + \frac{1\cdot 3}{2\cdot 4}e^{2i\theta} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}e^{3i\theta} + \cdots\cdots \infty$

$= (1+e^{i\theta})^{-\frac{1}{2}}$ $\quad\because (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots\cdots \infty$ B. Series

$= (1+\cos\theta + i\sin\theta)^{-\frac{1}{2}}$

$= \left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$

$= \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^{-\frac{1}{2}}$

DeMoivres

$= \left(2\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right)$

$$C+iS = \frac{\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}}{\sqrt{2\cos\frac{\theta}{2}}}$$

Comparing Real Part

$$C = \frac{\cos\frac{\theta}{4}}{\sqrt{2\cos\frac{\theta}{2}}} \quad \text{Ans}$$

x———————x

Q(11) $C = 1 + \frac{1}{2}\cos\theta + \frac{1\cdot 3}{2\cdot 4}\cos 2\theta + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 3\theta + \cdots\cdots \infty$

$S = \frac{1}{2}\sin\theta + \frac{1\cdot 3}{2\cdot 4}\sin 2\theta + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 3\theta + \cdots\cdots \infty$

$C+iS \quad 1 + \frac{1}{2}e^{i\theta} + \frac{1\cdot 3}{2\cdot 4}e^{2i\theta} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}e^{3i\theta} + \cdots\cdots \infty$

$= (1-e^{i\theta})^{-\frac{1}{2}}$ $\quad\because (1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$ B. Series

$= (1-\cos\theta - i\sin\theta)^{-\frac{1}{2}}$

$= \left(2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$

$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$

$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{-\frac{1}{2}}$

$= \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\left(\frac{\pi-\theta}{4}\right) + i\sin\left(\frac{\pi-\theta}{4}\right)\right)$

DeMoivres Th.

Comparing Real Part

$C = \left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\left(\frac{\pi-\theta}{4}\right)\right)$

$$C = \frac{\cos\left(\frac{\pi-\theta}{4}\right)}{\sqrt{2\sin\frac{\theta}{2}}} \quad \text{Ans}$$

x———————x

(10) $S = n\sin\theta + \frac{n(n+1)}{\lfloor 2}\sin 2\theta + \frac{n(n+1)(n+2)}{\lfloor 3}\sin 3\theta + \cdots\cdots\infty$

$C = 1 + n\cos\theta + \frac{n(n+1)}{\lfloor 2}\cos 2\theta + \frac{n(n+1)(n+2)}{\lfloor 3}\cos 3\theta + \cdots\cdots\infty$

$C+iS = 1 + n(\cos\theta + i\sin\theta) + \frac{n(n+1)}{\lfloor 2}(\cos 2\theta + i\sin 2\theta) + \frac{n(n+1)(n+2)}{\lfloor 3}(\cos 3\theta + i\sin 3\theta) + \cdots$

$= 1 + n e^{i\theta} + \frac{n(n+1)}{\lfloor 2} e^{2i\theta} + \frac{n(n+1)(n+2)}{\lfloor 3} e^{3i\theta} + \cdots\cdots\infty$

$= 1 + (-n)(-e^{i\theta}) + \frac{(-n)(-n-1)}{\lfloor 2}(-e^{i\theta})^2 + \frac{(-n)(-n-1)(-n-2)}{\lfloor 3}(-e^{i\theta})^3 + \cdots\cdots\infty$

$= (1 - e^{i\theta})^{-n}$ $\quad\because$ B. Series $(1-x)^{-n} = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \cdots\infty$

$= (1 - \cos\theta - i\sin\theta)^{-n}$

$= \left(2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-n}$ $\quad \cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]^{-n}$ $\quad \cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \sin\frac{\theta}{2}$, $\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \cos\frac{\theta}{2}$

$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{-n}$

$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\cos\left(-n\left(\frac{\pi-\theta}{2}\right)\right) - i\sin\left(-n\left(\frac{\pi-\theta}{2}\right)\right)\right]$ $\quad$ Demoivres Th.

$= \left(2\sin\frac{\theta}{2}\right)^{-n}\left[\cos\left(\frac{n\pi - n\theta}{2}\right) + i\sin\left(\frac{n\pi - n\theta}{2}\right)\right]$

Comparing Imaginary parts

$S = \left(2\sin\frac{\theta}{2}\right)^{-n}\sin\left(\frac{n\pi - n\theta}{2}\right)$

$S = \dfrac{\sin\frac{n}{2}(\pi-\theta)}{\left(2\sin\frac{\theta}{2}\right)^n}$ Ans.

1.5-12

(13)

$$C = \cos\alpha - \frac{\cos(\alpha+2\beta)}{\lfloor 3} + \frac{\cos(\alpha+4\beta)}{\lfloor 5} + \cdots\cdots\infty$$

$$S = \sin\alpha - \frac{\sin(\alpha+2\beta)}{\lfloor 3} + \frac{\sin(\alpha+4\beta)}{\lfloor 5} + \cdots\cdots\infty$$

$$C+iS = e^{i\alpha} - \frac{e^{i(\alpha+2\beta)}}{\lfloor 3} + \frac{e^{i(\alpha+4\beta)}}{\lfloor 5} + \cdots\cdots\infty$$

$$= e^{i\alpha}\left(1 - \frac{e^{i2\beta}}{\lfloor 3} + \frac{e^{i4\beta}}{\lfloor 5} - \cdots\cdots\infty\right)$$

$$= e^{i\alpha}\cdot\frac{e^{i\beta}}{e^{i\beta}}\left(1 - \frac{e^{i2\beta}}{\lfloor 3} + \frac{e^{i4\beta}}{\lfloor 5} - \cdots\cdots\infty\right)$$

$$= \frac{e^{i\alpha}}{e^{i\beta}}\left(e^{i\beta} - \frac{e^{i3\beta}}{\lfloor 3} + \frac{e^{i5\beta}}{\lfloor 5} - \cdots\cdots\infty\right)$$

$$= e^{i(\alpha-\beta)}\left(\sin e^{i\beta}\right) \qquad \because \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots\cdots\infty$$

$$= \left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right)\sin e^{i\beta}$$

$$= \left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right)\sin(\cos\beta + i\sin\beta)$$

$$= \left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right)\left[\sin(\cos\beta)\cos(i\sin\beta) + \cos(\cos\beta)\sin(i\sin\beta)\right]$$

$$= \left[\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right]\left[\sin(\cos\beta)\cosh(\sin\beta) + i\cos(\cos\beta)\sinh(\sin\beta)\right]$$

$$= \left[\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right]\left[\sin(\cos\beta)\cosh(\sin\beta) + i\cos(\cos\beta)\sinh(\sin\beta)\right]$$

$$C+iS = \left[\cos(\alpha-\beta)\sin(\cos\beta)\cosh(\sin\beta) - \sin(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)\right]$$
$$+ i\left[\sin(\alpha-\beta)\sin\cos\beta\cosh(\sin\beta) + \cos(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)\right]$$

Comparing Real Part

$$C = \cos(\alpha-\beta)\sin(\cos\beta)\cosh(\sin\beta) - \sin(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)$$

Ans.

1.5-13

Q14 $C = c\cos\theta + \frac{c^2}{2}\cos 2\theta + \frac{c^3}{3}\cos 3\theta + \cdots$

$S = c\sin\theta + \frac{c^2}{2}\sin 2\theta + \frac{c^3}{3}\sin 3\theta + \cdots$

$$C+iS = c(\cos\theta + i\sin\theta) + \frac{c^2}{2}(\cos 2\theta + i\sin 2\theta) + \frac{c^3}{3}(\cos 3\theta + i\sin 3\theta) + \cdots$$

$$= c e^{i\theta} + \frac{c^2}{2} e^{i2\theta} + \frac{c^3}{3} e^{i3\theta} + \cdots$$

$$= ce^{i\theta} + \frac{(ce^{i\theta})^2}{2} + \frac{(ce^{i\theta})^3}{3} + \cdots$$

$$= -\ln(1-ce^{i\theta}) \qquad \because \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$= -\log(1-ce^{i\theta}) \qquad \ln(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right.$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$= -\log(\overline{1-c\cos\theta} - ic\sin\theta)$$

$\because \log(x+iy) = \ln|x+iy| + i\operatorname{Arg} z$

$$= -\left[\ln\left|\overline{1-c\cos\theta} - ic\sin\theta\right| + i\arg(\overline{1-c\cos\theta} - ic\sin\theta)\right.$$

$$C+iS = -\left[\ln\sqrt{(1-c\cos\theta)^2 + c^2\sin^2\theta} + i\tan^{-1}\left(\frac{-c\sin\theta}{1-c\cos\theta}\right)\right]$$

Comparing Real Part

$$C = -\left[\ln\sqrt{(1-c\cos\theta)^2 + c^2\sin^2\theta}\right]$$

$$= -\frac{1}{2}\ln\{(1-c\cos\theta)^2 + c^2\sin^2\theta\}$$

$$= -\frac{1}{2}\ln(1 + c^2\cos^2\theta - 2c\cos\theta + c^2\sin^2\theta)$$

$$= -\frac{1}{2}\ln(1 + c^2(\cos^2\theta + \sin^2\theta) - 2c\cos\theta)$$

$$= -\frac{1}{2}\ln(1 + c^2 - 2c\cos\theta) \quad \text{Ans}$$

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(1.5-14)

(15) $S = \sin\theta - \frac{1}{2}\sin 3\theta + \frac{1}{3}\sin 5\theta + \cdots\cdots$

$C = \cos\theta - \frac{1}{2}\cos 3\theta + \frac{1}{3}\cos 5\theta + \cdots\cdots$

$C+iS = e^{i\theta} - \frac{1}{2}e^{i3\theta} + \frac{1}{3}e^{i5\theta}, \cdots\cdots$

$x+iy\,e^{i\theta}$. $= \frac{e^{i\theta}}{e^{i\theta}}\left(e^{i\theta} - \frac{1}{2}e^{i3\theta} + \frac{1}{3}e^{i5\theta} \cdots\cdots\right)$

$= \frac{1}{e^{i\theta}}\left(e^{2i\theta} - \frac{1}{2}e^{4i\theta} + \frac{1}{3}e^{6i\theta}\cdots\cdots\right)$

$= e^{-i\theta}\left[e^{2i\theta} - \frac{1}{2}(e^{2i\theta})^2 + \frac{1}{3}(e^{2i\theta})^3\cdots\cdots\right]$

$= e^{-i\theta}\left[\log(1+e^{2i\theta})\right]$ $\quad \because \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots\infty$

$= e^{-i\theta}\left[\log(1+\cos 2\theta + i\sin 2\theta)\right]$

$= e^{-i\theta}\left[\ln\sqrt{(1+\cos 2\theta)^2 + (\sin 2\theta)^2} + i\tan^{-1}\frac{\sin 2\theta}{1+\cos 2\theta}\right.$

$= e^{-i\theta}\left[\ln\sqrt{1+\cos^2 2\theta + 2\cos 2\theta + \sin^2 2\theta} + i\tan^{-1}\left(\frac{\cancel{2}\sin\theta\cancel{\cos\theta}}{\cancel{2}\cos^{\cancel{2}}\theta}\right)\right.$

$= e^{-i\theta}\left[\ln\sqrt{1+1+2\cos 2\theta} + i\tan^{-1}\tan\theta\right.$

$= e^{-i\theta}\left[\ln\sqrt{2(1+\cos 2\theta)} + i\theta\right]$

$= e^{-i\theta}\left[\ln\sqrt{2(2\cos^2\theta)} + i\theta\right]$

$= e^{-i\theta}\left[\ln(2\cos\theta) + i\theta\right]$

$= (\cos\theta - i\sin\theta)(\ln 2\cos\theta + i\theta)$

$= \cos\theta\ln 2\cos\theta + \theta\sin\theta + i(\theta\cos\theta - \sin\theta\ln(2\cos\theta))$

Comparing Imaginary parts

$S = \theta\cos\theta - \sin\theta\ln(2\cos\theta)$ Ans

GROUPS

B.S.c -Mathol- Chptor 2

Binary Operation:- (B.O)

B.O is a rule from $S\times S \to S$ i.e. which assigns to each element of $S\times S$ a unique element of S. where S is a non-empty set.

So a B.O on a nonempty set S can be defined as a fn which associates, with each $(a,b)\in S\times S$ a unique element of S. If this B.O, as a fn from $S\times S \to S$ is denoted by $*$ then the image of $(a,b)\in S\times S$ under $*$ is denoted by $a*b$. $\{a*b\in S\}$

If $*$ is a B.O in S then we say that S is closed under the B.O $*$.

i.e For $a,b\in S$, $a*$

$*$ may be '+', '.', '−' '÷'

Examples In the set Z of integers, ordinary addition denoted by '+', is a B.O in Z. ∵ the ordinary addition '+' associates with each ordered pair (a,b) of integers a unique integer which we denote by $a+b$

$2,3\in Z$. $2+3=5\in Z$. Hence Z is closed under '+'

Similarly ordinary multiplication '.' is a B.O in Z.

$2,3\in Z$ $2\cdot3=6\in Z$. Hence Z is closed under '.'

Similarly ordinary subtraction '−' is a B.O in Z

$2,3\in Z$ $2-3=-1\in Z$ Hence Z is closed under '−'

. ordinary division '$\div$' is not defined in Z

$\because$ as $2, 3 \in Z$ but $2 \div 3 = \frac{2}{3} \notin Z$

1) A B.O $\star$ may and may not be commutative.
If $a \star b = b \star a$ for all $a, b \in S$ then $\star$ is said to be a commutative binary operation.

2) A B.O $\star$ may and may not be Associative.
If $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in S$ then $\star$ is said to be an Associative B.O.

3) Existence of an Identity element.
An element 'e' of S is said to be an identity element in S w.r.t $\star$.

$\begin{cases} a\cdot1 = 1\cdot a = a \ \forall a\in R \\ a+0 = 0+a = a \ \forall a \in R \end{cases}$

if $a \star e = e \star a = a$ for all $a \in S$

4) Inverse of an Element.
Let $a \in S$. An element $a' \in S$ is said to be inverse element of a w.r.t $\star$ if

$$a \star a' = a' \star a = e$$

$\begin{cases} a\cdot\frac{1}{a} = 1 \\ a+(-a) = 0 \end{cases}$

Group:

A non-empty set G is said to be a group if

(i) there is a B.O $\star$ defined in G i.e for any two elements a, b of G, $a \star b \in G$.

(ii) the B.O in G is associative i.e

for $a, b, c \in G$ $(a \star b) \star c = a \star (b \star c)$

(iii) w.r.t the BO $\star$ there exists an identity element 'e' in G. i,e

$$a \star e = e \star a = a. \quad \forall a \in G$$

(iv) For each $a \in G$, there exists an inverse element $a' \in G$ such that

$$a \star a' = a' \star a = e$$ i,e each element of G has inverse element.

This Group is specified by the pair $(G, \star)$

Another Def. A pair $(G, \star)$ where G is non-empty set and $\star$ is a B.O in G is called a Group if the following conditions are satisfied in G.

(i) The B.O $\star$ is associative in G

(ii) Identity element exists in G w.r.t $\star$ $e \star a = a \star e = a$

(iii) Inverse element 'b' exist in G for every element 'a' of G $a \star b = b \star a = e$

Abelian Group:-

A Group $(G, \star)$ is said to be abelian if $a \star b = b \star a$ for all $a, b \in G$.

×————×

Note (1) A Groupoid $(S, \star)$ is an ordered pair consisting of a nonempty set S and a B.O $\star$ defined in S. (Closure property)

(2) A Groupoid $(S, \star)$ is called a Semi group if B.O $\star$ is Associative. If Identity element exists in a Semi group then it is called Monoid and for every element of Monoid, if there exists inverse element then Monoid is called a $\boxed{\text{Group}}$

Example 1 $G = [1, -1]$

B.O defined on G be the ordinary multiplication.

$\cdot$	1	-1
1	1	-1
-1	-1	1

Group table

(i) Associative $(1\cdot 1)\cdot 1 = 1(1\cdot 1)$ ✓

$(1\cdot -1)\cdot 1 = 1(-1\cdot 1)$ ✓

(ii) Identity $e*a = a*e = a$ (def)

$e = 1$ $\because 1\cdot(-1) = (-1)\cdot 1 = -1$

(iii) Inverse $ab = ba = e$ (def)

$(1)(1)^{-1} = 1(\frac{1}{1}) = 1 = e$

$(1)^{-1}(1) = (\frac{1}{1}) = 1 = e$

$(-1)(-1)^{-1} = (-1)(\frac{1}{-1}) = 1 = e$

$(-1)^{-1}(-1) = (\frac{1}{-1})(-1) = 1 = e$

All the three conditions are satisfied hence G is group

x———————x

Example 2 $(Z, +)$ is a group. '+' is ordinary addition

Z is set of integers i.e

$Z = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

(i) Let $4, 5, 6 \in Z$.

$(4+5)+6 = 4+(5+6)$

$15 = 15$ Hence Associative

(ii) Identity $e*a = a*e = a$ $e, a \in Z$.

$0+4 = 4+0 = 4$ Hence $e = 0 \in Z$.

(iii) Inverse $a*b = b*a = e$ $a, b \in Z$

$4+(-4) = (-4)+4 = 0$ $4, -4 \in Z$

Inverse of each element of z exists
Hence $(Z, +)$ is a group

Similarly $(Q, +), (R, +), (C, +)$ are groups.

In $(C, +)$ $0 + i0 = e$ $A = (a_1 + ib_1)$ $C = (a_3 + ib_3)$

$A, B, C \in C$. where $B = (a_2 + ib_2)$ $-A = -a_1 - ib$

∴ $(Q,+)$ is a group with '0' as identity element and '$-\frac{a}{b}$' $\in Q$ as additive inverse for each '$\frac{a}{b}$' $\in Q$. Associativity can be proved easily by taking any three elements, $\frac{a}{b}, \frac{c}{d}, \frac{p}{q}$

$(Q,\cdot)$ is not a group because '0' has no multiplicative inverse in Q. The identity of $(Q,\cdot)$ is '1'.

$(Q-\{0\}\cdot)$ is a group with '1' as identity element and $\frac{a}{b}$ as Multiplicative inverse for each $\frac{b}{a} \in Q$

$(R,+)$ is a group with 0 as identity element and '$-a$' $\in R$ as additive inverse for each 'a' $\in R$. Associativity can be proved easily.

$(R,\cdot)$ is not a group because '0' has no multiplicative inverse in R. The identity of $(R,\cdot)$ is '1'.

$(R-\{0\},\cdot)$ is a group with 1 as identity element and $\frac{1}{a} \in R-\{0\}$ as multiplicative inverse for each $a \in R-\{0\}$

x————————»

Neither $(N,+)$ nor $(N,\cdot)$ is a group ∵ N has no identity element w.r.t '+' and has no inverse element w.r.t '·'.

Neither $(W,+)$ nor $(W,\cdot)$ is a group ∵ For $a \in W$ there does not exist '$-a$' $\in W$ w.r.t '+' and For $a \in W$, there does not exist $\frac{1}{a} \in W$. (i.e Inverses does not exist)

$(C,+)$ and $(C-\{0\},\cdot)$ are groups $\quad a+ib \in C$.

For $(C,+)$ $\quad$ identity $= 0+i0$ $\quad$ Inverse $= -a-ib$ for $a+ib$

'+' is associative in C.

For $(C-\{0\},\cdot)$ $\quad$ Identity $= 1+0i$, For every $A = a+ib$ there exist $A^{-1} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$

'·' is associative in $C-\{0\}$

$$AA^{-1} = (a+ib)\frac{(a-ib)}{a^2+b^2} = 1 = 1+0i$$

Let M_2 be the set of all 2×2 matrices,

$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ $\quad$ a_1, b_1, a_2, b_2 are real numbers such that A is non singular i,e $|A| = a_1b_2 - b_1a_2 \neq 0$

$$B = \begin{bmatrix} a_1' & b_1' \\ a_2' & b_2' \end{bmatrix}$$

$$AB = \begin{bmatrix} a_1a_1' + b_1a_2' & a_1b_1' + b_1b_2' \\ a_2a_1' + b_2a_2' & a_2b_1' + b_2b_2' \end{bmatrix}$$

$|AB| \neq 0$ $\quad$ Hence $AB \in M_2$. So M_2 is closed under Matrix Multiplication

Also $(AB)C = A(BC)$ for all $A, B, C \in M_2$ can be verified easily.

Identity Matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ w.r.t '·' $\quad$ $A \cdot I = I \cdot A = A$

Moreover for each $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ in M_2 there exist A^{-1} s.t

$$A^{-1} = \begin{pmatrix} \frac{b_2}{a_1b_2 - b_1a_2} & \frac{-b_1}{a_1b_2 - b_1a_2} \\ \frac{-a_2}{a_1b_2 - b_1a_2} & \frac{a_1}{a_1b_2 - b_1a_2} \end{pmatrix}$$

Now $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So $(M_2, \cdot)$ is a group.

$G=\{1,\omega,\omega^2\}$ and '·' is defined by the table

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(i) Associative B.O

$$(1\cdot\omega)\cdot\omega^2 = 1\cdot(\omega\cdot\omega^2)$$
$$\omega\cdot\omega^2 = 1\cdot 1$$
$$1 = 1$$

(ii) Identity $\quad 1 = e \qquad e\cdot a = a\cdot e = a$

(iii) Inverse $\quad$ Inverse of ω is ω^2 $\qquad ab=ba=e \mid \because \omega^3=1,\ \omega\omega^2=1$

Inverse of ω^2 is ω $\qquad \omega\omega^2=\omega^2\omega=1$

Inverse of 1 is 1 $\qquad (1)(1)^{-1}=(1)^{-1}(1)=1$

Hence $(G,\cdot)$ is a group.

The set $\{1,-1,i,-i\}$ and '·' is defined by table.

$\cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	$+i$
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Associative B.O

$$1\cdot((-1)\cdot i) = (1\cdot(-1))\cdot i$$
$$1\cdot(-i) = (-1)\cdot i$$
$$-i = -i$$

Identity '1' is identity $\qquad a\cdot e = e\cdot a = a$

Inverse $\quad$ Inverse of 1 is 1

" " (-1) is (-1)

" " i is $-i$

" " $-i$ is i

Hence group

$$(i)^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$$

Order of a Group The number of elements in a group G is called order of that group. & denoted by $|G|$

Finite Group If number of elements in a group are finite then it is finite group.

Infinite Group If number of elements in a group are infinite then it is infinite.

x————x

$(\mathbb{Z},+), (\mathbb{Q},+), (\mathbb{R},+), (\mathbb{C},+), (\mathbb{Q}-0,\cdot), (\mathbb{R}-\{0\},\cdot)$ are all infinite groups.

The group of $\{1,\omega,\omega^2\}$ & $\{1,-1,i,-i\}$ are finite groups

x————x

Cancellation Law

If $a,b,c \in G$ then (i) $a\cdot b = a\cdot c \Rightarrow b=c$ left Cancell

(ii) $b\cdot a = c\cdot a \Rightarrow b=c$ Right Cancell

Proof Let $a\cdot b = a\cdot c$ given

$a^{-1}\cdot(a\cdot b) = a^{-1}\cdot(a\cdot c)$ pre multiplying by a^{-1}

$(a^{-1}a)\cdot b = (a^{-1}a)\cdot c$ $\because$ Associative Law.

$e\cdot b = e\cdot c$ $\because a^{-1}a=e$

$b = c$ $\because e$ is identity.

Hence $a\cdot b = a\cdot c \Rightarrow b=c$.

Similarly we can prove Right Cancellation Law.

<u>Example 8</u> $S=\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ be the set of residue classes modulo 5

To show $(S,+)$ is a group

We construct the group table

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$

For $\bar{a},\bar{b}\in S$

$\bar{a}+\bar{b}=r$

where r is the remainder after division of $a+b$ by 5

i) Associative $(a+b)+c = a+(b+c)$

$$(\bar{2}+\bar{3})+\bar{4} = \bar{2}+(\bar{3}+\bar{4})$$

$$\bar{0}+\bar{4} = \bar{2}+\bar{2}$$

$$\bar{4} = \bar{4}$$

ii) Identity $a*e=e*a=a$

Here $\bar{0}$ is identity

$\bar{0}+\bar{0}=0$ $\quad$ $\bar{0}+\bar{2}=\bar{2}$

$\bar{0}+\bar{1}=\bar{1}$ $\quad$ $\bar{0}+\bar{3}=\bar{3}$

$\bar{0}+\bar{4}=4.$

iii) Inverse. $a*b=b*a=e.$

$\bar{1}+\bar{4}=\bar{0}$ $\qquad$ $\bar{2}+\bar{3}=\bar{0}$ $\qquad$ $\bar{0}+\bar{0}=0.$

Hence Inverse element exists for each element of set S under addition modulo 5.

All conditions satisfied So $(S,+)$ is a group.

Example 9 Let $\bar{Z}'_5 = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ be the set of non-zero residue classes modulo 5 and multiplication defined is multiplication modulo 5. To Prove $(\bar{Z}'_5, \cdot)$ is a group

We construct group table

$\cdot$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

For $\bar{a}, \bar{b} \in \bar{Z}'$
$\bar{a} \cdot \bar{b} = r$
where r is the remainder obtained after dividing the usual product ab of a & b by 5

i) Associative

$$(\bar{2} \cdot \bar{3}) \cdot \bar{4} = \bar{2} \cdot (\bar{3} \cdot \bar{4})$$
$$\bar{1} \cdot \bar{4} = \bar{2} \cdot \bar{2}$$
$$\bar{4} = \bar{4}$$

ii) Identity $a \cdot e = e \cdot a = a$

Here identity w.r.t multiplication modulo 5 is $\bar{1}$

$\bar{1} \cdot \bar{1} = \bar{1}$, $\bar{1} \cdot \bar{2} = \bar{2}$, $\bar{1} \cdot \bar{3} = \bar{3}$ $\bar{1} \cdot \bar{4} = \bar{4}$

iii) Inverse $a \cdot b = b \cdot a = e$

$\bar{1} \cdot \bar{1} = \bar{1}$ $\bar{2} \cdot \bar{3} = \bar{1}$ $\bar{4} \cdot \bar{4} = \bar{1}$

Hence inverse for each element of set $\bar{Z}'_5$ exists

All conditions are satisfied

$\therefore (\bar{Z}'_5, \cdot)$ is a group.

<u>Theorem</u> (Solution of Linear Eq.)

For any two elements a, b in a group G, the eqs $ax=b$ & $ya=b$ have unique sol

<u>Proof</u> $\because (G, \cdot)$ is a group & $a, b \in G$, so inverse of each element of G exists in G. Let a^{-1} be inverse of a in G then

	$ax=b$	$ya=b$	
premultiply by a^{-1}	$a^{-1}ax=a^{-1}b$	$yaa^{-1}=ba^{-1}$	Post-multiply by a^{-1}
$a^{-1}a=e$	$(e)x=a^{-1}b$	$y(e)=ba^{-1}$	
	$x=a^{-1}b$	$y=ba^{-1}$	

So $x=a^{-1}b$ is a sol of $ax=b$ So $y=ba^{-1}$ is a sol of $ya=b$

Now we prove the uniqueness of the solution of $ax=b$

Let x', x'' be two solutions of $ax=b$.

Then $ax'=b$ & $ax''=b$

$ax'=ax''$ left cancellation Law

$x'=x''$

Similarly let y', y'' be the sol of $ya=b$

$\therefore$ $y'a=b$ & $y''a=b$

$y'a=y''a$ Right Cancellation Law

$y'=y''$

which proves the uniqueness of the theorem.

<u>Theorem</u> For a, b in a group G, $(ab)^{-1} = b^{-1}a^{-1}$

<u>Proof</u>

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} \quad \text{Associative Law}$$
$$= aea^{-1} \quad bb^{-1} = e$$
$$= aa^{-1} \quad e = \text{Identity}$$
$$(ab)(b^{-1}a^{-1}) = e$$

Pre-multiply by $(ab)^{-1}$

$$(ab)^{-1}(ab)(b^{-1}a^{-1}) = (ab)^{-1}.e$$
$$e\,(b^{-1}a^{-1}) = (ab)^{-1}$$
$$b^{-1}a^{-1} = (ab)^{-1} \quad \text{Proved.}$$

This result can be generalised for a product of finite no of elements of G.

i.e for $a_1, a_2, a_3, \ldots\ldots a_n \in G$.

$$(a_1 a_2 a_3 \ldots a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \ldots\ldots a_3^{-1} a_2^{-1} a_1^{-1}$$

<u>Th</u>

For an element a of a group G, the following exponentiation rules hold.

(i) $a^m = a.a.a\ldots\ldots a$ m factors

Put $m = 1$ $\quad a^1 = a$ 1 factor c-1 is satisfied

Put $m = K$ $\quad a^K = a.a.a\ldots$ K factors Supposed true

$$a^K(a) = (a.a \ldots a \text{ K factors}).a$$
$$a^{K+1} = a.a\ldots\ldots \quad K+1 \text{ factors}$$

K is replaced by K+1 Hence it is true

13 $\boxed{2\cdot 1-13}$

ii) $(a^{-1})^m = a^{-m}$ To Prove $\quad m \in Z$

For $m=1$ $\quad (a^{-1})^1 = a^{-1}$ $\quad$ C-1 is satisfied

For $m=K$ $\quad (a^{-1})^K = a^{-K}$ $\quad$ supposed true

Now consider $(a^{-1})^{K+1} = (a^{-1})^K.(a^{-1})^1 = (a^{-K})(a^{-1})$ $\quad$ using supposition

$(a^{-1})^{K+1} = (a^K)^{-1} a^{-1}$

$(a^{-1})^{K+1} = (a^{K+1})^{-1}$

$(a^{-1})^{K+1} = a^{-(K+1)}$ $\quad$ True for $m=K+1$

iii) $a^m . a^n = a^{m+n}$ $\quad m,n \in Z^+$, i.e $m>0$ $n>0$

Case 1. $m>0$ & $n>0$

Put $n=1$

$a^m . a^1 = a^{m+1}$ $\quad$ C-1 is satisfied

Put $n=K$ $\quad a^m . a^K = a^{m+K}$ $\quad$ Suppose true

Now Consider $a^m . a^{K+1} = a^m.(a^K a) = (a^m a^K).a$ $\quad$ Associative

$= (a^{m+K}).a$ $\quad$ using supposition

$a^m . a^{K+1} = a^{m+K+1}$ $\quad$ C-2 is satisfied

Hence $\boxed{a^m . a^n = a^{m+n}}$

Case 2 $\quad$ when $m<0$ & $n<0$

Let $m=-p$ $\quad n=-q$ $\quad p,q \in Z^+$

$\therefore$ $a^{m+n} = a^{-p-q} = a^{-(p+q)} = (a^{-1})^{p+q}$ $\quad$ from (ii) Th $\therefore (a^{-1})^m = a^{-m}$

$= (a^{-1})^p . (a^{-1})^q = a^{-p} . a^{-q}$

$\boxed{a^{m+n} = a^m . a^n}$ $\quad$ Proved

Case 3 Suppose $m=0$ & $n\neq 0$ To Prove $a^{m+n}=a^m a^n$

using $m=0$ $\therefore a^{m+n}=a^{0+n}=a^n=ea^n=a^0a^n$

$$a^{0+n}=a^0a^n$$

Similarly when $m\neq 0$ & $n=0$

then $a^{m+0}=a^m=a^me=a^ma^0$

$$a^{m+0}=a^ma^0$$

Case 4 Suppose $m>0$ & $n<0$ s.t $m+n\geq 0$

$a^{m+n}=a^{m+n}\cdot e$ $\quad\because e$ is identity

$=a^{m+n}(a^{-n}a^n)$

$=(a^{m+n}a^{-n})a^n$

$=(a^{m+n-n})a^n$ $\quad\left(\begin{matrix}\because n<0\\ \Rightarrow -n>0\end{matrix}\right)$

$a^{m+n}=a^ma^n$

Note $a^{-n}a^{+n}=\underbrace{a.a.a\ldots a}_{n\text{ times}}\underbrace{a^{-1}a^{-1}a^{-1}\ldots}_{n\text{ times}}$

$=a.a.a\ldots(aa^{-1})a^{-1}\ldots$

$=a.a\ldots a(e)a^{-1}a^{-1}\ldots$

$=a.a\ldots(aa^{-1})a^{-1}\ldots$

$=a.a\ldots a(e)a^{-1}$

$a^{-n}a^n=e.$

Similarly $a^{m+n}=a^m.a^n$ for $m<0$ & $n>0$ s.t $m+n\geq 0$

Case 5 Let $m>0$ $n<0$ s.t $m+n<0\Rightarrow -(m+n)>0$

Now $a^{m+n}=a^{m+n}\cdot e$

$=(a^{m+n}).(a^{-n}.a^n)$

$=a^{m+n}.(a^{-m-n}a^m).a^n$

$=(a^{m+n}a^{-m-n})a^ma^n$

$a^{m+n}=e(a^ma^n)=a^ma^n$

Note $a^{-(m+n)}.a^m=a^{-m-n}a^m$

$=a^{-m-n+m}$

$=a^{-n}$ $\quad\left(\begin{matrix}\because -m-n)0\\ \& m>0\\ \text{So by (1)}\end{matrix}\right)$

Hence we have $a^{m+n}=a^m.a^n$ $\forall\, m,n\in Z$

(15) 2·1–15

(iv) To Prove $(a^m)^n = a^{mn}$ $\forall m, n \in Z$

Case 1 when $n > 0$

Put $n = 1$ $(a^m)^1 = a^{m \cdot 1}$

$a^m = a^m$ C-1 is satisfied

Put $n = K$ $(a^m)^K = a^{mK}$ Supposed true

Now $(a^m)^{K+1} = (a^m)^K (a^m)^1$

$= a^{mK} a^m$ $\longrightarrow$ using supposition

$= a^{mK+m}$ $\longrightarrow$ using $a^{m+n} = a^m \cdot a^n$

$(a^m)^{K+1} = a^{m(K+1)}$ C-2 is satisfied

Hence $(a^m)^n = a^{mn}$ $\forall m \in Z$ & $n \in Z^+$

Case 2 When $n < 0$ Let $n = -r$ where $r \in Z^+$

$(a^m)^n = (a^m)^{-r}$

$= \left((a^m)^{-1}\right)^r$ ∵ In (ii) $(a^{-1})^m = a^{-m}$

$= (a^{-m})^r$

$= a^{-mr}$

$= a^{m(-r)}$

$(a^m)^n = a^{mn}$

Case 3 Let $n = 0$

$\Rightarrow (a^m)^n = (a^m)^0 = e = a^0 = a^{m0}$

Hence $(a^m)^n = a^{mn}$ $\forall m, n \in Z$,

Order of an element

Let 'a' be an element of a group G

A +ive integer 'n' is said to be the order of 'a' if $a^n = e$ where n is least +ive integer.

Example: $G = \{1, \omega, \omega^2\}$ $O(\omega) = 3$ $\because \omega^3 = 1 = e$

$G = \{1, -1, i, -i\}$ $O(-1) = 2$ $\because (-1)^2 = 1 = e$

It is denoted by $O(a) = n$ or $|a| = n$

If there does not exist 'n' s.t $a^n = e$ then a is said to be of Infinite Order

i.e If $n = 0$ is the only integer for which $a^n = e$ then a is said to be of infinite order

Example 10 $S = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ To Prove group under '×' modulo 8

×	$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{7}$
$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{7}$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{7}$	$\bar{5}$
$\bar{5}$	$\bar{5}$	$\bar{7}$	$\bar{1}$	$\bar{3}$
$\bar{7}$	$\bar{7}$	$\bar{5}$	$\bar{3}$	$\bar{1}$

(i) Associative $(\bar{3} \times \bar{5}) \times \bar{7} = \bar{3} \times (\bar{5} \times \bar{7})$

$\bar{7} \times \bar{7} = \bar{3} \times \bar{3}$

$\bar{1} = \bar{1}$

(ii) Identity $e = 1$

$\bar{1} \times \bar{1} = \bar{1}$

$1 \times \bar{3} = \bar{3}$

$1 \times \bar{5} = \bar{5}$

$1 \times \bar{7} = \bar{7}$

To Find Order $a^n = e$

$a^n = e$ n is order of 'a'

Order of $\bar{1}$ is 2 $\because (\bar{1})^2 = \bar{1}$

order of $\bar{3}$ is 2 $\because (\bar{3})^2 = \bar{3} \times \bar{3} = \bar{1}$

order of $\bar{5}$ is 2 $\because (\bar{5})^2 = \bar{5} \times \bar{5} = \bar{1}$

order of $\bar{7}$ is 2 $\because (\bar{7})^2 = \bar{7} \times \bar{7} = \bar{1}$

(iii) Inverse. For each element of S there exist inverse element.

as

$\bar{1} \times \bar{1} = \bar{1}$

$\bar{3} \times \bar{3} = \bar{1}$

$\bar{5} \times \bar{5} = \bar{1}$

$\bar{7} \times \bar{7} = \bar{1}$

each element of S is inverse of itself

Example II

2·1–17

Show that $O(a) = O(a^{-1})$

Let G be a group & $a \in G$.

Suppose $O(a) = m \Rightarrow a^m = e$ where m is least +ive integer

& $O(a^{-1}) = n \Rightarrow (a^{-1})^n = e$ " n is least +ive integer

Now $a^m = e$ (We have to prove that $m = n$)

$a^m a^{-m} = e a^{-m}$

$a^{m-m} = a^{-m}$

$e = (a^{-1})^m$ but $(a^{-1})^n = e$

$\Rightarrow m > n$ ——— (I) ($\because$ n is least +ive integer)

Similarly let $(a^{-1})^n = e$

$a^n (a^{-1})^n = a^n e$

$a^n a^{-n} = a^n$

$a^{n-n} = a^n$

$e = a^n$ but $a^m = e$

$\Rightarrow n > m$ ——— (II) ($\because$ m is least +ive integer)

So from (I) & (II)

$n = m$ Hence $O(a) = O(a^{-1})$

(ii) The orders of ab and ba are equal

$O(ab) = O(ba)$ To Prove.

Let G be a group & $a, b \in G$

Suppose $O(ab) = m \Leftrightarrow (ab)^m = e$

Thus $(ab)^m = e \Leftrightarrow ab \cdot ab \cdot ab \cdots \overset{m\text{ times}}{=} e$

pre multiply by a^{-1}

$\Leftrightarrow a^{-1}a\,b\cdot ab\cdot ab\cdots\cdots ab = a^{-1}e$

$\Leftrightarrow (a^{-1}a)\,b\cdot ab\cdot ab\cdots\cdots ab = a^{-1}$

$\Leftrightarrow e(ba)\cdot ba\cdots\cdots\cdots ab = a^{-1}$

post multiply by a

$\Leftrightarrow (ba)(ba)\cdots(ba)\cdot b = a^{-1}$

$\Leftrightarrow (ba)\cdot(ba)\cdots\cdots(ba)\cdot ba = a^{-1}a$

$\Leftrightarrow (ba)^m = e$

Hence the orders of ab & ba are equal.

Th The orders of a & bab^{-1} are equal.

$O(a) = O(bab^{-1})$

Let $a, b \in G$. Then the order of a is m i.e $a^m = e$.

$\therefore\ a^m = e$

$\Leftrightarrow a\,a\,a\cdots\cdots m$ times $= e$

$\Leftrightarrow aeaeae\cdots\cdots m$ times $= eee\cdots m$ times

$\Leftrightarrow a(b^{-1}b)a(b^{-1}b)a\cdots a(b^{-1}b) = e$

$\Leftrightarrow ab^{-1}(bab^{-1})(bab^{-1})\cdots(bab^{-1})b = e$

pre x by b

$\Leftrightarrow (bab^{-1})(bab^{-1})(bab^{-1})\cdots(bab^{-1})b = be$

$\Leftrightarrow (bab^{-1})(bab^{-1})(bab^{-1})\cdots(bab^{-1})b = b$

Post x by b^{-1}

$\Leftrightarrow (bab^{-1})(bab^{-1})(bab^{-1})\cdots\cdots(bab^{-1})bb^{-1} = bb^{-1}$

$\Leftrightarrow (bab^{-1})(bab^{-1})(bab^{-1})\cdots\cdots(bab^{-1})e = e$

$\Leftrightarrow (bab^{-1})^m = e$ $\quad\because m$ times (bab^{-1})

Hence order of bab^{-1} is m i.e $O(bab^{-1}) = m$.

So $O(a) = O(bab^{-1})$

$\boxed{2.1-19}$

Exercise # 2.1

Q#19 Let G be a group such that $(ab)^n = a^n b^n$ —— ①

For three consecutive no's $m-2, m-1, m$. ① holds i,e

$(ab)^{m-2} = a^{m-2} b^{m-2}$
$(ab)^{m-1} = a^{m-1} b^{m-1}$
$(ab)^m = a^m b^m$

$\because (ab)^m = a^m b^m$

$\Rightarrow (ab)^{m-1}(ab) = a^m b^m$

$\Rightarrow (a^{m-1} b^{m-1})(ab) = a^m b^m$

{We have to prove G abelian i,e $ab = ba$

$\Rightarrow a^{m-1}(b^{m-1}a)b = a^m b^m$ (Asso Law.)

Pre Multiply by a^{-m+1}

$\Rightarrow a^{-m+1} a^{m-1}(b^{m-1}a)b = a^{-m+1}(a^m b^m)$

$\Rightarrow e\,(b^{m-1}a)b = (a^{-m+1} a^m) b^m$

$(b^{m-1}a)b = ab^m$

Post Multiply by b^{-1}

$\Rightarrow (b^{m-1}a)bb^{-1} = (ab^m)b^{-1}$

$(b^{m-1}a)e = a(b^m b^{-1})$

$b^{m-1}a = ab^{m-1}$

2nd Method Easy

Let $m, m+1, m+2$, be three consecutive numbers

$(ab)^m = a^m b^m$ —— ①
$(ab)^{m+1} = a^{m+1} b^{m+1}$ —— ⑪
$(ab)^{m+2} = a^{m+2} b^{m+2}$ —— ⑪

$\Rightarrow (ab)^{m+1}(ab) = (a^{m+1} a\, b^{m+1} b)$

using ⑪ $\Rightarrow a^{m+1} b^{m+1}(ab) = (a^{m+1} a\, b^{m+1})b$

by Asso Law $\Rightarrow a^{m+1}(b^{m+1}a)\cancel{b} = a^{m+1}(a\, b^{m+1})\cancel{b}$

by Right Cancellation Law

$\Rightarrow \cancel{a}a^m(b^{m+1}a) = \cancel{a}a^m(ab^{m+1})$

by Left Cancellation Law

$\Rightarrow a^m b^m(b\,a) = a^{m+1} b^{m+1}$

using ① $\Rightarrow (ab)^m(ba) = (ab)^{m+1}$ using ⑪

$\Rightarrow (ab)^m(ba) = (ab)^m(ab)^1$

Left Cancellation Law $\Rightarrow$ $ba = ab$

Hence G is abelian

Thus $(ab)^m = a^m b^m \Rightarrow b^{m-1}a = ab^{m-1}$ —— ②

Put $m = m-1$ in ②

$\Rightarrow (ab)^{m-1} = a^{m-1} b^{m-1} \Rightarrow b^{m-1-1}a = ab^{m-1-1}$

$\Rightarrow b^{m-2}a = ab^{m-2}$ —— ③

$\therefore\ b^{m-1}a = (b\,b^{m-2})a$

$= b(b^{m-2}a)$ Asso Law.

$= b(a\,b^{m-2})$ using (3)

$b^{m-1}a = (ba)b^{m-2}$ ——— (A)

Similarly $ab^{m-1} = (ab)b^{m-2}$ ——— (B)

From (2) $b^{m-1}a = a\,b^{m-1}$

use (A) & (B) in (2) $(ba)\,b^{m-2} = (ab)\,b^{m-2}$

By Right Cancellation Law. $ab = cb \Rightarrow a = c$

$ba = ab$ Commutative Law proved.

Hence G is abelian

Q4 $S = \{\bar1, \bar2, \bar4, \bar5, \bar7, \bar8\}$ To Prove S is group under multiplication modulo 9.

$\cdot$	$\bar1$	$\bar2$	$\bar4$	$\bar5$	$\bar7$	$\bar8$
$\bar1$	$\bar1$	$\bar2$	$\bar4$	$\bar5$	$\bar7$	$\bar8$
$\bar2$	$\bar2$	$\bar4$	$\bar8$	$\bar1$	$\bar5$	$\bar7$
$\bar4$	$\bar4$	$\bar8$	$\bar7$	$\bar2$	$\bar1$	$\bar5$
$\bar5$	$\bar5$	$\bar1$	$\bar2$	$\bar7$	$\bar8$	$\bar4$
$\bar7$	$\bar7$	$\bar5$	$\bar1$	$\bar8$	$\bar4$	$\bar2$
$\bar8$	$\bar8$	$\bar7$	$\bar5$	$\bar4$	$\bar2$	$\bar1$

Associative

$\bar4\cdot(\bar5\cdot\bar7) = (\bar4\cdot\bar5)\cdot 7$

$\bar4\cdot\bar8 = \bar2\cdot\bar7$

$\bar5 = \bar5$

Identity $e * a = a * e = a$

$e = 1$ First row shows that identity is $\bar1$

Inverse $a * b = b * a = e$

For each element of S there exists inverse element.

$\bar1\cdot\bar1 = \bar1$ $\quad \bar2\cdot\bar5 = \bar1$ $\quad \bar4\cdot\bar7 = \bar1$ $\quad \bar8\cdot\bar8 = \bar1$

$\boxed{2 \cdot 1 - 21}$

Q* ⑤ Is (Z, o) a group where o is defined by

$$a \circ b = 0 \quad \text{for all } a, b \in Z.$$

Associative $\quad 2, 3, 4 \in Z$ $\quad$ for any $a, b, c \in Z$ $\quad (ab)c = a(bc)$

$$(2 \circ 3) \circ 4 = 2 \circ (3 \circ 4)$$

$$0 \circ 4 = 2 \circ 0$$

$$0 = 0$$

Identity $\quad a * e = e * a = a$

Let $e = 0$ $\quad 2 \circ 0 = 0$
$\quad 3 \circ 0 = 0$

Let $e = 1$ $\quad 2 \circ 1 = 0$
$\quad 4 \circ 1 = 0$

Neither 0 is identity
Nor 1 is identity
Here no element of Z can be taken as the identity element
So (Z, o) is not a group.

Q9 ⑧ Group G is such that $x \cdot x = e$ for all $x \in G$ where e is the identity element of G, then to Prove G is abelian

Let $a, b \in G$ then by def $\quad a \cdot a = e$
& $\quad b \cdot b = e$

Now $a, b \in G \Rightarrow ab \in G$ ($\because$ G is group so G is closed)

So $(ab) \cdot (ab) = e$

Now $a \cdot a = e \Rightarrow a = a^{-1}$ — ① ($\because$ By def of inverse if $ab = e$ then a is inverse of b)

$b \cdot b = e \Rightarrow b = b^{-1}$ — ②

$(ab) \cdot (ab) = e \Rightarrow ab = (ab)^{-1}$

$= b^{-1} a^{-1}$ $\quad \because (ab)^{-1} = b^{-1} a^{-1}$

$ab = ba$ (commutative) using ① & ②

Hence G is an Abelian Group.

Q10 If a group G has three elements then show that it is abelian

Sol Since the group G has three elements & by def of group one of those elements must be identity element.

$$\text{Let } G = \{e, a, b\} \qquad e \text{ is identity}$$

As $a \in G$ so $a \cdot a = a^2 \in G$ $\qquad \because G$ is a group (closure property)

then we have the following cases.

$$\left.\begin{array}{l} a^2 = e \\ \text{or } a^2 = a \\ \text{or } a^2 = b \end{array}\right\}$$

Case 1 Let $a^2 = e \Rightarrow a \cdot a = e \Rightarrow a = a^{-1}$ ——— ①

Also $a, b \in G \Rightarrow ab \in G$ $\quad \because G$ is a group

$\therefore ab = e$

or $ab = a$

or $ab = b$

If $ab = e \Rightarrow b = a^{-1} \Rightarrow b = a$ using ① which is a contradiction.

If $ab = a \Rightarrow ab = ae \Rightarrow b = e$ left can. law which is a contradiction.

If $ab = b \Rightarrow ab = eb \Rightarrow a = e$ again a contradiction $\because a$ & e are distinct.

Hence we conclude that $a^2 \neq e$

Case 2

If $a^2 = a \Rightarrow a \cdot a = ae \Rightarrow a = e$ Contradiction $\because a$ & e distin.

$\because$ we have proved in Case 1 & Case 2 that $a^2 \neq e$ & $a^2 \neq a$ $\therefore a^2 \neq a$

So we are left with only one possibility that $a^2 = b$

So G becomes $\{e, a, a^2\}$. Since G becomes cyclic group generated by a & we know that every cyclic group is abelian, so G is abelian.

Q12 Prove that if every non-identity element of a group G is of order 2 then G is abelian.

Sol For all non-identity elements $x \in G \Rightarrow x^2 = e$ (Given)

$\therefore$ For all $a, b \in G \Rightarrow ab \in G$ Closure property

So $a^2 = e$, $b^2 = e$, $(ab)^2 = e$ etc

Now $(ab)^2 = e$

$(ab)(ab) = e$

$a(ba)b = e$

$a\,a(ba)b = ae$ $\because a^2 = e$

$e(ba)b = a$

$(ba)bb = ab$ $\because b^2 = e$

$(ba)e = ab$

$(ba) = ab$

2nd Method

For all non-identity elements $x \in G \Rightarrow x^2 = e$

$\because$ For all $a, b \in G \Rightarrow a^2 = e \Rightarrow a^{-1}a^2 = a^{-1}e \Rightarrow a = a^{-1}$

For all $b \in G \Rightarrow b^2 = e \Rightarrow b^{-1}b^2 = b^{-1}e \Rightarrow b = b^{-1}$

$\because$ G is a group So for all $a, b \in G \Rightarrow ab \in G$ Closure Property

So $(ab)^2 = e$

$(ab)^{-1}(ab)^2 = (ab)^{-1}$

$(ab) = b^{-1}a^{-1}$

$ab = ba$ $\because b^{-1} = b$ & $a^{-1} = a$

Q11 (11) If every element of a group G is its own inverse show that G is abelian.

Let $a, b \in G \Rightarrow a = a^{-1}$ & $b = b^{-1}$

$\because a, b \in G$ so $ab \in G$ $\because G$ is a group (Closure P)

$$\text{So } ab = (ab)^{-1}$$
$$= b^{-1}a^{-1}$$
$$ab = ba \qquad \because a = a^{-1} \text{ \& } b = b^{-1}$$

Hence G is abelian

Q1 Which of the following statements are correct.

(i) A group can have more than one identity.

Incorrect.

$\because$ Identity element in a group G is unique.

(ii) The null set can be considered to be a group

Incorrect.

$\because$ A group is always a non-empty set.
secondly the null set can not contain identity element.

(iii) There may be groups where cancellation law fails

Incorrect

$\because$ cancellation law holds for all groups.

(iv) Every set of numbers which is a group under addition is also a group under multiplication & vice versa

Incorrect

$\because$ $(R, +)$ is group but $(R, \cdot)$ is not a group.

(v) The set R of all real numbers is a group with respect to subtraction

<u>Incorrect.</u>

∵ subtraction is not associative in the set of all real nos

Let $2,3,4 \in R$

$$2-(3-4) = (2-3)-4$$
$$2-(-1) = (-1)-4$$
$$3 \neq -5$$

(vii) To each element of a group there corresponds no inverse element.

<u>Incorrect.</u>

∵ Each element in a group G has its unique inverse.

vi) The set of all non-zero integers is a group w.r.t division

<u>Incorrect</u>

∵ Associative Law does not hold.

$2,3,4 \in Z'$

$$(2 \div 3) \div 4 = 2 \div (3 \div 4)$$
$$\frac{2}{3} \div 4 = 2 \div \frac{3}{4}$$
$$\frac{2}{3} \times \frac{1}{4} = 2 \times \frac{4}{3}$$
$$\frac{2}{12} \neq \frac{8}{3}$$

VIII) To each element of a group, there corresponds only one inverse element.

<u>Correct</u>

Each element in a group G has its unique inverse.

ix) For each element of a group, there correspond more than one inverse element.

<u>Incorrect.</u>

∵ Each element in a group G has its unique inverse.

:5

Q
6 $G = \{I, A, B, C\}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

<u>To Prove</u> These matrices form a Group under matrix multiplication

We construct Group table

·	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

We observe that the matrix multiplication is closed in this set $G = \{I, A, B, C\}$ ie the product of any two members of G belongs to G.

e.g

$A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = C \in G$

Hence Closed.

$I \cdot I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$I \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = A$

$I \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = B$

$I \cdot C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = C$

Hence $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is <u>identity element</u> of Set G.

<u>Inverse</u> element of each element of G exists

$ab = I$ (Def of inverse)

From table $I \cdot I = I$, $A \cdot A = I$, $B \cdot B = I$, $C \cdot C = I$

<u>Associative</u> Law holds for all members of Set G.

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$

$C \cdot C = A \cdot A$

$I = I$

All axioms of group are satisfied hence G is a Group

2·1-27

Table required for easy sol.

(10) Let $G = \{I, f, g, h\}$

Q where

where $z \in C$

$I(z) = z \quad f(z) = -z \quad g(z) = \frac{1}{z} \quad h(z) = -\frac{1}{z}$

To Prove G is a group under composition of functions defined by $(g \circ f)(z) = g(f(z))$

Associative

$$((g \circ f) \circ h)(z) = (g \circ (f \circ h))(z) \qquad f, g, h \in G$$

$$(g \circ f)(h(z)) = g((f \circ h)(z))$$
$$g(f(h(z))) = g(f(h(z)))$$
$$g\left(f\left(-\tfrac{1}{z}\right)\right) = g\left(f\left(-\tfrac{1}{z}\right)\right)$$
$$g\left(\tfrac{1}{z}\right) = g\left(\tfrac{1}{z}\right)$$
$$z = z$$

$(g \circ h)(z) = g(h(z)) \quad g, h \in G$
$= g(h(z))$
$= g\left(-\frac{1}{z}\right)$
$= -z$
$= f(z) \in G$ Hence G is closed under '$\circ$'

Hence Associative

Identity

$e \cdot a = a \cdot e = a$ Def of Identity

$(I \circ f)(z) = I(f(z)) = I(-z) = -z = f(z)$

$(I \circ g)(z) = I(g(z)) = I\left(\frac{1}{z}\right) = \frac{1}{z} = g(z)$

$(I \circ h)(z) = I(h(z)) = I\left(-\frac{1}{z}\right) = -\frac{1}{z} = h(z)$

$(I \circ I)(z) = I(I(z)) = I(z) = z = I(z)$

Hence $I(z)$ is Identity element of G.

Inverse

$ab = ba = e$

$(g \circ g)(z) = g(g(z)) = g\left(\frac{1}{z}\right) = z = I(z)$ Identity

$(h \circ h)(z) = h(h(z)) = h\left(-\frac{1}{z}\right) = z = I(z)$ Identity

$(f \circ f)(z) = f(f(z)) = f(z) = z = I(z)$ =

$(I \circ I)(z) = I(z)$

Hence f, g, h are inverses of themselves

All axioms satisfied hence G is a group.

(11) Q8 $C = 2^K$; $K = 0, \pm 1 \pm 2 \ldots\ldots$

To Prove C is a group under multiplication '$\cdot$'

Let $2^K, 2^l, 2^m \in C$ $\quad K, l, m = 0 \pm 1, \pm 2 \ldots$

$(2^K \cdot 2^l) \cdot 2^m = 2^K \cdot (2^l \cdot 2^m)$

$2^{K+l} \cdot 2^m = 2^K \cdot 2^{l+m}$

$2^{K+l+m} = 2^{K+l+m}$

Hence Associative Law holds

$2^K \cdot 2^l = 2^{K+l} \in C$

Hence multiplication is closed in C

Identity $\quad a \cdot e = e \cdot a = a.$

$2^0 \cdot 2^K = 2^{0+K} = 2^K$

$2^0 \cdot 2^l = 2^{0+l} = 2^l$

$2^0 \cdot 2^m = 2^{0+m} = 2^m$

Hence $2^0 \in C$ is Identity element of C.

Inverse $\quad a \cdot b = b \cdot a = e$

For each $2^K \in C$ there exist $2^{-K} \in C$

$2^K \cdot 2^{-K} = 2^{K-K} = 2^0$ Identity

Hence C is a group under multiplication

× ———— ×

(12) Q13 Let a, b & ab all have order 2 in the group G.

$\because O(a) = 2 \Rightarrow a^2 = e \Rightarrow a^{-1}a^2 = a^{-1}e \Rightarrow a = a^{-1}$

$O(b) = 2 \Rightarrow b^2 = e \Rightarrow b^{-1}b^2 = b^{-1}e \Rightarrow b = b^{-1}$

$O(ab) = 2 \Rightarrow (ab)^2 = e \Rightarrow (ab)^{-1}(ab)^2 = (ab)^{-1}e \Rightarrow ab = (ab)^{-1}$

$\because (ab) = (ab)^{-1} \Rightarrow ab = b^{-1}a^{-1} = ba$ Proved $\quad \because a = a^{-1}, \; b = b^{-1}$

(13) (i) The identity element is unique. (To Prove)

Let e & e' be the identity elements in a group G w.r.t B.O $\star$.

Then $e \star e' = e$ — (I) $\because e'$ is identity element

$e \star e' = e'$ — (II) $\because e$ is identity element

$\therefore e = e'$ ($\because$ LHS of (I) & (II) are same so RHS are same.)

Hence identity element is unique

(ii) The inverse of each element in a group G is unique

Let $a \in G$ and a' & $a'' \in G$ be the two inverses of a.

$\because G$ is a group so Associative Law holds in G

$\therefore$ $(a' \star a) \star a'' = a' \star (a \star a'')$ — (I)

LHS $(a' \star a) \star a''$

$= e \star a''$ $\because a'$ is inverse of a

$(a' \star a) \star a'' = a''$ — (II) e is the identity

RHS $a' \star (a \star a'')$

$= a' \star e$ $\because a''$ is inverse of a

$a' \star (a \star a'') = a'$ — (III) e is the identity

$\therefore$ $a'' = a'$ using (II) & (III) in (I)

x————x

Q 14

$(ab)^2 = a^2 b^2$ for all a, b in group G.

To Prove G is abelian.

$(ab)^2 = a^2 b^2$

$(ab)(ab) = (aa)(bb)$

$a(ba)b = a(ab)b$ (Associative Law.

$a^{-1}a(ba)b = a^{-1}a(ab)b$ $a^{-1} \in G$ $\because$ G is group.

$e(ba)b = e(ab)b$ e is identity. $\boxed{a \cdot e = a}$

$(ba)bb^{-1} = (ab)bb^{-1}$ $b^{-1} \in G$.

$(ba)e = (ab)e$

$ba = ab$ Hence G is abelian

Now To Prove $(ab)^2 = a^2 b^2$

$ab = ba$ (given.)

$(ab)^2 = (ab)(ab) = a(ba)b$ Associative Law

$= a(ab)b$ $\because ab = ba$.

$= (aa)(bb)$

$(ab)^2 = a^2 b^2$

Proved.

Q15 Let G be a group which has only one element of order 2. $\therefore a^2 = e$

Let x be any element of G. Now consider the element xax^{-1}.

$a^2 = e \Rightarrow (xax^{-1})^2 = (xax^{-1})(xax^{-1})$

$= xa(x^{-1}x)ax^{-1}$

$= xaeax^{-1}$

$= xa^2x^{-1}$

$\because a^2 = e$ $\quad = xex^{-1}$

$= xx^{-1}$

$(xax^{-1})^2 = e$

$\therefore xax^{-1}$ is of order 2.

but according to Question there is only one element of order 2

$\therefore xax^{-1} = a$

$(xax^{-1})x = ax$

$xa(x^{-1}x) = ax$

$xa(e) = ax$

$xa = ax$ for all $a \in G$.

proved

IInd Method

we know from Example 11 that order of a & bab^{-1} are equal

$\therefore$ order of a & xax^{-1} are '2'

but there is only one element of order 2 in G

$\therefore xax^{-1} = a$

$xa(x^{-1}x) = ax$

$xa = ax$

proved.

Q17 To Prove $(ab)^n = a^n b^n$ G is abelian (given)

We prove it by principal of Mathematical Induction

Put $n=1$ $(ab)^1 = a^1 b^1$ C-I is satisfied

Put $n=k$ $(ab)^k = a^k b^k$ supposed to be true

Multiply by ab

$$(ab)^k(ab) = a^k b^k (ab)$$

$$(ab)^{k+1} = a^k(b^k a)b \quad \text{Associative Law}$$

$$= a^k(a b^k)b \quad \because G \text{ is abelian}$$

$$= a^{k+1} b^{k+1}$$

C-II is satisfied

Now when $n=0$ in $(ab)^n$

$$(ab)^0 = e = e \cdot e = a^0 \cdot b^0$$

$$(ab)^0 = a^0 b^0$$

When $n=-p$ $p \in Z^+$

$$(ab)^n = (ab)^{-p} = [(ab)^{-1}]^p$$

$$= (b^{-1}a^{-1})^p = b^{-p}a^{-p}$$

$$= b^n a^n$$

$$(ab)^n = a^n b^n \quad \because G \text{ is abelian}$$

Subgroup

Let $(G, \cdot)$ be a group and H be a non empty subset of 'G', then 'H' is called a subgroup of 'G' if 'H' is itself a group w.r.t the binary operation defined in G.

Examples.

As $Z \subset R$ | Z = set of integers
$\&$ $(Z,+)$, $(R,+)$ are groups | R = Set of Real Nos
$\therefore$ $(Z,+)$ is a subgroup of $(R,+)$ | C = Set of Complex Nos
 | Q = Set of Rational Nos
As $Z \subset Q$ | $R' = R - \{0\}$
$\&$ $(Z,+)$, $(Q,+)$ are groups | $C' = C - \{0\}$
$\therefore$ $(Z,+)$ is a subgroup of $(Q,+)$

As $Q \subset R$
$\&$ $(Q,+)$, $(R,+)$ are groups
$\therefore$ $(Q,+)$ is a subgroup of $(R,+)$

As $R \subset C$
$\&$ $(R,+)$, $(C,+)$ are groups
$\therefore$ $(R,+)$ is a subgroup of $(C,+)$

$(R', \cdot)$ $(C', \cdot)$ are groups
$\therefore$ $(R', \cdot)$ is a sbgp of $(C', \cdot)$

$(1, \omega, \omega^2)$ is a sbgp of $(C', \cdot)$

$\{1, -1, i, -i\}$ is a sbgp of $(C', \cdot)$

Now Every group G has at least two sbgps namely 'G' itself $\&$ $\{e\}$ identity. These are called Trivial Sbgps

Any sbgp of G other than trivial sbgps is called <u>Non trivial</u> sbgp of G.

Imp
<u>Theorem</u> :-
Let $(G,\cdot)$ be a group. Then a non empty subset H of G is a sbgp if & only if
$$a,b \in H \Rightarrow ab^{-1} \in H.$$

<u>Proof</u>.
Let H is a sbgp of G & to prove that $\forall a,b \in H \Rightarrow ab^{-1} \in H$.
$\because$ H is sbgp So H is group also.
$\therefore$ If $b \in H$ then $b^{-1} \in H$.
So $a, b^{-1} \in H \Rightarrow ab^{-1} \in H$. by closure law.
<u>proved</u>

Conversely
To Prove H is a sbgp of G
Let $\forall a,b \in H \Rightarrow ab^{-1} \in H$. —— ①

Put $b=a$.
$a,a \in H \Rightarrow aa^{-1} \in H$
$\Rightarrow e \in H$. (Identity exist in H)

$\because$ H is a subset of G. $\therefore$ each element of H is also element of G.

$e,b \in H \Rightarrow eb^{-1} \in H$. using ①
$\Rightarrow b^{-1} \in H$. (Inverse exist in H)

$\because$ G is a group so Associative Law holds in G. & Since H is a subset of G, So Associative Law holds in H also.

Now for $a,b^{-1} \in H$ $\Rightarrow a(b^{-1})^{-1} \in H$ using ①
$\Rightarrow ab \in H$.
(Closure Law $a,b \in S \Rightarrow ab \in S$)
$\because a \in H$ & $b \in H$.
$\therefore$ H is closed under binary operation.
So H is a group and being subset of G it is Sbgp of G.

S98
Theorem The intersection of any collection of sbgps of a group is a sbgp of G.

| $H_1 \cap H_2 \cap H_3 \cdots H_n = \bigcap_{i=1}^{n} H_i$
| Cond for Sbgp S of G
| $a,b \in S \Rightarrow ab^{-1} \in S$

Proof Let $\{H_i, i \in I\}$ be a family of sbgps of G

Let $H = \bigcap_{i \in I} H_i$

Let $a,b \in H \Rightarrow a,b \in H_i, i \in I$

Since each H_i is a sbgp of G, $\therefore ab^{-1} \in H_i, i \in I$

$\therefore ab^{-1} \in \bigcap_{i \in I} H_i = H$

Hence H is a sbgp of G.

Theorem Let G be a group and H a subgp of G Then the set $aHa^{-1} = \{aha^{-1} : h \in H\}$ is a sbgp of G.

Proof

(To Prove aHa^{-1} is Sbgp)
(Cond for sbgp S of G
$a,b \in S \Rightarrow ab^{-1} \in S$

Let $x, y \in aHa^{-1}$

where $x = ah_1a^{-1}$, $h_1 \in H$

$y = ah_2a^{-1}$, $h_2 \in H$

(we see whether $xy^{-1} \in aHa^{-1}$)

$xy^{-1} = (ah_1a^{-1})(ah_2a^{-1})^{-1}$

$= (ah_1a^{-1})((a^{-1})^{-1}h_2^{-1}a^{-1})$

$(ab)^{-1} = b^{-1}a^{-1}$

$= ah_1a^{-1}a\,h_2^{-1}a^{-1}$

$= a h_1 e h_2^{-1} a^{-1}$

$= a h_1 h_2^{-1} a^{-1}$ $\quad \because h_1, h_2 \in H$ & H is sbgp of G
So $h_1 h_2^{-1} \in H$.

$xy^{-1} = a(h_1 h_2^{-1})a^{-1} \in aHa^{-1}$

So aHa^{-1} is a sbgp of G.

Theorem. The union $H \cup K$ of two sbgps H & K of a group G is a sbgp of G if & only if $H \subseteq K$ or $K \subseteq H$

Proof Let H, K be sbgps of G.
Suppose $H \subseteq K$ or $K \subseteq H$ $\quad$ [To Prove $H \cup K$ is sbgp of G]
So $H \cup K = K$ or $H \cup K = H$
$\forall a, b \in H \cup K \Rightarrow ab^{-1} \in H \cup K$ $\quad \because H$ & K are sbgps of G
So $H \cup K$ is a sbgp of G.

Conversely Suppose $H \cup K$ is a sbgp of G. $\quad$ [To Prove $H \subseteq K$, $K \subseteq H$
then $\exists a \in H \setminus K$, $b \in K \setminus H$
and both $a, b \in H \cup K$

$\because H \cup K$ is sbgp so $\forall a, b \in H \cup K \Rightarrow ab \in H \cup K$
Now $\because ab \in H \cup K$ $\therefore ab \in H$ or $ab \in K$
If $ab \in H$ then $\quad b = (a^{-1}a)b$
$= a^{-1}(ab) \in H$ $\quad a^{-1} \in H \because H$ is sbgp
$b \in H$ (Contradiction) $\therefore K \subseteq H$
If $ab \in K$ then $\quad a = abb^{-1}$
$a = (ab)b^{-1} \in K$ (contradiction)
$\therefore H \subseteq K$

Hence either $H \setminus K = \varphi \Rightarrow K \subseteq H$

or $K \setminus H = \varphi \Rightarrow H \subseteq K$

proved

Example 16 Find the subgroups of group $G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ with the following table

+	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

Sol Here binary operation is addition modulo 4.
Consider all non-empty subsets of group $G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ w.r.t addition modulo 4.
$\{\bar{0}\}, \{\bar{0}, \bar{1}\}, \{\bar{0}, \bar{2}\}, \{\bar{0}, \bar{3}\}, \{\bar{1}, \bar{2}\}, \{\bar{1}, \bar{3}\}, \{\bar{2}, \bar{3}\}, \{1\}$
$\{2\}, \{3\}, \{\bar{0}, \bar{1}, \bar{2}\}, \{\bar{0}, \bar{1}, \bar{3}\}, \{\bar{0}, \bar{2}, \bar{3}\}, \{\bar{1}, \bar{2}, \bar{3}\}, \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$
We know that $\{\bar{0}\}$ & $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ are trivial sbgps
we found that non-trivial sbgps is $\{\bar{0}, \bar{2}\}$

$\because$ In $\{\bar{0}, \bar{2}\}$ (i) $\bar{0}$ is identity $\quad \bar{0} + \bar{2} = 2 \qquad (a+e = a)$ def.

(ii) $\bar{0}$ & $\bar{2}$ are inverse of themselves $\qquad (a+b = e)$ def

$\bar{0} + \bar{0} = 0 \qquad \bar{2} + \bar{2} = 0$

(iii) $0 + (2+2) = (0+2) + 2$ Associative
$0 + 0 = \bar{2} + \bar{2}$
$0 = 0$

So $\{\bar{0},\bar{2}\}$ is a group.

but because $\{\bar{0},\bar{2}\}$ is a subset of group $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$

So it is subgp

Hence all the sbgps of group $G=\{\bar{0},\bar{1},\bar{2},\bar{3}\}$

are $\{\bar{0}\}$, $\{\bar{0},\bar{2}\}$, $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$

x———————x

IInd Method   Def of Inverse $a*b=e$

Easy Method

Inverse of $\bar{0}=\bar{0}$ $\because \bar{0}+\bar{0}=\bar{0}$

" " $\bar{1}=\bar{3}$ $\because \bar{1}+\bar{3}=\bar{0}$

" " $\bar{2}=\bar{2}$ $\because \bar{2}+\bar{2}=\bar{0}$

" " $\bar{3}=\bar{1}$ $\because \bar{3}+\bar{1}=\bar{0}$.

Cond for subgp $\forall a,b\in H \Rightarrow ab^{-1}\in H$.

We found only $\{0,2\}$ is non trivial sbgp among rest of subsets $\because \{\bar{0}\},\{\bar{0},\bar{1},\bar{2},\bar{3}\}$ are trivial sbgps.

$\because$ Let $H=\{\bar{0},\bar{2}\}$ $\bar{0},\bar{2}\in H \Rightarrow \bar{0}+(\text{Inverse of }\bar{2})$

$=\bar{0}+\bar{2}$

$=\bar{2}\in H$

Hence $\{\bar{0},\bar{2}\}$ is a sbgp of G.

Let $H=\{\bar{0},\bar{1},\bar{3}\}$

$\bar{0},\bar{1},\bar{3}\in H$ $\bar{0}+(\bar{1})^{-1}=\bar{0}+\bar{3}=\bar{3}\in H$

$\bar{1}+(\bar{3})^{-1}=\bar{1}+\bar{1}=\bar{2}\notin H$ Hence $\{\bar{0},\bar{1},\bar{3}\}$ is not a sbgp of G.

Rest of all subsets of G are not sbgps $\because$ Inverse element doest not exist in those subset.

Hence sbgps of G are $\{\bar{0}\}$, $\{\bar{0},\bar{2}\}$, $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$

Imp

Example 17 Find the subgroups of the Klein's Four Group

$G=\{e,a,b,c\}$ defined by

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Consider all non-empty subsets of $G=\{e,a,b,c\}$

$\{e\},\{e,a\},\{e,b\},\{e,c\},\{a,b\},\{a,c\},\{b,c\},\{a\}\{b\}\{c\}$

$\{e,a,b\},\{e,a,c\},\{e,b,c\},\{a,b,c\},\{e,a,b,c\}$

$\{e\}$ & $\{e,a,b,c\}$ are trivial sbgps.

From Table

$e^{-1}=e \quad \because e\cdot e=e$

$a^{-1}=a \quad \because a\cdot a=e$

$b^{-1}=b \quad \because b\cdot b=e$

$c^{-1}=c \quad \because c\cdot c=e$

Note $a^2=b^2=c^2=e$

{Cond for sbgp $a,b\in H \Rightarrow ab^{-1}\in H$

So we found only $\{e,a\},\{e,b\}$ & $\{e,c\}$ are non trivial sbgps.

$\because \quad e,a\in\{e,a\} \Rightarrow e\cdot a^{-1}=e\cdot a=a\in\{e,a\}$

$e,b\in\{e,b\} \Rightarrow e\cdot b^{-1}=e\cdot b=b\in\{e,b\}$

$e,c\in\{e,c\} \Rightarrow e\cdot c^{-1}=e\cdot c=c\in\{e,c\}$

$\boxed{2\cdot 2-8}$

(39) EX # 2.2

(9) $H = \{a+b\sqrt{2} \mid a,b \in \mathbb{Q}$ & both a, b are not simultaneously zero$\}$

Let $x, y \in H$ s.t.

$x = a+b\sqrt{2}$
$y = c+d\sqrt{2}$ where $a,b,c,d \in \mathbb{Q}$ and all a,b,c,d are not simultaneously zero.

To Prove H sbgp
$x, y \in H \Rightarrow xy^{-1} \in H$

Now $xy^{-1} = (a+b\sqrt{2})(c+d\sqrt{2})^{-1}$

$= (a+b\sqrt{2})\dfrac{1}{(c+d\sqrt{2})}$

$= (a+b\sqrt{2})\dfrac{1}{(c+d\sqrt{2})} \times \dfrac{(c-d\sqrt{2})}{(c-d\sqrt{2})}$

$= \dfrac{(a+b\sqrt{2}) \times (c-d\sqrt{2})}{c^2-d^2(2)}$

$= \dfrac{ac-\sqrt{2}ad+\sqrt{2}bc-2bd}{c^2-2d^2}$

$xy^{-1} = \dfrac{(ac-2bd)+(bc-ad)\sqrt{2}}{c^2-2d^2}$

$= \left(\dfrac{ac-2bd}{c^2-2d^2}\right) + \left(\dfrac{bc-ad}{c^2-2d^2}\right)\sqrt{2}$

$xy^{-1} = E+F\sqrt{2} \in H$ ($\because$ of the form of $a+b\sqrt{2}$)

where $E = \dfrac{ac-2bd}{c^2-2d^2}$ & $F = \dfrac{bc-ad}{c^2-2d^2}$, where $c^2-2d^2 \neq 0$

So H is a sbgp of the group of non-zero Real no's under multiplication.

Now $xy^{-1} = (a+b\sqrt{-5})(c+d\sqrt{-5})^{-1}$

$$= (a+b\sqrt{-5})\frac{1}{(c+d\sqrt{-5})}$$

$$= (a+b\sqrt{-5})\frac{1}{(c+d\sqrt{-5})} \times \frac{(c-d\sqrt{-5})}{(c-d\sqrt{-5})}$$

$$= \frac{(a+b\sqrt{-5}) \times (c-d\sqrt{-5})}{c^2+5d^2}$$

$$= \frac{ac - ad\sqrt{-5} + bc\sqrt{-5} + 5bd}{c^2+5d^2}$$

$$= \frac{ac+5bd + (bc-ad)\sqrt{-5}}{c^2+5d^2}$$

$$= \frac{(ac+5bd)}{(c^2+5d^2)} + \frac{(bc-ad)}{(c^2+5d^2)}\sqrt{-5}$$

$$xy^{-1} = E+F\sqrt{-5} \in H \qquad (\because \text{ of the form of } a+b\sqrt{-5})$$

where $E = \dfrac{ac+5bd}{c^2+5d^2}$ $\quad F = \dfrac{bc-ad}{c^2+5d^2}$

Since for $x, y \in H \Rightarrow xy^{-1} \in H$

Hence H is a sbgp of group of non zero complex no's under multiplication.

14. Let H is a sbgp of G To Prove $H \cdot H = H$

& Let $H \cdot H = \{h_1 h_2 : h_1, h_2 \in H\}$.

Let $h_1 h_2 \in H \cdot H$

Since H is a sbgp of gp G so H is closed under '·' i.e $\forall\, h_1, h_2 \in H \Rightarrow h_1 h_2 \in H$

$$\Rightarrow H \cdot H \subseteq H \quad \text{---} \quad (I)$$

Also for each $h \in H$

$$h = h \cdot e \in H \cdot H \qquad (\because h \in H,\ e \in H)$$

$$\Rightarrow H \subseteq H \cdot H \quad \text{---} \quad (II)$$

From (I) & (II) $\quad H \cdot H = H$

13. Let $(Z, +)$ be the group of integers $Z = 0, \pm 1, \pm 2 \cdots\cdots$

Let H_1 & H_2 be two subsets of Z

$H_1 = \{-1, -2, -3, \cdots\cdots\}$ $(-1) + (-2) = -3 \in H_1$

$H_2 = \{1, 2, 3, \cdots\cdots\}$ $2 + 3 = 5 \in H_2$

H_1 & H_2 are closed under $+$.

Now because inverse element does not exist in H_1 & H_2 so H_1 & H_2 are not sbgps

Def of Inverse
$a * b = e$

$\boxed{2\cdot 2-11}$ 42 - A

Q11 (i) $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in R \;\&\; ad \neq 0 \right\}$

To Prove H is a subgroup of $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R \text{ and } ad - bc \neq 0 \right\}$

Sol Let $A = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} \in H$, $B = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} \in H$

where $a_1 d_1 \neq 0$ & $a_2 d_2 \neq 0$

$$B^{-1} = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d_2}{a_2 d_2} & \frac{-b_2}{a_2 d_2} \\ \frac{0}{a_2 d_2} & \frac{a_2}{a_2 d_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{a_2} & \frac{-b_2}{a_2 d_2} \\ 0 & \frac{1}{d_2} \end{bmatrix}$$

$$AB^{-1} = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} \begin{bmatrix} \frac{1}{a_2} & \frac{-b_2}{a_2 d_2} \\ 0 & \frac{1}{d_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a_1}{a_2} + b_1 \times 0 & \frac{-a_1 b_2}{a_2 d_2} + \frac{b_1}{d_2} \\ 0 \times \frac{1}{a_2} + d_1 \times 0 & 0 \times \frac{-b_2}{a_2 d_2} + \frac{d_1}{d_2} \end{bmatrix} = \begin{bmatrix} \frac{a_1}{a_2} & \frac{-a_1 b_2}{a_2 d_2} + \frac{b_1}{d_2} \\ 0 & \frac{d_1}{d_2} \end{bmatrix} \in H$$

Thus $A, B \in H \Rightarrow AB^{-1} \in H$ Hence H is subgroup of G.

x ——————— x

(ii) $K = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in R \right\}$ To Prove K is a sbgp of G

$A = \begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix} \in K$ $B = \begin{bmatrix} 1 & b_2 \\ 0 & 1 \end{bmatrix} \in K$

$$AB^{-1} = \begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b_2 \\ 0 & 1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -b_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1 \times 1) + (b_1 \times 0) & 1 \times (-b_2) + (b_1 \times 1) \\ (0 \times 1) + (1 \times 0) & 0 \times (-b_2) + (1 \times 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -b_2 + b_1 \\ 0 & 1 \end{bmatrix} \in K$$

Thus $A, B \in K \Rightarrow AB^{-1} \in K$. Hence K is sbgp of G.

x ——————— x

16 2002 — 704 $\boxed{2\cdot 2-12}$

15 Let H, K be subgroups of an abelian group G.

Let $H\cdot K = \{hk : h\in H, k\in K\}$

To Prove HK is subgroup of G

$\forall x, y \in HK$
$xy^{-1} \in HK$
HK is sbgp

Let $x, y \in HK$ s.t

$x = h_1 k_1 \qquad y = h_2 k_2 \qquad h_1, h_2 \in H,\ k_1, k_2 \in K$

$$xy^{-1} = (h_1 k_1)(h_2 k_2)^{-1}$$

$$= (h_1 k_1)(k_2^{-1} h_2^{-1}) = (h_1 k_1)(h_2^{-1} k_2^{-1}) \quad \because G \text{ is abelian}$$

$$= h_1 (k_1 h_2^{-1}) k_2^{-1} \quad \text{Associative Law}$$

$$= h_1 (h_2^{-1} k_1) k_2^{-1} \quad \because G \text{ is abelian}$$

$$= (h_1 h_2^{-1})(k_1 k_2^{-1}) \in HK$$

$\therefore \quad xy^{-1} \in HK$

$h_1 h_2^{-1} \in H \because$ sbgp

$k_1 k_2^{-1} \in K \because$ sbgp

Since $\forall x, y \in HK \Rightarrow xy^{-1} \in HK$

Hence HK is a sbgp of G.

2·2-13 $\quad$ 13-A

(16) Let H be a sbgp of a group G and $a \in G$. if $(Ha)^{-1} = \{(ha)^{-1} : h \in H\}$ then Show that $(Ha)^{-1} = a^{-1}H$

Sol Let $x \in (Ha)^{-1}$, then for some $h \in H$

$$x = (ha)^{-1} = a^{-1}h^{-1} \in a^{-1}H \quad \because h^{-1} \in H$$

Hence $(Ha)^{-1} \subset a^{-1}H$ ——— (I)

Now let $y \in a^{-1}H$ then for some $h \in H$

$$y = a^{-1}h = (h^{-1}a)^{-1} \in (Ha)^{-1}$$

Thus $a^{-1}H \subset (Ha)^{-1}$ ——— (II)

From (I) & (II) we have $(Ha)^{-1} = a^{-1}H$

×——————×

(17) Let H, K be two sbgps of a finite group G. Prove that for any $g \in G$ $\quad g(H \cap K) = gH \cap gK$

Sol We know that

$$H \cap K \subset H \text{ and } H \cap K \subset K$$

$\Rightarrow g(H \cap K) \subset gH$ and $g(H \cap K) \subset gK \quad \forall g \in G.$

$\Rightarrow g(H \cap K) \subset gH \cap gK$ ——— (I)

Again Let $y \in gH \cap gK$

$\Rightarrow y \in gH$ and $y \in gK$

$\Rightarrow g(g^{-1}y) \in gH$ and $g(g^{-1}y) \in gK \quad \because gg^{-1} = e$

$\Rightarrow g^{-1}y \in H$ and $g^{-1}y \in K$

$\Rightarrow g^{-1}y \in H \cap K$

pre Multiply by g $\quad \Rightarrow gg^{-1}y \in g(H \cap K) \Rightarrow y \in g(H \cap K)$

Hence $gH \cap gK \subset g(H \cap K)$ ——— (II)

From (I) & (II) $\quad g(H \cap K) = gH \cap gK$

×——————×

Cyclic Groups:

A group $(G,\cdot)$ is said to be cyclic if every element of G can be written in power of a single element (say) 'a' of G

$$G = \{a^K ; K \in Z\}$$

such an element $a \in G$ is called Generator of G.

OR.

A group G is called Cyclic if and only if there exist an element $a \in G$ s.t $G = \{a^K / K \in Z\}$
i.e every element of G is written in the integral power of a. The element a is called generator of G and we write

$$G = [a] = gp(a)$$

Examples (i) $G = \{1, \omega, \omega^2\} = gp(\omega)$

$\omega^0 = 1$
$\omega^1 = \omega$
$\omega^2 = \omega^2$

Thus the group G is generated by ω.

(ii) $G = \{1, -1, i, -i\}$

$(i)^1 = i$, $(i)^2 = -1$, $(i)^3 = i^2 \cdot i = -i$ $(i)^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

Thus G is generated by i

$(-i)^1 = -i$, $(-i)^2 = -1$, $(-i)^3 = i$, $(-i)^4 = (-)^4 (i)^4 = 1$

Thus G is generated by $(-i)$

Thus we see that a cyclic gp can have more than one generator.

If $(G,+)$ is a group then

$G = gp(a) = \{Ka \,/\, K \in Z\}$ $\quad z = 0, \pm 1, \pm 2 ----$

(Z, +) is a group.

$$Z = \left\{ \begin{matrix} 0,1,2,3 ---- \\ -1,-2,-3, ---- \end{matrix} \right\} = \left\{ \begin{matrix} 0(1), 1(1), 2(1) ----- \\ -1(1), -2(1), -3(1) --- \end{matrix} \right\}$$

we see that every element of Z is a multiple of 1 i,e Z is generated by 1.

$$Z = [1] = gp(1)$$

Also $Z = \left\{ \begin{matrix} 0(-1), 1(-1), 2(-1) ---- \\ -1(-1), -2(-1), -3(-1) ---- \end{matrix} \right\}$

$$\Rightarrow Z = \{-1\} = gp(-1)$$

Group of integers under addition is a cyclic group. 1 & -1 are its generators

<u>Note</u> If the order of a is finite i,e if there exist least +ive integer 'n' s.t $a^n = e$ then G is said to be <u>Finite</u> cyclic group of order 'n'.

$$G = \langle a : a^n = e \rangle$$ (G is a cyclic group of order n generated by 'a'.)

(order of generator = order of cyclic group)

If the order of a infinite i,e if there does not exist least +ive integer n, s.t $a^n = e$ then G is said to be <u>Infinite</u> Cyclic group.

(45) $\boxed{2\cdot 2-16}$

Theorem Every Cyclic group is abelian

Let $x, y \in gp(a)$

then $x = a^m$ & $y = a^n$ where $m, n \in Z$

$\therefore xy = a^m \cdot a^n = a^{m+n}$

$= a^{n+m} = a^n \cdot a^m = yx$

$xy = yx$ Hence abelian

Theorem

Let G be any group. Let $a \in G$ have order n.

Then for any integer K $\boxed{o(a)=4, a^4 = e \text{ Also for } a^{12} = e \text{ iff } 4 \text{ divides } 12 \quad a^{13} \neq e \because 4 \text{ does not divide } 13}$

$a^K = e$ if and only if $K = qn$ or n divides K or $\frac{K}{n}$, $n < K$

where q is an integer.

Proof Suppose that n is the order of a i.e $a^n = e$

and for some integer 'K'; $a^K = e$

By division algorithm, there are integers q & r

s.t $K = nq + r$, $0 \leq r < n$ —— (1)

So that $e = a^K = a^{nq+r} = a^{nq} a^r$

$= (a^n)^q a^r = (e)^q a^r = e a^r = a^r$

$\therefore e = a^r$

$$\begin{array}{r} 5 \\ 3\overline{)16} \\ 15 \\ \hline 1 \end{array}$$

$16 = 3 \times 5 + 1$

$K = nq + r$

Since order of a is n, so that n is the smallest integer for which $a^n = e$, So $r = 0$ ($\because r < n$)

$\therefore K = nq$ from (1) proved

M·M·R (46) $\boxed{2\cdot 2-17}$

Conversely suppose $K = nq$

then $a^K = a^{nq} = (a^n)^q = (e)^q = e$

$a^K = e$ proved.

×——×

v.Imp. Every subgroup of a cyclic group is cyclic.

Pf Let $G = [a]$ be a cyclic group generated by a. (Then every element of G is a power of 'a'.)

Let H be a nontrivial subgp of G. Let K be the least positive integer such that $a^K \in H$

(We have to show that every element of H can be written in power of a^K $\because a^K$ is smallest member of H)

For this let $a^m \in H$

$\because$ H is sbgp of G so elements of H are in the powers of a.

If we divide m by K then there are integers q & r s.t

$$m = qK + r \qquad 0 \le r < K \quad (1)$$

$$a^m = a^{qK+r}$$

$$= a^{qK} a^r$$

$$a^{-(qK)} a^m = a^{-(qK)} a^{qK} a^r$$

premultiply by $a^{-(qK)}$

$$\left[(a^K)^q\right]^{-1} a^m = e \cdot a^r$$

$$\left[(a^K)^q\right]^{-1} a^m = a^r$$

$\therefore a^r \in H$

$5\overline{)16}$ 3, 15, 1

$16 = 3\times 5 + 1$

$m = qK + r$

$r < K$

$1 < 5$

$a, b \in H \Rightarrow ab^{-1} \in H$

$a^m, a^K \in H,\ a^m (a^K)^q \in H$

$\because a^m, (a^K)^q \in H \Rightarrow a^m \left[(a^K)^q\right]^{-1} \in H$

$\because$ H is sbgp

$\because a^r = \left[(a^K)^q\right]^{-1} a^m \in H$

$\boxed{2\cdot 2-18}$

But K is the least +ive integer s.t, $a^k \in H$.
& here $a^r \in H$ & $r < K$ so it is a
contradiction unless $r=0$

$\therefore$ from ① $m = qK \Rightarrow a^m = a^{qK} = (a^k)^q$

$a^m = (a^K)^q$ i.e all the elements of H are in the power of a^K $\because a^m \in H$

$\therefore a^K$ is generator of H

Hence H is cyclic

x ——— x

Th Let G be a cyclic group of order n generated by a. Then for each positive divisor 'd' of n, there is a unique subgroup (of G) of order d.

Proof Let $G=[a]$ be a cyclic group of order n

$\Rightarrow a^n = e \longrightarrow ①$

Let d be a positive divisor of n. then

$\exists$ an integer q s.t $\frac{n}{d} = q$

$n = dq \longrightarrow ②$

Take $b = a^q$

$b^d = a^{dq} = a^n = e$ $\longrightarrow$ using ② $\longrightarrow$ using ①

$b^d = e \quad \therefore o(b) = d$

$d < n \because d$ is divisor of n
So $O(H) < O(G)$
H is sbgp.
$\because$ sbgp of cyclic sbgp is cyclic

Hence $H=[b]$ be a cyclic sbgp of G and order of H is d.

(4B) $\boxed{2 \cdot 2 - 19}$

Now we prove the uniqueness of H.

Let K be another subgp of G. Order of K is 'd' then K is cyclic. $\because$ Sbgp of Cyclic Group is cyclic

K is generated by $c = a^p$

where p is least +ive integer s.t $a^p \in K$

$\because O(K) = d$

$\therefore (a^p)^d = a^{pd} = c^d = e$

$\because a^{pd} = e$ & $a^n = e$ using ①

$\Rightarrow pd = n$ —— ③

from ② $qd = n$

from ② & ③ $pd = qd$

so $p = q$

So $c = a^p = a^q = b$

Hence $K = [c] = [b] = H$

$K = H$ Hence H is unique.

Example 18 Let G be a cyclic group of order 12 generated by a. Then the elements of G are

$G = gp(a) = \{a, a^2, a^3, a^4, \text{------}\ a^{12} = e\}$

Order of G is 12 and +ive divisors of 12 are 1, 2, 3, 4, 6, 12

$\therefore$ Sbgps of G have orders 1, 2, 3, 4, 6, 12 $\left\{\begin{array}{l}\because \text{by Theorem 2.21} \\ \text{for each divisor } d \text{ of } 12 \\ G \text{ has a sbgp of order } d.\end{array}\right.$

Sbgp of order $1 = \{a^{12} = e\}$ $\quad \frac{n}{d} = \frac{12}{1} = q$

" " " $2 = [a^6] = \{a^6, a^{12} = e\}$ $\quad \frac{12}{2} = 6 = q$

" " " $3 = [a^4] = \{a^4, a^8, a^{12} = e\}$ $\quad \frac{12}{3} = 4 = q$

" " " $4 = [a^3] = \{a^3, a^6, a^9, a^{12} = e\}$

" " " $6 = [a^2]$ $\{a^2, a^4, a^6, a^8, a^{10}, a^{12} = e\}$

Sbgp of order $12 = [a]$

Cosets

Let H be a subgp of a group G and $a \in G$, then the set $aH = \{ah : h \in H\}$ is called Left Coset of H in G determined by 'a'.

Similarly $Ha = \{ha : h \in H\}$ is called Right Coset of H in G determined by 'a'.

The above def is for a group under multiplication.

If G is a group under addition i.e $(G, +)$

then $H+a = \{h+a : h \in H\}$ is called Right Coset of H in G determined by 'a'.

$a+H = \{a+h : h \in H\}$ is called Left Coset of H in G determined by a.

Note 1) $a \in G$ and $a \notin H$

but if $a \in H$ then $Ha = H$.

2) H itself is both a left Coset and a right coset in G determined by e as

$$eH = H = He$$

3; In an abelian group the left & right cosets of H coincides.

Example Let $G = \{e, a, a^2, a^3\}$ & $H = \{e, a^2\}$

$He = H$ $\qquad \boxed{a^4 = e}$

$Ha = \{e, a^2\}a = \{ea, a^3\} = \{a, a^3\}$

$Ha^2 = \{e, a^2\}a^2 = \{ea^2, a^4\} = \{a^2, e\} = H$

$Ha^3 = \{e, a^2\}a^3 = \{ea^3, a^5\} = \{a^3, a\} = Ha$

Hence $\{H, Ha\}$ is the set of Right Cosets of H in G determined by a.

 $\boxed{2\cdot 2-21}$

Example 19

$G=\{\bar 0,\bar 1,\bar 2,\bar 3,\bar 4,\bar 5\}$ be group of residue classes under $+$ modulo 6.

$H=\{\bar 0,\bar 2,\bar 4\}$ is a subgp of G

$\bar 0+H=\bar 0+\{\bar 0,\bar 2,\bar 4\}=\{\bar 0,\bar 2,\bar 4\}=H$

$\bar 1+H=\bar 1+\{\bar 0,\bar 2,\bar 4\}=\{\bar 1,\bar 3,\bar 5\}$

$\bar 2+H=\bar 2+\{\bar 0,\bar 2,\bar 4\}=\{\bar 2,\bar 4,\bar 0\}$

$\bar 3+H=\bar 3+\{\bar 0,\bar 2,\bar 4\}=\{\bar 3,\bar 5,\bar 1\}$

$\bar 4+H=\bar 4+\{\bar 0,\bar 2,\bar 4\}=\{\bar 4,\bar 0,\bar 2\}$

$\bar 5+H=\bar 5+\{\bar 0,\bar 2,\bar 4\}=\{\bar 5,\bar 1,\bar 3\}$

$+$	$\bar 0$	$\bar 1$	$\bar 2$	$\bar 3$	$\bar 4$	$\bar 5$
$\bar 0$	$\bar 0$	$\bar 1$	$\bar 2$	$\bar 3$	$\bar 4$	$\bar 5$
$\bar 1$	$\bar 1$	$\bar 2$	$\bar 3$	$\bar 4$	$\bar 5$	$\bar 0$
$\bar 2$	$\bar 2$	$\bar 3$	$\bar 4$	$\bar 5$	$\bar 0$	$\bar 1$
$\bar 3$	$\bar 3$	$\bar 4$	$\bar 5$	$\bar 0$	$\bar 1$	$\bar 2$
$\bar 4$	$\bar 4$	$\bar 5$	$\bar 0$	$\bar 1$	$\bar 2$	$\bar 3$
$\bar 5$	$\bar 5$	$\bar 0$	$\bar 1$	$\bar 2$	$\bar 3$	$\bar 4$

$\bar 0+H=\bar 2+H=\bar 4+H$

& $\bar 1+H=\bar 3+H=\bar 5+H$

$\therefore$ The left Cosets of H in G are only two and these are $\bar 0+H=\{\bar 0,\bar 2,\bar 4\}$ & $\bar 1+H=\{\bar 1,\bar 3,\bar 5\}$.

Example 20

$G=\{\bar 1,\bar 3,\bar 5,\bar 7\}$ be group of residue classes under '$\cdot$' modulo 8

$H_1=\{1,3\}$ $H_2=\{1,5\}$

$H_3=\{1,7\}$

are proper subgps of G.

$\cdot$	$\bar 1$	$\bar 3$	$\bar 5$	$\bar 7$
$\bar 1$	$\bar 1$	$\bar 3$	$\bar 5$	$\bar 7$
$\bar 3$	$\bar 3$	$\bar 1$	$\bar 7$	$\bar 5$
$\bar 5$	$\bar 5$	$\bar 7$	$\bar 1$	$\bar 3$
$\bar 7$	$\bar 7$	$\bar 5$	$\bar 3$	$\bar 1$

The left Coset of H_1 in G.

$\bar 1\cdot H_1=\bar 1\cdot\{\bar 1,\bar 3\}=\{\bar 1,\bar 3\}=H_1$

$\bar 3\cdot H_1=\bar 3\cdot\{\bar 1,\bar 3\}=\{\bar 3,\bar 1\}=H_1$

$\bar 5\cdot H_1=5\cdot\{\bar 1,\bar 3\}=\{\bar 5,\bar 7\}$

$7\cdot H_1=7\cdot\{\bar 1,\bar 3\}=\{\bar 7,\ \}$

Hence the set of left cosets of H_1 in G is $\{\bar 1\cdot H_1,\bar 5 H_1\}$

<u>Partition of a set A</u> we mean a collection of subsets $\{A_i : i\in I\}$ of A s.t

$$\cup\{A_i : i\in I\} = \boxed{A} \quad \& \quad A_i \cap A_j = \boxed{\phi} \quad \text{where } i,j\in I \ \& \ i\neq j$$

(Extra)

<u>Example</u> $G=\{e,\omega,\omega^2,\omega^3,\omega^4,\omega^5\}$ $\omega^6=e$

$H=\{e,\omega^2,\omega^4\}$ is a sbgp of G

Right Cosets

$He=\{e,\omega^2,\omega^4\}e=\{e,\omega^2,\omega^4\}$

$H\omega=\{e,\omega^2,\omega^4\}\omega=\{\omega,\omega^3,\omega^5\}$

$H\omega^2=\{e,\omega^2,\omega^4\}\omega^2=\{\omega^2,\omega^4,\omega^6\}=\{\omega^2,\omega^4,e\}$

$H\omega^3=\{e,\omega^2,\omega^4\}\omega^3=\{\omega^3,\omega^5,\omega^7\}=\{\omega^3,\omega^5,\omega\}$

$H\omega^4=\{e,\omega^2,\omega^4\}\omega^4=\{\omega^4,\omega^6,\omega^8\}=\{\omega^4,e,\omega^2\}$

$H\omega^5=\{e,\omega^2,\omega^4\}\omega^5=\{\omega^5,\omega^7,\omega^9\}=\{\omega^5,\omega,\omega^3\}$

Now similarly left Cosets

$eH=\{e,\omega^2,\omega^4\}$ $\omega H=\{\omega,\omega^3,\omega^5\}$

$\omega^2H=\{\omega^2,\omega^4,e\}$ $\omega^3H=\{\omega^3,\omega^5,\omega\}$

$\omega^4H=\{\omega^4,e,\omega^2\}$ $\omega^5H=\{\omega^5,\omega,\omega^3\}$

From above we see that

$eH=He$ $\omega H=H\omega$ $\omega^2H=H\omega^2$

$\omega^3H=H\omega^3$ $\omega^4H=H\omega^4$ $\omega^5H=H\omega^5$

The no of Right Coset of H in G is equal to the no of Left Coset of H in G, i.e. 2; He & $H\omega$

$He\cap H\omega=\phi$ —— (I)

$He\cup H\omega=G$ —— (II)

From (I) & (II) the no of Right (or Left) Coset of H in G is a partition of G.

$\boxed{2 \cdot 2 - 23}$

مختلف

i.e $\exists$ at least one element common {we are taking distinct not disjoint}

مکمل مختلف

So $x \in Ha \cap Hb$

$\Rightarrow x \in Ha$ & $x \in Hb$

Now $x \in Ha \Rightarrow x = h_1 a$
$x \in Hb \Rightarrow x = h_2 b$ for some $h_1, h_2 \in H$

So $h_1 a = h_2 b$

$\Rightarrow h_1^{-1} h_1 a = h_1^{-1} h_2 b$

$a = h_1^{-1} h_2 b$ —— (3)

Let $y \in Ha \Rightarrow y = h_3 a$ $\quad h_3 \in H$.
$= h_3 h_1^{-1} h_2 b$ using (3)
$= h' b \in Hb$ $\quad h' = h_3 h_1^{-1} h_2 \in H$ $\because H$ is a subgp
$y \in Hb$

So $Ha \subset Hb$ —— (4) $\quad$ ($\because y \in Ha$ and now $y \in Hb$ so $Ha \subset Hb$)

Similarly we can prove

$Hb \subset Ha$ —— (5)

from (4) & (5) $Ha = Hb$ which is a contradiction $\because Ha$ & Hb were distinct.

Hence $Ha \cap Hb = \phi$ $\boxed{\text{proved}}$

Thus $\{Ha : a \in G, h \in H\}$ defines a partition of G.

Def The number of distinct left (or right) cosets of a sbgp H of a group G is called the <u>Index of H in G</u> and is denoted by $[G:H]$

<u>Example 21</u> Find all the distinct left cosets of

$E = \{0, \pm 2, \pm 4, \ldots\} = \{2n : n \in Z\}$

in $Z = \{0, \pm 1, \pm 2, \ldots\}$

The only distinct left cosets of E in Z are $0+E$ & $1+E$

$2+E = 4+E \ldots = E$
$3+E = 5+E \ldots = 1+E$

$0+E = \{0+2n : n \in Z\} = \{0, \pm 2, \pm 4, \ldots\}$

$1+E = \{1+2n : n \in Z\} = \{\pm 1, \pm 3, \pm 5, \ldots\}$

$(0+E) \cup (1+E) = Z$

$(0+E) \cap (1+E) = \varphi$

$\therefore$ Index of E in Z is 2.

$1+2n \quad n \in 0, \pm 1, \pm 2 \ldots$
$1+2(0) = 1$
$1+2(1) = 3$
$1+2(-1) = -1$
$1+2(2) = 5$
$1+2(-2) = -3$
and so on

<u>Lagrange Theorem:-</u>

The order and index of a sbgp of a finite group divide the order of the group.

$\{$ i.e $\frac{|G|}{|H|}$

Proof Since G is finite so the set of right cosets of H in G is also finite. Let this set is

$\{Ha_1, Ha_2, Ha_3, \ldots Ha_n\}$

Since the set of right cosets of H in G is a partition of G

$\because$ Th 2·24

$\boxed{2\cdot 2-25}$

So $G = Ha_1 \cup Ha_2 \cup Ha_3 \cdots Ha_n = \bigcup_{i=1}^{n} Ha_i$

& $Ha_i \cap Ha_j = \varphi$ for $i \neq j$ $i,j = 1,2 \cdots n$

$\therefore |G| = |Ha_1| + |Ha_2| + |Ha_3| + \cdots |Ha_n|$ — (1) $\because$ cosets are disjoint

Now we find order of Ha_i i.e $|Ha_i| = ?$

So we define a mapping $\psi : H \longrightarrow Ha_i$

by $\psi(h) = ha_i$ $\forall\, h \in H$. $a_i \in G$

which is clearly onto $\because$ each element of Ha_i is the image of some element of H. So ψ is onto.

Let $\psi(h_1) = \psi(h_2)$ $h_1, h_2 \in H$

implies $h_1 a_i = h_2 a_i$

by cancellation law $h_1 = h_2$ $\Rightarrow \psi$ is one-one

$\because \psi$ is onto & one-one so it is bijective

$\therefore |H| = |Ha_i|$

& in (1) $|G| = |H| + |H| + |H| + \cdots |H|$

$= n|H|$

$n = \dfrac{|G|}{|H|}$ i.e order of sbgp H divides order of group G.

n is the index of H in G. {no of distinct cosets of H in G is called index of H in G

$|H| = \dfrac{|G|}{n}$ i.e index of H divides order of G.

$\boxed{2.2-26}$ 56-A

2001

(12) Let H and K be two subgroups of a group G, whose orders are relatively prime. Prove that

$$H \cap K = \{e\}$$

Sol Let $|H| = m$ and $|K| = n$ where $(m, n) = 1$

Now $H \cap K$ is subgp of G. ($\because$ Intersection of subgps is a subgp)

Also $H \cap K \subset H$ and $H \cap K \subset K$

which implies $H \cap K$ is a subgp of H and of K.

By Lagrange Theorem "order of subgp divides order of a group"

$\therefore$ $|H \cap K|$ divides H and $|H \cap K|$ divides $|K|$

or $|H \cap K|$ is a common divisor of m and n.

$\Rightarrow |H \cap K| = 1$ $\quad\because$ m & n are relatively prime only common divisor in 'm' & 'n' is 1

Hence $H \cap K = \{e\}$

x ——— x

Corollary (2·27) The order of an element of a finite group divides the order of the group.

Proof Let G be a group of finite order n.

Let $a \in G$ & order of a is m i,e {To prove 'm' divides 'n'}

$a^m = e$ m is least +ive integer

Now $a, a^2, a^3, ----- a^{m-1}, a^m = e$ are all distinct and form a sbgp of G. The order of this sbgp is m.

Now by Lagrange. Th "the order of sbgp divides the order of the group".

$\therefore$ m divides n i,e $\frac{n}{m}$

x————————x

Corollary (2·28) A group G whose order is a prime number is necessarily cyclic.

Proof Suppose that G is a group of prime order 'p'

i,e $|G| = P$, Let $a (\neq e) \in G$

Also Let H be a cyclic sbgp generated by 'a', order of H is 'K'. So $a^{K} = e$.

By Lagrange Th K divides P i,e $\frac{P}{K}$

$\because$ P is prime so either $K=1$ or $K=P$ { $a^K = e$ $K \neq 1$ $\because a \neq e$ so $K = P$ $\because$ P is prime

Hence $H = G$

& since H is cyclic so G is cyclic.

$\boxed{2\cdot 2 - 28}$

Q1 We know $G = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ be the group of residue classes under multiplication modulo 8. {Example 10}

Also commutative law holds in it i,e $a.b = b.a \ \forall a, b \in G$

$3\cdot 5 = 5\cdot 3$

$7 = 7$

So G is abelian

$\cdot$	$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{7}$
$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{7}$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{7}$	$\bar{5}$
$\bar{5}$	$\bar{5}$	$\bar{7}$	$\bar{1}$	$\bar{3}$
$\bar{7}$	$\bar{7}$	$\bar{5}$	$\bar{3}$	$\bar{1}$

Now G is not cyclic because every element of G can not be written in the power of one and the same element.

e.g $3^2 = 1$, $5^2 = 1$, $7^2 = 1$

Q2 A gp G of order 47 can not have proper sbgps because 47 is a prime no, so the only divisors of 47 are 1 & 47. By Lagrange Theorem the order of a sbgp of a finite group divides the order of a group. So sbgps of group G are only of order 1 & 47 i,e identity & group itself, which are not proper sbgps of G.

So G has no proper sbgps.

Q3 Let G be a group of order 89

(i) Since 12 does not divide 89, so G has no sbgp of order 12

(ii) Since 16 does not divide 89, so G has no sbgp of order 16

(iii) Since 24 does not divide 89, so G has no sbgp of order 24.

Q4 Let $G = [a]$ be an infinite cyclic group generated by a. it implies order of a does not exist. ($\because$ G is infinite cyclic)

G is infinite so $a \neq a^{-1}$

$\because$ if $a = a^{-1} \Rightarrow a^2 = e \Rightarrow o(a) = 2$ contradiction so $a \neq a^{-1}$.

For each $x \in G$ $x = a^n$ $\because$ G is infinite cyclic (i.e generated by a)

Also $x = (a^{-1})^{-n} = a^n$ (i.e generated by a^{-1})

$\therefore$ G has two distinct generators a & a^{-1}.

Now we show that G has exactly two distinct generators.

Let b is a generator of G s.t $b \neq a$ & $b \neq a^{-1}$

$\therefore$ $b \in gp[a]$ & $b \in gp[a^{-1}]$

So $\exists$ +ive integers m & n s.t

$b = a^m$ & $b = (a^{-1})^n = a^{-n}$

Thus $a^{m} = a^{-n}$

$\Rightarrow a^{m}.a^{n} = a^{-n}.a^{n} = e$ post $\times$ by a^{n}

$a^{m+n} = e$ a contradiction

$o(a) = m+n$ $\because$ order of a does not exist being G an infinite cyclic group.

Hence G has <u>exactly</u> two distinct generators

<u>Q5</u> Is $(Q,+)$ a cyclic group.

The group $(Q,+)$ of rational numbers under '+' is not cyclic. Because

If Q is cyclic group generated by 'a', then $a = \frac{p}{q}$, $q \neq 0$ and each element of Q must be expressed in the form $\{a, 2a, 3a, \text{------}\}$ i.e $na, n \in Z$

Now particularly $\frac{1}{2}a$ is a rational no

So must be written as na

$\Rightarrow na = \frac{1}{2}a$ By R. Cancellation Law

$n = \frac{1}{2} \notin Z$ (n must $\in Z$)

$\therefore$ There is no such 'n' for which

$na = \frac{1}{2}a \in Z$

Hence $(Q,+)$ is <u>not</u> cyclic.

<u>Note</u> $(Q,+)$ is abelian group but not cyclic (as proved)

Another example for Q1.

(56) $\boxed{2\cdot 2-31}$

Let G be a cyclic group of order 24 generated by 'a'

i.e $O(a) = 24 \Rightarrow a^{24} = e$ 24 is the least the integer for

i) $O(e) = 1 \quad \because e^1 = e$

$a^{24} = a^{48} = a^{72} = a^{96} \cdots = e$

ii) $O(a^9) = ?$

$(a^9)^8 = a^{72} = (a^{24})^3 = (e)^3 = e$

So $O(a^9) = 8$

iii) $O(a^{10}) = ?$

$(a^{10})^{12} = a^{120} = (a^{24})^5 = (e)^5 = e$

So $O(a^{10}) = 12$

Q Find all sbgps of the cyclic gp of order 60 generated by a.

Let G be a cyclic group of order 60 generated by 'a'.

Hence $G = \{a, a^2, a^3, \cdots\cdots a^{60} = e\}$

We know (by Th 2.21) that for each divisor d of 60 G has a sbgp of order d.

Set of all +ve divisors of 60 are

$\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

- Sbgp of G of order 1 $= \langle a^{60} \rangle = e$

" " " " " 2 $= \langle a^{30} \rangle = \{a^{30}, a^{60}\} = \{a^{30}, e\}$

" " " " " 3 $= \langle a^{20} \rangle = \{a^{20}, a^{40}, a^{60}\} = \{a^{20}, a^{40}, e\}$

Subgrop of G of order 4 $= [a^{15}] = [a^{15}, a^{30}, a^{45}, a^{60}=e]$

" " " " " 5 $= [a^{12}] = [a^{12}, a^{24}, a^{36}, a^{48}, a^{60}=e]$

" " " " " 6 $= [a^{10}] = [a^{10}, a^{20}, a^{30}, a^{40}, a^{50}, a^{60}]$

" " " " " 10 $= [a^{6}] = [a^{6}, a^{12}, a^{18}, a^{24}, a^{30}, a^{36}, a^{42}, a^{48}, a^{54}, a^{60}]$

" " " " 12 $= [a^{5}] = [a^{5}, a^{10}, a^{15}, a^{20}, a^{25}, a^{30}, a^{35}, a^{40}, a^{45}, a^{50}, a^{55}, a^{60}]$

" " " " 15 $= [a^{4}] = [a^{4}, a^{8}, a^{12}, a^{16}, a^{20}, a^{24}, a^{28}, a^{32}, a^{36}, a^{40}, a^{44}, a^{48}, a^{52}, a^{56}, a^{60}=e]$

" " " " 20 $= [a^{3}] = [a^{3}, a^{6}, a^{9}, a^{12}, \ldots\ldots a^{60}=e]$

" " " " 30 $= [a^{2}] = [a^{2}, a^{4}, a^{6}, a^{8}, a^{10} \ldots\ldots a^{60}=e]$

" " " " 60 $= [a] = [a, a^{2}, a^{3}, a^{4}, \ldots\ldots a^{60}=e] = G$

8) Let G be a cyclic group of order 18 generated by a.

Hence $G = \{a, a^2, a^3, a^4 \ldots\ldots a^{18} = e\}$

The divisors of 18 are 1, 2, 3, 6, 9, 18

We know (by Th 2·21) that for each divisor 'd' of 18 G has a sbgp of order 'd'.

$\therefore$ Sbgps of G have orders 1, 2, 3, 6, 9, 18

Sbgp of order 1 $= \{a^{18}\} = e$

" " " 2 $= \{a^{9}\} = \{a^{9}, a^{18}=e\}$

" " " 3 $= \{a^{6}\} = \{a^{6}, a^{12}, a^{18}=e\}$

" " " 6 $= \{a^{3}\} = \{a^{3}, a^{6}, a^{9}, a^{12}, a^{15}, a^{18}\}$

" " " 9 $= \{a^{2}\} = \{a^{2}, a^{4}, a^{6}, a^{8}, a^{10}, a^{12}, a^{14}, a^{16}, a^{18}\}$

" " " 18 $= \{a\} = \{a, a^{2}, a^{3} \ldots a^{18}=e\} = G$

(62) $\boxed{2\cdot 2-33}$

6) The elements of $(\bar{Z}'_{13}, \cdot)$ under multiplication 18 modulo 13 are $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}, \bar{12}\}$

$(\bar{2})^1 = \bar{2} \in \bar{Z}'_{13}$ $\quad$ $(\bar{2})^2 = \bar{4} \in \bar{Z}'_{13}$

$(\bar{2})^3 = \bar{8} \in \bar{Z}'_{13}$ $\quad$ $(\bar{2})^4 = \bar{3} \in \bar{Z}'_{13}$ $\quad$ $13\overline{)16}(1$, 13, 3 Remaind

$(\bar{2})^5 = \bar{6} \in \bar{Z}'_{13}$ $\quad$ $(\bar{2})^6 = \bar{12} \in \bar{Z}'_{13}$

$(\bar{2})^7 = \bar{11} \in \bar{Z}'_{13}$ $\quad$ $(\bar{2})^8 = \bar{9} \in \bar{Z}'_{13}$ $\quad$ $13\overline{)128}(9$, 117, 11

$(\bar{2})^9 = \bar{5} \in \bar{Z}'_{13}$ $\quad$ $(\bar{2})^{10} = \bar{10} \in \bar{Z}'_{13}$ $\quad$ $13\overline{)256}(19$, 247, 9

$(\bar{2})^{11} = \bar{7} \in \bar{Z}'_{13}$ $\quad$ $(\bar{2})^{12} = 1 \in \bar{Z}'_{13}$

$\because$ Each element of Z'_{13} can be written in the power of $\bar{2}$ so $\bar{Z}'_{13}$ is a cyclic gp generated by $\bar{2}$.

(i) $H_1 = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}, \bar{9}, \bar{11}\}$

$\bar{3} \times \bar{7} = \bar{8} \notin H_1 \Rightarrow H_1$ is not sbgp under $\cdot$ modulo 13.

(ii) $H_2 = \{\bar{1}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}, \bar{12}\}$

$\bar{2} \times \bar{8} = \bar{3} \notin H_2 \Rightarrow H_2$ is not sbgp under $\cdot$ modulo 13.

iii) $H_3 = \{\bar{1}, \bar{6}, \bar{8}, \bar{10}\}$

$\bar{6} \times \bar{8} = \bar{9} \notin H_2 \Rightarrow H_2$ is not a sbgp under $\cdot$ modulo 13

iv) $H_4 = \{\bar{1}, \bar{3}, \bar{9}\}$ $\quad$ $\forall a, b \in H \Rightarrow ab^{-1} \in H$. (required cond for sbgp)

$\bar{1}\cdot(\bar{3})^{-1} = \bar{1}\cdot\bar{9} = \bar{9} \in H_4$ $\quad$ $\therefore H_4$ is sbgp of $(\bar{Z}'_{13}, \cdot)$

$\bar{1}\cdot(\bar{9})^{-1} = \bar{1}\cdot\bar{3} = \bar{3} \in H_4$

$\boxed{2\cdot 2-34}$ 63

$H_4 = \{\bar{1}, \bar{3}, \bar{9}\}$ $\bar{1}$ is identity

$\bar{1}\cdot(\bar{3})^{-1} = \bar{1}\cdot\bar{9} = \bar{9} \in H_4$ $\because \bar{3}\cdot\bar{9} = \bar{1}$

$\bar{1}\cdot(\bar{9})^{-1} = \bar{1}\cdot\bar{3} = \bar{3} \in H_4$ $\bar{1}\cdot\bar{1} = \bar{1}$

$\bar{1}\cdot(\bar{1})^{-1} = \bar{1}\cdot\bar{1} = 1 \in H$ $\forall a,b \in H \Rightarrow ab^{-1} \in H.$

Hence H_4 is subgroup of $(\bar{Z}'_{13}, \cdot)$

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(54)

Permutations

Let X be a non-empty set. A bijective function $f: X \to X$ is called a permutation on X. If X consists of 'n' elements then we write S_n for set of all permutations on X. $\{\forall x \in X,\ (x)f = \text{image of } x \text{ under } f\}$

Th The Set S_n of all permutations on a set X with n elements is a group under the operation of composition of permutations.

Proof To show $(S_n, \circ)$ is a group we have to prove all the axioms for a group.

Let $f, g \in S_n$ be two bijective fns $\quad f: X \to X$, $g: X \to X$

We define $x(f \circ g) = ((x)f)g$

i) Since composition (product) of two bijective fns is always bijective fn. So $f \circ g$ is bijective fn. $\therefore f \circ g \in S_n$. hence S_n is closed under '$\circ$'.

ii) Let f, g, h be three bijective fns on X & $x \in X$

$$(f \circ g) \circ h = f \circ (g \circ h)$$
$$(x)[(f \circ g) \circ h] = (x)[f \circ (g \circ h)]$$
$$[(x)(f \circ g)]h = ((x)f)(g \circ h)$$
$$[((x)f)g]h = (((x)f)g)h$$

Hence '$\circ$' is Associative

(iii) The fn $I: X \to X$ defined by $(x)I = x \ \forall x$ is the identity element of S_n

For any $f \in S_n$

$$(x)(f \circ I) = ((x)f)I = (x)f \qquad \because (x)I = x$$

$$(x)(f \circ I) = (x)f \Rightarrow f \circ I = f \quad \forall f \in S$$

$$(x)(I \circ f) = ((x)I)f = (x)f$$

$$(x)(I \circ f) = (x)f \Rightarrow I \circ f = f \quad \forall f \in S_n$$

Hence I is the identity element of S_n

iv) Inverse of a bijective fn is also a bijective fn. So for any bijective fn $f \in S_n$ $(f: X \to X)$ $\exists$ its inverse $f^{-1} \in S_n$, $(f^{-1}: X \to X)$

By def of f^{-1}, for any $x, y \in X$

$$(x)f = y \Rightarrow x = (y)f^{-1}$$

$$(x)(f \circ f^{-1}) = ((x)f)f^{-1} = (y)f^{-1} = x = (x)I$$

$$(x)(f \circ f^{-1}) = (x)I \Rightarrow f \circ f^{-1} = I$$

$$y(f^{-1} \circ f) = ((y)f^{-1})f = (x)f = y = (y)I$$

$$y(f^{-1} \circ f) = (y)I \Rightarrow f^{-1} \circ f = I$$

All axioms are satisfied

Hence $(S_n, \circ)$ is a group

Cycles

Def A Cyclic Permutation or A Cycle is a permutation of the form $\alpha = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_k \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}$

i.e image of a_1 under $\alpha = a_2$ $\quad \alpha = (a_1 a_2 a_3 a_4 \cdots a_k)$

" " a_2 " " $= a_3$

" " a_3 " " $= a_4$

⋮

image of a_k under $\alpha = a_1$ where k is length of cycle.

e.g $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ cycle of length 5

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix} = (1\ 3\ 5\ 4)$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (3\ 4)$

Note (i) Product of two cycles need not be a cycle.

$\alpha = (1,2,5)$ $\beta = (2\ 1\ 4\ 5\ 6)$ $\quad \alpha, \beta$ are cycles.

$\alpha \circ \beta = \alpha\beta = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 2 \end{pmatrix}$ Not Cycle

(ii) Product of two mutually disjoint cycles is commutative.

$\alpha = (1\ 2\ 3)$ $\beta = (4\ 5\ 6)$ $\quad \beta\alpha = \begin{pmatrix} 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 4 & 2 & 3 & 1 \end{pmatrix}$

$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$ $\quad \beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$

$\alpha\beta = \beta\alpha$

Th Every permutation of degree n can be written as a product of cyclic permutations acting on mutually disjoint sets. OR

Every element of S_n is a product of disjoint cycles.

$\boxed{2\cdot 3 - 4}$

Proof: Let α be a permutation of degree n (i.e. permutation of n objects namely elements of X) & $X=\{1,2,3 \dots n\}$. Suppose that α acts on an element a_1. Further Suppose that under the action of α, $a_1 \to a_2$, $a_2 \to a_3$, $a_3 \to a_4$ ..... $a_{K-1} \to a_K$, $a_K \to a_1$ ($\because n$ is finite $\therefore \exists$ a natural no K s.t $a_K \to a_1$)

Thus a part of the effect of α on elements of set X is cycle.

$\alpha_1 = (a_1 a_2 a_3 \dots a_K)$

e.g. $\left\{ \begin{array}{l} \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} \\ \alpha = (1\,8)(3\,6\,4)(5\,7) \end{array} \right.$

Now if $K=n$ then $\alpha=\alpha_1$ & since α_1 is cyclic so theorem is proved. But

if $K<n$ then $\exists$ a 'b_1' which is not in $(a_1, a_2, a_3 \dots a_K)$

Suppose that under the action of α, $b_1 \to b_2$, $b_2 \to b_3$, $b_3 \to b_4 \dots\dots b_p \to b_1$. Thus a part of the effect of α on elements of Set X is cycle, $\alpha_2 = (b_1\ b_2\ b_3 \dots b_p)$

So we have extracted two cycles from α.

If $K+P=n$, then $\alpha = \alpha_1\alpha_2$

But if $K+P<n$, then $\exists$ a 'c_1' which is not in α_1 & α_2. Again Suppose that under the action of α, $c_1 \to c_2$, $c_2 \to c_3 \dots\dots c_q \to c_1$. Thus a part of the effect of α on elements of Set X is cycle, $\alpha_3 = (c_1\ c_2\ c_3 \dots\dots c_q)$

Continuing in this way, this process of extracting a cycle must end after a finite no of steps because n is finite.

$\therefore \exists$ a natural no 'r' s.t $n = K+P+q+\dots r$

and a part of the effect of α is a cycle

$$\alpha_t = (U_1, U_2\ U_3 \dots\dots U_r) \quad \text{where } U_i \notin (\alpha_1\alpha_2\alpha_3\dots)$$

$$\therefore \alpha = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \dots\dots \alpha_t$$ where each $\alpha_1, \alpha_2 \dots \alpha_t$ acts on mutually disjoint subsets of X.

Transpositions

A cycle of length 2 is called a Transposition. eg $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$

<u>Note</u> 1) Every cyclic permutation can be expressed as a product of transpositions. $\alpha = (a_1 a_2 a_3 \cdots a_k)$ --- $\alpha' = (a_1 a_2)(a_1 a_3) \cdots (a_1 a_k)$

2) Every permutation of degree n can be expressed as a product of transpositions.

3) For a given permutation the number of transpositions is always even or odd.

4) Inverse of a transposition is the same transposition $\alpha = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $\alpha^{-1} = \begin{pmatrix} b & a \\ a & b \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \alpha$

<u>Even Permutation.</u>

A permutation α in S_n is said to be an even permutation if it can be written as a product of an even number of transpositions (Inverse of even permutation is even permutation) (& Inverse of odd permutation is odd permutation)

<u>Odd Permutation</u>

A permutation α in S_n is said to be an odd permutation if it can be written as a product of an odd number of transpositions.

<u>Identity Permutation is an even permutation</u>, because the number of transpositions in the decomposition of the identity permutation is zero which is an even integer.

<u>Every transposition is an odd permutation.</u>

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Th (69)
i) The product of two even permutations is an even permutation.
ii) The product of two odd permutations is an even permutation.
iii) The product of an even permutation and an odd permutation is an odd permutation.

Proof Let α_1 & α_2 be any two permutations of degree n. i.e $\alpha_1, \alpha_2 \in S_n$
then α_1 & α_2 can be expressed as a product of m_1 & m_2 transpositions. (by theorem)

The product $\alpha_1\alpha_2$ contains m_1+m_2-2K transpositions,
where $K=0$ or $K=$ a natural no.

{The term $2K$ occurs because of the possible cancellation (simplification) of pairs of transpositions}
{i.e K is no. of common transpositions in α_1 & α_2}
e.g α_1 contain (p,q), α_2 contain (q,p)
then $\alpha_1 \cdot \alpha_2 = (p,q)(q,p) = \begin{pmatrix} p & q \\ p & q \end{pmatrix}$ = Identity

Case 1 No. of transpositions in $\alpha_1 \cdot \alpha_2 = m_1+m_2-2K$
If α_1 & α_2 are even permutations
$\Rightarrow m_1$ & m_2 are even
$\Rightarrow m_1+m_2-2K$ is an even integer.
$\Rightarrow \alpha_1\alpha_2$ is an even permutation.

Even+Even = Even
4+4 = 8

Case 2 If α_1 & α_2 are odd permutations
$\Rightarrow m_1$ & m_2 are odd
$\Rightarrow m_1+m_2-2K$ is an even integer
$\Rightarrow \alpha_1\alpha_2$ is an even permutation

$2K$ is always even
odd+odd
3+3 = 6 even

Case 3 If α_1 is odd & α_2 is even permutations
$\Rightarrow m_1$ is odd & m_2 is even
$\Rightarrow m_1+m_2-2K$ is an odd integer
$\Rightarrow \alpha_1\alpha_2$ is an odd permutation

Even + odd odd.
4+5 = 9

×——————×

2·3—7

Th For $n \geq 2$ the number of even permutations in S_n is equal to the number of odd permutations in S_n.

Proof Let all even permutations in S_n be $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_K$ —— ①
and Let all odd permutations in S_n be $\beta_1, \beta_2, \beta_3 \cdots \beta_l$ —— ②
So that $K+l = n!$

Let h be a transposition, then

③ — $h\alpha_1, h\alpha_2, h\alpha_3$ —— $h\alpha_K$ are all odd permutations and
④ — $h\beta_1, h\beta_2, h\beta_3$ —— $h\beta_l$ are all even permutations

from ① & ④ (even) $l \leq K$ — ⑤
from ② & ③ (odd) $K \leq l$ — ⑥
from ⑤ & ⑥ $\therefore$ $K = l$

see Q11
e.g. $\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 6 & 5 & 3 \end{pmatrix}$
$\alpha_1 = (1\,2\,4\,6\,3)$
$\alpha_1 = (1\,2)(1\,4)(1\,6)(1\,3)$
α_1 is even
Let $h = (7\,8)$
$h\alpha_1 = (1\,2)(1\,4)(1\,6)(1\,3)(7\,8)$
$h\alpha_1$ is odd.

Since $K + l = n!$
$\therefore$ $K + K = n!$
$2K = n!$
$K = \frac{n!}{2}$ $\therefore$ Number of all even permutations $= \frac{n!}{2}$ $\because K = l$
OR Number of all odd permutations

Th Set A_n of all even permutations in S_n forms a subgp of S_n and Set B_n of all odd permutations in S_n does not form a subgp of S_n.

Proof Let $\alpha_1, \alpha_2 \in A_n$ then we prove $\alpha_1\alpha_2^{-1} \in A_n$
since the inverse of an even permutation is also even permutation
So α_2^{-1} is an even permutation.
Now Since the product of two even permutations is also an even permutation (by theorem)
$\therefore$ $\alpha_1\alpha_2^{-1}$ is an even permutation
$\Rightarrow \alpha_1\alpha_2^{-1} \in A_n$. Hence A_n is subgp of S_n

Now Let $\beta_1, \beta_2 \in B_n$ then we prove $\beta_1\beta_2^{-1} \notin B_n$
Since the inverse of an odd permutation is also odd permutation.
So β_2^{-1} is odd permutation
Also since the product of two odd permutations is not odd, but even. (by theorem)
$\therefore$ $\beta_1\beta_2^{-1}$ is an even permutation
$\Rightarrow \beta_1\beta_2^{-1} \notin B_n$ Hence B_n is not subgp of S_n

2-3-8

(71)

Order of a Permutation

Let α be a permutation in S_n. The order of α is the least positive integer 'm' such that $\boxed{\alpha^m = I}$ Identity permutation

e.g $\alpha = (1\,2\,3)$

$$\alpha^2 = (1\,2\,3)(1\,2\,3) = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$$

$$\alpha^3 = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = I \quad \therefore \text{order} = 3$$

The order of transposition is 2. $\begin{pmatrix}a&b\\b&a\end{pmatrix}\begin{pmatrix}a&b\\b&a\end{pmatrix} = \begin{pmatrix}a&b\\a&b\end{pmatrix} = I$

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Th The order of a cyclic permutation of length 'm' is 'm'

Proof Let $\alpha = (a_1\, a_2 \text{---------} a_m)$ be a cyclic permutation of length m. Then under action of α^2

$$\alpha^2 = \alpha\alpha = \begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_2 & a_3 & a_4 & \cdots & a_1\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_2 & a_3 & a_4 & \cdots & a_1\end{pmatrix} = \begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_3 & a_4 & a_5 & \cdots & a_2\end{pmatrix}$$

$$\alpha^3 = \alpha^2\alpha = \begin{pmatrix}a_1 & a_2 & \cdots & a_m\\ a_3 & a_4 & \cdots & a_2\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_2 & a_3 & a_4 & \cdots & a_1\end{pmatrix} = \begin{pmatrix}a_1 & a_2 & \cdots & a_m\\ a_4 & a_5 & \cdots & a_3\end{pmatrix}$$

Similarly continuing we get

$$\alpha^m = \begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_1 & a_2 & a_3 & \cdots & a_m\end{pmatrix} = I$$

and m is least +ive integer for which $\alpha^m = I$

Hence order of 'α' is 'm'. $\therefore$ A cycle of length m has order 'm'.

To find the order of a permutation 'α'

First decompose α as a product of cyclic permutations $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_K$ of lengths $m_1, m_2, m_3 \cdots m_K$ resp ignoring Identity permutation. Then take LCM of $m_1, m_2, \cdots m_K$: we get order of α.

$\boxed{2 \cdot 3 - 9}$

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Ex 2·3

① (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 4 & 5 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 9 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 8 & 3 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 9 & 8 & 3 & 2 & 8 & 6 \end{pmatrix}$

② $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ $\qquad \alpha \circ \alpha^{-1} = I$

$\alpha^{-1} = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$

③ $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 4 & 1 & 7 & 2 & 6 \end{pmatrix}$ $\qquad \alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = I$

$\alpha^{-1} = \begin{pmatrix} 3 & 5 & 4 & 1 & 7 & 2 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 1 & 3 & 2 & 7 & 5 \end{pmatrix}$

④ $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ $\qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

$f^2 \circ g = ?$ $\qquad f^2 = f \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

$f^2 \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$f^2 \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

$f^3 \circ g = ?$ $\quad f^3 = f^2 \circ f = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$

$$f^3 = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}$$

$$f^3 \circ g = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$$

$$f^4 = f^3 \circ f = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix} =$$

$$g^2 = g \circ g = \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix} = I$$

$$(f \circ g)^2 = (f \circ g)\circ(f \circ g) = \begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix} =$$

Q5 Find $f \circ g$, $g \circ f$, $g \circ h$, $h \circ g$, f^2 g^2 Do yourself

$f = \begin{pmatrix}1&2&3&4&5&6\\2&4&6&5&3&1\end{pmatrix} = (1\ 2\ 4\ 5\ 3\ 6)$

Length of $f = 6$. $\quad O(f) = 6$. i.e $f^6 =$ Identity

$g = \begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix} = (1\ 6\ 3\ 4)\ (2\ 5)$

LCM of 2 & 4 is 4 $\quad$ order 4 $\quad$ order 2

So $O(g) = 4$ $\quad$ i.e $g^4 =$ Identity

$h = \begin{pmatrix}1&2&3&4&5&6\\5&4&1&2&6&3\end{pmatrix} = (1\ 5\ 6\ 3)\ (2\ 4)$

order 4 $\quad$ order 2

LCM of 2 & 4 is 4 So $O(h) = 4$ $\quad$ i.e $h^4 =$ Identity

Q6 $a = \begin{pmatrix}1&2&3&4&5\\2&1&4&5&3\end{pmatrix} = (1\ 2)(3\ 4\ 5)$

order 2 $\quad$ order 3

LCM of 2 & 3 is 6

So $O(a) = 6$ $\quad$ i.e $a^6 =$ Identity

7) $I=(1)=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$ $\quad a=(1\,2\,3\,4)=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$

$b=(1\,4\,3\,2)=\begin{pmatrix}1&4&3&2\\4&3&2&1\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}$

$c=(1\,3)(2\,4)=\begin{pmatrix}1&3&2&4\\3&1&4&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$

$I\cdot I=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}=I$

$I\cdot a=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=a$

$I\cdot b=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}=b$

$I\cdot c=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}=c$

Multiplication Table

$\cdot$	I	a	b	c
I	I	a	b	c
a	a	c	I	b
b	b	I	c	a
c	c	b	a	I

$a\cdot I=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=a$

Similarly $b\cdot I=b \quad c\cdot I=c$

$a\cdot a=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}=c$

$a\cdot b=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}=I$

$a\cdot c=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}=b$

$b\cdot a=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}=I$

$c\cdot a=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}=b$

$b\cdot b=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}=c$

$b\cdot c=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=a$

$c\cdot b=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=a$

$c\cdot c=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}=I$

$\boxed{2\cdot 3-12}$

(8) Find all elements of cyclic group generated by a.

$$a=\begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}$$

$$a^2=a\cdot a=\begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6\\5&2&6&4&1&3\end{pmatrix}$$

$$a^3=a^2\cdot a=\begin{pmatrix}1&2&3&4&5&6\\5&2&6&4&1&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6\\6&4&1&2&3&5\end{pmatrix}$$

$$a^4=a^3\cdot a=\begin{pmatrix}1&2&3&4&5&6\\6&4&1&2&3&5\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6\\1&2&3&4&5&6\end{pmatrix}$$

$$=I$$

Hence distinct elements of the cyclic group generated by a are I, a, a^2, a^3

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(9) (i) $(1\ 2\ 4)(1\ 3\ 6\ 5\ 4)$

$$\begin{pmatrix}1&2&4\\2&4&1\end{pmatrix}\begin{pmatrix}1&3&6&5&4\\3&6&5&4&1\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6\\2&4&3&1&5&6\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\3&2&6&1&&5\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6\\2&1&6&3&4&5\end{pmatrix}$$

$$=(1\ 2)(3\ 6\ 5\ 4)$$

(ii) $(1\ 2\ 3\ 4)(2\ 5\ 3\ 4\ 1)$

$$=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}2&5&3&4&1\\5&3&4&1&2\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\2&3&4&1&5\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\2&5&4&1&3\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5\\5&4&1&2&3\end{pmatrix}=\begin{pmatrix}1&5&3\\5&3&1\end{pmatrix}\begin{pmatrix}2&4\\4&2\end{pmatrix}=(1\ 5\ 3)(2\ 4)$$

(iii) $(1\ 4)(2\ 3\ 5)(3\ 5)(4\ 5)$ $\quad\begin{pmatrix}1&4&2&3&5\\4&1&3&5&2\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}4&5\\5&4\end{pmatrix}$

$$=\left[\begin{pmatrix}1&2&3&4&5\\4&2&3&1&5\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\1&3&5&4&2\end{pmatrix}\right]\left[\begin{pmatrix}1&2&3&4&5\\1&2&5&4&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix}\right]$$

$$=\begin{pmatrix}1&2&3&4&5\\4&3&5&1&2\end{pmatrix}\cdot\begin{pmatrix}1&2&3&4&5\\1&2&4&5&3\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\5&4&3&1&2\end{pmatrix}=\begin{pmatrix}1&5&2&4&3\\5&2&4&1&3\end{pmatrix}$$

$\boxed{2\cdot 3-13}$

(10) (i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$ Express Permutation as product of disjoint cycles.

$= \begin{pmatrix} 1 & 8 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} 3 & 6 & 4 \\ 6 & 4 & 3 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 7 & 5 \end{pmatrix} = (18)(364)(57)$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 5 & 8 & 7 \\ 8 & 7 & 5 \end{pmatrix} = (134)(26)(587)$

(11) Express permutation as product of transpositions

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 1 & 6 & 5 & 3 & 8 & 9 & 7 \end{pmatrix}$

First we express permutation as product of disjoint cycles.

$= \begin{pmatrix} 1 & 2 & 4 & 6 & 3 \\ 2 & 4 & 6 & 3 & 1 \end{pmatrix}\begin{pmatrix} 7 & 8 & 9 \\ 8 & 9 & 7 \end{pmatrix}$

$= (12463)(789)$

$= (12)(14)(16)(13)(78)(79)$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 5 & 6 & 7 & 8 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1234) = (12)(13)(14)$

(13) $(123)(256)(4351)$ Permutations even or odd.

First we express permutations as product of disjoint cycles.

$\left[\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 4 & 6 & 2 \end{pmatrix}\right]\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 3 & 1 & 6 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 4 & 6 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 3 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 4 & 3 & 6 & 2 \end{pmatrix}$

$= \begin{pmatrix} 2 & 5 & 6 \\ 5 & 6 & 2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = (256)(34)$

Express disjoint cycles into Transposition $= (25)(26)(34)$ 3 Transposition So Odd.

$\boxed{2\cdot 3 - 14}$

(B)

$(147)(345)(87)(8345)$ Even or odd.

$$\begin{pmatrix}1&4&2&3&5&6&7&8\\4&1&2&3&5&6&1&8\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&4&5&3&6&7&8\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&3&4&5&6&8&7\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&4&5&8&6&7&3\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6&7&8\\5&2&4&7&3&6&1&8\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&4&5&8&6&3&7\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6&7&8\\8&2&5&3&4&6&1&7\end{pmatrix} = \begin{pmatrix}1&8&7\\8&7&1\end{pmatrix}\begin{pmatrix}3&5&4\\5&4&3\end{pmatrix}$$

$$=(187)(354) = (18)(17)(35)(34) \quad \because 4 \text{ So even}$$

$\because$ No of Transpositions are 4 so even.

(14) Find the order of each permutation.

(i) $a=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} = \begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}3&4\\4&3\end{pmatrix} = \underbrace{(12)}_{\text{order } 2}\underbrace{(34)}_{\text{order } 2}$

The LCM of the orders of the cycles on disjoint sets is 2.

So $O(a)=2$

(ii) $a=\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix} = \begin{pmatrix}1&4&3\\4&3&1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = \underbrace{(143)}_{\text{order } 3} \quad \therefore O(a)=3$

(iii) $a=\begin{pmatrix}1&2&3&4&5\\4&3&2&5&1\end{pmatrix} = \begin{pmatrix}1&4&5\\4&5&1\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix} = \underbrace{(145)}_{\text{order } 3}\underbrace{(23)}_{\text{order } 2}$

The LCM of the orders of the cycles on disjoint sets i.e $3 \& 2 = 6$

So $O(a)=6$

(v) $a=\begin{pmatrix}1&2&3&4&5&6\\2&3&1&5&6&4\end{pmatrix} = \underbrace{\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}}_{\text{order } 3}\underbrace{\begin{pmatrix}4&5&6\\5&6&4\end{pmatrix}}_{\text{order } 3}$

The LCM of the orders of the cycles on disjoint sets i.e $3 \& 3 = 3$

So $O(a)=3$

3-A-1

① Ch 03

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M.M MATRICES Matrics

Matrix:
Any rectangular arrangement of real or complex nos, subject to certain rules of operations is called a 'Matrix'. If there are 'm' rows & 'n' columns then order of matrix is $m \times n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

→ Rows
↓ Columns

Tabular Form

In abbreviated form, $A = [a_{ij}]_{m\times n}$ $\quad i = 1,2,3 \ldots m$, $j = 1,2,3 \ldots n$

Types Of Matrices:-

Square Matrix If $m = n$ i.e No of Rows = No of Columns

e.g $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$

Rectangular Matrix if $m \neq n$

Diagonal Matrix If all entries of a square matrix are zero except main diagonal entries.

e.g $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ $\quad a_{ij} = 0$ when $i \neq j$, $\quad a_{ij} \neq 0$ when $i = j$

Identity Matrix If all entries of a square matrix are zero except main diagonal entries which are '1'.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Zero or Null Matrix If all entries of a matrix are zero. $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Scalar Matrix The matrix obtained by multiplying a non-zero scalar to each of its entries is called Scalar Matrix.

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ $\quad KA = \begin{bmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \end{bmatrix}$

2

<u>Upper Triangular Matrix</u>:- A square matrix whose [illegible] below the main diagonal are all zero.

$$A = \begin{bmatrix} 1 & 7 & 9 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 7 & 5 \end{bmatrix}$$

Upper Triangular Matrix — Lower Triangular Matrix

<u>Lower Triangular Matrix</u>:- A square matrix whose elements above the main diagonal are all zero.

<u>Matrix Addition</u>: A & B are conformable for addition if

Order of A == Order of B

$$A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Matrix Multiplication</u> A & B are conformable for multiplication if No of Column of A = No of Rows of B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3} \qquad B = \begin{bmatrix} 7 & 8 & 4 \\ 9 & 0 & 5 \\ 1 & 2 & 6 \end{bmatrix}_{3\times3}$$

AB is conformable for multiplication $\therefore$ 2×3, 3×3 same AB

BA is not conformable for multiplication. 3×3, 2×3 → Not BA

$$AB = \begin{bmatrix} 1\times7+2\times9+3\times1 & 1\times8+2\times0+3\times2 & 1\times4+2\times5+3\times6 \\ 4\times7+5\times9+6\times1 & 4\times8+5\times0+6\times2 & 4\times4+5\times5+6\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 7+18+3 & 8+0+6 & 4+10+18 \\ 28+45+6 & 32+0+12 & 16+25+36 \end{bmatrix}$$

$$AB = \begin{bmatrix} 28 & 14 & 32 \\ 79 & 44 & 77 \end{bmatrix}_{2\times3}$$

Theorem – If the matrices A, B and C are conformable for the indicated sums and products then

i) $A(BC) = (AB)C$ Associative Law

ii) $A(B+C) = AB+AC$

iii) $(A+B)C = AC+BC$ } Distributive Law

iv) $K(AB) = (KA)B = A(KB)$

{We prove the theorems by showing that element in the ith row & jth col of LHS = the element in the ith row & jth col of R.H.S.}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \end{bmatrix}$$

3rd Col $\begin{bmatrix} \sum_{\lambda=1}^{3} a_{1\lambda} b_{\lambda 3} \\ \sum_{\lambda=1}^{3} a_{2\lambda} b_{\lambda 3} \end{bmatrix}$

Proof (i) $A(BC) = (AB)C$

Let $A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{n\times p}$

$C = [c_{ij}]_{p\times q}$

Order of $A = m\times n$

Order of $BC = n\times p_{p\times q} = n\times q$

Order of $A(BC) = m\times n_{n\times q} = \boxed{m\times q}$

Similarly

Order of $AB = m\times n_{n\times p} = m\times p$

Order of $C = p\times q$

Order of $(AB)C = m\times p_{p\times q} = \boxed{m\times q}$

Now ith row of A is $\{a_{i1}, a_{i2}, a_{i3} \ldots\ldots a_{in}\}$

& jth column of BC is $\begin{Bmatrix} \sum_{\lambda=1}^{p} b_{1\lambda} c_{\lambda j} \\ \sum_{\lambda=1}^{p} b_{2\lambda} c_{\lambda j} \\ \vdots \\ \sum_{\lambda=1}^{p} b_{n\lambda} c_{\lambda j} \end{Bmatrix}$

∴ the element in the ith row & jth col of $A(BC)$ is

$$a_{i1}\left(\sum_{\lambda=1}^{p} b_{1\lambda} c_{\lambda j}\right) + a_{i2}\left(\sum_{\lambda=1}^{p} b_{2\lambda} c_{\lambda j}\right) + \cdots + a_{in}\left(\sum_{\lambda=1}^{p} b_{n\lambda} c_{\lambda j}\right)$$

$$= \sum_{\mu=1}^{n} a_{i\mu}\left(\sum_{\lambda=1}^{p} b_{\mu\lambda} c_{\lambda j}\right)$$

$$= \sum_{\mu=1}^{n} \sum_{\lambda=1}^{p} a_{i\mu}\left(b_{\mu\lambda} c_{\lambda j}\right)$$

$$= \sum_{\mu=1}^{n} \sum_{\lambda=1}^{p} \left(a_{i\mu} b_{\mu\lambda}\right) c_{\lambda j} \quad \because \text{Associative Law holds in Real Nos.}$$

$$= \sum_{\mu=1}^{n} \left(a_{i\mu} b_{\mu\lambda}\right) \sum_{\lambda=1}^{p} c_{\lambda j}$$

$$= \sum_{\lambda=1}^{p} \left(\sum_{\mu=1}^{n} a_{i\mu} b_{\mu\lambda}\right) c_{\lambda j}$$

$$= \left(\sum_{\mu=1}^{n} a_{i\mu} b_{\mu 1}\right) c_{1j} + \left(\sum_{\mu=1}^{n} a_{i\mu} b_{\mu 2}\right) c_{2j} + \cdots \left(\sum_{\mu=1}^{n} a_{i\mu} b_{\mu p}\right) c_{pj}$$

= Element in the ith row & jth col of $(AB)C$.

Hence $A(BC) = (AB)C$

$$A(B+C) = AB + AC$$

Let $A = [a_{ij}]_{m\times n}$ $\quad B = [b_{ij}]_{n\times p}$ $\quad C = [c_{ij}]_{n\times p}$

order of $(B+C) = n\times p$

order of $A(B+C) = m\times n\; n \times p$
$= \boxed{m\times p}$

order of $AB = m\times n\; n\times p = m\times p$

order of $AC = m\times n\; n\times p = m\times p$

order of $AB + AC = \boxed{m\times p}$

order of $A(B+C)$ = order of $AB + AC$

Now
ith row of A is $\{a_{i1}, a_{i2}, \cdots a_{in}\}$

jth column of $(B+C)$ is $\begin{Bmatrix} b_{1j} + c_{1j} \\ b_{2j} + c_{2j} \\ \vdots \\ b_{nj} + c_{nj} \end{Bmatrix}$

The element in the ith row & jth column of $A(B+C)$ is

$$a_{i1}(b_{1j}+c_{1j})+a_{i2}(b_{2j}+c_{2j})+\cdots\cdots a_{in}(b_{nj}+c_{nj})$$

$$=\sum_{\lambda=1}^{n} a_{i\lambda}(b_{\lambda j}+c_{\lambda j})$$

$$=\sum_{\lambda=1}^{n}\left[a_{i\lambda}b_{\lambda j}+a_{i\lambda}c_{\lambda j}\right] \quad \because a, b, c \text{ are real nos}$$

$$=\sum_{\lambda=1}^{n} a_{i\lambda}b_{\lambda j}+\sum_{\lambda=1}^{n} a_{i\lambda}c_{\lambda j}$$

= element in the ith row & jth col of AB + element in ith row & jth col of AC

= element in the ith row & jth column of $(AB+AC)$

Thus $\underline{A(B+C)=AB+AC}$

(iii) $(A+B)C = AC+BC$

Let $A=[a_{ij}]_{m\times n}$ $B=[b_{ij}]_{m\times n}$ $C=[c_{ij}]_{n\times p}$

order of $A+B = m\times n$

order of $(A+B)C = m\times n\ n\times p$
$= \boxed{m\times p}$

order of $AC = m\times n\ n\times p = m\times p$

order of $BC = m\times n\ n\times p = m\times p$

order of $AB+BC = \boxed{m\times p}$

Now ith row of $(A+B)$ is $(a_{i1}+b_{i1},\ a_{i2}+b_{i2},\ a_{i3}+b_{i3}\cdots\cdots a_{in}+b_{in})$

jth column of C is $\begin{Bmatrix} c_{1j}\\ c_{2j}\\ \vdots\\ c_{nj}\end{Bmatrix}$

The element in the ith row & jth column of $(A+B)C$ is

$$(a_{i1}+b_{i1})c_{1j}+(a_{i2}+b_{i2})c_{2j}+\cdots\cdots+(a_{in}+b_{in})c_{nj}$$

$$=\sum_{\lambda=1}^{n}(a_{i\lambda}+b_{i\lambda})c_{\lambda j}$$

$$= \sum_{\lambda=1}^{n} \left\{ a_{i\lambda} c_{\lambda j} + b_{i\lambda} c_{\lambda j} \right\}$$

$$= \sum_{\lambda=1}^{n} a_{i\lambda} c_{\lambda j} + \sum_{\lambda=1}^{n} b_{i\lambda} c_{\lambda j}$$

= element in the ith row & jth column of AC + element in the ith row & jth column of BC

= element in the i^{th} row & j^{th} column of $AC+BC$.

Hence $(A+B)C = AC+BC$

(iv) $K(AB) = (KA)B = A(KB)$

Let. $A = [a_{ij}]_{m\times n}$ $\quad B = [b_{ij}]_{n\times p}$

order of $AB = m\times n \ n\times p = m\times p$ $\qquad$ order of $K(AB) = m\times p$

order of $(KA) = m\times n$ $\qquad$ order of $KB = n\times p$.

order of $A(KB) = m\times n \ n\times p = m\times p$

order of $K(AB) = m\times p$ = order of $K(AB)$ = order of $A(KB)$

Element in the ith row of A is $(a_{i1}, a_{i2}, \cdots a_{in})$

Element in the j^{th} column of B is $\begin{Bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{Bmatrix}$

Element in the i^{th} row & j^{th} column of AB $= \sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j}$

Element in the i^{th} row & j^{th} column of $K(AB) = K\left(\sum_{\lambda=1}^{n} a_{i\lambda} b_{\lambda j}\right)$

$$= \sum_{\lambda=1}^{n} (K a_{i\lambda}) b_{\lambda j}$$

= element in the ith row & jth column of $(KA)B$

$$= \sum_{\lambda=1}^{n} a_{i\lambda} (K b_{\lambda j})$$

which is element in the ith row & jth col of $A(KB)$

(i) $(A+B)^t = A^t + B^t$

$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$ then $A' = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}$

Proof Let $A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{m\times n}$

order of $A+B = m\times n$

order of $(A+B)^t = \boxed{n\times m}$

order of $A^t = n\times m$

order of $B^t = n\times m$

order of $A^t + B^t = \boxed{n\times m}$

LHS

Now the element in the i^{th} row & j^{th} column of $(A+B)^t$ = the element in the j^{th} row & i^{th} column of $(A+B)$

= the element in the j^{th} row & i^{th} col of A + the element in the j^{th} row & i^{th} col of B

$= a_{ji} + b_{ji}$ —— (1)

RHS Now the element in the i^{th} row & j^{th} column of $A^t + B^t$ = element in the i^{th} row & j^{th} col of A^t + element in the i^{th} row & j^{th} col of B^t

= element in the j^{th} row & i^{th} column of A + element in the j^{th} row & i^{th} column of B

$= a_{ji} + b_{ji}$ —— (2)

(1) = (2) Hence proved

(ii) $(A^t)^t = A$

Let $A = [a_{ij}]_{m\times n}$

order of $A^t = n\times m$

order of $(A^t)^t = \boxed{m\times n}$

order of $A = \boxed{m\times n}$

Now element in the i^{th} row & j^{th} column of $(A^t)^t$ = element in the j^{th} row & i^{th} column of A^t

= element in the i^{th} row & j^{th} column of A.

(iii) $(KA)^t = KA^t$

order of $KA = m\times n$
order of $KA^t = \boxed{n\times m}$
order of $(KA)^t = \boxed{n\times m}$

Let $A = [a_{ij}]_{m\times n}$

Now element in the i^{th} row & j^{th} column of $(KA)^t$ = element in the j^{th} row & i^{th} column of KA

= element in the i^{th} row & j^{th} column of KA^t

$(AB)^t = B^t A^t$

Let $A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{n\times p}$

order of $AB = m\times n_{\,n\times p} = m\times p$

order of $(AB)^t = \boxed{p\times m}$

order of $B^t = p\times n$
order of $A^t = n\times m$
order of $B^t A^t = p\times n_{\,n\times m}$
$= \boxed{p\times m}$

Now element in the i^{th} row and j^{th} col of $(AB)^t$ = element in the j^{th} row & i^{th} column of (AB)

$= \sum_{\lambda=1}^{n} a_{j\lambda} b_{\lambda i}$ —— (I)

RHS and the element in the i^{th} row & j^{th} col of $B^t A^t$ = Sum of the products of the corresponding elements of i^{th} row of B^t & j^{th} col of A^t

= Sum of the products of the corresponding elements of i^{th} col of B & j^{th} row of A

$= \sum_{\lambda=1}^{n} a_{j\lambda} b_{\lambda i}$ —— (II)

(I) = (II) proved

Ex3.1

(1) Let $A=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}$

(ii) $A-B=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}-\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}$

$=\begin{bmatrix}2+1 & -3-3 & -5-5\\ -1-1 & 4+3 & 5+5\\ 1+1 & 3-3 & -4-5\end{bmatrix}$

$A-B=\begin{bmatrix}3 & -6 & -10\\ -2 & 7 & 10\\ 2 & -6 & -9\end{bmatrix}$

(iii) $2A+3B=2\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}+3\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}$

$=\begin{bmatrix}4 & -6 & -10\\ -2 & 8 & 10\\ 2 & -6 & -8\end{bmatrix}+\begin{bmatrix}-3 & 3 & 15\\ 3 & -9 & -15\\ -3 & 9 & 15\end{bmatrix}$

$=\begin{bmatrix}4-3 & -6+3 & -10+15\\ -2+3 & 8-9 & 10-15\\ 2-3 & -6+9 & -8+15\end{bmatrix}$

$=\begin{bmatrix}1 & -3 & +5\\ 1 & -1 & -5\\ -1 & 3 & 7\end{bmatrix}$

(v) $AB=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}$

$=\begin{bmatrix}-2-3+5 & 6+9-15 & 10+15-25\\ 1+4-5 & -3-12+15 & -5+20+25\\ -1-3+4 & 3+9-12 & 5+15-20\end{bmatrix}$

$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

Q. (ii) $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= \begin{bmatrix} ax+hy+gz & hx+by+fz & gx+fy+cz \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x(ax+hy+gz)+y(hx+by+fz)+z(gx+fy+cz) \end{bmatrix}$$

(iv)

$$\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^3 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^2 \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8 & 2-6 \\ 4-12 & 8+9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9-16 & 18+12 \\ -8+68 & -16-51 \end{bmatrix} = \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$$

(vi)

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 1-1+0 & 1+0-1 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) $A = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$ $\quad B = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}$

$$AB = \begin{pmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\cdot\sin^2\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \cos\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{pmatrix}$$

$$= \cos(\theta-\phi)\begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{pmatrix}$$

(given) $\theta - \phi = K\frac{\pi}{2}$ where K is odd $K = 1, 3, 5, 7 \ldots$

$$\therefore AB = \cos\left(K\frac{\pi}{2}\right)\begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{pmatrix}$$

$$= 0\cdot\begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{pmatrix}$$

$$= O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \because \cos K\frac{\pi}{2} = 0. \text{ where } K \text{ is odd.}$$

(4)
$$\begin{pmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{pmatrix}\begin{pmatrix} \mu_1^2 & \mu_1\mu_2 & \mu_1\mu_3 \\ \mu_1\mu_2 & \mu_2^2 & \mu_2\mu_3 \\ \mu_1\mu_3 & \mu_2\mu_3 & \mu_3^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1^2\mu_1^2 + \lambda_1\lambda_2\mu_1\mu_2 + \lambda_1\lambda_3\mu_1\mu_3 & \lambda_1^2\mu_1\mu_2 + \lambda_1\lambda_2\mu_2^2 + \lambda_1\lambda_3\mu_2\mu_3 & \lambda_1^2\mu_1\mu_3 + \lambda_1\lambda_2\mu_2\mu_3 + \lambda_1\lambda_3\mu_3^2 \\ \lambda_1\lambda_2\mu_1^2 + \lambda_2^2\mu_1\mu_2 + \lambda_2\lambda_3\mu_1\mu_3 & \lambda_1\lambda_2\mu_1\mu_2 + \lambda_2^2\mu_2^2 + \lambda_2\lambda_3\mu_2\mu_3 & \lambda_1\lambda_2\mu_1\mu_3 + \lambda_2^2\mu_2\mu_3 + \lambda_2\lambda_3\mu_3^2 \\ \lambda_1\lambda_3\mu_1^2 + \lambda_2\lambda_3\mu_1\mu_2 + \lambda_3^2\mu_1\mu_3 & \lambda_1\lambda_3\mu_1\mu_2 + \lambda_2\lambda_3\mu_2^2 + \lambda_3^2\mu_2\mu_3 & \lambda_1\lambda_3\mu_1\mu_3 + \lambda_2\lambda_3\mu_2\mu_3 + \lambda_3^2\mu_3^2 \end{pmatrix}$$

$$= \begin{bmatrix} \lambda_1\mu_1(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_1\mu_2(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_1\mu_3(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) \\ \lambda_2\mu_1(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_2\mu_2(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_2\mu_3(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) \\ \lambda_3\mu_1(\lambda_1\mu_1+\mu_2\lambda_2+\lambda_3\mu_3) & \lambda_3\mu_2(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_3\mu_3(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) \end{bmatrix}$$

$= 0$ only when $\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3 = 0$
i.e when lines are $\perp$ to each other.

x————x

(5) —

$$(A+B)^2 = (A+B)(A+B)$$
$$= A^2+AB+BA+B^2$$
$$(A+B)^2 \neq A^2+2AB+B^2 \quad \because AB \neq BA$$

Also
$$(A-B)(A+B)$$
$$= A^2+AB-BA-B^2$$
$$\neq A^2-B^2 \quad \because AB \neq BA.$$

Equality will hold only if $AB = BA$ i.e A & B commute

x————x

(6) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$A^2 - 4A - 5I = 0$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 0$$

$$\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 0$$

$$\begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Some Special Types of Square Matrix.

(i) **Periodic Matrix**

A square matrix A is said to be a periodic matrix of period K, if $\boxed{A^{K+1} = A}$

(ii) **Idempotent Matrix** if $\boxed{A^2 = A}$

(iii) **Nilpotent Matrix** if $\boxed{A^P = 0}$ P is index

(iv) **Involutory Matrix** if $\boxed{A^2 = I}$

(v) **Symmetric Matrix** if $\boxed{A^t = A}$

(vi) **Skew Symmetric Matrix** if $\boxed{A^t = -A}$

(vii) **Hermition Matrix** if $\boxed{(\bar{A})^t = A}$

viii) **Skew Hermition Matrix** if $\boxed{(\bar{A})^t = -A}$

(IX) Let A be a matrix over C (complex numbers) If the elements of A are replaced by their complex conjugate, the resulting matrix is called Conjugate of A and denoted by $\bar{A}$

$$A = \begin{bmatrix} 2+3i & -i \\ 4 & 5+3i \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} 2-3i & i \\ 4 & 5-3i \end{bmatrix}$$

Some Results (i) If A is symmetric then $A+A^t$ is Symmetric & $A-A^t$ is Skew Symmetric

(ii) Every square matrix can be written as a sum of Symmetric & Skew Symmetric matrix

$$A = \frac{1}{2}(A+A^t) + \frac{1}{2}(A-A^t)$$

(iii) If A is square matrix over C then $A+(\bar{A})^t$ is Hermition & $A-(\bar{A})^t$ is Skew Hermition

(iv) Every square matrix can be written as a sum of Hermition & Skew Hermition Matrix

$$A = \frac{1}{2}\left[A+(\bar{A})^t\right] + \frac{1}{2}\left[A-(\bar{A})^t\right]$$

$\boxed{31-6}$

(7) $$A = \begin{pmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{pmatrix}$$

$A^{K+1} = A$.
K is period

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6-12 & -2-4 & -6-18+18 \\ -3-6+18 & 6+4 & 18+18-27 \\ 2-6 & -4 & -12+9 \end{bmatrix}$$

$$A^2 = \begin{pmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+18-12 & 10-12 & 30-54+18 \\ 9-30+18 & -18+20 & -54+90-27 \\ -4+12-6 & 8-8 & 24-36+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} = A$$

$A^3 = A$ or $A^{2+1} = A$. hence period = 2.

(8) $$A = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}$$

Nilpotent $\boxed{A^P = 0}$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-4 & -3-9+12 & -4-12+16 \\ -1-3+4 & 3+9-12 & 4+12-16 \\ 1+3-4 & -3-9+12 & -4-12+16 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Involutory $A^2 = I$

(9)

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ -12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^2 = I$ hence Involutory

×————×

(10)

21

$$A = \frac{1}{2}(2A)$$

$$= \frac{1}{2}[A + A]$$

$$= \frac{1}{2}[A + A^t - A^t + A]$$

$$A = \frac{1}{2}[A + A^t] + \frac{1}{2}[A - A^t]$$

$$A = B + C \quad \text{——} (1)$$

Now we prove that B is symmetric & C is skew symmetric

$$B = \frac{1}{2}[A + A^t]$$

$$B^t = \frac{1}{2}[A + A^t]^t = \frac{1}{2}[A^t + (A^t)^t] = \frac{1}{2}[A^t + A]$$

$$B^t = \frac{1}{2}[A + A^t] = B$$

So B is symmetric

Now $C = \frac{1}{2}[A - A^t]$

$$C^t = \frac{1}{2}[A - A^t]^t = \frac{1}{2}[A^t - (A^t)^t] = \frac{1}{2}[A^t - A]$$

$$C^t = -\frac{1}{2}[A - A^t] = -C$$

So C is skew symmetric

(11)

$$A = \frac{1}{2}(2A)$$
$$= \frac{1}{2}(A+A)$$
$$= \frac{1}{2}\left[A+(\bar{A})^t-(\bar{A})^t+A\right]$$
$$= \frac{1}{2}\left[A+(\bar{A})^t\right]+\frac{1}{2}\left[A-(\bar{A})^t\right]$$

$$A = B + D \quad \text{——} \quad (1)$$

We prove that B is Hermition & D is skew Hermition.

$$B = \frac{1}{2}\left[A+(\bar{A})^t\right]$$

$$\bar{B} = \frac{1}{2}\overline{\left[A+(\bar{A})^t\right]} = \frac{1}{2}\left[\bar{A}+(\bar{\bar{A}})^t\right] = \frac{1}{2}\left[\bar{A}+(A)^t\right]$$

$$(\bar{B})^t = \frac{1}{2}\left[\bar{A}+(A)^t\right]^t = \frac{1}{2}\left[(\bar{A})^t+(A^t)^t\right]$$
$$= \frac{1}{2}\left[(\bar{A})^t+A\right] = \frac{1}{2}\left[A+(\bar{A})^t\right] = B$$

$(\bar{B})^t = B$ Hence B is Hermition.

$$D = \frac{1}{2}\left[A-(\bar{A})^t\right]$$
$$\bar{D} = \frac{1}{2}\overline{\left[A-(\bar{A})^t\right]} = \frac{1}{2}\left[\bar{A}-(\bar{\bar{A}})^t\right] = \frac{1}{2}\left[\bar{A}-A^t\right]$$
$$(\bar{D})^t = \frac{1}{2}\left[\bar{A}-A^t\right]^t = \frac{1}{2}\left[(\bar{A})^t-(A^t)^t\right] = \frac{1}{2}\left[(\bar{A})^t-A\right]$$
$$= -\frac{1}{2}\left[A-(\bar{A})^t\right] = -D$$

$(\bar{D})^t = -D$ Hence D is skew Hermition.

(12) A & B are symmetric Matrices

So $A^t = A$ —(1) & $B^t = B$ —(2)

Suppose A & B commute i.e. $AB = BA$ & Prove $(AB)^t = AB$

$$(AB)^t = B^t A^t = BA = AB$$

Hence AB is symmetric using (1) & (2) supposition

Now suppose $(AB)^t = AB$

& Prove $AB = BA$

$(AB)^t = AB$

$B^tA^t = AB$

$BA = AB$ using ① & ②

Hence A & B commute.

(13) Let A is symmetric i.e $A^t = A$ — ①

We prove $B = P^tAP$ is symmetric

$$B^t = (P^tAP)^t \qquad (AB)^t = B^tA^t$$

$$= P^tA^t(P^t)^t$$

$$B^t = P^tAP \qquad \because A^t = A \;\&\; (P^t)^t = P.$$

$$B^t = B$$ Hence B is symmetric

Now let A is skew symmetric i.e $A^t = -A$ — ②

We prove $B = P^tAP$ is skew symmetric

$$B^t = (P^tAP)^t$$

$$= P^tA^t(P^t)^t$$

$$= P^t(-A)P \qquad \because \text{using ②}$$

$$= -P^tAP$$

$$B^t = -B$$

Hence B is skew symmetric.

(14) To Prove AA^t & A^tA are symmetric for a square matrix A

Let A be a square matrix

Now $(AA^t)^t = (A^t)^t (A^t)$ $\qquad (AB)^t = B^tA^t$

$= AA^t$

So AA^t is symmetric.

Now $(A^tA)^t = A^t(A^t)^t$

A^tA

So A^tA is also symmetric.

(15) $a_nA^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I$

Let A is symmetric So $A^t = A$.

First we prove that $(A^n)^t = (A^t)^n$. where n is +ve integer

Now $(A^n)^t = (A.A \dots n \text{ times})^t$

$= A^t.A^t \dots n \text{ times}$

$= (A^t)^n$

Now we prove $B = a_nA^n + a_{n-1}A^{n-1} + \dots a_1A + a_0I$ is symmetric

$B^t = [a_nA^n + a_{n-1}A^{n-1} + \dots a_1A + a_0I]^t$

$= (a_nA^n)^t + (a_{n-1}A^{n-1})^t + \dots (a_1A)^t + (a_0I)^t$

$= a_n(A^n)^t + a_{n-1}(A^{n-1})^t + \dots a_1(A)^t + a_0(I)^t$

$= a_n(A^t)^n + a_{n-1}(A^t)^{n-1} + \dots a_1A^t + a_0I$

$= a_nA^n + a_{n-1}(A)^{n-1} + \dots a_1A + a_0I$ $\qquad \because A^t = A.$

$B^t = B$

Hence $a_nA^n + a_{n-1}A^{n-1} + \dots a_1A + a_0I$ is Symmetric

3.1 – 11

(16) Let $B = A + (\bar{A})^t$

$\bar{B} = \overline{A + (\bar{A})^t} = \bar{A} + (\bar{\bar{A}})^t = \bar{A} + (A)^t$

$(\bar{B})^t = [\bar{A} + (A)^t]^t = [(\bar{A})^t + (A^t)^t] = [(\bar{A})^t + A]$

$(\bar{B})^t = [A + (\bar{A})^t] = B$

Hence $B = A + (\bar{A})^t$ is Hermition.

Let $C = A(\bar{A})^t$

$\bar{C} = \overline{A(\bar{A})^t} = \bar{A}(\bar{\bar{A}})^t = \bar{A}(A)^t$

$(\bar{C})^t = (\bar{A}A^t)^t = (A^t)^t(\bar{A})^t = (A)(\bar{A})^t$

$(\bar{C})^t = C$

Hence $C = A(\bar{A})^t$ is Hermition.

Let $D = (\bar{A})^t A$

$\bar{D} = \overline{(\bar{A})^t A} = (\bar{\bar{A}})^t \bar{A} = (A)^t \bar{A}$

$(\bar{D})^t = [(A)^t \bar{A}]^t = (\bar{A})^t (A^t)^t = (\bar{A})^t A$

$(\bar{D})^t = D$ Hence D is Hermition.

x———————x

Imp
(17) Let A be a square Matrix

Now $A = \frac{1}{2}(2A)$

$= \frac{1}{2}(A+A)$

$= \frac{1}{2}[A + (\bar{A})^t - (\bar{A})^t + A]$

$= \frac{1}{2}[A + (\bar{A})^t] + \frac{1}{2}[A - (\bar{A})^t]$

$A = \frac{1}{2}[A + (\bar{A})^t] + i \cdot \frac{1}{2i}[A - (\bar{A})^t]$

$A = P + iQ$

Now we prove that P & Q are Hermition

$$P = \frac{1}{2}(A + \bar{A}^t)$$

$$(\bar{P})^t = \frac{1}{2}\left[\overline{A + \bar{A}^t}\right]^t$$

$$= \frac{1}{2}\left[\bar{A} + (\bar{\bar{A}})^t\right]^t$$

$$= \frac{1}{2}\left[\bar{A} + (A)^t\right]^t = \frac{1}{2}\left[(\bar{A})^t + (A^t)^t\right]$$

$$= \frac{1}{2}\left[(\bar{A})^t + A\right] = \frac{1}{2}\left[A + (\bar{A})^t\right] = P$$

$(\bar{P})^t = P$ Hence P is Hermition.

Now $Q = \frac{1}{2i}\left(A - (\bar{A})^t\right)$

$$(\bar{Q}) = \overline{\frac{1}{2i}\left(A - \bar{A}^t\right)} = -\frac{1}{2i}\left[\bar{A} - (\bar{\bar{A}})^t\right]$$

$$= -\frac{1}{2i}\left(\bar{A} - A^t\right)$$

$$(\bar{Q})^t = -\frac{1}{2i}\left(\bar{A} - A^t\right)^t = -\frac{1}{2i}\left[(\bar{A})^t - (A^t)^t\right]$$

$$= -\frac{1}{2i}\left[(\bar{A})^t - A\right] = \frac{1}{2i}\left[A - (\bar{A})^t\right] = Q$$

$(\bar{Q})^t = Q$ Hence Q is Hermition.

Now we prove the uniqueness of P & Q.

Let $A = R + iS$ —① (where R & S are Hermition)

Now $(\bar{A})^t = \left[\overline{R + iS}\right]^t = \left[\bar{R} - i\bar{S}\right]^t = (\bar{R})^t - i(\bar{S})^t$

$(\bar{A})^t = R - iS$ —② ∵ R & S are Hermition.

Adding ① & ②

$$A + (\bar{A})^t = 2R$$

$$\frac{1}{2}\left[A + (\bar{A})^t\right] = R$$

$$\boxed{P = R}$$

Subtracting ② from ①

$$2iS = A - (\bar{A})^t$$

$$S = \frac{1}{2i}\left[A - (\bar{A})^t\right] = Q$$

Hence $P + iQ$ is unique.

(17) (Easy) $\bar{A} = A \because A$ is real

$A^t = A \because A$ is symmetric

$\bar{B} = B \because B$ is real

$B^t = -B \because B$ is skew symmetric

To Prove $A+iB$ is Hermition

Let $Q = A+iB$

$$\bar{Q} = \overline{A+iB}$$
$$= \bar{A} - i\bar{B}$$
$$\bar{Q} = A - iB \quad \because \bar{A}=A \ \& \ \bar{B}=B$$
$$(\bar{Q})^t = (A-iB)^t$$
$$= A^t - iB^t$$
$$= A - i(-B) \quad \because A^t = A, \ B^t = -B$$
$$= A + iB$$
$$(\bar{Q})^t = Q$$

2nd Method

Let P is a Hermition Matrix i.e. $(\bar{P})^t = P$

then P is a square Matrix over $\mathbb{C}$

$$\therefore P = [\alpha_{ij} + i\beta_{ij}] \quad \text{where } \alpha_{ij}, \beta_{ij} \in \mathbb{R}$$
$$= [\alpha_{ij}] + i[\beta_{ij}]$$
$$P = A + i B \quad \text{where } A = [\alpha_{ij}], \ B = [\beta_{ij}]$$
$$\bar{P} = \overline{A+iB} \quad \therefore \bar{A} = A \text{ over } R, \ \bar{B} = B$$
$$(\bar{P})^t = (\bar{A} - i\bar{B})^t$$
$$= (A - iB)^t$$
$$(\bar{P})^t = A^t - iB^t$$
$$P = A^t - iB^t \quad \because (\bar{P})^t = P$$
$$A+iB = A^t - iB^t \quad \because P = A+iB$$

Equating we get

$A = A^t$ $\therefore A$ is Symmetric

& $B = -B^t$ & B is skew Symmetric

Hence A is symmetric & Real

B is skew symmetric & Real.

(18) Let $A = [a_{ij} + i\alpha_{ij}]$.

$B = [b_{ij} + i\beta_{ij}]$ & $K = p + iq$

(i) To Prove $\bar{\bar{A}} = A$

$$\bar{A} = [a_{ij} - i\alpha_{ij}]$$
$$\bar{\bar{A}} = [a_{ij} + i\alpha_{ij}]$$

So $\bar{\bar{A}} = A$.

(ii) To Prove $\overline{KA} = \bar{K}\bar{A}$

$$KA = [(p+iq)(a_{ij} + i\alpha_{ij})]$$
$$\overline{KA} = [\overline{(p+iq)(a_{ij}+i\alpha_{ij})}] = [\overline{(p+iq)}\,\overline{(a_{ij}+i\alpha_{ij})}]$$
$$= [(p-iq)(a_{ij} - i\alpha_{ij})] = (p-iq)[a_{ij} - i\alpha_{ij}]$$
$$\overline{KA} = \bar{K} \cdot \bar{A}$$

(iii) $\overline{A+B} = \bar{A} + \bar{B}$ To Prove

Let $A+B = [(a_{ij} + i\alpha_{ij}) + (b_{ij} + i\beta_{ij})]$

$\overline{A+B} = \overline{[(a_{ij} + i\alpha_{ij}) + (b_{ij} + i\beta_{ij})]} = \overline{(a_{ij} + i\alpha_{ij})} + \overline{(b_{ij} + i\beta_{ij})}$

$= (a_{ij} - i\alpha_{ij}) + (b_{ij} - i\beta_{ij})$ $\quad \because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{A+B} = \bar{A} + \bar{B}$

(iv) $A = [a_{ij} + i\alpha_{ij}]$

then $\bar{A} = [\overline{a_{ij} + i\alpha_{ij}}] = [a_{ij} - i\alpha_{ij}]$

$(\bar{A})^t = [a_{ij} - i\alpha_{ij}]^t$

$(\bar{A})^t = [a_{ji} - i\alpha_{ji}]$ ——— ①

Now $A^t = [a_{ij} + i\alpha_{ij}]^t$

$A^t = [a_{ji} + i\alpha_{ji}]$

$\overline{(A^t)} = [\overline{a_{ji} + i\alpha_{ji}}]$

$\overline{(A^t)} = [a_{ji} - i\alpha_{ji}]$ ——— ②

$(\bar{A})^t = \overline{(A^t)}$

×———×

(19) If A is a matrix over the field of real numbers and $AA^t = 0$, show that $A = 0$

Sol Let $A = [a_{ij}]_{m\times n}$, $A^t = [a_{ji}]_{n\times m}$, $AA^t = 0$ (given)

Now i^{th} row of A is $[a_{i1} \; a_{i2} \; a_{i3} \cdots a_{in}]$

j^{th} column of A^t is $\begin{bmatrix} a_{j1} \\ a_{j2} \\ a_{j3} \\ \vdots \\ a_{jn} \end{bmatrix}$

$(i,j)^{th}$ element of $AA^t = a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} + \cdots a_{in}a_{jn}$

$(i,i)^{th}$ element of $AA^t = a_{i1}^2 + a_{i2}^2 + a_{i3}^2 + \cdots a_{in}^2$ $\because$ Put $i=j$ For Diagonal element

$0 = a_{i1}^2 + a_{i2}^2 + a_{i3}^2 + \cdots a_{in}^2$ $\because AA^t = 0 \Rightarrow (i,i)^{th}$ element $= 0$

$\therefore$ each $a_{i1} = 0$, $a_{i2} = 0$, $\cdots$ $a_{in} = 0$ $\quad i = 1,2,3,\cdots m$

Hence each element of $A = 0$

So $A = [a_{ij}] = 0$

Ruff

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots\cdots & a_{2n} \\ a_{m1} & a_{m2} & a_{m3} \cdots\cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$A^t = \begin{bmatrix} a_{11} & a_{21} \cdots\cdots & a_{m1} \\ a_{12} & a_{22} & a_{m2} \\ a_{13} & a_{23} & a_{m3} \\ \vdots & | & \vdots \\ a_{1n} & a_{2n} & a_{mn} \end{bmatrix}_{n\times m}$$

Let $i=1, j=2$

i^{th} row of $A = [a_{11} \; a_{12} \; a_{13} \cdots a_{1n}]$

j^{th} col of $A^t = \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \\ \vdots \\ a_{2n} \end{bmatrix}$

$(i,j)^{th}$ element of $AA^t = a_{11}a_{21} + a_{12}a_{22} + \cdots a_{1n}a_{2n}$

$\because a_{i1} = 0$, $a_{i2} = 0$, $\cdots a_{in} = 0$ $\quad i = 1,2,3,\cdots m$

i.e.

$a_{11} = 0 \quad a_{12} = 0 \cdots a_{1n} = 0$

$a_{21} = 0 \quad a_{22} = 0 \quad a_{2n} = 0$

$a_{31} = 0 \quad a_{32} = 0 \quad a_{3n} = 0$

$\vdots$

$a_{m1} = 0 \quad a_{m2} = 0 \quad a_{mn} \neq 0$

Note $AA^t = 0 \Rightarrow AA^t =$ zero Matrix

---

(20) If A is Matrix over C and $A(\bar{A})^t = 0$, show that $\bar{A} = 0 = A$

Sol Let $A = [\alpha_{ij} + i\beta_{ij}]_{m\times n}$, $\bar{A} = [\alpha_{ij} - i\beta_{ij}]$, $(\bar{A})^t = [\alpha_{ji} - i\beta_{ji}]_{n\times m}$

Now i^{th} row of A is $[\alpha_{i1} + i\beta_{i1} \quad \alpha_{i2} + i\beta_{i2} \cdots \alpha_{in} + i\beta_{in}]$

j^{th} col of $(\bar{A})^t$ is $\begin{bmatrix} \alpha_{j1} - i\beta_{j1} \\ \alpha_{j2} - i\beta_{j2} \\ \vdots \\ \alpha_{jn} - i\beta_{jn} \end{bmatrix}$

$(i,j)^{th}$ element of $A(\bar{A})^t = (\alpha_{i1} + i\beta_{i1})(\alpha_{j1} - i\beta_{j1}) + (\alpha_{i2} + i\beta_{i2})(\alpha_{j2} - i\beta_{j2}) + \cdots + (\alpha_{in} + i\beta_{in})(\alpha_{jn} - i\beta_{jn})$

$(i,i)^{th}$ element of $A(\bar{A})^t = (\alpha_{i1} + i\beta_{i1})(\alpha_{i1} - i\beta_{i1}) + (\alpha_{i2} + i\beta_{i2})(\alpha_{i2} - i\beta_{i2}) + \cdots + (\alpha_{in} + i\beta_{in})(\alpha_{in} - i\beta_{in})$ $\because$ putting $i=j$ for Diagonal elements

$0 = (\alpha_{i1}^2 + \beta_{i1}^2) + (\alpha_{i2}^2 + \beta_{i2}^2) + \cdots + (\alpha_{in}^2 + \beta_{in}^2)$ $\because A(\bar{A})^t = 0 \Rightarrow (i,i)^{th}$ element $= 0$

$\therefore$ each $\alpha_{i1} = 0 = \alpha_{i2} = \alpha_{i3} \cdots \alpha_{in}$ $\quad i = 1,2,3,\cdots m$

$\beta_{i1} = 0 = \beta_{i2} = \beta_{i3} \cdots \beta_{in}$ $\quad$ i.e. each

$\therefore$ each element of $A = 0$

$A = [0 + i0] = 0$, and $\bar{A} = [0 - i0] = 0$

Q23 $A = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad B = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$

Find AB using indicated partitioning.

Sol: Let $A_{11} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ and $B_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_{12} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\therefore AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \quad \text{—} (1)$$

Now

$$A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0+0 & 0+0 & 0+0 \\ 0+0 & 0+0 & 0+0 \end{bmatrix} + \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{11}B_{12} + A_{12}B_{22} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2+2 & 1-4 \\ -4+1 & -2-2 \end{bmatrix} + \begin{bmatrix} 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$$

$$A_{21}B_{11} + A_{22}B_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0+0 & 0+0 & 0+0 \\ 0+0 & 0+0 & 0+0 \\ 0+0 & 0+0 & 0+0 \end{bmatrix} + \begin{bmatrix} 0+0+0 & 0+0+0 & 1+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+1 & 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_{21}B_{12} + A_{22}B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} + \begin{bmatrix} 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from (1)
$$AB = \begin{bmatrix} 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Inverse of a square matrix

Let A be a square matrix of order n. If there exists a matrix B of same order n s.t

$$AB = I_n = BA$$

then matrix B is called the inverse of A i.e

$$B = A^{-1} \quad \text{so} \quad AA^{-1} = A^{-1}A = I$$

Theorem Inverse of a square matrix if its exists is unique.

Proof Let B & C are two inverses of a square matrix A.

So, by def $BA = AB = I$
& $CA = AC = I$

Now (Associative Law) $(BA)C = B(AC)$

$(I)C = B(I)$

$C = B$

Hence inverse is unique.

Q2: $A = \left[\begin{array}{cc:c} 1 & 2 & 1 \\ 3 & 4 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right]$ $\quad B = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 1 \\ 4 & 3 & 6 & 1 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]$

Let $A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $A_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ $A_{22} = [2]$

and $B_{11} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix}$, $B_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $B_{21} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ $B_{22} = [1]$

$$AB = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} \quad \text{---(1)}$$

Now

$A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 9 & 8 & 15 \\ 19 & 18 & 33 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 15 \\ 19 & 18 & 33 \end{bmatrix}$

$A_{11}B_{12} + A_{12}B_{22} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

$A_{21}B_{11} + A_{22}B_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix} + [2]\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

$A_{21}B_{12} + A_{22}B_{22} = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + [2][1] = [0] + [2] = [2]$

Hence $AB = \left[\begin{array}{ccc:c} 9 & 8 & 15 & 4 \\ 19 & 18 & 33 & 7 \\ \hdashline 0 & 0 & 0 & 2 \end{array}\right] = \begin{bmatrix} 9 & 8 & 15 & 4 \\ 19 & 18 & 33 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ Ans.

Theorem

Let A & B be non-singular matrices of same order then AB is non-singular & $(AB)^{-1} = B^{-1}A^{-1}$

Proof:- If we show that $(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$ then obviously AB is non-singular (∵ inverse exists) & that its inverse is $B^{-1}A^{-1}$.

Now $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ ∵ Associative Law

$= A(I)A^{-1}$

$= AA^{-1}$

$= I$ ——— ①

Also $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$ ∵ Associative Law.

$= B^{-1}(I)B$

$= B^{-1}B$

$= I$ ——— ②

So from ① & ② $(AB)^{-1} = B^{-1}A^{-1}$ & AB is non-singular

Ex 4.2

Q1. Inverse of a Diagonal Matrix is a Diagonal Matrix.

Let $D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & d_n \end{bmatrix}$ & $D^{-1} = \begin{bmatrix} d_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & d_2^{-1} & 0 & \cdots & 0 \\ 0 & 0 & d_3^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n^{-1} \end{bmatrix}$

If D^{-1} is inverse of D, then we should have

$DD^{-1} = I_n = D^{-1}D$. where I_n is Identity matrix of order n

(27)

Ex 3.2

Q1 Show that inverse of diagonal Matrix with all non-zero diagonal elements is a diagonal matrix.

Sol Let $A = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ be a diagonal Matrix of order 'n'

and

Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix}$ be the inverse of diagonal Matrix 'A'.

By def of Inverse $AB = I_n$

$$\therefore \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = I_n$$

$$\begin{bmatrix} d_1 b_{11} & d_1 b_{12} & \cdots & d_1 b_{1n} \\ d_2 b_{21} & d_2 b_{22} & \cdots & d_2 b_{2n} \\ \vdots & \vdots & & \vdots \\ d_n b_{n1} & d_n b_{n2} & \cdots & d_n b_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$\Rightarrow d_i b_{ij} = 0 \quad \Rightarrow b_{ij} = \frac{0}{d_i} = \boxed{0}$

for $i \neq j = 1, 2, \ldots, n$

&

$\Rightarrow d_i b_{ii} = 1 \quad \Rightarrow b_{ii} = \boxed{\frac{1}{d_i}}$

for $i = 1, 2, \ldots, n$

Hence.

$B = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$ is the diagonal Matrix

$\therefore$ Inverse of diagonal Matrix A is diagonal Matrix B.

Q2 Show that inverse of a scalar Matrix is a scalar Matrix.

Sol

Let $A = \begin{bmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & c \end{bmatrix}$ be a Scalar Matrix of order 'n'

and

$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix}$ be the inverse of Scalar Matrix A

By def of Inverse

$$AB = I_n$$

$$\therefore \begin{bmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & c \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = I_n$$

$$\begin{bmatrix} b_{11}c & b_{12}c & \cdots & b_{1n}c \\ b_{21}c & b_{22}c & \cdots & b_{2n}c \\ \vdots & \vdots & & \vdots \\ b_{n1}c & b_{n2}c & \cdots & b_{nn}c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$\Rightarrow b_{ij} c = 0 \quad \Rightarrow b_{ij} = \frac{0}{c} = \boxed{0}$

for $i \neq j = 1, 2, \ldots, n$

& $b_{ii} c = 1 \quad \Rightarrow b_{ii} = \boxed{\frac{1}{c}}$

for $i = 1, 2, \ldots, n$

Hence

$B = \begin{bmatrix} \frac{1}{c} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{c} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{c} \end{bmatrix}$

is the Scalar Matrix

$\therefore$ Inverse of Scalar Matrix is Scalar.

② (i) $(A^n)^{-1} = (A^{-1})^n$ To Prove

Sol: $(A^n)^{-1} = (A \cdot A \cdot A \cdots n \text{ times})^{-1}$

$= A^{-1}A^{-1}A^{-1} \cdots n \text{ times}$

$(A^n)^{-1} = (A^{-1})^n$ Proved.

(iv) $(\overline{A})^{-1} = \overline{(A^{-1})}$

$(\overline{A})(\overline{A^{-1}}) = \overline{(AA^{-1})} = \overline{I} = I$

Pre Multiply by $(\overline{A})^{-1}$

$(\overline{A})^{-1}(\overline{A})(\overline{A^{-1}}) = (\overline{A})^{-1} I$

$I(\overline{A^{-1}}) = (\overline{A})^{-1}$

$\overline{(A^{-1})} = (\overline{A})^{-1}$

(v) $(\overline{A^t})^{-1} = \overline{(A^{-1})^t}$

$(\overline{A^t})(\overline{(A^{-1})^t}) = \overline{A^t(A^{-1})^t} = \overline{(A^{-1}A)^t} = \overline{(I)^t} = \overline{I} = I$

Pre multiply both sides by $(\overline{A^t})^{-1}$

$(\overline{A^t})^{-1}(\overline{A^t})\overline{(A^{-1})^t} = (\overline{A^t})^{-1} I$

$\overline{(A^{-1})^t} = (\overline{A^t})^{-1}$ proved.

(iii) $(A^{-1})^t = (A^t)^{-1}$ To Prove

Sol $(A^{-1})^t(A^t) = (AA^{-1})^t = I^t = I$

$[(A^{-1})^t(A^t)](A^t)^{-1} = I(A^t)^{-1}$ Post x by $(A^t)^{-1}$

$(A^{-1})^t(I) = (A^t)^{-1}$

$(A^{-1})^t = (A^t)^{-1}$ proved

(ii) $(KA)^{-1} = K^{-1}A^{-1}$ To Prove

Sol $(KA)(K^{-1}A^{-1})$

$= (AK)(K^{-1}A^{-1})$

$= A(KK^{-1})A^{-1}$

$= A.I\,A^{-1}$

$= AA^{-1}$

$= I$

$(KA)^{-1}[(KA)(K^{-1}A^{-1})] = (KA)^{-1} I$ Pre Multiply by $(KA)^{-1}$

$(I)\,K^{-1}A^{-1} = (KA)^{-1}$

$K^{-1}A^{-1} = (KA)^{-1}$

Q4 Given that $A^2B = B^2A$ —① and $A^3 = B^3$ —②, A, B are diff.

Suppose A^2+B^2 is invertible

then $A-B = I(A-B)$ $\because I$ is identity Matrix

$= (A^2+B^2)^{-1}(A^2+B^2)(A-B)$ $\because I = (A^2+B^2)^{-1}(A^2+B^2)$

$= (A^2+B^2)^{-1}(A^3 - A^2B + B^2A - B^3)$

$= (A^2+B^2)^{-1}(0)$ using ① & ②

$A - B = 0$

$\Rightarrow A = B$, a contradiction as A & B are given diff matrices.

Hence our supposition is wrong and A^2+B^2 is not invertible.

Note

$(A^t)^{-1} = (A^{-1})^t$

$\overline{(A^{-1})} = (\overline{A})^{-1}$

$(A^{-1})^n = (A^n)^{-1}$

$\overline{(A^t)} = (\overline{A})^t$

$(A^t)^n = (A^n)^t$

$\overline{(A^n)} = (\overline{A})^n$

All these inverse, transpose, conjugate, exponent are interchangeable.

Q. If A is invertible & $AB = 0$ show that $B = 0$

Sol $\because$ A is invertible so A^{-1} exists

$$AB = 0 \quad \text{given}$$
$$A^{-1}AB = A^{-1}0$$
$$IB = 0$$
$$B = 0$$

Elementary Row Operations (ERO)

(i) These operations on a matrix are called ERO

(i) Interchange of any two rows denoted by R_{ij} i.e i^{th} & j^{th} rows are interchanged.

(ii) Multiplication of a row by a scalar denoted by $K(R_i)$ i.e each element of i^{th} row is multiplied by K.

(iii) Addition of a multiple of one row to any other row, denoted by $(K)R_i + R_j$

Row Equivalent Matrices:

A matrix B is said to be row equivalent to a matrix A of same order if B can be obtained from A by applying finite no of ERO on A.

$B \overset{R}{\sim} A$ $\quad$ B is row equivalent to A.

Elementary Matrix: The matrix E obtained by applying one ERO to I is called an Elementary Matrix.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K \end{bmatrix} \quad \text{by } KR_3$$

Echelon Form of a Matrix

A matrix is said to be in Echelon form if it has the following structure

i) All the zero rows are below the non-zero rows of A.

ii) The number of zeros occuring before the first nonzero entry in each non-zero row is greater than the number of zeros that appear before the first nonzero element in any preceeding row.

For example

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 0, & -1 & -2 \end{bmatrix} \qquad A = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A, B, C are not in Row Echelon Form.
D, E, F, G, H are in Reduced Echelon Form.
Rest are Echelon Form

$$D = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The <u>first nonzero</u> element in each row of an Echelon Matrix Form is called <u>PIVOT</u> element of that row. A column containg <u>PIVOT</u> is called <u>PIVOT Column</u>.

<u>Reduced Echelon Form</u> has the following structure

$$X = \begin{bmatrix} 0 & 5 & 2 & 1 \\ 0 & 0 & 9 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

X is in Echelon
Y is in Reduced Echelon

i) It is in Echelon Form

ii) Pivot element of each row is '1' i.e first nonzero element is '1' in each row.

iii) Every entry in the pivot column is zero except the pivot element '1'

<u>Note</u> It becomes easy to reduce a matrix in Echelon form if we obtain Pivot element to be '1', Also some authors include this condition in Echelon form instead of Reduced Echelon form.
Echelon form is different for a matrix depending upon the sequence of ERO applied but Reduced Echelon form is the same.

Th If a square matrix A is reduced to the identity matrix by a sequence of elementary row operations, the same sequence of operations performed on the identity matrix produces the inverse of A.

Proof

$$I \overset{R}{\sim} A$$

$$\Rightarrow I = (E_n E_{n-1} E_{n-2} \ldots E_2 E_1) A$$

where E_i are suitable elementary matrices.

Post Multiply both sides by A^{-1}

$$I A^{-1} = (E_n E_{n-1} E_{n-2} \ldots E_2 E_1) A A^{-1}$$

$$A^{-1} = (E_n E_{n-1} E_{n-2} \ldots E_2 E_1) I$$

x ———————— x

Q4 Find the inverse of

(i) $\begin{bmatrix} 1 & 0 & K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ over R

Matrix A. $\qquad\qquad$ I_3

$$\begin{bmatrix} 1 & 0 & K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reducing 'A' to Identity matrix by ERO

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{by } -KR_3 + R_1$$

I_3

Hence $\begin{bmatrix} 1 & 0 & -K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the inverse of 'A'.

(ii) $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ by R_{13}

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ by R_{23}

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ by $(-1)R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ by $(-1)R_3$

I_3

Hence $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ is inverse of matrix A.

×——————×

(iii) $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ $\qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by $(-1)R_1$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -6 \\ 0 & 6 & -7 \end{bmatrix}$ $\qquad \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$ by $-2R_1+R_2$, $-4R_1+R_3$

$$\underset{\sim}{R}\begin{bmatrix}1 & -2 & 3\\ 0 & 5 & -6\\ 0 & 1 & -1\end{bmatrix} \qquad \begin{bmatrix}-1 & 0 & 0\\ 2 & 1 & 0\\ 2 & -1 & 1\end{bmatrix} \qquad -R_2+R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1 & -2 & 3\\ 0 & 1 & -1\\ 0 & 5 & -6\end{bmatrix} \qquad \begin{bmatrix}-1 & 0 & 0\\ 2 & -1 & 1\\ 2 & 1 & 0\end{bmatrix} \qquad R_{23}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & -1\end{bmatrix} \qquad \begin{bmatrix}3 & -2 & 2\\ 2 & -1 & 1\\ -8 & 6 & -5\end{bmatrix} \qquad \begin{matrix}2R_2+R_1\\ -5R_2+R_3\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix} \qquad \begin{bmatrix}3 & -2 & 2\\ 2 & -1 & 1\\ 8 & -6 & 5\end{bmatrix} \qquad (-1)R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \begin{bmatrix}-5 & 4 & -3\\ 10 & -7 & 6\\ 8 & -6 & 5\end{bmatrix} \qquad \begin{matrix}-R_3+R_1\\ R_3+R_2\end{matrix}$$

$$\underset{\sim}{I_3} \qquad \text{Hence} \begin{bmatrix}-5 & 4 & -3\\ 10 & -7 & 6\\ 8 & -6 & 5\end{bmatrix} = A^{-1}$$

(iv)

$$A=\begin{bmatrix}2 & 1 & -1\\ 0 & 2 & 1\\ 5 & 2 & -3\end{bmatrix} \qquad \underset{\sim}{I_3}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}2 & 1 & -1\\ 0 & 2 & 1\\ 1 & 0 & -1\end{bmatrix} \qquad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix} \qquad -2R_1+R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & 1\\ 2 & 1 & -1\end{bmatrix} \qquad \begin{bmatrix}-2 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} \qquad R_{13}$$

$$\sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & +1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & -2 \end{bmatrix} \qquad -2R_1+R_3$$

$$\sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 & 1 \\ 5 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \qquad R_{23}$$

$$\sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 & 1 \\ 5 & 0 & -2 \\ -10 & 1 & 4 \end{bmatrix} \qquad -2R_2+R_3$$

$$\sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 & 1 \\ 5 & 0 & -2 \\ 10 & -1 & -4 \end{bmatrix} \qquad (-1)R_3$$

$$\sim_R \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3} \qquad \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} \qquad \begin{matrix} R_3+R_1 \\ -R_3+R_2 \end{matrix}$$

Hence $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} = A^{-1}$

(vi) $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\sim_R \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I_3} \qquad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_{13}$$

Hence $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A^{-1}$

v)
$$A=\begin{pmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} \qquad I_3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & i & 2 \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (-i)R_1$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & i & 2 \\ 0 & -2i & -2 \\ 0 & i & 3 \end{pmatrix} \qquad \begin{pmatrix} -i & 0 & 0 \\ 2i & 1 & 0 \\ -i & 0 & 1 \end{pmatrix} \qquad \begin{matrix} -2R_1+R_2 \\ R_1+R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & i & 2 \\ 0 & 1 & -i \\ 0 & i & 3 \end{pmatrix} \qquad \begin{pmatrix} -i & 0 & 0 \\ -1 & i/2 & 0 \\ -i & 0 & 1 \end{pmatrix} \qquad \left(\frac{i}{2}\right)R_2$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -i \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -1 & i/2 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \qquad \begin{matrix} -iR_2+R_1 \\ -iR_2+R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -1 & i/2 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad \frac{1}{2}R_3$$

$$\underset{\sim}{R}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ -1 & \frac{3i}{4} & \frac{i}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad \begin{matrix} -R_3+R_1 \\ iR_3+R_2 \end{matrix}$$

I_3

Hence $A^{-1}=\begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ -1 & \frac{3i}{4} & \frac{i}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

⑥ (i) $\begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

$\underset{R}{\sim} \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{pmatrix}$ $\quad \begin{matrix} -2R_1+R_2 \\ -3R_1+R_3 \end{matrix}$

$\underset{R}{\sim} \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 7 & -7 & 6 \end{pmatrix}$ $\quad \frac{1}{3}R_2$

$\underset{R}{\sim} \begin{pmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -4/3 & \frac{4}{3} \\ 0 & 0 & 7/3 & -\frac{10}{3} \end{pmatrix}$ $\quad \begin{matrix} 2R_2+R_1 \\ -7R_2+R_3 \end{matrix}$

$\underset{R}{\sim} \begin{pmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 1 & -10/7 \end{pmatrix}$ $\quad \frac{3}{7}R_3$

$\underset{R}{\sim} \begin{pmatrix} 1 & 0 & 0 & 15/7 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & -10/7 \end{pmatrix}$ $\quad \begin{matrix} \frac{4}{3}R_3+R_2 \\ -\frac{1}{3}R_3+R_1 \end{matrix}$ Required Reduced Echelon Form

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$A = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$

$\underset{R}{\sim} \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix}$ $\quad R_{12}$

$$\overset{R}{\sim}\begin{pmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix} \quad \frac{1}{2}R_1$$

$$\overset{R}{\sim}\begin{pmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix} \quad -2R_1+R_3$$

$$\overset{R}{\sim}\begin{pmatrix} 1 & 0 & -7/2 & \frac{5}{2} \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} -\frac{1}{2}R_2+R_1 \\ -2R_2+R_3 \end{matrix}$$

Required Reduced Echelon Form

(iii) $A=\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 3 \end{pmatrix}$

$$\overset{R}{\sim}\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 0 \end{pmatrix} \quad \begin{matrix} -2R_1+R_2 \\ -3R_3+R_3 \end{matrix}$$

$$\overset{R}{\sim}\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 5 & -12 & 0 \end{pmatrix} \quad \frac{1}{3}R_2$$

$$\overset{R}{\sim}\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & -2 & -5/3 \end{pmatrix} \quad -5R_2+R_3$$

$$\overset{R}{\sim}\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & 1 & 5/6 \end{pmatrix} \quad \frac{-1}{2}R_3$$

Echelon Form.

(iv)

$$A=\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 11 & -5 & 3\\ 2 & -5 & 3 & 1\\ 4 & 1 & 1 & 5\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 11 & -5 & 3\\ 0 & -11 & 5 & -3\\ 0 & -11 & 5 & -3\end{bmatrix}\qquad \begin{matrix}-2R_3+R_3\\ -4R_1+R_4\end{matrix}$$

$$\sim\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 1 & -\frac{5}{11} & \frac{3}{11}\\ 0 & -11 & 5 & -3\\ 0 & -11 & 5 & -3\end{bmatrix}\qquad \frac{1}{11}R_2$$

$$\sim\begin{bmatrix}1 & 0 & \frac{4}{11} & 13/11\\ 0 & 1 & -5/11 & 3/11\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\qquad \begin{matrix}-3R_2+R_1\\ 11R_2+R_3\\ 11R_2+R_4\end{matrix}$$

Required Reduced Echelon Form

Q6

$$A=\begin{bmatrix}1 & 2 & 1\\ 3 & 1 & 2\\ 0 & 1 & 2\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 1\\ 0 & -5 & -1\\ 0 & 1 & 2\end{bmatrix}\qquad -3R_1+R_2$$

$$\sim\begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 2\\ 0 & -5 & -1\end{bmatrix}\qquad R_{23}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{bmatrix} \qquad \begin{matrix} 5R_2+R_3 \\ -2R_2+R_1 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \tfrac{1}{9}R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{matrix} 3R_3+R_1 \\ -2R_3+R_2 \end{matrix} \qquad \text{Hence } A \overset{R}{\sim} I_3$$

Elementary Column Operations.

Equivalent Matrices:

A $m \times n$ matrix B is said to be equivalent to an $m \times n$ matrix A if B can be obtained from A by applying some elementary row & column operations on A. We denote it as $B \sim A$.

$$\text{If } B \sim A \text{ then } B = PAQ.$$

- where P and Q are non-singular matrices of order m & n.

'P' is obtained from I_m by some row operations applied on A to get B.

'Q' is obtained from I_n by the same column operations applied on A to get B.

Normal or Canonical form of a matrix.

A matrix is said to be in normal or canonical form when it has the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

where I_r is Identity matrix of order 'r' & the remaining submatrices are zero matrices. The following are normal matrices.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} I_3 & 0 \end{bmatrix} \qquad \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\& \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of a Matrix

No of non-zero rows in Echelon or Reduced Echelon Form of a matrix is called Rank of a Matrix.

No of non-zero columns in Echelon or reduced Echelon Form of a matrix is called Column Rank of a matrix.

The row rank & the column rank of a matrix are equal.

Q7
$$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & -16 \\ 0 & 9 \end{bmatrix} \quad \begin{matrix} -5R_1+R_3 \\ 2R_1+R_4 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & -16 \\ 0 & 9 \end{bmatrix} \quad -\tfrac{1}{2}R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} 16R_2+R_3 \\ -9R_2+R_4 \end{matrix}$$

No of non zero Rows = 2

So Rank = 2

(ii)

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 3 & -3 \\ 0 & 6 & -5 \end{bmatrix} \qquad \begin{matrix} -2R_1 + R_2 \\ +2R_1 + R_3 \\ +R_1 + R_4 \end{matrix}$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & 6 & -5 \end{bmatrix} \qquad -\frac{1}{3} R_2$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix} \qquad \begin{matrix} -3R_2 + R_3 \\ -6R_2 + R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 7 \end{bmatrix} \qquad \frac{1}{3} R_3$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad -7R_3 + R_4$$

No of non zero rows are 3 (Ist, 2nd, IIIrd)

So Rank A = 3

(iii) $\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$

$\underset{R}{\sim} \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{pmatrix} \quad \begin{matrix} -R_1 + R_2 \\ -2R_1 + R_3 \\ -3R_1 + R_4 \end{matrix}$

$\underset{R}{\sim} \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} 3R_2 + R_3 \\ R_2 + R_4 \end{matrix}$

Non zero rows are Ist & 2nd

Hence Rank = 2

(iv) $\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}$

$\underset{R}{\sim} \begin{pmatrix} 1 & 3 & 2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 1 & 4 & -1 & -1 \\ 0 & 1 & 1 & -4 & 5 \end{pmatrix} \quad \begin{matrix} -R_1 + R_2 \\ -R_1 + R_3 \\ -2R_1 + R_4 \end{matrix}$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & -2 & 4 \end{bmatrix} \quad \begin{matrix} -R_2+R_3 \\ -R_2+R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad 2R_3+R_4$$

3 Nonzero Rows 1st, 2nd, 3rd

Hence Rank = 3.

(8) Matrix A | Row Operation | Col operation.

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -4 & 1 \\ 0 & 3 & 2 \end{bmatrix}_{3\times3} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & -2 & -5 \\ 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -2R_1+R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} +1 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3+R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2+R_1$$

 3.2—19

$$\begin{array}{cccc} A & \text{Row Operations} & \text{Col operations} & \\ \text{9.(iii)}\ \begin{bmatrix} 1 & -1 & 3 \\ 2 & -4 & 1 \\ 0 & 3 & 2 \end{bmatrix}_{3\times 3} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} & \\ \begin{bmatrix} 1 & -1 & 3 \\ 0 & -2 & -5 \\ 0 & 3 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{by } -2R_1+R_2 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 3 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{by } 1C_1+C_2,\ -3C_1+C_3 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 3 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{by } R_3+R_2 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 11 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 6 & -3 & -2 \end{bmatrix} & \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & -3R_2+R_3 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 11 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 6 & -3 & -2 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} & 3C_2+C_3 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 6 & -3 & -2 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3/11 \\ 0 & 0 & 1/11 \end{bmatrix} & \frac{1}{11}C_3 \\ I_3 & P & Q & \end{array}$$

Hence Normal form of $\begin{bmatrix} 1 & -1 & 3 \\ 2 & -4 & 1 \\ 0 & 3 & 2 \end{bmatrix}$ is I_3

x———————x

3.2-20

Normal or Canonical form :-

A matrix is said to be in normal form when it has the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where I_r is identity matrix of order 'r' and remaining submatrices are zero matrices.

e.g $\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} I_3 & 0_{3\times1} \end{bmatrix}$

Q9 Reduce in Canonical form

i) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}_{2\times2}$

	For Row Operation	For col operations	
$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	by R_{12}
$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	by $-2R_1+R_2$
$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$	by $-2C_1+C_2$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}$	by $\frac{1}{-3}C_2$
I_2	P	Q	

Note P, Q are not unique. i.e may be diff, if we operate in a different manner.

(ii) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times3}$

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times3}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	by R_{12}
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	by $-2R_1+R_2$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	by $-C_1+C_2$, by $-C_1+C_3$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$-R_2$

by $-C_2+C_3$

$\left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} I_2 & 0_{2\times1} \end{bmatrix}$ $\quad P \quad Q$

 3.2-21

4(iv)

$$\underset{3\times 4}{\overset{A}{\begin{bmatrix}1&2&0&-1\\3&4&1&2\\-2&3&2&5\end{bmatrix}}}\quad \underset{3\times 3}{\overset{I_3}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}}\quad \underset{4\times 4}{\overset{I_4}{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}}$$

$$\begin{bmatrix}1&2&0&-1\\0&-2&1&5\\0&7&2&3\end{bmatrix}\quad\begin{bmatrix}1&0&0\\-3&1&0\\2&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad \text{by } -3R_1+R_2,\ 2R_1+R_3$$

$$\begin{bmatrix}1&0&0&0\\0&-2&1&5\\0&7&2&3\end{bmatrix}\quad\begin{bmatrix}1&0&0\\-3&1&0\\2&0&1\end{bmatrix}\quad\begin{bmatrix}1&-2&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad \text{by } -2C_1+C_2,\ 1C_1+C_4$$

$$\begin{bmatrix}1&0&0&0\\0&1&-2&5\\0&2&7&3\end{bmatrix}\quad\begin{bmatrix}1&0&0\\-3&1&0\\2&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&-2&1\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\quad \text{by } C_{23}$$

$$\begin{bmatrix}1&0&0&0\\0&1&-2&5\\0&0&11&-7\end{bmatrix}\quad\begin{bmatrix}1&0&0\\-3&1&0\\8&-2&1\end{bmatrix}\quad\begin{bmatrix}1&0&-2&1\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\quad \text{by } -2R_2+R_3$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&11&-7\end{bmatrix}\quad\begin{bmatrix}1&0&0\\-3&1&0\\8&-2&1\end{bmatrix}\quad\begin{bmatrix}1&0&-2&1\\0&0&1&0\\0&1&2&-5\\0&0&0&1\end{bmatrix}\quad 2C_2+C_3,\ -5C_2+C_4$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-7\end{bmatrix}\quad\begin{bmatrix}1&0&0\\-3&1&0\\8&-2&1\end{bmatrix}\quad\begin{bmatrix}1&0&-2/11&1\\0&0&1/11&0\\0&1&2/11&-5\\0&0&0&1\end{bmatrix}\quad \tfrac{1}{11}C_3$$

$$\underset{[I_3\ \ 0]}{\left[\begin{array}{ccc:c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]}\quad\underset{P}{\begin{bmatrix}1&0&0\\-3&1&0\\8&-2&1\end{bmatrix}}\quad\underset{Q}{\begin{bmatrix}1&0&&-14/11\\0&0&&7/11\\0&1&&-41/11\\0&0&&1\end{bmatrix}}\quad \text{by } 7C_3+C_4 \qquad \frac{14}{11}-5=\frac{-41}{11}$$

Hence the required normal form is $[I_3\ \ 0]$

x ——————————— x

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 3 & 1 \\ 2 & -2 & 1 & 0 & 2 \\ 1 & 1 & -1 & -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4×5

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 2 & -2 & -4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} -R_1+R_2 \\ -2R_1+R_3 \\ -R_1+R_4 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 2 & 2 & -4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} C_1+C_2 \\ -C_1+C_3 \\ -C_1+C_4 \\ -C_1+C_5 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 2 & 2 & -4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad C_{23}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -8 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} R_2+R_3 \\ -2R_2+R_4 \end{array}$$

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$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&2&-8&-2\end{pmatrix}\begin{pmatrix}1&0&0&0\\-1&1&0&0\\-3&1&1&0\\1&-2&0&1\end{pmatrix}\begin{pmatrix}1&-1&1&1&-1\\0&0&1&0&0\\0&1&0&-2&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \quad -2C_2+C_4$$

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&2&-8&-2\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\-1&1&0&0\\1&-2&0&1\\-3&1&1&0\end{pmatrix}\begin{pmatrix}1&-1&1&1&-1\\0&0&1&0&0\\0&1&0&-2&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \quad R_{34}$$

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&-4&-1\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\-1&1&0&0\\\frac{1}{2}&-1&0&\frac{1}{2}\\-3&1&1&0\end{pmatrix}\begin{pmatrix}1&-1&1&1&-1\\0&0&1&0&0\\0&1&0&-2&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \quad \frac{1}{2}R_3$$

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\-1&1&0&0\\\frac{1}{2}&-1&0&\frac{1}{2}\\-3&1&1&0\end{pmatrix}\begin{pmatrix}1&-1&1&5&0\\0&0&1&4&1\\0&1&0&-2&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \quad \begin{matrix}4C_3+C_4\\C_3+C_5\end{matrix}$$

$$\begin{pmatrix}I_3&0\\0&0\end{pmatrix} \qquad P \qquad Q$$

Systems of linear equations (Chapter No. 4)

Consider m linear eqs. in n unknowns

Mathematical Method

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m \end{array}\right\}$$

The above system of linear eqs. can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or $\quad AX = B$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is the matrix of Coefficients of Variables.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \& \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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Note Let the system $AX = B$ is given.

① If $B \neq 0$

Then this system is called non homogeneous system of linear eqs.

② If $B = 0$

then $AX = 0$

& then this system is called homogeneous system of linear eqs.

③ If the system $AX = B$ has soln. Then this system is called consistent.

④ If the system $AX = B$ has no soln. then this system is called inconsistent.

Type ① when no. of eqs. is equal to the no. of variables & system $AX = B$ is non homogeneous then unique soln. of the system exists if matrix A is non singular after applying row operations.

Type ② when no. of eqs. is not equal (may be equal) to the no. of variables & system is non homogenous Then the system has a soln. if

$$\text{rank } A = \text{rank } A_b$$

Type ③ A system of m homogeneous linear eqs. $AX = 0$ in n unknowns has a non trivial soln. if

$$\text{rank } A < n$$

where n is no. of Columns of A

Guassian elimination method

In this method we reduce the augmented matrix into echelon form. In this way, the value of last variable is calculated & then by backward substitution, the values of remaining unknowns can be calculated.

Guass Jordan method:

In this method, we reduce the augmented matrix into reduced echelon form by applying row operations. In this way, the values of all the unknowns is calculated directly without any backward substitution.

※ Exercise No. 4 ※

Solve the following systems of linear equations, the field of scalars being R:

Mathematical Method

Q1

$$2x_1 + x_3 = 1$$
$$2x_1 + 4x_2 - x_3 = -2$$
$$x_1 - 8x_2 - 3x_3 = 2$$

Sol: Given system is

$$2x_1 + x_3 = 1$$
$$2x_1 + 4x_2 - x_3 = -2$$
$$x_1 - 8x_2 - 3x_3 = 2$$

Take augmented matrix

$$A_b = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 4 & -1 & -2 \\ 1 & -8 & -3 & 2 \end{bmatrix}$$

We reduce A to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & -8 & -3 & 2 \\ 2 & 4 & -1 & -2 \\ 2 & 0 & 1 & 1 \end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -8 & -3 & 2 \\ 0 & 20 & 5 & -6 \\ 0 & 16 & 7 & -3 \end{bmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -8 & -3 & 2 \\ 0 & 1 & \frac{1}{4} & -\frac{3}{10} \\ 0 & 16 & 7 & -3 \end{bmatrix} \qquad \frac{1}{20}R_2$$

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$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & -1 & -\frac{2}{5} \\ 0 & 1 & \frac{1}{4} & -\frac{3}{10} \\ 0 & 0 & 3 & \frac{9}{5} \end{bmatrix} \qquad \begin{matrix} R_3 - 16R_2 \\ R_1 + 8R_2 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & -1 & -2/5 \\ 0 & 1 & 1/4 & -3/10 \\ 0 & 0 & 1 & 3/5 \end{bmatrix} \qquad \frac{1}{3}R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & -9/20 \\ 0 & 0 & 1 & 3/5 \end{bmatrix} \qquad \begin{matrix} R_1 + R_3 \\ R_2 - \frac{1}{4}R_3 \end{matrix}$$

Since matrix A is non singular
So unique solution exists
& soln. is

$$\left.\begin{aligned} x_1 &= \frac{1}{5} \\ x_2 &= -\frac{9}{20} \\ x_3 &= \frac{3}{5} \end{aligned}\right\}$$

Q2. $x_1 + x_2 + x_3 = a$
$x_1 + (1+a)x_2 + x_3 = 2a$ where $a \neq 0$
$x_1 + x_2 + (1+a)x_3 = 3a$

Sol. Given system is

$$x_1 + x_2 + x_3 = a$$
$$x_1 + (1+a)x_2 + x_3 = 2a$$
$$x_1 + x_2 + (1+a)x_3 = 3a$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 1+a & 1 & 2a \\ 1 & 1 & 1+a & 3a \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

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$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & 1 & a\\ 0 & a & 0 & a\\ 0 & 0 & a & 2a\end{bmatrix} \qquad \begin{matrix}R_2-2R_1\\ R_3-R_1\end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & 1 & a\\ 0 & 1 & 0 & 1\\ 0 & 0 & a & 2a\end{bmatrix} \qquad \frac{1}{a}R_2$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & 1 & a-1\\ 0 & 1 & 0 & 1\\ 0 & 0 & a & 2a\end{bmatrix} \qquad R_1-R_2$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & 1 & a-1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\end{bmatrix} \qquad \frac{1}{a}R_3$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & 0 & a-3\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\end{bmatrix} \qquad R_1-R_3$$

As matrix A is non singular

So unique solution exists

& soln. is

$$x_1 = a-3$$
$$x_2 = 1$$
$$x_3 = 2$$

<u>Q3</u>

$$\begin{aligned}x_1 - x_2 + x_3 - x_4 + x_5 &= 1\\ 2x_1 + x_2 + 3x_3 \quad + 4x_5 &= 3\\ 3x_1 - 2x_2 + 2x_3 + x_4 + x_5 &= 1\\ x_2 + x_4 + x_5 &= 0\end{aligned}$$

<u>Soln.</u> Given system is

$$\begin{aligned}x_1 - x_2 + x_3 - x_4 + x_5 &= 1\\ 2x_1 + x_2 + 3x_3 \quad + 4x_5 &= 3\\ 3x_1 - 2x_2 + 2x_3 + x_4 + x_5 &= 1\\ x_2 + x_4 + x_5 &= 0\end{aligned}$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 \\ 2 & 1 & 3 & 0 & 4 & 3 \\ 3 & -2 & 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 3 & 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & 4 & -2 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 & -2 & -2 \\ 0 & 3 & 1 & 2 & 2 & 1 \end{bmatrix} \qquad R_{24}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 3 & -3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \qquad \begin{matrix} R_1 + R_2 \\ R_3 - R_2 \\ R_4 - 3R_2 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 3 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \qquad (-1)R_3$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 3 & 2 \\ 0 & 0 & 0 & 2 & -4 & -1 \end{bmatrix} \qquad R_4 - R_3$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 3 & 2 \\ 0 & 0 & 0 & 1 & -2 & -1/2 \end{bmatrix} \qquad \tfrac{1}{2} R_4$$

$$\underset{\sim}{R}\left[\begin{array}{cccc:c:c} 1 & 0 & 0 & 0 & 5 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 3 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -3 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -2 & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 - R_4 \\ R_2 - R_4 \\ R_3 + R_4 \end{array}$$

Here rank A = rank A_b

So soln. exists & soln. is

$$\left.\begin{array}{l} x_1 + 5x_5 = \frac{1}{2} \\ x_2 + 3x_5 = \frac{1}{2} \\ x_3 - 3x_5 = \frac{1}{2} \\ x_4 - 2x_5 = -\frac{1}{2} \end{array}\right\}$$

So req. soln. is

$$\left.\begin{array}{l} x_1 = \frac{1}{2} - 5x_5 \\ x_2 = \frac{1}{2} - 3x_5 \\ x_3 = \frac{1}{2} + 3x_5 \\ x_4 = -\frac{1}{2} + 2x_5 \end{array}\right\} \text{ where } x_5 \text{ is arbitrary.}$$

Q4

$$x_1 + x_2 - x_3 = 1$$
$$x_2 + x_3 - x_4 = 1$$
$$x_3 + x_4 - x_5 = 1$$
$$x_5 + x_4 - x_3 = 1$$
$$x_4 + x_3 - x_2 = 1$$

Sol. Given system is

$$x_1 + x_2 - x_3 = 1$$
$$x_2 + x_3 - x_4 = 1$$
$$x_3 + x_4 - x_5 = 1$$
$$x_5 + x_4 - x_3 = 1$$
$$x_4 + x_3 - x_2 = 1$$

Take augmented matrix.

$$A_b = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{array}{l} R_1 - R_2 \\ R_5 + R_2 \end{array}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 3 & -2 & 2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & -2 & 2 & 0 \end{bmatrix} \qquad \begin{array}{l} R_1 + 2R_3 \\ R_2 - R_3 \\ R_4 + R_3 \\ R_5 - 2R_3 \end{array}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 3 & -2 & 2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 2 & 0 \end{bmatrix} \qquad \tfrac{1}{2} R_4$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix} \qquad \begin{array}{l} R_1 - 3R_4 \\ R_2 + 2R_4 \\ R_3 - R_4 \\ R_5 + 2R_4 \end{array}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \frac{1}{2}R_5$$

$$\underset{\sim}{R}\left[\begin{array}{ccccc:c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Since matrix A is non singular, so unique soln. exists & soln. is

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 1 \\ x_5 = 1 \end{array}\right\}$$

Q5
$$x_1 - 2x_2 - 7x_3 + 7x_4 = 5$$
$$-x_1 + 2x_2 + 8x_3 - 5x_4 = -7$$
$$3x_1 - 4x_2 - 17x_3 + 13x_4 = 14$$
$$2x_1 - 2x_2 - 11x_3 + 8x_4 = 7$$

Sol:- Given system is

$$x_1 - 2x_2 - 7x_3 + 7x_4 = 5$$
$$-x_1 + 2x_2 + 8x_3 - 5x_4 = -7$$
$$3x_1 - 4x_2 - 17x_3 + 13x_4 = 14$$
$$2x_1 - 2x_2 - 11x_3 + 8x_4 = 7$$

Take augmented matrix

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$$A_b = \begin{bmatrix} 1 & -2 & -7 & 7 & 5 \\ -1 & 2 & 8 & -5 & -7 \\ 3 & -4 & -17 & 13 & 14 \\ 2 & -2 & -11 & 8 & 7 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & -7 & 7 & 5 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 2 & 4 & -8 & -1 \\ 0 & 2 & 3 & -6 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 + 2R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{array}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & -7 & 7 & 5 \\ 0 & 2 & 4 & -8 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 2 & 3 & -6 & -3 \end{bmatrix} \quad R_{23}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -2 & -7 & 7 & 5 \\ 0 & 1 & 2 & -4 & -1/2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 2 & 3 & -6 & -3 \end{bmatrix} \quad \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & -3 & -1 & 4 \\ 0 & 1 & 2 & -4 & -1/2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & -1 & 2 & -2 \end{bmatrix} \quad \begin{array}{l} R_1 + 2R_2 \\ R_4 - 2R_2 \end{array}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5 & -2 \\ 0 & 1 & 0 & -8 & 7/2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix} \quad \begin{array}{l} R_1 + 3R_3 \\ R_2 - 2R_3 \\ R_4 + R_3 \end{array}$$

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$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 5 & -2 \\ 0 & 1 & 0 & -8 & 7/2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \frac{1}{4}R_4$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -9/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \qquad \begin{array}{l} R_1 - 5R_4 \\ R_2 + 8R_4 \\ R_3 - 2R_4 \end{array}$$

Since matrix A is non singular
So unique soln. exists & soln. is

$$\left.\begin{array}{l} x_1 = 3 \\ x_2 = -9/2 \\ x_3 = 0 \\ x_4 = -1 \end{array}\right\}$$

Q6
$$\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ 6x_1 + x_2 + x_3 &= -4 \\ 2x_1 - 3x_2 - x_3 &= 0 \\ x_1 - x_2 &= 1 \end{aligned}$$

Soln. Given system is

$$\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ 6x_1 + x_2 + x_3 &= -4 \\ 2x_1 - 3x_2 - x_3 &= 0 \\ x_1 - x_2 &= 1 \end{aligned}$$

Take augmented matrix

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$$A_b = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 6 & 1 & 1 & -4 \\ 2 & -3 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & -11 & -5 & 2 \\ 0 & -7 & -3 & 2 \\ 0 & -3 & -1 & 2 \end{bmatrix} \qquad \begin{matrix} R_2 - 6R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & -6 \\ 0 & -7 & -3 & 2 \\ 0 & -3 & -1 & 2 \end{bmatrix} \qquad R_2 - 4R_4$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 3 & 11 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & -10 & -40 \\ 0 & 0 & -4 & -16 \end{bmatrix} \qquad \begin{matrix} R_1 - 2R_2 \\ R_3 + 7R_2 \\ R_4 + 3R_2 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 3 & 11 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -4 & -16 \end{bmatrix} \qquad -\frac{1}{10}R_3$$

$$\overset{R}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{matrix} R_1 - 3R_3 \\ R_2 + R_3 \end{matrix}$$

As rank A = rank A_b. So soln. exists & soln. is

$$\left.\begin{matrix} x_1 = -1 \\ x_2 = -2 \\ x_3 = 4 \end{matrix}\right\}$$

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Q7
$$2x_1 + x_2 + 5x_3 = 4$$
$$3x_1 - 2x_2 + 2x_3 = 2$$
$$5x_1 - 8x_2 - 4x_3 = 5$$

Sol. Given system is
$$2x_1 + x_2 + 5x_3 = 4$$
$$3x_1 - 2x_2 + 2x_3 = 2$$
$$5x_1 - 8x_2 - 4x_3 = 5$$

Take augmented matrix

$$A_b = \begin{bmatrix} 2 & 1 & 5 & 4 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 5 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations

$$\underset{\sim}{R} \begin{bmatrix} 2 & 1 & 5 & 4 \\ 1 & -3 & -3 & -2 \\ 5 & -8 & -4 & 5 \end{bmatrix} \quad R_2 - R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -3 & -3 & -2 \\ 2 & 1 & 5 & 4 \\ 5 & -8 & -4 & 5 \end{bmatrix} \quad R_{12}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -3 & -3 & -2 \\ 0 & 7 & 11 & 8 \\ 0 & 7 & 11 & 15 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -3 & -3 & -2 \\ 0 & 1 & \frac{11}{7} & \frac{8}{7} \\ 0 & 7 & 11 & 15 \end{bmatrix} \quad \frac{1}{7}R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & \frac{13}{7} & \frac{12}{7} \\ 0 & 1 & \frac{11}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 7 \end{bmatrix} \quad \begin{matrix} R_1 + 3R_2 \\ R_3 - 7R_2 \end{matrix}$$

Since rank $A \neq$ rank A_b

So soln. does not exists

For what value of λ have the following homogeneous equations non trivial solutions? Find these solns (Prob 8–10)

<u>Q8</u>

$$(1-\lambda)x_1 + x_2 = 0$$
$$x_1 + (1-\lambda)x_2 = 0$$

<u>Sol.</u> Given system is

$$(1-\lambda)x_1 + x_2 = 0$$
$$x_1 + (1-\lambda)x_2 = 0$$

Take matrix A of Co-efficients of Variables

$$A = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1-\lambda \\ 1-\lambda & 1 \end{bmatrix} \quad R_{12}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1-\lambda \\ 0 & 2\lambda-\lambda^2 \end{bmatrix} \longrightarrow \text{(A)} \quad R_2 - (1-\lambda)R_1$$

For non trivial soln., rank $A < 2$

$$\Rightarrow 2\lambda - \lambda^2 = 0$$
$$\lambda(2-\lambda) = 0$$
$$\Rightarrow \boxed{\lambda = 0, 2}$$

Put $\lambda = 0$ in (A)

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

& non trivial soln. is

$$x_1 + x_2 = 0$$

or $\quad x_2 = -x_1$

So for $\lambda = 0$, non trivial soln. is $x_2 = -x_1$; x_1 is arbitrary

Now Put $\lambda = 2$ in (A)

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

& non trivial soln. is

$$x_1 - x_2 = 0$$

or $\quad x_2 = x_1$

So for $\lambda = 2$, non trivial soln. is $x_2 = x_1$; x_1 is arbitrary

---

Q9 $\quad (3-\lambda)x_1 - x_2 + x_3 = 0$

$$x_1 - (1-\lambda)x_2 + x_3 = 0$$

$$x_1 - x_2 + (1-\lambda)x_3 = 0$$

Sol. Given system is

$$(3-\lambda)x_1 - x_2 + x_3 = 0$$

$$x_1 - (1-\lambda)x_2 + x_3 = 0$$

$$x_1 - x_2 + (1-\lambda)x_3 = 0$$

Take matrix A of Coefficients of Variables

$$A = \begin{bmatrix} 3-\lambda & -1 & 1 \\ 1 & -(1-\lambda) & 1 \\ 1 & -1 & 1-\lambda \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations only.

$$\overset{R}{\sim}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 1 & -(1-\lambda) & 1 \\ 3-\lambda & -1 & 1 \end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & \lambda & \lambda \\ 0 & 2-\lambda & -\lambda^2+4\lambda-2 \end{bmatrix} \longrightarrow (\alpha) \qquad \begin{matrix} R_2-R_1 \\ R_3-(3-\lambda)R_1 \end{matrix}$$

Suppose $\lambda \neq 0$

So $$\overset{R}{\sim}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & 1 & 1 \\ 0 & 2-\lambda & -\lambda^2+4\lambda-2 \end{bmatrix} \qquad \frac{1}{\lambda}R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 2-\lambda \\ 0 & 1 & 1 \\ 0 & 0 & -\lambda^2+5\lambda-4 \end{bmatrix} \longrightarrow (A) \qquad \begin{matrix} R_1+R_2 \\ R_3+(\lambda-2)R_2 \end{matrix}$$

For non trivial soln.

rank $A < 3$ (no. of columns)

$$\Rightarrow \quad -\lambda^2+5\lambda-4 = 0$$

$$\lambda^2-5\lambda+4 = 0$$

$$(\lambda-1)(\lambda-4) = 0$$

$$\Rightarrow \quad \boxed{\lambda = 1, 4}$$

Put $\lambda = 1$ in matrix (A)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

+ non trivial soln. is

$$x_1 + x_3 = 0$$

$$x_2 + x_3 = 0$$

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or $x_1 = -x_3$

$x_2 = -x_3$

So for $\lambda = 1$, non trivial soln. is

$$\left.\begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \end{aligned}\right\} \text{ where } x_3 \text{ is arbitrary}$$

Now put $\lambda = 4$ in matrix (A)

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

& non trivial soln. is

$x_1 - 2x_3 = 0$

$x_2 + x_3 = 0$

or $x_1 = 2x_3$

$x_2 = -x_3$

Hence for $\lambda = 4$, non trivial soln. is

$$\left.\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \end{aligned}\right\} \text{ where } x_3 \text{ is arbitrary}$$

In case $\lambda = 0$, put in matrix (A)

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_{23}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \frac{1}{2}R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 + R_2$$

So non trivial soln. is

$$x_2 - x_3 = 0$$

or $x_2 = x_3$

Hence for $\lambda = 0$, non trivial soln. is

$$\left.\begin{aligned} x_1 &= 0 \\ x_2 &= x_3 \end{aligned}\right\} \quad \text{where } x_3 \text{ is arbitrary}$$

Q10 $\quad (1-\lambda)x_1 + x_2 + x_3 = 0$

$x_1 - \lambda x_2 + x_3 = 0$

$x_1 - x_2 + (1-\lambda)x_3 = 0$

Sol. Given system is

$$(1-\lambda)x_1 + x_2 + x_3 = 0$$
$$x_1 - \lambda x_2 + x_3 = 0$$
$$x_1 - x_2 + (1-\lambda)x_3 = 0$$

Take matrix A of Cofficients of Variables

$$A = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 1-\lambda \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1-\lambda \\ 1 & -\lambda & 1 \\ 1-\lambda & 1 & 1 \end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & 1-\lambda & \lambda \\ 0 & -\lambda & 2\lambda-\lambda^2 \end{bmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3-(1-\lambda)R_1 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & -1 & 1-\lambda \\ 0 & 1 & -\lambda+\lambda^2 \\ 0 & -\lambda & 2\lambda-\lambda^2 \end{bmatrix} \qquad R_2 - R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 1-2\lambda+\lambda^2 \\ 0 & 1 & -\lambda+\lambda^2 \\ 0 & 0 & \lambda^3-2\lambda^2+2\lambda \end{bmatrix} \longrightarrow (A) \qquad \begin{matrix} R_1 + R_2 \\ R_3 + \lambda R_2 \end{matrix}$$

For non trivial soln.

rank $A < 3$ (no. of Columns)

$$\Rightarrow \quad \lambda^3 - 2\lambda^2 + 2\lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda + 2) = 0$$

$$\lambda = 0 \quad , \quad \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

take only real value of λ

So $\boxed{\lambda = 0}$

Put in last matrix (A)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

& non trivial soln. is

$$x_1 + x_3 = 0$$
$$x_2 = 0$$

So for $\lambda = 0$, non trivial soln. is

$$\left.\begin{matrix} x_1 = -x_3 \\ x_2 = 0 \end{matrix}\right\} \text{ where } x_3 \text{ is arbitrary}$$

In each of the following cases, use Guass Jordan[21] method to reduce the given system to the reduced echelon form, indicating the operations you perform & determine the soln. if any.
(Problems 11 to 18):

Q11
$$6x_1 - 6x_2 + 6x_3 = 6$$
$$2x_1 - 4x_2 - 6x_3 = 12$$
$$10x_1 - 5x_2 + 5x_3 = 30$$

Soln. Given system is
$$6x_1 - 6x_2 + 6x_3 = 6$$
$$2x_1 - 4x_2 - 6x_3 = 12$$
$$10x_1 - 5x_2 + 5x_3 = 30$$

Take augmented matrix

$$A_b = \begin{bmatrix} 6 & -6 & 6 & 6 \\ 2 & -4 & -6 & 12 \\ 10 & -5 & 5 & 30 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -2 & -3 & 6 \\ 2 & -1 & 1 & 6 \end{bmatrix} \qquad \frac{1}{6}R_1, \frac{1}{2}R_2, \frac{1}{5}R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & -4 & 5 \\ 0 & 1 & -1 & 4 \end{bmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & -1 & -4 & 5 \end{bmatrix} \qquad R_{23}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -5 & 9 \end{bmatrix} \quad \begin{matrix} R_1+R_2 \\ R_3+R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -9/5 \end{bmatrix} \quad -\frac{1}{5}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 11/5 \\ 0 & 0 & 1 & -9/5 \end{array}\right] \quad R_2+R_3$$

Since matrix A is non singular

So unique soln. exists & soln. is

$$\left.\begin{aligned} x_1 &= 5 \\ x_2 &= \frac{11}{5} \\ x_3 &= -\frac{9}{5} \end{aligned}\right\}$$

Q12
$$5x_1 - 2x_2 + x_3 = 0$$
$$3x_1 + 2x_2 + 7x_3 = 0$$
$$x_1 + x_2 + 3x_3 = 0$$

Sol. Given system is

$$5x_1 - 2x_2 + x_3 = 0$$
$$3x_1 + 2x_2 + 7x_3 = 0$$
$$x_1 + x_2 + 3x_3 = 0$$

Take matrix A of Cofficients of Variables

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 2 & 7 \\ 1 & 1 & 3 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & 3\\3 & 2 & 7\\5 & -2 & 1\end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & 3\\0 & -1 & -2\\0 & -7 & -14\end{bmatrix} \qquad \begin{matrix}R_2-3R_1\\R_3-5R_1\end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & 3\\0 & 1 & 2\\0 & -7 & -14\end{bmatrix} \qquad (-1)R_2$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \qquad \begin{matrix}R_1-R_2\\R_3+7R_2\end{matrix}$$

Given system is equivalent to

$$x_1 + x_3 = 0$$
$$x_2 + 2x_3 = 0$$

Hence infinite solutions of given system are

$$\left.\begin{matrix}x_1 = -x_3\\x_2 = -2x_3\end{matrix}\right\} \text{ where } x_3 \text{ is arbitrary}$$

Q13

$$5x_1 - 2x_2 + x_3 = 3$$
$$3x_1 + 2x_2 + 7x_3 = 5$$
$$x_1 + x_2 + 3x_3 = 2$$

Sol. Given system is

$$5x_1 - 2x_2 + x_3 = 3$$
$$3x_1 + 2x_2 + 7x_3 = 5$$
$$x_1 + x_2 + 3x_3 = 2$$

Take augmented matrix

$$A_b = \begin{bmatrix}5 & -2 & 1 & 3\\3 & 2 & 7 & 5\\1 & 1 & 3 & 2\end{bmatrix}$$

we reduce it to reduced echelon form by
applying row operations.

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & 2 & 7 & 5 \\ 5 & -2 & 1 & 3 \end{bmatrix} \qquad R_{13}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & -7 & -14 & -7 \end{bmatrix} \qquad \begin{matrix} R_2-3R_1 \\ R_3-5R_1 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -7 & -14 & -7 \end{bmatrix} \qquad (-1)R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} R_1-R_2 \\ R_3+7R_2 \end{matrix}$$

Since rank A = rank Ab

So soln. exists & soln. is

$$x_1 + x_3 = 1$$
$$x_2 + 2x_3 = 1$$

or

$$\left.\begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= 1 - 2x_3 \end{aligned}\right\} \text{ where } x_3 \text{ is arbitrary.}$$

Q14

$$5x_1 - 2x_2 + x_3 = 2$$
$$3x_1 + 2x_2 + 7x_3 = 3$$
$$x_1 + x_2 + 3x_3 = 2$$

Sol:- Given system is

$$5x_1 - 2x_2 + x_3 = 2$$
$$3x_1 + 2x_2 + 7x_3 = 3$$
$$x_1 + x_2 + 3x_3 = 2$$

We reduce it to reduced echelon form by applying row operations. 25

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & 2 & 7 & 3 \\ 5 & -2 & 1 & 2 \end{bmatrix} \qquad R_{13}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & -7 & -14 & -8 \end{bmatrix} \qquad \begin{matrix} R_2 - 3R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & -7 & -14 & -8 \end{bmatrix} \qquad (-1)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 13 \end{bmatrix} \qquad \begin{matrix} R_1 - R_2 \\ R_3 + 7R_2 \end{matrix}$$

Since rank A $\neq$ rank Ab

So soln. does not exist

Q15 $\quad 2x_1 - x_2 + 3x_3 = 3$

$3x_1 + x_2 - 5x_3 = 0$

$4x_1 - x_2 + x_3 = 3$

Soln. Given system is

$$2x_1 - x_2 + 3x_3 = 3$$
$$3x_1 + x_2 - 5x_3 = 0$$
$$4x_1 - x_2 + x_3 = 3$$

Take augmented matrix

$$A_b = \begin{bmatrix} 2 & -1 & 3 & 3 \\ 3 & 1 & -5 & 0 \\ 4 & -1 & 1 & 3 \end{bmatrix}$$

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We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim}\begin{bmatrix} -1 & -2 & 8 & 3 \\ 3 & 1 & -5 & 0 \\ 4 & -1 & 1 & 3 \end{bmatrix} \qquad R_1 - R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & -3 \\ 3 & 1 & -5 & 0 \\ 4 & -1 & 1 & 3 \end{bmatrix} \qquad (-1)R_1$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & -3 \\ 0 & -5 & 19 & 9 \\ 0 & -9 & 33 & 15 \end{bmatrix} \qquad \begin{matrix} R_2 - 3R_1 \\ R_3 - 4R_1 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & -3 \\ 0 & -5 & 19 & 9 \\ 0 & 1 & -5 & -3 \end{bmatrix} \qquad R_3 - 2R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & -3 \\ 0 & 1 & -5 & -3 \\ 0 & -5 & 19 & 9 \end{bmatrix} \qquad R_{23}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & -6 & -6 \end{bmatrix} \qquad \begin{matrix} R_1 - 2R_2 \\ R_3 + 5R_2 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad -\frac{1}{6}R_3$$

$$\overset{R}{\sim}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \begin{matrix} R_1 - 2R_3 \\ R_2 + 5R_3 \end{matrix}$$

Since matrix A is non singular, so unique soln. exists

∴ Soln. is

$$\left.\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 1 \end{aligned}\right\}$$

<u>Q16</u>

$$x_1 + 3x_2 + 5x_3 - 4x_4 = 1$$
$$x_1 + 2x_2 + x_3 - x_4 + x_5 = -1$$
$$x_1 - 2x_2 + 3x_3 + 2x_4 - x_5 = 3$$
$$x_1 + 5x_2 + 3x_3 + x_4 + x_5 = -11$$
$$x_1 + 3x_2 - x_3 + x_4 + 2x_5 = -3$$

<u>Sol.</u> Given system is

$$x_1 + 3x_2 + 5x_3 - 4x_4 = 1$$
$$x_1 + 2x_2 + x_3 - x_4 + x_5 = -1$$
$$x_1 - 2x_2 + 3x_3 + 2x_4 - x_5 = 3$$
$$x_1 + 5x_2 + 3x_3 + x_4 + x_5 = -11$$
$$x_1 + 3x_2 - x_3 + x_4 + 2x_5 = -3$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 1 & 2 & 1 & -1 & 1 & -1 \\ 1 & -2 & 3 & 2 & -1 & 3 \\ 1 & 5 & 3 & 1 & 1 & -11 \\ 1 & 3 & -1 & 1 & 2 & -3 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & -1 & -4 & 3 & 1 & -2 \\ 0 & -5 & -2 & 6 & -1 & 2 \\ 0 & 2 & -2 & 5 & 1 & -12 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \\ R_5 - R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & -5 & -2 & 6 & -1 & 2 \\ 0 & 2 & -2 & 5 & 1 & -12 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{bmatrix} \qquad (-1)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 18 & -9 & -6 & 12 \\ 0 & 0 & -10 & 11 & 3 & -16 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{bmatrix} \qquad \begin{matrix} R_3 + 5R_2 \\ R_4 - 2R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 6 & -3 & -2 & 4 \\ 0 & 0 & -10 & 11 & 3 & -16 \\ 0 & 0 & -6 & 5 & 2 & -4 \end{bmatrix} \qquad \frac{1}{3}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 6 & -3 & -2 & 4 \\ 0 & 0 & 2 & 5 & -1 & -8 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} R_4 + 2R_3 \\ R_5 + R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 6 & -3 & -2 & 4 \\ 0 & 0 & 2 & 5 & -1 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \frac{1}{2}R_5$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 2 & -13 & 0 & 20 \\ 0 & 0 & 2 & 5 & -1 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad R_3 - 2R_4$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 2 & -13 & 0 & 20 \\ 0 & 0 & 0 & 18 & -1 & -28 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad R_4 - R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & -4 & 0 & 1 \\ 0 & 1 & 4 & -3 & -1 & 2 \\ 0 & 0 & 1 & -\frac{13}{2} & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 18 & -1 & -28 \end{bmatrix} \quad \begin{matrix} \frac{1}{2}R_3 \\ R_{45} \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -28 \end{bmatrix} \quad \begin{matrix} R_1 + 4R_4 \\ R_2 + 3R_4 \\ R_3 + \frac{13}{2}R_4 \\ R_5 - 18R_4 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & -49 \\ 0 & 1 & 0 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 28 \end{bmatrix} \quad \begin{matrix} R_1 - 5R_3 \\ R_2 - 4R_3 \\ (-1)R_5 \end{matrix}$$

$$\overset{R}{\sim}\left[\begin{array}{ccccc:c} 1 & 0 & 0 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 28 \end{array}\right]$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & 1 & 9/5 & \frac{2}{5} \\ 0 & -7 & -11 & -2 \end{bmatrix} \qquad -\frac{1}{5}R_2$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & -2/5 & 9/5 \\ 0 & 1 & 9/5 & 2/5 \\ 0 & 0 & 8/5 & 4/5 \end{bmatrix} \qquad \begin{matrix} R_1 - 3R_2 \\ R_3 + 7R_2 \end{matrix}$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & -2/5 & 9/5 \\ 0 & 1 & 9/5 & \frac{2}{5} \\ 0 & 0 & 1 & 1/2 \end{bmatrix} \qquad \frac{5}{8}R_3$$

$$\underset{R}{\sim}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \qquad \begin{matrix} R_1 + \frac{2}{5}R_3 \\ R_2 - \frac{9}{5}R_3 \end{matrix}$$

Since matrix A is non singular

So unique soln. exists & soln. is

$$x_1 = 2$$
$$x_2 = -1/2$$
$$x_3 = 1/2$$

<u>Q18</u>

$$5x_1 + 4x_3 + 2x_4 = 3$$
$$x_1 - x_2 + 2x_3 + x_4 = 1$$
$$4x_1 + x_2 + 2x_3 = 1$$
$$x_1 + x_2 + x_3 + x_4 = 0$$

<u>Sol.</u> Given system is

$$5x_1 + 4x_3 + 2x_4 = 3$$
$$x_1 - x_2 + 2x_3 + x_4 = 1$$
$$4x_1 + x_2 + 2x_3 = 1$$
$$x_1 + x_2 + x_3 + x_4 = 0$$

Take augmented matrix

$$A_b = \begin{bmatrix} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 5 & 0 & 4 & 2 & 3 \end{bmatrix} \quad , R_{14}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & -3 & -2 & -4 & 1 \\ 0 & -5 & -1 & -3 & 3 \end{bmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 4R_1 \\ R_4 - 5R_1 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & -3 & -2 & -4 & 1 \\ 0 & -5 & -1 & -3 & 3 \end{bmatrix} \quad R_2 - R_3.$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 7 & 8 & 1 \\ 0 & 0 & 14 & 17 & 3 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ R_3 + 3R_2 \\ R_4 + 5R_2 \end{matrix}$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 8/7 & 1/7 \\ 0 & 0 & 14 & 17 & 3 \end{bmatrix} \quad \frac{1}{7} R_3$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -5/7 & 2/7 \\ 0 & 1 & 0 & 4/7 & -3/7 \\ 0 & 0 & 1 & 8/7 & 1/7 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_1 + 2R_3 \\ R_2 - 3R_3 \\ R_4 - 14R_3 \end{matrix}$$

$$\overset{R}{\sim}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad \begin{array}{l} R_1 + \frac{5}{7} R_4 \\ R_2 - \frac{4}{7} R_4 \\ R_3 - \frac{8}{7} R_4 \end{array}$$

Since matrix A is non singular

So unique soln. exists & soln. is

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = -1 \\ x_3 = -1 \\ x_4 = 1 \end{array}\right\}$$

Q19 Show that the system

$$2x_1 - x_2 + 3x_3 = a$$
$$3x_1 + x_2 - 5x_3 = b$$
$$-5x_1 - 5x_2 + 21x_3 = c$$

is inconsistent if $c \neq 2a - 3b$

Soln Given system is

$$2x_1 - x_2 + 3x_3 = a$$
$$3x_1 + x_2 - 5x_3 = b$$
$$-5x_1 - 5x_2 + 21x_3 = c$$

Take augmented matrix

$$A_b = \begin{bmatrix} 2 & -1 & 3 & a \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations.

$$\overset{R}{\sim}\begin{bmatrix} -1 & -2 & 8 & a-b \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{bmatrix} \qquad R_1 - R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & b-a \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{bmatrix} \qquad (-1)R_1$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & -5 & 19 & 3a-2b \\ 0 & 5 & -19 & -5a+5b+c \end{bmatrix} \qquad \begin{matrix} R_2-3R_1 \\ R_3+5R_1 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & 1 & -19/5 & \frac{2b-3a}{5} \\ 0 & 5 & -19 & -5a+5b+c \end{bmatrix} \qquad -\frac{1}{5}R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & -2/5 & \frac{a+b}{5} \\ 0 & 1 & -19/5 & \frac{2b-3a}{5} \\ 0 & 0 & 0 & -2a+3b+c \end{bmatrix} \qquad \begin{matrix} R_1-2R_2 \\ R_3-5R_2 \end{matrix}$$

The given system is inconsistent

if rank $A \neq$ rank A_b

i.e., only possible when $-2a+3b+c \neq 0$ or $c \neq 2a-3b$

So the given system is inconsistent if $c \neq 2a-3b$.

Q20 A soap manufacturer decides to spend 600,000 rupees on radio, magazine & T.V. advertising. If he spends as much on T.V. advertising as on magazines & radio together, and the amount spent on magazines & T.V. combined equals five times that spent on radio. What is the amount to be spent on each type of advertising?

Sol:-

Sol. Let x_1, x_2, x_3 be the amounts in rupees spent on radio, magazines & TV advertising resp. then by given Conditions

$$x_1 + x_2 + x_3 = 600,000 \quad \text{——— (1)}$$

$$x_3 = x_1 + x_2$$

$$\text{or } x_1 + x_2 - x_3 = 0 \quad \text{——— (2)}$$

Also $\quad x_2 + x_3 = 5x_1$

$$\text{or } \quad 5x_1 - x_2 - x_3 = 0. \quad \text{——— (3)}$$

Now we will solve eqs. (1), (2) & (3)

$$x_1 + x_2 + x_3 = 600,000$$

$$x_1 + x_2 - x_3 = 0$$

$$5x_1 - x_2 - x_3 = 0$$

Take augmented matrix

$$A_b = \begin{bmatrix} 1 & 1 & 1 & 600,000 \\ 1 & 1 & -1 & 0 \\ 5 & -1 & -1 & 0 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations.

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 & 600,000 \\ 0 & 0 & -2 & -600,000 \\ 0 & -6 & -6 & -3000000 \end{bmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 & 600,000 \\ 0 & -6 & -6 & -3000000 \\ 0 & 0 & -2 & -600,000 \end{bmatrix} \quad R_{23}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 & 600,000 \\ 0 & 1 & 1 & 500,000 \\ 0 & 0 & -2 & -600,000 \end{bmatrix} \quad -\frac{1}{6} R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 100,000 \\ 0 & 1 & 1 & 500,000 \\ 0 & 0 & -2 & -600,000 \end{bmatrix} \qquad R_1 - R_2$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 100,000 \\ 0 & 1 & 1 & 500,000 \\ 0 & 0 & 1 & 300,000 \end{bmatrix} \qquad -\frac{1}{2}R_3$$

$$\overset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 100,000 \\ 0 & 1 & 0 & 200,000 \\ 0 & 0 & 1 & 300,000 \end{array}\right] \qquad R_2 - R_3$$

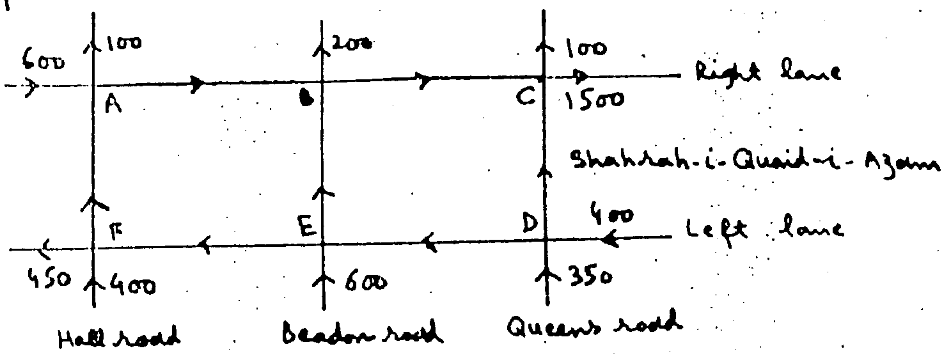
Since matrix A is non singular
So unique soln. exists & soln. is

$x_1 = 100,000$

$x_2 = 200,000$

$x_3 = 300,000$

Q21 Traffic Counters submitted the following information for March 23 from 7 P.M. to 8 P.M. on the following roads of Lahore:



(i) Construct a mathematical model that describes this system, carefully labelling the variables you introduce

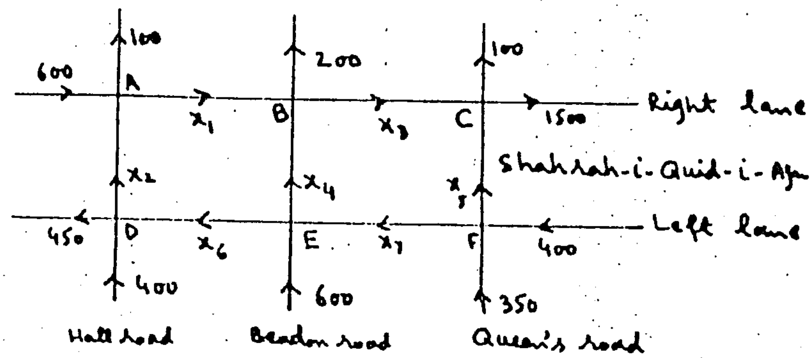
(ii) Show that there must be atleast 50 vehicles travelling on the section of left lane to Hall road from Beadon road during the count.

(iii) The city planners are inclined to take this traffic count as typical rush hour evening traffic in this area. In their planning of the annual closure of left lane between Queen's road & Beadon road for repair, how much traffic can be expected on right lane between Queen's road & Beadon road?

Sol:

(i)

Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be the no. of vehicles along different sections of various roads, as shown in the figure



Equating the incoming flow to the outgoing flow at each junction, we have following mathematical model:

At junction A : $x_2 + 600 = x_1 + 100 \Rightarrow x_1 - x_2 = 500$ —— ①

At junction B : $x_1 + x_4 = x_3 + 200 \Rightarrow x_1 - x_3 + x_4 = 200$ —— ②

At junction C : $x_3 + x_5 = 1500 + 100 \Rightarrow x_3 + x_5 = 1600$ —— ③

At junction D : $x_6 + 400 = x_2 + 450 \Rightarrow x_2 - x_6 = -50$ —— ④

At junction E : $x_7 + 600 = x_4 + x_6 \Rightarrow x_4 + x_6 - x_7 = 600$ —— ⑤

At junction F : $x_5 + x_7 = 400 + 350 \Rightarrow x_5 + x_7 = 750$ —— ⑥

So we have the following system of eqs:

$$x_1 - x_2 = 500$$
$$x_1 - x_3 + x_4 = 200$$
$$x_3 + x_5 = 1600$$
$$x_2 - x_6 = -50$$
$$x_4 + x_6 - x_7 = 600$$
$$x_5 + x_7 = 750$$

& its augmented matrix is

$$A_b = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 500 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1600 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -50 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 750 \end{bmatrix}$$

(ii) From eq. (4) we have

$$x_6 = x_2 + 50$$

which shows that if $x_2 = 0$ then least no. of vehicles travelling on the section of left left lane to Hall road from Beadon road during the Count is 50.

(iii) Because of closure of left lane b/w Queen's road & Beadon road for repair, we have

$$x_7 = 0$$

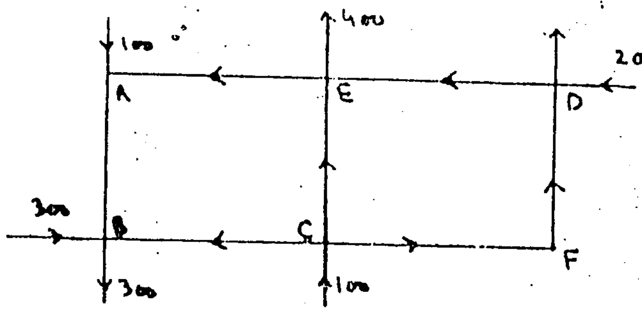
then we can obtain x_3, the no. of vehicles expected on right lane b/w Queen's road & on Beadon road from eqs. (3) & (6)

(6) $\Rightarrow$ $x_5 = 750$ Put in (3)

$$x_3 + 750 = 1600$$

or $x_3 = 1600 - 750 = 850$

Q:2 One part of Lahore's network of traffic is given with the no. of vehicles that enter & leave during a typical rush hour as shown below. All the lanes are one way in the direction indicated by the arrows. 39



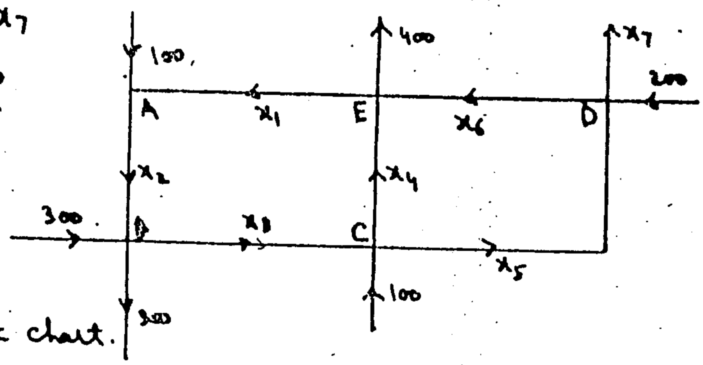
(i) Construct the linear mathematical model that describes this system.

(ii) If the stretch EA is closed for repair, what will be the traffic flow along the other stretches?

(iii) If only 100 vehicles are allowed to pass during the rush hour through EA, how will that effect on other branches?

Sol:-

Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be the no. of vehicles along different sections of various roads in the rush hour as shown by the traffic chart.



Since the no. of incoming & outgoing vehicles at each junction in the network must be equal So the req. mathematical model can be constructed as:

At junction A : $x_1 + 100 = x_2 \Rightarrow x_1 - x_2 = -100$ —— ①

At junction B : $x_2 + 300 = 300 + x_3 \Rightarrow x_2 - x_3 = 0$ —— ②

At junction C : $x_3 + 100 = x_4 + x_5 \Rightarrow x_3 - x_4 - x_5 = -100$ —— ③

At junction D : $x_5 + 200 = x_6 + x_7 \Rightarrow x_5 - x_6 - x_7 = -200$ —— ④

At junction E : $x_4 + x_6 = 400 + x_1 \Rightarrow x_1 - x_4 - x_6 = -400$ —— ⑤

& also $x_7 + 700 = 700 \Rightarrow x_7 = 0$ —— ⑥

Now augmented matrix of this system is

$$A_b = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -100 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & -200 \\ 1 & 0 & 0 & -1 & 0 & -1 & -400 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

As rank A $\neq$ rank A_b

So let $x_6 = a$

(ii) When the section EA is closed for repair then $x_1 = 0$

& so from eqs. ① & ② $x_2 = 100$ & $x_3 = 100$

From eq. ⑤ $x_4 = 400 - a$ where $a \le 400$ —— ⑦

From eq. ③ $x_5 = a$ —— ⑧

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 700 \quad \text{—— ⑨}$$

$$\Rightarrow 0 + 100 + 100 + 400 - a + a + a = 700$$

or $600 + a = 700 \Rightarrow a = 100$

So $\boxed{x_5 = a = 100}$

& ⑦ $\Rightarrow x_4 = 400 - a = 400 - 100 = 300$

(iii) Here in this case $x_1 = 100$

① $\Rightarrow 100 - x_2 = -100 \Rightarrow x_2 = 200$

② $\Rightarrow 200 - x_3 = 0 \Rightarrow x_3 = 200$

⑤ $\Rightarrow x_4 + x_6 = 500$ —— ⑩

From (3), we have $x_4 + x_5 = 300$ ——— (11)

$$\Rightarrow \quad x_4 + \alpha = 300$$

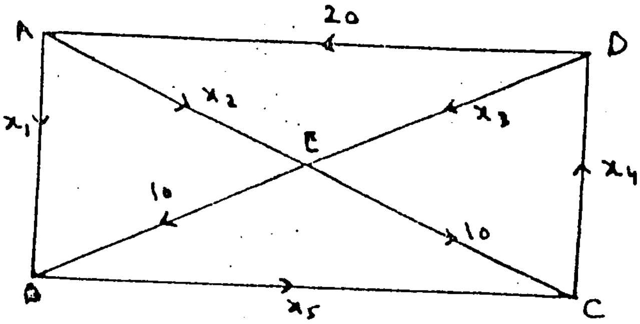
$$\text{or} \quad x_4 = 300 - \alpha \quad ; \quad 0 \le \alpha \le 300$$

from (4) $\quad x_5 + 200 = \alpha \Rightarrow x_5 = \alpha - 200 \; ; \; 0 \le \alpha \le 200$

Hence $0 \le \alpha \le 300$ & $0 \le \alpha \le 200 \Rightarrow 0 \le \alpha \le 200$

Assign arbitrary value to α s.t. $0 \le \alpha \le 200$ we get infinite no. of solutions.

Q23 Set up a system of linear eqs. to represent the network shown in the diagram & solve the system.



If $x_1 = x_3 = 0$, find the flow.

Sol. Here x_1, x_2, x_3, x_4, x_5 be the no. of vehicles along different sections of various roads as shown. Equating the incoming flow to the outgoing flow at each junction, we have following mathematical model.

At junction A $\quad x_1 + x_2 = 20$

At junction B $\quad x_1 + 10 = x_5$

At junction C $\quad x_5 + 10 = x_4$

At junction D $\quad x_3 + 20 = x_4$

At junction E $\quad x_2 + x_3 = 20$

Thus we have the following system

$$x_1 + x_2 = 20$$
$$x_1 - x_5 = -10$$
$$x_4 - x_5 = 10$$
$$x_3 - x_4 = -20$$
$$x_2 + x_3 = 20$$

The augmented matrix of this system is

$$A_b = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{bmatrix}$$

we reduce it to reduced echelon form by applying row operations.

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & -1 & 0 & 0 & -1 & -30 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{bmatrix} \quad R_2 - R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{bmatrix} \quad (-1)R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ R_5 - R_2 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{bmatrix} \quad R_{34}$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{bmatrix} \quad R_5 - R_3$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 + R_4 \\ R_5 - R_4 \end{matrix}$$

Here rank A = rank Ab

So soln. exists & soln. is

$$x_1 - x_5 = -10$$
$$x_2 + x_5 = 30$$
$$x_3 - x_5 = -10$$
$$x_4 - x_5 = 10$$

or
$$\left.\begin{aligned} x_1 &= x_5 - 10 \\ x_2 &= 30 - x_5 \\ x_3 &= x_5 - 10 \\ x_4 &= x_5 + 10 \end{aligned}\right\} \text{ where } x_3 \text{ is arbitrary}$$

when $x_1 = x_3 = 0$ then we see that $x_5 = 0$

Hence

$x_1 = 0$, $x_2 = 20$, $x_3 = 0$, $x_4 = 20$, $x_5 = 10$

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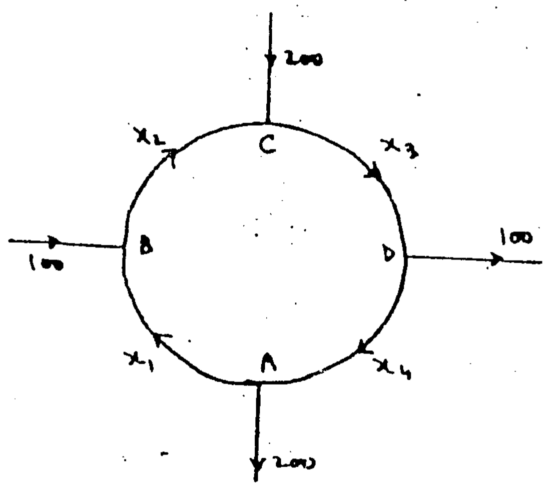
Q2.4 The flow of traffic at the Kalma Chowk on Ferozepur road, Lahore is shown below:



(i) Solve the system

(ii) Find the traffic flow when $x_4 = 300$

Sol. Here x_1, x_2, x_3, x_4 be the no. of vehicles along different sections of various roads as shown. Equating the incoming traffic to the outgoing traffic at each junction, we have the following mathematical model.

At Junction A $\quad x_1 + 200 = x_4$

At junction B $\quad x_1 + 100 = x_2$

At junction C $\quad x_2 + 200 = x_3$

At junction D $\quad x_3 = x_4 + 100$

Thus we have the following system.

$$x_1 - x_4 = -200$$
$$x_1 - x_2 = -100$$
$$x_2 - x_3 = -200$$
$$x_3 - x_4 = 100$$

The augmented matrix of this system is

$$A_b = \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 1 & -1 & 0 & 0 & -100 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix}$$

We reduce it to reduced echelon form by applying row operations

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix} \quad R_2 - R_1$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix} \quad (-1)R_2$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & -1 & 1 & -100 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix} \quad R_3 - R_2$$

$$\overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 1 & -1 & 100 \end{bmatrix} \quad (-1)R_3$$

$$\overset{R}{\sim} \left[\begin{array}{cccc:c} 1 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_4 - R_3$$

Here rank A = rank A_b

So soln. exists &

soln. is

$$x_1 - x_4 = -200$$

$$x_2 - x_4 = -100$$

$$x_3 - x_4 = 100$$

or
$$\left.\begin{aligned} x_1 &= x_4 - 200 \\ x_2 &= x_4 - 100 \\ x_3 &= x_4 + 100 \end{aligned}\right\} \text{ where } x_4 \text{ is arbitrary}$$

(ii) If $x_4 = 300$ then

$$x_1 = 300 - 200 = 100$$

$$x_2 = 300 - 100 = 200$$

$$x_3 = 300 + 100 = 400$$

End of linear eqs.

thank God

Determinants (Chapter No. 5)

Mathematical Method

5

Consider the simultaneous eqs.

$$a_1 x + b_1 = 0 \quad \text{--------} \quad (1)$$

$$a_2 x + b_2 = 0 \quad \text{--------} \quad (2)$$

Let us eliminate x from these two eqs.

From (1) $x = -\frac{b_1}{a_1}$

Put in (2)

$$a_2\left(-\frac{b_1}{a_1}\right) + b_2 = 0$$

$$-a_2 b_1 + a_1 b_2 = 0$$

or $a_1 b_2 - a_2 b_1 = 0$

The expression on the left i.e., $a_1 b_2 - a_2 b_1$ is symbolically written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$ & is called a determinant

As this determinant has two rows & two columns, so it is said to be a determinant of order 2.

Again

Consider the simultaneous eqs.

$$a_1 x + b_1 y + c_1 = 0 \quad \text{--------} \quad (1)$$

$$a_2 x + b_2 y + c_2 = 0 \quad \text{--------} \quad (2)$$

$$a_3 x + b_3 y + c_3 = 0 \quad \text{--------} \quad (3)$$

Let us eliminate x & y from these three eqs.

from (2) & (3)

$$\frac{x}{b_2 c_3 - b_3 c_2} = \frac{-y}{a_2 c_3 - a_3 c_2} = \frac{1}{a_2 b_3 - a_3 b_2}$$

$$\Rightarrow x = \frac{b_2 c_3 - b_3 c_2}{a_2 b_3 - a_3 b_2} \quad \& \quad y = -\frac{a_2 c_3 - a_3 c_2}{a_2 b_3 - a_3 b_2}$$

Put values in (1)

$$a_1\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}\right)+b_1\left(-\frac{a_2c_3-a_3c_2}{a_2b_3-a_3b_2}\right)+c_1=0$$

or

$$a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)=0$$

or

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

As it consists of three row & three columns, so it is said to be a determinant of order 3.

Properties of determinants:

Following are some important properties of determinants.

(i) The value of a determinant is the same as the value of its transpose.

(ii) The interchange of two adjacent rows or columns changes the sign of the determinant.

(iii) If a row or column of a determinant is passed over m rows or columns then its value is multiplied by $(-1)^m$.

(iv) If any two rows or columns of a determinant are identical then value of determinant is zero.

(v) If all the elements in a row or column of a determinant are zero then value of the determinant is zero.

(vi) If a non zero scalar is multiplied by a determinant then this scalar will be multiplied by any one of the rows or columns of that det.

(vii) If each element in a row or Column of a determinant is the sum of two elements then this determinant will be written as the sum of two determinants as

$$\begin{vmatrix} a_1 & b_1+t_1 & c_1 \\ a_2 & b_2+t_2 & c_2 \\ a_3 & b_3+t_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & t_1 & c_1 \\ a_2 & t_2 & c_2 \\ a_3 & t_3 & c_3 \end{vmatrix}$$

(viii) Addition of some scalar multiple of a row or Column to any other row or Column does not change the value of that determinant.

Minors & Cofactors:

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

be a given determinant of order n.

The minor of an element a_{ij} of Δ is the det. M_{ij} obtained by deleting the rows & Columns in which a_{ij} lies. Clearly M_{ij} is a determinant of order $n-1$.

The Cofactor A_{ij} of an element a_{ij} of Δ is

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Note

(i) $\Delta = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots\cdots + a_{in}A_{in}$ for any i

(ii) $\Delta = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots\cdots + a_{nj}A_{nj}$ for any j

(iii) If the elements of a line are multiplied by the cofactors of the corresponding elements of any other parallel line & the results so obtained are added the answer will be zero.

Adjoint of a square matrix:

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n. Denoting the co-factors by A_{ij} of the elements a_{ij} of A, we define

$$\operatorname{Adj} A = [A_{ij}]^t_{n\times n} = [A_{ji}]_{n\times n}$$

Inverse of a square matrix:

Let A be a non singular square matrix of order n then inverse of A is defined as

$$A^{-1} = \frac{\operatorname{Adj} A}{|A|}$$

Note If A & B are square matrices of order n then

(i) $\det(AB) = \det(A).\det(B)$

(ii) $\det(BA) = \det(B).\det(A)$

(iii) $\det(A^{-1}) = (\det(A))^{-1}$ if A is non singular

(iv) $\det(A^t) = \det(A)$

(v) $\det(A^n) = [\det(A)]^n$ where $n \in Z^+$

(vi) $\det(KA) = K^n.\det(A)$

Exercise No. 5.1

5 Mathematical Method

Q1 Let M_2 be the set of all 2×2 matrices. Set up the transformation $A \longrightarrow \det(A)$, $A \in M_2$. What is the range of this mapping? Is the mapping one-to-one?

Sol.

Let $f : A \longrightarrow \det(A)$; $A \in M_2$

be defined by

$$f(A) = \det(A)$$

Suppose the field for all $A \in M_2$ be the set of Complex no's. C, then the range of f is C. But if the field is taken as the set of real no's. R then range of f is also R

This mapping f is not one-to-one as shown by the following example

Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 6 & 0 \\ 9 & 1 \end{bmatrix}$

then clearly $A \neq B$

Now

$$\det(A) = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 8-2 = 6$$

& $$\det(B) = \begin{vmatrix} 6 & 0 \\ 9 & 1 \end{vmatrix} = 6-0 = 6$$

Hence $\det(A) = \det(B)$

So we have proved that

$$A \neq B \Rightarrow \det(A) = \det(B)$$

Hence by def., f is not one-to-one.

Q2 For 2×2 matrices A & B which of the following equations hold ?

(i) $\det(A+B) = \det(A) + \det(B)$

Sol.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} f & g \\ h & k \end{bmatrix}$

then

$$A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} f & g \\ h & k \end{bmatrix}$$

or

$$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

$$\Rightarrow \det(A+B) = \begin{vmatrix} a+f & b+g \\ c+h & d+k \end{vmatrix}$$

$$= (a+f)(d+k) - (b+g)(c+h)$$

$$= ad + ak + fd + fk - bc - bh - gc - gh \quad \text{———} \ (1)$$

Now

$$\det A + \det B = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} f & g \\ h & k \end{vmatrix}$$

$$= ad - bc + fk - gh \quad \text{———} \ (2)$$

from (1) & (2)

$$\det(A+B) \neq \det A + \det B$$

(ii) $\det(A+B)^2 = [\det(A+B)]^2$

Sol.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} f & g \\ h & k \end{bmatrix}$

then

$$A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} f & g \\ h & k \end{bmatrix}$$

or

$$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

Now

$$(A+B)^2 = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}\begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

or

$$(A+B)^2 = \begin{bmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{bmatrix}$$

So

$$\det(A+B)^2 = \begin{vmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{vmatrix} \quad \text{———} (1)$$

Now

$$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

then

$$\det(A+B) = \begin{vmatrix} a+f & b+g \\ c+h & d+k \end{vmatrix}$$

&

$$[\det(A+B)]^2 = \begin{vmatrix} a+f & b+g \\ c+h & d+k \end{vmatrix}\begin{vmatrix} a+f & b+g \\ c+h & d+k \end{vmatrix}$$

or

$$[\det(A+B)]^2 = \begin{vmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{vmatrix} \quad \text{———} (2)$$

from (1) & (2)

$$\det(A+B)^2 = [\det(A+B)]^2$$

(iii) $\det(A+B)^2 = \det(A^2+B^2)$

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} f & g \\ h & k \end{bmatrix}$

then

$$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

Now

$$(A+B)^2 = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}\begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{bmatrix}$$

So

$$\det(A+B)^2 = \begin{vmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{vmatrix} \quad \text{--- } ①$$

Now

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

& $$B^2 = \begin{bmatrix} f & g \\ h & k \end{bmatrix}\begin{bmatrix} f & g \\ h & k \end{bmatrix}$$

$$= \begin{bmatrix} f^2+gh & fg+gk \\ hf+kh & gh+k^2 \end{bmatrix}$$

So

$$A^2+B^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} + \begin{bmatrix} f^2+gh & fg+gk \\ hf+kh & gh+k^2 \end{bmatrix}$$

$$A^2+B^2 = \begin{bmatrix} a^2+f^2+bc+gh & ab+bd+fg+gk \\ ac+cd+hf+kh & d^2+k^2+bc+gh \end{bmatrix}$$

So

$$\det(A^2+B^2) = \begin{vmatrix} a^2+f^2+bc+gh & ab+bd+fg+gk \\ ac+cd+hf+kh & d^2+k^2+bc+gh \end{vmatrix} \quad \text{--- } ②$$

from ① & ②

$$\det(A+B)^2 \neq \det(A^2+B^2)$$

(iv) $\det(A+B)^2 = \det(A^2+2AB+B^2)$

Sol.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} f & g \\ h & k \end{bmatrix}$

then

$$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

Now

$$(A+B)^2 = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}\begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{bmatrix}$$

So

$$\det(A+B)^2 = \begin{vmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{vmatrix} \quad \text{——— (1)}$$

Now

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} f & g \\ h & k \end{bmatrix}\begin{bmatrix} f & g \\ h & k \end{bmatrix} = \begin{bmatrix} f^2+gh & fg+gk \\ hf+kh & gh+k^2 \end{bmatrix}$$

&

$$2AB = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} f & g \\ h & k \end{bmatrix}$$

$$= 2\begin{bmatrix} af+bh & ag+bk \\ cf+dh & gc+dk \end{bmatrix}$$

$$= \begin{bmatrix} 2af+2bh & 2ag+2bk \\ 2cf+2dh & 2gc+2dk \end{bmatrix}$$

So

$$A^2+2AB+B^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} + \begin{bmatrix} 2af+2bh & 2ag+2bk \\ 2cf+2dh & 2gc+2dk \end{bmatrix} + \begin{bmatrix} f^2+gh & fg+gk \\ hf+kh & gh+k^2 \end{bmatrix}$$

$$A^2+2AB+B^2 = \begin{bmatrix} a^2+bc+2af+2bh+f^2+gh & ab+bd+2ag+2bk+fg+gk \\ ac+cd+2cf+2dh+hf+kh & bc+d^2+2gc+2dk+gh+k^2 \end{bmatrix}$$

So

$$\det(A^2+2AB+B^2) = \begin{vmatrix} a^2+bc+2af+2bh+f^2+gh & ab+bd+2ag+2bk+fg+gk \\ ac+cd+2cf+2dh+hf+kh & bc+d^2+2gc+2dk+gh+k^2 \end{vmatrix} \quad \text{—②}$$

from ① & ②

$$\det(A+B)^2 \neq \det(A^2+2AB+B^2)$$

Q3 Find the Value of each of the following determinants:

(i)
$$\begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}.$$

Sol:

$$\text{Let } \Delta = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}$$

Expanding from R_1

$$= 1\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= 1(28-30) - 0 + 2(18-20)$$

$$= -2 + 2(-2)$$

$$= -2-4$$

$$\Delta = -6$$

(ii) $\begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 4 \\ -1 & 0 & 3 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 4 \\ -1 & 0 & 3 \end{vmatrix}$

Expanding from R_1

$$= 2\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ -1 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ -1 & 0 \end{vmatrix}$$

$$= 2(6-0) + (9+4) + (0+2)$$

$$= 2(6) + 13 + 2$$

$$= 12 + 15$$

$$\Delta = 27$$

(iii) $\begin{vmatrix} 6 & -6 & 6 \\ 2 & 4 & -6 \\ 15 & -5 & 5 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} 6 & -6 & 6 \\ 2 & 4 & -6 \\ 15 & -5 & 5 \end{vmatrix}$

Expanding from R_1

$$= 6\begin{vmatrix} 4 & -6 \\ -5 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & -6 \\ 15 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 4 \\ 15 & -5 \end{vmatrix}$$

$$= 6(20-30) + 6(10+90) + 6(-10-60)$$

$$= 6(-10) + 6(100) + 6(-70)$$

$$= -60 + 600 - 420$$

$$\Delta = -60 + 180 = 120$$

Q4 Evaluate

(i) $$\begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 1 & -1 & 5 & 0 \\ -2 & 4 & 0 & 5 \end{vmatrix} \qquad R_3 - R_1$$

$$= \begin{vmatrix} 0 & 5 & -12 & 4 \\ 0 & 11 & -38 & 10 \\ 1 & -1 & 5 & 0 \\ 0 & 2 & 10 & 5 \end{vmatrix} \qquad \begin{matrix} R_1 - 2R_3 \\ R_2 - 7R_3 \\ R_4 + 2R_3 \end{matrix}$$

Expanding from C_1

$$= 0 - 0 + \begin{vmatrix} 5 & -12 & 4 \\ 11 & -38 & 10 \\ 2 & 10 & 5 \end{vmatrix} - 0$$

$$= \begin{vmatrix} 5 & -12 & 4 \\ 11 & -38 & 10 \\ 2 & 10 & 5 \end{vmatrix}$$

Expanding from R_1

$$= 5\begin{vmatrix} -38 & 10 \\ 10 & 5 \end{vmatrix} + 12\begin{vmatrix} 11 & 10 \\ 2 & 5 \end{vmatrix} + 4\begin{vmatrix} 11 & -38 \\ 2 & 10 \end{vmatrix}$$

$$= 5(-190 - 100) + 12(55 - 20) + 4(110 + 76)$$

$$= 5(-290) + 12(35) + 4(186)$$

$$= -1450 + 420 + 744$$

$$= -1450 + 1164$$

$$\Delta = -286$$

(ii) $\begin{vmatrix} 3 & 7 & 5 & 2 \\ 2 & 4 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ 1 & 1 & 3 & 4 \end{vmatrix}$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} 3 & 7 & 5 & 2 \\ 2 & 4 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ 1 & 1 & 3 & 4 \end{vmatrix}$$

Expanding from R_3

$$= -2\begin{vmatrix} 7 & 5 & 2 \\ 4 & 1 & 1 \\ 1 & 3 & 4 \end{vmatrix}$$

Expanding from R_1

$$= -2\left\{7\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} - 5\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix}\right\}$$

$$= -2\left\{7(4-3) - 5(16-1) + 2(12-1)\right\}$$

$$= -2\left\{7(1) - 5(15) + 2(11)\right\}$$

$$= -2(7 - 75 + 22)$$

$$= -2(-46)$$

$$= 92$$

(iii) $\begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & -3 \\ 0 & -7 & 3 & 1 \end{vmatrix}$

Sol:

$$\text{Let } \Delta = \begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & -3 \\ 0 & -7 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & -3 & 1 \\ 0 & -7 & 3 & 1 \end{vmatrix} \qquad R_3 - R_1$$

$$= \begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 1 \\ -7 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & -4 & 8 \end{vmatrix} \qquad \begin{matrix} R_2 - 5R_1 \\ R_3 + 7R_1 \end{matrix}$$

{Expanding from C_1

$$= 1\begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix}$$

$$= 16 - 16$$

$$\Delta = 0$$

(iv) $\begin{vmatrix} 9 & 93 & 12 & -6 \\ 1 & 92 & 84 & -6 \\ 2 & 185 & 100 & -12 \\ 4 & 270 & 196 & -24 \end{vmatrix}$

Sol:

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Let $\Delta = \begin{vmatrix} 9 & 93 & 12 & -6 \\ 1 & 92 & 84 & -6 \\ 2 & 185 & 100 & -12 \\ 4 & 270 & 196 & -24 \end{vmatrix}$

taking -6 Common from C_4

$$= -6 \begin{vmatrix} 9 & 93 & 12 & 1 \\ 1 & 92 & 84 & 1 \\ 2 & 185 & 100 & 2 \\ 4 & 270 & 196 & 4 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 9 & 93 & 12 & 1 \\ -8 & -1 & 72 & 0 \\ -16 & -1 & 76 & 0 \\ -32 & -102 & 148 & 0 \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{matrix}$$

Expanding from C_4

$$= -6 \begin{vmatrix} -8 & -1 & 72 \\ -16 & -1 & 76 \\ -32 & -102 & 148 \end{vmatrix}$$

taking $-8, -1, 4$ Common from C_1, C_2, C_3

$$= (-6)(-8)(-1)(4) \begin{vmatrix} 1 & 1 & 18 \\ 2 & 1 & 19 \\ 4 & 102 & 37 \end{vmatrix}$$

$$= -192 \begin{vmatrix} 1 & 1 & 18 \\ 2 & 1 & 19 \\ 4 & 102 & 37 \end{vmatrix}$$

$$= -192 \begin{vmatrix} 1 & 1 & 18 \\ 0 & -1 & -17 \\ 0 & 98 & -35 \end{vmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$$

Expanding from C_1

$$= -192 \begin{vmatrix} -1 & -17 \\ 98 & -35 \end{vmatrix}$$

$$\Delta = -192(35+1666)$$

$$= -192(1701)$$

$$\Delta = -326592$$

(v) $\begin{vmatrix} 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 \\ -1 & 2 & -2 & 0 & -2 \\ 1 & -2 & 0 & 0 & 2 \\ -1 & 0 & -2 & 0 & 0 \end{vmatrix}$ $\quad C_2-C_1, \; C_3+C_1, \; C_4+C_1, \; C_5+C_1$

$= 0 \qquad (\because C_4 = 0)$

So $\Delta = 0$

Q5 Without expanding, show that

(i) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$

Sol.

Sol:-

Consider $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$

$= \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & k \end{vmatrix}$ $\quad R_{12}$

$= \begin{vmatrix} e & d & f \\ b & a & c \\ h & g & k \end{vmatrix}$ $\quad C_{12}$

$= \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$ By taking transpose

So
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$$

(ii) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$

Sol:

Let $\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$ By taking transpose

$= (-1)^3 \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$ By taking -1 Common from R_1, R_2, R_3

$\Delta = -\Delta$

$\Delta + \Delta = 0$

$$2\Delta = 0$$

$$\Delta = 0$$

$$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$$

(iii) $$\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix} = 0$$

Sol.

Let $$\Delta = \begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & c & 1 \\ b+c+a & a & 1 \\ c+a+b & b & 1 \end{vmatrix} \qquad C_1+C_2$$

$$= \begin{vmatrix} a+b+c & c & 1 \\ a+b+c & a & 1 \\ a+b+c & b & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & c & 1 \\ 1 & a & 1 \\ 1 & b & 1 \end{vmatrix} \qquad \text{taking } a+b+c \text{ Common from } C_1$$

$$= (a+b+c)(0) \qquad \because C_1 = C_3$$

$$\Delta = 0$$

or $$\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix} = 0$$

Q6 Prove that

(i) $$\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$$

Sol.

Let $$\Delta = \begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a^2 & b^2 & c^2 \end{vmatrix}$$ Multiplying R_2 by abc

$$= \frac{1}{abc}(0)$$ $\because R_1 = R_2$

$\Delta = 0$

So $$\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$$

(ii) $$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Sol.

Let $$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

2.

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} \quad C_1+(C_2+C_3)$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

$$\Delta = 0 \qquad \because C_1 = 0$$

So $$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

(iii) $$\begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix} = 0$$

Sol.

let $$\Delta = \begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a & a^2 & a^2 \\ b & b^2 & b^2 \\ c & c^2 & c^2 \end{vmatrix} \quad \text{Multiplying } C_3 \text{ by } abc$$

$$\Delta = \frac{1}{abc}(0)$$

$$\Delta = 0$$

So.
$$\begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix} = 0$$

(iv)
$$\begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix} = 0$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix}$$

$$= \begin{vmatrix} \sin^2\theta+\cos^2\theta & 1 & \cos^2\theta \\ \sin^2\phi+\cos^2\phi & 1 & \cos^2\phi \\ \sin^2\psi+\cos^2\psi & 1 & \cos^2\psi \end{vmatrix} \qquad R_1 + R_2$$

$$= \begin{vmatrix} 1 & 1 & \cos^2\theta \\ 1 & 1 & \cos^2\phi \\ 1 & 1 & \cos^2\psi \end{vmatrix}$$

$$\Delta = 0 \qquad \because C_1 = C_2$$

or
$$\begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix} = 0$$

(V)
$$\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix} = 0$$

Sol.

Let $\Delta = \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}$

$$= \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha - \sin^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta - \sin^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma - \sin^2\gamma \end{vmatrix} \qquad C_3 - C_1$$

$$= \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos 2\alpha \\ \sin^2\beta & \cos 2\beta & \cos 2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos 2\gamma \end{vmatrix} \qquad \because \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\Delta = 0 \qquad \because C_2 = C_3$$

So
$$\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix} = 0$$

(vi)
$$\begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0$$

Sol.

Let $\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix}$

or

$$\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \sin\alpha\cos\delta + \cos\alpha\sin\delta \\ \cos\beta & \sin\beta & \sin\beta\cos\delta + \cos\beta\sin\delta \\ \cos\gamma & \sin\gamma & \sin\gamma\cos\delta + \cos\gamma\sin\delta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & \sin\alpha\cos\delta \\ \cos\beta & \sin\beta & \sin\beta\cos\delta \\ \cos\gamma & \sin\gamma & \sin\gamma\cos\delta \end{vmatrix} + \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha\sin\delta \\ \cos\beta & \sin\beta & \cos\beta\sin\delta \\ \cos\gamma & \sin\gamma & \cos\gamma\sin\delta \end{vmatrix}$$

$$= \cos\delta \begin{vmatrix} \cos\alpha & \sin\alpha & \sin\alpha \\ \cos\beta & \sin\beta & \sin\beta \\ \cos\gamma & \sin\gamma & \sin\gamma \end{vmatrix} + \sin\delta \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \cos\beta & \sin\beta & \cos\beta \\ \cos\gamma & \sin\gamma & \cos\gamma \end{vmatrix}$$

$$= \cos\delta(0) + \sin\delta(0)$$ (∵ two Columns are identical)

$$= 0 + 0$$

$$\Delta = 0$$

So

$$\begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0$$

(vii)

$$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \end{vmatrix} = 0$$

Sol.

$$\Delta = \begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & 1-\cos^2\alpha & -\sin\alpha\sin\beta \\ \cos\beta & -\sin\alpha\sin\beta & 1-\cos^2\beta \end{vmatrix} \qquad \begin{matrix} C_2 - (\cos\alpha)C_1 \\ C_3 - (\cos\beta)C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin^2\alpha & -\sin\alpha\sin\beta \\ \cos\beta & -\sin\alpha\sin\beta & \sin^2\beta \end{vmatrix}$$

$$= (\sin\alpha)(-\sin\beta) \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin\alpha & \sin\alpha \\ \cos\beta & -\sin\beta & -\sin\beta \end{vmatrix}$$

taking $\sin\alpha$ Common from C_2 & $-\sin\beta$ Common from C_3

$$= -\sin\alpha\sin\beta(0) \qquad \because C_2 = C_3$$

$$\Delta = 0$$

So $$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \end{vmatrix} = 0$$

(viii)
$$\begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix} = 0$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+2ab & a^2+b^2 & ab \\ c^2+d^2+2cd & c^2+d^2 & cd \\ g^2+h^2+2gh & g^2+h^2 & gh \end{vmatrix}$$

$$= \begin{vmatrix} 2ab & a^2+b^2 & ab \\ 2cd & c^2+d^2 & cd \\ 2gh & g^2+h^2 & gh \end{vmatrix} \qquad C_1 - C_2$$

$$= 2\begin{vmatrix} ab & a^2+b^2 & ab \\ cd & c^2+d^2 & cd \\ gh & g^2+h^2 & gh \end{vmatrix} \qquad \text{taking 2 common from } C_1$$

$$= 2(0) \qquad (\because C_1 = C_3)$$

$$\Delta = 0$$

So
$$\begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix} = 0$$

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(ix) $$\begin{vmatrix} (a^m+a^{-m})^2 & (a^m-a^{-m})^2 & abc \\ (b^n+b^{-n})^2 & (b^n-b^{-n})^2 & abc \\ (c^p+c^{-p})^2 & (c^p-c^{-p})^2 & abc \end{vmatrix} = 0$$

Soln

Let $$\Delta = \begin{vmatrix} (a^m+a^{-m})^2 & (a^m-a^{-m})^2 & abc \\ (b^n+b^{-n})^2 & (b^n-b^{-n})^2 & abc \\ (c^p+c^{-p})^2 & (c^p-c^{-p})^2 & abc \end{vmatrix}$$

$$= abc \begin{vmatrix} a^{2m}+a^{-2m}+2 & a^{2m}+a^{-2m}-2 & 1 \\ b^{2n}+b^{-2n}+2 & a^{2m}+a^{-2m}-2 & 1 \\ c^{2p}+c^{-2p}+2 & c^{2p}+c^{-2p}-2 & 1 \end{vmatrix}$$

$$= abc \begin{vmatrix} a^{2m}+a^{-2m} & a^{2m}+a^{-2m} & 1 \\ b^{2n}+b^{-2n} & b^{2n}+b^{-2n} & 1 \\ c^{2p}+c^{-2p} & c^{2p}+c^{-2p} & 1 \end{vmatrix}$$ C_1-2C_3, C_2+2C_3

$= (abc)(0)$ $\because C_1 = C_2$

$\Delta = 0$

(x) $$\begin{vmatrix} \frac{1}{2!} & 1 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} = 0$$

Soln

let $$\Delta = \begin{vmatrix} \frac{1}{2!} & 1 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \frac{1}{2} & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{2} \end{vmatrix}$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(\frac{1}{24}\right) \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 6 \\ 1 & 4 & 12 \end{vmatrix}$$ taking $\frac{1}{2}, \frac{1}{6}, \frac{1}{24}$ common from R_1, R_2, R_3

$$= \frac{1}{288} \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 6 \\ 1 & 4 & 12 \end{vmatrix}$$

$$= \frac{1}{288} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 1 & 2 & 12 \end{vmatrix}$$

$$= \frac{6}{288} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$ taking 6 common from C_3

$$= \frac{1}{48}(0)$$ $\because C_2 = C_3$

$$\Delta = 0$$

(xi)
$$\begin{vmatrix} a^2 & b\sin\alpha & c\sin\alpha \\ b\sin\alpha & 1 & \cos\alpha \\ c\sin\alpha & \cos\alpha & 1 \end{vmatrix} = 0$$

where a, b, c are the magnitudes of the sides of a triangle & α is the measure of the angle opposite to the side with magnitude a

$$= \begin{vmatrix} a^2 - c^2\sin^2\alpha & b\sin\alpha - c\sin\alpha\cos\alpha & 0 \\ b\sin\alpha - c\sin\alpha\cos\alpha & 1 - \cos^2\alpha & 0 \\ c\sin\alpha & \cos\alpha & 1 \end{vmatrix} \quad \begin{matrix} R_1 - c\sin\alpha\, R_3 \\ R_2 - \cos\alpha\, R_3 \end{matrix}$$

Expanding from C_3

$$= \begin{vmatrix} a^2 - c^2\sin^2\alpha & \sin\alpha(b - c\cos\alpha) \\ \sin\alpha(b - \cos\alpha) & \sin^2\alpha \end{vmatrix}$$

$$= \sin^2\alpha\,(a^2 - c^2\sin^2\alpha) - \sin^2\alpha\,(b - c\cos\alpha)^2$$

$$= a^2\sin^2\alpha - c^2\sin^4\alpha - \sin^2\alpha\,(b^2 + c^2\cos^2\alpha - 2bc\cos\alpha)$$

$$= a^2\sin^2\alpha - c^2\sin^4\alpha - b^2\sin^2\alpha - c^2\sin^2\alpha\cos^2\alpha + 2bc\sin^2\alpha\cos\alpha$$

$$= a^2\sin^2\alpha - c^2\sin^4\alpha - b^2\sin^2\alpha - c^2\sin^2\alpha\,(1 - \sin^2\alpha) + 2bc\sin^2\alpha\cos\alpha$$

$$= a^2\sin^2\alpha - \cancel{c^2\sin^4\alpha} - b^2\sin^2\alpha - c^2\sin^2\alpha + \cancel{c^2\sin^4\alpha} + 2bc\sin^2\alpha\cos\alpha$$

$$= [a^2 - b^2 - c^2 + 2bc\cos\alpha]\sin^2\alpha$$

$$= \left[a^2 - b^2 - c^2 + 2bc\left(\frac{b^2 + c^2 - a^2}{2bc}\right)\right]\sin^2\alpha$$

$$= [\cancel{a^2} - \cancel{b^2} - \cancel{c^2} + \cancel{b^2} + \cancel{c^2} - \cancel{a^2}]\sin^2\alpha$$

$$= (0)\sin^2\alpha$$

$$\Delta = 0$$

(XII)

$$\begin{vmatrix} a & b & c & d & 1 \\ b & c & d & a & 1 \\ c & d & a & b & 1 \\ d & a & b & c & 1 \\ b & a & d & c & 1 \end{vmatrix} = 0$$

Sol:-

$$\text{Let } \Delta = \begin{vmatrix} a & b & c & d & 1 \\ b & c & d & a & 1 \\ c & d & a & b & 1 \\ d & a & b & c & 1 \\ b & a & d & c & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c+d & b & c & d & 1 \\ b+c+d+a & c & d & a & 1 \\ c+d+a+b & d & a & b & 1 \\ d+a+b+c & a & b & c & 1 \\ b+a+d+c & a & d & c & 1 \end{vmatrix} \qquad C_1 + (C_2 + C_3 + C_4)$$

$$= \begin{vmatrix} a+b+c+d & b & c & d & 1 \\ a+b+c+d & c & d & a & 1 \\ a+b+c+d & d & a & b & 1 \\ a+b+c+d & a & b & c & 1 \\ a+b+c+d & a & d & c & 1 \end{vmatrix}$$

$$= (a+b+c+d) \begin{vmatrix} 1 & b & c & d & 1 \\ 1 & c & d & a & 1 \\ 1 & d & a & b & 1 \\ 1 & a & b & c & 1 \\ 1 & a & d & c & 1 \end{vmatrix}$$

$$= (a+b+c+d)(0) \qquad \because C_1 = C_5$$

$$= 0$$

So $\Delta = 0$

3.

(xiii) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & a^2+2a+1 & a^2+4a+4 & a^2+6a+9 \\ b^2 & b^2+2b+1 & b^2+4b+4 & b^2+6b+9 \\ c^2 & c^2+2c+1 & c^2+4c+4 & c^2+6c+9 \\ d^2 & d^2+2d+1 & d^2+4d+4 & d^2+6d+9 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \\ C_4 - C_1 \end{matrix}$$

$$= \begin{vmatrix} a^2 & 2a+1 & 2 & 6 \\ b^2 & 2b+1 & 2 & 6 \\ c^2 & 2c+1 & 2 & 6 \\ d^2 & 2d+1 & 2 & 6 \end{vmatrix} \quad \begin{matrix} C_3 - 2C_2 \\ C_4 - 3C_2 \end{matrix}$$

$$= 3\begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} \quad \text{taking 3 common from } C_4$$

$$\Delta = 0 \qquad \because C_3 = C_4$$

Q7 Without expansion, prove that

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Sol.

Let $$\Delta = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$$ Multiplying R_1, R_2, R_3 by a, b, c

$$= \frac{\cancel{abc}}{\cancel{abc}}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$ taking abc common from C_1

$$\Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

So

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Q8 Prove that

$$\begin{vmatrix} 1 & \alpha & \beta\gamma \\ 1 & \beta & \gamma\alpha \\ 1 & \gamma & \alpha\beta \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}$$

Sol.

Let $$\Delta = \begin{vmatrix} 1 & \alpha & \beta\gamma \\ 1 & \beta & \gamma\alpha \\ 1 & \gamma & \alpha\beta \end{vmatrix}$$

Q9 Prove that

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}$$

$$= \frac{1}{abcd}\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ -ab & ab & -cd & cd \\ -ac & bd & ac & -bd \\ -ad & -bc & bc & ad \end{vmatrix}$$ Multiplying C_1, C_2, C_3, C_4 by a, b, c, d resp.

$$= \frac{1}{abcd}\begin{vmatrix} a^2+b^2+c^2+d^2 & b^2 & c^2 & d^2 \\ 0 & ab & -cd & cd \\ 0 & bd & ac & -bd \\ 0 & -bc & bc & ad \end{vmatrix}$$ $C_1 + (C_2+C_3+C_4)$

Expanding from C_1

$$= \frac{(a^2+b^2+c^2+d^2)}{abcd}\begin{vmatrix} ab & -cd & cd \\ bd & ac & -bd \\ -bc & bc & ad \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2+d^2)\cancel{bcd}}{a\cancel{bcd}}\begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix}$$ taking b, c, d common from C_1, C_2, C_3

$$= \frac{(a^2+b^2+c^2+d^2)}{a}\begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix}$$

Expanding from R_1

$$\Delta = \frac{(a^2+b^2+c^2+d^2)}{a}\left\{a(a^2+b^2)+d(ad-bc)+c(bd+ac)\right\}$$

$$= \frac{(a^2+b^2+c^2+d^2)}{a}\left\{a^3+ab^2+ad^2-\cancel{bc}d+\cancel{bc}d+ac^2\right\}$$

$$= (a^2+b^2+c^2+d^2)(a^2+b^2+c^2+d^2)$$

$$\Delta = (a^2+b^2+c^2+d^2)^2$$

Q10 Prove that

(i) $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

Sol.

Let $$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix} \quad \begin{matrix} C_1-C_3 \\ C_2-C_3 \end{matrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$\Delta = (a+b+c)^2 \begin{vmatrix} (b+c-a) & 0 & a^2 \\ 0 & (c+a-b) & b^2 \\ (c-a-b) & (c-a-b) & (a+b)^2 \end{vmatrix}$$ taking $(a+b+c)$ Common from C_1 & C_2

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$ $R_3-(R_1+R_2)$

$$= -2(a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ b & a & -ab \end{vmatrix}$$

Expanding from R_1

$$= -2(a+b+c)^2 \left[(b+c-a)\left\{(-ab)(c+a-b) - ab^2\right\} - 0 + a^2\left\{0 - b(c+a-b)\right\} \right]$$

$$= -2(a+b+c)^2 \left\{ (b+c-a)(-abc - a^2b + \cancel{ab^2} - \cancel{ab^2}) + a^2(-bc - ab + b^2) \right\}$$

$$= -2(a+b+c)^2 \left\{ (-ab)(b+c-a)(c+a) + (-ab)(ac + a^2 - ab) \right\}$$

$$= -2(a+b+c)^2(-ab)\left\{ (b+c-a)(c+a) + ac + a^2 - ab \right\}$$

$$= 2ab(a+b+c)^2 \left\{ bc + \cancel{ab} + c^2 + \cancel{ac} - \cancel{ac} - \cancel{a^2} + ac + \cancel{a^2} - \cancel{ab} \right\}$$

$$= 2ab(a+b+c)^2(bc + c^2 + ac)$$

$$= 2abc(a+b+c)^2(b+c+a)$$

$$\Delta = 2abc(a+b+c)^3$$

(ii)
$$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$$

Sol:

$$\text{Let } \Delta = \begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix} \quad \text{Multiplying } R_1, R_2, R_3 \text{ by } c, a, b \text{ resp.}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ b^2-c^2-a^2 & b^2-c^2-a^2 & c^2+a^2 \end{vmatrix} \quad \begin{matrix} C_1 - C_3 \\ C_2 - C_3 \end{matrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ -2a^2 & -2c^2 & 0 \end{vmatrix} \quad R_3 - (R_1+R_2)$$

$$= \frac{-2}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ a^2 & c^2 & 0 \end{vmatrix} \quad \text{taking } -2 \text{ Common from } R_3$$

Expanding from R_1

$$= -\frac{2}{abc}\left\{(a^2+b^2-c^2)(0-a^2c^2) - 0 + c^2\left(0 - a^2(b^2+c^2-a^2)\right)\right\}$$

$$-\frac{2}{abc}\left\{-a^2c^2(a^2+b^2-c^2) - a^2c^2(b^2+c^2-a^2)\right\}$$

$$= -\frac{2}{abc}(-a^2c^2)\left\{\cancel{a^2}+b^2-\cancel{c^2}+b^2+\cancel{c^2}-\cancel{a^2}\right\}$$

$$= \frac{2ac}{b}(2b^2)$$

$$\Delta = 4abc$$

Q11 Prove that

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)(x-a)^3$$

Sol

Let $\Delta = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$

$$= \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix} \qquad C_1 + (C_2 + C_3 + C_4)$$

$$= (x+3a)\begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix} \qquad \text{taking } (x+3a) \text{ Common from } C_1$$

$$= (x+3a)\begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

$$\Delta = (x+3a)(x-a)^3 \begin{vmatrix} 1 & a & a & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$ taking $(x-a)$ Common from R_2, R_3, R_4

Expanding from C_1

$$= (x+3a)(x-a)^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta = (x+3a)(x-a)^3$$

Q12 Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

Sol.

Let $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \alpha-\gamma & \beta-\gamma & \gamma \\ \beta(\gamma-\alpha) & \alpha(\gamma-\beta) & \alpha\beta \end{vmatrix}$$ $C_1 - C_3$, $C_2 - C_3$

Expanding from R_1

$$= \begin{vmatrix} \alpha-\gamma & \beta-\gamma \\ \beta(\gamma-\alpha) & \alpha(\gamma-\beta) \end{vmatrix}$$

$$= (\alpha-\gamma)(\beta-\gamma) \begin{vmatrix} 1 & 1 \\ -\beta & -\alpha \end{vmatrix}$$ taking $\alpha-\gamma$, $\beta-\gamma$ Common from C_1, C_2 resp.

$$\Delta = (\alpha-\gamma)(\beta-\gamma)(-\alpha+\beta)$$

$$\Delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

Q13 Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Sol:-

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \alpha-\gamma & \beta-\gamma & \gamma \\ \alpha^3-\gamma^3 & \beta^3-\gamma^3 & \gamma^3 \end{vmatrix} \quad \begin{matrix} C_1 - C_3 \\ C_2 - C_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ (\alpha-\gamma) & \beta-\gamma & \gamma \\ (\alpha-\gamma)(\alpha^2+\alpha\gamma+\gamma^2) & (\beta-\gamma)(\beta^2+\beta\gamma+\gamma^2) & \gamma^3 \end{vmatrix}$$

$$= (\alpha-\gamma)(\beta-\gamma) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \gamma \\ \alpha^2+\alpha\gamma+\gamma^2 & \beta^2+\beta\gamma+\gamma^2 & \gamma^3 \end{vmatrix}$$ taking $(\alpha-\gamma)$, $(\beta-\gamma)$ Common from C_1 & C_2 resp.

Expanding from R_1

$$= (\alpha-\gamma)(\beta-\gamma) \begin{vmatrix} 1 & 1 \\ \alpha^2+\alpha\gamma+\gamma^2 & \beta^2+\beta\gamma+\gamma^2 \end{vmatrix}$$

$$= (\alpha-\gamma)(\beta-\gamma)\left\{ \beta^2+\beta\gamma+\cancel{\gamma^2}-\alpha^2-\alpha\gamma-\cancel{\gamma^2} \right\}$$

$$= (\alpha-\gamma)(\beta-\gamma)\left\{ \beta^2-\alpha^2+\beta\gamma-\gamma\alpha \right\}$$

$$\Delta = (\alpha-\gamma)(\beta-\gamma)\{(\beta-\alpha)(\beta+\alpha)+\gamma(\beta-\alpha)\}$$

$$= (\alpha-\gamma)(\beta-\gamma)(\beta-\alpha)(\beta+\alpha+\gamma)$$

$$\Delta = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Q14 Show that

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3)(a-1)^3$$

Sol.

Let $\Delta = \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}$

$$= \begin{vmatrix} a+3 & 1 & 1 & 1 \\ a+3 & a & 1 & 1 \\ a+3 & 1 & a & 1 \\ a+3 & 1 & 1 & a \end{vmatrix} \qquad C_1+(C_2+C_3+C_4)$$

$$=(a+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} \qquad \text{taking } a+3 \text{ common from } C_1$$

$$=(a+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & a-1 \end{vmatrix} \qquad \begin{matrix} R_2-R_1 \\ R_3-R_1 \\ R_4-R_1 \end{matrix}$$

$$=(a+3)\begin{vmatrix} a-1 & 0 & 0 \\ 0 & a-1 & 0 \\ 0 & 0 & a-1 \end{vmatrix} \qquad \text{Expanding from } C_1$$

$$\Delta = (a+3)(a-1)^3\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}$$ taking $a-1$ Common from R_1, R_2, R_3

$$\Delta = (a+3)(a-1)^3$$

(ii) $$\begin{vmatrix}1+a&1&1&1\\1&1+b&1&1\\1&1&1+c&1\\1&1&1&1+d\end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix}1+a&1&1&1\\1&1+b&1&1\\1&1&1+c&1\\1&1&1&1+d\end{vmatrix}$$

$$= abcd\begin{vmatrix}1+\frac{1}{a}&\frac{1}{a}&\frac{1}{a}&\frac{1}{a}\\\frac{1}{b}&1+\frac{1}{b}&\frac{1}{b}&\frac{1}{b}\\\frac{1}{c}&\frac{1}{c}&1+\frac{1}{c}&\frac{1}{c}\\\frac{1}{d}&\frac{1}{d}&\frac{1}{d}&1+\frac{1}{d}\end{vmatrix}$$ taking a, b, c, d Common from R_1, R_2, R_3, R_4 resp.

Adding R_2, R_3, R_4 in R_1

$$= abcd\begin{vmatrix}1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}&1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}&1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}&1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\\\frac{1}{b}&1+\frac{1}{b}&\frac{1}{b}&\frac{1}{b}\\\frac{1}{c}&\frac{1}{c}&1+\frac{1}{c}&\frac{1}{c}\\\frac{1}{d}&\frac{1}{d}&\frac{1}{d}&1+\frac{1}{d}\end{vmatrix}$$

$$= (abcd)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix}1&1&1&1\\\frac{1}{b}&1+\frac{1}{b}&\frac{1}{b}&\frac{1}{b}\\\frac{1}{c}&\frac{1}{c}&1+\frac{1}{c}&\frac{1}{c}\\1/d&1/d&1/d&1+1/d\end{vmatrix}$$ taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$ Common from R_1

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1/b & 1 & 0 & 0 \\ 1/c & 0 & 1 & 0 \\ 1/d & 0 & 0 & 1 \end{vmatrix} \qquad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \\ C_4 - C_1 \end{matrix}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{Expanding from } R_1$$

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

(iii) $$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2$$

Sol:-

$$\text{Let } \Delta = \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ -x & -x & 0 & 1 \\ -x & 0 & y & 1 \\ (-x+y+xy) & y & y & 1-y \end{vmatrix} \qquad \begin{matrix} C_1 - (1+x)C_4 \\ C_2 - C_4 \\ C_3 - C_4 \end{matrix}$$

Expanding from R_1

$$= -\begin{vmatrix} -x & -x & 0 \\ -x & 0 & y \\ -x+y+xy & y & y \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & -x & 0 \\ -x & 0 & y \\ -x+xy & y & y \end{vmatrix}$$

$$= x\begin{vmatrix} 0 & 1 & 0 \\ -x & 0 & y \\ -x+xy & y & y \end{vmatrix}.$$ taking $-x$ Common from R_1

Expanding from R_1

$$= -x\begin{vmatrix} -x & y \\ -x+xy & y \end{vmatrix}$$

$$= -x\left(-\cancel{x}y + \cancel{x}y - xy^2\right)$$

$$= -x\left(-xy^2\right)$$

$$\Delta = x^2y^2$$

Q15
$$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$

Sol.

$$\text{let } \Delta = \begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^3-1 & 3(a^2-1) & 3(a-1) & 0 \\ a^2-1 & a^2+2a-3 & 2(a-1) & 0 \\ a-1 & 2(a-1) & a-1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$ R_1-R_4, R_2-R_4, R_3-R_4

Expanding from C_4

$$\Delta = \begin{vmatrix} a^3-1 & 3(a^2-1) & 3(a-1) \\ a^2-1 & a^2+2a-3 & 2(a-1) \\ a-1 & 2(a-1) & a-1 \end{vmatrix}$$

$$= \begin{vmatrix} (a-1)(a^2+a+1) & 3(a-1)(a+1) & 3(a-1) \\ (a-1)(a+1) & (a+3)(a-1) & 2(a-1) \\ a-1 & 2(a-1) & a-1 \end{vmatrix}$$

$$= (a-1)^3 \begin{vmatrix} a^2+a+1 & 3a+3 & 3 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$ taking $a-1$ Common from C_1, C_2, C_3

$$= (a-1)^3 \begin{vmatrix} a^2+a-2 & 3a-3 & 0 \\ a-1 & a-1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$ $C_1 - 3C_3$, $C_2 - 2C_3$

Expanding from C_3

$$= (a-1)^3 \begin{vmatrix} (a+2)(a-1) & 3(a-1) \\ a-1 & a-1 \end{vmatrix}$$

$$= (a-1)^5 \begin{vmatrix} a+2 & 3 \\ 1 & 1 \end{vmatrix}$$ taking $a-1$ Common from C_1 & C_2

$$= (a-1)^5 (a+2-3)$$

$$= (a-1)(a-1)$$

$$\Delta = (a-1)^6$$ —— Ans.

Q16 If
$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

& A, B, C, ... are Cofactors of a, b, c, ... in Δ, then show that

(i) $BC - F^2 = a\Delta$

(ii) $CA - G^2 = b\Delta$

(iii) $AB - H^2 = c\Delta$

(iv) $GH - AF = f\Delta$

(v) $HF - BG = g\Delta$

(vi) $FG - CH = h\Delta$

(vii) $aG + hF + gC = 0$

(viii) $hG + bF + fC = 0$

(ix) $gG + fF + cC = \Delta$

Sol.

Here $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Expanding from R_1

$$= a(bc - f^2) - h(ch - gf) + g(hf - gb)$$

$$= abc - af^2 - ch^2 + ghf + ghf - g^2b$$

$$\Delta = abc + 2ghf - af^2 - g^2b - ch^2 \quad \text{——— (1)}$$

Now

$$A = (-1)^{1+1}\begin{vmatrix} b & f \\ f & c \end{vmatrix} = (-1)^2(bc - f^2) = bc - f^2$$

$$B = (-1)^{2+2}\begin{vmatrix} a & g \\ g & c \end{vmatrix} = (-1)^4(ac - g^2) = ac - g^2$$

$$C = (-1)^{3+3}\begin{vmatrix} a & h \\ h & b \end{vmatrix} = (-1)^6(ab - h^2) = ab - h^2.$$

$$F = (-1)^{2+3}\begin{vmatrix} a & h \\ g & f \end{vmatrix} = (-1)^5(af - gh) = gh - af$$

$$G = (-1)^{1+3}\begin{vmatrix} h & b \\ g & f \end{vmatrix} = (-1)^4(hf - gb) = hf - gb$$

$$H = (-1)^{1+2}\begin{vmatrix} h & f \\ g & c \end{vmatrix} = (-1)^3(ch - gf) = gf - ch$$

Now

(i) $BC - F^2 = (ac - g^2)(ab - h^2) - (gh - af)^2$

$$= a^2bc - ach^2 - abg^2 + \cancel{g^2h^2} - \cancel{g^2h^2} - a^2f^2 + 2afgh$$
$$= a^2bc - ach^2 - abg^2 - a^2f^2 + 2afgh$$
$$= a(abc + 2fgh - af^2 - g^2b^2 - ch^2)$$

So $BC - F^2 = a\Delta$

(ii) $CA - G^2 = b\Delta$

Sol.

$CA - G^2 = (ab - h^2)(bc - f^2) - (hf - gb)^2$

$$= ab^2c - abf^2 - h^2bc + \cancel{h^2f^2} - \cancel{h^2f^2} - g^2b^2 + 2fghb$$
$$= b(abc + 2ghf - af^2 - g^2b - ch^2)$$

$CA - G^2 = b\Delta$

(iii) $AB - H^2 = c\Delta$

Sol.

$AB - H^2 = (bc - f^2)(ac - g^2) - (gf - hc)^2$

$$= abc^2 - g^2bc - f^2ac + \cancel{g^2f^2} - \cancel{g^2f^2} - c^2h^2 + 2ghfc$$
$$= c(abc + 2ghf - af^2 - g^2b - ch^2)$$

$AB - H^2 = c\Delta$

(iv) $GH - AF = f\Delta$

Sol.

$GH - AF = (hf - bg)(fg - ch) - (bc - f^2)(gh - af)$

$$= f^2gh - ch^2f - bfg^2 + \cancel{bcgh} - \cancel{bcgh} + abcf + f^2gh - af^3$$

$$GH - AF = 2f^2gh - ch^2f - bfg^2 + abcf - af^3$$ 46

$$= f(abc + 2fgh - af^2 - g^2b - ch^2)$$

$$GH - AF = f\Delta$$

(v) $HF - BG = g\Delta$

Sol.

$$HF - BG = (fg - ch)(gh - af) - (ac - g^2)(hf - gb)$$

$$= g^2fh - af^2g - gch^2 + \cancel{ac}fh - \cancel{ac}fh + acgb + g^2hf - g^3b$$

$$= abcg + 2g^2hf - af^2g - gch^2 - g^3b$$

$$= g(abc + 2ghf - af^2 - ch^2 - g^2b)$$

$$HF - BG = g\Delta$$

(vi) $FG - CH = h\Delta$

Sol.

$$FG - CH = (gh - af)(hf - gb) - (ab - h^2)(gf - ch)$$

$$= gh^2f - g^2hb - ahf^2 + a\cancel{b}gf - a\cancel{b}gf + abch + h^2gf - ch^3$$

$$= abch + 2gh^2f - ahf^2 - g^2hb - ch^3$$

$$= h(abc + 2ghf - af^2 - g^2b - ch^2)$$

$$FG - CH = h\Delta$$

(vii) $aG + hF + gC = 0$

Sol.

$$aG + hF + gC = a(hf - bg) + h(gh - af) + g(ab - h^2)$$

$$= a\cancel{h}f - a\cancel{b}g + \cancel{gh^2} - a\cancel{h}f + \cancel{gab} - \cancel{gh^2}$$

$$= 0$$

(viii) $hG + bF + fC = 0$

Sol.

$$hG + bF + fC = h(hf - bg) + b(gh - af) + f(ab - h^2) \qquad 47$$
$$= \cancel{h^2 f} - \cancel{bgh} + \cancel{bgh} - \cancel{abf} + \cancel{abf} - \cancel{h^2 f}$$
$$= 0$$

(ix) $gG + fF + cC = \Delta$

Sol:

$$gG + fF + cC = g(hf - bg) + f(gh - af) + c(ab - h^2)$$
$$= ghf - g^2 b + ghf - af^2 + abc - ch^2$$
$$= abc + 2ghf - af^2 - g^2 b - ch^2$$
$$gG + fF + cC = \Delta$$

Q17 If Δ of problem 16 is zero, show that

(i) $F^2 + G^2 = C(A+B)$

(ii) $G^2 + H^2 = A(B+C)$

(iii) $H^2 + F^2 = B(A+C)$

(iv) $ABC = FGH$

Sol:

(i) As we have proved that

$$\left.\begin{aligned} BC - F^2 &= a\Delta \\ \& \quad CA - G^2 &= b\Delta \end{aligned}\right]$$

But $\Delta = 0$

So

$$\left.\begin{aligned} BC - F^2 &= 0 \\ CA - G^2 &= 0 \end{aligned}\right]$$

or

$$\left.\begin{aligned} F^2 &= BC \\ G^2 &= CA \end{aligned}\right]$$

or

$$F^2 + G^2 = BC + CA$$
$$F^2 + G^2 = C(B + A)$$

or $F^2 + G^2 = C(A+B)$

(ii) $G^2 + H^2 = A(B+C)$

Soln. As we have proved that

$$\left.\begin{aligned} CA - G^2 &= b\Delta \\ \& \quad AB - H^2 &= c\Delta \end{aligned}\right]$$

But $\Delta = 0$

So

$$\left.\begin{aligned} CA - G^2 &= 0 \\ AB - H^2 &= 0 \end{aligned}\right]$$

or

$$\left.\begin{aligned} G^2 &= CA \\ H^2 &= AB \end{aligned}\right]$$

Adding

$$\begin{aligned} G^2 + H^2 &= CA + AB \\ &= A(C+B) \\ G^2 + H^2 &= A(B+C) \end{aligned}$$

(iii) $H^2 + F^2 = B(A+C)$

Soln. As we have proved that

$$\left.\begin{aligned} AB - H^2 &= c\Delta \\ \& \quad BC - F^2 &= a\Delta \end{aligned}\right]$$

But $\Delta = 0$

So

$$\left.\begin{aligned} AB - H^2 &= 0 \\ BC - F^2 &= 0 \end{aligned}\right]$$

or

$$\left.\begin{aligned} H^2 &= AB \\ \& \quad F^2 &= BC \end{aligned}\right]$$

Adding $H^2 + F^2 = B(A+C)$

(iv) $ABC = FGH$

Sol. As we know that

$$\left.\begin{aligned} BC &= F^2 \\ CA &= G^2 \\ AB &= H^2 \end{aligned}\right\}$$

Multiplying these eqs.

$$(BC)(CA)(AB) = F^2G^2H^2$$

$$A^2B^2C^2 = F^2G^2H^2$$

or $$ABC = FGH$$

Q18 Prove that

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

Q19 Show that

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix}^2 = 125$$

where ω is a fifth root of 1.

Sol.

Since ω is the fifth root of 1

So $\omega^5 = 1$ ——— (1)

& $1+\omega+\omega^2+\omega^3+\omega^4 = 0$ ——— (2)

Now

let $$\Delta = \begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix}^2$$

$$= \begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+\omega+\omega^2+\omega^3 & \omega+\omega^3+\omega^5+\omega^7 & \omega^2+\omega^5+\omega^3+\omega^6 & \omega^3+\omega^2+\omega^6+\omega^5 \\ 1+\omega^2+\omega^4+\omega & \omega+\omega^4+\omega^7+\omega^5 & \omega^2+\omega^6+\omega^5+\omega^4 & \omega^3+\omega^3+\omega^8+\omega^3 \\ 1+\omega^3+\omega+\omega^4 & \omega+\omega^5+\omega^4+\omega^8 & \omega^2+\omega^7+\omega^2+\omega^7 & \omega^3+\omega^4+\omega^5+\omega^6 \\ 1+\omega^4+\omega^3+\omega^2 & \omega+\omega^6+\omega^6+\omega^6 & \omega^2+\omega^8+\omega^4+\omega^5 & \omega^3+\omega^5+\omega^7+\omega^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1+\omega+\omega^2+\omega^3 & 1+\omega+\omega^2+\omega^3 & 1+\omega+\omega^2+\omega^3 & 1+\omega+\omega^2+\omega^3 \\ 1+\omega+\omega^2+\omega^4 & 1+\omega+\omega^2+\omega^4 & 1+\omega+\omega^2+\omega^4 & 4\omega^3 \\ 1+\omega+\omega^3+\omega^4 & 1+\omega+\omega^3+\omega^4 & 4\omega^2 & 1+\omega+\omega^3+\omega^4 \\ 1+\omega^2+\omega^3+\omega^4 & 4\omega & 1+\omega^2+\omega^3+\omega^4 & 1+\omega^2+\omega^3+\omega^4 \end{vmatrix} \quad \text{By (1)}$$

$$\Delta = \begin{vmatrix} -\omega^4 & -\omega^4 & -\omega^4 & -\omega^4 \\ -\omega^3 & -\omega^3 & -\omega^3 & 4\omega^3 \\ -\omega^2 & -\omega^2 & 4\omega^2 & -\omega^2 \\ -\omega & 4\omega & -\omega & -\omega \end{vmatrix}$$

$$= (-1)^4 \omega^4 . \omega^3 . \omega^2 . \omega \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -4 \\ 1 & 1 & -4 & 1 \\ 1 & -4 & 1 & 1 \end{vmatrix}$$

$$= \omega^{10} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & -5 & 0 \\ 0 & -5 & 0 & 0 \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

$$= (\omega^5)^2 \begin{vmatrix} 0 & 0 & -5 \\ 0 & -5 & 0 \\ -5 & 0 & 0 \end{vmatrix} \qquad \text{expanding from } C_1$$

$$= (1)^2 \begin{vmatrix} 0 & 0 & -5 \\ 0 & -5 & 0 \\ -5 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -5 \\ 0 & -5 & 0 \\ -5 & 0 & 0 \end{vmatrix}$$

Expanding from R_1

$$= 0 - 0 + (-5) \begin{vmatrix} 0 & -5 \\ -5 & 0 \end{vmatrix}$$

$$= -5(0 - 25)$$

$$\Delta = 125$$

Q20 Prove that the determinant

$$\begin{vmatrix} 2b_1+c_1 & c_1+3a_1 & 2a_1+3b_1 \\ 2b_2+c_2 & c_2+3a_2 & 2a_2+3b_2 \\ 2b_3+c_3 & c_3+3a_3 & 2a_3+3b_3 \end{vmatrix}$$

is a multiple of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ & find the other factor.

Sol.

Consider $$\begin{vmatrix} 2b_1+c_1 & c_1+3a_1 & 2a_1+3b_1 \\ 2b_2+c_2 & c_2+3a_2 & 2a_2+3b_2 \\ 2b_3+c_3 & c_3+3a_3 & 2a_3+3b_3 \end{vmatrix}$$

$$= \begin{vmatrix} 0a_1+2b_1+1.c_1 & 3a_1+0b_1+1.c_1 & 2a_1+3b_1+0c_1 \\ 0a_2+2b_2+1.c_2 & 3a_2+0b_2+1.c_2 & 2a_2+3b_2+0c_2 \\ 0a_3+2b_3+1.c_3 & 3a_3+0b_3+1.c_3 & 2a_3+3b_3+0c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 3 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \left\{ 0 - 3(0-3) + 2(2-0) \right\}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} (9+4)$$

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$$= 13.\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Hence given determinant is a multiple of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

& other factor is 13.

Q21 35,282 ; 44,759 ; 58,916 ; 80,652 & 92,469 are all multiples of 13. Show that the determinant

$$\begin{vmatrix} 3 & 5 & 2 & 8 & 2 \\ 4 & 4 & 7 & 5 & 9 \\ 5 & 8 & 9 & 1 & 6 \\ 8 & 0 & 6 & 5 & 2 \\ 9 & 2 & 4 & 6 & 9 \end{vmatrix}$$ is also a multiple of 13.

Sol.-

$$\text{Let } \Delta = \begin{vmatrix} 3 & 5 & 2 & 8 & 2 \\ 4 & 4 & 7 & 5 & 9 \\ 5 & 8 & 9 & 1 & 6 \\ 8 & 0 & 6 & 5 & 2 \\ 9 & 2 & 4 & 6 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 5 & 2 & 8 & 35,282 \\ 4 & 4 & 7 & 5 & 44,759 \\ 5 & 8 & 9 & 1 & 58,916 \\ 8 & 0 & 6 & 5 & 80,652 \\ 9 & 2 & 4 & 6 & 92,469 \end{vmatrix} \qquad R_5 + 10R_4 + 100R_3 + 1000R_2 + 10000R_1$$

Since 35,282, 44,759, 58,916, 80,652 & 92,469 are all multiples of 13, so 13 is a common factor of these numbers.

Hence

$$\Delta = 13\begin{vmatrix} 3 & 5 & 2 & 8 & 2714 \\ 4 & 4 & 7 & 5 & 3443 \\ 5 & 8 & 9 & 1 & 4532 \\ 8 & 0 & 6 & 5 & 6204 \\ 9 & 2 & 4 & 6 & 7113 \end{vmatrix} \quad \text{taking 13 Common from } C_5$$

So given determinant Δ is a multiple of 13.

Q22 Prove that

$$\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2-2ax+x^2 & a^2-2ay+y^2 & a^2-2az+z^2 \\ b^2-2bx+x^2 & b^2-2by+y^2 & b^2-2bz+z^2 \\ c^2-2cx+x^2 & c^2-2cy+y^2 & c^2-2cz+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$\Delta = \Delta_1 \cdot \Delta_2 \quad \text{———} \quad (1)$$

Now

$$\Delta_1 = \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} a^2-c^2 & a-c & 0 \\ b^2-c^2 & b-c & 0 \\ c^2 & c & 1 \end{vmatrix} \qquad \begin{matrix} R_1-R_3 \\ R_2-R_3 \end{matrix}$$

Expanding from C_3

$$= -2\begin{vmatrix} a^2-c^2 & a-c \\ b^2-c^2 & b-c \end{vmatrix}$$

$$= -2(a-c)(b-c)\begin{vmatrix} a+c & 1 \\ b+c & 1 \end{vmatrix}$$

taking $a-c$ Common from R_1
& $b-c$ Common from R_2

$$= -2(a-c)(b-c)(a+\cancel{c}-b-\cancel{c})$$

$$\Delta_1 = 2(a-b)(b-c)(c-a)$$

&

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x-z & y-z & z \\ x^2-z^2 & y^2-z^2 & z^2 \end{vmatrix} \qquad \begin{matrix} C_1-C_3 \\ C_2-C_3 \end{matrix}$$

Expanding from R_1

$$= \begin{vmatrix} x-z & y-z \\ x^2-z^2 & y^2-z^2 \end{vmatrix}$$

$$\Delta_2 = (x-z)(y-z)\begin{vmatrix} 1 & 1 \\ x+z & y+z \end{vmatrix}$$ taking $x-z$, $y-z$ Common from R_1 & R_2.

$$= (x-z)(y-z)(y+\cancel{z}-x-\cancel{z})$$

$$= (x-z)(y-z)(y-x)$$

$$\Delta_2 = (x-y)(y-z)(z-x)$$

Putting values of Δ_1, Δ_2 in ①

$$\Delta = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

Q23 Show that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$ taking transpose of second det.

$$= \begin{vmatrix} -a^2+bc+bc & -\cancel{ab}+\cancel{ab}+c^2 & -\cancel{ac}+b^2+\cancel{ac} \\ -\cancel{ab}+c^2+\cancel{ab} & -b^2+ca+ca & -\cancel{bc}+\cancel{bc}+a^2 \\ -\cancel{ac}+\cancel{ac}+b^2 & -\cancel{bc}+a^2+\cancel{bc} & -c^2+ab+ab \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$ ———— Ans I

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Again Consider

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$\Delta = \Delta_1 \cdot \Delta_2 \quad \text{———} \quad (1)$$

Now

$$\Delta_1 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Expanding from R_1

$$= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= -a^3 - b^3 - c^3 + 3abc$$

$$\Delta_1 = -(a^3 + b^3 + c^3 - 3abc)$$

&

$$\Delta_2 = \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

Expanding from R_1

$$= (-a)(a^2 - bc) - c(-ab + c^2) + b(-b^2 + ac)$$

$$= -a^3 + abc + abc - c^3 - b^3 + abc$$

$$= -a^3 - b^3 - c^3 + 3abc$$

$$\Delta_2 = -(a^3 + b^3 + c^3 - 3abc)$$

Put in (1)

$$\Delta = (a^3 + b^3 + c^3 - 3abc)^2$$

Hence

$$\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2$$

Q.24 Find, by the adjoint method, the inverse of each[58] of the following matrices:

(i) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Sol.

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

We know that

$$A^{-1} = \frac{\text{Adj}A}{|A|} \text{ ——— } (1)$$

Now

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Expanding from R_1

$$= 2(4-1) - 1(2-1) + 1(1-2)$$

$$= 2(3) - (1) + (-1)$$

$$= 6 - 1 - 1$$

$$|A| = 4 \neq 0 \quad \text{So } A^{-1} \text{ exists}$$

Now

$$\text{Adj}A = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} 4-1 & -(2-1) & 1-2 \\ -(2-1) & 4-1 & -(2-1) \\ 1-2 & -(2-1) & 4-1 \end{bmatrix}^t$$

$$= \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^t$$

$$\text{Adj}\,A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

So from (1)

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\text{So } A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

(ii) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Sol:-

Let $D = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

We know that

$$D^{-1} = \frac{\text{Adj}\,D}{|D|} \quad \text{———} \quad (1)$$

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Now

$$|D| = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Expanding from R_1

$$= a(bc - f^2) - h(ch - gf) + g(hf - gb)$$

$$= abc - af^2 - ch^2 + ghf + ghf - g^2 b$$

$|D| = abc + 2ghf - af^2 - ch^2 - g^2 b \neq 0$ (say) then D^{-1} exists.

Now

$$\text{Adj } D = \begin{bmatrix} \begin{vmatrix} b & f \\ f & c \end{vmatrix} & -\begin{vmatrix} h & f \\ g & c \end{vmatrix} & \begin{vmatrix} h & b \\ g & f \end{vmatrix} \\ -\begin{vmatrix} h & g \\ f & c \end{vmatrix} & \begin{vmatrix} a & g \\ g & c \end{vmatrix} & -\begin{vmatrix} a & h \\ g & f \end{vmatrix} \\ \begin{vmatrix} h & g \\ b & f \end{vmatrix} & -\begin{vmatrix} a & g \\ h & f \end{vmatrix} & \begin{vmatrix} a & h \\ h & b \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} bc - f^2 & -(ch - gf) & hf - gb \\ -(ch - gf) & ac - g^2 & -(af - gh) \\ hf - gb & -(af - gh) & ab - h^2 \end{bmatrix}^t$$

$$= \begin{bmatrix} bc - f^2 & gf - ch & hf - gb \\ gf - ch & ac - g^2 & gh - af \\ hf - gb & gh - af & ab - h^2 \end{bmatrix}^t$$

$$\text{Adj } D = \begin{bmatrix} bc - f^2 & gf - ch & hf - gb \\ gf - ch & ac - g^2 & gh - af \\ hf - gb & gh - af & ab - h^2 \end{bmatrix}$$

So from ①

$$D^{-1} = \frac{1}{(abc+2ghf-af^2-ch^2-g^2b)}\begin{bmatrix} bc-f^2 & gf-ch & hf-gb \\ gf-ch & ac-g^2 & gh-af \\ hf-gb & gh-af & ab-h^2 \end{bmatrix}$$

(iii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

Sol.

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

we know that

$$A^{-1} = \frac{\text{Adj} A}{|A|} \text{ ——— ①}$$

Now

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{vmatrix}$$

Expanding from R_1

$$= 1(-2+2)+1(-4+3)+2(4-3)$$

$$= 0-1+2$$

$|A| = 1 \neq 0$, so A^{-1} exists.

Now

$$\text{Adj} A = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^t$$

$$\text{Adj} A = \begin{bmatrix} -2+2 & -(-4+3) & 4-3 \\ -(2-4) & -2-6 & -(2+3) \\ 1-2 & -(-1-4) & 1+2 \end{bmatrix}^t$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 2 & -8 & -5 \\ -1 & 5 & 3 \end{bmatrix}^t$$

$$\text{Adj} A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}$$

So from ①

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}$$

Exercise No. 5.2

Q1 Solve for x, each of the following equations:

(i)
$$\begin{vmatrix} 1 & 2+x & 3 \\ 2 & 1 & 3+x \\ 3 & 2+x & 1 \end{vmatrix} = 0$$

Sol. Given

$$\begin{vmatrix} 1 & 2+x & 3 \\ 2 & 1 & 3+x \\ 3 & 2+x & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2+x & 3 \\ 0 & -3-2x & -3+x \\ 0 & -4-2x & -8 \end{vmatrix} = 0 \qquad \text{By } R_2 - 2R_1,\ R_3 - 3R_1$$

Expanding from C_1

$$\begin{vmatrix} -3-2x & -3+x \\ -4-2x & -8 \end{vmatrix} = 0$$

$$-8(-3-2x) - (-3+x)(-4-2x) = 0$$
$$24 + 16x - (12 + 6x - 4x - 2x^2) = 0$$
$$24 + 16x - 12 - 2x + 2x^2 = 0$$
$$2x^2 + 14x + 12 = 0$$
$$x^2 + 7x + 6 = 0$$
$$x^2 + 6x + x + 6 = 0$$
$$x(x+6) + 1(x+6) = 0$$
$$(x+6)(x+1) = 0$$
$$\Rightarrow \boxed{x = -6, -1}$$

(iii) $\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 9 & 9-x^2 \end{vmatrix} = 0$

Sol.

Given $\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 9 & 9-x^2 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1-x^2 & 0 & 0 \\ 2 & 1 & -3 & -1 \\ 2 & 1 & 5 & 3-x^2 \end{vmatrix} = 0$ By $C_2 - C_1$, $C_3 - 2C_1$, $C_4 - 3C_1$

Expanding from R_1

$\begin{vmatrix} 1-x^2 & 0 & 0 \\ 1 & -3 & -1 \\ 1 & 5 & 3-x^2 \end{vmatrix} = 0$

Expanding from R_1

$(1-x^2)\begin{vmatrix} -3 & -1 \\ 5 & 3-x^2 \end{vmatrix} = 0$

$(1-x^2)(-9+3x^2+5) = 0$

$(1-x^2)(3x^2-4) = 0$

$1-x^2 = 0 \quad , \quad 3x^2-4 = 0$

$x^2 = 1 \quad , \quad x^2 = 4/3$

$x = \pm 1, \pm \frac{2}{\sqrt{3}}$

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(iii) $$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} = 0$$

<u>Soln.</u> Given

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 2-x & 2^2-x^2 & 2^3-x^3 \\ 0 & 3-x & 3^2-x^2 & 3^3-x^3 \\ 0 & 4-x & 4^2-x^2 & 4^3-x^3 \end{vmatrix} = 0$$ $R_2 - R_1$, $R_3 - R_1$, $R_4 - R_1$

Expanding from C_1

$$\begin{vmatrix} 2-x & 2^2-x^2 & 2^3-x^3 \\ 3-x & 3^2-x^2 & 3^3-x^3 \\ 4-x & 4^2-x^2 & 4^3-x^3 \end{vmatrix} = 0$$

$$(2-x)(3-x)(4-x)\begin{vmatrix} 1 & 2+x & 4+2x+x^2 \\ 1 & 3+x & 9+3x+x^2 \\ 1 & 4+x & 16+4x+x^2 \end{vmatrix} = 0$$ taking $(2-x),(3-x),(4-x)$ common from R_1, R_2, R_3 resp.

$$(2-x)(3-x)(4-x)\begin{vmatrix} 1 & 2+x & 4+2x+x^2 \\ 0 & 1 & 5+x \\ 0 & 2 & 12+2x \end{vmatrix} = 0$$ $R_2 - R_1$, $R_3 - R_1$

Expanding from C_1

$$(2-x)(3-x)(4-x)\begin{vmatrix} 1 & 5+x \\ 2 & 12+2x \end{vmatrix} = 0$$

$$(2-x)(3-x)(4-x)(12+\cancel{2x}-10-\cancel{2x}) = 0$$

$$(2-x)(3-x)(4-x)(2) = 0$$

$$(2-x)(3-x)(4-x) = 0$$

$$\Rightarrow \quad \boxed{x = 2, 3, 4}$$

(iv)

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x+1 & 3 & 4 & 5 \\ 1 & 2 & x+1 & 4 & 5 \\ 1 & 2 & 3 & x+1 & 5 \\ 1 & 2 & 3 & 4 & x+1 \end{vmatrix} = 0$$

Sol.-

Given

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x+1 & 3 & 4 & 5 \\ 1 & 2 & x+1 & 4 & 5 \\ 1 & 2 & 3 & x+1 & 5 \\ 1 & 2 & 3 & 4 & x+1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & x-1 & 0 & 0 & 0 \\ 0 & 0 & x-2 & 0 & 0 \\ 0 & 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & 0 & x-4 \end{vmatrix} = 0 \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \\ R_5 - R_1 \end{matrix}$$

Expanding from C_1

$$\begin{vmatrix} x-1 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & x-4 \end{vmatrix} = 0$$

$(x-1)(x-2)(x-3)(x-4) = 0$ ($\because$ det. of a diagonal matrix is equal to the product of diagonal elements.)

$\Rightarrow \boxed{x = 1, 2, 3, 4}$

(v)
$$\begin{vmatrix} x & a & a & a & a \\ a & x & a & a & a \\ a & a & x & a & a \\ a & a & a & x & a \\ a & a & a & a & x \end{vmatrix} = 0$$

Sol.
Given

$$\begin{vmatrix} x & a & a & a & a \\ a & x & a & a & a \\ a & a & x & a & a \\ a & a & a & x & a \\ a & a & a & a & x \end{vmatrix} = 0$$

$$\begin{vmatrix} x+4a & a & a & a & a \\ x+4a & x & a & a & a \\ x+4a & a & x & a & a \\ x+4a & a & a & x & a \\ x+4a & a & a & a & x \end{vmatrix} = 0 \qquad C_1 + (C_2+C_3+C_4+C_5)$$

$$(x+4a)\begin{vmatrix} 1 & a & a & a & a \\ 1 & x & a & a & a \\ 1 & a & x & a & a \\ 1 & a & a & x & a \\ 1 & a & a & a & x \end{vmatrix} = 0.$$ taking $x+4a$ Common from C_1

$$(x+4a)\begin{vmatrix} 1 & a & a & a & a \\ 0 & x-a & 0 & 0 & 0 \\ 0 & 0 & x-a & 0 & 0 \\ 0 & 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & 0 & x-a \end{vmatrix} = 0$$

R_2-R_1
R_3-R_1
R_4-R_1
R_5-R_1

Expanding from C_1

$$(x+4a)\begin{vmatrix} x-a & 0 & 0 & 0 \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix} = 0$$

$$(x+4a)(x-a)(x-a)(x-a)(x-a) = 0$$

$$\Rightarrow \boxed{x = -4a, a, a, a, a}$$

(vi) $$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0$$

Sol. Given

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0$$

$$\begin{vmatrix} 4+x & 1 & 1 & 1 \\ 4+x & 1+x & 1 & 1 \\ 4+x & 1 & 1+x & 1 \\ 4+x & 1 & 1 & 1+x \end{vmatrix} = 0 \qquad C_1 + (C_2 + C_3 + C_4)$$

$$(4+x)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0 \qquad \text{taking } 4+x \text{ common from } C_1$$

$$(4+x)\cdot\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix} = 0 \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

Expanding from C_1

$$(4+x)\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = 0$$

$$\Rightarrow \quad (4+x)\cdot x^3 = 0$$

$$\Rightarrow$$

$$\boxed{x = -4, 0, 0, 0}$$

Q2 Evaluate each of the following determinants:

(i)
$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$$

$$= \begin{vmatrix} a+(n-1)d & b & b & \cdots & b \\ a+(n-1)d & a & b & \cdots & b \\ a+(n-1)d & b & a & \cdots & b \\ \vdots & & & & \\ a+(n-1)d & b & b & \cdots & b \end{vmatrix} \quad \text{By } C_1+(C_2+C_3+\cdots+C_n)$$

$$= [a+(n-1)d] \begin{vmatrix} 1 & b & b & \cdots & b \\ 1 & a & b & \cdots & b \\ 1 & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & b & b & \cdots & b \end{vmatrix} \quad \text{taking } a+(n-1)d \text{ common from } C_1$$

$$= [a+(n-1)d] \begin{vmatrix} 1 & b & b & \cdots & b \\ 0 & a-b & 0 & \cdots & 0 \\ 0 & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a-b \end{vmatrix} \quad \begin{matrix} R_2-R_1 \\ R_3-R_1 \\ \vdots \\ R_n-R_1 \end{matrix}$$

Expanding from C_1

$$\Delta = [a+(n-1)d]\begin{vmatrix} a-b & 0 & \cdots & 0 \\ 0 & a-b & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & a-b \end{vmatrix}$$

$$(a+(n-1)d)(a-b)^{n-1}$$

(ii)
$$\begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}$$

<u>Sol:-</u>

$$\text{Let } \Delta = \begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1-n & 1 & \cdots & 0 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix} \quad R_1+(R_2+R_3+\cdots+R_n)$$

$\Delta = 0 \qquad \because R_1 = 0$

(iii) $\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

$= \begin{vmatrix} \Sigma n & \Sigma n & \Sigma n & \cdots & \Sigma n & \Sigma n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & & & & & \\ n-2 & n-1 & n & \cdots & n-4 & n-3 \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$ Adding R_2 to R_n in R_1 & denoting $1+2+\cdots+n$ by Σn

$= \Sigma n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & n-1 & n & & n-4 & n-3 \\ n-1 & n & 1 & & n-3 & n-2 \\ n & 1 & 2 & & n-2 & n-1 \end{vmatrix}$ taking Σn common from C_1

$$= \Sigma n \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 2 & \cdots & n-2 & -1 \\ 3 & 1 & 2 & \cdots & 2 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & 1 & 2 & \cdots & -2 & -1 \\ n-1 & 1 & 2-n & \cdots & -2 & -1 \\ n & 1-n & 2-n & \cdots & -2 & -1 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \\ C_4 - C_1 \\ \vdots \\ C_n - C_1 \end{matrix}$$

Expanding from R_1

$$= \Sigma n \begin{vmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 2 & \cdots & 2 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & \cdots & -2 & -1 \\ 1 & 2-n & \cdots & -2 & -1 \\ 1-n & 2-n & \cdots & -2 & -1 \end{vmatrix}$$

$$= \Sigma n \begin{vmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -n & -1 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & -n & -1 \\ 0 & -n & \cdots & -2 & -1 \\ -n & -n & \cdots & -2 & -1 \end{vmatrix} \quad \begin{matrix} C_1 + C_{n-1} \\ C_2 + 2C_{n-1} \\ \vdots \\ C_{n-2} + (n-2)C_{n-1} \end{matrix}$$

$$= \Sigma n . (-1)^{n-1} . (n)^{n-2} \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

taking $-n$ Common from C_{n-2} & -1 Common from $C_1, C_2, \ldots, C_{n-1}$

Going through each of the preceding columns, $(n-1)$th column shifted to the place of first column, there will be $n-2$ changes of sign. The second last column which now is at the $(n-1)$th position shifted similarly at the position of second column, there will be $n-3$ changes of sign etc.

So we have

$$\Delta = \Sigma n (-1)^{n-1} \cdot (n)^{n-2+r} \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\Delta = \Sigma n . (-1)^{(n-1)+r} . (n)^{n-2} . 1$$

$$\text{Now } (n-1)+r = (n-1)+(n-2)+(n-3)+ \cdots +3+2+1$$

$$= \frac{(n-1)(n-1+1)}{2}$$

$$= \frac{(n-1)(n)}{2}$$

$$= \frac{n(n-1)}{2}$$

So last eq. becomes

$$\Delta = \Sigma n . (-1)^{\frac{n(n-1)}{2}} . (n)^{n-2}$$

(iv)
$$\begin{vmatrix} x+1 & x & \cdots & x \\ x & x+2 & \cdots & x \\ \vdots & \vdots & & \vdots \\ x & x & \cdots & x+n \end{vmatrix}$$

Sol.

Let $\Delta = \begin{vmatrix} x+1 & x & \cdots & x & x \\ x & x+2 & \cdots & x & x \\ \vdots & \vdots & & \vdots & \vdots \\ x & x & \cdots & x+(n-1) & x \\ x & x & \cdots & x & x+n \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & -n \\ 0 & 2 & \cdots & 0 & -n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & n-1 & -n \\ x & x & \cdots & x & x+n \end{vmatrix} \quad \begin{matrix} R_1 - R_n \\ R_2 - R_n \\ \vdots \\ R_{n-1} - R_n \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & n-1 & 0 \\ x & x & \cdots & x & nx\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{x}\right) \end{vmatrix} \quad C_n + \left(nC_1 + \frac{n}{2}C_2 + \frac{n}{3}C_3 + \cdots + \frac{n}{n-1}C_{n-1}\right)$$

$$\Delta = 1.2.3 \cdots n-1 . nx\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{x}\right) = n!.x.\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{x}\right)$$

Q3 If A & B are 3×3 matrices such that $\det(A^2B^3) = 108$ & $\det(A^3B^2) = 72$.
Find $\det(2A)$ & $\det(B^{-1})$.

Sol. Given

$$\left.\begin{array}{l}\det(A^2B^3) = 108 \\ \& \det(A^3B^2) = 72\end{array}\right]$$

By product theorem

$$\left.\begin{array}{l}\det(A^2).\det(B^3) = 108 \\ \det(A^3).\det(B^2) = 72\end{array}\right]$$

or

$$(\det A)^2.(\det B)^3 = 108 \quad \text{——— (1)}$$

$$\& \quad (\det A)^3.(\det B)^2 = 72 \quad \text{——— (2)}$$

Dividing (1) by (2)

$$\frac{\det B}{\det A} = \frac{108}{72}$$

$$\text{or} \quad \frac{\det B}{\det A} = \frac{3}{2}$$

$$\Rightarrow \det B = \frac{3}{2}\det A$$

Put in (2)

$$(\det A)^3.\left(\frac{3}{2}\det A\right)^2 = 72$$

$$\frac{9}{4}(\det A)^5 = 72$$

$$\Rightarrow (\det A)^5 = \frac{72\times 4}{9}$$

$$(\det A)^5 = 32$$

$$\Rightarrow \boxed{\det A = 2}$$

Now $\det(2A) = 2^3.\det A$

$\det(2A) = 8 \times 2$

So

$$\boxed{\det(2A) = 16}$$

Now

$$\det(B^{-1}) = (\det B)^{-1}$$

$$= \left(\tfrac{3}{2}\det A\right)^{-1}$$

$$= \left(\tfrac{3}{2}\times 2\right)^{-1}$$

$$= (3)^{-1}$$

So $\det(B^{-1}) = \frac{1}{3}$

Q4 Let A be an $n\times n$ matrix. Show that

(i) $\det A^m = (\det A)^m$ for any +ve integer m

(ii) If $\det A^m = 1$ then $\det A = \pm 1$

(iii) If $\det A^m = 0$ then $\det A = 0$

Sol:-

(i) We will prove

$\det A^m = (\det A)^m$ by applying induction on m.

Step ① Let $m = 1$

So $\det A^1 = (\det A)^1$

or $\det A = \det A$

Hence it is true for $m = 1$

Step ② Suppose it is true for $m = K$

i.e., $\det A^K = (\det A)^K$ ——— ① for $K \geq 1$

Step ③ Now we prove it for $m = K+1$

Now

$$\det(A^{k+1}) = \det(A^k \cdot A)$$
$$= (\det A^k)\cdot(\det A) \qquad \text{By product theorem}$$
$$= (\det A)^k \cdot (\det A) \qquad \text{using } ①$$

$\therefore \det(A^{k+1}) = (\det A)^{k+1}$

So it is true for $m = k+1$

Hence

$\det(A^m) = (\det A)^m$ for all +ve integers m.

(ii) If $\det A^m = 1$ then $\det A = \pm 1$

Sol.-

Since $\det A^m = 1$

So

$$(\det A)^m = 1$$

So $\boxed{\det A = \pm 1}$ where m is an even integer

(iii) If $\det A^m = 0$ then $\det A = 0$

Sol.-

Since $\det A^m = 0$

$$\Rightarrow (\det A)^m = 0$$

$$\Rightarrow \det A = 0$$

Q5 For any non singular matrix C, show that

(i) $\det(C^{-1}) = (\det C)^{-1}$

(ii) $\det(CAC^{-1}) = \det A$

Sol.

(i) Since C is a non singular matrix, so C^{-1} exists such that

$$CC^{-1} = I$$

$$\Rightarrow \det(CC^{-1}) = \det(I)$$

or $\det(C).\det(C^{-1}) = \det(I)$ (By product Theorem)

$$\det C \,.\, \det(C^{-1}) = 1$$

$$\det(C^{-1}) = \frac{1}{\det C}$$

$$\det(C^{-1}) = (\det C)^{-1}$$

(ii) $\det(CAC^{-1}) = \det A$

Sol. using product theorem

$$\det(CAC^{-1}) = \det C \,.\, \det A \,.\, \det C^{-1}$$

$= \det C \,.\, \det C^{-1} \,.\, \det A$ $\because$ det. of an element is an element of field & so they commute

$= \det(CC^{-1}) \,.\, \det A$ By product theorem

$$= \det I \,.\, \det A$$

$$= 1 \,.\, \det A$$

So $\det(CAC^{-1}) = \det A$

8.

Q6 For what value of α is the matrix

$$A = \begin{bmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{bmatrix} \text{ singular ?}$$

Sol.

$$\text{Given } A = \begin{bmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{bmatrix}$$

Since A is singular

So $\det A = 0$

$$\Rightarrow \begin{vmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{vmatrix} = 0$$

$$\begin{vmatrix} -\alpha & 3\alpha-1 & 4\alpha+1 \\ 1 & 0 & 0 \\ 2-\alpha & 3\alpha-1 & 4\alpha+1 \end{vmatrix} = 0 \qquad \begin{matrix} C_2-2C_1 \\ C_3-3C_1 \end{matrix}$$

Expanding from R_2

$$-\begin{vmatrix} 3\alpha-1 & 4\alpha+1 \\ 3\alpha-1 & 4\alpha+1 \end{vmatrix} = 0$$

$$-(3\alpha-1)(4\alpha+1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

obviously matrix A is singular for all values of α.

$$\frac{\text{Adj}A}{\det A}\cdot A = \frac{-1}{28}\begin{bmatrix} -2 & -2 & -9 \\ 20 & -8 & 6 \\ -2 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & -2 \end{bmatrix}$$

$$= \frac{-1}{28}\begin{bmatrix} -2-8-18 & 2-2+0 & -6-12+18 \\ 20-32+12 & -20-8 & 60-48-12 \\ -2-8+10 & 2-2+0 & -6-12-10 \end{bmatrix}$$

$$= \frac{-1}{28}\begin{bmatrix} -28 & 0 & 0 \\ 0 & -28 & 0 \\ 0 & 0 & -28 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\text{Adj}A}{\det A}\cdot A = I \quad \text{———} \quad (2)$$

from (1) & (2)

$$A\cdot\frac{\text{Adj}A}{\det A} = \frac{\text{Adj}A}{\det A}\cdot A = I$$

$$\Rightarrow \quad A^{-1} = \frac{\text{Adj}A}{\det A}$$

Q8 Evaluate

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 2 & -2 \\ 2 & 4 & 2 & 1 \\ 3 & 1 & 5 & -3 \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 2 & -2 \\ 2 & 4 & 2 & 1 \\ 3 & 1 & 5 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & -6 \\ 2 & 4 & 4 & -3 \\ 3 & 1 & 8 & -9 \end{vmatrix} \qquad \begin{matrix} C_3 + C_1 \\ C_4 - 2C_1 \end{matrix}$$

Expanding from R_1

$$= \begin{vmatrix} 3 & 4 & -6 \\ 4 & 4 & -3 \\ 1 & 8 & -9 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -20 & 21 \\ 0 & -28 & 33 \\ 1 & 8 & -9 \end{vmatrix} \qquad \begin{matrix} R_1 - 3R_3 \\ R_2 - 4R_3 \end{matrix}$$

Expanding from C_1

$$= \begin{vmatrix} -20 & 21 \\ -28 & 33 \end{vmatrix}$$

$$= -660 + 588$$

$$\Delta = -72$$

M. 6

(Vector spaces) (Chapter No. 6)

Vector Space

Let F be a field & V a non empty set on which an operation of addition is defined. Suppose for every $a \in F$ & $v \in V$, av is an element of V then V is called a vector space over F if the following conditions are satisfied

(i) V is an abelian gr under addition

(ii) $a(bu) = (ab)u$ $a,b \in F$, $u \in V$

(iii) $(a+b)u = au + bu$ $a,b \in F$, $u \in V$

(iv) $a(v+w) = av + aw$ $a \in F$, $v,w \in V$

(v) $1v = v$ where 1 is unity of F

Note ① The elements of F are called scalars & the elements of V are called vectors

② If V is a vector space over F, we write it as V(F)

③ We do not multiply elements of V since only add them we multiply an element of F by an element of V

Theorem Let V be a vector space over F then

(i) $a0 = 0$ for $a \in F$

(ii) $0v = 0$ for $v \in V$

(iii) $(-a)v = a(-v) = -av$ for $a \in F$, $v \in V$

(iv) If $av = 0$ then either $a = 0$ or $v = 0$

(v) $a(v-u) = av - au$ for $a \in F$, $u, v \in V$

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Examples

(1) 9f $\lambda = 6327.9$ E

Proof

$\because 0 = 0+0$ ($\because 0$ is add... identity of V)

$\therefore a0 = a(0+0)$ $\forall a, u \in V$

$a0 = a0 + a0$ (by iv)

$a0 + 0 = a0 + a0$ $\because 0$ is identity of V

$\therefore 0 = a0$ $\because$ V is a gr. so cancellation law holds

or $\boxed{a0 = 0}$

(ii) $\because 0 = 0 + 0$ $\because 0$ is additive identity of F

$0u = (0+0)u$ $\forall a, u \in V$

$\Rightarrow 0u = 0u + 0u$

$0u + 0 = 0u + 0u$ ($\because 0$ is identity of V)

So $0 = 0u$ ($\because$ V is a gr. so cancellation law holds)

$\therefore \boxed{0u = 0}$

(iii) $\because (-a)u + au = (-a+a)u$

$= 0u$

$= 0$

So $(-a)u$ & au are additive inverses of each other

$\therefore \boxed{(-a)u = -au}$

Also $a(-u) + au = a(-u+u)$ by (iv)

$= a0$

$= 0$

So $a(-u)$ & au are additive inverses of each other

$a(-u) = -au$

So $(-a)u = a(-u) = -au$

(iv)

Let $a \neq 0$ so that a^{-1} exists s.t. $aa^{-1} = a^{-1}a = 1$

Now

$u = 1 \cdot u$ (by V)

$= (a^{-1}a)u$

$= a^{-1}(au)$ (by (ii))

$= a^{-1}(0)$

$= 0$

So if $a \neq 0$ then $u = 0$

Similarly if $u \neq 0$ then $a = 0$

So $au = 0 \Rightarrow$ either $a = 0$ or $u = 0$

(v) $a(u-v) = a(u+(-v))$

$= au + a(-v)$ by (iv)

$= au + (-av)$ just proved

$= au - av$

So $a(u-v) = au - av$

Subspace

A non empty subset W of a vector space V is called a subspace of V if W itself is a vector space over F under the same operation as defined in V

· Empty

V. Therefore W is a subspace of V

Corollary A non empty subset W of a vector space V is a subspace of V iff
for $w_1, w_2 \in W$ & $a, b \in F \Rightarrow aw_1 + bw_2 \in W$

Proof.

We shall establish that conditions (i) & (ii) of above theorem imply the condition of this corollary & viceversa.

For this suppose that W is a subspace of V then by (ii) condition

$aw_1, bw_2 \in W$

& therefore by (i) condition $aw_1 + bw_2 \in W$

Conversely

Let W be a non empty subset of V in which the condition

$w_1, w_2 \in W$ & $a, b \in F \Rightarrow aw_1 + bw_2 \in W$ holds

We shall prove that W is a subspace of V

Take $a = b = 1$ in above condition

$1 \cdot w_1 + 1 \cdot w_2 = w_1 + w_2 \in W$

Now take $b = 0$

So $aw_1 + 0w_2 = aw_1 \in W$

Hence by previous theorem

W is a subspace of V

theram tat U of W la subpaces of a vecter space V over F then $U \wedge W$ is also a

Proof,

let $a, b \in F$ a $w_1, w_2 \in U \wedge W$

$\Rightarrow w_1, w_2 \in U$ ' d $w_1, w_2 \in W$

But U + W are subspaces of V

So $aw_1 + bw_2 \in U$

+ $aw_1 + bw_2 \in W$

Hence $aw_1 + bw_2 \in U \cap W$

So $a, b \in F$ + $w_1, w_2 \in U \cap W \Rightarrow aw_1 + bw_2 \in U \wedge W$

Hence $U \wedge W$ is a subspace of V.

---

thareni The intersection of any no. of subspaces of a vector space V is a subspace of V

Proof

Let $\{U_\alpha : \alpha \in I\}$ be any subcallection of subspaces of a vector space V over the field F Then we have to prove that $\underset{\alpha \in I}{\cup} U_\alpha$ is also a subspace of V

For this let $a, b \in F$ + $u_1, u_2 \in \underset{\alpha \in T}{\cap} U_\alpha$

$\Rightarrow u_1, u_2 \in U_\alpha$ for all $\alpha \in I$

But each U_α is a sutaspace of V

So $au_1 + bu_2 \in U_\alpha$ fa each $\alpha \in I$

I

$\Rightarrow au_1 + bu_2 \in \bigcap_{\alpha \in I} U_\alpha$

So $\bigcap_{\alpha \in I} U_\alpha$ is a subspace of V.

Sum of two subspaces

Let U & W be two subspaces of a vector space V. We define U+W as

$U+W = \{u+w : u \in U \text{ \& } w \in W\}$

Theorem If U, W are subspaces of a vector space V then U+W is a subspace of V containing both U & W. Further U+W is the smallest subspace of V containing both U & W.

Proof Given U & W are subspaces of V then we define

$U+W = \{u+w \mid u \in U \text{ \& } w \in W\}$

we will prove that U+W is a subspace of V

For this let $a, b \in F$ & $v_1, v_2 \in U+W$

$\Rightarrow v_1 = u_1 + w_1$

& $v_2 = u_2 + w_2$

where $u_1, u_2 \in U$ & $w_1, w_2 \in W$

Now since U is a subspace of V so $au_1 + bu_2 \in U$

& since W is a subspace of V, so $aw_1 + bw_2 \in W$

Hence $(au_1 + bu_2) + (aw_1 + bw_2) \in U+W$

or $(au_1 + aw_1) + (bu_2 + bw_2) \in U+W$

$a(u_1+w_1) + b(u_2+w_2) \in U+W$

or $ax_1 + bx_2 \in U+W$

So for $a, b \in F$ & $x_1, x_2 \in U+W \Rightarrow ax_1 + bx_2 \in U+W$

Hence $U+W$ is a subspace of V

Next we prove that $U+W$ is a subspace of V containing both U & W. i.e., $U \subseteq U+W$ & $W \subseteq U+W$

Since $u \in U$ & $0 \in W$

$\Rightarrow u+0 = u \in U+W$ for all $u \in U$

So $U \subseteq U+W$

Similarly $W \subseteq U+W$

Hence $U+W$ is a subspace of V containing both U & W

Now we will prove that $U+W$ is the smallest subspace of V containing both U & W

Let S be any subspace of V containing both U & W then for every $u \in U$ & $w \in W$,

we have $u \in S$ & $w \in S$ so that $u+w \in S$

But $u+w \in U+W$

So $U+W \subseteq S$

Hence $U+W$ is the smallest subspace of V containing both U & W.

Direct sum:.

A vector space V is called the direct sum of its subspaces U & W if

(i) $V = U + W$

(ii) $U \cap W = \{0\}$

Linear Combination:

Let V be a vector space over a field F & let $u_1, u_2, u_3, \ldots, u_n \in V$

Any vector in V of the form.

$a_1u_1 + a_2u_2 + \cdots + a_nu_n$ where $a_i \in F$

is called a linear Combination of $v_1, v_2, \ldots, u_n$

Linear span:

Let S be a non empty subset of a vector space V Then the set of all linear Combination of finite no. of elements of S is called the linear span of S & is denoted by $\langle S \rangle$

Note (i) $\langle S \rangle$ is said to be spanned or generated by S & S is called a spanning set for $\langle S \rangle$

Theorem. Let S be a non empty set of vectors in a vector space V over a field F Than. $\langle S \rangle$ is a subspace of V Containing S & it is the smallest subspace of V containing S

Proof

Let $a, b \in F$ & $u, u \in \langle S \rangle$ then.

$u = a_1u_1 + a_2u_2 + \cdots a_nu_n = \sum_{i=1}^{n} a_iu_i$

$u = b_1u_1 + b_1u_2 + \cdots + b_nu_m = \sum_{j=1}^{m} b_ju_j$

where $u_i, u_j \in S$ & $a_i, b_j \in F$

Now $au + bv = a\left(\sum_{i=1}^{n} a_i u_i\right) + b\left(\sum_{j=1}^{m} b_j v_j\right)$

$$= \sum_{i=1}^{n} a(a_i u_i) + \sum_{j=1}^{m} b(b_j v_j)$$

$$= \sum_{i=1}^{n} (a a_i) u_i + \sum_{j=1}^{m} (b b_j) v_j \qquad \because a(bu) = (ab)u$$

Which shows that $au + bv$ is a linear Combination of vectors in S. So $au + bv \in \langle S \rangle$

So $a, b \in F$, $u, v \in \langle S \rangle \Rightarrow au + bv \in \langle S \rangle$

Hence $\langle S \rangle$ is a subspace of V.

(Now)

Since for all $u \in S$

$$u = 1.u \qquad \Rightarrow \qquad u \in \langle S \rangle$$

So $S \subseteq \langle S \rangle$

Hence $\langle S \rangle$ is a subspace of V Containing S.

Now we prove that $\langle S \rangle$ is the smallest subspace of V Containing S.

If W is any other subspace of V Containing S. then it Contains all vectors of the form $\sum_{i=1}^{n} a_i u_i$; where $a_i \in F$ & $u_i \in S$

$\Rightarrow \langle S \rangle \subseteq W$

Thus $\langle S \rangle$ is the smallest subspace of V Containing S.

<u>Theorem</u>:- If S, T are subsets of V then $S \subset T$ implies $\langle S \rangle \subset \langle T \rangle$

<u>Proof</u>:-

Let $S = \{v_1, v_2, ------, v_r\}$

& $T = \{v_1, v_2, ------, v_r, v_{r+1}, -----, v_n\}$

then obviously

$S \subset T$

We want to show that $\langle S \rangle \subset \langle T \rangle$. For this

let $v \in \langle S \rangle$ then by definition of $\langle S \rangle$, v is a linear combination of vectors $v_1, v_2, \ldots, v_r$ of S

i.e., $v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r$

Now v can also be written as

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r + 0 v_{r+1} + 0 v_{r+2} + \cdots + 0 v_n$$

Which shows that v is a linear combination of vectors of T

Hence $v \in \langle T \rangle$

So $\langle S \rangle \subset \langle T \rangle$

<u>Finite dimensional vector space</u>:

A vector space V is said to be finite dimensional if there is a finite subset S in V such that $V = \langle S \rangle$

Exercise No. 6.1

<u>Q1</u> Let V be the set of all infinite sequences in a field F with addition & scalar multiplication defined as below:

For $u = \{a_n\} = a_1, a_2, \ldots, a_n, \ldots \in V$

$v = \{b_n\} = b_1, b_2, \ldots, b_n, \ldots \in V$

$u + v = \{a_n\} + \{b_n\} = a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n, \ldots$

& $ku = k\{a_n\} = ka_1, ka_2, \ldots, ka_n, \ldots$

where a_n, b_n & k are all in F, $n = 1, 2, 3, \ldots$

Show that V is a vector space over F.

<u>Sol.</u>

Here $V = \{(a_1, a_2, \ldots) \mid a_i \in F\}$

First we prove that $(V, +)$ is an abelian gr.

(a)

(i) <u>Closure law</u>

~~Let $v_1 = (a_1, a_2, \ldots)$~~

& $v_2 = (b_1, b_2, \ldots\ldots)$

$\Rightarrow v_1 + v_2 = (a_1, a_2, \ldots\ldots) + (b_1, b_2, \ldots\ldots)$

$= (a_1 + b_1, a_2 + b_2, \ldots\ldots) \in V$

(ii) Association law

Let $v_1 = (a_1, a_2, \ldots\ldots)$

$v_2 = (b_1, b_2, \ldots\ldots)$

& $v_3 = (c_1, c_2, \ldots\ldots) \in V$

then we prove $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

Now

$v_1 + (v_2 + v_3) = (a_1, a_2, \ldots\ldots) + \left[(b_1, b_2, \ldots\ldots) + (c_1, c_2, \ldots\ldots)\right]$

$= (a_1, a_2, \ldots\ldots) + \left[(b_1 + c_1), (b_2 + c_2), \ldots\ldots\right]$

$= \left[a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots\ldots\right]$

$= \left[(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots\ldots\right]$

$= \left[(a_1 + b_1), (a_2 + b_2), \ldots\ldots\right] + (c_1, c_2, \ldots\ldots)$

$= \left[(a_1, a_2, \ldots\ldots) + (b_1, b_2, \ldots\ldots)\right] + (c_1, c_2, \ldots\ldots)$

$= (v_1 + v_2) + v_3$

(iii) Identity law

Here $0 = (0, 0, \ldots\ldots)$ is the additive identity in V because for any $v = (a_1, a_2, \ldots\ldots) \in V$

$v + 0 = (a_1, a_2, \ldots\ldots) + (0, 0, \ldots\ldots)$

$= (a_1 + 0, a_2 + 0, \ldots\ldots)$

$= (a_1, a_2, \ldots\ldots)$

$v + 0 = v$

Similarly $0 + v = v$

(iv) Inverse law

Every $v = (a_1, a_2, \ldots\ldots) \in V$ has additive inverse $-v = (-a_1, -a_2, \ldots\ldots) \in V$ because

$v + (-v) = (a_1, a_2, \ldots\ldots) + (-a_1, -a_2, \ldots\ldots)$

$= (a_1 - a_1, a_2 - a_2, \ldots\ldots)$

$u+(-u) = (0,0,\ldots\ldots)$

$u+(-u) = 0$

Similarly $-u+u = 0$

(v) Commutative law

Let $u_1 = (a_1, a_2, \ldots\ldots)$

& $u_2 = (b_1, b_2, \ldots\ldots)$

then we prove $u_1+u_2 = u_2+u_1$

Now

$$
\begin{aligned}
u_1+u_2 &= (a_1, a_2, \ldots\ldots) + (b_1, b_2, \ldots\ldots) \\
&= (a_1+b_1, a_2+b_2, \ldots\ldots) \\
&= (b_1+a_1, b_2+a_2, \ldots\ldots) \\
&= (b_1, b_2, \ldots\ldots) + (a_1, a_2, \ldots\ldots)
\end{aligned}
$$

$u_1+u_2 = u_2+u_1$

Hence $(V,+)$ is an abelian gr.

(b) Scalar multiplication:

(i) Let $a \in F$ & $u = (a_1, a_2, \ldots\ldots) \in V$ then

$$
\begin{aligned}
au &= a(a_1, a_2, \ldots\ldots) \\
&= (aa_1, aa_2, \ldots\ldots) \in V
\end{aligned}
$$

(ii)

Let $a, b \in F$ & $u = (u_1, u_2, \ldots\ldots) \in V$

then we prove $a(bu) = (ab)u$

$$
\begin{aligned}
\text{Now } a(bu) &= a[b(a_1, a_2, \ldots\ldots)] \\
&= a[(ba_1, ba_2, \ldots\ldots)] \\
&= [a(ba_1), a(ba_2), \ldots\ldots] \\
&= [(ab)a_1, (ab)a_2, \ldots\ldots] \\
&= (ab)(a_1, a_2, \ldots\ldots)
\end{aligned}
$$

$\therefore \quad a(bu) = (ab)u$

(iii)

Let $a, b \in F$ & $u = (a_1, a_2, \ldots\ldots)$

then we prove $(a+b)v = av + bv$

Now.

$(a+b)v = (a+b)(a_1, a_2, \ldots\ldots)$

$= [(a+b)a_1, (a+b)a_2, \ldots\ldots]$

$= [(aa_1+ba_1), (aa_2+ba_2), \ldots\ldots]$

$= (aa_1, aa_2, \ldots\ldots) + (ba_1, ba_2, \ldots\ldots)$

$= a(a_1, a_2, \ldots\ldots) + b(a_1, a_2, \ldots\ldots)$

$(a+b)v = av + bv$

(iv) Let $a \in F$ & $v_1 = (a_1, a_2, \ldots\ldots)$, $v_2 = (b_1, b_2, \ldots) \in V$

then we show $a(v_1+v_2) = av_1 + av_2$

Now

$a(v_1+v_2) = a[(a_1, a_2, \ldots\ldots) + (b_1, b_2, \ldots\ldots)]$

$= a[(a_1+b_1, a_2+b_2, \ldots\ldots]$

$= [a(a_1+b_1), a(a_2+b_2), \ldots\ldots]$

$= [(aa_1+ab_1, aa_2+ab_2, \ldots\ldots)]$

$= (aa_1, aa_2, \ldots\ldots) + (ab_1, ab_2, \ldots\ldots)$

$= a(a_1, a_2, \ldots\ldots) + a(b_1, b_2, \ldots\ldots)$

$\therefore a(v_1+v_2) = av_1 + av_2$

(v) Let $1 \in F$ & $v = (a_1, a_2, \ldots\ldots) \in V$ then we prove

$1.v = v$

Now $1.v = 1.(a_1, a_2, \ldots\ldots)$

$= (1.a_1, 1.a_2, \ldots\ldots)$

$= (a_1, a_2, \ldots\ldots)$

$1.v = v$

Since all Conditions are satisfied.

So V is a vector space over F.

Q2 Let V be the set of all ordered pairs of real no's. [15]
Check whether V is a vector space over R w.r.t. the indicated operations. If not, state the axioms which fail to hold.

(i) $(a,b)+(c,d) = (a+c, b+d)$

& $K(a,b) = (Ka, b)$

Sol. Here $V = \{(a,b) \mid a,b \in R\}$

& $(a,b)+(c,d) = (a+c, b+d)$

$K(a,b) = (Ka, b)$

First we prove that $(V,+)$ is an abelian gr.

(a)
(i) Closure law

Let $u_1 = (a,b)$ & $u_2 = (c,d)$ $\in V$

then $u_1+u_2 = (a,b)+(c,d)$

$= (a+c, b+d) \in V$

(ii) Associative law

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$ then

we prove $u_1+(u_2+u_3) = (u_1+u_2)+u_3$

Now

$$
\begin{aligned}
u_1+(u_2+u_3) &= (a,b)+\big((c,d)+(e,f)\big) \\
&= (a,b)+(c+e, d+f) \\
&= \big(a+(c+e), b+(d+f)\big) \\
&= \big((a+c)+e, (b+d)+f\big) \\
&= (a+c, b+d)+(e,f) \\
&= \big((a,b)+(c,d)\big)+(e,f) \\
u_1+(u_2+u_3) &= (u_1+u_2)+u_3
\end{aligned}
$$

(iii) Identity law

Here $O = (0,0)$ is the additive identity in V

because for $u = (a,b) \in V$

$$
\begin{aligned}
0+u &= (0,0)+(a,b) \\
&= (0+a, 0+b) \\
&= (a,b)
\end{aligned}
$$

$0+u = u$. Similarly $u+0 = u$.

(iv) Inverse law

Every element $v = (a,b) \in V$ has its additive inverse $-v = (-a,-b)$ in V because

$v + (-v) = (a,b) + (-a,-b) = (a-a, b-b) = (0,0) = 0$

& $-v + v = (-a,-b) + (a,b) = (-a+a, -b+b) = (0,0) = 0$

(V) Commutative law

Let $v_1 = (a,b)$, $v_2 = (c,d) \in V$ then we prove $v_1 + v_2 = v_2 + v_1$

Now

$$v_1 + v_2 = (a,b) + (c,d) = (a+c, b+d) = (c+a, d+b)$$
$$= (c,d) + (a,b) = v_2 + v_1$$

Hence $(V,+)$ is an abelian gr.

(b) Scalar multiplication

(i) Let $a \in R$ & $v_1 = (a_1,b_1) \in V$

then $av_1 = a(a_1,b_1) = (aa_1, b_1) \in V$

(ii) Let $a,b \in R$ & $v_1 = (a_1,b_1) \in V$

Then we prove $a(bv_1) = (ab)v_1$

Now $a(bv_1) = a(b(a_1,b_1)) = a(ba_1, b_1) = (a(ba_1), b_1)$

$= ((ab)a_1, b_1) = (ab)(a_1,b_1) = (ab)v_1$

(iii) Let $a,b \in R$ & $v_1 = (a_1,b_1) \in V$ then we prove

$(a+b)v_1 = av_1 + bv_1$

Now $(a+b)v_1 = (a+b)(a_1,b_1) = ((a+b)a_1, b_1) = ((aa_1+ba_1), b_1)$

& $av_1 + bv_1 = a(a_1,b_1) + b(a_1,b_1) = (aa_1,b_1) + (ba_1,b_1) = (aa_1+ba_1, 2b_1)$

So $(a+b)v_1 \neq av_1 + bv_1$

Since this condition is not satisfied.

So V is not a vector space over R.

(ii) $(a,b) + (c,d) = (a,b)$ & $k(a,b) = (ka, kb)$

Sol. Let $V = \{(a,b) \mid a,b \in R\}$

Here $(a,b) + (c,d) = (a,b)$

& $k(a,b) = (ka, kb)$

First we prove $(V,+)$ is an abelian gr.

(a)

(i) Closure law

Let $u_1 = (a,b)$, $u_2 = (c,d) \in V$ then

$u_1 + u_2 = (a,b) + (c,d)$

$= (a,b) \in V$

(ii) Associative law

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$ then

we prove $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$

Now

$$\begin{aligned} u_1 + (u_2 + u_3) &= (a,b) + \left[(c,d) + (e,f)\right] \\ &= (a,b) + (c,d) \\ &= (a,b) \end{aligned}$$

$$\begin{aligned} \& \ (u_1 + u_2) + u_3 &= \left[(a,b) + (c,d)\right] + (e,f) \\ &= (a,b) + (e,f) \\ &= (a,b) \end{aligned}$$

So $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$

(iii) Identity law

There is no identity element in V

As this condition is not satisfied. So V is not a vector space over R.

(iii) $(a,b) + (c,d) = (a+c, b+d)$ & $k(a,b) = (k^2 a, k^2 b)$

Sol. Let $V = \{(a,b) \mid a,b \in R\}$

Here $(a,b) + (c,d) = (a+c, b+d)$

& $k(a,b) = (k^2 a, k^2 b)$

First we prove that $(V,+)$ is an abelian gr.

(a)(i) Closure law

Let $u_1 = (a,b)$, $u_2 = (c,d) \in V$ then

$u_1 + u_2 = (a,b) + (c,d) = (a+c, b+d) \in V$

(ii) Associative law

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$

then we prove $v_1+(v_2+v_3) = (v_1+v_2)+v_3$.

Now

$$\begin{aligned} v_1+(v_2+v_3) &= (a,b)+[(c,d)+(e,f)] \\ &= (a,b)+(c+e,d+f) \\ &= (a+(c+e),\ b+(d+f)) \\ &= ((a+c)+e,\ (b+d)+f) \\ &= (a+c,b+d)+(e,f) \\ &= [(a,b)+(c,d)]+(e,f) \end{aligned}$$

$\Rightarrow v_1+(v_2+v_3) = (v_1+v_2)+v_3$

(iii) Identity law

Here $0=(0,0)$ is the additive identity in V because for $v=(a,b)\in V$

$0+v = (0,0)+(a,b) = (0+a,0+b) = (a,b) = v$

& $v+0 = (a,b)+(0,0) = (a+0,b+0) = (a,b) = v$

(iv) Inverse law

Each element $v=(a,b)\in V$ has its additive inverse $-v=(-a,-b)$ because

$v+(-v) = (a,b)+(-a,-b) = (a-a,b-b) = (0,0) = 0$

& $-v+v = (-a,-b)+(a,b) = (-a+a,-b+b) = (0,0) = 0$

(v) Commutative law

Let $v_1=(a,b),\ v_2=(c,d)\in V$ then we prove $v_1+v_2 = v_2+v_1$

Now

$$\begin{aligned} v_1+v_2 &= (a,b)+(c,d) \\ &= (a+c,b+d) \\ &= (c+a,d+b) \\ &= (c,d)+(a,b) \\ &= v_2+v_1 \end{aligned}$$

Hence $(V,+)$ is an abelian gr.

(b) Scalar multiplication

(i) Let $\alpha\in R$ & $v_1=(a_1,b_1)\in V$

Then $\alpha v_1 = \alpha(a_1,b_1) = (\alpha^2 a_1, \alpha^2 b_1)\in V$

(ii) Let $a, b \in R$ & $u_1 = (a_1, b_1) \in V$

then we prove $a(bu_1) = (ab)u_1$

Now $a(bu_1) = a(b(a_1, b_1))$

$= a(b^2a_1, b^2b_1)$

$= (a^2(b^2a_1), a^2(b^2b_1))$

$= ((a^2b^2)a_1, (a^2b^2)b_1)$

$= (ab)(a_1, b_1)$

$\therefore a(bu_1) = (ab)u_1$

(iii) Let $a, b \in R$ & $u_1 = (a_1, b_1) \in V$ then we prove

$(a+b)u_1 = au_1 + bu_1$

Now $(a+b)u_1 = (a+b)(a_1, b_1)$

$= ((a+b)^2a_1, (a+b)^2b_1)$

&

$au_1 + bu_1 = a(a_1, b_1) + b(a_1, b_1)$

$= (a^2a_1, a^2b_1) + (b^2a_1, b^2b_1)$

$= (a^2a_1 + b^2a_1, a^2b_1 + b^2b_1)$

$= ((a^2+b^2)a_1, (a^2+b^2)b_1)$

So $(a+b)u_1 \neq au_1 + bu_1$

As this condition is not satisfied.

So V is not a vector space over R.

(iv) $(a, b) + (c, d) = (a+c, b+d)$ & $K(a, b) = (Ka, 0)$

Sol.-

Let $V = \{(a,b) \mid a, b \in R\}$

Here $(a,b) + (c,d) = (a+c, b+d)$

& $K(a,b) = (Ka, 0)$

First we prove that $(V, +)$ is an abelian gr.

(a)(i) Closure law

Let $u_1 = (a,b)$, $u_2 = (c,d) \in V$ then

$u_1 + u_2 = (a,b) + (c,d)$

$= (a+c, b+d) \in V$

(ii) Associative law

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$

then we prove

$$u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$$

Now

$$\begin{aligned} u_1 + (u_2 + u_3) &= (a,b) + [(c,d) + (e,f)] \\ &= (a,b) + (c+e, d+f) \\ &= (a + (c+e), b + (d+f)) \\ &= ((a+c)+e, (b+d)+f) \\ &= (a+c, b+d) + (e,f) \\ &= [(a,b) + (c,d)] + (e,f) \end{aligned}$$

$\Rightarrow u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$

(iii) Identity law

Here $0 = (0,0)$ is the additive identity in V because for $u = (a,b) \in V$

$0 + u = (0,0) + (a,b) = (0+a, 0+b) = (a,b) = u$

& $u + 0 = (a,b) + (0,0) = (a+0, b+0) = (a,b) = u$

(iv) Inverse law

Each element $u = (a,b) \in V$ has its additive inverse $-u = (-a,-b)$ in V because

$u + (-u) = (a,b) + (-a,-b) = (a-a, b-b) = (0,0) = 0$

& $-u + u = (-a,-b) + (a,b) = (-a+a, -b+b) = (0,0) = 0$

(v) Commutative law

Let $u_1 = (a,b)$ & $u_2 = (c,d) \in V$ then we prove

$$u_1 + u_2 = u_2 + u_1$$

Now

$$\begin{aligned} u_1 + u_2 &= (a,b) + (c,d) \\ &= (a+c, b+d) \\ &= (c+a, d+b) \\ &= (c,d) + (a,b) \\ &= u_2 + u_1 \end{aligned}$$

Hence $(V,+)$ is an abelian gr.

(b) Scalar multiplication.

(i) Let $a \in R$ & $u_1 = (a_1, b_1) \in V$ then

$a u_1 = a(a_1, b_1) = (a a_1, 0) \in V$

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(ii) Let $a, b \in R$ & $u_1 = (a_1, b_1) \in V$ then we prove

$$a(bu_1) = (ab)u_1$$

Now

$$\begin{aligned} a(bu_1) &= a\big(b(a_1, b_1)\big) \\ &= a(ba_1, 0) \\ &= (a(ba_1), 0) \\ &= ((ab)a_1, 0) \\ &= (ab)(a_1, b_1) \end{aligned}$$

$\therefore a(bu_1) = (ab)u_1$

(iii) Let $a, b \in R$ & $u_1 = (a_1, b_1) \in V$ then we prove

$$(a+b)u_1 = au_1 + bu_1$$

Now

$$\begin{aligned} (a+b)u_1 &= (a+b)(a_1, b_1) \\ &= \big((a+b)a_1, 0\big) \\ &= (aa_1 + ba_1, 0) \\ &= (aa_1, 0) + (ba_1, 0) \\ &= a(a_1, b_1) + b(a_1, b_1) \end{aligned}$$

$\therefore (a+b)u_1 = au_1 + bu_1$

(iv) Let $a \in R$ & $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in V$ then we prove $a(u_1 + u_2) = au_1 + au_2$

Now

$$\begin{aligned} a(u_1 + u_2) &= a\big((a_1, b_1) + (a_2, b_2)\big) \\ &= a(a_1 + a_2, b_1 + b_2) \\ &= \big(a(a_1 + a_2), 0\big) \\ &= (aa_1 + aa_2, 0) \\ &= (aa_1, 0) + (aa_2, 0) \\ &= a(a_1, b_1) + a(a_2, b_2) \end{aligned}$$

$\therefore a(u_1 + u_2) = au_1 + au_2$

(v) Let $1 \in R$ & $u = (a, b) \in V$ then we prove $1.u = u$

Now

$$\begin{aligned} 1.u &= 1.(a, b) \\ &= (1.a, 0) = (a, 0) \neq (a, b) = u \end{aligned}$$

So $1.u \neq u$

As this condition is not satisfied.

So V is not a vector space over R

Q3 Check whether each of the following is a real vector space. 22

(i) The set $C[a,b]$ of all continuous real valued functions defined on $[a,b]$ with the usual operations on functions as

For $f, g \in C[a,b]$ & $\alpha \in R$

$$(f+g)(x) = f(x) + g(x)$$

$$\& \quad (\alpha f)(x) = \alpha \cdot f(x).$$

Sol.

Let $C[a,b] = \{f : f \text{ is continuous real valued fn. defined on } [a,b]\}$

then clearly $C[a,b]$ is a subset of the vector space V of all real valued continuous functions defined on R.

To show that $C[a,b]$ is a vector space over R, we have to show that $C[a,b]$ is a subspace of V

For this

let $f, g \in C[a,b]$

then both f & g are real valued continuous fns. defined on $[a,b]$.

Then $(f+g)(x) = f(x) + g(x)$ for $x \in [a,b]$

Now $f+g$ being the sum of two continuous real valued functions defined on $[a,b]$ is also a continuous real valued function defined on $[a,b]$

So $f+g \in C[a,b]$.

Hence $f, g \in C[a,b] \Rightarrow f+g \in C[a,b]$

Now let $\alpha \in R$ & $f \in C[a,b]$

then f is a real valued continuous fn. defined on $[a,b]$.

& $(\alpha f)(x) = \alpha \cdot f(x)$

Clearly scalar multiple of a continuous real valued function is also a continuous real valued function defined on $[a,b]$.

So $\alpha \in \mathbb{R}$, $f \in C[a,b] \Rightarrow \alpha f \in C[a,b]$

Hence $C[a,b]$ is a subspace of the vector space V of all real valued functions defined on $\mathbb{R}$

Hence $C[a,b]$ is a vector space over $\mathbb{R}$

(ii) The set of all functions $f \in C[a,b]$ such that $f(a) = f(b)$

Sol:

Let $C'[a,b] = \{ f \mid f \in C[a,b]$ & $f(a) = f(b) \}$

then clearly $C'[a,b]$ is a subset of $C[a,b]$

To show that $C'[a,b]$ is a vector space over $\mathbb{R}$, we have to show that $C'[a,b]$ is a subspace of the vector space $C[a,b]$.

For this

let $f, g \in C'[a,b]$

then $f, g \in C[a,b]$ s.t.

$$f(a) = f(b) \quad \& \quad g(a) = g(b)$$

Now $f+g$ being sum of two real valued continuous functions defined on $[a,b]$ is a real valued continuous function defined on $[a,b]$.

Hence $f+g \in C[a,b]$

Moreover $(f+g)(a) = f(a) + g(a)$

$$= f(b) + g(b)$$

$$\text{i.e. } (f+g)(a) = (f+g)(b)$$

$$\Rightarrow f+g \in C'[a,b]$$

So $f, g \in C'[a,b] \Rightarrow f+g \in C'[a,b]$

Now

let $\alpha \in \mathbb{R}$ & $f \in C'[a,b]$ then clearly αf is a real valued continuous function defined on $[a,b]$,

So $\alpha f \in C'[a,b]$

Moreover

$(af)(a) = a f(a)$

$= af(b)$ $\quad \because f \in C'[a,b]$ so $f(a) = f(b)$

$\therefore (af)(a) = (af)(b)$

Hence $af \in C'[a,b]$

So $a \in R$, $f \in C'[a,b] \Rightarrow af \in C'[a,b]$

Hence $C'[a,b]$ is a subspace of $C[a,b]$

So $C[a,b]$ is a vector space over R

(iii) The set of all solutions of the diff. eq.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Sol.

Let W be the set of all solutions of the given diff. eq. Then V is a subset of the vector space V of all real functions defined on R.

To show that W is a vector space over R, we have to show that W is a subspace of V over R.

For this

let $f, g \in W$ & $a, b \in R$

then f & g are solns. of diff. eq. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Then $\frac{df}{dx^2} - 5\frac{df}{dx} + 6f = 0$

& $\frac{d^2g}{dx^2} - 5\frac{dg}{dx} + 6g = 0$

Now

$$\frac{d^2}{dx^2}(af+bg) - 5\frac{d}{dx}(af+bg) + 6(af+bg)$$

$$= \frac{d^2}{dx^2}(af) + \frac{d^2}{dx^2}(bg) - 5\frac{d}{dx}(af) - 5\frac{d}{dx}(bg) + 6af + 6bg$$

$$= a\frac{d^2f}{dx^2} + b\frac{d^2g}{dx^2} - 5a\frac{df}{dx} - 5b\frac{dg}{dx} + 6af + 6bg$$

$$= a\left(\frac{d^2f}{dx^2} - 5\frac{df}{dx} + 6f\right) + b\left(\frac{d^2g}{dx^2} - 5\frac{dg}{dx} + 6g\right)$$

$$= a(0) + b(0)$$

$$= 0$$

So $af+bg$ is a soln. of $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Hence $af+bg \in W$

So W is a subspace of V over R

Hence W is a vector space over R.

(iv) The set of all 2×2 real matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

Sol.

Let $V = \left\{ \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} : a, b \in R \right\}$

First we prove that V is an abelian gr. under matrix addition.

This set V has no additive identity

So V is not a gr. under addition.

As this condition is not satisfied.

So V is not a vector space over R.

Q4 Check whether each of the following subsets is a subspace of the indicated vector space:

(i) Q, the set of rational no's. in R

Sol.

Let $a, b \in R$ & $q_1, q_2 \in Q$

then q_1 & q_2 are rational numbers

Since $a, b \in R$ & $q_1, q_2 \in Q$

So $aq_1 + bq_2$ may not be a rational no.

Hence $aq_1 + bq_2 \notin Q$

So $a, b \in R$, $q_1, q_2 \in Q \Rightarrow aq_1 + bq_2 \notin Q$

Hence Q is not a subspace of R

(ii) All 2×2 non singular real matrices in M_{22}.

Sol.

Let V be the set of all non singular real matrices

in M_{22}.
As additive identity $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ of V does not belong to V. So V itself is not a vector space over R. Hence V is not a subspace of M_{22}.

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iii) The set $B[a,b]$ of all bounded real functions defined on $[a,b]$ in the space of all real functions defined on $[a,b]$.

Soln

Here $B[a,b]$ is the set of all bounded real functions defined on $[a,b]$

Let $a,b \in R$ & $f,g \in B[a,b]$

Then f & g are bounded real functions defined on $[a,b]$ then $af+bg$ is also a bounded real valued function defined on $[a,b]$.

So $af+bg \in B[a,b]$

Hence $a,b \in R$ & $f,g \in B[a,b] \Rightarrow af+bg \in B[a,b]$

So $B[a,b]$ is the subspace of the vector space of all real functions defined on $[a,b]$.

Q5 Show that the union of two subspaces of a vector space need not be a subspace. Let X & Y be subspaces of a vector space V. Prove that $X \cup Y$ is a subspace of V if & only if either $X \subset Y$ or $Y \subset X$.

Soln

Consider the Euclidean space R^3 where

$R^3 = \{(x,y,z) : x,y,z \in R\}$

Let $X = \{(x_1,0,0) : x_1 \in R\}$

& $Y = \{(0,x_2,0) : x_2 \in R\}$

Then clearly X & Y are subspaces of R^3.

We shall show that $X \cup Y$ is not a subspace of R^3.

Then $x = (x_1, 0, 0)$ & $y = (0, x_2, 0) \in Y$

$\Rightarrow x, y \in X \cup Y$

But $x + y = (x_1, 0, 0) + (0, x_2, 0)$

$= (x_1, x_2, 0) \notin X \cup Y$

As closure property under addition does not hold in $X \cup Y$

So $X \cup Y$ is not a subspace of R^3.

Next

Suppose $X \cup Y$ is a subspace of V & suppose neither $X \subset Y$ nor $Y \subset X$

Then there are elements x & y such that

$x \in X$ but $x \notin Y$ & $y \in Y$ but $y \notin X$.

Now $x, y \in X \cup Y$ & since $X \cup Y$ is a vector space

So $x + y \in X \cup Y$

$\Rightarrow$ either $x + y \in X$ or $x + y \in Y$

Suppose $x + y \in X$

then $y = (x+y) - x \in X$ (Since X is a vector space)

viz Contradiction

Similarly if $x + y \in Y$

Then $x = (x+y) - y \in Y$ (Since Y is a vector space)

viz again Contradiction

Hence our supposition is wrong.

Hence either $X \subset Y$ or $Y \subset X$

Conversely

Let $X \subset Y$ or $Y \subset X$

$\Rightarrow X \cup Y = Y$ or $X \cup Y = X$

Since X & Y are subspaces of V

Hence $X \cup Y$ is also a subspace of V

Q6 Which of the following are subspaces of R^3?

(i) $W = \{(x,y,z) : x+y+z = 0\}$

Sol.

$W = \{(x,y,z) : x+y+z = 0\}$

Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $x_1+y_1+z_1 = 0$ & $x_2+y_2+z_2 = 0$

Now let $a, b \in R$ then

$$aw_1 + bw_2 = a(x_1,y_1,z_1) + b(x_2,y_2,z_2)$$
$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1+bx_2,\ ay_1+by_2,\ az_1+bz_2)$$

Now $aw_1 + bw_2 \in W$ if $ax_1+bx_2+ay_1+by_2+az_1+bz_2 = 0 \quad \forall a,b \in R$

Now

$$ax_1+bx_2+ay_1+by_2+az_1+bz_2 = ax_1+ax_2+ax_3+bx_2+by_2+bz_2$$
$$= a(x_1+y_1+z_1) + b(x_2+y_2+z_2)$$
$$= a(0) + b(0)$$
$$= 0$$

So $aw_1 + bw_2 \in W$

Hence for $a,b \in R$, $w_1, w_2 \in W \Rightarrow aw_1 + bw_2 \in W$

Hence W is a subspace of R^3.

(ii) $W = \{(x,y,z) : x \geq 0\}$

Sol. Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $x_1 \geq 0$ & $x_2 \geq 0$

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(x_1,y_1,z_1) + b(x_2,y_2,z_2)$$
$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1+bx_2,\ ay_1+by_2,\ az_1+bz_2)$$

Now $aw_1+bw_2 \in W$ if $ax_1+bx_2 \geq 0$ $\forall a,b \in R$

As $a,b \in R$ & $x_1,x_2 \geq 0$

So ax_1+bx_2 may not be ≥ 0

Hence $aw_1+bw_2 \notin W$ $\forall a,b \in R$

So W is not a subspace of R^2

(iii) $W = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$

Sol.

Let $w_1, w_2 \in W$

$\Rightarrow$ $w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $x_1^2+y_1^2+z_1^2 \leq 1$ & $x_2^2+y_2^2+z_2^2 \leq 1$

Now let $a,b \in R$ then

$$aw_1+bw_2 = a(x_1,y_1,z_1)+b(x_2,y_2,z_2)$$
$$= (ax_1, ay_1, az_1)+(bx_2, by_2, bz_2)$$
$$= (ax_1+bx_2,\ ay_1+by_2,\ az_1+bz_2)$$

Now $aw_1+bw_2 \in W$ if $(ax_1+bx_2)^2+(ay_1+by_2)^2+(az_1+bz_2)^2 \leq 1$

As $(ax_1+bx_2)^2+(ay_1+by_2)^2+(az_1+bz_2)^2$

$$= a^2(x_1^2+y_1^2+z_1^2)+b^2(x_2^2+y_2^2+z_2^2)+2ab(x_1x_2+y_1y_2+z_1z_2)$$

If we take

$$a^2 = \frac{1}{x_1^2+y_1^2+z_1^2} \quad \& \quad b^2 = \frac{1}{x_2^2+y_2^2+z_2^2} \quad \&$$

$a, b, x_1, y_1, z_1, x_2, y_2, z_2$ are all +ve then above expression is $\nleq 1$

So $aw_1+bw_2 \notin W$ $\forall a,b \in R$

Hence W is not a subspace of R^3

(iv) $W = \{(x,y,z) : x,y,z$ are rationals$\}$

Sol.

Let $w_1, w_2 \in W$ then

$w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where x_1, y_1, z_1 & x_2, y_2, z_2 are rationals.

Let $a, b \in R$ then

$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2$ are rationals

Since $a, b \in R$, so a, b may not be rationals. Hence

$ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2$ may not be rationals

So $aw_1 + bw_2 \notin W \quad \forall a, b \in R$

Hence W is not a subspace of R^3

(V):- $W = \{(x, 0, z) : x, z \in R\}$

Sol:-

Let $w_1, w_2 \in W$ then

$w_1 = (x_1, 0, z_1)$

& $w_2 = (x_2, 0, z_2)$ where $x_1, z_1, x_2, z_2 \in R$

Let $a, b \in R$ then

$aw_1 + bw_2 = a(x_1, 0, z_1) + b(x_2, 0, z_2)$

$= (ax_1, 0, az_1) + (bx_2, 0, bz_2)$

$= (ax_1 + bx_2, 0, az_1 + bz_2)$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2, az_1 + bz_2 \in R$

But as $a, b, x_1, z_1, x_2, z_2 \in R$.

So $ax_1 + bx_2, az_1 + bz_2 \in R. \quad \forall a, b \in R$

Hence $aw_1 + bw_2 \in W$

So W is a subspace of R^3

(Vi) $W = \{(x, y, z) : y^2 = x + z\}$

Sol:- Let $w_1, w_2 \in W$.

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_1)$ where $y_1^2 = x_1^2 + z_1^2$ & $y_2^2 = x_2^2 + z_2^2$

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2)$$

Now $aw_1 + bw_2 \in W$ if $(ay_1 + by_2)^2 = (ax_1 + bx_2)^2 + (az_1 + bz_2)^2$

Now

$$(ay_1 + by_2)^2 = a^2y_1^2 + b^2y_2^2 + 2aby_1y_2$$
$$= a^2(x_1^2 + z_1^2) + b^2(x_2^2 + z_2^2) + 2aby_1y_2$$
$$= (a^2x_1^2 + b^2x_2^2) + (a^2z_1^2 + b^2z_2^2) + 2ab\sqrt{x_1^2 + z_1^2}\ \sqrt{x_2^2 + z_2^2}$$
$$= (a^2x_1^2 + b^2x_2^2) + (a^2z_1^2 + b^2z_2^2) + 2ab\sqrt{(x_1^2 + z_1^2)(x_2^2 + z_2^2)}$$

&

$$(ax_1 + bx_2)^2 + (az_1 + bz_2)^2 = a^2x_1^2 + b^2x_2^2 + 2abx_1x_2 + a^2z_1^2 + b^2z_2^2 + 2abz_1z_2$$
$$= (a^2x_1^2 + b^2x_2^2) + (a^2z_1^2 + b^2z_2^2) + 2ab(x_1x_2 + z_1z_2)$$

So $(ay_1 + by_2)^2 \neq (ax_1 + bx_2)^2 + (az_1 + bz_2)^2$

Hence $aw_1 + bw_2 \in W$

So W is not a subspace of R^3

Q7 Let V be the vector space of all real valued functions defined on R. State which of the following are subspaces of V.

(i) The set of all even functions.

(ii) The set of all differentiable functions.

(iii) The set $W = \{ f \mid f(x) = Kf(-x),\ K \in R \text{ fixed} \}$

(iv) The set $W = \{ f \in V : \int_0^1 f(x)\,dx = 0 \}$

Sol.

(i) Here $V = \{ f : f \text{ is a real valued fn. defined on } R \}$

& $W = \{ f : f \text{ is an even function} \}$

Let $f, g \in W$

then both f & g are even functions.

i.e.,

$f(-x) = f(x)$

& $g(-x) = g(x)$

Let $a, b \in R$ then we prove $af+bg \in W$

Now

$$
\begin{aligned}
(af+bg)(-x) &= (af)(-x) + (bg)(-x) \\
&= a \cdot f(-x) + b\,g(-x) \\
&= af(x) + bg(x) \qquad \because f, g \text{ are even fns.} \\
&= (af)(x) + (bg)(x) \\
&= (af+bg)(x)
\end{aligned}
$$

So $af+bg$ is an even fn. Hence $af+bg \in W$

Hence

$a, b \in R$, $f, g \in W \Rightarrow af+bg \in W$

So W is a subspace of V.

(ii) $W = \{ f : f$ is a differentiable function $\}$

Sol. Let $f, g \in W$

Then both f & g are differentiable functions i.e., f' & g' exists.

Now let $a, b \in R$ then we prove $af+bg \in W$

As

$$
\begin{aligned}
(af+bg)' &= (af)' + (bg)' \\
&= af' + bg'
\end{aligned}
$$

Since f', g' exists. Hence $(af+bg)'$ exists.

So $af+bg$ is differentiable. Hence $af+bg \in W$

Hence $a, b \in R$ & $f, g \in W \Rightarrow af+bg \in W$

So W is a subspace of V

(iii) $W = \{ f : f(x) = Kf(-x),\ K \in R$ fixed $\}$

Sol. Let $f, g \in W$ then

$f(x) = Kf(-x)$

& $g(x) = Kg(-x)$ where $K \in R$ is fixed

Let $a, b \in R$ then we prove $af + bg \in W$

Now

$$(af+bg)(x) = (af)(x) + (bg)(x)$$
$$= af(x)$$
$$= aKf(-x) + bKg(-x)$$
$$= Kaf(-x) + Kbg(-x) \qquad \because a, K \in R$$
$$= K\left[(af)(-x) + (bg)(-x)\right] \qquad \text{So } aK = Ka$$
$$= K\left[(af+bg)(-x)\right]$$

So $af + bg \in W$

Hence $a, b \in R$, $f, g \in W \Rightarrow af + bg \in W$

So W is a subspace of V.

(iv) $W = \left\{ f \in V : \int_0^1 f(x)\,dx = 0 \right\}$

Sol. Let $f, g \in W$

Then $\int_0^1 f(x)\,dx = 0$ & $\int_0^1 g(x)\,dx = 0$

Now for $a, b \in R$, we show that $af + bg \in W$

As

$$\int_0^1 (af+bg)(x)\,dx = \int_0^1 \left[(af)(x) + (bg)(x)\right]dx$$
$$= \int_0^1 \left[af(x) + bg(x)\right]dx$$
$$= \int_0^1 af(x)\,dx + \int_0^1 bg(x)\,dx$$
$$= a\int_0^1 f(x)\,dx + b\int_0^1 g(x)\,dx$$
$$= a.0 + b.0$$
$$= 0, \quad \text{so } af + bg \in W$$

Hence for $a, b \in R$ & $f, g \in W \Rightarrow af + bg \in W$

So W is a subspace of V.

Q8 Let V be the vector space of all real polynomials of degree $\leq n$ together with the zero polynomial. Determine whether or not W is a subspace of V, where W consists of the zero polynomial and all polynomials

(i) With integral Cofficients & of degree $\leq n$.

(ii) of degree ≤ 3

(iii) With only even powers of x & of degree $\leq n$.

Sol.

(i) Here $V = \{a_0 + a_1x + a_2x^2 + \cdots\cdots + a_nx^n : a_i\text{'s} \in R\}$

& $W = \{a_0 + a_1x + a_2x^2 + \cdots\cdots + a_nx^n : a_i\text{'s} \in Z\}$

Let $w_1, w_2 \in W$ then

$w_1 = a_0 + a_1x + a_2x^2 + \cdots\cdots + a_nx^n$

& $w_2 = b_0 + b_1x + b_2x^2 + \cdots\cdots + b_mx^m$ where $a_i, b_i \in Z$ & $m < n$

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(a_0 + a_1x + \cdots\cdots + a_nx^n) + b(b_0 + b_1x + \cdots\cdots + b_mx^m)$$
$$= aa_0 + aa_1x + \cdots\cdots + aa_nx^n + bb_0 + bb_1x + \cdots\cdots + bb_mx^m$$
$$= (aa_0 + bb_0) + (aa_1 + bb_1)x + \cdots\cdots + (aa_m + bb_m)x^m + \cdots + aa_nx^n$$

Since $a, b \in R$ & $a_i, b_i \in Z$

So $aa_i + bb_i$ may not be integers

Hence $aw_1 + bw_2 \notin W \quad \forall a, b \in R$

So W is not a subspace of V.

(ii) $W = \{a_0 + a_1x + \cdots\cdots + a_nx^n : n \leq 3\}$

Sol.

Let $w_1, w_2 \in W$ then

$w_1 = a_0 + a_1x + a_2x^2 + a_3x^3$

& $w_2 = b_0 + b_1x + b_2x^2$

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(a_0 + a_1x + a_2x^2 + a_3x^3) + b(b_0 + b_1x + b_2x^2)$$

$$aw_1+bw_2 = aa_0+aa_1x+aa_2x^2+aa_3x^3+bb_0+bb_1x+bb_2x^2$$
$$= (aa_0+bb_0)+(aa_1+bb_1)x+(aa_2+bb_2)x^2+aa_3x^3$$
which is a polynomial of degree ≤ 3

Hence $aw_1+bw_2 \in W$

So W is a subspace of V

(iii) $W = \{a_0+a_1x^2+a_2x^4+\cdots\cdots+a_nx^{2n} : n\in Z^+$

Sol.

Let $w_1, w_2 \in W$ then

$w_1 = a_0+a_1x^2+a_2x^4+\cdots\cdots+a_nx^{2n}$

& $w_2 = b_0+b_1x^2+b_2x^4+\cdots\cdots+b_mx^{2m}$ where $m<n$ & $m,n\in Z^+$

Let $a,b\in R$ then

$$aw_1+bw_2 = a(a_0+a_1x^2+a_2x^4+\cdots+a_nx^{2n})+b(b_0+b_1x^2+b_2x^4+\cdots+b_mx^{2m})$$
$$= aa_0+aa_1x^2+\cdots+aa_nx^{2n}+bb_0+bb_1x^2+\cdots+bb_mx^{2m}$$
$$= (aa_0+bb_0)+(aa_1+bb_1)x^2+\cdots+(aa_m+bb_m)x^{2m}+\cdots+aa_nx^{2n}$$
which is a polynomial with only even powers of x.

Hence $aw_1+bw_2 \in W$

So W is a subspace of V.

Q9 Express the vector $(2,-5,3)$ in R^3 as a linear combination of the vectors $(1,-3,2), (2,-4,-1)$ & $(1,-5,7)$.

Sol.

Let $(2,-5,3) = a(1,-3,2)+b(2,-4,-1)+c(1,-5,7)$
$$=(a,-3a,2a)+(2b,-4b,-b)+(c,-5c,7c)$$

So $(2,-5,3) = (a+2b+c, -3a-4b-5c, 2a-b+7c)$

$\Rightarrow$

$$a+2b+c = 2 \quad \text{——— (1)}$$
$$-3a-4b-5c = -5 \quad \text{——— (2)}$$
$$2a-b+7c = 3 \quad \text{——— (3)}$$

Multiplying (1) by 3 & adding in (2)

$$\cancel{3a} + 6b + 3c = 6$$
$$-\cancel{3a} - 4b - 5c = -5$$
$$2b - 2c = 1$$
or $$b - c = \tfrac{1}{2} \quad \text{——— (4)}$$

Now multiplying (1) by 2 & subt. (3) from (1)

$$\cancel{2a} + 4b + 2c = 4$$
$$\underset{-}{\cancel{2a}} \underset{+}{-} b \underset{-}{+} 7c = \underset{-}{3}$$
$$5b - 5c = 1$$
or $$b - c = \frac{1}{5} \quad \text{——— (5)}$$

From (4) & (5), we cannot find values of b & c

Thus $(2, -5, 3)$ cannot be expressed as a linear combination of $(1, -3, 2)$, $(2, -4, -1)$ & $(1, -5, 7)$

<u>Q1.</u> For what value of K will the vector $(1, -2, K)$ in R^3 be a linear combination of the vectors $(3, 0, -2)$ & $(2, -1, -5)$?

<u>Sol.</u>

Let $$(1, -2, K) = a(3, 0, -2) + b(2, -1, -5)$$
$$= (3a, 0, -2a) + (2b, -b, -5b)$$
or $$(1, -2, K) = (3a + 2b, -b, -2a - 5b)$$

$$\Rightarrow \quad 3a + 2b = 1 \quad \text{——— (1)}$$
$$-b = -2 \quad \text{——— (2)}$$
& $$-2a - 5b = K \quad \text{——— (3)}$$

from (2) $\boxed{b = 2}$

Put in (1)

$$3a + 2(2) = 1$$
$$3a + 4 = 1$$
$$3a = -3$$
$$\boxed{a = -1}$$

Putting values of a & b in ③

$$-2(-1)-5(2) = K$$
$$2-10 = K$$
$$-8 = K$$
or $\boxed{K = -8}$

So for $K = -8$, the vector $(1,-2,K)$ is a linear Combination of $(3,0,-2)$ & $(2,-1,-5)$.

Q11 Let U & W be the subspaces of R^3 defined by $U = \{(x,y,z) : x = y = z\}$
& $W = \{(0,y,z) : y,z \in R\}$
Show that
$$R^3 = U \oplus W$$

Sol.

To show that $R^3 = U \oplus W$, we have to prove that $R^3 = U + W$
& $U \cap W = \{0\}$

Here
$$U = \{(x,y,z) : x = y = z\}$$
& $W = \{(0,y,z) : y,z \in R\}$

Let
$$(x,y,z) \in R^3$$
then $(x,y,z) = (x,x,x) + (0,y-x,z-x) \in U+W$
So $(x,y,z) \in U+W$
Hence $R^3 \subseteq U+W$ ——— ①

Conversely
let $u \in U$ & $w \in W$ then $u+w \in U+W$
$\Rightarrow u = (x,x,x)$ & $w = (0,y,z)$
then $u+w = (x,x,x) + (0,y,z)$
$= (x, x+y, x+z) \in R^3$

So $u+w \in R^3$

Hence $U+W \subseteq R^3$ ——— (2)

From (1) & (2)

$$R^3 = U+W$$

Now we want to show $U \cap W = \{0\}$

Let $\alpha \in U \cap W$

$\Rightarrow \alpha \in U$ & $\alpha \in W$

$\Rightarrow \alpha = (x,x,x)$ & $\alpha = (0,y,z)$

$\Rightarrow (x,x,x) = (0,y,z)$

So $x = 0$, $x = y$, $x = z$

$\Rightarrow x = y = z = 0$

So $\alpha = (0,0,0)$

Hence $U \cap W = \{0\}$

So

$$R^3 = U \oplus W$$

Q12 Show that each of the following sets of vectors generate R^3

(i) $\{(1,2,3), (0,1,2), (0,0,1)\}$

(ii) $\{(1,1,1), (0,1,1), (0,1,-1)\}$

Sol:-

(i) Given set is $\{(1,2,3), (0,1,2), (0,0,1)\}$

Let $(x,y,z) \in R^3$ & suppose

$(x,y,z) = a(1,2,3) + b(0,1,2) + c(0,0,1)$

$= (a, 2a, 3a) + (0, b, 2b) + (0,0,c)$

So $(x,y,z) = (a, 2a+b, 3a+2b+c)$

$\Rightarrow a = x$ ——— (1)

$2a+b = y$ ——— (2)

$3a+2b+c = z$ ——— (3)

from (1) $\boxed{a = x}$

Put in (2)

$2x + b = y$

$\boxed{b = y - 2x}$

Put values of a & b in (3)

$3x + 2(y-2x) + c = z$

$3x + 2y - 4x + c = z$

or $-x + 2y + c = z$

or $\boxed{c = x - 2y + z}$

So

$(x,y,z) = x(1,2,3) + (y-2x)(0,1,2) + (x-2y+z)(0,0,1)$

Hence given vectors generate R^3.

(ii) Given set is $\{(1,1,1),(0,1,1),(0,1,-1)\}$

Let $(x,y,z) \in R^3$ & suppose

$(x,y,z) = a(1,1,1) + b(0,1,1) + c(0,1,-1)$

$= (a,a,a) + (0,b,b) + (0,c,-c)$

or $(x,y,z) = (a, a+b+c, a+b-c)$

$\Rightarrow$

$a = x$ ——— (1)

$a+b+c = y$ ——— (2)

$a+b-c = z$ ——— (3)

from (1) $\boxed{a = x}$

Adding (2) & (3)

$2a + 2b = y + z$

$2x + 2b = y + z$

or $2b = y + z - 2x$

or $\boxed{b = \frac{y+z-2x}{2}}$

Put values of a & b in (2)

$x + \frac{y+z-2x}{2} + c = y$

or $c = y - x - \dfrac{y+z-2x}{2}$

$= \dfrac{2y - \cancel{2x} - y - z + \cancel{2x}}{2}$

$$\boxed{c = \frac{y-z}{2}}$$

So

$(x,y,z) = x(1,1,1) + \left(\frac{y+z-2x}{2}\right)(0,1,1) + \left(\frac{y-z}{2}\right)(0,1,-1)$

Hence given vectors generate R^3.

Q13 Determine whether the set $S = \{(1,1,2),(1,0,1),(2,1,3)\}$ spans R^3

Sol: Given set is $S = \{(1,1,2),(1,0,1),(2,1,3)\}$

Let $(x,y,z) \in R^3$ & suppose

$(x,y,z) = a(1,1,2) + b(1,0,1) + c(2,1,3)$

$= (a,a,2a) + (b,0,b) + (2c,c,3c)$

So $(x,y,z) = (a+b+2c,\ a+c,\ 2a+b+3c)$

$\Rightarrow$

$a + b + 2c = x$ ——— (1)

$a + c = y$ ——— (2)

$2a + b + 3c = z$ ——— (3)

Subt. (3) from (1)

$-a - c = x - z$

or $a + c = z - x$ ——— (4)

& $a + c = y$ ——— (2)

The eqs. (2) & (4) cannot be solved for a & c

Hence we cannot find values of a, b & c.

So the set $S = \{(1,1,2),(1,0,1),(2,1,3)\}$ does not span R^3

Q14 Show that the yz-plane $W = \{(0,y,z) : y, z \in R\}$ is spanned by

(i) $(0,1,1)$ and $(0,2,-1)$

(ii) $(0,1,2), (0,2,3)$ & $(0,3,1)$

Sol:-

(i) Let $W = \{(0,y,z) : y, z \in R\}$

Suppose

$(0,y,z) \in W$ &

$$(0,y,z) = a(0,1,1) + b(0,2,-1)$$
$$= (0,a,a) + (0,2b,-b)$$

So $(0,y,z) = (0, a+2b, a-b)$

$$\Rightarrow \quad a+2b = y \quad \text{——— (1)}$$
$$a-b = z \quad \text{——— (2)}$$

Subt. (2) from (1)

$$3b = y-z$$
$$\boxed{b = \frac{y-z}{3}}$$

Put value in (2)

$$a - \frac{y-z}{3} = z$$

or $$a = \frac{y-z}{3} + z$$
$$= \frac{y-z+3z}{3}$$

So $$\boxed{a = \frac{y+2z}{3}}$$

So

$$(0,y,z) = \left(\frac{y+2z}{3}\right)(0,1,1) + \left(\frac{y-z}{3}\right)(0,2,-1)$$

Hence yz-plane is spanned by $(0,1,1)$ & $(0,2,-1)$

(ii) Sol:- Given vectors are $(0,1,2), (0,2,3)$ & $(0,3,1)$

Let $(0,y,z) \in W$ & suppose

$$(0,y,z) = a(0,1,2) + b(0,2,3) + c(0,3,1)$$

$= (0, a, 2a) + (0, 2b, 3b) + (0, 3c, c)$

So $(0, y, z) = (0, a+2b+3c, 2a+3b+c)$

$\Rightarrow$

$a + 2b + 3c = y$ ——— ①

& $2a + 3b + c = z$ ——— ②

Put $a = 0$

So $2b + 3c = y$ ——— ③

$3b + c = z$ ——— ④

Multiplying ④ by 3 & subt. from ③

$2b + \cancel{3c} = y$ ——— ③

$\underline{-9b \pm \cancel{3c} = -3z}$ ——— ⑤

$-7b = y - 3z$

$$\boxed{b = \frac{3z - y}{7}}$$

Put in ③

$2\left(\frac{3z-y}{7}\right) + 3c = y$

$\frac{6z - 2y}{7} + 3c = y$

$3c = y - \frac{6z - 2y}{7}$

$= \frac{7y - 6z + 2y}{7}$

$3c = \frac{9y - 6z}{7}$

$$\boxed{c = \frac{3y - 2z}{7}}$$

Hence $(0, y, z) = 0(0,1,2) + \left(\frac{3z-y}{7}\right)(0,2,3) + \left(\frac{3y-2z}{7}\right)(0,3,1)$

So yz- plane is spanned by $(0,1,2), (0,2,3)$ & $(0,3,1)$

Q15 Find an eq. (or equations) of the subspace W of R^3 generated by each of following sets of vectors.

(i) $\{(1,-3,5), (-2,6,-10)\}$

(ii) $\{(1,-3,2), (-2,0,3)\}$

(iii) ~~$\{(1,-2,1), (-2,0,3), (3,-2,-2)\}$~~

(i) $\{(1,-3,5),(-2,6,-10)\}$

Sol. Since W is spanned by the vectors $(1,-3,5)$, $(-2,6,-10)$.

So each vector $(x,y,z) \in W$ is a linear combination of these vectors. i.e., there exist scalars a,b s.t.

$$(x,y,z) = a(1,-3,5) + b(-2,6,-10)$$
$$= (a,-3a,5a) + (-2b,6b,-10b)$$

or $(x,y,z) = (a-2b, -3a+6b, 5a-10b)$

$$\Rightarrow \left.\begin{aligned} a-2b &= x \\ -3a+6b &= y \\ 5a-10b &= z \end{aligned}\right\} \text{——— } (1)$$

We reduce the augmented matrix of system (1) to echelon form as:

$$A_b = \begin{bmatrix} 1 & -2 & x \\ -3 & 6 & y \\ 5 & -10 & z \end{bmatrix}$$

$$\sim \left[\begin{array}{cc:c} 1 & -2 & x \\ 0 & 0 & y+3x \\ 0 & 0 & z-5x \end{array}\right] \qquad \begin{array}{l} R_2+3R_1 \\ R_3-5R_1 \end{array}$$

The system (1) is Consistent if

rank A = Rank A_b

$\Rightarrow$ $y+3x = 0$ & $z-5x = 0$

$$\text{or } \left.\begin{aligned} x &= t \\ y &= -3t \\ z &= 5t \end{aligned}\right\} \qquad t \in R$$

These are the req. eqs. of the subspace W of R^3

(ii) $\{(1,-3,2),(-2,0,3)\}$

Sol. Since W is spanned by the vectors $(1,-3,2)$ & $(-2,0,3)$. So each vector $(x,y,z) \in W$ is a

linear combination of these vectors.
i.e., there exist scalars $a, b \in \mathbb{R}$ s.t.

$$(x, y, z) = a(1, -3, 2) + b(-2, 0, 3)$$
$$= (a, -3a, 2a) + (-2b, 0, 3b)$$

or $(x, y, z) = (a-2b, -3a, 2a+3b)$

$\Rightarrow$

$$\left.\begin{aligned} a - 2b &= x \\ -3a &= y \\ 2a + 3b &= z \end{aligned}\right\} \quad \text{——— (1)}$$

We reduce the augmented matrix of system (1) to echelon form as follows:

$$A_b = \begin{bmatrix} 1 & -2 & x \\ -3 & 0 & y \\ 2 & 3 & z \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & x \\ 0 & -6 & y+3x \\ 0 & 7 & z-2x \end{bmatrix} \qquad \begin{matrix} R_2 + 3R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & x \\ 0 & 1 & -\frac{1}{6}(y+3x) \\ 0 & 1 & \frac{1}{7}(z-2x) \end{bmatrix} \qquad -\frac{1}{6}R_2, \ \frac{1}{7}R_3$$

$$\sim \left[\begin{array}{cc:c} 1 & -2 & x \\ 0 & 1 & -\frac{1}{6}(y+3x) \\ 0 & 0 & \frac{1}{7}(z-2x) + \frac{1}{6}(y+3x) \end{array}\right] \qquad R_3 - R_2$$

The system (1) is consistent if the rank A = rank A_b

$$\Rightarrow \quad \frac{1}{7}(z-2x) + \frac{1}{6}(y+3x) = 0$$

or
$$6(z-2x) + 7(y+3x) = 0$$
$$6z - 12x + 7y + 21x = 0$$
$$9x + 7y + 6z = 0$$

which is the req. eq. of the subspace W of $\mathbb{R}^3$.

(iii) $\{(1,-2,1),(-2,0,3),(3,-2,-2)\}$

Sol. Since W is spanned by the vectors $(1,-2,1)$, $(-2,0,3)$ & $(3,-2,-2)$. So each vector $(x,y,z)\in W$ is a linear combination of these vectors.

i.e., there exist scalars $a,b,c\in R$ s.t.

$$(x,y,z) = a(1,-2,1)+b(-2,0,3)+c(3,-2,-2)$$
$$= (a,-2a,a)+(-2b,0,3b)+(3c,-2c,-2c)$$

or $(x,y,z) = (a-2b+3c,\ -2a-2c,\ a+3b-2c)$

$$\Rightarrow \left.\begin{aligned} a-2b+3c &= x\\ -2a-2c &= y\\ a+3b-2c &= z\end{aligned}\right\} \text{———} (1)$$

We reduce the augmented matrix of this system to echelon form as:

$$A_b = \begin{bmatrix} 1 & -2 & 3 & x\\ -2 & 0 & -2 & y\\ 1 & 3 & -2 & z\end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & x\\ 0 & -4 & 4 & y+2x\\ 0 & 5 & -5 & z-x\end{bmatrix} \quad \begin{matrix} R_2+2R_1\\ R_3-R_1\end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & x\\ 0 & 1 & -1 & -\frac{1}{4}(y+2x)\\ 0 & 1 & -1 & \frac{1}{5}(z-x)\end{bmatrix} \quad -\frac{1}{4}R_2,\ \frac{1}{5}R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & x\\ 0 & 1 & -1 & -\frac{1}{4}(y+2x)\\ 0 & 0 & 0 & \frac{1}{5}(z-x)+\frac{1}{4}(y+2x)\end{bmatrix} \quad R_3-R_2$$

The system (1) is consistent if

rank A = rank A_b

$$\Rightarrow \frac{1}{5}(z-x)+\frac{1}{4}(y+2x) = 0$$

$\Rightarrow \quad 4(z-x)+5(y+2x) = 0$

or $\quad 4z-4x+5y+10x = 0$

or $\quad 6x+5y+4z = 0$

Which is the req. eq. of subspace W of R^3

Q16 Show that the complex no's. $2+3i$ & $1-2i$ generate the vector space C over R.

Sol. Here $C = \{ x+iy : x,y \in R \}$

Any vector of C has the form $x+iy$; $x,y \in R$

Suppose

$$x+iy = a(2+3i)+b(1-2i)$$
$$= 2a+3ai+b-2bi$$

or $\quad x+iy = (2a+b)+i(3a-2b)$

$\Rightarrow$

$$2a+b = x \quad \text{———} \quad (1)$$
$$3a-2b = y \quad \text{———} \quad (2)$$

Multiplying (1) by 2 & adding in (2)

$$4a+\cancel{2b} = 2x \quad \text{———} \quad (1)$$
$$3a-\cancel{2b} = y \quad \text{———} \quad (2)$$

$$7a = 2x+y$$

$$\boxed{a = \frac{2x+y}{7}}$$

Put in (1)

$$2\left(\frac{2x+y}{7}\right)+b = x$$

$$\frac{4x+2y}{7}+b = x$$

$$b = x-\frac{4x+2y}{7}$$

$$= \frac{7x-4x-2y}{7}$$

$$\boxed{b = \frac{3x-2y}{7}}$$

Hence

$$x+iy = \left(\frac{2x+y}{7}\right)(2+3i) + \left(\frac{3x-2y}{7}\right)(1-2i)$$

So given vectors $2+3i$ & $1-2i$ generate C over R.

Q17 Let S & T be subsets of a vector space V. Show that

(i) $\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$,

(ii) $\langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Give an example to show that equality need not hold in either case.

Sol.

(i) Let $S = \{u_1, u_2, \ldots, u_r\}$ &

$T = \{v_1, v_2, \ldots, v_t\}$

$\Rightarrow$ $S \cup T = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_t\}$

We want to show $\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$

Let $v \in \langle S \rangle \cup \langle T \rangle$

$\Rightarrow$ $v \in \langle S \rangle$ or $v \in \langle T \rangle$

Let $v \in \langle S \rangle$ then v is a linear combination of vectors of S

i.e., $v = k_1 u_1 + k_2 u_2 + \ldots + k_r u_r$

We can also write v as

$v = k_1 u_1 + k_2 u_2 + \ldots + k_r u_r + 0v_1 + 0v_2 + \ldots + 0v_t$

Which shows that v is a linear combination of vectors of $S \cup T$

$\Rightarrow v \in \langle S \cup T \rangle$

So $v \in \langle S \rangle \Rightarrow v \in \langle S \cup T \rangle$

Similarly $v \in \langle T \rangle \Rightarrow v \in \langle S \cup T \rangle$

Hence

$$\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$$

Now we give an example to show that equality does not hold in above result.

Example

In R^2, Let $S = \{(1,0)\}$, $T = \{(0,1)\}$ then

$$\langle S \rangle = \{ k(1,0) : k \in R \} = x\text{-axis}$$

& $$\langle T \rangle = \{ l(0,1) : l \in R \} = y\text{-axis}$$

Therefore

$$\langle S \rangle \cup \langle T \rangle = \{ k(1,0) : k \in R \} \cup \{ l(0,1) : l \in R \} \neq R^2$$

Now

$S \cup T = \{(1,0),(0,1)\}$ and

Any vector $(x,y) \in R^2$ is a linear Combination of $(1,0)$ & $(0,1)$, because

$$(x,y) = x(1,0) + y(0,1)$$

Therefore

$$\langle S \cup T \rangle = R^2$$

This shows that $\langle S \rangle \cup \langle T \rangle \neq \langle S \cup T \rangle$

(ii) $\langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Soln

First we prove that if $S \subset T$ then $\langle S \rangle \subset \langle T \rangle$

Let $S = \{v_1, v_2, \ldots\ldots, v_r\}$

& $T = \{v_1, v_2, \ldots\ldots, v_r, v_{r+1}, \ldots\ldots, v_n\}$

then obviously $S \subset T$

we want to show that $\langle S \rangle \subset \langle T \rangle$

Let $v \in \langle S \rangle$

then v is a linear Combination of vectors of S.

i.e., $v = a_1 v_1 + a_2 v_2 + \ldots\ldots + a_r v_r$

Now v can also be written as

$$v = a_1 v_1 + a_2 v_2 + \ldots\ldots + a_r v_r + 0 v_{r+1} + \ldots\ldots + 0 v_n$$

Which shows v is a linear Combination of vectors of T. Hence $v \in \langle T \rangle$

So $\langle S \rangle \subset \langle T \rangle$

Now as $S \cap T \subset S$ & $S \cap T \subset T$

So $\langle S \cap T \rangle \subset \langle S \rangle$ & $\langle S \cap T \rangle \subset \langle T \rangle$

$\Rightarrow \langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Now we will give an example to show that equality does not hold in above result.

<u>Example</u>

In R^2, let $S = \{(0,0),(1,0)\}$ & $T = \{(0,0),(0,3)\}$

then $S \cap T = \{(0,0)\}$

So $\langle S \cap T \rangle = \{ k(0,0) : k \in R \}$

$= \{(0,0)\}$ ——————— (i)

Now

$$\langle S \rangle = \{ a(0,0) + b(0,1) : a,b \in R \}$$
$$= \{ (0,b) : b \in R \}$$

&

$$\langle T \rangle = \{ p(0,0) + q(0,3) : p,q \in R \}$$
$$= \{ (0,3q) : q \in R \}$$
$$= \{ (0,c) : c \in R \}$$

Now

$$\langle S \rangle \cap \langle T \rangle = \{ (0,b) : b \in R \} \cap \{ (0,c) : c \in R \}$$
$$= \{ (0,y) : y \in R \} \quad \text{———— (ii)}$$

From (i) & (ii)

$$\langle S \cap T \rangle \neq \langle S \rangle \cap \langle T \rangle$$

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Linearly dependent & linearly independent vectors:-

The vectors $v_1, v_2, \ldots, v_m$ of a vector space V over F are said to be linearly dependent if there exist elements $a_1, a_2, \ldots, a_m \in F$, not all zeros such that

$$a_1v_1 + a_2v_2 + \ldots + a_mv_m = 0$$

On the other hand if

$$a_1v_1 + a_2v_2 + \ldots + a_mv_m = 0 \quad \text{where all } a_i = 0$$

then vectors $v_1, v_2, \ldots, v_m$ are said to be linearly independent.

Note

① If any two vectors out of $v_1, v_2, \ldots, v_m$ are equal say $v_3 = v_4$
then $v_1, v_2, \ldots, v_m$ are linearly dependent
because $0.v_1 + 0.v_2 + 1.v_3 + (-1)v_4 + 0.v_5 + \ldots + 0v_m = 0$

② If any one of the vector out of $v_1, v_2, \ldots, v_m$ is zero, say $v_2 = 0$ then $v_1, v_2, \ldots, v_m$ are linearly dependent because

$$0.v_1 + 1.v_2 + 0v_3 + \ldots + 0v_m = 0$$

③ A single non zero vector is always linearly independent because
let $v \neq 0$ be the single vector
then $av = 0 \Rightarrow a = 0$
Thus v is linearly independent.

④ Two vectors v_1 & v_2 are linearly dependent if one of them is a multiple of other.

<u>Theorem</u> Let V be a vector space over a field F [51] & $S = \{v_1, v_2, \ldots\ldots v_m\}$ be a set of vectors in V Then

(i) If S is linearly independent then any subset of S is also linearly independent.

(ii) If S is linearly dependent, then the set $\{v, v_1, v_2, \ldots\ldots, v_m\}$ is linearly dependent for all $v \in V$ i.e, every superset of S is also linearly dependent.

<u>Proof</u>:-

(i) Here $S = \{v_1, v_2, \ldots\ldots, v_m\}$

Let $\{v_1, v_2, \ldots\ldots, v_i\}$ where $i < m$ is a subset of S

& let $a_1v_1 + a_2v_2 + \ldots\ldots + a_iv_i = 0$ where $a_i \in F$

or $a_1v_1 + a_2v_2 + \ldots\ldots + a_iv_i + 0v_{i+1} + \ldots\ldots + 0v_m = 0$

But $S = \{v_1, v_2, \ldots\ldots, v_m\}$ is linearly dependent

So $a_1 = a_2 = a_3 = \ldots\ldots = a_i = 0$

Hence $\{v_1, v_2, \ldots\ldots, v_i\}$ is linearly dependent.

(ii) As $S = \{v_1, v_2, \ldots\ldots, v_m\}$ is linearly dependent

So $a_1v_1 + a_2v_2 + \ldots\ldots + a_mv_m = 0$ where $a_i \neq 0$ for some i

Now

$0.v + a_1v_1 + a_2v_2 + \ldots\ldots + a_mv_m = 0$ where $a_i \neq 0$ for some i

So $\{v, v_1, v_2, \ldots\ldots, v_m\}$ is linearly dependent.

<u>Theorem</u> A set $S = \{v_1, v_2, \ldots\ldots, v_n\}$ of n vectors $(n \geq 2)$ in a vector space V is linearly dependent iff. atleast one of the vectors in S is a linear combination of the remaining vectors of the set.

<u>Proof</u>:-

(i) Suppose the set

$S = \{ V_1, V_2, \text{-------}, V_n \}$ is linearly independent

Then there exist scalars $a_1, a_2, \text{-------}, a_n$ atleast one of them say a_i is non zero, such that

$$a_1V_1 + a_2V_2 + \text{--------} + a_iV_i + \text{-------} + a_nV_n = 0$$

or $a_iV_i = -a_1V_1 - a_2V_2 - \text{-------} - a_{i-1}V_{i-1} - a_{i+1}V_{i+1} - \text{-----} - a_nV_n$

or

$$V_i = -\frac{a_1}{a_i}V_1 - \frac{a_2}{a_i}V_2 - \text{------} - \frac{a_{i-1}}{a_i}V_{i-1} - \frac{a_{i+1}}{a_i}V_{i+1} - \text{-----} - \frac{a_n}{a_i}V_n$$

which shows that V_i is a linear Combination of remaining vectors of the set.

Conversly

let some vector V_j of the given set is a linear Combination of the remaining vectors. i.e.,

$$V_j = a_1V_1 + a_2V_2 + \text{-----} + a_{j-1}V_{j-1} + a_{j+1}V_{j+1} + \text{-------} + a_nV_n$$

Then above eq. Can be written as

$$a_1V_1 + a_2V_2 + \text{------} + a_{j-1}V_{j-1} + (-1)V_j + a_{j+1}V_{j+1} + \text{-----} + a_nV_n = 0$$

Here there is atleast one Cofficient namely -1 of V_j which is non zero & so the set

$\{V_1, V_2, \text{-----}, V_{j-1}, V_j, V_{j+1}, \text{-----}, V_n\}$ is linearly dependent.

(ii) Suppose that the set $S = \{V_1, V_2, \text{-------}, V_n\}$ is linearly dependent then there exist scalars $a_1, a_2, \text{-----}, a_n \in F$ not all zero such that

$$a_1V_1 + a_2V_2 + \text{----------} + a_nV_n = 0 \quad \text{———} \quad (1)$$

Let a_K be the last non zero scalar in (1) then the terms $a_{K+1}V_{K+1}, a_{K+2}V_{K+2}, \text{--------}, a_nV_n$ are all zero

So eq. (1) becomes

$$a_1V_1 + a_2V_2 + \text{-------} + a_KV_K = 0 \qquad \text{where } a_K \neq 0$$

or $a_KV_K = -a_1V_1 - a_2V_2 - \text{---------} - a_{K-1}V_{K-1}$

or $$V_K = -\frac{a_1}{a_K}V_1 - \frac{a_2}{a_K}V_2 - \text{-----------} - \frac{a_{K-1}}{a_K}V_{K-1}$$

Which shows that V_K is a linear Combination of the vectors preceding it. 53

Conversely

Suppose that in $S = \{V_1, V_2, \ldots\ldots, V_n\}$, some of the vectors say V_K is a linear Combination of the vectors preceding it

i.e., $V_K = b_1V_1 + b_2V_2 + \ldots\ldots + b_{K-1}V_{K-1}$

or $b_1V_1 + b_2V_2 + \ldots\ldots + b_{K-1}V_{K-1} + (-1)V_K = 0$

or $b_1V_1 + b_2V_2 + \ldots\ldots + b_{K-1}V_{K-1} + (-1)V_K + 0V_{K+1} + \ldots + 0V_n = 0$

Here atleast one Cofficient -1 of V_K is non zero

Hence $S = \{V_1, V_2, \ldots, V_n\}$ is linearly dependent.

Basis of a vector space:

A linearly independent set which generates or spans a vector space V is called a basis for V.

Theorem Any finite dimensional vector space contains a basis.

Proof:

Let V be a finite dimensional vector space then V should be linear span of some finite set. Let $\{V_1, V_2, \ldots V_r\}$ be a finite spanning set of V. In case $V_1, V_2, \ldots, V_n$ are linearly independent, then they form a basis for V & the proof is complete.

Suppose $V_1, V_2, \ldots, V_r$ are not linearly independent i.e; they are linearly dependent, so one of the vectors V_i is a linear Combination of the preceding vectors. We drop this vector V_i from the set & obtain a set of $r-1$ vectors

$v_1, v_2, \ldots, v_{r-1}$. Clearly any linear combination of $v_1, v_2, \ldots, v_r$ is also a linear combination of $v_1, v_2, \ldots, v_{r-1}$. So $\{v_1, v_2, \ldots, v_{r-1}\}$ is also a spanning set for V.

Continuing in this way, we arrive at a linearly independent spanning set $\{v_1, v_2, \ldots, v_n\}$; $1 \le n \le r$ & so it forms a basis for V.

Thus every finite dimensional vector space contains a basis.

<u>Note</u> If a vector space V is generated by $v_1, v_2, \ldots, v_m$ then any linearly independent set in V cannot have more than m no. of elements.

<u>Dimension of a vector space</u>:

The no. of elements in a basis of a vector space V over F is called dimension of V. It is denoted by $\dim V$.

<u>Theorem</u> All bases of a finite dimensional vector space contains the same no. of elements.

<u>Proof</u>: Let a vector space V over F has two bases A & B with m & n no. of elements.

Since A spans V & B is a linearly independent subset in V, so B cannot have more than m no. of elements.

i.e., $n \le m$ ——— ①

Now since B spans V & A is a linearly independent subset in V, so A cannot have more than n no. of elements

i.e., $m \leq n$ ——— (2)

from (1) & (2)

$$m = n$$

Hence no. of elements in A = no. of elements in B which is req. proof.

<u>Theorem</u> Let V be a vector space such that $\dim V = n < \infty$.

A set of vectors $\{v_1, v_2, \ldots\ldots, v_n\} \subset V$ is a basis for V iff. each vector in V is uniquely expressible as a linear combination of vectors $v_1, v_2, \ldots\ldots, v_n$.

<u>Proof</u>:- Let the set $\{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for V. Then every vector $v \in V$ can be expressed in atleast one way as a linear combination of the basis vectors. i.e.,

$v = a_1 v_1 + a_2 v_2 + \ldots\ldots + a_n v_n$ where $a_i \in F$ $1 \leq i \leq n$

Suppose v can also be expressed as

$v = b_1 v_1 + b_2 v_2 + \ldots\ldots + b_n v_n$ where $b_i \in F$ $1 \leq i \leq n$

Comparing above eqs.

$a_1 v_1 + a_2 v_2 + \ldots\ldots + a_n v_n = b_1 v_1 + b_2 v_2 + \ldots\ldots + b_n v_n$

or $(a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \ldots\ldots + (a_n - b_n) v_n$

Since $v_1, v_2, \ldots\ldots, v_n$ are linearly independent

So $a_1 - b_1 = 0$, $a_2 - b_2 = 0$, $\ldots\ldots$, $a_n - b_n = 0$

$\Rightarrow a_1 = b_1$, $a_2 = b_2$, $\ldots\ldots$, $a_n = b_n$

Hence every vector $v \in V$ can be expressed in a unique way as a linear combination of $v_1, v_2, \ldots, v_n$.

<u>Conversely</u>

Let every vector $v \in V$ is uniquely expressible as a linear combination of $v_1, v_2, \ldots\ldots, v_n$.

Then these vectors span V. We will prove that they are linearly independent.

Suppose that for scalars $a_1, a_2, \dots\dots, a_n$

$a_1V_1 + a_2V_2 + \dots\dots\dots + a_nV_n = 0$ ——— (1)

As $0V_1 + 0V_2 + \dots\dots\dots + 0V_n = 0$ ——— (2)

Since the representation is unique

So $a_1 = 0, a_2 = 0, \dots\dots\dots, a_n = 0$

Hence $V_1, V_2, \dots\dots\dots, V_n$ are linearly independent.

Since they also span V.

Hence $\{V_1, V_2, \dots\dots\dots, V_n\}$ form a basis for V

<u>Theorem</u> Let $S = \{V_1, V_2, \dots\dots\dots, V_n\}$ be a basis for an n-dimensional vector space V over a field F. Then every set with more than n vectors is linearly dependent.

<u>Proof:</u>

Let $B = \{u_1, u_2, \dots\dots\dots, u_r\}$ be a set of r vectors in V where $r > n$.

We shall show that B is linearly dependent.

To show that B is linearly dependent, we must find scalars $c_1, c_2, \dots\dots, c_r$, not all zero, such that

$c_1u_1 + c_2u_2 + \dots\dots\dots + c_ru_r = 0$ ——— (1)

Since

the set $S = \{V_1, V_2, \dots\dots, V_n\}$ is linearly independent

So each u_i can be uniquely expressed as a linear combination of $V_1, V_2, \dots\dots, V_n$. Hence

$$\left.\begin{aligned} u_1 &= a_{11}V_1 + a_{12}V_2 + \dots\dots\dots + a_{1n}V_n \\ u_2 &= a_{21}V_1 + a_{22}V_2 + \dots\dots\dots + a_{2n}V_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ u_r &= a_{r1}V_1 + a_{r2}V_2 + \dots\dots\dots + a_{rn}V_n \end{aligned}\right\} \text{——— (2)} \quad \text{where } a_{ij} \in F$$

Putting values of $u_1, u_2, \ldots\ldots, u_r$ from (2) in (1) so

$$c_1(a_{11}v_1 + a_{12}v_2 + \cdots\cdots + a_{1n}v_n) + c_2(a_{21}v_1 + a_{22}v_2 + \cdots\cdots + a_{2n}v_n) + \cdots\cdots + c_r(a_{r1}v_1 + a_{r2}v_2 + \cdots\cdots + a_{rn}v_n) = 0$$

or

$$(c_1a_{11} + c_2a_{21} + \cdots\cdots + c_ra_{r1})v_1 + (c_1a_{12} + c_2a_{22} + \cdots\cdots + c_ra_{r2})v_2 + \cdots \cdots\cdots + (c_1a_{1n} + c_2a_{2n} + \cdots + c_ra_{rn})v_n = 0$$

Since $v_1, v_2, \ldots\ldots, v_n$ are linearly independent

$$\Rightarrow \left.\begin{array}{l} a_{11}c_1 + a_{21}c_2 + \cdots\cdots + a_{r1}c_r = 0 \\ a_{12}c_1 + a_{22}c_2 + \cdots\cdots + a_{r2}c_r = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ a_{1n}c_1 + a_{2n}c_2 + \cdots\cdots + a_{rn}c_r = 0 \end{array}\right\}$$

Which is a homogeneous system of n eqs. in r unknowns $c_1, c_2, \ldots\ldots, c_r$.

Since $n < r$, so this system has a non trivial soln.

Hence atleast one of $c_1, c_2, \ldots\ldots, c_r$ is non zero & so from eq. (1) set $B = \{u_1, u_2, \ldots\ldots, u_r\}$ is linearly dependent.

<u>Theorem</u> Let $v_1, v_2, \ldots\ldots, v_n$ be linearly independent in a vector space V over a field F. If v is any non zero vector in V. Then the set $\{v_1, v_2, \ldots\ldots, v_n, v\}$ is linearly independent iff. v is not in the linear span $\langle v_1, v_2, \ldots\ldots, v_n\rangle$.

<u>Proof.</u> Let $v \notin \langle v_1, v_2, \ldots\ldots, v_n\rangle$

then we prove $\{v_1, v_2, \ldots\ldots, v_n, v\}$ is linearly independent.

Consider

$$a_1v_1 + a_2v_2 + \cdots\cdots + a_nv_n + av = 0 \quad \text{—— (1)}$$

where $a_1, a_2, \ldots, a_n, a \in F$

Suppose that $a \neq 0$ then from ①, we have

$$V = -\frac{1}{a}(a_1V_1 + a_2V_2 + \cdots\cdots + a_nV_n)$$

Which shows that $V \in \langle V_1, V_2, \ldots, V_n \rangle$

which is a Contradiction.

Hence $a = 0$

Also Since $V_1, V_2, \ldots, V_n$ are linearly independent

So eq. ① with $a = 0$ implies $a_1, a_2, \ldots, a_n = 0$

Hence $\{V_1, V_2, \ldots, V_n, V\}$ is linearly independent set.

<u>Conversely</u>

let $\{V_1, V_2, \ldots, V_n, V\}$ be linearly independent set

then we prove $V \notin \langle V_1, V_2, \ldots, V_n \rangle$

Suppose $V \in \langle V_1, V_2, \ldots, V_n \rangle$

then V Can be expressed as a linear Combination

of $V_1, V_2, \ldots, V_n$. i.e.,

$$V = a_1V_1 + a_2V_2 + \cdots\cdots + a_nV_n \qquad \text{where } a_i \in F$$

or

$$a_1V_1 + a_2V_2 + \cdots\cdots + a_nV_n + (-1)V = 0$$

Which shows that $\{V_1, V_2, \ldots, V_n, V\}$ is linearly dependent set.

which is a Contradiction

Hence $V \notin \langle V_1, V_2, \ldots, V_n \rangle$

<u>Theorem</u> Let V be an n-dimensional vector space over a field F. Then every set $S = \{V_1, V_2, \ldots, V_n\}$ of n linearly independent vectors in V is a basis for V.

<u>Proof.</u>

Let $V \in V$ be any non zero vector then the set $\acute{S} = \{V_1, V_2, \ldots, V_n, V\}$ is a linearly dependent set. So we Can find scalars $a_1, a_2, \ldots, a_n, a$ not all zeros such that

$a_1v_1 + a_2v_2 + \cdots + a_nv_n + av = 0$ ——— (1)

Since $S = \{v_1, v_2, \ldots, v_n\}$ is linearly independent

So $a_i = 0$ for $i = 1, 2, 3, \ldots, n$

Hence $a \neq 0$

And so, eq. (1) can be written as

$av = -a_1v_1 - a_2v_2 - \cdots - a_nv_n$

or

$$v = \left(-\frac{a_1}{a}\right)v_1 + \left(-\frac{a_2}{a}\right)v_2 + \cdots + \left(-\frac{a_n}{a}\right)v_n$$

which shows that v is a linear combination of vectors $v_1, v_2, \ldots, v_n$.

So S spans V.

Hence S is a linearly independent spanning set for V.

So $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for V.

Theorem (i) Any linearly independent set of vectors in a finite dimensional vector space V can be extended to a basis for V.

(ii) If W is a subspace of a finite dimensional vector space V then $\dim W \leq \dim V$.

Moreover if $\dim W = \dim V$ then $W = V$

Proof:

Since V is finite dimensional so let $\dim V = n$

Let $S = \{v_1, v_2, \ldots, v_r\}$ (where $r < n$) be a linearly independent set of vectors in V.

Since $\dim V = n$, so the set S cannot span V.

There is a vector say $v_{r+1} \in V$ such that $v_{r+1} \notin \langle S \rangle$ & the set $S \cup \{v_{r+1}\}$ is linearly independent. This process can be repeated $n-r$ times to get

a larger set $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ 60
which is linearly independent & so this will form a basis for V.

(ii) Proof:-
Since V is finite dimensional
suppose $\dim V = n$
Then any set of $n+1$ or more vectors is linearly dependent.
Moreover since a basis of W consists of linearly independent vectors, so it cannot contain more than n elements
Hence $\dim W \leq n = \dim V$
$\Rightarrow \dim W \leq \dim V$
If $\dim W = \dim V$
Then every basis of W is also a basis for V.
Hence $W = V$

Theorem A vector space V is the direct sum of its subspaces U & W iff. each $v \in V$ can be uniquely written as
$$v = u + w \qquad \text{for } u \in U, w \in W$$
Proof:-
Suppose that V is the direct sum of its subspaces U & W then by def.
(i) $V = U + W$
(ii) $U \cap W = \{0\}$
We want to show that each $v \in V$ can be uniquely written as
$$v = u + w \qquad \text{for } u \in U, w \in W$$

Let $v \in V$ then

$v = u + w$ —— ① for $u \in U$, $w \in W$ by (i)

If possible let

$v = u_1 + w_1$ for $u_1 \in U$, w_1

then

$$u + w = u_1 + w_1$$

or $u - u_1 = w_1 - w \in U \cap W$

But $U \cap W = \{0\}$ by (ii)

So $u - u_1 = w_1 - w = 0$

$\Rightarrow u = u_1$ & $w = w_1$

So the expression for v in ① is unique.

Hence each $v \in V$ can be uniquely written as

$v = u + w$ for $u \in U$, $w \in W$

<u>Conversely</u>

let each $v \in V$ is uniquely written as

$v = u + w$ for $u \in U$, $w \in W$

So (i) is satisfied. Now we prove condition (ii)

For this let $v \in U \cap W$ then v can be written as

$v = u + 0$ $u \in U$

$v = 0 + w$ $w \in W$

Since the expression for v is unique

So $u + 0 = 0 + w$

$\Rightarrow u = 0$, $w = 0$

Hence $v = u + w = 0$

So $U \cap W = \{0\}$

Hence condition (ii) is satisfied.

So V is the direct sum of U & W

<u>Theorem</u> If U & W are finite dimensional subspaces of a vector space V over a field F then

(i) $\dim(U+W) = \dim U + \dim W - \dim(U\cap W)$

(ii) If $U\cap W = \{0\}$ then $V = U \oplus W$

and $\dim V = \dim U + \dim W$

<u>Proof.</u>

Suppose that $U\cap W \neq \{0\}$.

Let $\{v_1, v_2, \ldots, v_r\}$ be a basis for $U\cap W$,

$\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s\}$ be a basis for U

& $\{v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_t\}$ be a basis for W

Thus dimensions of $U\cap W$, U & W are r, $r+s$ & $r+t$ resp.

Clearly

$\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s, w_1, w_2, \ldots, w_t\}$ spans $U+W$.

Now we show that $\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s, w_1, \ldots, w_t\}$ is linearly independent. Suppose that

$$a_1v_1 + a_2v_2 + \cdots + a_rv_r + b_1u_1 + b_2u_2 + \cdots + b_su_s + c_1w_1 + \cdots + c_tw_t = 0 \quad (1)$$

where a's, b's, c's $\in F$

or $$a_1v_1 + a_2v_2 + \cdots + a_rv_r + b_1u_1 + b_2u_2 + \cdots + b_su_s = -(c_1w_1 + \cdots + c_tw_t) \quad (2)$$

from eq. (2) we see that

$-(c_1w_1 + c_2w_2 + \cdots + c_tw_t) \in U\cap W$

But $\{v_1, v_2, \ldots, v_r\}$ is a basis for $U\cap W$. So

$$-(c_1w_1 + c_2w_2 + \cdots + c_tw_t) = d_1v_1 + d_2v_2 + \cdots + d_rv_r$$

where d's $\in F$

or

$$d_1v_1 + d_2v_2 + \cdots + d_rv_r + c_1w_1 + c_2w_2 + \cdots + c_tw_t = 0$$

Since $\{v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_t\}$ being basis of W is linearly independent

$\Rightarrow d_1 = d_2 = \cdots = d_r = 0, \quad c_1 = c_2 = \cdots = c_t = 0$

Then eq. (1) becomes

$$a_1v_1 + a_2v_2 + \cdots + a_rv_r + b_1u_1 + \cdots + b_su_s = 0$$

Again $\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s\}$ being basis of U is linearly independent.

So $a_1 = a_2 = \ldots = a_r = 0$, $b_1 = b_2 = \ldots = b_s = 0$

Hence eq. ① shows that the set

$\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots u_s, w_1, w_2, \ldots, w_t\}$ is linearly independent set. & so it forms a basis for $U+W$. So

$$\dim(U+W) = r+s+t$$
$$= (r+s)+(r+t) - r$$

So $\boxed{\dim(U+W) = \dim U + \dim W - \dim(U \cap W)}$

(ii) Now Suppose that $U \cap W = \{0\}$

Let $\{u_1, u_2, \ldots, u_s\}$ be a basis for U

& $\{w_1, w_2, \ldots, w_t\}$ be a basis for W

Then clearly the set $\{u_1, u_2, \ldots, u_s, w_1, w_2, \ldots, w_t\}$ spans $U+W = U \oplus W$

Consider $a_1u_1 + \ldots + a_su_s + b_1w_1 + \ldots + b_tw_t = 0$ — ③

then $a_1u_1 + \ldots + a_su_s = -(b_1w_1 + \ldots + b_tw_t)$

This shows that each vector is in $U \cap W$.

Hence $a_1 = a_2 = \ldots = a_s = 0$, $b_1 = b_2 = \ldots = b_t = 0$

So ③ shows that $\{u_1, \ldots, u_s, w_1, \ldots, w_t\}$ is linearly independent & so forms a basis for $U \oplus W$

Hence $\dim(U \oplus W) = s + t$

or $\dim(V) = \dim U + \dim W$ $\because V = U \oplus W$

Exercise No. 6.2

Q1 Determine whether the following vectors in R^4 are linearly independent or linearly dependent:

(i) $(1,3,-1,4),(3,8,-5,7),(2,9,4,23)$

Sol.

Let $a(1,3,-1,4)+b(3,8,-5,7)+c(2,9,4,23)=0 \quad a,b,c\in F$

or $(a,3a,-a,4a)+(3b,8b,-5b,7b)+(2c,9c,4c,23c)=0$

$(a+3b+2c,\ 3a+8b+9c,\ -a-5b+4c,\ 4a+7b+23c)=0$

$\Rightarrow$

$$a+3b+2c=0 \quad \text{———} (1)$$
$$3a+8b+9c=0 \quad \text{———} (2)$$
$$-a-5b+4c=0 \quad \text{———} (3)$$
$$4a+7b+23c=0 \quad \text{———} (4)$$

From (1) & (2)

$$\frac{a}{27-16}=\frac{-b}{9-6}=\frac{c}{8-9}$$

or $\dfrac{a}{11}=\dfrac{b}{-3}=\dfrac{c}{-1}=K$

$\Rightarrow \quad a=11K$

$b=-3K$

$c=-K$

Putting these values in (3) & (4), we see eqs. (3) & (4) are satisfied.

Hence given vectors in R^4 are linearly dependent.

(ii) $(1,-2,4,1),(2,1,0,-3),(1,-6,1,4)$

Sol.

Let $a(1,-2,4,1)+b(2,1,0,-3)+c(1,-6,1,4)=0$ where $a,b,c\in F$

or $(a,-2a,4a,a)+(2b,b,0,-3b)+(c,-6c,c,4c)=0$

or $(a+2b+c,\ -2a+b-6c,\ 4a+c,\ a-3b+4c)=0$

$\Rightarrow$

$$a + 2b + c = 0 \quad \text{———} \; (1)$$
$$-2a + b - 6c = 0 \quad \text{———} \; (2)$$
$$4a + c = 0 \quad \text{———} \; (3)$$
$$a - 3b + 4c = 0 \quad \text{———} \; (4)$$

From (1) & (2)

$$\frac{a}{-12-1} = \frac{-b}{-6+2} = \frac{c}{1+4}$$

$$\frac{a}{-13} = \frac{b}{4} = \frac{c}{5} = K$$

$$\Rightarrow \quad a = -13K$$
$$b = 4K$$
$$c = 5K$$

Putting these values in (3) & (4), we see that eqs. are not satisfied. They are satisfied only when $K = 0$

$$\Rightarrow \quad a = b = c = 0.$$

Hence given vectors in R^4 are linearly independent.

Q2 Let $V = P_3(x)$ be the vector space of all polynomials of degree ≤ 3 over R together with the zero polynomial. Determine whether $u, v, w \in V$ are linearly dependent or linearly independent.

(i) $u = x^3 - 4x^2 + 2x + 3$, $v = x^3 + 2x^2 + 4x - 1$, $w = 2x^3 - x^2 - 3x + 3$

Sol.-

Let $au + bv + cw = 0$ where $a, b, c \in F$

or $a(x^3 - 4x^2 + 2x + 3) + b(x^3 + 2x^2 + 4x - 1) + c(2x^3 - x^2 - 3x + 3) = 0$

or $(a + b + 2c)x^3 + (-4a + 2b - c)x^2 + (2a + 4b - 3c)x + (3a - b + 3c) = 0$

$$\Rightarrow \quad a + b + 2c = 0 \quad \text{———} \; (1)$$
$$-4a + 2b - c = 0 \quad \text{———} \; (2)$$
$$2a + 4b - 3c = 0 \quad \text{———} \; (3)$$
$$3a - b + 3c = 0 \quad \text{———} \; (4)$$

From (1) & (2)

$$\frac{a}{-1-4} = \frac{-b}{-1+8} = \frac{c}{2+4}$$

$$\frac{a}{-5} = \frac{b}{-7} = \frac{c}{6} = K$$

$\Rightarrow$ $a = -5K$

$b = -7K$

$c = 6K$

Putting these values in ③ + ④ we see eqs. are not satisfied. They are satisfied only when $K = 0$

$\Rightarrow$ $a = b = c = 0$

Hence vectors u, v, w are linearly independent.

(ii) $u = x^3 - 3x^2 - 2x + 3$, $v = x^3 - 4x^2 - 3x + 4$, $w = 2x^3 - 7x^2 - 7x + 9$

Sol.

Let $au + bv + cw = 0$ where $a, b, c \in F$

or $a(x^3 - 3x^2 - 2x + 3) + b(x^3 - 4x^2 - 3x + 4) + c(2x^3 - 7x^2 - 7x + 9) = 0$

or $(a + b + 2c)x^3 + (-3a - 4b - 7c)x^2 + (-2a - 3b - 7c)x + (3a + 4b + 9c) = 0$

$\Rightarrow$ $a + b + 2c = 0$ ——— ①

$-3a - 4b - 7c = 0$ ——— ②

$-2a - 3b - 7c = 0$ ——— ③

$3a + 4b + 9c = 0$ ——— ④

From ① + ②

$$\frac{a}{-7+8} = \frac{-b}{-7+6} = \frac{c}{-4+3}$$

or $$\frac{a}{1} = \frac{b}{1} = \frac{c}{-1} = K$$

$\Rightarrow$ $a = K$

$b = K$

$c = -K$

Putting these values in ③ + ④, we see eqs. are not satisfied. They are satisfied only when $K = 0$

$\Rightarrow$ $a = b = c = 0$.

Hence given vectors u, v, w are linearly independent.

Q3 Show that the vectors $(1-i, i)$ & $(2, -1+i)$ in C^2 are linearly dependent over C but linearly independent over R.

Sol.

Let $a(1-i, i) + b(2, -1+i) = 0$

or $(a(1-i), ai) + (2b, b(-1+i)) = 0$

or $(a(1-i) + 2b, ai + b(-1+i)) = 0$

$\Rightarrow \quad a(1-i) + 2b = 0 \quad \text{———} \quad (1)$

$\quad\quad ai + (-1+i)b = 0 \quad \text{———} \quad (2)$

These eqs. are satisfied in R only when $a=b=0$

Hence given vectors are linearly independent over R

Now we find the values of a & b from the set C which satisfy eqs. (1) & (2)

From (1)

$$a(1-i) = -2b$$

$$\frac{a}{b} = \frac{-2}{1-i}$$

$$= \frac{-2}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{-2(1+i)}{1+1}$$

$$= \frac{-2(1+i)}{2}$$

$$\frac{a}{b} = -(1+i)$$

$$\text{or} \quad \frac{a}{1+i} = \frac{b}{-1} = K$$

$$\Rightarrow \quad a = (1+i)K$$

$$b = -K$$

Putting these values in eq. (2), we see eq. (2) is satisfied.

Hence given vectors are linearly dependent over C

Q4 Show that the vectors $(3+\sqrt{2}, 1+\sqrt{2})$ & $(7, 1+2\sqrt{2})$ in R^2 are linearly dependent over R but linearly independent over Q.

Sol.

Let $a(3+\sqrt{2}, 1+\sqrt{2}) + b(7, 1+2\sqrt{2}) = 0$

or $(a(3+\sqrt{2}), a(1+\sqrt{2})) + (7b, b(1+2\sqrt{2})) = 0$

or $(a(3+\sqrt{2})+7b, a(1+\sqrt{2}) + b(1+2\sqrt{2})) = 0$

$\Rightarrow$

$$(3+\sqrt{2})a + 7b = 0 \quad \text{———} \; (1)$$

$$\& \quad (1+\sqrt{2})a + (1+2\sqrt{2})b = 0 \quad \text{———} \; (2)$$

These eqs. are satisfied in Q only when $a=b=0$

Hence the given vectors are linearly independent over Q.

Now we will find values of a & b in R which satisfy the eqs. (1) + (2)

From (1)

$$(3+\sqrt{2})a = -7b$$

$$\text{or} \quad \frac{a}{b} = \frac{-7}{3+\sqrt{2}}$$

$$\Rightarrow \quad \frac{a}{7} = \frac{-b}{3+\sqrt{2}} = K$$

$$\Rightarrow \quad a = 7K$$

$$\& \quad b = -(3+\sqrt{2})K$$

Putting these values in eq. (2), we see eq. (2) is satisfied.

Hence given vectors are dependent over R.

Q5 Suppose that u, v & w are linearly independent vectors. Prove that

(i) $u+v-2w$, $u-v-w$, $u+w$ are linearly independent.

(ii) $u+v-3w$, $u+3v-w$, $u+w$ are linearly independent.

(i) <u>Sol.</u>

Let $a(u+v-2w)+b(u-v-w)+c(u+w) = 0$

or $(a+b+c)u+(a-b)v+(-2a-b+c)w = 0$

But u, v & w are linearly independent

So
$$a+b+c = 0 \quad \text{———} \ (1)$$
$$a-b = 0 \quad \text{———} \ (2)$$
$$-2a-b+c = 0 \quad \text{———} \ (3)$$

From (1) & (3)

$$\frac{a}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}}$$

or

$$\frac{a}{1+1} = \frac{-b}{1+2} = \frac{c}{-1+2}$$

$$\frac{a}{2} = \frac{b}{-3} = \frac{c}{1} = k$$

$$\Rightarrow \quad a = 2k$$
$$b = -3k$$
$$\& \quad c = k$$

Putting these values in (2), we see that eq. (2) is not satisfied. It is satisfied only when $k = 0$

$$\Rightarrow \quad a = b = c = 0$$

Hence given vectors are linearly independent.

(ii) <u>Sol.</u>

Let $a(u+v-3w)+b(u+3v-w)+c(u+w) = 0$

or $(a+b+c)u+(a+3b)v+(-3a-b+c)w = 0$

But u, v & w are linearly independent

so
$$a+b+c = 0 \quad \text{———} \ (1)$$
$$a+3b = 0 \quad \text{———} \ (2)$$
$$-3a-b+c = 0 \quad \text{———} \ (3)$$

From ① & ③

$$\frac{a}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix}}$$

$$\frac{a}{1+1} = \frac{-b}{1+3} = \frac{c}{-1+3}$$

$$\frac{a}{2} = \frac{b}{-4} = \frac{c}{2} = K$$

$$\Rightarrow \quad a = 2K$$
$$b = -4K$$
$$c = 2K$$

Putting these values in eq. ②, we see eq. ② is not satisfied. It is satisfied only if $K=0$

$$\Rightarrow \quad a = b = c = 0$$

Hence given vectors are linearly independent.

Q6 Determine K so that the vectors $(1,-1,K-1),(2,K,-4)$ $(0,2+K,-8)$ in R^3 are linearly dependent.

Sol.- Suppose that given vectors are linearly dependent then one of them must be a linear combination of the other two

Let $(1,-1,K-1) = a(2,K,-4) + b(0,2+K,-8)$ where $a,b \in F$

$$= (2a, aK, -4a) + (0, b(2+K), -8b)$$

or $(1,-1,K-1) = (2a, aK+b(2+K), -4a-8b)$

$$\Rightarrow \quad 2a = 1 \quad \text{———} ①$$
$$aK + b(2+K) = -1 \quad \text{———} ②$$
$$-4a - 8b = K-1 \quad \text{———} ③$$

from ① $a = \frac{1}{2}$

Put in ② & ③

② $\Rightarrow$ $\frac{K}{2} + b(2+K) = -1$ ——— ④

③ $\Rightarrow$ $-4(\frac{1}{2}) - 8b = K-1$

$$-2 - 8b = K-1$$

$$-8b = K-1+2$$

$$-8b = K+1$$

$$b = -\frac{K+1}{8}$$

Put this value in (4)

$$\frac{K}{2} + (2+K)\left(-\frac{K+1}{8}\right) = -1$$

$$4K - (2+K)(K+1) = -8$$

$$4K - 2K - 2 - K^2 - K + 8 = 0$$

$$-K^2 + K + 6 = 0$$

or $$K^2 - K - 6 = 0$$

$$K^2 - 3K + 2K - 6 = 0$$

$$K(K-3) + 2(K-3) = 0$$

$$(K-3)(K+2) = 0$$

$$\Rightarrow \boxed{K = 3, -2}$$

So for $K = 3, -2$; given vectors are linearly dependent.

Q7 Using the technique of casting out vectors which are linear combination of others, find a linearly independent subset of the given set spanning the same subspace:

(i) $\{(1,-3,1), (2,1,-4), (-2,6,-2), (-1,10,-7)\}$ in R^3

Sol.

Given set is $\{(1,-3,1), (2,1,-4), (-2,6,-2), (-1,10,-7)\}$

We see that

$$(-2,6,-2) = -2(1,-3,1)$$

So we cast out $(-2,6,-2)$ & obtain the subset

$$\{(1,-3,1), (2,1,-4), (-1,10,-7)\}$$

Suppose $(-1,10,-7) = a(1,-3,1) + b(2,1,-4)$ for $a, b \in R$

or $(-1,10,-7) = (a+2b, -3a+b, a-4b)$

$\Rightarrow$ $a+2b = -1$ ———— (1)

$-3a+b = 10$ ———— (2)

$a-4b = -7$ ———— (3)

Multiplying ① by 2 & adding in ③

$$2a + \cancel{4b} = -2 \quad \text{——— ④}$$
$$a - \cancel{4b} = -7 \quad \text{——— ③}$$

$$3a = -9$$
$$\boxed{a = -3}$$

Put in ③

$$-3 - 4b = -7$$
$$-4b = -7 + 3$$
$$-4b = -4$$
$$\boxed{b = 1}$$

So $(-1, 10, -7) = -3(1, -3, 1) + 1(2, 1, -4)$

Hence $(-1, 10, -7)$ is a linear Combination of $(1, -3, 1)$ & $(2, 1, -4)$

We Cast out $(-1, 10, -7)$ & obtain a subset

$A = \{(1, -3, 1), (2, 1, -4)\}$

Since $(2, 1, -4)$ is not a multiple of $(1, -3, 1)$ So the set $A = \{(1, -3, 1), (2, 1, -4)\}$ is req. linearly independent set which spans the same subspace as the given set of four vectors.

(ii) $\{1, \sin^2 x, \cos 2x, \cos^2 x\}$ in the space of all functions from R to R.

Sol.

Given set is $\{1, \sin^2 x, \cos 2x, \cos^2 x\}$

As

$$\cos 2x = \cos^2 x - \sin^2 x$$

or $\cos 2x = 1 \cdot \cos^2 x + (-1) \cdot \sin^2 x$

So $\cos 2x$ is a linear Combination of $\cos^2 x$ & $\sin^2 x$

We cast out $\cos 2x$ & obtain the subset

$\{1, \sin^2 x, \cos^2 x\}$

Also $\cos^2 x = 1 - \sin^2 x$

or $\cos^2 x = 1.1 + (-1)\sin^2 x$

So $\cos^2 x$ is a linear Combination of 1 & $\sin^2 x$

We Cast out $\cos^2 x$ & obtain the subset

$\{1, \sin^2 x\}$

Since none of 1 & $\sin^2 x$ is multiple of other

So $\{1, \sin^2 x\}$ is req. linearly independent set which spans the same subspace as the given set of four vectors.

(iii) $\{1, 3x-4, 4x+3, x^2+2, x-x^2\}$ in the space $P_2(x)$ of polynomials.

Sol.

Given set is $\{1, 3x-4, 4x+3, x^2+2, x-x^2\}$

We see that

$$3x-4 = \frac{3}{4}(4x+3) - \frac{25}{4}(1)$$

So $3x-4$ is a linear Combination of $4x+3$ & 1

We Cast out $3x-4$ & obtain the subset

$\{1, 4x+3, x^2+2, x-x^2\}$

Now

Let $x-x^2 = a(1) + b(4x+3) + c(x^2+2)$

or $x-x^2 = (a+3b+2c) + (4b)x + cx^2$

$$\Rightarrow \quad a+3b+2c = 0 \quad \text{——— } (1)$$

$$4b = 1 \quad \text{——— } (2)$$

$$c = -1 \quad \text{——— } (3)$$

(3) $\Rightarrow$ $\boxed{c = -1}$

(2) $\Rightarrow$ $\boxed{b = \frac{1}{4}}$

(1) $\Rightarrow$ $a + \frac{3}{4} - 2 = 0$

$a - \frac{5}{4} = 0$

$\boxed{a = 5/4}$

So $x - x^2 = 5/4(1) + \frac{1}{4}(4x+3) - 1(x^2+2)$

Hence $x-x^2$ is a linear Combination of $1, 4x+3$ & x^2+2

We Cast out $x-x^2$ & obtain the subset

$A = \{1, 4x+3, x^2+2\}$

We check whether A is linearly independent or not

Let $a(1) + b(4x+3) + c(x^2+2) = 0$

or $(a+3b+2c) + 4bx + cx^2 = 0$

$\Rightarrow \quad a+3b+2c = 0$ ——— ①

$4b = 0$ ——— ②

$c = 0$ ——— ③

③ $\Rightarrow \boxed{c=0}$

② $\Rightarrow \boxed{b=0}$

Put in ①

$a+0+0=0 \Rightarrow \boxed{a=0}$

Hence the set $A = \{1, 4x+3, x^2+2\}$ is linearly independent set which spans the same subspace as the given set of five vectors.

Q8 verify that the polynomials $2-x^2$, x^3-x, $2-3x^2$ & $3-x^3$ form a basis for $P_3(x)$. Express each of (i) $1+x$ & (ii) $x+x^2$ as a linear Combination of these basis vectors.

Sol.

We want to show that $\{2-x^2, x^3-x, 2-3x^2, 3-x^3\}$ form a basis for $P_3(x)$.

First we prove that this set is linearly independent.

Let $a(2-x^2) + b(x^3-x) + c(2-3x^2) + d(3-x^3) = 0 \quad a,b,c,d \in F$

or $(2a+2c+3d) - bx + (-a-3c)x^2 + (b-d)x^3 = 0$

$\Rightarrow \quad 2a+2c+3d = 0$ ——— ①

$-b = 0$ ——— ②

$-a-3c = 0$ ——— ③

$b-d = 0$ ——— ④

(2) $\Rightarrow$ $\boxed{b = 0}$

(4) $\Rightarrow$ $\boxed{b = d}$

(3) $\Rightarrow$ $a = -3c$

Put in (1)

$2(-3c) + 2c + 3(0) = 0$

$-6c + 2c = 0$

$-4c = 0 \Rightarrow \boxed{c = 0}$

Put in (3)

$-a - 3(0) = 0 \Rightarrow \boxed{a = 0}$

Hence the given set of polynomials is linearly independent

As dimension of $P_3(x)$ is 4

& no. of vectors in set $\{2-x^2, x^3-x, 2-3x^2, 3-x^3\}$

So the given set $\{2-x^2, x^3-x, 2-3x^2, 3-x^3\}$ form a basis for $P_3(x)$.

Now we express $1+x$ as a linear combination of given vectors.

Let $1+x = a(2-x^2) + b(x^3-x) + c(2-3x^2) + d(3-x^3)$

or $1+x = (2a+2c+3d) - bx + (-a-3c)x^2 + (b-d)x^3$

where $a, b, c, d \in R$

$$\Rightarrow \quad 2a + 2c + 3d = 1 \quad \text{——— (1)}$$

$$-b = 1 \quad \text{——— (2)}$$

$$-a - 3c = 0 \quad \text{——— (3)}$$

$$b - d = 0 \quad \text{——— (4)}$$

(2) $\Rightarrow$ $\boxed{b = -1}$

Put in (4)

$-1 - d = 0 \Rightarrow \boxed{d = -1}$

(3) $\Rightarrow$ $a = -3c$

Put in (1)

$2(-3c) + 2c + 3(-1) = 1$

$-4c - 3 = 1$

$-4c = 4$

$\Rightarrow \boxed{c = -1}$

Put in ③

$-a - 3(-1) = 0$

$-a + 3 = 0 \Rightarrow \boxed{a = 3}$

So $1 + x = 3(2 - x^2) - (x^3 - x) - (2 - 3x^2) - (3 - x^3)$

(ii) Now we express $x + x^2$ as a linear Combination of given vectors

Let $x + x^2 = a(2 - x^2) + b(x^3 - x) + c(2 - 3x^2) + d(3 - x^3)$

where $a, b, c, d \in R$

or $x + x^2 = (2a + 2c + 3d) - bx + (-a - 3c)x^2 + (b - d)x^3$

$\Rightarrow 2a + 2c + 3d = 0$ ——— ①

$-b = 1$ ——— ②

$-a - 3c = 1$ ——— ③

$b - d = 0$ ——— ④

② $\Rightarrow \boxed{b = -1}$

④ $\Rightarrow -1 - d = 0$ or $\boxed{d = -1}$

③ $\Rightarrow a = -1 - 3c$

Put in ①

$2(-1 - 3c) + 2c + 3(-1) = 0$

$-2 - 6c + 2c - 3 = 0$

$-4c - 5 = 0$

$4c = -5 \Rightarrow \boxed{c = -\frac{5}{4}}$

Put in ③

$-a - 3(-5/4) = 1$

$-a + \frac{15}{4} = 1$

$a = \frac{15}{4} - 1$

$\boxed{a = \frac{11}{4}}$

So $x + x^2 = \frac{11}{4}(2 - x^2) - (x^3 - x) - \frac{5}{4}(2 - 3x^2) - (3 - x^3)$

Q9 Determine whether or not the given set of vectors is a basis for R^n. 77

(i) $\{(1,1),(3,1)\}$

Sol. Given set is $\{(1,1),(3,1)\}$
First we will check their independency
For scalars $a, b \in R$
Let $a(1,1) + b(3,1) = 0$
or $(a+3b, a+b) = 0$
$\Rightarrow$

$$a+3b = 0 \quad \text{--- (1)}$$
$$a+b = 0 \quad \text{--- (2)}$$

Subt. (2) from (1)
$2b = 0 \Rightarrow \boxed{b=0}$
Put in (1)
$a+0 = 0 \Rightarrow \boxed{a=0}$

Hence given set of vectors is linearly independent.
Since dimension of R^2 is 2
& the linearly independent vectors are also 2.
So the given set of vectors $\{(1,1),(3,1)\}$ form a basis for R^2

(ii) $\{(2,1),(1,-1)\}$

Sol. Given set is $\{(2,1),(1,-1)\}$
First we will check their independency.
For scalars $a, b \in R$
Let $a(2,1) + b(1,-1) = 0$
or $(2a+b, a-b) = 0$
$\Rightarrow$ $2a+b = 0$ --- (1)
& $a-b = 0$ --- (2)
Adding (1) & (2)
$3a = 0 \Rightarrow \boxed{a=0}$

Put in ②

$0 - b = 0 \Rightarrow \boxed{b = 0}$

Hence given set of vectors $\{(2,1),(1,-1)\}$ is linearly independent.

Since dimension of R^2 is 2

& no. of linearly independent vectors are also 2

So $\{(2,1),(1,-1)\}$ forms a basis for R^2

Q10 Determine whether or not the given set of vectors is a basis for R^3:

(i) $\{(1,2,-1),(0,3,1),(1,-5,3)\}$

Sol. Given set is $\{(1,2,-1),(0,3,1),(1,-5,3)\}$

First we will check their independency.

For scalars $a,b,c \in R$

Let $a(1,2,-1) + b(0,3,1) + c(1,-5,3) = 0$

or $(a+c,\ 2a+3b-5c,\ -a+b+3c) = 0$

$$\Rightarrow \quad a + c = 0 \quad \text{——— ①}$$

$$2a + 3b - 5c = 0 \quad \text{——— ②}$$

$$-a + b + 3c = 0 \quad \text{——— ③}$$

From ② & ③

$$\frac{a}{9+5} = \frac{-b}{6-5} = \frac{c}{2+3}$$

$$\frac{a}{14} = \frac{b}{-1} = \frac{c}{5} = k$$

$$\Rightarrow \quad a = 14k$$

$$b = -k$$

$$c = 5k$$

Putting these values in ①, we see eq. ① is not satisfied. It is satisfied only when $k = 0$

$\Rightarrow a = 0,\ b = 0,\ c = 0$

Hence given set of vectors $\{(1,2,-1),(0,3,1),(1,-5,3)\}$ is linearly independent.

Since dimension of R^3 is 3

& the no. of linearly independent vectors in R^3 is also 3.

So the set $\{(1,2,-1),(0,3,1),(1,-5,3)\}$ forms a basis for R^3.

(ii) $\{(2,4,-3),(0,1,1),(0,1,-1)\}$

Sol:

Given set is $\{(2,4,-3),(0,1,1),(0,1,-1)\}$

First we will check their independency.

For scalars $a,b,c \in R$

Let $a(2,4,-3)+b(0,1,1)+c(0,1,-1)=0$

or $(2a, 4a+b+c, -3a+b-c)=0$

$\Rightarrow$ $2a = 0$ ———— ①

$4a+b+c = 0$ ———— ②

$-3a+b-c = 0$ ———— ③

① $\Rightarrow$ $\boxed{a=0}$

Put in ② & ③

$b+c=0$ ———— ②

$b-c=0$ ———— ③

Adding ② & ③

$2b=0 \Rightarrow \boxed{b=0}$

Put in ②

$0+c=0 \Rightarrow \boxed{c=0}$

Hence given set of vectors $\{(2,4,-3),(0,1,1),(0,1,-1)\}$ is linearly independent.

Since dimension of R^3 is 3

& no. of linearly independent vectors in R^3 is also 3

So the set $\{(2,4,-3),(0,1,1),(0,1,-1)\}$ forms a basis for R^3

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Q11 Let V be the real vector space of all functions defined on R into R. Determine whether the given vectors are linearly independent or linearly dependent in V:

(i) x, $\cos x$

Sol:- Given vectors are $x, \cos x$

Suppose that for scalars $a, b \in R$

$$ax + b\cos x = 0 \quad \text{——— (1)} \quad \text{for all } x \in R$$

Put $x = 0$ in (1)

$$a(0) + b\cos 0 = 0$$

$$0 + b = 0 \quad \Rightarrow \boxed{b = 0}$$

Now Put $x = \pi/2$ in (1)

$$a(\pi/2) + b\cos \pi/2 = 0$$

$$a(\pi/2) + 0 = 0$$

$$\boxed{a = 0}$$

Hence given vectors are linearly independent.

(ii) $\sin^2 x$, $\cos^2 x$, $\cos 2x$

Sol:- Given vectors are $\sin^2 x$, $\cos^2 x$, $\cos 2x$

Suppose that for scalars $a, b, c \in R$.

$$a\sin^2 x + b\cos^2 x + c\cos 2x = 0 \quad \text{——— (1)} \quad \text{for all } x \in R$$

Put $x = 0$ in (1)

$$a\sin^2(0) + b\cos^2 0 + c\cos 0 = 0$$

$$b + c = 0 \quad \text{——— (2)}$$

Put $x = \pi/2$ in (1)

$$a\sin^2\frac{\pi}{2} + b\cos^2 \pi/2 + c\cos \pi = 0$$

$$a - c = 0 \quad \text{——— (3)}$$

Put $x = \pi/4$ in (1)

$$a\sin^2 \pi/4 + b\cos^2 \pi/4 + c\cos \pi/2 = 0$$

or $$\frac{a}{2} + \frac{b}{2} = 0$$

or $\quad a+b=0$ ———— (4)

(ii) $\Rightarrow$ $b=-c$

(i) $\Rightarrow$ $a=c$

So non zero soln. of above eqs. is

$a = c$

$b = -c$

$c = c$ or $(c, -c, c)$ where $c \in R$

So given vectors are linearly dependent

(iii) $\sin x, \cos x, \sinh x, \cosh x$

Sol.

Given vectors are $\sin x, \cos x, \sinh x, \cosh x$

Suppose that for scalars $a, b, c, d \in R$

$a\sin x + b\cos x + c\sinh x + d\cosh x = 0$ ———— (A)

Put $x = 0$

$a\sin 0 + b\cos 0 + c\sinh 0 + d\cosh 0 = 0$

$b + d = 0$ ———— (1)

Diff. (A) w.r.t. x

$a\cos x - b\sin x + c\cosh x + d\sinh x = 0$ ———— (B)

Put $x = 0$

$a\cos 0 - b\sin 0 + c\cosh 0 + d\sinh 0 = 0$

$a + c = 0$ ———— (2)

Diff. (B) w.r.t. x

$-a\sin x - b\cos x + c\sinh x + d\cosh x = 0$ ———— (C)

Put $x = 0$

$-a\sin 0 - b\cos 0 + c\sinh 0 + d\cosh 0 = 0$

or $\quad -b + d = 0$ ———— (3)

Diff. (C) w.r.t. x

$-a\cos x + b\sin x + c\cosh x + d\sinh x = 0$

Put $x = 0$

$-a\cos 0 + b\sin 0 + c\cosh 0 + d\sinh 0 = 0$

$-a + c = 0$ ——— (4)

Adding (1) & (3)

$2d = 0 \Rightarrow \boxed{d = 0}$

Put in (1)

$b + 0 = 0 \Rightarrow \boxed{b = 0}$

Adding (2) & (4)

$2c = 0 \Rightarrow \boxed{c = 0}$

Put in (4)

$-a + 0 = 0 \Rightarrow \boxed{a = 0}$

Hence given vectors are linearly independent.

(iv) $\sin x$, $\sin x + \cos x$, $\sin x - \cos x$

Sol. Given vectors are $\sin x$, $\sin x + \cos x$, $\sin x - \cos x$

Suppose that for scalars $a, b, c \in R$

$a \sin x + b(\sin x + \cos x) + c(\sin x - \cos x) = 0$

or $(a+b+c)\sin x + (b-c)\cos x = 0$ ——— (A)

Put $x = 0$ in (A)

$(a+b+c)\sin 0 + (b-c)\cos 0 = 0$

or $b - c = 0$ ——— (1)

Put $x = \pi/2$ in (A)

$(a+b+c)\sin \pi/2 + (b-c)\cos \pi/2 = 0$

or $a + b + c = 0$ ——— (2)

Put $x = \pi$ in (A)

$(a+b+c)\sin \pi + (b-c)\cos \pi = 0$

or $(b-c)(-1) = 0$

or $b - c = 0$ ——— (3)

(3) $\Rightarrow b = c$

Put in (2)

$a + c + c = 0 \Rightarrow \boxed{a = -2c}$

So a non zero soln. of homo eqs. is $(-2c, c, c)$

Cont.

Hence the given vectors are linearly dependant. 83

(v) e^{ax}, e^{bx}, e^{cx} ; a, b, c being distinct real no's.

Sol.

Given vectors are e^{ax}, e^{bx}, e^{cx}

Suppose that for scalars $\alpha, \beta, \gamma \in R$

$$\alpha e^{ax} + \beta e^{bx} + \gamma e^{cx} = 0$$

Q12. Determine a basis for each of the following subspace of R^3:

(i) The plane $x-2y+5z=0$

Sol.

Given eq. of plane is $x-2y+5z=0$

or $x = 2y-5z$ where y, z are free variables

The above eq. in vector form can be written as

$$(x,y,z) = (2y-5z, y, z)$$
$$= (2y-5z, y+0, 0+z)$$
$$= (2y, y, 0) + (-5z, 0, z)$$
$$\Rightarrow (x,y,z) = y(2,1,0) + z(-5,0,1)$$

Thus given plane is spanned by vectors $(2,1,0)$ & $(-5,0,1)$. Since none of the vector is multiple of other. So the set $\{(2,1,0),(-5,0,1)\}$ is linearly independent.

Hence $\{(2,1,0),(-5,0,1)\}$ forms a basis for given subspace of R^3.

(ii) The line $\frac{x}{-2} = \frac{y}{1} = \frac{z}{6}$

Sol.

Given eq. of line is

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{6} = t$$

$$\Rightarrow \quad x = -2t$$
$$y = t$$
$$\& \quad z = 6t$$

The above eq. in vector form can be written as

$$(x,y,z) = (-2t, t, 6t)$$

or $(x,y,z) = t(-2,1,6)$

Hence the given line is spanned by the vector $(-2,1,6)$. Also $(-2,1,6)$ being a non

zero single vector is linearly independent
So $\{(-2,1,6)\}$ forms a basis for the given subspace of R^3.

(iii) All vectors of the form (a,b,c) where
$3a-2b+c=0$

Soln

Given eq. is

$$3a-2b+c=0$$

or $c=-3a+2b$ where a,b are free variables

The above eq. can be written in vector form as

$$(a,b,c) = (a,b,-3a+2b)$$
$$= (a+0,\ 0+b,\ -3a+2b)$$
$$= (a,0,-3a)+(0,b,2b)$$

or $(a,b,c) = a(1,0,-3)+b(0,1,2)$

So given subspace is spanned by $(1,0,-3)$ & $(0,1,2)$.

Now since none of the vector is multiple of other. So the vectors $(1,0,-3)$ & $(0,1,2)$ are linearly independent

So the set $\{(1,0,-3),(0,1,2)\}$ forms a basis for the given subspace of R^3.

Q13 Find the dimension of the subspace
$\{(x_1,x_2,x_3,x_4): x_2=x_3\}$ of R^4. Also determine a basis.

Soln.

Let $W=\{(x_1,x_2,x_3,x_4): x_2=x_3\}$

Suppose (x_1,x_2,x_2,x_4) be a general vector of W then we can write it as

$$(x_1,x_2,x_2,x_4) = (x_1,0,0,0)+(0,x_2,x_2,0)+(0,0,0,x_4)$$
$$= x_1(1,0,0,0)+x_2(0,1,1,0)+x_4(0,0,0,1)$$

Which shows that $(x_1, x_2, x_2, x_4) \in W$ is a linear Combination of vectors $(1,0,0,0), (0,1,1,0)$ & $(0,0,0,1)$.
So the set $S = \{(1,0,0,0), (0,1,1,0), (0,0,0,1)\}$ spans W.
Now we check the independency of set S.
Suppose that for scalars $a, b \in R$

$$a(1,0,0,0) + b(0,1,1,0) + c(0,0,0,1) = (0,0,0,0)$$

or $(a, b, b, c) = (0,0,0,0)$

$$\Rightarrow \quad a = 0$$
$$b = 0$$
$$c = 0$$

Hence the set $S = \{(1,0,0,0), (0,1,1,0), (0,0,0,1)\}$ is also linearly independent.
Hence S is a basis for W
So dimension of W is 3

Q14 A subspace U of R^4 is spanned by the vectors $(1,0,2,3)$ & $(0,1,-1,2)$ & a subspace W is spanned by $(1,2,3,4), (-1,-1,5,0)$ & $(0,0,0,1)$. Find the dimensions of U & W.

Sol.

Let $S = \{(1,0,2,3), (0,1,-1,2)\}$
Since these vectors span U
So we only check their independency.
Suppose that for scalars $a, b \in R$

$$a(1,0,2,3) + b(0,1,-1,2) = 0$$

or $(a, b, 2a-b, 3a+2b) = (0,0,0,0)$

$$\Rightarrow \left.\begin{aligned} a &= 0 \\ b &= 0 \\ 2a-b &= 0 \\ 3a+2b &= 0 \end{aligned}\right] \Rightarrow a = b = 0$$

Which shows that set S is linearly independent

Hence $S = \{(1,0,2,3),(0,1,-1,2)\}$ forms a basis for U

So dimension of $U = 2$

(ii) $(1,2,3,4),(-1,-1,5,0),(0,0,0,1)$

Sol.

Let $S = \{(1,2,3,4),(-1,-1,5,0),(0,0,0,1)\}$

Since these vectors span W (given)

So we only check their independency.

Suppose that for scalars $a,b \in R$

$a(1,2,3,4)+b(-1,-1,5,0)+c(0,0,0,1) = 0$

or $(a-b, 2a-b, 3a+5b, 4a+c) = (0,0,0,0)$

$\Rightarrow$

$$a-b = 0 \quad \text{------} \quad (1)$$
$$2a-b = 0 \quad \text{------} \quad (2)$$
$$3a+5b = 0 \quad \text{------} \quad (3)$$
$$4a+c = 0 \quad \text{------} \quad (4)$$

Subt. (2) from (1)

$-a = 0 \Rightarrow \boxed{a=0}$

(1) $\Rightarrow \boxed{b=0}$

(4) $\Rightarrow \boxed{c=0}$

Hence the set $S = \{(1,2,3,4),(-1,-1,5,0),(0,0,0,1)\}$ is linearly independent.

So $S = \{(1,2,3,4),(-1,-1,5,0),(0,0,0,1)\}$ forms a basis for W

Hence dimension of $W = 3$

Q15 Suppose that U & W are distinct four dimensional subspaces of a vector space V of dimension six. Find the possible dimension of $U \cap W$.

Sol. We are given that

$\dim U = 4$

$\dim W = 4$

& $\dim V = 6$

Since $U+W$ is a subspace of V

So $\dim(U+W) \leq \dim V = 6$

$\Rightarrow \dim(U+W) \leq 6$

Now as $U \subseteq U+W$ & $W \subseteq U+W$

So $\dim U \leq \dim(U+W) \leq 6$

or $4 \leq \dim(U+W) \leq 6$

Hence $\dim(U+W)$ is 4 or 5 or 6

Since U & W are distinct, so they must be different by atleast one generator

So $\dim(U+W) > 4$

Hence $\dim(U+W)$ is 5 or 6

As we know

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

or $\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$

(i) If $\dim(U+W) = 5$

then $\dim(U \cap W) = 4+4-5 = 3$

(ii) If $\dim(U+W) = 6$

then $\dim(U \cap W) = 4+4-6 = 2$

Hence the possible dimensions of $U \cap W$ are 2 or 3

Q16 Find a basis & dimension of the subspace W of R^4 spanned by

(i) $(1,4,-1,3), (2,1,-3,-1)$ & $(0,2,1,-5)$

(ii) $(1,-4,-2,1), (1,-3,-1,2)$ & $(3,-8,-2,7)$

Sol:-

as the subspace W of R^4 is spanned by $(1,4,-1,3)$, $(2,1,-3,-1)$ & $(0,2,1,-5)$

Now we only check their independency

For this let for $a,b,c \in F$

$$a(1,4,-1,3)+b(2,1,-3,-1)+c(0,2,1,-5) = (0,0,0,0)$$

$$(a+2b,\ 4a+b+2c,\ -a-3b+c,\ 3a-b-5c) = (0,0,0,0)$$

$$\Rightarrow \quad a+2b = 0 \quad \text{———} \quad (1)$$

$$4a+b+2c = 0 \quad \text{———} \quad (2)$$

$$-a-3b+c = 0 \quad \text{———} \quad (3)$$

$$3a-b-5c = 0 \quad \text{———} \quad (4)$$

from (1) & (2)

$$\frac{a}{4-0} = \frac{-b}{2-0} = \frac{c}{1-8}$$

$$\frac{a}{4} = \frac{b}{-2} = \frac{c}{-7} = k$$

$$\Rightarrow \left.\begin{aligned} a &= 4k \\ b &= -2k \\ c &= -7k \end{aligned}\right]$$

Put in (3) & (4), we see eqs. are not satisfied & they are satisfied only when $k=0$

i.e., when $a=b=c=0$

Which shows that given vectors are linearly independent

So $\{(1,4,-1,3),(2,1,-3,-1),(0,2,1,-5)\}$ form a basis of W

Hence $\dim W = 3$

(ii) $(1,-4,-2,1),(1,-3,-1,2),(3,-8,-2,7)$

Sol. Given vectors are

$(1,-4,-2,1)$, $(1,-3,-1,2)$ & $(3,-8,-2,7)$

As the subspace W of R^4 is spanned by $(1,-4,-2,1)$, $(1,-3,-1,2)$ & $(3,-8,-2,7)$.

So we only check their independency.

For this let for scalars a,b,c

$a(1,-4,-2,1)+b(1,-3,-1,2)+c(3,-8,-2,7)=(0,0,0,0)$

or $(a+b+3c,\ -4a-3b-8c,\ -2a-b-2c,\ a+2b+7c)=(0,0,0,0)$

$$\Rightarrow\quad a+b+3c=0 \quad\text{---------}\ (1)$$

$$-4a-3b-8c=0 \quad\text{---------}\ (2)$$

$$-2a-b-2c=0 \quad\text{---------}\ (3)$$

$$a+2b+7c=0 \quad\text{---------}\ (4)$$

from (1) & (2)

$$\frac{a}{-8+9}=\frac{-b}{-8+12}=\frac{c}{-3+4}$$

$$\frac{a}{1}=\frac{b}{-4}=\frac{c}{1}=k$$

$$\Rightarrow\quad \left.\begin{aligned} a&=k\\ b&=-4k\\ c&=k\end{aligned}\right]$$

Put these values in (3) & (4), we see these eqs. are satisfied so the given vectors are linearly dependent.

Now we take only first two vectors $(1,-4,-2,1)$ & $(1,-3,-1,2)$ & check their independency.

Suppose for $a,b\in F$

$a(1,-4,-2,1)+b(1,-3,-1,2)=(0,0,0,0)$

or $(a+b,\ -4a-3b,\ -2a-b,\ a+2b)=(0,0,0,0)$

$$\Rightarrow\quad a+b=0 \quad\text{---------}\ (1)$$

$$-4a-3b=0 \quad\text{---------}\ (2)$$

$$-2a-b=0 \quad\text{---------}\ (3)$$

$$a+2b=0 \quad\text{---------}\ (4)$$

Adding ① & ③

$-a = 0$ or $\boxed{a = 0}$

Put in ①

$0 + b = 0 \Rightarrow \boxed{b = 0}$

Hence the vectors are linearly independent & so the set $\{(1,-4,-2,1),(1,-3,-1,2)\}$ form a basis of W

Hence $\dim W = 2$

Q17 Let U & W be 2-dimensional subspaces of R^3. Show that $U \cap W \neq \{0\}$

Sol. Given that

$\dim U = 2$
& $\dim W = 2$

If $U = W$ then $\dim U = \dim W = 2$

So $U \cap W \neq \{0\}$

Hence we suppose that $U \neq W$. This means that U & W are not spanned by the same set.

So $\dim(U+W) > 2$

Since U & W are subspaces of R^3

Hence $\dim(U+W) \leq 3$

So $2 < \dim(U+W) \leq 3$

Hence $\dim(U+W) = 3$

As $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

So $\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$

$= 2 + 2 - 3$

$\therefore \dim(U \cap W) = 1$

It shows that $U \cap W$ contains a non zero element & so $U \cap W \neq \{0\}$

(Exercise 6.3)

Q1 Check which of the following define linear transformations from R^3 to R^2?

(i) $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$

(i) then we prove $T(u_1 + u_2) = T(u_1) + T(u_2)$

$$\begin{aligned}
\text{Now } T(u_1+u_2) &= T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big) \\
&= T(x_1+y_1, x_2+y_2, x_3+y_3) \\
&= \big((x_1+y_1)-(x_2+y_2), (x_1+y_1)-(x_3+y_3)\big) \\
&= (x_1+y_1-x_2-y_2, x_1+y_1-x_3-y_3) \\
&= (x_1-x_2+y_1-y_2, x_1-x_3+y_1-y_3) \\
&= (x_1-x_2, x_1-x_3) + (y_1-y_2, y_1-y_3) \\
&= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
&= T(u_1) + T(u_2)
\end{aligned}$$

(ii) Let $a \in R$ & $u_1 = (x_1, x_2, x_3) \in R^3$

then we prove $T(au_1) = aT(u_1)$

$$\begin{aligned}
\text{Now } T(au_1) &= T\big(a(x_1, x_2, x_3)\big) \\
&= T(ax_1, ax_2, ax_3) \\
&= (ax_1 - ax_2, ax_1 - ax_3) \\
&= a(x_1 - x_2, x_1 - x_3) \\
&= aT(x_1, x_2, x_3) \\
&= aT(u_1)
\end{aligned}$$

Hence T is a linear transformation from R^3 to R^2

(ii) $T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$

let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (|x_1+y_1|, (x_2+y_2) - (x_3+y_3))$$
$$\therefore T(u_1+u_2) = (|x_1+y_1|, x_2+y_2-x_3-y_3) \quad \text{———— (1)}$$

Now

$$T(u_1)+T(u_2) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$
$$= (|x_1|, x_2-x_3) + (|y_1|, y_2-y_3)$$
$$= (|x_1|+|y_1|, x_2-x_3+y_2-y_3)$$
$$\therefore T(u_1)+T(u_2) = (|x_1|+|y_1|, x_2+y_2-x_3-y_3) \quad \text{———— (2)}$$

From (1) & (2)

$$T(u_1+u_2) \neq T(u_1)+T(u_2)$$

Hence T is not a linear transformation from R^3 to R^2.

(iii) $T(x_1, x_2, x_3) = (x_1 + 1, x_2 + x_3)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (x_1+1, x_2+x_3)$

let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$

$$T(u_1+u_2) = T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_1+y_1+1, x_2+y_2+x_3+y_3)$$
$$T(u_1+u_2) = (x_1+y_1+1, x_2+x_3+y_2+y_3) \quad \text{——— (1)}$$

Now

$$T(u_1)+T(u_2) = T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= (x_1+1, x_2+x_3) + (y_1+1, y_2+y_3)$$
$$= (x_1+1+y_1+1, x_2+x_3+y_2+y_3)$$
$$T(u_1)+T(u_2) = (x_1+y_1+2, x_2+x_3+y_2+y_3) \quad \text{——— (2)}$$

From (1) & (2)

$$T(u_1+u_2) \neq T(u_1)+T(u_2)$$

Hence T is not a linear transformation from R^3 to R^2.

(iv) $T(x_1,x_2,x_3) = (0, x_3)$

<u>Sol.</u> Given transformation is

$$T(x_1,x_2,x_3) = (0, x_3)$$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove.

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (0, x_3+y_3)$$
$$= (0,x_3)+(0,y_3)$$
$$= T(x_1,x_2,x_3)+T(y_1,y_2,y_3)$$

$\therefore T(u_1+u_2) = T(u_1)+T(u_2)$

(ii) Let $\alpha \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove

$$T(\alpha u_1) = \alpha T(u_1)$$

Now $T(\alpha u_1) = T(\alpha(x_1,x_2,x_3))$

$$T(au_1) = T(ax_1, ax_2, ax_3)$$
$$= (0, ax_3)$$
$$= a(0, x_3)$$
$$= aT(x_1, x_2, x_3)$$
$$T(au_1) = aT(u_1)$$

Hence T is a linear transformation from R^3 to R^2.

(v) $T(x_1, x_2, x_3) = \left(\frac{x_1+x_2}{x_3}, x_3\right)$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = \left(\frac{x_1+x_2}{x_3}, x_3\right)$$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big)$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= \left(\frac{x_1+y_1+x_2+y_2}{x_3+y_3}, x_3+y_3\right) \quad \text{——— (1)}$$

Now

$$T(u_1) + T(u_2) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$
$$= \left(\frac{x_1+x_2}{x_3}, x_3\right) + \left(\frac{y_1+y_2}{y_3}, y_3\right)$$
$$= \left(\frac{x_1+x_2}{x_3} + \frac{y_1+y_2}{y_3}, x_3+y_3\right)$$
$$= \left(\frac{y_3(x_1+x_2)+x_3(y_1+y_2)}{x_3y_3}, x_3+y_3\right) \quad \text{——— (2)}$$

From (1) & (2)

$$T(u_1+u_2) \neq T(u_1) + T(u_2)$$

Hence T is not a linear transformation from R^3 to R^2.

(vi) $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1)$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned}
T(u_1+u_2) &= T((x_1,x_2,x_3)+(y_1,y_2,y_3)) \\
&= T(x_1+y_1, x_2+y_2, x_3+y_3) \\
&= (3(x_1+y_1) - 2(x_2+y_2) + (x_3+y_3), (x_3+y_3) - 3(x_2+y_2) - 2(x_1+y_1)) \\
&= (3x_1 - 2x_2 + x_3 + 3y_1 - 2y_2 + y_3, x_3 - 3x_2 - 2x_1 + y_3 - 3y_2 - 2y_1) \\
&= (3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1) + (3y_1 - 2y_2 + y_3, y_3 - 3y_2 - 2y_1) \\
&= T(x_1,x_2,x_3) + T(y_1,y_2,y_3) \\
&= T(u_1) + T(u_2)
\end{aligned}$$

(ii) Let $a \in R$ & $u_1 = (x_1, x_2, x_3) \in R^3$ then we prove

$T(au_1) = aT(u_1)$

$$\begin{aligned}
\text{Now } T(au_1) &= T(a(x_1,x_2,x_3)) \\
&= T(ax_1, ax_2, ax_3) \\
&= (3ax_1 - 2ax_2 + ax_3, ax_3 - 3ax_2 - 2ax_1) \\
&= a(3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1) \\
&= aT(x_1,x_2,x_3) \\
&= aT(u_1)
\end{aligned}$$

Hence T is a linear transformation from R^3 to R^2.

Q2 Show that each of the following defines linear transformation from R^3 to R^3.

(i) $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1)$

Sol. Given transformation is

$T(x_1,x_2,x_3) = (x_1-x_2, x_2-x_3, x_1)$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$$\begin{aligned}
T(u_1+u_2) &= T\big((x_1,x_2,x_3)+(y_1,y_2,y_3)\big) \\
&= T(x_1+y_1, x_2+y_2, x_3+y_3) \\
&= \big((x_1+y_1)-(x_2+y_2), (x_2+y_2)-(x_3+y_3), x_1+y_1\big) \\
&= (x_1+y_1-x_2-y_2, x_2+y_2-x_3-y_3, x_1+y_1) \\
&= (x_1-x_2+y_1-y_2, x_2-x_3+y_2-y_3, x_1+y_1) \\
&= (x_1-x_2, x_2-x_3, x_1)+(y_1-y_2, y_2-y_3, y_1) \\
&= T(x_1,x_2,x_3) + T(y_1,y_2,y_3) \\
&= T(u_1)+T(u_2)
\end{aligned}$$

Now

$$\begin{aligned}
T(au_1) &= T\big(a(x_1,x_2,x_3)\big) \\
&= T(ax_1, ax_2, ax_3) \\
&= (ax_1-ax_2, ax_2-ax_3, ax_1) \\
&= a(x_1-x_2, x_2-x_3, x_1) \\
&= aT(x_1,x_2,x_3) \\
&= aT(u_1)
\end{aligned}$$

Hence T is a linear transformation from R^3 to R^3

(ii) $T(x_1,x_2,x_3) = (x_1+x_2, -x_1-x_2, x_3)$

Sol. Given transformation is

$T(x_1,x_2,x_3) = (x_1+x_2, -x_1-x_2, x_3)$

let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_1+y_1+x_2+y_2, -(x_1+y_1)-(x_2+y_2), x_3+y_3)$$
$$= (x_1+y_1+x_2+y_2, -x_1-y_1-x_2-y_2, x_3+y_3)$$
$$= (x_1+x_2+y_1+y_2, -x_1-x_2-y_1-y_2, x_3+y_3)$$
$$= (x_1+x_2, -x_1-x_2, x_3) + (y_1+y_2, -y_1-y_2, y_3)$$
$$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= T(u_1) + T(u_2)$$

(ii) Let $a \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove

$T(au_1) = aT(u_1)$

Now

$$T(au_1) = T(a(x_1,x_2,x_3))$$
$$= T(ax_1, ax_2, ax_3)$$
$$= (ax_1+ax_2, -ax_1-ax_2, ax_3)$$
$$= a(x_1+x_2, -x_1-x_2, x_3)$$
$$= aT(x_1,x_2,x_3)$$
$$= aT(u_1)$$

Hence T is a linear transformation from R^3 to R^3

(iii) $T(x_1,x_2,x_3) = (x_2, -x_1, -x_3)$

Sol. Given transformation is

$T(x_1,x_2,x_3) = (x_2, -x_1, -x_3)$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$

$$T(u_1+u_2) = T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_2+y_2, -(x_1+y_1), -(x_3+y_3))$$
$$= (x_2+y_2, -x_1-y_1, -x_3-y_3).$$
$$= (x_2, -x_1, -x_3) + (y_2, -y_1, -y_3)$$
$$= T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$
$$= T(u_1) + T(u_2)$$

(ii) Let $a \in R$ & $u_1 = (x_1, x_2, x_3) \in R^3$ then we prove
$T(au_1) = aT(u_1)$

Now

$$T(au_1) = T(a(x_1, x_2, x_3))$$
$$= T(ax_1, ax_2, ax_3)$$
$$= (ax_2, -ax_1, -ax_3)$$
$$= a(x_2, -x_1, -x_3)$$
$$= aT(x_1, x_2, x_3)$$
$$= aT(u_1)$$

Hence T is a linear transformation from R^3 to R^3.

(iv) $T(x_1, x_2, x_3) = (x_1-3x_2-2x_3, x_2-4x_3, x_3)$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1-3x_2-2x_3, x_2-4x_3, x_3)$$

Let $u_1 = (x_1, x_2, x_3)$
& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove
(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= ((x_1+y_1) - 3(x_2+y_2) - 2(x_3+y_3), (x_2+y_2) - 4(x_3+y_3), x_3+y_3)$$
$$= (x_1-3x_2-2x_3+y_1-3y_2-2y_3, x_2-4x_3+y_2-4y_3, x_3+y_3)$$

$T(u_1+u_2) = (x_1-3x_2-2x_3, x_2-4x_3, x_3) + (y_1-3y_2-2y_3, y_2-4y_3, y_3)$
$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$
$= T(u_1) + T(u_2)$

(ii) let $\alpha \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove
$T(\alpha u_1) = \alpha T(u_1)$

Now

$T(\alpha u_1) = T(\alpha(x_1,x_2,x_3))$
$= T(\alpha x_1, \alpha x_2, \alpha x_3)$
$= (3\alpha x_1 - 3\alpha x_2 - 2\alpha x_3, \alpha x_2 - 4\alpha x_3, \alpha x_3)$
$= \alpha(3x_1-3x_2-2x_3, x_2-4x_3, x_3)$
$= \alpha T(x_1,x_2,x_3)$
$= \alpha T(u_1)$

Hence T is a linear transformation from R^3 to R^3.

(v) $T(x_1,x_2,x_3) = (x_1+x_3, x_1-x_3, x_2)$

Sol. Given Transformation is
$T(x_1,x_2,x_3) = (x_1+x_3, x_1-x_3, x_2)$
Let $u_1 = (x_1,x_2,x_3)$
& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove
$T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$
$= T(x_1+y_1, x_2+y_2, x_3+y_3)$
$= ((x_1+y_1)+(x_3+y_3), (x_1+y_1)-(x_3+y_3), x_2+y_2)$
$= (x_1+x_3+y_1+y_3, x_1-x_3+y_1-y_3, x_2+y_2)$
$= (x_1+x_3, x_1-x_3, x_2) + (y_1+y_3, y_1-y_3, y_2)$
$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$
$= T(u_1) + T(u_2)$

(ii) Let $a \in R$ & $u_1 = (x_1, x_2, x_3) \in R^3$ then we prove

$T(au_1) = aT(u_1)$

Now

$$\begin{aligned} T(au_1) &= T(a(x_1, x_2, x_3)) \\ &= T(ax_1, ax_2, ax_3) \\ &= (ax_1 + ax_3, ax_1 - ax_3, ax_2) \\ &= a(x_1 + x_3, x_1 - x_3, x_2) \\ &= aT(x_1, x_2, x_3) \\ &= aT(u_1) \end{aligned}$$

Hence T is a linear transformation from R^3 to R^3.

Q3 Show that each of the following transformations is not linear.

(i) $T : R^2 \longrightarrow R$ defined by $T(x_1, x_2) = x_1 x_2$

Sol. Given transformation is

$T(x_1, x_2) = x_1 x_2$

let $u_1 = (x_1, x_2)$

& $u_2 = (y_1, y_2) \in R^2$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2) + (y_1, y_2)) \\ &= T(x_1 + y_1, x_2 + y_2) \\ &= (x_1 + y_1)(x_2 + y_2) \quad \text{———} \ (1) \end{aligned}$$

&

$$\begin{aligned} T(u_1) + T(u_2) &= T(x_1, x_2) + T(y_1, y_2) \\ &= x_1 x_2 + y_1 y_2 \quad \text{———} \ (2) \end{aligned}$$

from (1) & (2) $T(u_1 + u_2) \neq T(u_1) + T(u_2)$.

Hence T is not a linear transformation from R^2 to R

(ii) $T: R^2 \longrightarrow R^3$ defined by $T(x_1, x_2) = (x_1+1, 2x_2, x_1+x_2)$

Sol: Given transformation is

$$T(x_1, x_2) = (x_1+1, 2x_2, x_1+x_2)$$

Let $u_1 = (x_1, x_2)$

& $u_2 = (y_1, y_2) \in R^2$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1+u_2) &= T\big((x_1, x_2) + (y_1, y_2)\big) \\ &= T(x_1+y_1, x_2+y_2) \\ &= \big(x_1+y_1+1, 2(x_2+y_2), (x_1+y_1)+(x_2+y_2)\big) \\ &= (x_1+y_1+1, 2x_2+2y_2, x_1+x_2+y_1+y_2) \quad \text{——— (1)} \end{aligned}$$

$$\begin{aligned} T(u_1)+T(u_2) &= T(x_1, x_2) + T(y_1, y_2) \\ &= (x_1+1, 2x_2, x_1+x_2) + (y_1+1, 2y_2, y_1+y_2) \\ &= (x_1+y_1+2, 2x_2+2y_2, x_1+x_2+y_1+y_2) \quad \text{——— (2)} \end{aligned}$$

from (1) & (2)

$$T(u_1+u_2) \neq T(u_1) + T(u_2)$$

Hence T is not a linear transformation from R^2 to R^3.

(iii) $T: R^3 \longrightarrow R^2$ defined by $T(x_1, x_2, x_3) = (|x_1|, 0)$

Sol: Given transformation is

$$T(x_1, x_2, x_3) = (|x_1|, 0)$$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1+u_2) &= T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big) \\ &= T(x_1+y_1, x_2+y_2, x_3+y_3) \end{aligned}$$

$T(u_1+u_2) = (|x_1+y_1|, 0)$ ——— ①

&

$T(u_1)+T(u_2) = T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$

$= (|x_1|, 0) + (|y_1|, 0)$

$= (|x_1|+|y_1|, 0)$ ——— ②

From ① & ②

$T(u_1+u_2) \neq T(u_1) + T(u_2)$

Hence T is not a linear transformation from R^3 to R^2.

(iv) $T: R^2 \longrightarrow R^2$ defined by $T(x_1,x_2) = (x_1^2, x_2^2)$

Sol. Given transformation is

$T(x_1,x_2) = (x_1^2, x_2^2)$

Let $u_1 = (x_1,x_2)$

& $u_2 = (y_1,y_2) \in R^2$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$T(u_1+u_2) = T((x_1,x_2)+(y_1,y_2))$

$= T(x_1+y_1, x_2+y_2)$

$= ((x_1+y_1)^2, (x_2+y_2)^2)$ ———

&

$T(u_1)+T(u_2) = T(x_1,x_2) + T(y_1,y_2)$

$= (x_1^2, x_2^2) + (y_1^2, y_2^2)$

$= (x_1^2+y_1^2, x_2^2+y_2^2)$ ——— ②

from ① & ②

$T(u_1+u_2) \neq T(u_1) + T(u_2)$

Hence T_1 is not a linear transformation from R^2 to R^2

(v) $T : R^3 \longrightarrow R^3$ defined by $T(x_1, x_2, x_3) = (x_1, x_2, x_3) + (1,1,1)$

Sol: Given transformation is

$T(x_1, x_2, x_3) = (x_1, x_2, x_3) + (1,1,1)$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1+u_2) &= T\big((x_1,x_2,x_3)+(y_1,y_2,y_3)\big) \\ &= T(x_1+y_1,\ x_2+y_2,\ x_3+y_3) \\ &= (x_1+y_1,\ x_2+y_2,\ x_3+y_3) + (1,1,1) \\ &= (x_1+y_1+1,\ x_2+y_2+1,\ x_3+y_3+1) \quad \text{——— (1)} \end{aligned}$$

&

$$\begin{aligned} T(u_1)+T(u_2) &= T(x_1,x_2,x_3) + T(y_1,y_2,y_3) \\ &= (x_1,x_2,x_3)+(1,1,1)+(y_1,y_2,y_3)+(1,1,1) \\ &= (x_1+1,\ x_2+1,\ x_3+1) + (y_1+1,\ y_2+1,\ y_3+1) \\ &= (x_1+y_1+2,\ x_2+y_2+2,\ x_3+y_3+2) \quad \text{——— (2)} \end{aligned}$$

from (1) & (2)

$T(u_1+u_2) \neq T(u_1) + T(u_2)$

Hence T is not a linear transformation from R^3 to R^3.

Q3 Determine which of the following transformations are linear :

(a) $T : M_{22} \longrightarrow R$ defined by

(i) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d$

Sol: Given transformation is

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d$ ...

Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

& $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_{22}$ then we prove

(i) $T(A_1+A_2) = T(A_1)+T(A_2)$

Now

$$T(A_1+A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right)$$

$$= a_1+a_2+d_1+d_2 \quad \text{———— } ①$$

&

$$T(A_1)+T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= a_1+d_1 + a_2+d_2$$

$$= a_1+a_2+d_1+d_2 \quad \text{———— } ②$$

from ① & ②

$$T(A_1+A_2) = T(A_1)+T(A_2)$$

(ii) Let $a \in R$ & $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in M_{22}$ then we prove

$$T(aA_1) = aT(A_1)$$

Now $T(aA_1) = T\left(a\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$

$$= T\left(\begin{bmatrix} aa_1 & ab_1 \\ ac_1 & ad_1 \end{bmatrix}\right)$$

$$= aa_1 + ad_1$$

$$= a(a_1+d_1)$$

$$= aT\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$$

$$= aT(A_1)$$

Hence T is a linear <u>transformation</u> from M_{22} to R

(ii) $T: M_{22} \longrightarrow R$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Sol. Given transformation is

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

& $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

then we prove

$$T(A_1 + A_2) = T(A_1) + T(A_2)$$

Now

$$T(A_1 + A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right)$$

$$= (a_1+a_2)(d_1+d_2) - (b_1+b_2)(c_1+c_2) \quad \text{——— } (1)$$

Now

$$T(A_1) + T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= a_1d_1 - b_1c_1 + a_2d_2 - b_2c_2 \quad \text{——— } (2)$$

from (1) & (2)

$$T(A_1 + A_2) \neq T(A_1) + T(A_2)$$

Hence T is a linear transformation from M_{22} to R.

(b) $T: P_2(x) \longrightarrow P_2(x)$ defined by

(i) $T(a + bx + cx^2) = a + (b+c)x + (2a-3b)x^2$

Sol. Given transformation is

$$T(a + bx + cx^2) = a + (b+c)x + (2a-3b)x^2$$

Let $u = a+bx+cx^2$

& $v = p+qx+\lambda x^2 \in P_2(x)$ then we prove

(i) $T(u+v) = T(u)+T(v)$

Now

$$\begin{aligned}
T(u+v) &= T\big((a+bx+cx^2)+(p+qx+\lambda x^2)\big) \\
&= T\big((a+p)+(b+q)x+(c+\lambda)x^2\big) \\
&= (a+p)+(b+q+c+\lambda)x+(2a+2p-3b-3q)x^2 \\
&= (a+p)+(b+c+q+\lambda)x+(2a-3b+2p-3q)x^2 \\
&= \big(a+(b+c)x+(2a-3b)x^2\big)+\big(p+(q+\lambda)x+(2p-3q)x^2\big) \\
&= T(a+bx+cx^2)+T(p+qx+\lambda x^2) \\
&= T(u)+T(v)
\end{aligned}$$

(ii) Let $k \in R$ & $u = a+bx+cx^2$ then we prove

$T(ku) = kT(u)$

Now

$$\begin{aligned}
T(ku) &= T\big(k(a+bx+cx^2)\big) \\
&= T(ka+kbx+kcx^2) \\
&= ka+(kb+kc)x+(2ka-3kb)x^2 \\
&= k\big(a+(b+c)x+(2a-3b)x^2\big) \\
&= kT(a+bx+cx^2) \\
&= kT(u)
\end{aligned}$$

Hence T is a linear transformation from $P_2(x)$ to $P_2(x)$.

(iii). $T: P_2(x) \longrightarrow P_2(x)$ defined by

$T(a+bx+cx^2) = (a+1)+bx+cx^2$

Sol. Given transformation is

$T(a+bx+cx^2) = (a+1)+bx+cx^2$

Let $u = a+bx+cx^2$

& $v = p+qx+\lambda x^2 \in P_2(x)$ then we prove

(i) $T(u+v) = T(u) + T(v)$

Now

$$T(u+v) = T\big((a+bx+cx^2)+(p+qx+rx^2)\big)$$
$$= T\big((a+p)+(b+q)x+(c+r)x^2\big)$$
$$= (a+p+1)+(b+q)x+(c+r)x^2 \quad \text{——— (1)}$$

&

$$T(u)+T(v) = T(a+bx+cx^2)+T(p+qx+rx^2)$$
$$= (a+1)+bx+cx^2+(p+1)+qx+rx^2$$
$$= (a+p+2)+(b+q)x+(c+r)x^2 \quad \text{——— (2)}$$

From (1) & (2)

$$T(u+v) \neq T(u)+T(v)$$

Hence T is not a linear transformation from $P_2(x)$ to $P_2(x)$.

Q5. If A is an $m\times n$ matrix, show that $T(x) = Ax$ is a linear transformation from R^n to R^m where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$$

Sol: Given transformation is

$$T(x) = Ax$$

Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \& \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in R^n$$

Then we prove

$$T(x+y) = T(x) + T(y)$$

Now

$T(x+y) =$

Q5 Determine which of the following linear transformations are one-to-one:

(i) $T: R^2 \longrightarrow R^3$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_1 + 2x_2)$

Sol. Given linear transformation is

$$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_1 + 2x_2)$$

Let $x = (x_1, x_2)$

& $y = (y_1, y_2) \in R^2$

Then $T(x) = (x_1 + x_2, x_1 - x_2, x_1 + 2x_2)$

& $T(y) = (y_1 + y_2, y_1 - y_2, y_1 + 2y_2)$

Suppose $T(x) = T(y)$

$\Rightarrow (x_1 + x_2, x_1 - x_2, x_1 + 2x_2) = (y_1 + y_2, y_1 - y_2, y_1 + 2y_2)$

$$\Rightarrow \quad x_1 + x_2 = y_1 + y_2 \quad \text{——— (1)}$$

$$x_1 - x_2 = y_1 - y_2 \quad \text{——— (2)}$$

$$x_1 + 2x_2 = y_1 + 2y_2 \quad \text{——— (3)}$$

Adding (1) & (2)

$$2x_1 = 2y_1$$

$$\boxed{x_1 = y_1}$$

Put in (1)

$$x_1 + x_2 = x_1 + y_2$$

$$\Rightarrow \boxed{x_2 = y_2}$$

Hence $(x_1, x_2) = (y_1, y_2)$

or $x = y$

Hence $T(x) = T(y) \Rightarrow x = y$

Hence T is one-to-one

(ii) $T: R^3 \longrightarrow R^2$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_3)$

Sol. Given linear transformation is

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_3)$$

Let $x = (x_1, x_2, x_3)$

& $y = (y_1, y_2, y_3) \in R^3$

then $T(x) = (x_1 - x_2, x_3)$

& $T(y) = (y_1 - y_2, y_3)$

Suppose $T(x) = T(y)$

$\Rightarrow (x_1 - x_2, x_3) = (y_1 - y_2, y_3)$

$\Rightarrow \quad x_1 - x_2 = y_1 - y_2$ ——— (1)

$x_3 = y_3$ ——— (2)

From (1) we cannot conclude that

$x_1 = y_1$ & $x_2 = y_2$

Hence $T(x) = T(y) \not\Rightarrow x = y$

So T is not one-to-one.

(iii) $T: R^2 \longrightarrow R^3$ defined by $T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$

Sol. Given linear transformation is

$T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$

Let $x = (x_1, x_2)$

& $y = (y_1, y_2) \in R^2$

then $T(x) = (x_1, x_1 + x_2, x_1 - x_2)$

& $T(y) = (y_1, y_1 + y_2, y_1 - y_2)$

Suppose $T(x) = T(y)$

$\Rightarrow (x_1, x_1 + x_2, x_1 - x_2) = (y_1, y_1 + y_2, y_1 - y_2)$

or $\quad x_1 = y_1$ ——— (1)

$x_1 + x_2 = y_1 + y_2$ ——— (2)

$x_1 - x_2 = y_1 - y_2$ ——— (3)

(1) $\Rightarrow \boxed{x_1 = y_1}$

Subt. (2) & (3)

$2x_2 = 2y_2 \Rightarrow \boxed{x_2 = y_2}$

Hence $(x_1, x_2) = (y_1, y_2)$

or $x = y$

So $T(x) = T(y) \Rightarrow x = y$

Hence T is one-to-one.

Q7 Let C be the vector space of Complex number over the field of reals & $T : C \longrightarrow C$ be defined by $T(z) = \bar{z}$ where $\bar{z}$ denotes the Complex Conjugate of z. Show that T is linear.

Sol. Given transformation is

$$T(z) = \bar{z}$$

Let $z_1, z_2 \in C$ then we prove

(i) $T(z_1 + z_2) = T(z_1) + T(z_2)$

Now

$$\begin{aligned} T(z_1+z_2) &= \overline{z_1+z_2} \\ &= \bar{z}_1 + \bar{z}_2 \\ &= T(z_1) + T(z_2) \end{aligned}$$

(ii) Let $a \in R$ & $z_1 \in C$ then we prove

$T(az_1) = aT(z_1)$

Now

$$\begin{aligned} T(az_1) &= \overline{az_1} \\ &= a\bar{z}_1 \qquad \because a \in R \\ &= aT(z_1) \end{aligned}$$

Hence T is a linear transformation from C to C.

Q8 Let V be the vector space $P_n(x)$ of polynomials $p(x)$ with real cofficients & of degree not exceeding n together with the zero polynomial. Let $T : V \longrightarrow V$

be defined by $T(p(x)) = p(x+1)$

Show that T is linear.

Sol. Given transformation is

$$T(p(x)) = p(x+1)$$

Let $p_1(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

& $p_2(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \in V$

then we prove $T(p_1(x) + p_2(x)) = T(p_1(x)) + T(p_2(x))$

Now

$$T(p_1(x)+p_2(x)) = T\left((a_0+a_1x+\cdots+a_nx^n)+(b_0+b_1x+\cdots+b_nx^n)\right)$$
$$= T\left((a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n\right)$$
$$= (a_0+b_0)+(a_1+b_1)(x+1)+\cdots+(a_n+b_n)(x+1)^n$$
$$= \left[a_0+a_1(x+1)+\cdots+a_n(x+1)^n\right]+\left[b_0+b_1(x+1)+\cdots+b_n(x+1)^n\right]$$
$$= T(a_0+a_1x+\cdots+a_nx^n) + T(b_0+b_1x+\cdots+b_nx^n)$$
$$= T(p_1(x)) + T(p_2(x))$$

(ii) Let $a \in R$ & $p_1(x) = a_0 + a_1 x + \cdots + a_n x^n$

then we prove $T(ap_1(x)) = aT(p_1(x))$

Now

$$T(ap_1(x)) = T\left(a(a_0+a_1x+\cdots+a_nx^n)\right)$$
$$= T(aa_0+aa_1x+\cdots+aa_nx^n)$$
$$= aa_0+aa_1(x+1)+\cdots+aa_n(x+1)^n$$
$$= a\left(a_0+a_1(x+1)+\cdots+a_n(x+1)^n\right)$$
$$= aT(a_0+a_1x+\cdots+a_nx^n)$$
$$= aT(p_1(x))$$

Hence T is a linear transformation from V to V.

Q9 Let $V_1 = (1,1,1)$, $V_2 = (1,1,0)$ & $V_3 = (1,0,0)$ be a basis for R^3. Find a linear transformation $T: R^3 \longrightarrow R^2$ s.t. $T(V_1) = (1,0)$, $T(V_2) = (2,-1)$ & $T(V_3) = (4,3)$

Sol. Let $x = (x_1, x_2, x_3)$ be any vector of R^3 then for scalars a_1, a_2, a_3

$$x = a_1V_1 + a_2V_2 + a_3V_3$$

$$= a_1(1,1,1) + a_2(1,1,0) + a_3(1,0,0)$$

$$(x_1, x_2, x_3) = (a_1 + a_2 + a_3,\ a_1 + a_2,\ a_1)$$

$$\Rightarrow \quad a_1 + a_2 + a_3 = x_1 \quad \text{——— (1)}$$

$$a_1 + a_2 = x_2 \quad \text{——— (2)}$$

$$a_1 = x_3 \quad \text{——— (3)}$$

(3) $\Rightarrow$ $\boxed{a_1 = x_3}$

Put in (2)

$$x_3 + a_2 = x_2 \Rightarrow \boxed{a_2 = x_2 - x_3}$$

Put in (1)

$$\cancel{x_3} + x_2 - \cancel{x_3} + a_3 = x_1$$

$$\boxed{a_3 = x_1 - x_2}$$

So

$$x = x_3V_1 + (x_2 - x_3)V_2 + (x_1 - x_2)V_3$$

Applying T on both sides

$$T(x) = T(x_3V_1 + (x_2 - x_3)V_2 + (x_1 - x_2)V_3$$

$$= x_3T(V_1) + (x_2 - x_3)T(V_2) + (x_1 - x_2)T(V_3) \quad \text{(As T is linear)}$$

$$= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$$

$$= (x_3, 0) + (2x_2 - 2x_3,\ -x_2 + x_3) + (4x_1 - 4x_2,\ 3x_1 - 3x_2)$$

$$= (x_3 + 2x_2 - 2x_3 + 4x_1 - 4x_2,\ -x_2 + x_3 + 3x_1 - 3x_2)$$

$$T(x_1, x_2, x_3) = (4x_1 - 2x_2 - x_3,\ 3x_1 - 4x_2 + x_3)$$

Which is req. linear transformation from R^3 to R^2

Q10 Let $T: R^2 \to R$ be the linear transformation for which $T(1,1) = 3$ & $T(0,1) = -2$. Find $T(x_1, x_2)$

Sol. First we prove that the vectors $(1,1)$ & $(0,1)$ form a basis for R^2.

Suppose for scalars $a, b \in R$

$$a(1,1)+b(0,1) = 0$$
$$(a,a)+(0,b) = 0$$
$$(a, a+b) = 0$$
$$\Rightarrow \quad a = 0 \quad \text{——— (1)}$$
$$a+b = 0 \quad \text{——— (2)}$$

(1) $\Rightarrow$ $\boxed{a=0}$

(2) $\Rightarrow$ $\boxed{b=0}$

Hence vectors $(1,1)$ & $(0,1)$ are linearly independent.

As there are two linearly independent vectors in R^2

So $(1,1)$ & $(0,1)$ form a basis for R^2

Suppose $(x_1,x_2) \in R^2$ be an arbitrary vector

then $(x_1,x_2) = a(1,1) + b(0,1)$ where $a, b \in R$

or $(x_1,x_2) = (a, a+b)$

$$\Rightarrow \quad a = x_1 \quad \text{——— (1)}$$
$$a+b = x_2 \quad \text{——— (2)}$$

(1) $\Rightarrow$ $\boxed{a = x_1}$

Put in (2)

$x_1 + b = x_2 \Rightarrow \boxed{b = x_2 - x_1}$

So

$$(x_1,x_2) = x_1(1,1) + (x_2-x_1)(0,1)$$

Applying T on both sides

$$T(x_1,x_2) = T\big(x_1(1,1)+(x_2-x_1)(0,1)\big)$$
$$= x_1T(1,1) + (x_2-x_1)T(0,1) \quad (\because T \text{ is linear})$$
$$= x_1(3) + (x_2-x_1)(-2)$$
$$= 3x_1 - 2x_2 + 2x_1$$

$T(x_1,x_2) = 5x_1 - 2x_2$ which is T in terms of co-ords.

Q11 Let $D: P_2(x) \longrightarrow P_2(x)$ be the differentiation operator & $D(p(x)) = p'(x)$ for all $p(x) \in P_2(x)$. Find $N(D)$.

Sol. Given operator is.

$D(p(x)) = p'(x)$

Here $N(D)$ will consist of those polynomials in $P_2(x)$ for which $D(p(x)) = 0$

Since we know that

$$D(p(x)) = 0 \quad \text{if} \quad p(x) = \text{Constt polynomial}$$

So $N(D)$ will consist of all Constt. polynomials.

Q12 Define $T: R^3 \longrightarrow R^3$ by $T(x_1, x_2, x_3) = (-x_3, x_1, x_1 + x_3)$. Find $N(T)$. Is T one-to-one?

Sol. Given transformation is

$T(x_1, x_2, x_3) = (-x_3, x_1, x_1 + x_3)$

Here $N(T) = \{(x_1, x_2, x_3) \in R^3 : T(x_1, x_2, x_3) = (0, 0, 0)\}$

Now $T(x_1, x_2, x_3) = (0, 0, 0)$

$\Rightarrow (-x_3, x_1, x_1 + x_3) = (0, 0, 0)$

$$\Rightarrow \quad -x_3 = 0 \quad \text{——— (1)}$$
$$x_1 = 0 \quad \text{——— (2)}$$
$$x_1 + x_3 = 0 \quad \text{——— (3)}$$

(1) $\Rightarrow x_3 = 0$

(2) $\Rightarrow x_1 = 0$

Which shows that $N(T)$ will consist of all vectors of the form $(0, x_2, 0)$. Which is x_2-axis.

i.e., $N(T) = \{(0, x_2, 0) \in R^3 : x_2 \in R\}$

Since $N(T) = (0, x_2, 0) \neq (0, 0, 0)$. So T is not one-to-one.

Q13 Suppose U, V & W are vector spaces over the same field F. Let $T: U \to V$ & $S: V \to W$ be linear transformations. The transformation $S \circ T : U \to W$ is defined by $(S \circ T)(u) = S(T(u))$ for all $u \in U$. Show that $S \circ T$ is a linear transformation.

Sol:

Here $S \circ T : U \to W$ be defined as

$$(S \circ T)(u) = S(T(u)) \quad \text{for all } u \in U$$

Let $u_1, u_2 \in U$ then we prove

$$(S \circ T)(u_1 + u_2) = (S \circ T)(u_1) + (S \circ T)(u_2)$$

Now

$$\begin{aligned}
(S \circ T)(u_1 + u_2) &= S(T(u_1 + u_2)) && \text{By def. of } S \circ T \\
&= S(T(u_1) + T(u_2)) && \because T \text{ is linear} \\
&= S(T(u_1)) + S(T(u_2)) && \because S \text{ is linear} \\
&= (S \circ T)(u_1) + (S \circ T)(u_2)
\end{aligned}$$

(ii) Let $a \in F$ & $u \in U$ then we prove

$$(S \circ T)(au) = a(S \circ T)(u)$$

Now

$$\begin{aligned}
(S \circ T)(au) &= S(T(au)) && \text{By def of } S \circ T \\
&= S(aT(u)) && \because T \text{ is linear} \\
&= aS(T(u)) && \because S \text{ is linear} \\
&= a(S \circ T)(u)
\end{aligned}$$

Hence $S \circ T$ is a linear transformation from U to W.

Q14 Let U & V be two vector spaces over the same field F. Denote the set of all linear transformations from U into V by $L(U,V)$. Show that $L(U,V)$ is a vector space over F with vector space operations as defined in example 31

Sol: Consider the set $L(U,V)$. Let $S,T \in L(U,V)$ then $S: U \longrightarrow V$ & $T: U \longrightarrow V$ be two linear transformations. Define

$S+T : U \longrightarrow V$ & $aS : U \longrightarrow V$ by

$$(S+T)(u) = S(u) + T(u)$$

$$(aS)(u) = aS(u) \qquad \text{for all } u \in U \text{ \& } a \in F$$

First we show that $L(U,V)$ is an abelian gr. under +.

(i) Closure law

Let $S,T \in L(U,V)$, then we show $S+T \in L(U,V)$.

Now $(S+T)(u_1+u_2) = S(u_1+u_2) + T(u_1+u_2)$ By def. of $S+T$

$= S(u_1) + S(u_2) + T(u_1) + T(u_2)$ $\because$ S,T are linear

$= S(u_1) + T(u_1) + S(u_2) + T(u_2)$

$= (S+T)(u_1) + (S+T)(u_2)$

Let $K \in F$ & $u \in U$

$(S+T)(Ku) = S(Ku) + T(Ku)$

$= KS(u) + KT(u)$ $\because$ S,T are linear

$= K\big(S(u) + T(u)\big)$

$= K(S+T)(u)$

Hence $S+T$ is linear & so $S+T \in L(U,V)$.

(ii) Associative law

Let $R,S,T \in L(U,V)$ then we prove

$$R + (S+T) = (R+S) + T$$

Now Consider for $u \in U$

$$[R+(S+T)](u) = R(u) + (S+T)(u) \qquad \text{(By def. of sum)}$$
$$= R(u) + [S(u) + T(u)] \qquad \text{" " "}$$
$$= [R(u) + S(u)] + T(u) \qquad \because R(u), S(u), T(u) \in F$$
$$= [(R+S)(u)] + T(u)$$
$$= [(R+S)(u) + T(u)]$$
$$= [(R+S)+T](u)$$

$\Rightarrow \quad R+(S+T) = (R+S)+T$

So $+$ is associative in $L(U,V)$.

(iii) Identity law

Clearly the zero transformation $\underline{O}$ defined by

$$\underline{O}(u) = 0 \qquad \text{for all } u \in U$$

is a linear transformation from U to V & it is the additive identity in $L(U,V)$

(iv) Inverse law

For each $T \in L(U,V)$, we define

$-T \in L(U,V)$ by

$$(-T)(u) = -T(u)$$

Then $-T$ is the additive inverse of T.

(v) Commutative law

Let $S, T \in L(U,V)$ then we show $S+T = T+S$

Now Consider

$$(S+T)(u) = S(u) + T(u) \qquad \text{By def. of sum}$$
$$= T(u) + S(u) \qquad \because S(u), T(u) \in F$$
$$= (T+S)(u)$$

$\Rightarrow S+T = T+S$

Hence $+$ is Commutative in $L(U,V)$

So $L(U,V)$ is an abelian gr. under $+$.

Now we check scalar multiplication axioms.

(i) Let $a \in F$ & $S \in L(U,V)$ then we prove $aS \in L(U,V)$.

Now $(aS)(u_1+u_2) = a[S(u_1+u_2)]$

$= a[S(u_1)+S(u_2)]$ $\because S$ is linear

$= aS(u_1) + aS(u_2)$

Suppose $K \in F$ & $u \in U$ then

$(aS)(Ku) = a[S(Ku)]$

$= a[KS(u)]$ $\because S$ is linear

$= (aK)S(u)$

$= (Ka)S(u)$

$= K(aS)(u)$

Hence aS is linear & so $aS \in L(U,V)$.

(ii) Let $a,b \in F$ & $S \in L(U,V)$ then we prove $a(bS) = (ab)S$

Now $[a(bS)](u) = a.(bS)(u)$

$= a[b.S(u)]$

$= (ab).S(u)$

$= [(ab)S](u)$

$\Rightarrow a(bS) = (ab)S$

(iii) Let $a,b \in F$ & $S \in L(U,V)$ then we prove $(a+b)S = aS+bS$

Now $[(a+b)S](u) = (a+b).S(u)$

$= a.S(u) + b.S(u)$

$= (aS)(u) + (bS)(u)$

$= [aS+bS](u)$

(iv) Let $a \in F$ & $S,T \in L(U,V)$ then we prove $a(S+T) = aS+aT$

Now $[a(S+T)](u) = a[(S+T)(u)]$

$= a[S(u)+T(u)]$

$= a.S(u) + a.T(u)$

$= (aS)(u) + (aT)(u)$

So $[\alpha(S+T)](u) = [\alpha S + \alpha T](u)$

$\Rightarrow \quad \alpha(S+T) = \alpha S + \alpha T$

(v) Let $1 \in F$ & $S \in L(U,V)$ then we prove $1.S = S$

Now $(1.S)(u) = 1.S(u)$

$= S(u)$

$\Rightarrow \quad 1.S = S$

Since all the Conditions are satisfied. So $L(U,V)$ is a vector space over F.

Q15 Find a basis & dimension of each of $R(T)$ & $N(T)$, where

(i) $T: R^3 \to R^3$ is defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3,\ x_2 + x_3,\ x_1 + x_2 - 2x_3)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3,\ x_2 + x_3,\ x_1 + x_2 - 2x_3)$$

Since R^3 is generated by $(1,0,0), (0,1,0)$ & $(0,0,1)$. So $R(T)$ will be generated by $T(1,0,0), T(0,1,0)$ & $T(0,0,1)$

Here $T(1,0,0) = (1,0,1)$

$T(0,1,0) = (2,1,1)$

& $T(0,0,1) = (-1,1,-2)$

Hence $R(T)$ is generated by $(1,0,1), (2,1,2)$ & $(-1,1,-2)$

Since $(2,1,1) = 3(1,0,1) + 1(-1,1,-2)$

So casting out the vector $(2,1,1)$, the set $\{(1,0,1), (-1,1,-2)\}$ also spans $R(T)$. Since none of the two vectors is a multiple of other, so the set $\{(1,0,1), (-1,1,-2)\}$ is linearly independent & so forms a basis for $R(T)$.

Hence $\dim R(T) = 2$

Now we find $\dim N(T)$.

A vector $(x_1, x_2, x_3) \in N(T)$ if $T(x_1, x_2, x_3) = 0$

i.e., if $(x_1 + 2x_2 - x_3,\ x_2 + x_3,\ x_1 + x_2 - 2x_3) = (0,0,0)$

$$\Rightarrow \quad x_1 + 2x_2 - x_3 = 0 \quad \text{——— (1)}$$

$$x_2 + x_3 = 0 \quad \text{——— (2)}$$

$$x_1 + x_2 - 2x_3 = 0 \quad \text{——— (3)}$$

Adding (1) & (2)

$$x_1 + 3x_2 = 0$$

or $\boxed{x_1 = -3x_2}$

Put in (3)

$$-3x_2 + x_2 - 2x_3 = 0$$

$$-2x_2 - 2x_3 = 0$$

$$x_2 + x_3 = 0$$

or $\boxed{x_3 = -x_2}$

If $x_2 = 1$

then $x_1 = -3$, $x_2 = 1$, $x_3 = -1$

So the vector $(-3, 1, -1)$ spans $N(T)$. Also $(-3, 1, -1)$ is linearly independent. So $\{(-3, 1, -1)\}$ forms a basis for $N(T)$.

Hence $\dim N(T) = 1$

(ii) $T : R^3 \longrightarrow R^4$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_3, 4x_1 + x_2, x_1 + x_3, x_3 - 4x_2)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (2x_1 + x_3, 4x_1 + x_2, x_1 + x_3, x_3 - 4x_2)$$

Since R^3 is generated by $(1,0,0), (0,1,0)$ & $(0,0,1)$. So $R(T)$ will be generated by $T(1,0,0), T(0,1,0)$ & $T(0,0,1)$

Here
$$T(1,0,0) = (2,4,1,0)$$
$$T(0,1,0) = (0,1,0,-4)$$
$$T(0,0,1) = (1,0,1,1)$$

Hence $R(T)$ will be generated by $(2,4,1,0), (0,1,0,-4), (1,0,1,1)$

Now we check whether these vectors are linearly independent. For this let

$a(2,4,1,0) + b(0,1,0,-4) + c(1,0,1,1) = (0,0,0,0)$ where $a, b, c \in F$

or $(2a + c, 4a + b, a + c, -4b + c) = (0,0,0,0)$

$$\Rightarrow \quad 2a + c = 0 \quad \text{——— (1)}$$
$$4a + b = 0 \quad \text{——— (2)}$$
$$a + c = 0 \quad \text{——— (3)}$$
$$-4b + c = 0 \quad \text{——— (4)}$$

(1) − (3) $\Rightarrow \boxed{a = 0}$

(3) $\Rightarrow 0 + c = 0 \Rightarrow \boxed{c = 0}$

(2) $\Rightarrow 0 + b = 0 \Rightarrow \boxed{b = 0}$

Hence vectors $(2,4,1,0), (0,1,0,-4)$ & $(1,0,1,1)$ are linearly independent. Hence $\{(2,4,1,0), (0,1,0,-4), (1,0,1,1)\}$ form a basis for $R(T)$

Hence $\dim R(T) = 3$

Q16 Show that linear transformations preserve linear dependence.

Sol. Let $T: U \to V$ be a linear transformation, where U & V are vector spaces over the same field F. Suppose a set $\{u_1, u_2, \ldots, u_n\}$ in U is linearly dependent we want to show that $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ is a linearly dependent set in V.

Since $\{u_1, u_2, \ldots, u_n\}$ is linearly dependent, so there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$$

Applying T on both sides

$$T(a_1u_1 + a_2u_2 + \cdots + a_nu_n) = T(0)$$

or $a_1T(u_1) + a_2T(u_2) + \cdots + a_nT(u_n) = 0 \quad (\because T \text{ is linear})$

Since $a_1, a_2, \ldots, a_n$ are not all zero, so the above eq. shows that $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ are linearly dependent in V. Hence T preserve linear dependence.

Q17 Find the rank of each matrix in problem 8 of exercise 3.2 by the method of 6.42

(i) $\begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$

Sol:-

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$ then

$$\text{rank } A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} -2-0 \\ -1-15 \\ 3+6 \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} -2 \\ -16 \\ 9 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix} -2 & 0 \\ -16 & 0 \end{vmatrix} \\ \begin{vmatrix} -2 & 0 \\ 9 & 0 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rank $A = 2$

(ii) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$

Sol:-

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$ then

$$\text{rank } A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -1 & -2 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} -3 & 6 \\ 3 & -3 \\ 6 & -5 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix} -3 & 6 \\ 3 & -3 \end{vmatrix} \\ \begin{vmatrix} -3 & 6 \\ 6 & -5 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 9-18 \\ 15-36 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} -9 \\ -21 \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} \begin{vmatrix} -9 & 0 \\ -21 & 0 \end{vmatrix} \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} 0 \end{bmatrix}$$

$$= 3$$

(iii) $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

Sol.

Let $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$ then

$$\text{rank } A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 1 & -4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -7 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 2 & -3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -7 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 3 & -8 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} 1 & 2 & 1 & -1 \\ -3 & -6 & -3 & 3 \\ -1 & -2 & -1 & 1 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ -3 & -6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 2 + 0$$

$$= 2$$

(iv) $\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$

Sol.

Let $A = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$ then

$$\text{rank } A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix}1 & 3\\1 & 4\end{vmatrix} & \begin{vmatrix}1 & -2\\1 & 1\end{vmatrix} & \begin{vmatrix}1 & 5\\1 & 3\end{vmatrix} & \begin{vmatrix}1 & 4\\1 & 5\end{vmatrix} \\ \begin{vmatrix}1 & 3\\1 & 4\end{vmatrix} & \begin{vmatrix}1 & -2\\1 & 2\end{vmatrix} & \begin{vmatrix}1 & 5\\1 & 4\end{vmatrix} & \begin{vmatrix}1 & 4\\1 & 3\end{vmatrix} \\ \begin{vmatrix}1 & 3\\2 & 7\end{vmatrix} & \begin{vmatrix}1 & -2\\2 & -3\end{vmatrix} & \begin{vmatrix}1 & 5\\2 & 6\end{vmatrix} & \begin{vmatrix}1 & 4\\2 & 13\end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & 4 & -1 & -1 \\ 1 & 1 & -4 & 5 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix}1 & 3\\1 & 4\end{vmatrix} & \begin{vmatrix}1 & -2\\1 & -1\end{vmatrix} & \begin{vmatrix}1 & 1\\1 & -1\end{vmatrix} \\ \begin{vmatrix}1 & 3\\1 & 1\end{vmatrix} & \begin{vmatrix}1 & -2\\1 & -4\end{vmatrix} & \begin{vmatrix}1 & 1\\1 & 5\end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} \begin{vmatrix}1 & 1\\-2 & -2\end{vmatrix} & \begin{vmatrix}1 & -2\\-2 & 4\end{vmatrix} \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= 3 + 0$$

$$\text{rank } A = 3$$

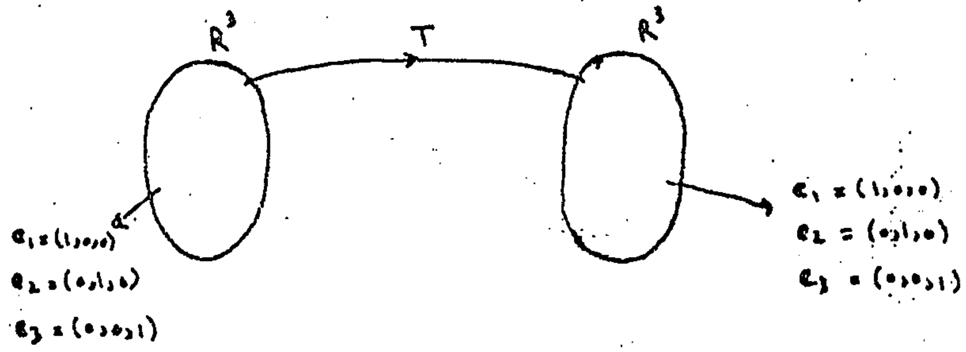
Exercise No. 6.4

Q1 Find the matrix of each of the following linear transformations from R^3 to R^3 with respect to the standard basis for R^3:

(i) $T(x_1, x_2, x_3) = (x_1, x_2, 0)$

Sol:- Given linear transformation is

$$T(x_1, x_2, x_3) = (x_1, x_2, 0)$$



Now

$$T(1,0,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$

$$T(0,1,0) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$$

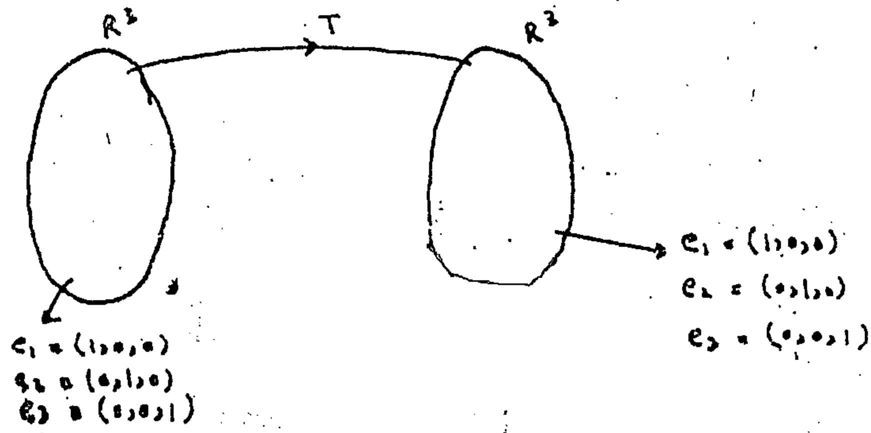
$$T(0,0,1) = (0,0,0) = 0(1,0,0) + 0(0,1,0) + 0(0,0,1)$$

Hence matrix of linear transformation T is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

Sol:-



Here $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

then. $T(1,0,0) = (1,-1,0) = 1(1,0,0) + (-1)(0,1,0) + 0(0,0,1)$

$T(0,1,0) = (1,-1,0) = 1(1,0,0) + (-1)(0,1,0) + 0(0,0,1)$

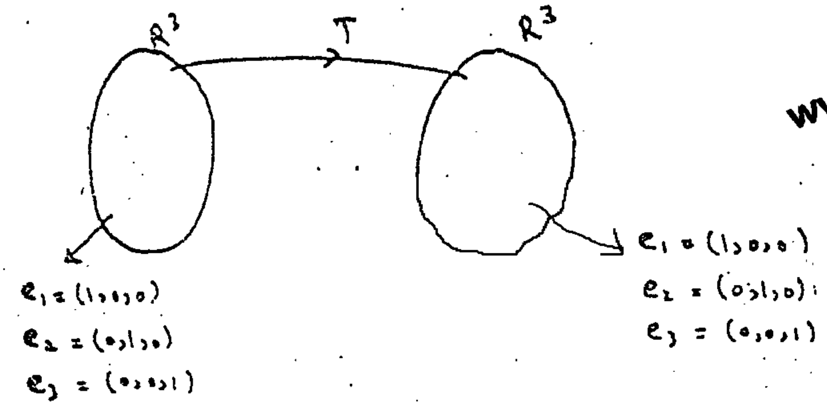
$T(0,0,1) = (0,0,1) = 0(1,0,0) + 0(0,1,0) + 1(0,0,1)$

Hence matrix of linear transformation T is

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii) $T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$

Sol.



Here $T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$

then $T(1,0,0) = (0,-1,0) = 0(1,0,0) + (-1)(0,1,0) + 0(0,0,1)$

$T(0,1,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

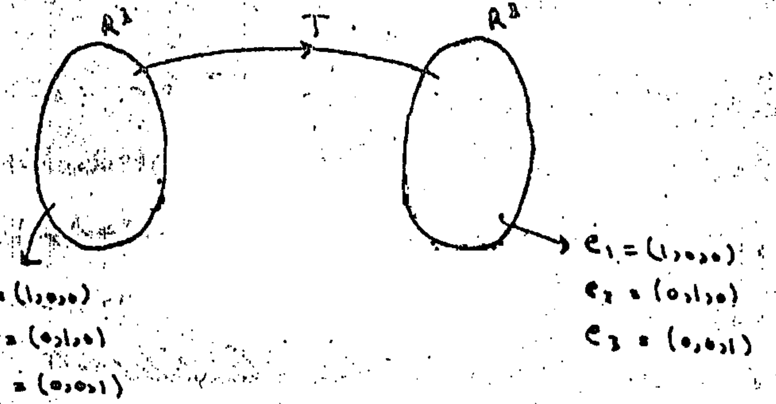
$T(0,0,1) = (0,0,-1) = 0(1,0,0) + 0(0,1,0) + (-1)(0,0,1)$

Hence matrix of linear transformation T is

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(iv) $T(x_1, x_2, x_3) = (x_1, x_2 + x_3, x_1 + x_2 + x_3)$

Sol.



Here $T(x_1, x_2, x_3) = (x_1, x_2 + x_3, x_1 + x_2 + x_3)$

then $T(1,0,0) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$T(0,1,0) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$

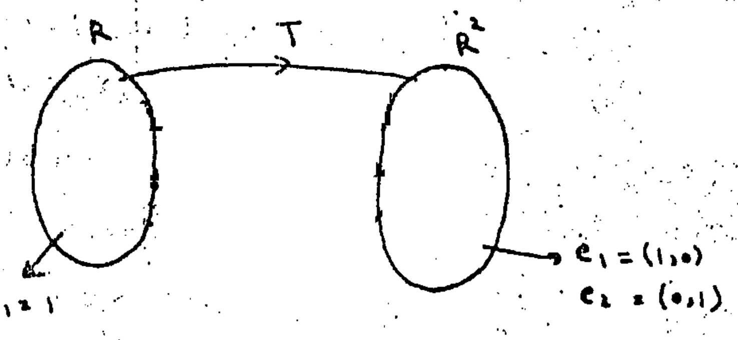
Hence matrix of linear transformation T is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q2 Find the matrix of each of the following linear transformations with respect to the standard bases of the given spaces:

(i) $T : R \longrightarrow R^2$ defined by $T(x) = (3x, 5x)$

Sol.



Here $T(x) = (3x, 5x)$

then $T(1) = (3,5) = 3(1,0) + 5(0,1)$

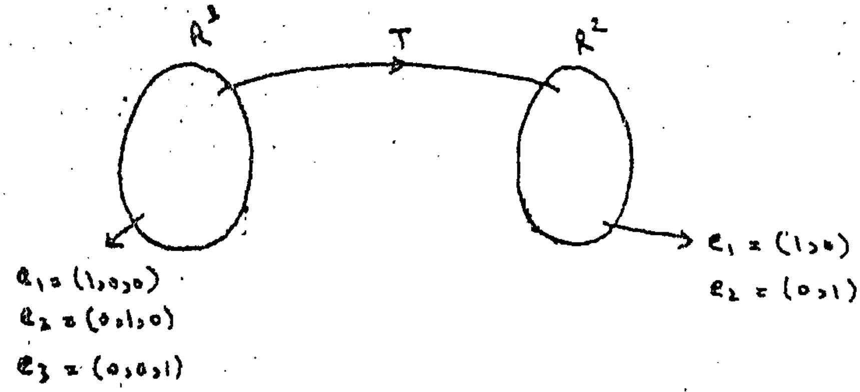
Hence matrix of linear transformation T is

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(ii) $T : R^3 \longrightarrow R^2$ defined by

$$T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3,\ 5x_1 + 3x_2 - 2x_3)$$

Sol.



Here $T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3,\ 5x_1 + 3x_2 - 2x_3)$

Then $T(1,0,0) = (3,5) = 3(1,0) + 5(0,1)$

$T(0,1,0) = (-4,3) = -4(1,0) + 3(0,1)$

$T(0,0,1) = (9,-5) = 9(1,0) + (-5)(0,1)$

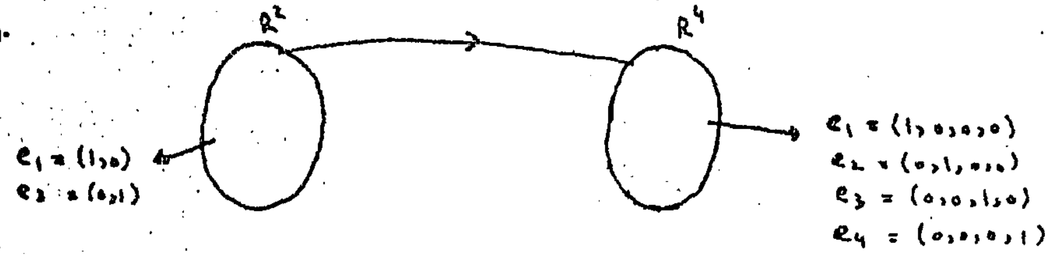
Hence matrix of linear transformation T is

$$\begin{bmatrix} 3 & -4 & 9 \\ 5 & 3 & -5 \end{bmatrix}$$

(iii) $T : R^2 \longrightarrow R^4$ defined by

$$T(x_1, x_2) = (3x_1 + 4x_2,\ 5x_1 - 2x_2,\ x_1 + 7x_2,\ 4x_1)$$

Sol.



Here $T(x_1, x_2) = (3x_1 + 4x_2,\ 5x_1 - 2x_2,\ x_1 + 7x_2,\ 4x_1)$

Then $T(1,0) = (3,5,1,4) = 3(1,0,0,0) + 5(0,1,0,0) + 1(0,0,1,0) + 4(0,0,0,1)$

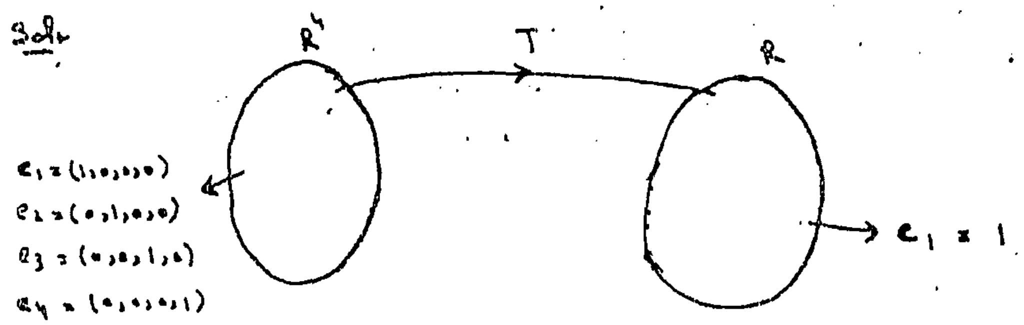
$T(0,1) = (4,-2,7,0) = 4(1,0,0,0) - 2(0,1,0,0) + 7(0,0,1,0) + 0(0,0,0,1)$

Hence matrix of linear transformation T is $\begin{bmatrix} 3 & 4 \\ 5 & -2 \\ 1 & 7 \\ 4 & 0 \end{bmatrix}$

(iv) $T : R^4 \longrightarrow R$ defined by

$T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 - 7x_3 + x_4$

Sol:



Here $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 - 7x_3 + x_4$

then
$$T(1,0,0,0) = 2 = 2 \cdot 1$$
$$T(0,1,0,0) = 3 = 3 \cdot 1$$
$$T(0,0,1,0) = -7 = -7 \cdot 1$$
$$T(0,0,0,1) = 1 = 1 \cdot 1$$

Hence matrix of linear transformation T is

$$\begin{bmatrix} 2 & 3 & -7 & 1 \end{bmatrix}$$

Q3 Each of the following is the matrix of a linear transformation $T : R^n \longrightarrow R^m$. Determine m, n & express T in terms of Co-ordinates.

(i)
$$\begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Sol: Given linear transformation is

$T : R^n \longrightarrow R^m$

Here n = no. of Columns = 5

& m = no. of rows = 3

Now

let $x = (x_1, x_2, x_3, x_4, x_5) \in R^5$

then T is defined as

$$= \begin{bmatrix} 3 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 3x_1 + x_2 + 0x_3 + 2x_4 + x_5 \\ x_1 + 0x_2 + 0x_3 + x_4 + x_5 \\ 0x_1 - x_2 + x_3 + x_4 + x_5 \end{bmatrix}$$

or

$$T(x_1, x_2, x_3, x_4, x_5) = (3x_1 + x_2 + 2x_4 + x_5,\ x_1 + x_4 + x_5,\ -x_2 + x_3 + x_4 + x_5)$$

which is T in terms of Co-ordinates.

(ii) $\begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Sol. Given linear transformation is

$$T : R^n \longrightarrow R^m$$

Here n = no. of Columns = 2

& m = no. of rows = 3

Now let $x = (x_1, x_2) \in R^2$

then T is defined as

$$T(x) = Ax$$

$$= \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or $T(x) = \begin{bmatrix} 6x_1 - x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix}$

or $T(x_1, x_2) = (6x_1 - x_2, x_1 + 2x_2, x_1 + 3x_3)$
which is T in terms of Co-ords.

(iii) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}$

Sol. Given linear transformation is
$T : R^n \longrightarrow R^m$

Here n = no. of Columns = 3
& m = no. of rows = 3
Let $x = (x_1, x_2, x_3) \in R^3$ then
T is defined as

$$T(x) = Ax$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 5x_2 + 6x_3 \\ -2x_1 + 3x_2 - x_3 \end{bmatrix}$$

or

$$T(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3, 2x_1 + 5x_2 + 6x_3, -2x_1 + 3x_2 - x_3)$$
Which is T in terms of Co-ords.

Q4 The matrix of a linear transformation
$T : R^3 \longrightarrow R^3$ is

$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ Find T in terms of Co-ords.
& its matrix w.r.t. the basis
$V_1 = (0, 1, 2)$, $V_2 = (1, 1, 1)$, $V_3 = (1, 0, -2)$.

Sol. Given linear transformation is

$T : R^3 \longrightarrow R^3$

Let $x = (x_1, x_2, x_3) \in R^3$ then T is defined as

$T(x) = Ax$

or

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

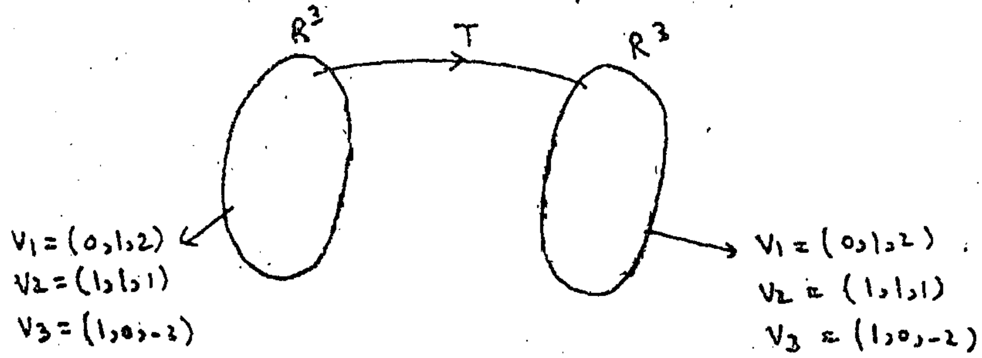
$$T(x) = \begin{bmatrix} 0x_1 + x_2 + x_3 \\ x_1 + 0x_2 - x_3 \\ -x_1 - x_2 + 0x_3 \end{bmatrix}$$

or

$T(x_1, x_2, x_3) = (x_2 + x_3, x_1 - x_3, -x_1 - x_2)$

which is T in terms of co-ords.

Now we find matrix of T



Here $T(x_1, x_2, x_3) = (x_2 + x_3, x_1 - x_3, -x_1 - x_2)$

then $T(0,1,2) = (3,-2,-1) = \overset{-11}{a}(0,1,2) + \overset{9}{b}(1,1,1) + \overset{-6}{c}(1,0,-2)$

$(3,-2,-1) = (b+c, a+b, 2a+b-2c)$

$\Rightarrow$ $b + c = 3$ ——— ①

$a + b = -2$ ——— ②

$2a + b - 2c = -1$ ——— ③

① − ② $\Rightarrow$ $c - a = 5$ ——— Ⓐ

② − ③ $\Rightarrow$ $-a + 2c = -1$ ——— Ⓑ

Subt. Ⓑ from Ⓐ

$-c = 6$

or $\boxed{c = -6}$

Put in ①

$b - 6 = 3$

or $\boxed{b = 9}$

Put in ②

$a + 9 = -2$

$\boxed{a = -11}$

Now

$$T(1,1,1) = (2,0,-2) = \overset{-2}{a}(0,1,2) + \overset{2}{b}(1,1,1) + \overset{0}{c}(1,0,-2)$$
$$= (b+c,\ a+b,\ 2a+b-2c)$$

$\Rightarrow$ $b + c = 2$ ——— ①

$a + b = 0$ ——— ②

$2a + b - 2c = -2$ ——— ③

① − ② $\Rightarrow$ $c - a = 2$

$-a + 2c = 2$

Sub. we get

$-c = 0 \Rightarrow \boxed{c = 0}$

Put in ①

$b + 0 = 2 \Rightarrow \boxed{b = 2}$

Put in ②

$a + 2 = 0 \Rightarrow \boxed{a = -2}$

Now

$$T(1,0,-2) = (-2,3,-1) = \overset{14}{a}(0,1,2) + \overset{-11}{b}(1,1,1) + \overset{9}{c}(1,0,-2)$$
$$= (b+c,\ a+b,\ 2a+b-2c)$$

$\Rightarrow$ $b + c = -2$ ——— ①

$a + b = 3$ ——— ②

$2a + b - 2c = -1$ ——— ③

① − ② $\Rightarrow$

Adding ① & ②

$2a = 0 \Rightarrow \boxed{a = 0}$

Put in ①

$0 - b = 0 \Rightarrow \boxed{b = 0}$

Hence vectors $(1,1)$ & $(-1,1)$ are linearly independent & since they are two in number, so they form a basis for R^2.

Let (x_1, x_2) be an arbitrary vector of R^2

Then (x_1, x_2) can be expressed as

$$(x_1, x_2) = a(1,1) + b(-1,1)$$

$$= (a-b, a+b)$$

$\Rightarrow$

$$a - b = x_1 \quad \text{———— Ⓐ}$$

$$a + b = x_2 \quad \text{———— Ⓑ}$$

$$2a = x_1 + x_2$$

or $\boxed{a = \dfrac{x_1 + x_2}{2}}$

Put in ①

$$\frac{x_1 + x_2}{2} - b = x_1$$

$$b = \frac{x_1 + x_2}{2} - x_1$$

$$= \frac{x_1 + x_2 - 2x_1}{2}$$

$$\boxed{b = \frac{x_2 - x_1}{2}}$$

So $(x_1, x_2) = \left(\dfrac{x_1 + x_2}{2}\right)(1,1) + \left(\dfrac{x_2 - x_1}{2}\right)(-1,1)$

Applying T on both sides

$$T(x_1, x_2) = T\left\{\left(\frac{x_1 + x_2}{2}\right)(1,1) + \left(\frac{x_2 - x_1}{2}\right)(-1,1)\right\}$$

$$= \left(\frac{x_1 + x_2}{2}\right)T(1,1) + \left(\frac{x_2 - x_1}{2}\right)T(-1,1) \qquad \because T \text{ is linear}$$

$c - a = -5$

$① - ③ \Rightarrow -a + 2c = 4$

Subt. we get

$-c = -9 \Rightarrow \boxed{c = 9}$

Put in ①

$b + 9 = -2 \Rightarrow \boxed{b = -11}$

Put in ②

$a - 11 = 3 \Rightarrow \boxed{a = 14}$

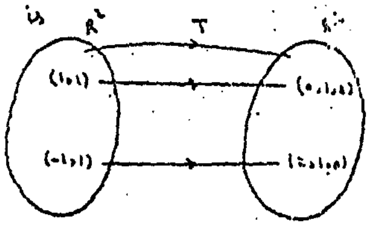
Hence matrix of linear transformation T

$$\begin{bmatrix} -11 & -2 & 14 \\ 9 & 2 & -11 \\ -6 & 0 & 9 \end{bmatrix}$$

Q5 A linear transformation $T : R^2 \longrightarrow R^3$ maps the vector $(1,1)$ into $(0,1,2)$ & the vector $(-1,1)$ into $(2,1,0)$. What matrix does T represent with respect to the standard bases for R^2 & R^3?

Sol. Given linear transformation is

$T : R^2 \longrightarrow R^3$



We are given that

$T(1,1) = (0,1,2)$

& $T(-1,1) = (2,1,0)$

First we check linear independency of vectors $(1,1)$ & $(-1,1)$

suppose for scalars a, b

$a(1,1) + b(-1,1) = 0$

$(a-b, a+b) = (0,0)$

$\Rightarrow a - b = 0$ —— ①

$a + b = 0$ —— ②

$T(x_1,x_2) = \left(\frac{x_1+x_2}{2}\right)(0,1,2) + \left(\frac{x_2-x_1}{2}\right)(2,1,0)$

$= \left(x_2 - x_1, \frac{x_1+x_2}{2} + \frac{x_2-x_1}{2}, x_1+x_2\right)$

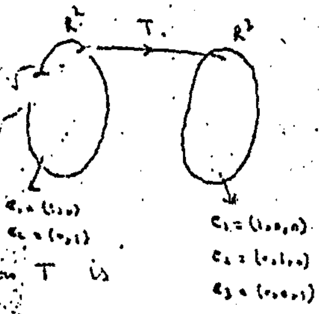
or $T(x_1,x_2) = (x_2 - x_1, x_2, x_1+x_2)$

which is T in terms of Co-ords.

Now as $T(x_1,x_2) = (x_2-x_1, x_2, x_1+x_2)$

then $T(1,0) = (-1,0,1)$.

$= -1(1,0,0) + 0(0,1,0) + 1(0,0,1)$



& $T(0,1) = (1,1,1)$

$= 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

Hence matrix of linear transformation T is

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

GOOD LUCK

Exercise 6.4

Q1 (i) $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $T(x_1, x_2, x_3) = (x_1, x_2, 0)$

Standard basis for $\mathbb{R}^3 = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$

$$T(1,0,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$
$$T(0,1,0) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$$
$$T(0,0,1) = (0,0,0) = 0(1,0,0) + 0(0,1,0) + 0(0,0,1)$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

Standard basis for $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$T(1,0,0) = (1,-1,0) = 1(1,0,0) - 1(0,1,0) + 0(0,0,1)$$
$$T(0,1,0) = (1,-1,0) = 1(1,0,0) - 1(0,1,0) + 0(0,0,1)$$
$$T(0,0,1) = (0,0,1) = 0(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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(iii) $$T(x_1,x_2,x_3) = (x_2, -x_1, -x_3)$$

Standard basis for $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$T(1,0,0) = (0,-1,0) = 0(1,0,0) - 1(0,1,0) + 0(0,0,1)$$

$$T(0,1,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$

$$T(0,0,1) = (0,0,-1) = 0(1,0,0) + 0(0,1,0) - 1(0,0,1)$$

Hence the standard matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(iv) $$T(x_1,x_2,x_3) = (x_1, x_2+x_3, x_1+x_2+x_3)$$

Standard basis for $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$T(1,0,0) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

$$T(0,1,0) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

$$T(0,0,1) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

---

2 (i) $T : \mathbb{R} \to \mathbb{R}^2$ defined by $T(x) = (3x, 5x)$

Standard basis for $\mathbb{R} = \{1\}$

Standard basis for $\mathbb{R}^2 = \{(1,0), (0,1)\}$

$$T(1) = (3, 5) = 3(1,0) + 5(0,1)$$

So the matrix of T is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(ii) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3, 5x_1 + 3x_2 - 2x_3)$

Standard basis for $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

Standard basis for $\mathbb{R}^2 = \{(1,0), (0,1)\}$

$$T(1,0,0) = (3,5) = 3(1,0) + 5(0,1)$$

$$T(0,1,0) = (-4,3) = -4(1,0) + 3(0,1)$$

$$T(0,0,1) = (9,-2) = 9(1,0) - 2(0,1)$$

So the matrix of T is $\begin{bmatrix} 3 & -4 & 9 \\ 5 & 3 & -2 \end{bmatrix}$

(iii) $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$T(x_1, x_2) = (3x_1 + 4x_2, 5x_1 - 2x_2, x_1 + 7x_2, 4x_1).$$

Standard basis for $\mathbb{R}^2 = \{(1,0), (0,1)\}$

Standard basis for $\mathbb{R}^4 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$$T(1,0) = (3,5,1,4) = 3(1,0,0,0) + 5(0,1,0,0) + (0,0,1,0) + 4(0,0,0,1)$$

$$T(0,1) = (4,-2,7,0) = 4(1,0,0,0) - 2(0,1,0,0) + 7(0,0,1,0) + 0(0,0,0,1)$$

So tha matrix of T is $\begin{bmatrix} 3 & 4 \\ 5 & -2 \\ 1 & 7 \\ 4 & 0 \end{bmatrix}$.

(iv) $T: \mathbb{R}^4 \to \mathbb{R}$ defined by

$$T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 - 7x_3 + x_4$$

Standard basis for $\mathbb{R}^4 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

Standard basis for $\mathbb{R} = \{1\}$

$$T(1,0,0,0) = 2 = 2(1)$$

$$T(0,1,0,0) = 3 = 3(1)$$

$$T(0,0,1,0) = -7 = -7(1)$$

$$T(0,0,0,1) = 1 = 1(1)$$

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So the matrix of T is $\begin{bmatrix} 2 & 3 & -7 & 1 \end{bmatrix}$.

Qn 3 (i) Let $A = \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Since matrix A is of order 3×2, so $m = 3$, $n = 2$.

i.e. $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 - x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

or $$T(x_1, x_2) = (6x_1 - x_2, x_1 + 2x_2, x_1 + 3x_2).$$

(ii) Let $$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}$$

Since matrix A is of order 3×3, so $m = 3$, $n = 3$.

i.e., $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 5x_2 + 6x_3 \\ -2x_1 + 3x_2 - x_3 \end{bmatrix}$$

or $$T(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3,\ 2x_1 + 5x_2 + 6x_3,\ -2x_1 + 3x_2 - x_3)$$

(iii) Let $$A = \begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Since matrix A is of order 3×5, so $m = 3$, $n = 5$.

i.e., $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ is defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 + 2x_4 \\ x_1 + x_4 + x_5 \\ -x_2 + x_3 + x_4 + x_5 \end{bmatrix}$$

or

$$T(x_1, x_2, x_3, x_4, x_5) = (3x_1 + x_2 + 2x_4,\ x_1 + x_4 + x_5,\ -x_2 + x_3 + x_4 + x_5).$$

Q4. The matrix of $$T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

T in terms of coordinates is

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_1 - x_3 \\ -x_1 - x_2 \end{bmatrix}$$

or

$$T(x_1, x_2, x_3) = (x_2 + x_3,\ x_1 - x_3,\ -x_1 - x_2)$$

Given basis for $\mathbb{R}^3 = \{v_1 = (0,1,2),\ v_2 = (1,1,1),\ v_3 = (1,0,-2)\}$

$T(v_1) = T(0,1,2) = (3,-2,-1)$

$$(3,-2,-1) = a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$(3,-2,-1) = a_1(0,1,2) + a_2(1,1,1) + a_3(3,-2,-1)$$

$$(3,-2,-1) = (a_2 + a_3\ ,\ a_1 + a_2\ ,\ 2a_1 + a_2 - 2a_3)$$

$a_2 + a_3 = 3$, $a_1 + a_2 = -2$, $2a_1 + a_2 - 2a_3 = -1$

$a_2 = 3 - a_3$, $a_1 + (3 - a_3) = -2$, $2(-5 + a_3) + (3 - a_3) - 2a_3 = -1$

$a_1 = -2 - 3 + a_3$, $-10 + 2a_3 + 3 - a_3 - 2a_3 = -1$

$a_1 = -5 + a_3$, $-a_3 - 7 = -1$

$-a_3 = 6$

$\boxed{a_3 = -6}$

$a_2 = 3 - (-6)$, $a_1 = -5 - 6$

$\boxed{a_2 = 9}$, $\boxed{a_1 = -11}$

So $\quad T(v_1) = -11 v_1 + 9v_2 - 6v_3 \quad - (1)$

Similarly $\quad T(v_2) = -2v_1 + 2v_2 + 0v_3 \quad - (2)$

and $\quad T(v_3) = 14v_1 - 11v_2 + 9v_3 \quad - (3)$

Hence the matrix of T w.r.t. new basis is

$$\begin{bmatrix} -11 & -2 & 14 \\ 9 & 2 & -11 \\ -6 & 0 & 9 \end{bmatrix}$$

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(5) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$T(1,1) = (0,1,2)$ and $T(-1,1) = (2,1,0)$

Standard basis for $\mathbb{R}^2 = \{(1,0), (0,1)\}$

Standard basis for $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

Since $(1,1) = 1(1,0) + 1(0,1)$

and $(-1,1) = -1(1,0) + 1(0,1)$

So $(0,1,2) = T(1,1) = T(1,0) + T(0,1)$ — (1)

and $(2,1,0) = T(-1,1) = -T(1,0) + T(0,1)$ — (2)

Subtract (2) from (1), we get

$(-2,0,2) = 2T(1,0) + 0$

$\Rightarrow \quad T(1,0) = (-1,0,1) = -1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

Add (1) and (2) to get

$(2,2,2) = 2T(0,1)$

$\Rightarrow \quad T(0,1) = (1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

⑥ Q. $T: M_{23} \rightarrow M_{32}$ defined by $T(A) = A^T \in M_{32}$.

Standard basis for $M_{23} = \left\{ \overset{v_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}, \overset{v_5}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}, \overset{v_6}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \right\}$

Standard basis for $M_{32} = \left\{ w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, w_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, w_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, w_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$T(v_1) = w_2 = 0w_1 + w_2 + 0w_3 + 0w_4 + 0w_5 + 0w_6$$

$$T(v_2) = w_4 = 0w_1 + 0w_2 + 0w_3 + w_4 + 0w_5 + 0w_6$$

$$T(v_3) = w_6 = 0w_1 + 0w_2 + 0w_3 + 0w_4 + 0w_5 + w_6$$

$$T(v_4) = w_1 = w_1 + 0w_2 + 0w_3 + 0w_4 + 0w_5 + 0w_6$$

$$T(v_5) = w_3 = 0w_1 + 0w_2 + w_3 + 0w_4 + 0w_5 + 0w_6$$

$$T(v_6) = w_5 = 0w_1 + 0w_2 + 0w_3 + 0w_4 + w_5 + 0w_6$$

Hence the matrix of T w.r.t. standard basis is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

If we change the order of basis elements then there will be different matrix of T.

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(IPS) INNER PRODUCT SPACES

Inner product spaces are simply vector spaces over the field 'F' of real or complex numbers and with an Inner Product defined on them.

Def:

Let V be the vectorspace over the field F of real or complex numbers.

A mapping (function) $\langle\, ,\, \rangle : V\times V \to F$ is said to be an INNER PRODUCT (IP) on V if the following conditions are satisfied:

i) $\langle v, v\rangle \geq 0$

& $\langle v, v\rangle = 0$ iff $v=0$ $\forall\, v\in V$

ii) $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}$ $\forall\, v_1, v_2 \in V$

where $\overline{\langle v_2, v_1\rangle}$ is complex conjugate of $\langle v_2, v_1\rangle$

iii) $\langle av_1+bv_2, v_3\rangle = a\langle v_1, v_3\rangle + b\langle v_2, v_3\rangle$ $\forall\, v_1, v_2, v_3\in V$ $a, b\in F$

The pair $(V, \langle\, ,\rangle)$ is called Inner Product Space (IPS) where V is a vectorspace over the field 'F' of real or complex numbers and $\langle\, ,\rangle$ is an inner product on V.

Note If F is taken as the field of real numbers then condition (ii) becomes $\langle v_1, v_2\rangle = \langle v_2, v_1\rangle$

$\because$ if z is real then $\bar{z} = z$
$\therefore$ $\overline{\langle v_2, v_1\rangle} = \langle v_2, v_1\rangle$ only in real numbers.

We shall consider innerproduct space over R only

using cond ii
So condition (iii) becomes

$$\langle v_3, av_1+bv_2\rangle = \langle av_1+bv_2, v_3\rangle \quad \text{using cond(ii)}$$

$$= a\langle v_1, v_3\rangle + b\langle v_2, v_3\rangle \quad \text{by (iii)}$$

$$\langle v_3, av_1+bv_2\rangle = a\langle v_3, v_1\rangle + b\langle v_3, v_2\rangle \quad \text{by (iii)}$$

Example 1 Let $u, v \in R^n$ where $u = (u_1, u_2, \cdots u_n)$, $v = (v_1, v_2 - - - v_n)$

then the dot product $\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots u_nv_n$ is an inner product or R^n, verify.

Sol We show that $\langle u, v\rangle$ satisfies the three conditions.

c_1: $\langle u, u\rangle = u_1u_1 + u_2u_2 + \cdots u_nu_n$

$= u_1^2 + u_2^2 + \cdots u_n^2 > 0$ $\because$ sum of squares

Let $\langle u, u\rangle = 0$

$\Leftrightarrow u_1^2 + u_2^2 - \cdots + u_n^2 = 0$

$\Leftrightarrow$ each $u_i^2 = 0$, $i = 1, 2 \cdots n$

$\Leftrightarrow$ each $u_i = 0$

So $u = (0, 0, \cdots 0) = 0$

Hence $\langle u, u\rangle \geq 0$

c_2: $\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots u_nv_n$

$= v_1u_1 + v_2u_2 + \cdots v_nu_n$ $\because u_iv_i \in R$

$\langle u, v\rangle = \langle v, u\rangle$

c_3: $\langle au + bv, w\rangle$

$w = (w_1, w_2, \cdots w_n) \in R^n$

$(au + bv) = a(u_1, u_2, \cdots u_n) + b(v_1, v_2 - - v_n) = au_1 + bv_1, au_2 + bv_2, \cdots, au_n + bv_n$

$= (au_1 + bv_1)w_1 + (au_2 + bv_2)w_2 + \cdots (au_n + bv_n)w_n$ (as defined)

$= au_1w_1 + bv_1w_1 + au_2w_2 + bv_2w_2 + \cdots au_nw_n + bv_nw_n$

$= a(u_1w_1 + u_2w_2 + \cdots u_nw_n) + b(v_1w_1 + v_2w_2 \cdots v_nw_n)$

$= a\langle u, w\rangle + b\langle v, w\rangle$

Hence $\langle u, v\rangle$ is an Inner Product on R^n.

Note $(R^n, \langle u, v\rangle)$ is called Euclidean Inner Product Space on R^n

where $\langle u, v\rangle$ = dot product

$= u_1v_1 + u_2v_2 \cdots u_nv_n$

Example 2 Let V be the vector space of all $n\times 1$ matrices over R (Such matrices are called Column Vectors)

$c_1, c_2 \in V$ where $c_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $c_2 = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$c_1^t = [x_1, x_2 \cdots x_n]$ $\quad c_2^t = [y_1, y_2 \cdots y_n]$

Show that $\langle c_1, c_2\rangle$ is IP, where $\langle c_1, c_2\rangle = \det(c_1^t c_2) = |c_1^t c_2|$

Sol

$$c_1^t c_2 = [x_1, x_2 \cdots x_n]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$1\times(n)$
$(n)\times 1$
$= (x)$
$=$ element

$$c_1^t c_2 = [(x_1y_1 + x_2y_2 + \cdots x_ny_n)]$$

$$|c_1^t c_2| = |(x_1y_1 + x_2y_2 + \cdots x_ny_n)| = x_1y_1 + x_2y_2 \cdots x_ny_n \quad \text{--- (1)}$$

c_1: $\langle c_1, c_1\rangle = |c_1^t c_1|$

$$= x_1x_1 + x_2x_2 + \cdots x_nx_n \quad = x_1^2 + x_2^2 + \cdots x_n^2 > 0$$

Let $\langle c_1, c_1\rangle = 0$

$\Leftrightarrow x_1^2 + x_2^2 + \cdots x_n^2 = 0$

$\Leftrightarrow$ each $x_i = 0 \quad i = 1, 2, \cdots n.$

$\Leftrightarrow c_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \qquad \therefore \langle c_1, c_1\rangle = 0 \Rightarrow c_1 = 0$

c_2: $\langle c_1, c_2\rangle = |c_1^t c_2| = x_1y_1 + x_2y_2 + \cdots x_ny_n$

$$= y_1x_1 + y_2x_2 + \cdots y_nx_n$$

$$= |c_2^t c_1|$$

$$= \langle c_2, c_1\rangle$$

c_3: $\langle ac_1 + bc_2, c_3\rangle = |(ac_1 + bc_2)^t c_3|$

Let $a, b \in R$ & $c_3 = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

$ac_1 + bc_2 = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ \vdots \\ ax_n + by_n \end{bmatrix}$

$$= |(ax_1 + by_1)z_1 + (ax_2 + by_2)z_2 + \cdots (ax_n + by_n)z_n|$$

$$= (ax_1 + by_1)z_1 + (ax_2 + by_2)z_2 + \cdots (ax_n + by_n)z_n$$

$$= a(x_1z_1 + x_2z_2 + \cdots x_nz_n) + b(y_1z_1 + y_2z_2 + \cdots y_nz_n)$$

$$= a\,|c_1^t c_3| + b\,|c_2^t c_3|$$

$$= a\langle c_1, c_3\rangle + b\langle c_2, c_3\rangle$$

Hence $\langle c_1, c_2\rangle$ is an I.P on V

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Example 3 Let $u, v \in R^2$, $u = (x_1, x_2)$ $v = (y_1, y_2)$

Then show that $\langle u, v\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ is an IP on R^2.

Sol (i) $\langle u, u\rangle = x_1x_1 - x_1x_2 - x_2x_1 + 3x_2x_2$

$= x_1^2 - 2x_1x_2 + 3x_2^2$

$= x_1^2 - 2x_1x_2 + x_2^2 + 2x_2^2$

$= (x_1 - x_2)^2 + 2x_2^2 \geq 0$

Let $\langle u, u\rangle = 0$

$\Leftrightarrow (x_1 - x_2)^2 + 2x_2^2 = 0$

$\Leftrightarrow (x_1 - x_2)^2 \;\&\; 2x_2^2 = 0$

$\Leftrightarrow (x_1 - x_2) = 0 \;\&\; \sqrt{2}x_2 = 0$

$\Leftrightarrow x_1 = x_2 \;\&\; x_2 = 0$

$\Leftrightarrow x_1 = 0, \;\&\; x_2 = 0$

$\Leftrightarrow u = 0$

(ii) $\langle u, v\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$

$= y_1x_1 - y_2x_1 - y_1x_2 + 3y_2x_2$

$= y_1x_1 - y_1x_2 - y_2x_1 + 3y_2x_2$

$= \langle v, u\rangle \quad \forall\, u, v \in R^2$

(iii) $\langle au + bv, w\rangle = \langle (ax_1 + by_1, ax_2 + by_2), (z_1, z_2)\rangle$

Let $w = (z_1, z_2)$

$\because au + bv$
$= a(x_1, x_2) + b(y_1, y_2)$
$= (ax_1 + by_1, ax_2 + by_2)$

$= (ax_1 + by_1)z_1 - (ax_1 + by_1)z_2 - (ax_2 + by_2)z_1 + 3(ax_2 + by_2)z_2$

$= ax_1z_1 + by_1z_1 - ax_1z_2 - by_1z_2 - ax_2z_1 - by_2z_1 + 3ax_2z_2 + 3by_2z_2$

$= a(x_1z_1 - x_1z_2 - x_2z_1 + 3x_2z_2) + b(y_1z_1 - y_1z_2 - y_2z_1 + 3y_2z_2)$

$= a\langle u, w\rangle + b\langle v, w\rangle$

Thus all the three conditions are satisfied

So $(R^2, \langle \cdot, \cdot \rangle)$ is an IPS.

Norm (or Length) of a vector.

Let V be an IPS and $v \in V$, then the real number $\sqrt{\langle v, v\rangle}$ is called the norm of v, it is denoted by $\|v\|$.

$$\therefore \|v\| = \sqrt{\langle v, v\rangle} \quad \& \quad \|v\|^2 = \langle v, v\rangle$$

If $\|v\| = 1$

i.e $\sqrt{\langle v, v\rangle} = 1$

i.e $\langle v, v\rangle = 1$ then v is called Unit vector or Normalized Vector.

Any non-zero vector $u \in V$ can be normalized by multiplying it with $\frac{1}{\|u\|}$

$\therefore$ $\frac{u}{\|u\|}$ is a unit vector i.e vector 'u' has been Normalized.

Example 5 Find the norm of $v = (3,4) \in R^2$ with respect to the Euclidean inner product and the inner product defined in Example 3.

Sol Euclidean inner product on R^n is defined as

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots u_nv_n$$

where $u = (u_1, u_2, \cdots u_n) \in R^n$
$v = (v_1, v_2, \cdots v_n) \in R^n$

$$\|v\| = \sqrt{\langle v, v\rangle}$$
$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2} \quad \text{on } R^n$$
$$\|v\| = \sqrt{v_1^2 + v_2^2} \quad \text{on } R^2$$
$$= \sqrt{9+16}$$
$$= 5$$

$v = (3,4) \in R^2$
$\therefore v_1 = 3, \; v_2 = 4$

ii) $$\langle u, v\rangle = \langle (u_1, u_2), (v_1, v_2)\rangle$$
$$= u_1v_1 - u_1v_2 - u_2v_1 + 3u_2v_2$$ as defined in Example 3
$$\|v\| = \sqrt{\langle v, v\rangle}$$
$$= \sqrt{v_1v_1 - v_1v_2 - v_2v_1 + 3v_2v_2}$$
$$= \sqrt{3\cdot3 - 3\cdot4 - 4\cdot3 + 3\cdot4\cdot4} = \sqrt{9-12-12+48} = \sqrt{33} \text{ Ans.}$$

Example 4 Let V be the vectorspace of all real-valued continuous functions on the interval $a \le t \le b$, then for $f, g \in V$

$$\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt$$ is an inner product on V.

Sol (i) $\langle f, f\rangle = \int_a^b f(t) f(t)\, dt$

$= \int_a^b f^2(t)\, dt \geq 0$ $\because$ the definite integral gives the area bounded by the curve, which is always +ve.

Let $\langle f, f\rangle = 0$

$\Leftrightarrow \int_a^b f^2(t)\, dt = 0$

$\Leftrightarrow f^2(t) = 0$

$\Leftrightarrow f(t) = 0$

(ii) $\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt$

$= \int_a^b g(t)\, f(t)\, dt$

$= \langle g, f\rangle$

(iii) $\langle a_1 f + b_1 g, h\rangle = \int_a^b (a_1 f(t) + b_1 g(t))\, h(t)\, dt$

$= \int_a^b (a_1 f(t) h(t) + b_1 g(t) h(t))\, dt$

$= a_1 \int_a^b f(t)\, h(t)\, dt + b_1 \int_a^b g(t) h(t)\, dt$

$= a_1 \langle f, h\rangle + b_1 \langle g, h\rangle \quad \forall a_1, b_1 \in R$
$\forall f, g, h \in V$

Thus all the three conditions of an I.P are satisfied

So $(V, \langle \cdot, \cdot \rangle)$ is an IPS

Example 6

Normalize $v = (1,2,1) \in R^3$ w.r.t Euclidean I.P

Sol $\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ $\quad \because \|v\| = \sqrt{\langle v, v\rangle}$

$= \sqrt{1^2 + 2^2 + 1^2}$

$= \sqrt{6}$

Normalized Vector $= \dfrac{v}{\|v\|} = \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}$

Th (v. imp) The Cauchy-Schwarz Inequality

Let u, v be the elements of an inner product space V or R

then $|\langle u,v\rangle| \leq \|u\| \, \|v\|$

Proof If $v = 0$ then

LHS $\langle u,v\rangle = \langle u,0\rangle \quad \because v=0$

$= \langle u, 0\cdot w\rangle \quad \forall w \in V$

$= 0\langle u, w\rangle$ by cond(iii)

$= 0$

RHS $\|u\|\|v\| = \|u\|\|0\|$

$= 0 \quad \therefore$ LHS $=$ RHS.

If $v \neq 0$ then for all real 't' $\in R$

$0 \leq \|u - tv\|^2$ $\quad a=1, b=-t$

$= \langle u-tv, u-tv\rangle$

$= \langle u, u-tv\rangle - t\langle v, u-tv\rangle$ by cond iii

$= \langle u,u\rangle - t\langle u,v\rangle - t\langle v,u\rangle + t^2\langle v,v\rangle$ by cond iii

$= \|u\|^2 - 2t\langle u,v\rangle + t^2\|v\|^2$

For real t: $t\cdot\bar{t} = |t|^2$, $t^2 = |t|^2$

Let $t = \dfrac{\langle u,v\rangle}{\langle v,v\rangle}$, $\Rightarrow t^2 = \dfrac{|\langle u,v\rangle|^2}{\|v\|^4}$ $\quad \because (\langle u,v\rangle)(\langle u,v\rangle) = |\langle u,v\rangle|^2$

$0 \leq \|u\|^2 - 2\dfrac{\langle u,v\rangle}{\langle v,v\rangle}\langle u,v\rangle + \dfrac{|\langle u,v\rangle|^2}{\|v\|^4}\|v\|^2$

$0 \leq \|u\|^2 - 2\dfrac{|\langle u,v\rangle|^2}{\|v\|^2} + \dfrac{|\langle u,v\rangle|^2}{\|v\|^2}$

by LCM $0 \leq \|u\|^2\|v\|^2 - |\langle u,v\rangle|^2 \xrightarrow{\text{Taking Square Root}} |\langle u,v\rangle| \leq \|u\|\|v\|$

proved.

<u>Theorem</u> The norm in an inner product space V satisfies the following axioms

(i) $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$, $v \in V$

(ii) $\|Kv\| = |K|\|v\|$ for all $v \in V$ and $K \in R$

(iii) $\|u+v\| \leq \|u\| + \|v\|$ for all $u, v \in V$

<u>Proof</u> (i) $\|v\| = \sqrt{\langle v, v\rangle}$

$\geq 0 \quad \because \langle v, v\rangle \geq 0$ by cond(i)

$\therefore \|v\| \geq 0$

Further $\|v\| = 0 \iff \sqrt{\langle v, v\rangle} = 0$

$\iff \langle v, v\rangle = 0$

$\iff v = 0$

(ii) $\|Kv\|^2 = \langle Kv, Kv\rangle \quad K \in R$

$= K\langle v, Kv\rangle$

$= K^2\langle v, v\rangle$

$= K^2\|v\|^2 \quad \because \sqrt{\langle v, v\rangle} = \|v\|$

$= |K|^2 \|v\|^2 \quad \because K \in R$

(iii) $\|u+v\|^2 = \langle u+v, u+v\rangle$

$= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \qquad z+\bar{z} = 2\operatorname{Re} z$

$= \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \qquad \because$ cond ii $\because \langle u, v\rangle = \langle v, u\rangle$

$\leq \|u\|^2 + 2|\langle u, v\rangle| + \|v\|^2 \qquad (\operatorname{Re} z \leq |z|)$

$\leq \|u\|^2 + 2\|u\|\cdot\|v\| + \|v\|^2$ By Cauchy Schwarz $\qquad \langle u, v\rangle \leq |\langle u, v\rangle|$

$= (\|u\| + \|v\|)^2$

$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$

$\|u+v\| \leq \|u\| + \|v\|$

ORTHOGONALITY

Let 'θ' be the Angle between two vectors $u, v \in V$ (IPS), then

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|} \quad 0 \le \theta \le \pi$$

∵ By Cauchy Schwarz

$|\langle u,v\rangle| \le \|u\|\|v\|$

$\frac{|\langle u,v\rangle|}{\|u\|\|v\|} \le \frac{\|u\|\|v\|}{\|u\|\|v\|}$

$\frac{|\langle u,v\rangle|}{\|u\|\|v\|} \le 1$

If $\theta = 90$

$$\cos(90) = \frac{\langle u, v\rangle}{\|u\|\|v\|}$$

$$0 = \frac{\langle u, v\rangle}{\|u\|\|v\|}$$

$$0 = \langle u, v\rangle \quad ∴$$

∴ u & v are orthogonal iff $\langle u, v\rangle = 0$
($u \perp v$, read u is orthogonal to v)

Note
1) If $u \perp v \Rightarrow \langle u, v\rangle = 0$
If $v \perp u \Rightarrow \langle v, u\rangle = 0$

∴ $\langle u, v\rangle = \langle v, u\rangle$
Hence relation of orthogonality is symmetric.

2) The vector '0' is orthogonal to every $v \in V$

∵ $\langle 0, v\rangle = \langle 0v, v\rangle = 0\cdot\langle v, v\rangle = 0$ by cond iii

3) If u is orthogonal to itself then $u = 0$

∵ $\langle u, u\rangle = 0$ by def of orthogonality

$\|u\|^2 = 0 \Rightarrow \|u\| = 0 \Rightarrow u = 0$

Example 7 Show that $x \perp y$ where $x, y \in R^3$
$x = (1, -1, 2)$ $y = (-1, 1, 1)$

Sol $\langle x, y\rangle = (1)(-1) + (-1)(1) + (2)(1)$
$= -1 - 1 + 2$
$\langle x, y\rangle = 0$ So $x \perp y$

Show that $x \perp y$ where $x, y \in R^4$
$x = (1, -1, 1, -1)$. $y = (-1, 2, 2, -1)$.
$\langle x, y\rangle = (1)(-1) + (-1)(2) + (1)(2) + (-1)(-1)$
$= -1 - 2 + 2 + 1$
$= 0$ So $x \perp y$.

<u>Example 8</u> If u is orthogonal to v then every scalar multiple of u is also orthogonal to v. (Ku is multiple of u)

<u>Sol</u> If $u \perp v \Rightarrow \langle u, v\rangle = 0$

then $\langle Ku, v\rangle = K\langle u, v\rangle$ by cond iii

$= K(0)$

$\langle Ku, v\rangle = 0$

Hence $Ku \perp v$

<u>Example 9</u> Find a unit vector orthogonal to both $(1,1,2)$ and $(0,1,3)$ in R^3.

<u>Sol</u> Let $(x,y,z) \in R^3$ be a vector orthogonal to given vectors

$\therefore \langle (x,y,z), (1,1,2)\rangle = 0$ $\because (x,y,z) \perp (1,1,2)$

$x(1)+y(1)+z(2) = 0$ ——— (i)

Also $\langle (x,y,z), (0,1,3)\rangle = 0$ $\because (x,y,z) \perp (0,1,3)$

$x(0)+y(1)+z(3) = 0$ ——— (ii)

using (ii) in (i) $y = -3z$

$x - 3z + 2z = 0$

$x - z = 0$

$x = z$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -3z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$\Rightarrow (1,-3,1)$ is $\perp$ to both u & v

$\|(1,-3,1)\| = \sqrt{1^2+3^2+1^2} = \sqrt{11}$

Unit Vector $= \left(\frac{1}{\sqrt{11}}, \frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$

2nd Method

w = vector $\perp$ u & v $= \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$ $e_1 = (1,0,0) = i$, $e_2 = (0,1,0) = j$, $e_3 = (0,0,1) = k$

$w = e_1(3-2) - e_2(3-0) + e_3(1-0)$

$= e_1 - 3e_2 + e_3 = 1, -3, 1$

Unit vector $= \frac{w}{\|w\|} = \frac{1}{\sqrt{11}}(e_1 - 3e_2 + e_3) = \frac{1}{\sqrt{11}}e_1, \frac{-3}{\sqrt{11}}e_2, \frac{1}{\sqrt{11}}e_3$

Orthogonal Complement:

Let 'W' be a subset of an Inner Product Space 'V' over R. The orthogonal complement of 'W' denoted by $W^{\perp}$ and read as 'W perp', consists of those vectors in V which are orthogonal to every $w \in W$. Thus

$$W^{\perp} = \{v \in V : \langle v, w\rangle = 0 \quad \forall\, w \in W\}$$

Prove that $W^{\perp}$ is subspace of V.

Let $u, v \in W^{\perp}$ and $a, b \in R$ then for $w \in W$ $\quad \langle u, w\rangle = 0$, $\langle v, w\rangle = 0 \quad \because u, v \in W^{\perp}$

$$\text{Since } \langle au+bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$$
$$= a\cdot 0 + b\cdot 0$$
$$= 0$$

So $au+bv \in W^{\perp}$. Thus $W^{\perp}$ is a subspace of V.

Orthogonal Systems

A set 'S' of vectors in an I.P. Space 'V' over R is said to be an orthogonal system if its distinct vectors are orthogonal i.e if $\langle u_i, u_j\rangle = 0 \quad \forall\, u_i, u_j \in S \quad i \neq j$

Orthonormal System

A set S of vectors in an I.P. Space 'V' over R is said to be an orthonormal system if

$$\langle u_i, u_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example 10 Show that the system $\{u_1 = (1,-1,1,-1)$
$u_2 = (3,1,-1,1)$
$u_3 = (0,2,1,-1)$
$u_4 = (0,0,1,1)\}$ is an orthonormal system in R^4!

Sol
$$\langle u_1, u_2\rangle = 1\cdot 3 + (1)(-1) + 1(-1) + (-1)(1)$$
$$= 3-1-1-1$$
$$= \boxed{0}$$
$$\langle u_1, u_3\rangle = 1\cdot 0 + (-1)(2) + 1(1) + (-1)(-1)$$
$$= 0-2+1-1$$
$$= \boxed{0}$$
$$\langle u_1, u_4\rangle = 1\cdot 0 + (-1)(0) + 1(1) + (-1)(1)$$
$$= 0+0+1-1$$
$$= \boxed{0}$$
$$\langle u_2, u_3\rangle = (3)0 + (1)(2) + (-1)(1) + 1(-1)$$
$$= 0+2-1-1$$
$$= \boxed{0}$$
$$\langle u_2, u_4\rangle = (3)(0) + 1(0) + (-1)(1) + 1(1)$$
$$= 0+0-1+1$$
$$= \boxed{0}$$
$$\langle u_3, u_4\rangle = (0)(0) + (-1)(0) + (1)(1) + (-1)(1)$$
$$= 0+0+1-1$$
$$= \boxed{0}$$
$$\langle u_1, u_1\rangle = (1)(1) + (-1)(-1) + (1)(1) + (-1)(-1)$$
$$= 1+1+1+1$$
$$= 4 \neq 1$$
$$\langle u_2, u_2\rangle = (3)(3) + (1)(1) + (-1)(-1) + (1)(1)$$
$$= 9+1+1+1$$
$$= 12 \neq 1$$

So this is orthogonal system not orthonormal system.

Def.

Example 11 Let V be the vectorspace of real-valued continuous fns on the interval $-\pi \le t \le \pi$ with inner product defined by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\cdot g(t)\,dt \quad \forall f, g \in V$$

then $\{1, \cos t, \cos 2t, \ldots \sin t, \sin 2t \ldots\}$ is an orthogonal system in V.

Sol

$$\langle 1, \cos mt\rangle = \int_{-\pi}^{\pi} 1\cdot \cos mt\,dt = \left|\frac{\sin mt}{m}\right|_{-\pi}^{\pi} = \frac{1}{m}\left(\sin(m\pi) - \sin(-m\pi)\right) = \frac{1}{m}(0-0) = \boxed{0}$$

$$\langle 1, \sin mt\rangle = \int_{-\pi}^{\pi} 1\cdot \sin mt\,dt = \left|\frac{-\cos mt}{m}\right|_{-\pi}^{\pi} = \frac{1}{m}\left(\cos m\pi + \cos(-m\pi)\right) = \frac{1}{m}\left(\cancel{\cos m\pi} - \cancel{\cos m\pi}\right) = \boxed{0}$$

when $m \ne n$

$$\langle \cos mt, \sin nt\rangle = \int_{-\pi}^{\pi} \cos(mt)\sin(nt)\,dt = \frac{1}{2}\int_{-\pi}^{\pi} 2\cos(mt)\sin(nt)\,dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left\{\sin(m+n)t - \sin(m-n)t\right\}dt \qquad \because 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta) \ \boxed{\text{where } m > n}$$

$$= \frac{1}{2}\left|-\frac{\cos(m+n)t}{(m+n)} - \left(-\frac{\cos(m-n)t}{m-n}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left\{\left(-\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n}\right) - \left(-\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n}\right)\right.$$

$$= \frac{1}{2}\left\{\cancel{-\frac{\cos(m+n)\pi}{m+n}} + \cancel{\frac{\cos(m-n)\pi}{m-n}} + \cancel{\frac{\cos(m+n)(-\pi)}{m+n}} - \cancel{\frac{\cos(m-n)(-\pi)}{m-n}}\right.$$

$$= \boxed{0} \qquad \because \cos(-\pi) = \cos\pi$$

when $m = n$

$$\langle \cos mt, \sin mt\rangle = \int_{-\pi}^{\pi} \cos mt \sin mt\,dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\cos mt\sin mt\,dt = \frac{1}{2}\int_{-\pi}^{\pi}\sin 2mt\,dt \qquad \because 2\cos\theta\sin\theta = \sin 2\theta$$

$$= -\frac{1}{2}\left|\frac{\cos 2mt}{2m}\right|_{-\pi}^{\pi} = \frac{1}{4m}\left(\cancel{\cos 2m\pi} + \cancel{\cos 2m(-\pi)}\right)$$

$$= \boxed{0}$$

PTO

when $m \neq n$

$$\langle \cos mt, \cos nt \rangle = \int_{-\pi}^{\pi} \cos mt \cos nt \, dt$$

(In orthogonal system distinct vectors are orthogonal. So we will not take $\langle \cos mt, \cos mt \rangle$ $\langle \sin mt, \sin mt \rangle$)

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\cos mt \cos nt \, dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)t + \cos(m-n)t \, dt$$

$$= \frac{1}{2}\left| \frac{\sin(m+n)t}{m+n} + \frac{\sin(m-n)t}{m-n} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n} \right]$$

$$= \boxed{0}$$

$\because \sin\pi = 0 = \sin(-\pi)$

when $m \neq n$

$$\langle \sin mt, \sin nt \rangle = \int_{-\pi}^{\pi} \sin mt, \sin nt \, dt$$

$$= -\frac{1}{2}\int_{-\pi}^{\pi} -2\sin mt \sin nt \, dt$$

$$= -\frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)t - \cos(m-n)t \, dt$$

$$= -\frac{1}{2}\left| \frac{\sin(m+n)t}{m+n} - \frac{\sin(m-n)}{m-n} t \right|_{-\pi}^{\pi}$$

$$= -\frac{1}{2}\left(\sin(m+n)\pi - \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} + \frac{\sin(m-n)(-\pi)}{m-n} \right)$$

$$= \boxed{0}$$

Thus the set $\{1, \cos t, \cos 2t, \ldots \sin t, \sin 2t, \ldots\}$ is an orthogonal system in V.

Extra when $m = n$

$$\langle \cos t, \cos nt \rangle = \int_{-\pi}^{\pi} \cos^2 mt \, dt = \int_{-\pi}^{\pi} \left(\frac{1+\cos 2mt}{2}\right) dt = \frac{1}{2}\left| t + \frac{\sin 2mt}{2m} \right|_{-\pi}^{\pi} =$$

$$= \frac{1}{2}\left[\pi + \frac{\sin 2m\pi}{2m} - (-\pi) - \frac{\sin 2m(-\pi)}{2m} \right] = \frac{1}{2}(\pi+\pi) = \boxed{\pi}$$

when $m = n$

$$\langle \sin mt, \sin nt \rangle = \int_{-\pi}^{\pi} \sin^2 mt \, dt = \int_{-\pi}^{\pi} \left(\frac{1-\cos 2mt}{2}\right) dt = \frac{1}{2}\left| t - \frac{\sin 2mt}{2m} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{\sin 2m\pi}{2m} - (-\pi) + \frac{\sin 2m(-\pi)}{2m} \right) = \frac{1}{2}(\pi+\pi) = \boxed{\pi}$$

The Gram-Schmidt Process

V is an I.P. Space over 'R'.

$\{v_1, v_2 \dots v_n\}$ is a basis of $V(R)$.

An orthonormal basis $\{u_1, u_2 \dots u_n\}$ of V can be constructed as,

Step 1 Let $u_1 = v_1 \cdot \frac{1}{\|v_1\|}$

$$\|u_1\| = \|v_1\| \cdot \frac{1}{\|v_1\|} = 1$$

u_1 is written in the linear combination of v_1 & $\|v_1\|$ where $\|v_1\|$ is scalar. Similarly $v_1 = u_1 \|v_1\|$ so $[u_1] = [v_1]$

Step 2 Let $u_2 = \frac{w_2}{\|w_2\|}$ where $w_2 = v_2 - \langle v_2, u_1\rangle u_1$

$$\langle u_1, u_2\rangle = \langle u_1, \frac{w_2}{\|w_2\|}\rangle$$

$$= \frac{1}{\|w_2\|}\langle u_1, w_2\rangle$$

$$= \frac{1}{\|w_2\|}\langle u_1, v_2 - \langle v_2, u_1\rangle u_1\rangle$$

$$= \frac{1}{\|w_2\|}\left(\langle u_1, v_2\rangle - \langle v_2, u_1\rangle\langle u_1, u_1\rangle\right) \quad \text{by cond iii}$$

$$= \frac{1}{\|w_2\|}\left[\langle u_1, v_2\rangle - \langle u_1, v_2\rangle \cdot 1\right]$$

$\because \|u_1\| = \sqrt{\langle u_1, u_1\rangle}$

$1 = \sqrt{\langle u_1, u_1\rangle}$

squaring $1 = \langle u_1, u_1\rangle$

$$\langle u_1, u_2\rangle = 0 \quad \& \quad \|u_2\| = \frac{\|w_2\|}{\|w_2\|} = 1$$

Hence $[u_1, u_2]$ is orthonormal

Hence $[u_1, u_2]$ is O.N. system

Step 3 Let $u_3 = \frac{w_3}{\|w_3\|}$ where $w_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2$

Similarly $[u_1, u_2, u_3]$ is orthonormal.

and so on, then $u_n = \frac{w_n}{\|w_n\|}$ where $w_n = v_n - \langle v_n, u_1\rangle u_1 - \langle v_n, u_2\rangle u_2 \dots$
$\dots - \langle v_n, u_{n-1}\rangle u_{n-1}$

$\therefore$ The Set $\{u_1, u_2, \dots u_n\}$ is orthonormal

$\because$ We know orthonormal is linearly independent and

Span $\{u_1, u_2, \dots u_n\}$ = Span $\{v_1, v_2, \dots v_n\}$

Hence $\{u_1, u_2, \dots u_n\}$ is orthonormal basis of V.

Theorem Every orthonormal system $\{u_1, u_2, \ldots u_n\}$ is linearly independent. Moreover, for all $v \in V$, the vector $w = v - \sum_{k=1}^{n} \langle v, u_k \rangle u_k$ is orthogonal to each u_i, $1 \le i \le n$. i.e $\{u_1, u_2 \ldots u_n\}$

Proof Suppose $a_1 u_1 + a_2 u_2 + \cdots a_n u_n = 0$ where a_i are scalars $a_i \in R$

Taking innerproduct of both sides with 'u_i'

$$\langle a_1 u_1 + a_2 u_2 \cdots + a_n u_n , u_i \rangle = \langle 0, u_i \rangle$$

$$a_1 \langle u_1, u_i \rangle + a_2 \langle u_2, u_i \rangle + \cdots a_{i-1} \langle u_{i-1}, u_i \rangle$$

$$+ a_i \langle u_i, u_i \rangle + a_{i+1} \langle u_{i+1}, u_i \rangle + \cdots a_n \langle u_n, u_i \rangle = 0$$

$\because$ Cond iii $\because 0$ is orthogonal to every $v \in V$ $\therefore \langle 0, v \rangle = 0$

$$0 + 0 + \cdots 0 + a_i \cdot 1 + 0 + \cdots 0 = 0$$

$$a_i \cdot 1 = 0 \qquad \because 1 \neq 0 \quad a_i = 0$$

$$a_i = 0 \qquad \forall i = 1, 2, \ldots n$$

$\because \{u_1, u_2 \ldots u_n\}$ is orthonormal system $\therefore \langle u_i, u_j \rangle = 0$, $i \neq j$ Distinct; $\langle u_i, u_i \rangle = 1$ $i = j$

$\therefore \{u_1, u_2 \ldots u_n\}$ is linearly independent.

Now to prove w is orthogonal to each u_i, $1 \le i \le n$

Consider
$$\langle w, u_i \rangle = \left\langle v - \sum_{k=1}^{n} \langle v, u_k \rangle u_k , u_i \right\rangle$$

$$= \langle v, u_i \rangle - \left\langle \sum_{k=1}^{n} \langle v, u_k \rangle u_k , u_i \right\rangle \qquad \because \text{Cond iii}$$

$$= \langle v, u_i \rangle - \left\langle \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 , \ldots \langle v, u_n \rangle u_n , u_i \right\rangle$$

$$= \langle v, u_i \rangle - (\langle v, u_1 \rangle) \langle u_1, u_i \rangle - (\langle v, u_2 \rangle) \langle u_2, u_i \rangle, \ldots$$
$$- (\langle v, u_{i-1} \rangle) \langle u_{i-1}, u_i \rangle - \langle v, u_i \rangle \langle u_i, u_i \rangle$$
$$- (\langle v, u_{i+1} \rangle) \langle u_{i+1}, u_i \rangle \cdots - (\langle v, u_n \rangle) \langle u_n, u_i \rangle$$

$\because \langle v, u \rangle \in F = R$ $\therefore$ by Cond iii

$$= \langle v, u_i \rangle - 0 - 0 \cdots 0 - \langle v, u_i \rangle \cdot 1 - 0 \cdots 0$$

$\left(\because \langle u_i, u_j \rangle = 0 \right.$ $\langle u_i, u_i \rangle = 1$

$$= \langle v, u_i \rangle - \langle v, u_i \rangle$$

$$\langle w, u_i \rangle = \boxed{0}$$

Hence w is orthogonal to each u_i, $1 \le i \le n$

x————x

Example 12 Show that $\{(1,1,1),(0,1,1),(0,0,1)\}$ is a basis of R^3. Using Gram-Schmidt orthogonalization process, transform this basis into an orthonormal basis.

Sol. Let $(x,y,z) \in R^3$

Suppose $(x,y,z) = a(1,1,1) + b(0,1,1) + c(0,0,1)$ linear combination of a, b, c & given set of vectors

$(x,y,z) = (a,\ a+b,\ a+b+c)$

$\therefore \boxed{a = x}$

$a+b = y \quad \Rightarrow b = y-a \quad \Rightarrow \boxed{b = y-x}$

$a+b+c = z \quad \Rightarrow c = z-a-b \quad \Rightarrow c = z-x-(y-x)$

$c = z-x-y+x$

$\boxed{c = z-y}$

$\because$ the given set of vectors $\{(1,1,1),(0,1,1),(0,0,1)\}$ can be written as a linear combination of a, b, c, as values of a, b, c exist.

$\therefore$ the set $\{(1,1,1),(0,1,1),(0,0,1)\}$ is a spanning set for R^3

Now check Linear Independence of $\{(1,1,1),(0,1,1),(0,0,1)\}$

$a(1,1,1) + b(0,1,1) + c(0,0,1) = 0$

$(a,\ a+b,\ a+b+c) = 0$

$\boxed{a = 0}$

$a+b = 0 \quad \Rightarrow 0+b = 0 \quad \Rightarrow \boxed{b=0}$

$a+b+c = 0 \quad \Rightarrow 0+0+c=0 \quad \Rightarrow \boxed{c=0}$

2nd Method

Since the Matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ Rank of $A = 3$ = No of vectors $= 3$

is in Echelon form

So the given three vectors $(1,1,1),(0,1,1),(0,0,1)$ are Linearly Independent.

Hence given vectors are Lin Independent.

Since the given set of vectors is a Spanning Set for R^3 and is Linearly Independent. Hence

Hence the given set of vectors $\{(1,1,1),(0,1,1),(0,0,1)\}$ is a basis of R^3.

P.T.O.

by Gram Schmidt Orthogonalisation.

Let $v_1 = (1,1,1)$, $v_2 = (0,1,1)$, $v_3 = (0,0,1)$

Now $u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{(1,1,1)}{\sqrt{1^2+1^2+1^2}} = \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$

$u_2 = \dfrac{\omega_2}{\|\omega_2\|}$ where $\omega_2 = v_2 - \langle v_2, u_1\rangle u_1$

$$= (0,1,1) - \left\langle (0,1,1), \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)\right\rangle \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)$$

$$= (0,1,1) - \left(0+\tfrac{1}{\sqrt3}+\tfrac{1}{\sqrt3}\right)\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)$$

$$= (0,1,1) - \tfrac{2}{\sqrt3}\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)$$

$$= (0,1,1) - \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right)$$

$$\omega_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$\|\omega_2\| = \sqrt{\tfrac{4}{9}+\tfrac{1}{9}+\tfrac{1}{9}} = \sqrt{\tfrac{6}{9}} = \sqrt{\tfrac{2}{3}}$$

$\therefore u_2 = \dfrac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}{\sqrt{\frac{2}{3}}}$

$u_2 = \sqrt{\tfrac{3}{2}}\left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = \left(-\tfrac{2}{\sqrt6}, \tfrac{1}{\sqrt6}, \tfrac{1}{\sqrt6}\right)$

$u_3 = \dfrac{\omega_3}{\|\omega_3\|}$ where $\omega_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2$

$$= (0,0,1) - \left\langle (0,0,1), \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)\right\rangle\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right) -$$

$$- \left\langle (0,0,1), \left(-\tfrac{2}{\sqrt6}, \tfrac{1}{\sqrt6}, \tfrac{1}{\sqrt6}\right)\right\rangle\left(-\tfrac{2}{\sqrt6}, \tfrac{1}{\sqrt6}, \tfrac{1}{\sqrt6}\right)$$

$$= (0,0,1) - \left(\tfrac{1}{\sqrt3}\right)\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right) - \left(\tfrac{1}{\sqrt6}\right)\left(-\tfrac{2}{\sqrt6}, \tfrac{1}{\sqrt6}, \tfrac{1}{\sqrt6}\right)$$

$$= (0,0,1) - \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) - \left(-\tfrac{2}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

$$= \left(0 - \tfrac{1}{3} + \tfrac{2}{6},\ 0 - \tfrac{1}{3} - \tfrac{1}{6},\ 1 - \tfrac{1}{3} - \tfrac{1}{6}\right)$$

$$= \left(0,\ \tfrac{-2-1}{6},\ \tfrac{6-2-1}{6}\right)$$

$$\omega_3 = \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$\|\omega_3\| = \sqrt{0+\tfrac{1}{4}+\tfrac{1}{4}} = \tfrac{1}{\sqrt2}$$

$\therefore u_3 = \dfrac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\frac{1}{\sqrt2}}$

$= \sqrt2\left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right) = \left(0, -\tfrac{1}{\sqrt2}, \tfrac{1}{\sqrt2}\right)$

Thus the orthonormal basis is $\{u_1, u_2, u_3\} = \left\{\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right), \left(-\tfrac{2}{\sqrt6}, \tfrac{1}{\sqrt6}, \tfrac{1}{\sqrt6}\right)\left(0, -\tfrac{1}{\sqrt2}, \tfrac{1}{\sqrt2}\right)\right\}$

Exercise 7.1

Q1 Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ belong to R^2.
verify that $\langle u,v\rangle = u_1v_1-2u_1v_2-2u_2v_1+5u_2v_2$ is an I.P on R^2

(i) $\langle u,u\rangle = u_1u_1-2u_1u_2-2u_2u_1+5u_2u_2$
$= u_1^2-4u_1u_2+5u_2^2$
$= u_1^2-4u_1u_2+(2u_2)^2-(2u_2)^2+5u_2^2$ completing square.
$= (u_1-2u_2)^2+u_2^2 \geq 0$

If $\langle u,u\rangle = 0$
$\Leftrightarrow (u_1-2u_2)^2+u_2^2 = 0$
$\Leftrightarrow (u_1-2u_2)^2=0$ & $u_2^2=0$
$\Leftrightarrow u_1-2u_2=0$ & $\boxed{u_2=0}$
$\Leftrightarrow u_1-2(0)=0 \quad \because u_2=0$
$\Leftrightarrow \boxed{u_1=0}$
$\Leftrightarrow u=0 \quad \because u=(u_1,u_2)$

(ii) $\langle u,v\rangle = u_1v_1-2u_1v_2-2u_2v_1+5u_2v_2$
$= v_1u_1-2v_2u_1-2v_1u_2+5v_2u_2$
$= \langle v,u\rangle$

(iii) $\langle au+bv, w\rangle = (au_1+bv_1)w_1-2(au_1+bv_1)w_2-2(au_2+bv_2)w_1+5(au_2+bv_2)w_2$

$au+bv=(au_1+bv_1, au_2+bv_2)$
$w=(w_1,w_2)$

$= au_1w_1+bv_1w_1-2au_1w_2-2bv_1w_2-2au_2w_1-2bv_2w_1+5au_2w_2+5bv_2w_2$
$= a(u_1w_1-2u_1w_2-2u_2w_1+5u_2w_2)+b(v_1w_1-2v_1w_2-2v_2w_1+5v_2w_2)$
$= a\langle u,w\rangle + b\langle v,w\rangle$

All the three conditions for I.P are satisfied
Hence $\langle u,v\rangle$ as defined above is an I.P on R^2.

Q1 (ii) For what value of K $\langle u,v\rangle = u_1v_1 - 3u_1v_2 - 3u_2v_1 + Ku_2v_2$ is an I.P on R^2.

Sol

$\langle u,u\rangle = u_1u_1 - 3u_1u_2 - 3u_2u_1 + Ku_2u_2$ $\qquad u=(u_1,u_2)$, $v=(v_1,v_2)$

$= u_1^2 - 6u_1u_2 + Ku_2^2$

For an I.P condition (i) i.e $\langle u,u\rangle$ must be +ive

for this $\langle u,u\rangle$ must be in perfect square form

$\therefore \langle u,u\rangle = u_1^2 - 6u_1u_2 + 9u_2^2$ is perfect square form for $K \geq 9$

$= (u_1 - 3u_2)^2$ $\qquad$ for $K=9$

$= (u_1-3u_2)^2 + u_2^2$ $\qquad$ for $K=10$

Cond (ii) & (iii) are also satisfied for $K \geq 9$.

Q2 Find the norm of $(2,3) \in R^2$ w.r.t Euclidean inner product on R^2

(i) $\langle u,v\rangle = u_1v_1 + u_2v_2$ $\qquad u=(u_1,u_2)$ $\quad v=(v_1,v_2)$

$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u_1^2+u_2^2}$

$\|(2,3)\| = \sqrt{2^2+3^2} = \sqrt{13}$ Ans

(ii) I.P defined as $\langle u,v\rangle = u_1v_1 - 2u_1v_2 - 2u_2v_1 + 5u_2v_2$

$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u_1u_1 - 2u_1u_2 - 2u_2u_1 + 5u_2u_2}$

$= \sqrt{u_1^2 - 4u_1u_2 + 5u_2^2}$

$\|(2,3)\| = \sqrt{2^2 - 4\cdot2\cdot3 + 5\cdot3^2}$

$= \sqrt{4 - 24 + 45}$

$= \sqrt{25} = \boxed{5}$ Ans

Q3 Find the norm of $\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, \frac{1}{6}\right) \in R^4$ w.r.t Euclidean I.P on R^4

$\langle u,v\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$ $\qquad$ Euclidean I.P on R^4

$\|u\| = \sqrt{\langle u,u\rangle}$

$= \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$

$\left\|\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, \frac{1}{6}\right)\right\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{1}{16} + \frac{1}{9} + \frac{1}{36}}$

$= \sqrt{\frac{36+9+16+4}{144}} = \sqrt{\frac{65}{144}} = \boxed{\frac{\sqrt{65}}{12}}$ Ans

Q4 Let V denote the vectorspace of 2×2 matrices over R. If $A, B \in V$ and $Tr(A)$ (called the Trace of A) denote the sum of diagonal elements of A, show that $\langle A,B\rangle = Tr(B^t A)$ is an I.P on V. Also find norm of $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$.

Sol Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $\quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$$B^t A = \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

$$B^t A = \begin{bmatrix} b_1a_1 + b_3a_3 & b_1a_2 + b_3a_4 \\ b_2a_1 + b_4a_3 & b_2a_2 + b_4a_4 \end{bmatrix}$$

$$\langle A,B\rangle = Tr(B^tA)$$
$$= b_1a_1 + b_2a_2 + b_3a_3 + b_4a_4 \quad \text{— (sum of diagonal element) } (1)$$

(i) $\langle A, A\rangle = a_1a_1 + a_2a_2 + a_3a_3 + a_4a_4$
$= a_1^2 + a_2^2 + a_3^2 + a_4^2 \geq 0$

if $\langle A, A\rangle = 0 \iff a_1^2 + a_2^2 + a_3^2 + a_4^2 = 0$

$\iff a_1^2 = 0,\ a_2^2 = 0,\ a_3^2 = 0,\ a_4^2 = 0$

$\iff a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

(ii) $\langle A, B\rangle = b_1a_1 + b_2a_2 + b_3a_3 + b_4a_4$
$= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$
$= \langle B, A\rangle$

L, M are const. $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$

(iii) $\langle LA + MB, C\rangle = Tr(C^t(LA+MB))$
$= c_1(La_1 + Mb_1) + c_2(La_2 + Mb_2) + c_3(La_3 + Mb_3) + c_4(La_4 + Mb_4)$ using (1)
$= L(c_1a_1 + c_2a_2 + c_3a_3 + c_4a_4) + M(c_1b_1 + c_2b_2 + c_3b_3 + c_4b_4)$
$= L\langle A, C\rangle + M\langle B, C\rangle$ Hence $\langle A,B\rangle$ is I.P on V.

$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$

$\left\|\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\right\| = \sqrt{1^2 + 2^2 + 3^2 + (-4)^2} = \sqrt{1+4+9+16} = \boxed{\sqrt{30}}$

Q5 Let V be the vectorspace $F(x)$ of polynomials over R. Show that $\langle f,g\rangle = \int_0^1 f(t)g(t)dt$ defines an inner product on V.

Sol (i) $\langle f,f\rangle = \int_0^1 f(t)f(t)\,dt$

$= \int_0^1 f^2(t)\,dt \geq 0$ as definite integral represents area of plane region.

If $\langle f,f\rangle = 0$

$\Leftrightarrow \int_0^1 f^2(t)dt = 0$

$\Leftrightarrow f^2(t) = 0$

$\Leftrightarrow f(t) = 0$

(ii) $\langle f,g\rangle = \int_0^1 f(t)g(t)dt$

$= \int_0^1 g(t)f(t)dt$

$= \langle g,f\rangle$

(iii) $\langle af+bg,h\rangle = \int_0^1 (af(t)+bg(t))h(t)dt$

$= a\int_0^1 f(t)h(t)dt + b\int_0^1 g(t)h(t)dt$

$= a\langle f,h\rangle + b\langle g,h\rangle$

Hence $\langle f,g\rangle$ is I.P in V.

⑥ Let $u_1=\begin{bmatrix}1\\2\\1\end{bmatrix}$ $u_2=\begin{bmatrix}2\\1\\2\end{bmatrix}$ $u_3=\begin{bmatrix}2\\1\\-4\end{bmatrix}$

Using Example 2, find i.e $\langle u_1,u_2\rangle = \det(u_1^t u_2)$

(i) the inner product of each pair of above column vectors

(ii) the norm of each vector

(iii) a vector orthogonal to u_1 & u_2

(iv) a vector orthogonal to u_1 & u_3

Sol (i) $\langle u_1,u_2\rangle = \det(u_1^t u_2) = \det\left(\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix}\right) = \det\left[1(2)+2(1)+1(2)\right] = \det[6] = 6$

$\langle u_1,u_3\rangle = \det(u_1^t u_3) = \det\left(\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}2\\1\\-4\end{bmatrix}\right) = \det\left[1(2)+2(1)+1(-4)\right] = 0$

$\langle u_2,u_3\rangle = \det(u_2^t u_3) = \det\left(\begin{bmatrix}2 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\1\\-4\end{bmatrix}\right) = \det\left[2(2)+1(1)+2(-4)\right] = -3$

(ii) $\|u_1\| = \sqrt{\langle u_1,u_1\rangle} = \sqrt{\det(u_1^t u_1)} = \sqrt{\det\left(\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}\right)} = \sqrt{\det\left[1^2+2^2+1^2\right]} = \sqrt{\det[6]} =$

$\|u_2\| = \sqrt{\langle u_2,u_2\rangle} = \sqrt{\det(u_2^t u_2)} = \sqrt{\det\left(\begin{bmatrix}2 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix}\right)} = \sqrt{\det\left[2^2+1^2+2^2\right]} = \sqrt{\det[9]} =$

$\|u_3\| = \sqrt{\langle u_3,u_3\rangle} = \sqrt{\det(u_3^t u_3)} = \sqrt{\det\left(\begin{bmatrix}2 & 1 & -4\end{bmatrix}\begin{bmatrix}2\\1\\-4\end{bmatrix}\right)} = \sqrt{\det\left[2^2+1^2+4^2\right]} = \sqrt{\det[21]} =$

(iii) Suppose $u = \begin{bmatrix}x\\y\\z\end{bmatrix}$ be a vector orthogonal to u_1 & u_2

$\therefore \langle u,u_1\rangle = 0 \Rightarrow \det(u^t u_1) = 0 \Rightarrow \det\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix} = 0$

$\Rightarrow x+2y+z = 0$ —(I)

& $\langle u,u_2\rangle = 0 \Rightarrow \det(u^t u_2) = 0 \Rightarrow \det\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix} = 0$

$\Rightarrow 2x+y+2z = 0$ —(II)

from (I)&(II) $x+2y+z=0$, $2x+y+2z=0$ $\quad \dfrac{x}{4-1} = \dfrac{-y}{2-2} = \dfrac{z}{1-4} = k$

$x = 3k$
$y = 0$
$z = -3k$

$u = \begin{bmatrix}x\\y\\z\end{bmatrix} = 3k\begin{bmatrix}1\\0\\-1\end{bmatrix} \Rightarrow \begin{bmatrix}1\\0\\-1\end{bmatrix}$ is orthogonal to u_1 & u_2

2nd Method

$\begin{vmatrix} i & j & k\\ 1 & 2 & 1\\ 2 & 1 & 2\end{vmatrix} = 3i - 0j + -3k = 3(i - 0j - k)$

$\therefore \begin{bmatrix}1\\0\\-1\end{bmatrix}$ is required vec

(iv) Let $u = \begin{bmatrix}x\\y\\z\end{bmatrix}$ be a vector orthogonal to u_1 & u_3

$\therefore \langle u,u_1\rangle = x+2y+z = 0$ —(I)

$\langle u,u_3\rangle = 2x+y-4z = 0$ —(II)

$\dfrac{x}{-8-1} = \dfrac{-y}{-4-2} = \dfrac{z}{1-4} = k$

$x = -9k$
$y = +6k$
$z = -3k$

$u = \begin{bmatrix}x\\y\\z\end{bmatrix} = -3k\begin{bmatrix}3\\-2\\1\end{bmatrix} \Rightarrow \begin{bmatrix}3\\-2\\1\end{bmatrix}$ is orthogonal to u_1 & u_3.

2nd Method

$\begin{vmatrix} i & j & k\\ 1 & 2 & 1\\ 2 & 1 & -4\end{vmatrix}$

$= -9i - j(-6) + k(1-$

$= -3(3i - 2j + k)$

$\therefore \begin{bmatrix}3\\-2\\1\end{bmatrix}$ is orthogonal u_1 & u_3

(i) Show that (1,1) (0,1) is a basis of R^2. Using the Gram-Schmidt process, find an orthonormal basis of R^2.

Sol: Let $(x,y) \in R^2$ $\quad S = \{(1,1), (0,1)\}$

Consider $(x,y) = a(1,1) + b(0,1)$ — linear combination of a, b & given vectors

$(x,y) = (a, a+b)$

$\Rightarrow \boxed{a = x}$

& $a+b = y \Rightarrow x+b = y \Rightarrow \boxed{b = y-x}$

∴ the linear combination exists, as (1,1) & (0,1) can be written as a linear combination of a & b. So the set 'S' is spanning set for R^2.

Now for Linear Independence.

$a(1,1) + b(0,1) = 0$

$\boxed{a = 0}$

$a+b = 0 \Rightarrow 0+b = 0 \Rightarrow \boxed{b = 0}$

Hence given vectors are Lin. independent.

2nd Method

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

It is in echelon form

Rank of A = 2 = No. of vectors in S

∴ S is Lin. independent.

S is spanning set & Linearly Independent So S is basis for R^2

Orthonormal Basis of R^2 by Gram Schmidt Process.

Let $v_1 = (1,1) \quad v_2 = (0,1)$

Now $u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{(1,1)}{\sqrt{1^2+1^2}} = \left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$u_2 = \dfrac{w_2}{\|w_2\|}$ where $w_2 = v_2 - \langle v_2, u_1\rangle u_1$

$= (0,1) - \left\langle (0,1), \left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)\right\rangle \left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$= (0,1) - \left(0\cdot\frac{1}{\sqrt2} + 1\cdot\frac{1}{\sqrt2}\right)\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$= (0,1) - \frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$= (0,1) - \left(\frac{1}{2}, \frac{1}{2}\right)$

$= \left(0 - \frac{1}{2}, 1 - \frac{1}{2}\right)$

$w_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\|w_2\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt2}$

∴ $u_2 = \dfrac{\left(-\frac{1}{2}, \frac{1}{2}\right)}{\frac{1}{\sqrt2}}$

$u_2 = \sqrt2\left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

Thus the orthonormal basis is $\{u_1, u_2\} = \left\{\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right), \left(-\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)\right\}$

$$\|\omega_3\| = \sqrt{1^2+0^2+0^2} = 1$$

$$u_3 = \frac{\omega_3}{\|\omega_3\|} = \frac{(1,0,0)}{1} = (1,0,0)$$

Therefore orthonormal basis is $\{u_1, u_2, u_3\} = \{(0,0,1), (0,1,0), (1,0,0)\}$

(ii) Taking $v_1 = (1,0,1)$, $v_2 = (0,1,1)$, $v_3 = (0,0,1)$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,0,1)}{\sqrt{1^2+0^2+1^2}} = \boxed{\left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)}$$

$$u_2 = \frac{\omega_2}{\|\omega_2\|} \quad \text{where} \quad \omega_2 = v_2 - \langle v_2, u_1\rangle u_1$$

$$= (0,1,1) - \left\langle (0,1,1), \left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)\right\rangle \left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)$$

$$= (0,1,1) - \frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)$$

$$= (0,1,1) - \left(\frac12, 0, \frac12\right)$$

$$\omega_2 = \left(-\frac12, 1, \frac12\right)$$

$$\|\omega_2\| = \sqrt{\frac14 + 1 + \frac14} = \sqrt{\frac64} = \sqrt{\frac32}$$

$$\therefore u_2 = \frac{\omega_2}{\|\omega_2\|} = \frac{\left(-\frac12, 1, \frac12\right)}{\sqrt{\frac32}} = \frac{\sqrt2}{\sqrt3}\left(-\frac12, 1, \frac12\right) = \left(-\frac{1}{\sqrt6}, \frac{\sqrt2}{\sqrt3}, \frac{1}{\sqrt6}\right)$$

$$= \boxed{\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right)}$$

$$u_3 = \frac{\omega_3}{\|\omega_3\|} \quad \text{where} \quad \omega_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2$$

$$= (0,0,1) - \left\langle (0,0,1), \left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)\right\rangle \left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)$$

$$- \left\langle (0,0,1), \left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right)\right\rangle \left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right)$$

$$= (0,0,1) - \frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right) - \frac{1}{\sqrt6}\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right)$$

$$= (0,0,1) - \left(\frac12, 0, \frac12\right) - \left(-\frac16, \frac26, \frac16\right)$$

$$= \left(0-\frac12, 0, \frac12\right) - \left(-\frac16, \frac26, \frac16\right)$$

$$\omega_3 = \left(-\frac12+\frac16, -\frac26, \frac12-\frac16\right) = \left(-\frac13, -\frac13, \frac13\right)$$

$$\|\omega_3\| = \sqrt{\frac19+\frac19+\frac19} = \sqrt{\frac39} = \sqrt{\frac13}$$

$$u_3 = \frac{\omega_3}{\|\omega_3\|} = \frac{\left(-\frac13, -\frac13, \frac13\right)}{\frac{1}{\sqrt3}} = \sqrt3\left(-\frac13, -\frac13, \frac13\right) = \boxed{\left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)}$$

$\therefore \{u_1, u_2, u_3\}$ is orthonormal basis

Show that $(1,0,1),(0,1,1),(0,0,1)$ is a basis of R^3. Using the Gram Schmidt process, find an orthonormal basis of R^3 by taking $u_1=(0,0,1)$

Let $S=\{(1,0,1),(0,1,1),(0,0,1)\} \subset R^3$

Let $(x,y,z)\in R^3$

Consider $(x,y,z)=a(1,0,1)+b(0,1,1)+c(0,0,1)$

$=(a,\ b,\ a+b+c)$

$\boxed{a=x}$

$\boxed{b=y}$

$a+b+c=z \Rightarrow \boxed{c=z-x-y}$ So Linear combination of vectors of 'S' and a, b, c exists. So S is spanning set for R^3.

Now $a(1,0,1)+b(0,1,1)+c(0,0,1)=0$

$a+0+0=0 \Rightarrow \boxed{a=0}$

$0+b+0=0 \Rightarrow \boxed{b=0}$ Hence 'S' is Lin Independent.

$a+b+c=0 \Rightarrow \boxed{c=0}$

So S is a basis of R^3 as S is spanning & Lin Independent.

To check Linear Independence.

2nd Method

$A=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}$

Rank of $A=3=$ No. of vectors in S.

$\therefore$ S is independent.

Gram Schmidt Orthonormalisation Process.

Taking $v_1=(0,0,1)\quad v_2=(0,1,1),\quad v_3=(1,0,1)$

$u_1=\frac{v_1}{\|v_1\|}=\frac{(0,0,1)}{\sqrt{0^2+0^2+1^2}}=(0,0,1)$

$u_2=\frac{w_2}{\|w_2\|}$ where $w_2=v_2-\langle v_2,u_1\rangle u_1$

$=(0,1,1)-\langle(0,1,1),(0,0,1)\rangle(0,0,1)$

$=(0,1,1)-(0+0+1)(0,0,1)$

$=(0,1,1)-(0,0,1)$

$w_2=(0,1,0)$

$\|w_2\|=\sqrt{0^2+1^2+0^2}=1$

$\therefore u_2=\frac{(0,1,0)}{1}=(0,1,0)$

$u_3=\frac{w_3}{\|w_3\|}$ where $w_3=v_3-\langle v_3,u_1\rangle u_1-\langle v_3,u_2\rangle u_2$

$=(1,0,1)-\langle(1,0,1),(0,0,1)\rangle(0,0,1)-\langle(1,0,1),(0,1,0)\rangle(0,1,0)$

$=(1,0,1)-1(0,0,1)-0(0,1,0)$

$=(1,0,1)-(0,0,1)$

$w_3=(1,0,0)$

(9) Show that $\{(1,-1,0),(2,-1,-2),(1,-1,-2)\}$ is a basis of R^3. Find an orthonormal basis of R^3 using Graham Schmidt Process.

Sol $S = \{(1,-1,0), (2,-1,-2), (1,-1,-2)\}$

Let $(x,y,z) \in R^3$

Consider $(x,y,z) = a(1,-1,0) + b(2,-1,-2) + c(1,-1,-2)$ — (I)

from

$x = a+2b+c$ — (II)

$y = -a-b-c$ — (III)

$z = -2b-2c$ — (IV)

from (II) & (III) $\boxed{x+y = b}$ Put in (IV)

$z = -2x-2y-2c \Rightarrow 2c = -2x-2y-z$

$\boxed{c = -x-y-\frac{z}{2}}$ Put in

from (III) $y = -a-(x+y)-(-x-y-\frac{z}{2})$

$a = -y-\cancel{x}-\cancel{y}+\cancel{x}+\cancel{y}+\frac{z}{2} \Rightarrow \boxed{a = \frac{z}{2}-y}$

Linear combination of a, b, c & vectors of S exist. So S is spanning set.

Now
$a(1,-1,0)+b(2,-1,-2)+c(1,-1,-2) = 0$

$a+2b+c = 0$ — (V)

$-a-b-c = 0$ — (VI)

$-2b+(-2c) = 0 \Rightarrow b = \frac{\cancel{2}c}{-\cancel{2}} = \boxed{-c}$ — (VII)

$-a-b-c=0 \Rightarrow -a-(-c)-c = 0$

$\Rightarrow -a+c-c = 0$

$\Rightarrow \boxed{a = 0}$ Put in (V)

$a+2b+c = 0 \Rightarrow 0+2(-c)+c = 0$

$\Rightarrow \boxed{c = 0}$

by Putting $c=0$ in (VII) $\boxed{b = 0}$

Hence S is linearly Independent.

$\because$ S is spaning & linearly Independent
so S is basis of R^3.

2nd Method

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ 1 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix} \text{ Echelon form}$$

Rank of A = 3 = No. of vectors in S.

So S is linearly Independent

Gram Schmidt Process.

$S = \{(1,-1,0),\ (2,-1,-2),\ (1,-1,-2)\}$

$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,-1,0)}{\sqrt{1^2+1^2+0}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

Let $u_2 = \frac{w_2}{\|w_2\|}$ where $w_2 = v_2 - \langle v_2, u_1\rangle u_1$

$= (2,-1,-2) - \left\langle (2,-1,-2), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\right\rangle \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$= (2,-1,-2) - \left(\frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0\right)\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$= (2,-1,-2) - \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$= (2,-1,-2) - \left(\frac{3}{2}, -\frac{3}{2}, 0\right)$

$w_2 = \left(\frac{1}{2}, +\frac{1}{2}, -2\right)$

$\|w_2\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-2)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}+4} = \sqrt{\frac{18}{4}}$

$= \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}}$

$\therefore u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(\frac{1}{2}, \frac{1}{2}, -2\right)}{\frac{3}{\sqrt{2}}} = \frac{\sqrt{2}}{3}\left(\frac{1}{2}, \frac{1}{2}, -2\right) = \left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{2\sqrt{2}}{3}\right)$

$u_3 = \frac{w_3}{\|w_3\|}$ where $w_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2$

$= (1,-1,-2) - \left\langle (1,-1,-2), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\right\rangle \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) -$

$- \left\langle (1,-1,-2), \left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{2\sqrt{2}}{3}\right)\right\rangle \left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{2\sqrt{2}}{3}\right)$

$= (1,-1,-2) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0\right)\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) - \left(\cancel{\frac{1}{3\sqrt{2}}} - \cancel{\frac{1}{3\sqrt{2}}} + \frac{4\sqrt{2}}{3}\right)\left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{2\sqrt{2}}{3}\right)$

$= (1,-1,-2) - \left(\frac{2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) - \left(\frac{4\sqrt{2}}{3}\right)\left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{2\sqrt{2}}{3}\right)$

$= (1,-1,-2) - \left(\frac{2}{2}, -\frac{2}{2}, 0\right) - \left(\frac{4}{9}, \frac{4}{9}, -\frac{16}{9}\right)$

$= (1-1, -1+1, -2-0) - \left(\frac{4}{9}, \frac{4}{9}, -\frac{16}{9}\right)$

$w_3 = (0, 0, -2) - \left(\frac{4}{9}, \frac{4}{9}, -\frac{16}{9}\right) = \left(-\frac{4}{9}, -\frac{4}{9}, -\frac{2}{9}\right)$

$\|w_3\| = \sqrt{\frac{16}{81} + \frac{16}{81} + \frac{4}{81}} = \sqrt{\frac{36}{81}} = \frac{6}{9} = \frac{2}{3}$

$\therefore u_3 = \frac{w_3}{\|w_3\|} = \frac{\left(-\frac{4}{9}, -\frac{4}{9}, -\frac{2}{9}\right)}{2/3} = \frac{3}{2}\left(-\frac{4}{9}, -\frac{4}{9}, -\frac{2}{9}\right) = \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$

$\therefore$ Orthonormal basis is $\{u_1, u_2, u_3\} = \left\{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{2\sqrt{2}}{3}\right), \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)\right\}$

(10) Let $\{e_1, e_2, \cdots e_n\}$ be an orthonormal basis of an inner product space V over R. Show that for any $v \in V$

$$v = \langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2 + \cdots + \langle v, e_n\rangle e_n$$

Sol Since $\{e_1, e_2, \cdots e_n\}$ is orthonormal basis so $\langle e_i, e_j\rangle = 0, i \neq j$

and $\|e_i\| = \langle e_i, e_i\rangle = 1, i = j$

Also since $\{e_1, e_2, \cdots e_n\}$ is basis for V, so any vector $v \in V$ can be written as linear combination of basis vectors.

Therefore $v = a_1 e_1 + a_2 e_2 + \cdots a_n e_n$ ——— (1)

Taking inner product of (1) with $e_1, e_2, \ldots e_n$

$$\langle v, e_1\rangle = \langle a_1 e_1 + a_2 e_2 + \cdots a_n e_n, e_1\rangle$$

$$\langle v, e_1\rangle = a_1\langle e_1, e_1\rangle + a_2\langle e_2, e_1\rangle + \cdots a_n\langle e_n, e_1\rangle$$

$$= a_1(1) + a_2(0) + \cdots a_n(0)$$

$$\boxed{\langle v, e_1\rangle = a_1}$$

$$\langle v, e_2\rangle = a_1\langle e_1, e_2\rangle + a_2\langle e_2, e_2\rangle + \cdots a_n\langle e_n, e_2\rangle$$

$$= a_1(0) + a_2(1) + \cdots a_n(0)$$

$$\boxed{\langle v, e_2\rangle = a_2}$$

Similarly $\boxed{\langle v, e_3\rangle = a_3}$

and so on $\boxed{\langle v, e_n\rangle = a_n}$

$\therefore$ (1) becomes $v = \langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2 + \cdots \langle v, e_n\rangle e_n$

Matrix of Linear Transformation:-

Let V & W be two finite dimensional vectorspaces over the same field F. $\dim V = n$ & $\dim W = m$

Let $B = \{v_1, v_2, \dots v_n\}$ and $E = \{w_1, w_2 \dots w_m\}$ be any bases for V & W resp.

Any vector v in V can be expressed in a unique way as a linear combination of $v_1, v_2 \dots v_n$ i.e basis B.

$\therefore \quad v = a_1 v_1 + a_2 v_2 + \dots a_n v_n$ $\qquad a_i \in F,\ i = 1, 2, \dots n$

a_i = coordinate vector of v relative to B

Let $T : V \to W$ be a linear transformation.

The images $T(v_1), T(v_2), \dots T(v_n)$ are elements of W

Each image can be expressed uniquely as a linear combination of the basis vectors $w_1, w_2, \dots w_m$.

where $a_{ij} \in F$

$$
\begin{aligned}
T(v_1) &= a_{11} w_1 + a_{21} w_2 + \dots a_{i1} w_i \dots + a_{m1} w_m \\
T(v_2) &= a_{12} w_1 + a_{22} w_2 + \dots a_{i2} w_i \dots + a_{m2} w_m \\
&\vdots \\
T(v_j) &= a_{1j} w_1 + a_{2j} w_2 + \dots + a_{ij} w_i + \dots + a_{mj} w_m \\
&\vdots \\
T(v_n) &= a_{1n} w_1 + a_{2n} w_2 + \dots a_{in} w_i + \dots + a_{mn} w_m
\end{aligned}
$$

The $m \times n$ Matrix of T is called Matrix of linear Transformation

$$
\begin{bmatrix}
a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\
a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{i1} & a_{i2} & \dots & \boxed{a_{ij}} & \dots & a_{in} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{m1} & a_{m2} & & a_{mj} & & a_{mn}
\end{bmatrix}
$$

a_{ij} is coeff of w_i in the image of $T(v_j)$

Q10(ii) If $T : V \to V$ is linear transformation show that $\langle T(e_j), e_i \rangle$ is the ij^{th} entry of the matrix representing T in the given basis $\{e_1, e_2 \dots e_n\}$.

Sol $T : V \to V$ $\qquad$ Let $\{v_1, v_2 \dots v_n\} = \{w_1, w_2 \dots w_n\} = \{e_1, e_2 \dots e_n\}$

$$
\begin{aligned}
T(v_j) &= a_{1j} w_1 + a_{2j} w_2 + \dots a_{ij} w_i + \dots a_{nj} w_n \\
T(e_j) &= a_{1j} e_1 + a_{2j} e_2 + \dots a_{ij} e_i + \dots a_{nj} e_n
\end{aligned}
$$

Take I.P with e_i

$$
\begin{aligned}
\langle T(e_j), e_i \rangle &= a_{1j} \langle e_1, e_i \rangle + a_{2j} \langle e_2, e_i \rangle + \dots a_{ij} \langle e_i, e_i \rangle + \dots a_{nj} \langle e_n, e_i \rangle \\
&= 0 + 0 + \dots a_{ij}(1) + 0 \dots 0
\end{aligned}
$$

$\because \{e_1, e_2 \dots e_n\}$ is orthonormal basis

$\langle T(e_j), e_i \rangle = a_{ij} = ij^{th}$ entry in the matrix of T.

7.1-12

2(11) Let W be a subspace of an inner product space V. Show that there is an orthonormal basis of W which is part of an orthonormal basis of V.

Sol W is subspace of inner product space $V(R)$

Let $\{w_1, w_2, \dots w_t\}$ be a basis for W

$\dim(W) = t$

Let $\dim(V) = n$

As $\{w_1, w_2 \dots w_t\}$ is a linearly independent set of vectors in V (being basis for W.)

So by theorem "Any linearly independent set of vectors in a finite dimensional vectorspace 'V' can be extended to a basis for V"

Another basis for $V = \{w, w_2, \dots w_t, u_1, u_2, \dots u_{n-t}\}$

$\Rightarrow$ Basis for $W \subseteq$ Basis for V ——— (1)

Transforming by Gram Schmidt process into orthonormal basis for W & V

(1) becomes orthonormal basis for $W \subseteq$ orthonormal basis for V.

Orthogonal Matrices

A square matrix A over the field R is called Orthogonal if $A^tA = I = AA^t$

OR $A^t = A^{-1}$

A^t is inverse of A $\because A^tA = I$

From the def it follows that

i) A is orthogonal iff A^{-1} is orthogonal

ii) If A is orthogonal then A is symmetric iff $A^2 = I$

iii) If A & B are orthogonal then AB and BA are also orthogonal.

iv) If A is orthogonal then the col vectors are of unit length and their dot products are zero i.e Columns of A form an orthonormal set

v) If A is orthogonal then the rows of A form an orthonormal set.

vi) A^{-1} is orthogonal iff A^t is orthogonal.

imp. **Theorem** The following conditions for a square Matrix A are equivalent:

i) A is orthogonal

ii) The rows of A form an orthonormal set.

iii) The col of A form an orthonormal set.

Proof Let $A = [a_{ij}]_{n\times n}$ be an Orthogonal Matrix i.e $AA^t = I = A^tA$

Let R_i & C_i denote the i^{th} row & i^{th} col of A where $1 \le i \le n$

Let $c_{ij} = (i,j)$ element of AA^t

$=$ Sum of products of the corresponding elements in the i^{th} row of A and j^{th} col of A^t.

$=$ Sum of products of the corresponding elements in the i^{th} row of A and j^{th} row of A

$= a_{i1}a_{j1} + a_{i2}a_{j2} + \dots a_{in}a_{jn}$

i^{th} row of $A = (a_{i1}, a_{i2}, \dots a_{in})$
j^{th} row of $A = (a_{j1}, a_{j2}, \dots a_{jn})$
for detail see examples

$c_{ij} = \langle R_i, R_j \rangle$ ——— (1)
(I.P of two rows)

Matrix

Now $[c_{ij}] = AA^t$

$[c_{ij}] = I$ $\qquad\because A$ is orthogonal so $AA^t = I$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\\c_{31}&c_{32}&c_{33}\end{pmatrix}$$
when $i=j$, $c_{ij}=1$; when $i\neq j$, $c_{ij}=0$

$\therefore \langle R_i, R_j\rangle = c_{ij} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$

$\because c_{ij} = \langle R_i, R_j\rangle$

$\therefore \langle R_i, R_j\rangle = \begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$ $\Rightarrow$ Rows of A are Orthonormal

$\Rightarrow$ (i) $\Rightarrow$ (ii) (from statement of Theorem)

<u>Conversely</u> if rows of A form an orthonormal set, then

$$\langle R_i, R_j\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases}$$

$\because \langle R_i, R_j\rangle = c_{ij}$ $\therefore c_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases}$

$\overset{\text{Matrix}}{\Rightarrow} [c_{ij}] = I$

$\because AA^t = [c_{ij}]$ $\therefore AA^t = I \Rightarrow$ A is orthogonal Matrix

So $(ii) \Rightarrow (i)$ Hence $(i) \Leftrightarrow (ii)$

Now Consider the Matrix $A^tA = [d_{ij}]$

$d_{ij} = (i,j)$ element of A^tA

= Sum of the products of the corresponding elements in i^{th} <u>row of A^t</u> and j^{th} col of A.

= Sum of the products of the corresponding elements in i^{th} <u>col of A</u> and j^{th} col of A

$= a_{1i}a_{1j} + a_{2i}a_{2j} + \cdots a_{ni}a_{nj}$

$\because i^{th}$ row of $A^t = [a_{i1}\ a_{i2} \ldots a_{in}]$

i^{th} col of $A = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni}\end{bmatrix}$

j^{th} col of $A = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{nj}\end{bmatrix}$

= Inner product of i^{th} & j^{th} col matrices

$d_{ij} = \langle C_i, C_j\rangle$

$[d_{ij}] = A^tA = I$ $\quad\because$ A is orthogonal Matrix

$\Rightarrow d_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases}$

$\because d_{ij} = \langle C_i, C_j\rangle$

$\therefore \langle C_i, C_j\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases}$ $\Rightarrow$ cols of A C_i & C_j are orthonormal

$\Rightarrow (i) \Rightarrow (iii)$

<u>Conversely</u> if columns of A form orthonormal set, then

$$\langle C_i, C_j\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases}$$

$\because \langle C_i, C_j\rangle = d_{ij}$ $\therefore d_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases}$

$\Rightarrow [d_{ij}] = I$

$\because [d_{ij}] = A^tA$ $\therefore A^tA = I$ $\Rightarrow$ A is orthogonal Matrix

So $(iii) \Rightarrow (i)$ Hence $(i) \Leftrightarrow (iii)$

Now $(i) \Leftrightarrow (ii)$ & $(i) \Leftrightarrow (iii)$ $\Rightarrow (ii) \Leftrightarrow (iii)$

<u>Note</u>: As $(ii) \Leftrightarrow (iii)$ so A is orthogonal iff A^t is orthogonal

Example 13 Show that the rows (Columns) of the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ form an orthonormal set.

Sol To show that the rows (colms) of the matrix A form an orthonormal set we have only to show that A is orthogonal (Th 7.12 on page 277)

(If A is orthogonal then $AA^t = I$)

$$\therefore AA^t = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta+\sin^2\theta+0 & \cancel{\cos\theta\sin\theta}-\cancel{\sin\theta\cos\theta}+0 & 0+0+0 \\ \cancel{\sin\theta\cos\theta}-\cancel{\cos\theta\sin\theta}+0 & \sin^2\theta+\cos^2\theta+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$ Thus A is orthogonal
So Rows of A form an orthonormal set.

Example 14 Find an orthogonal matrix A whose first row is $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$

Sol First Method Let $R_1 = \boxed{\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]}$

A vector (row) orthogonal to R_1 is R_2^*

$$= \begin{vmatrix} e_1 & e_2 \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} e_2 & e_3 \\ \frac{2}{3} & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} e_3 & e_1 \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= \cancel{e_1\left(\tfrac{2}{3}\right)} - e_2\left(\tfrac{1}{3}\right) + e_2\left(\tfrac{2}{3}\right) - e_3\left(\tfrac{2}{3}\right) + e_3\left(\tfrac{1}{3}\right) - \cancel{e_1\left(\tfrac{2}{3}\right)}$$

$$= e_2\left(\tfrac{2}{3}-\tfrac{1}{3}\right) + e_3\left(-\tfrac{2}{3}+\tfrac{1}{3}\right) = e_2\left(\tfrac{1}{3}\right) + e_3\left(-\tfrac{1}{3}\right)$$

$$R_2^* = \left(0, \tfrac{1}{3}, -\tfrac{1}{3}\right)$$

$$R_2 = \frac{R_2^*}{\|R_2^*\|} = \frac{\left(0,\frac{1}{3},-\frac{1}{3}\right)}{\sqrt{2}/3} = \frac{3}{\sqrt{2}}\left(0,\tfrac{1}{3},-\tfrac{1}{3}\right) = \boxed{\left(0,\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}}\right)}$$

A vector (row) orthogonal to R_1 & R_2^*

$$R_3^* = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = e_1\left(-\tfrac{2}{9}-\tfrac{2}{9}\right) - e_2\left(-\tfrac{1}{9}\right) + e_3\left(\tfrac{1}{9}\right)$$

$$= -\tfrac{4}{9}e_1 + \tfrac{1}{9}e_2 + \tfrac{1}{9}e_3$$

$$= \left(-\tfrac{4}{9}, \tfrac{1}{9}, \tfrac{1}{9}\right)$$

$$R_3 = \frac{R_3^*}{\|R_3^*\|} = \frac{\left(-\frac{4}{9},\frac{1}{9},\frac{1}{9}\right)}{\frac{\sqrt{18}}{9}} = \frac{9}{3\sqrt{2}}\left(-\tfrac{4}{9},\tfrac{1}{9},\tfrac{1}{9}\right) = \boxed{\left(\tfrac{-4}{3\sqrt{2}},\tfrac{1}{3\sqrt{2}},\tfrac{1}{3\sqrt{2}}\right)}$$

$$\therefore \text{Orthogonal Matrix } A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$$

(To Check Find A^tA. we get $A^tA = I$)

Example 14 2nd Method

Let $R_1 = \boxed{\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)}$

A vector orthogonal to R_1 is R_2^*

$$R_2^* = \begin{vmatrix} e_1 & e_2 \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} e_2 & e_3 \\ \frac{2}{3} & \frac{2}{3} \end{vmatrix} + \begin{vmatrix} e_3 & e_1 \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= e_1\left(\tfrac{2}{3}\right) - e_2\left(\tfrac{1}{3}\right) + e_2\left(\tfrac{2}{3}\right) - e_3\left(\tfrac{2}{3}\right) + e_3\left(\tfrac{1}{3}\right) - e_1\left(\tfrac{2}{3}\right)$$

$$R_2^* = e_2\left(\tfrac{1}{3}\right) + e_3\left(-\tfrac{1}{3}\right) = \left(0, \tfrac{1}{3}, -\tfrac{1}{3}\right)$$

$$\|R_2^*\| = \sqrt{0 + \tfrac{1}{9} + \tfrac{1}{9}} = \sqrt{\tfrac{2}{9}} = \frac{\sqrt{2}}{3}$$

$$R_2 = \frac{R_2^*}{\|R_2^*\|} = \frac{\left(0, \frac{1}{3}, -\frac{1}{3}\right)}{\frac{\sqrt{2}}{3}} = \frac{3}{\sqrt{2}}\left(0, \tfrac{1}{3}, -\tfrac{1}{3}\right) = \boxed{\left(0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)}$$

A vector orthogonal to R_1 & R_2 is R_3^*

$$R_3^* = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{vmatrix} = e_1\left(\frac{-2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}}\right) - e_2\left(\frac{-1}{3\sqrt{2}}\right) + e_3\left(\frac{1}{3\sqrt{2}}\right) = \left(\frac{-4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$$

$$R_3 = \frac{R_3^*}{\|R_3^*\|} = \frac{\left(\frac{-4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)}{1} = \boxed{\frac{-4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}}$$

Hence orthogonal Matrix $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$

Note

We can check our Answer by $\langle R_1, R_2\rangle = 0 = \langle R_1, R_3\rangle$

$0 = \langle R_2, R_3\rangle$

$\langle R_1, R_1\rangle = \langle R_2, R_2\rangle = 1 = \langle R_3, R_3\rangle$

OR $AA^t = I$

3rd Method

Let $V_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ & $\omega_2 = (x, y, z)$

Let ω_2 is orthogonal to V_1 ∴ $\langle V_1, \omega_2\rangle = 0$

$\Rightarrow \frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = 0$ or $x + 2y + 2z = 0$ —(i)

If we take $\boxed{x = 0}$ then $2y + 2z = 0 \Rightarrow \boxed{z = -y}$

∴ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ ∴ $\omega_2 = (x, y, z) = (0, 1, -1)$

$$V_2 = \frac{\omega_2}{\|\omega_2\|} = \frac{(0, 1, -1)}{\sqrt{1+1}} = \boxed{\left(0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)}$$

$V_1 \Rightarrow$ order of A is 3×3

∵ A must be square.

V_1 is first row of A

V_2 is 2nd row of A

(we may take $y = 0$ or $z = 0$

Now let $\omega_3 = (x, y, z)$ be orthogonal to V_1 & V_2

∴ $\langle V_1, \omega_3\rangle = 0 \Rightarrow \frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = 0 \Rightarrow x + 2y + 2z = 0$ —(ii)

$\langle V_2, \omega_3\rangle = 0 \Rightarrow \frac{y}{\sqrt{2}} - \frac{z}{\sqrt{2}} = 0 \Rightarrow y - z = 0 \Rightarrow y = z$ Put in (ii)

$\Rightarrow x = -4z$ from (ii)

∴ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ ∴ $\omega_3 = (-4, 1, 1)$

$$V_3 = \frac{\omega_3}{\|\omega_3\|} = \frac{(-4, 1, 1)}{\sqrt{18}} = \left(\frac{-4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$$

Hence $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{4}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$

Ex 7.2

Q1 If A is an orthogonal matrix show that $|A| = 1$ or -1

Sol Since A is orthogonal Matrix

$\therefore \quad AA^t = I$

$\det(AA^t) = \det I$

By Product Theorem $\Rightarrow \det(A)\cdot\det(A^t) = 1 \qquad \because (\det(AB) = \det A \cdot \det B)$

$\Rightarrow \det(A)\cdot\det(A) = 1 \qquad \because \det(A^t) = \det(A)$

$\Rightarrow [\det(A)]^2 = 1$

$\det(A) = \pm 1$ Proved

2nd Method

Since A is orthogonal Matrix

$\therefore \quad A^t = A^{-1}$

$\det(A^t) = \det(A^{-1})$

$\det A = \dfrac{1}{\det(A)} \qquad \because \det(A^t) = \det A.$

$[\det A]^2 = 1$

$\det A = \pm 1$ Proved.

Prove that $\det A^{-1} = \dfrac{1}{\det A}$

We know $A^{-1}A = I$

$\det(A^{-1}A) = \det I$

By Product Theorem $\det A^{-1} \det A = 1$

$\det A^{-1} = \dfrac{1}{\det A}$

proved.

Q2 Find an orthogonal matrix whose first row is $\left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right)$

Sol Let $R_1 = \left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right)$ (given)

A vector orthogonal to R_1 is $R_2^{\star} = \begin{vmatrix} e_1 & e_2 \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{vmatrix}$

$= \frac{2}{\sqrt5}e_1 - \frac{e_2}{\sqrt5}$

$= \left(\frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)$

$\|R_2^{\star}\| = \sqrt{\frac{4}{5}+\frac{1}{5}} = \sqrt{\frac{5}{5}} = 1$

$R_2 = \dfrac{R_2^{\star}}{\|R_2^{\star}\|} = \dfrac{\left(\frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)}{1} = \left(\frac{2}{\sqrt5}, \frac{-1}{\sqrt5}\right)$

$\therefore \quad A = \begin{bmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \end{bmatrix}$

To Check $AA^t = I$

$AA^t = \begin{bmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5}+\frac{4}{5} & \frac{2}{5}-\frac{2}{5} \\ \frac{2}{5}-\frac{2}{5} & \frac{4}{5}+\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ proved

Q3 Find an orthogonal matrix whose first row is multiple of (1,1,1)

Sol $w_1 = (1,1,1)$ given

$\|w_1\| = \sqrt{3}$

Normalised first Row $V_1 = \dfrac{w_1}{\|w_1\|} = \dfrac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{3}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

We can not use $\begin{vmatrix} e_1 & e_2 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3}\end{vmatrix} + \begin{vmatrix} e_2 & e_3 \\ -\frac{1}{\sqrt3} & \frac{1}{\sqrt3}\end{vmatrix} + \begin{vmatrix} e_3 & e_1 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3}\end{vmatrix}$ because it becomes 0. and when it becomes 0 it is not applicable.

Let w_2 is orthogonal to V_1, where $w_2 = (x, y, z)$

$\therefore \langle V_1, w_2 \rangle = 0$

$\Rightarrow \frac{x}{\sqrt3} + \frac{y}{\sqrt3} + \frac{z}{\sqrt3} = 0$

$\Rightarrow x + y + z = 0$

Put $x = 0$

$\Rightarrow y + z = 0 \qquad \Rightarrow z = -y$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\therefore w_2 = (0, 1, -1)$

$\|w_2\| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt2$

$V_2 = \dfrac{w_2}{\|w_2\|} = \dfrac{(0,1,-1)}{\sqrt2} = \left(0, \frac{1}{\sqrt2}, -\frac{1}{\sqrt2}\right)$

A vector orthogonal to V_1 & V_2 is

$$w_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{vmatrix}$$

$$= e_1\left(-\frac{1}{\sqrt6} - \frac{1}{\sqrt6}\right) - e_2\left(\frac{-1}{\sqrt6}\right) + e_3\left(\frac{1}{\sqrt6}\right)$$

$$w_3 = \left(-\frac{2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right)$$

$$\|w_3\| = \sqrt{\frac{4}{6} + \frac{1}{6} + \frac{1}{6}} = 1$$

$$V_3 = \frac{w_3}{\|w_3\|} = \left(-\frac{2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right)$$

Hence the required orthogonal matrix is $\begin{pmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ -\frac{2}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} \end{pmatrix}$

Q4. Find an orthogonal matrix whose first row is $\left(0, \frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right)$

Let $R_1 = \boxed{\left[0, \frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right]}$

A vector (row) orthogonal to R_1 is $R_2^* = \begin{vmatrix} e_1 & e_2 \\ 0 & \frac{1}{\sqrt5} \end{vmatrix} + \begin{vmatrix} e_2 & e_3 \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{vmatrix} + \begin{vmatrix} e_3 & e_1 \\ \frac{2}{\sqrt5} & 0 \end{vmatrix}$

$= e_1\left(\frac{1}{\sqrt5}\right) + \frac{2}{\sqrt5}e_2 - \frac{1}{\sqrt5}e_3 + \left(0 - \frac{2}{\sqrt5}e_1\right)$

$= e_1\left(\frac{1}{\sqrt5} - \frac{2}{\sqrt5}\right) + \frac{2}{\sqrt5}e_2 - \frac{1}{\sqrt5}e_3$

$R_2^* = -\frac{1}{\sqrt5}e_1 + \frac{2}{\sqrt5}e_2 - \frac{1}{\sqrt5}e_3 = \left(-\frac{1}{\sqrt5}, \frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)$

$\|R_2^*\| = \sqrt{\frac{1}{5} + \frac{4}{5} + \frac{1}{5}} = \sqrt{\frac{6}{5}}$

$R_2 = \frac{R_2^*}{\|R_2^*\|} = \frac{\left(-\frac{1}{\sqrt5}, \frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)}{\sqrt{\frac{6}{5}}} = \frac{\sqrt5}{\sqrt6}\left(-\frac{1}{\sqrt5}, \frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right) = \boxed{\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, -\frac{1}{\sqrt6}\right)}$

A vector orthogonal to R_1 & R_2 is $R_3^* = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ -\frac{1}{\sqrt6} & \frac{2}{\sqrt6} & -\frac{1}{\sqrt6} \end{vmatrix}$

$= e_1\left(\frac{-1}{\sqrt{30}} - \frac{4}{\sqrt{30}}\right) - e_2\left(\frac{2}{\sqrt{30}}\right) - e_3\left(+\frac{1}{\sqrt{30}}\right)$

$R_3^* = \left(\frac{-5}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$

$\|R_3^*\| = \sqrt{\frac{25}{30} + \frac{4}{30} + \frac{1}{30}} = 1$

$R_3 = \frac{R_3^*}{\|R_3^*\|} = \boxed{\left(\frac{-5}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)}$

Hence orthogonal Matrix $A = \begin{bmatrix} 0 & \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ -\frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} \\ -\frac{5}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{bmatrix}$

Q5 Show that the products and inverses of orthogonal matrices are orthogonal. Hence show that orthogonal matrices form a group under multiplication.

Sol Let A & B be orthogonal matrices. So $A^tA = I \Rightarrow A^t = A^{-1}$

$B^tB = I \Rightarrow B^t = B^{-1}$

Now To Prove Product of orthogonal Matrix is Orthogonal i.e (AB) is orthogonal, we show $(AB)^t(AB) = I$

Therefore
$$(AB)^t(AB) = B^tA^tAB$$
$$= B^t(A^tA)B$$
$$= B^t\,I\,B$$
$$= B^tB$$
$$(AB)^t(AB) = I$$

So AB is an orthogonal Matrix.

Now To Prove Inverse of orthogonal Matrix is orthogonal. i.e A^{-1} is orthogonal, we show $(A^{-1})^t(A^{-1}) = I$

Therefore
$$(A^{-1})^t(A^{-1}) = (A^{-1})^t(A^t) \quad (\because (A^{-1}) = A^t \text{ since } A \text{ is orthogonal})$$
$$= (AA^{-1})^t \quad \because (AB)^t = B^tA^t$$
$$= (I)^t$$
$$(A^{-1})^t(A^{-1}) = I$$

Hence A^{-1} is orthogonal.

To Prove Set of orthogonal matrices form a group under multiplication.

Let G = Set of orthogonal matrices

i) Let $A, B \in G$ then $AB \in G$ $\because$ product of orthogonal is orthogonal

Hence G is closed under B.O of multiplication.

ii) In Matrices Associative law holds.

iii) Identity Matrix is always orthogonal Matrix. so $I \in G$.

iv) Inverse of each orthogonal matrix is orthogonal (as solved above)

So Inverse of each orthogonal matrix belonging to G is in G.

So Inverse exists in G.

All conditions are satisfied, Hence G is a group under multiplication

Eigen Values and Eigen Vectors

Let A be a square matrix over 'R', of order 'n' then a scalar number 'λ' is called Eigen Value of A, if there exists a non-zero column vector $v \in R^n$ such that

$$\boxed{Av = \lambda v}$$

Here 'v' is called Eigen Vector of A corresponding to Eigen Value 'λ'

I is Identity Matrix

Now $Av = \lambda v \Rightarrow Av = \lambda I v$

$\Rightarrow (A - \lambda I)v = 0$

Since $v \neq 0$ (given)

So $A - \lambda I$ is singular Matrix by theorem of homogenous system of eqs

$\therefore$ $\boxed{|A - \lambda I| = 0}$ ——— (1)

$|A - \lambda I|$ is called characteristic polynomial of matrix A

Every Root of eq (1) i.e value of λ λ is called Eigen Value of A or Characteristic Value of A

The value of 'V' corresponding to value of 'λ' given by $(A - \lambda I)v = 0$ is called Eigen Vector corresponding to λ.

(standard)

Example 15 Find the eigen values and corresponding eigenvectors of the matrix.
$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Sol $$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \quad \text{———} \ (1)$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)(2-\lambda) - 2\right] - 2\left[(2-\lambda) - 1\right] + 1\left[2 - (3-\lambda)\right] = 0$$

$$\Rightarrow (2-\lambda)\left[6 - 5\lambda + \lambda^2 - 2\right] - 2(1-\lambda) + (-1+\lambda) = 0$$

$$\Rightarrow 12 - 10\lambda + 2\lambda^2 - 4 - 6\lambda + 5\lambda^2 - \lambda^3 + 2\lambda - 2 + 2\lambda - 1 + \lambda = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\therefore \lambda = 5, 1, 1$ (Eigen Values)

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & \downarrow & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & 0 \end{array}$$

$\lambda^2 - 6\lambda + 5 = 0$

$\lambda = \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 5}}{2}$

$= \frac{6 \pm 4}{2} = 5, 1$

Let Eigen Vector 'V' for $\lambda = 5$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)V = 0$

put $\lambda = 5$ in ①

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x + 2y + z = 0$

$x - 2y + z = 0$

$x + 2y - 3z = 0$

Homogeneous Eqs i.e. $AX = 0$

System of homogeneous 'm' eqs in 'n' unknowns has non trivial sol if Rank $A^* < n$

for Solution of Homogeneous Eqs Reduce in Echelon form

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 1 \\ -3 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$ by R_{12}

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{bmatrix}$ by $3R_1 + R_2$ and $-R_1 + R_3$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ by $\frac{-1}{4}R_2$, by $\frac{1}{4}R_3$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $-R_2 + R_3$ (Echelon form)

$A^* = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Rank of $(A^*) = 2 <$ No. of unknowns $= 3$ $\therefore$ System has non-trivial sol.

$x - 2y + z = 0$ ——— ⑪

$y - z = 0$ $\Rightarrow y = z$ — ⑪ Put in ⑪

from ⑪ $x - 2y + y = 0$

$x - y = 0 \Rightarrow x = y$

(Giving Arbitrary Values) Let $x = y = z = a$

then $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$

Eigen Vector V corresponding to $\lambda = 5$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [1\ 1\ 1]^t$ where $a = 1$ is arbitrary

Now For Eigen Value $\lambda=1$, Eigen vector $v=\begin{bmatrix}x\\y\\z\end{bmatrix}$

$$(A-\lambda I)v = 0$$

Put $\lambda=1$ in (1)

$$\begin{bmatrix}1 & 2 & 1\\1 & 2 & 1\\1 & 2 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Reduce in Echelon form

$$\begin{bmatrix}1 & 2 & 1\\1 & 2 & 1\\1 & 2 & 1\end{bmatrix} \xrightarrow{R} \begin{bmatrix}1 & 2 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \text{ by } -R_1+R_2,\ -R_1+R_3 \quad \text{Echelon form}$$

Now

$$\begin{bmatrix}1 & 2 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow x + 2y + z = 0$

Let $y=a$ & $z=b$ (giving arbitrary values)

So $x+2a+b=0 \Rightarrow x=-2a-b$

$$v=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2a-b\\a\\b\end{bmatrix}$$

$$=\begin{bmatrix}-2a\\a\\0\end{bmatrix}+\begin{bmatrix}-b\\0\\b\end{bmatrix}$$

$$=a\begin{bmatrix}-2\\1\\0\end{bmatrix}+b\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

linearly independent $\because \begin{bmatrix}1&2&1\\0&0&0\\0&0&0\end{bmatrix}$ is Echelon form.

Scalar multiples of $[1\ 1\ 1]^t$ are all eigen vectors corresponding to $\lambda=5$ and they form a subspace of R^3 generated by $(1\ 1\ 1)^t$ which is also a basis of subspace (called the Eigen space of A corresponding to $\lambda=5$)

Here we have two linearly independent vectors $[-2\ 1\ 0]^t$ and $[-1\ 0\ 1]^t$ corresponding to eigen vector $\lambda=1$.

Any linear combination of $[-2\ 1\ 0]^t$ and $[-1\ 0\ 1]^t$ is also an Eigen vector corresponding to $\lambda=1$.

The Set of linear combinations of these eigen vectors form a subspace of R^3. This subspace is called Eigenspace of A corresponding to $\lambda=1$

The basis of eigen space $=\{[-2\ 1\ 0]^t, [-1\ 0\ 1]^t\}$

Theorem

Non-zero eigen vectors of a matrix A corresponding to distinct eigen values are linearly independent.

Proof Let $v_1, v_2, ---v_n$ be non zero eigen vectors of matrix A corresponding to distinct eigen values $\lambda_1, \lambda_2, \lambda_3, ---\lambda_n$ respectively.

We prove the theorem by induction.

For $n=1$

then v_1 is linearly independent as $v_1 \neq 0$ so $a=0$.

For $n=K$ suppose the theorem is true i.e vectors $v_1, v_2 --- v_K$ are linearly independent. i.e $a_1v_1 + a_2v_2 + --- a_K v_K = 0$ — ①

For $n=K+1$ Consider $a_1v_1 + a_2v_2 + --- a_K v_K + a_{K+1} v_{K+1} = 0$ — ②

$$A(a_1v_1 + a_2v_2 + ----- a_n v_K + a_{K+1} v_{K+1}) = A.0$$

$$\Rightarrow a_1 Av_1 + a_2 Av_2 + ---- a_K Av_K + a_{K+1} Av_{K+1} = 0$$

$$\Rightarrow a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + ----- a_K \lambda_K v_K + a_{K+1} \lambda_{K+1} v_{K+1} = 0 \quad \text{—③} \quad \because Av = \lambda v \text{ By def.}$$

Multiply ② by λ_{K+1} $\Rightarrow a_1 \lambda_{K+1} v_1 + a_2 \lambda_{K+1} v_2 + ------ a_K \lambda_{K+1} v_K + a_{K+1} \lambda_{K+1} v_{K+1} = 0$ — ④

Subtracting ④ from ③ $\Rightarrow a_1(\lambda_1 - \lambda_{K+1})v_1 + a_2(\lambda_2 - \lambda_{K+1})v_2 + ---- a_K(\lambda_K - \lambda_{K+1})v_K + 0 = 0$

From ① the vectors $v_1, v_2 ---- v_K$ are linearly independent (As supposed)

So $a_1(\lambda_1 - \lambda_{K+1}) = 0$

$a_2(\lambda_2 - \lambda_{K+1}) = 0$

$\vdots$

$a_K(\lambda_K - \lambda_{K+1}) = 0$

Since $\lambda_1, \lambda_2 --- \lambda_K$ are all distinct So $\lambda_i - \lambda_{K+1} \neq 0$ $i=1,2--K$

$\Rightarrow a_i = 0$ $i=1,2---K$

using $a_i = 0$, $i=1,2--K$ in ② $\Rightarrow a_{K+1} v_{K+1} = 0$

$\Rightarrow a_{K+1} = 0$ $\because v_{K+1}$ is nonzero

Thus vectors $v_1, v_2 --- v_{K+1}$ are all linearly independent

Hence by Induction $v_1, v_2, ---- v_n$ are linearly independent.

Theorem Any two eigen vectors corresponding to two distinct eigen values of an orthogonal matrix are orthogonal.

Proof Let A be an $n\times n$ orthogonal matrix, having eigen vectors v_1 & v_2 corresponding to distinct eigen values λ_1 & λ_2.

Then by def $Av_1 = \lambda_1 v_1$ —— (1)

and $Av_2 = \lambda_2 v_2$ $\qquad \lambda_1 \neq \lambda_2$ ∵ distinct eigenvalues (given)

taking transpose $(Av_2)^t = (\lambda_2 v_2)^t$

$v_2^t A^t = \lambda_2 v_2^t$ —— (2)

$\because Av_1 = \lambda_1 v_1$

$\times$ by $v_2^t A^t$

$\therefore Av_1 v_2^t A^t = \lambda v_1 v_2^t A^t$

$= \lambda_1 v_1 \lambda_2 v_2$ using (2)

From (1) & (2) $(v_2^t A^t) A v_1 = (\lambda_2 v_2^t)(\lambda_1 v_1)$

$v_2^t (A^t A) v_1 = \lambda_1 \lambda_2 v_2^t v_1$

$v_2^t I v_1 = \lambda \lambda_2 v_2^t v_1$

$v_2^t v_1 = \lambda \lambda_2 v_2^t v_1$

$v_2^t v_1 (1 - \lambda_1 \lambda_2) = 0$ —— (3)

$v_2^t v_1 (\lambda_2^2 - \lambda_1 \lambda_2) = 0$ $\qquad$ Since by theorem if λ is an eigen value of an orthogonal matrix then $|\lambda| = 1$ i,e $\lambda^2 = 1$ $\therefore \lambda_2^2 = 1$

$v_2^t v_1 \lambda_2 (\lambda_2 - \lambda_1) = 0$

As $\lambda_2 \neq \lambda_1$ & $\lambda_2 \neq 0$ $\because |\lambda_2| = 1$

$\therefore v_2^t v_1 = 0$

$\Rightarrow \langle v_2, v_1 \rangle = 0$ $\qquad$ ∵ (see Example 2) I.P of v_1 & v_2 Col vector is $\langle v_1 \cdot v_2 \rangle = v_1^t v_2$

Hence v_1, v_2 are orthogonal vectors.

Theorem If λ is an eigen value of an orthogonal matrix, then $|\lambda|=1$

Proof Let A be a square matrix of order 'n'. Also A is orthogonal matrix and 'λ' be an eigen value of A. Then there exists a non-zero column vector $v \in R^n$ such that

$$Av = \lambda v \quad \text{———} (1)$$

taking transpose

$$(Av)^t = (\lambda v)^t$$

$$v^t A^t = \lambda v^t \text{———} (2) \qquad \because \lambda \text{ is scalar.}$$

From (1) & (2) $\quad (v^t A^t) Av = (\lambda v^t) \lambda v \qquad$ ($\times$ both sides of (2) by Av, then put $Av = \lambda v$ on R.H.S.

$$v^t (A^t A) v = \lambda^2 v^t v$$

$$v^t I v = \lambda^2 v^t v \qquad \because A \text{ is orthogonal so } A^t A = I$$

$$v^t v = \lambda^2 v^t v$$

$$v^t v (1 - \lambda^2) = 0$$

$$1 - \lambda^2 = 0 \qquad \because v \neq 0 \text{ so } v^t v \neq 0$$

$$1 = \lambda^2$$

$$1 = |\lambda|^2$$

$$1 = |\lambda| \quad \text{Proved}$$

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Ex 7.3

Q1 For each of the following matrices, find the characteristic polynomial, all eigen values and a basis of each eigen space.

(i) $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$

Characteristic Polynomial is $|A-\lambda I|$

$A - \lambda I = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{pmatrix}$ ——— (1)

$|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$ $\quad \because |A-\lambda I| = 0$

$\Rightarrow (3-\lambda)\{(4-\lambda)(3-\lambda)-2\} - \{6-2\lambda-2\} + \{2-4+\lambda\} = 0$

$\Rightarrow (3-\lambda)\{12-7\lambda+\lambda^2-2\} - 4+2\lambda+2-4+\lambda = 0$

$\Rightarrow (3-\lambda)(10-7\lambda+\lambda^2) - 6 + 3\lambda = 0$

$\Rightarrow 30-21\lambda+3\lambda^2-10\lambda+7\lambda^2-\lambda^3-6+3\lambda = 0$

$\Rightarrow -\lambda^3+10\lambda^2-28\lambda+24 = 0$

$|A-\lambda I| = \boxed{\lambda^3 - 10\lambda^2 + 28\lambda - 24}$ Characteristic Polynomial.

Eigen Values (i.e Roots) are $\lambda = 2, 2, 6$

$$\begin{array}{r|rrrr} 1 & 1 & -10 & 28 & -24 \\ & \downarrow & 1 & -9 & 19 \\ \hline & 1 & -9 & 19 & -5 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -10 & 28 & -24 \\ & \downarrow & 2 & -16 & 24 \\ \hline & 1 & -8 & 12 & 0 \end{array}$$

$\lambda^2 - 8\lambda + 12 = 0$

$\lambda = \dfrac{8 \pm \sqrt{64-48}}{2}$

$\lambda = \dfrac{8\pm 4}{2} = 2, 6$

Let Eigen Vector 'V' for $\lambda = 2$ is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$(A-\lambda I)V = 0$

Put $\lambda = 2$ in (1)
$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Now reduce matrix in Echelon form $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \overset{R}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ by $-2R_1+R_2$, $-R_1+R_3$

Now $\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\Rightarrow \quad x + y + z = 0$

For sol Giving arbitrary values to y & z, for Let $y = a$, $z = b$

$x + a + b = 0$

$x = -a - b$

then $v = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-a-b\\ a\\ b\end{bmatrix} = \begin{bmatrix}-a\\ a\\ 0\end{bmatrix} + \begin{bmatrix}-b\\ 0\\ b\end{bmatrix}$

$= +a\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix} + b\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$

The Eigen vectors corresponding to $\lambda = 2$ are $[-1 \ 1 \ 0]^t$, $[-1 \ 0 \ 1]^t$

Basis of EigenSpace corresponding to $\lambda = 2$ is $\{[-1 \ 1 \ 0]^t, [-1 \ 0 \ 1]^t\}$

Now for $\lambda = 6$ EigenVector $v = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$(A - \lambda I)v = 0$

Put $\lambda = 6$ in (1) $\begin{bmatrix}-3 & 1 & 1\\ 2 & -2 & 2\\ 1 & 1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

Now reduce Matrix in Echelon form $\begin{bmatrix}-3 & 1 & 1\\ 2 & -2 & 2\\ 1 & 1 & -3\end{bmatrix} \overset{R}{\sim} \begin{bmatrix}1 & 1 & -3\\ 2 & -2 & 2\\ -3 & 1 & 1\end{bmatrix}_{\text{by } R_{13}} \overset{R}{\sim} \begin{bmatrix}1 & 1 & -3\\ 0 & -4 & 8\\ 0 & 4 & -8\end{bmatrix}_{\text{by } -2R_1+R_2,\ 3R_1+R_3} \overset{R}{\sim} \begin{bmatrix}1 & 1 & -3\\ 0 & 1 & -2\\ 0 & 4 & -8\end{bmatrix}_{\text{by } \frac{-1}{4}R_2}$

$\overset{R}{\sim} \begin{bmatrix}1 & 1 & -3\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix}_{\text{by } -4R_2+R_3}$

$x + y - 3z = 0$

$y - 2z = 0 \Rightarrow y = 2z$

Let $z = a$ be arbitrary value

$x + 2a - 3a = 0 \Rightarrow x = a$

$y = 2a$

$z = a$

$\therefore v = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}a\\ 2a\\ a\end{bmatrix} = a\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$

$v = [1 \ 2 \ 1]^t$

Eigen Vector corresponding to $\lambda = 6$ is $[1 \ 2 \ 1]^t$

Basis of EigenSpace corresponding to $\lambda = 6$ is $\{[1 \ 2 \ 1]^t\}$

x——————x

Q 1(ii)

$$A=\begin{pmatrix}1 & 2 & 2\\ 1 & 2 & -1\\ -1 & 1 & 4\end{pmatrix}$$

$$A-\lambda I=\begin{pmatrix}1 & 2 & 2\\ 1 & 2 & -1\\ -1 & 1 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$A-\lambda I=\begin{pmatrix}1-\lambda & 2 & 2\\ 1 & 2-\lambda & -1\\ -1 & 1 & 4-\lambda\end{pmatrix} \quad \text{———— (1)}$$

$|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix}1-\lambda & 2 & 2\\ 1 & 2-\lambda & -1\\ -1 & 1 & 4-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(4-\lambda)+1\right]-2\left[(4-\lambda)-1\right]+2\left[1+(2-\lambda)\right]=0$$

$$\Rightarrow (1-\lambda)\left[8-6\lambda+\lambda^2+1\right]-2\left[\cancel{3-\lambda}\right]+2\left[\cancel{3-\lambda}\right]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-6\lambda+9)=0$$

$$\Rightarrow \lambda^2-6\lambda+9-\lambda^3+6\lambda^2-9\lambda=0$$

$$\Rightarrow -\lambda^3+7\lambda^2-15\lambda+9=0$$

$$\Rightarrow \lambda^3-7\lambda^2+15\lambda-9=0$$

$|A-\lambda I| = \boxed{\lambda^3-7\lambda^2+15\lambda-9}$ Characteristic Polynomial

Eigen Values (i.e Roots) are $\lambda = 1, 3, 3$

1	1	−7	15	−9
	↓	1	−6	9
	1	−6	9	0

$\lambda^2-6\lambda+9=0$

$\lambda = \dfrac{6\pm\sqrt{36-36}}{2}$

$\lambda = \dfrac{6\pm 0}{2} = 3, 3$

For $\lambda=1$

Let Eigen Vector 'V' $=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$(A-\lambda I)V=0$

Put $\lambda=1$ in (1)

$$\begin{bmatrix}0 & 2 & 2\\ 1 & 1 & -1\\ -1 & 1 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now reduce matrix in Echelon form

$$\begin{bmatrix}0 & 2 & 2\\ 1 & 1 & -1\\ -1 & 1 & 3\end{bmatrix} \underset{R}{\sim} \begin{bmatrix}1 & 1 & -1\\ 0 & 2 & 2\\ -1 & 1 & 3\end{bmatrix} \text{by } R_{12} \underset{R}{\sim} \begin{bmatrix}1 & 1 & -1\\ 0 & 2 & 2\\ 0 & 2 & 2\end{bmatrix} \text{by } R_1+R_3$$

$$\underset{R}{\sim} \begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 0 & 2 & 2\end{bmatrix} \text{by } \tfrac{1}{2}R_2 \underset{R}{\sim} \begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix} \text{by } -2R_2+R_3 \text{ Echelon form}$$

Now $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + y - z = 0$

$y + z = 0 \quad \Rightarrow y = -z$

Let $z = a$ be arbitrary value

$x + (-a) - a = 0 \quad \Rightarrow x = 2a$

$y = -a$

$z = a$

$\therefore v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2a \\ -a \\ a \end{bmatrix} = a\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Eigen Vector 'v' $= [2 \;\; -1 \;\; 1]^t$ corresponding to $\lambda = 1$

Basis of Eigen Space corresponding to Eigen Value $\lambda = 1$ is $\{[2 \;\; -1 \;\; 1]^t\}$

Now for $\lambda = 3$ Eigen Vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)v = 0$

Put $\lambda = 3$ in (1) $\begin{bmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now Reduce Matrix in Echelon form $\begin{bmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \overset{R}{\sim} \begin{bmatrix} 1 & -1 & -1 \\ -2 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ by R_{12} $\overset{R}{\sim} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ by $2R_1 + R_2$, $R_1 + R_3$

Now $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x - y - z = 0$

Let $y = a$, $z = b$ be arbitrary values.

$\therefore \Rightarrow x = a + b.$

Hence $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ b \end{bmatrix}$

$= a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Therefore the Eigen Vectors corresponding to Eigen Value $\lambda = 3$ are $[1 \;\; 1 \;\; 0]^t$ & $[1 \;\; 0 \;\; 1]^t$

Basis of Eigen Space corresponding to $\lambda = 3$ is $\{[1 \;\; 1 \;\; 0]^t, [1 \;\; 0 \;\; 1]^t\}$

Q1(iii) $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix} \quad \text{———} (1)$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(4-\lambda)(-5-\lambda)+18\right] + 3\left[3(4-\lambda) - 18\right] + 3\left[-18 + 6(5+\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[-20+\lambda+\lambda^2+18\right] + 3\left[12-3\lambda-18\right] + 3\left[12+6\lambda\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2+\lambda-2\right] + 3\left[-6-3\lambda\right] + (36+18\lambda) = 0$$

$$\Rightarrow \cancel{\lambda^2} + \lambda - 2 - \lambda^3 - \cancel{\lambda^2} + 2\lambda - 18 - 9\lambda + 36 + 18\lambda = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda + 16 = 0$$

$|A - \lambda I| = \boxed{\lambda^3 - 12\lambda - 16}$ Characteristic Polynomial

Eigen Values (i.e) Roots are $\lambda = 4, -, -2$

4	1	0	-12	-16
	↓	4	16	16
	1	4	4	(0

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-16}}{2}$$

$$= \frac{-4 \pm 0}{2} = -2, -2$$

For $\lambda = 4$ Let Eigen Vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)v = 0$

Put $\lambda = 4$ in (1) $\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Now reduce Matrix in Echelon Form

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \overset{R}{\sim} \begin{pmatrix} 1 & 1 & -1 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \text{by } -\tfrac{1}{3}R_1 \quad \overset{R}{\sim} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{pmatrix} \text{by } \begin{matrix} -3R_1+R_2 \\ -6R_1+R_3 \end{matrix}$$

$$\overset{R}{\sim}\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & -\frac{1}{2}\\ 0 & -12 & 6\end{bmatrix} -\frac{1}{12}R_2$$

Now $\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\Rightarrow x + y - z = 0$

$y - \frac{z}{2} = 0 \Rightarrow y = \frac{z}{2}$

Let $z = a$ be arbitrary value

$\therefore y = \frac{a}{2}$

and $x + \frac{a}{2} - a = 0 \Rightarrow x = a - \frac{a}{2}$

$x = \frac{a}{2}$

$\therefore v = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}a/2\\ a/2\\ a\end{bmatrix} = a\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ 1\end{bmatrix}$

Note If we take $z = 2a$ then $y = a$ and $x + a - 2a = 0$, $x = a$

$\therefore v = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}a\\ a\\ 2a\end{bmatrix} = a\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}$ Ans in Book

Eigen Vector $v = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 1\end{bmatrix}^t$ corresponding to $\lambda = 4$

Basis of Eigen Space corresponding to Eigen Value $\lambda = 4$ is $\left\{\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 1\end{bmatrix}^t\right\}$

Now for $\lambda = -2$ Eigen Vector $v = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$(A - \lambda I)v = 0$

Put $\lambda = -2$ in (1) $\begin{bmatrix}3 & -3 & 3\\ 3 & -3 & 3\\ 6 & -6 & 6\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

Now reduce Matrix in Echelon Form $\begin{bmatrix}3 & -3 & 3\\ 3 & -3 & 3\\ 6 & -6 & 6\end{bmatrix} \overset{R}{\sim} \begin{bmatrix}1 & -1 & 1\\ 3 & -3 & 3\\ 6 & -6 & 6\end{bmatrix}\frac{1}{3}R_1 \overset{R}{\sim} \begin{bmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{matrix}\\ -3R_1+R_2\\ -6R_1+R_3\end{matrix}$ Echelon form

Now $\begin{bmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\Rightarrow x - y + z = 0$

Let $y = a$, $z = b$ be arbitrary values

$\therefore x - a + b = 0 \Rightarrow x = a - b$

Hence $v = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}a-b\\ a\\ b\end{bmatrix} = \begin{bmatrix}a\\ a\\ 0\end{bmatrix} + \begin{bmatrix}-b\\ 0\\ b\end{bmatrix}$

$= a\begin{bmatrix}1\\ 1\\ 0\end{bmatrix} + b\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$

Therefore the Eigen Vectors corresponding to Eigen Value $\lambda = -2$ are $\begin{bmatrix}1 & 1 & 0\end{bmatrix}^t$ and $\begin{bmatrix}-1 & 0 & 1\end{bmatrix}^t$

Basis of Eigen Space corresponding to $\lambda = -2$ is $\left\{\begin{bmatrix}1 & 1 & 0\end{bmatrix}^t, \begin{bmatrix}-1 & 0 & 1\end{bmatrix}^t\right\}$

Q1 (iv) $A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix} \quad \text{———} \quad (1)$$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-3-\lambda)\left[(5-\lambda)(-2-\lambda)+6\right] - 1\left[-7(-2-\lambda) - 6\right] - 1\left[-42 + 6(5-\lambda)\right] = 0$$

$$\Rightarrow (-3-\lambda)\left[-10-5\lambda+2\lambda+\lambda^2+6\right] - 1\left[+14+7\lambda-6\right] - \left[-42+30-6\lambda\right] = 0$$

$$\Rightarrow (-3-\lambda)(-4-3\lambda+\lambda^2) - 8 - 7\lambda + 12 + 6\lambda = 0$$

$$\Rightarrow 12 + 9\lambda - \cancel{3\lambda^2} + 4\lambda + \cancel{3\lambda^2} - \lambda^3 + 4 - \lambda = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda + 16 = 0$$

$|A - \lambda I| = \boxed{\lambda^3 - 12\lambda - 16}$ Charactristic Polynomial

Eigen Values are $\lambda = 4, -2, -2,$

$$\begin{array}{c|cccc} 4 & 1 & 0 & -12 & -16 \\ & \downarrow & 4 & 16 & 16 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$\lambda^2 + 4\lambda + 4 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{16-16}}{2}$

$= \dfrac{-4 \pm 0}{2} = -2, -2$

For $\lambda = 4$ Eigen Vectors $V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$(A - \lambda I)V = 0$

Put $\lambda = 4$ in (1)
$$\begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now reduce Matrix in Echelon form
$$\begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix} \overset{R}{\sim} \begin{pmatrix} -6 & 6 & -6 \\ -7 & 1 & -1 \\ -7 & 1 & -1 \end{pmatrix}_{R_{13}} \overset{R}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ -7 & 1 & -1 \\ -7 & 1 & -1 \end{pmatrix}_{-\frac{1}{6}R_1}$$

$$\overset{R}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ 0 & -6 & 6 \\ 0 & -6 & 6 \end{pmatrix}_{\substack{7R_1+R_2 \\ 7R_1+R_3}} \overset{R}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -6 & 6 \end{pmatrix}_{-\frac{1}{6}R_2} \overset{R}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}_{6R_1+R_3} \text{Echelon}$$

Now $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x - y + z = 0$

$y - z = 0 \Rightarrow y = z.$

Let $z = a$ be the arbitrary value of

$y = a$

$x - a + a = 0 \Rightarrow x = 0..$

Hence $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vector corresponding to $\lambda = 4$ is $[1\ 1\ 1]^t$

Basis of Eigen Space corresponding to $\lambda = 4$ is $\{[1\ 1\ 1]^t\}$

For $\lambda = -2$ Eigen Vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)v = 0$

Put $\lambda = -2$ in ① $\begin{bmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now reduce Matrix in Echelon Form $\begin{bmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix} \overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix} -1R_1 \overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \end{bmatrix} \begin{matrix} 7R_1+R_2 \\ 6R_1+R_3 \end{matrix}$

$\overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 6 \end{bmatrix} \frac{1}{6}R_2 \overset{R}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} -6R_2+R_3$ Echelon form

Now $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x - y + z = 0$

$z = 0$

$\Rightarrow \quad x - y + 0 = 0 \Rightarrow x = y$

Let $y = a \Rightarrow x = a$

Hence $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector corresponding to $\lambda = -2$ is $[1\ 1\ 0]^t$

Basis of Eigen Space corresponding to $\lambda = -2$ is $\{[1\ 1\ 0]^t$

x————x

Imp Q2 Show that eigen values of a diagonal matrix are its diagonal elements and the eigen vectors are the standard basis vectors.

Sol Let $A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$ Diagonal Matrix

In diagonal matrix diagonal elements are non zero.

$$A-\lambda I = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_{11}-\lambda & 0 & 0 & \cdots & 0 \\ 0 & a_{22}-\lambda & 0 & \cdots & 0 \\ 0 & 0 & a_{33}-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}-\lambda \end{pmatrix} \quad (1)$$

$|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & 0 & 0 & \cdots & 0 \\ 0 & a_{22}-\lambda & 0 & \cdots & 0 \\ 0 & 0 & a_{33}-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\cdots\cdots(a_{nn}-\lambda) = 0$

Since this is determinant of triangular matrix, so it is expanded by Product of diagonal elements

$\Rightarrow (a_{11}-\lambda)=0, \ (a_{22}-\lambda)=0, \ \cdots\cdots (a_{nn}-\lambda)=0$

$\Rightarrow \quad \lambda = a_{11}, a_{22}, a_{33} \cdots\cdots a_{nn}$ i.e. Eigen Values are diagonal elements

Proved.

Now For $\lambda = a_{11}$ Eigen Vector $v_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$

$(A-\lambda I)v_1 = 0$

$$\Rightarrow \begin{bmatrix} a_{11}-a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22}-a_{11} & 0 & \cdots & 0 \\ 0 & 0 & a_{33}-a_{11} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn}-a_{11} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow (a_{11}-a_{11})x_1 + 0.x_2 + 0.x_3 + \cdots 0.x_n = 0 \quad \Rightarrow 0.x_1 = 0 \quad \Rightarrow x_1$ is non-zero,

$0.x_1 + (a_{22}-a_{11})x_2 + 0.x_3 + \cdots 0.x_n = 0 \quad \Rightarrow (a_{22}-a_{11})x_2 = 0 \quad \Rightarrow x_2 = 0$ $\because (a_{22}-a_{11}) \neq 0$ diagonal elements are non-zero

$0.x_1 + 0.x_2 + (a_{33}-a_{11})x_3 + \cdots 0.x_n = 0 \quad \Rightarrow (a_{33}-a_{11})x_3 = 0 \quad \Rightarrow x_3 = 0$ $\because (a_{33}-a_{11})$ is diagonal element

$\vdots$

$0.x_1 + 0.x_2 + 0.x_3 + \cdots (a_{nn}-a_{11})x_n = 0 \quad \Rightarrow (a_{nn}-a_{11})x_n = 0 \quad \Rightarrow x_n = 0$ $\because (a_{nn}-a_{11}) \neq 0$ being diagonal element

Let $\boxed{x_1 = a}$ giving arbitrary value

Hence $v_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow v_1 = [1 \ 0 \ 0 \cdots 0]^t$

Similarly for $\lambda = a_{22}, a_{33}, \cdots a_{nn}$

we have $v_2 = [0 \ 1 \ 0 \cdots 0]^t$, $v_3 = [0 \ 0 \ 1 \cdots 0]^t, \ldots, v_n = [0 \ 0 \cdots 1]^t$

Hence the Eigen Vectors $v_1, v_2, v_3, \ldots v_n$ are Standard basis for R^n

Q3. Show that A and A^t have the same eigen values. Give an example where A and A^t have different eigen vectors (A is a square Matrix)

Sol Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$ Square Matrix

Eigen Values of A are given by $|A-\lambda I| = 0$

$$\therefore A-\lambda I = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & 0 & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn}-\lambda \end{vmatrix} = 0 \quad \text{——— (I)}$$

Now $A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & & & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn}-\lambda \end{pmatrix}$

Eigen values of A^t are given by $|A^t - \lambda I| = 0$

$$|A^t - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22}-\lambda & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33}-\lambda & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \text{——— (II)}$$

We know that the value of determinant is unchanged if rows and columns are interchanged (i.e transpose)

$$\therefore \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = \begin{vmatrix} a_{11}-\lambda & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22}-\lambda & a_{32} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & & \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$|A-\lambda I| = |A^t - \lambda I|$$

Hence Eigen Values of A & A^t are same.

7-3-11

Example of different Eigen Vectors of A & A^t

For Eigen Value $|A-\lambda I|=0$
For Eigen Vector $(A-\lambda I)v=0$

Let $A=\begin{bmatrix}2&4\\1&5\end{bmatrix}$

$$A-\lambda I=\begin{bmatrix}2&4\\1&5\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}2-\lambda&4-0\\1-0&5-\lambda\end{bmatrix}$$

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix}2-\lambda&4\\1&5-\lambda\end{vmatrix}=0$$

$$\Rightarrow (5-\lambda)(2-\lambda)-4=0$$

$$\Rightarrow 10-7\lambda+\lambda^2-4=0$$

$$\Rightarrow \lambda^2-7\lambda+6=0$$

$$\Rightarrow \lambda=\frac{7\pm\sqrt{49-24}}{2}=\frac{7\pm 5}{2}$$

$$\Rightarrow \lambda=6,1 \quad \text{Eigen Values}$$

For $\lambda=1$ Eigen Vector of A is given by $(A-\lambda I)v=0$

Put $\lambda=1$ in (1)

$$\Rightarrow \begin{bmatrix}1&4\\1&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$\Rightarrow x+4y=0 \quad \Rightarrow x=-4y \quad \Rightarrow x=-4a$ where a is arbitrary

$x+4y=0 \quad \Rightarrow -4a+4y=0 \Rightarrow y=\frac{4a}{4}=a$

Say $\boxed{a=1}$

$$\therefore v=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-4a\\a\end{bmatrix}=a\begin{bmatrix}-4\\1\end{bmatrix} \qquad \boxed{v=[-4\ \ 1]^t} \quad \text{—— (3)}$$

For $\lambda=1$ the Eigen Vector of A^t is given by $(A^t-\lambda I)v=0$

$$A^t-\lambda I=\begin{bmatrix}2&1\\4&5\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}2-\lambda&1\\4&5-\lambda\end{bmatrix} \quad \text{—— (2)}$$

$$|A^t-\lambda I|=0 \Rightarrow \begin{vmatrix}2-\lambda&1\\4&5-\lambda\end{vmatrix}=(2-\lambda)(5-\lambda)-4 \Rightarrow \lambda=6,1 \text{ Eigen Values}$$

$(A^t-\lambda I)v=0$

Put $\lambda=1$ in (2)

$$\Rightarrow \begin{bmatrix}1&1\\4&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{matrix}x+y=0\\4x+4y=0\end{matrix} \Rightarrow x=-y=-a \text{ arbitrary Value.}$$

Say $\boxed{a=1}$

$$\therefore v=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-a\\a\end{bmatrix}=a\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\therefore \boxed{v=[-1\ \ 1]^t} \quad \text{—— (4)}$$

from (3) & (4) Eigen Vector for A & A^t are different.

(4) Suppose v is an eigen vector of a square matrix A corresponding to the eigen value λ. Show that for $n>0$, v is also an eigen vector of A^n corresponding to λ^n.

Sol

(given) $AV = \lambda V$ —(i)

$\because$ By def V is an eigen vector of A corresponding to eigen value λ

To Prove $\underline{A^n V = \lambda^n V}$

$\because$ we have to show that V is the eigen vector of A^n corresponding to eigen value λ^n

We prove by induction

For $n=1$ $\quad A^1 V = \lambda^1 V \quad$ true by (i) $\quad$ C-1 is satisfied

For $n=K$ $\quad A^K V = \lambda^K V \quad$ true supposed. —(ii)

Consider $A^{K+1} V = A(A^K V)$

$= A(\lambda^K V)$ $\quad$ using (ii)

$= \lambda^K (AV)$ $\quad$ $\because \lambda^K$ is scalar

$= \lambda^K (\lambda V)$ $\quad$ using (i)

$A^{K+1} V = \lambda^{K+1} V$ $\quad$ C-2 is satisfied

Hence the result is true for all +ve integral values of 'n'.

x———————x

(5) If λ is an eigen value of a non-singular matrix A, then λ^{-1} is an eigen value of A^{-1}.

Sol We know

(given) $\because \lambda$ is an eigen value of A

A is non singular matrix implies A^{-1} exists.

Let V is eigen vector corresponding to eigen value λ.

$AV = \lambda V$

$A^{-1}(AV) = A^{-1}(\lambda V)$

$(A^{-1}A)V = \lambda(A^{-1}V)$

$I\,V = \lambda(A^{-1}V)$

$V = \lambda(A^{-1}V)$

$\lambda^{-1} V = A^{-1} V$ $\quad$ $\because \lambda$ is scalar no.

Hence λ^{-1} is the eigen value of A^{-1}.

x———————x

⑥ If A & B are square matrices show that AB & BA have same eigen values.

Sol $|A-\lambda I|=0$ gives the Eigen values 'λ' of A.

$\therefore |AB-\lambda I|=0$ gives the Eigen values 'λ' of AB

$\Rightarrow \left|B^{-1}B(AB)-B^{-1}B(\lambda I)\right|=0$ $\quad$ × by $B^{-1}B$

$\Rightarrow \left|B^{-1}(BA)B-\lambda(B^{-1}B)I\right|=0$

$\Rightarrow \left|B^{-1}(BA-\lambda I)B\right|=0$

$\Rightarrow |B^{-1}|\,|BA-\lambda I|\cdot|B|=0$ $\quad$ (By Product Th)

$\Rightarrow \frac{1}{|B|}\,|BA-\lambda I|\,|B|=0$ $\quad \because \det A^{-1}=\frac{1}{\det A}$

$\Rightarrow |BA-\lambda I|=0$ gives Eigen Values 'λ' of BA.

⑥ 2nd Method

If λ be an eigen value of A, corresponding to eigen vector V then $AV=\lambda V$

Let λ be an eigen value of AB. then $(AB)V=\lambda V$

$B(AB)V=B(\lambda V)$ $\quad$ × by B

$(BA)(BV)=\lambda(BV)$

which shows that BV is an eigen vector of BA with eigenvalue λ. Thus eigen values of BA & AB are same

⑦ If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigenvalues of a square matrix of order 'n' then $K\lambda_1, K\lambda_2, \ldots K\lambda_n$, where K is a scalar, are eigen values of KA.

Sol If λ be an eigen value of A corresponding to eigen vector V then $AV=\lambda V$

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigenvalues of A of order 'n' corresponding to eigen vector V then $AV=(\lambda_1, \lambda_2 \cdots \lambda_n)V$

$KA.V=K(\lambda_1, \lambda_2 \cdots \lambda_n)V$

$(KA)V=(K\lambda_1, K\lambda_2 \cdots K\lambda_n)V$

which shows $K\lambda_1, K\lambda_2 \ldots K\lambda_n$ are eigen values of KA.

⑧ Suppose v is an eigen vector of $n \times n$ matrices A & B. Show that v is also an eigen vector of $aA - bE$, where a & b are any scalar.

Sol v is the eigenvector of A & B corresponding to eigen value λ_1 & λ_2

then $Av = \lambda_1 v$ and $Bv = \lambda_2 v$

$\Rightarrow aAv = a\lambda_1 v$ —— (i) $\quad bBv = b\lambda_2 v$ —— (ii)

Adding (i) & (ii)

$$aAv + bBv = a\lambda_1 v + b\lambda_2 v$$

$$(aA + bB)v = (a\lambda_1 + b\lambda_2)v$$

$\Rightarrow$ v is the eigen vector of matrix $aA + bB$

x ———————— x

⑨ Let λ be an eigen value of a square matrix A. Let V_λ denote the set of all eigen vectors of A corresponding to eigen value λ. Show that V_λ is a subspace of V (V_λ is the eigen space of A corresponding to λ)

Sol To Prove V_λ is subspace of V.

Let v_1, v_2, belong to V_λ

a_1, a_2 be scalars then

$$Av_1 = \lambda v_1 \text{ —— (i)}$$
$$Av_2 = \lambda v_2 \text{ —— (ii)}$$

$$\text{and } A(a_1v_1 + a_2v_2) = a_1 Av_1 + a_2 Av_2$$
$$= a_1 \lambda v_1 + a_2 \lambda v_2$$
$$= \lambda(a_1 v_1 + a_2 v_2)$$

So $a_1v_1 + a_2v_2$ is an eigen vector of 'A' corresponding to eigen value 'λ'.

Thus $a_1v_1 + a_2v_2$ belongs to V_λ.

Hence V_λ is a subspace of V. (Since if $w_1, w_2 \in W$ and $a, b \in F$ imply $aw_1 + bw_2 \in W$ then W is a subspace of V. where $W \subseteq V$)

x ———————— x

Q10 For the linear transformation $T: R^2 \to R^2$, find all eigenvalues, eigenvectors and a basis for each eigen space.

(i) $T(x,y) = (3x+3y, x+5y)$

We first find matrix of given linear transformation w.r.t standard basis, $e_1 = (1,0)$, $e_2 = (0,1)$

Standard Basis for $R^2 = \{e_1=(1,0), e_2=(0,1)\}$

$T(e_1) = T(1,0) = (3\cdot1+3\cdot0, 1+5\cdot0)$

$= (3, 1) \qquad = 3e_1 + e_2$

$T(e_2) = T(0,1) = (3\cdot0+3\cdot1, 0+5\cdot1)$

$= (3, 5) \qquad = 3e_1 + 5e_2$

Matrix of linear transformation i.e 'T' is $\begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} = A$ (say)

$$A - \lambda I = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{pmatrix} \qquad \text{—— (1)}$$

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(5-\lambda) - 3 = 0$

$\Rightarrow 12 - 8\lambda + \lambda^2 = 0$

$\Rightarrow \lambda = 6, 2$ Eigen Values

$\lambda^2 - 8\lambda + 12 = 0$

$\lambda = \dfrac{8 \pm \sqrt{64 - 4\cdot1\cdot12}}{2}$

$= \dfrac{8 \pm \sqrt{16}}{2}$

$= 6, 2$

For $\underline{\lambda = 6}$ let Eigen Vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

put $\lambda = 6$ in (1)

$\Rightarrow -3x + 3y = 0 \qquad \Rightarrow x = \cancel{3}y/\cancel{3}$

$x - y = 0$

Let $x = a$ any arbitrary value.

$\therefore v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = a\begin{bmatrix} & \end{bmatrix}$

$\therefore$ Eigen Vector for $\lambda = 6$ is $[1 \quad 1]^t$

The set $\{[1 \quad 1]^t\}$ generates the subspace of R^2 known as Eigen Space of A.

Basis of Eigen Space $\{[1 \quad 1]^t\}$

For $\underline{\lambda = 2}$

$(A - \lambda I)v = 0$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + 3y = 0 \Rightarrow x = -3y \qquad$ Let $y = a$ any arbitrary value

$x + 3y = 0$

$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3a \\ a \end{bmatrix} = a\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\therefore$ Eigen Vector for $\lambda = 2$ is $[-3 \quad 1]^t$

Any linear combination of $[-3 \quad 1]^t$ is also Eigen Vector for $\lambda = 2$. The set $\{[-3 \quad 1]^t\}$ i,e i,e set of linear combinations is a subspace of R^2 called Eigen Space of A having basis $\{[-3 \quad 1]^t\}$

7-3-16

10 (ii) For the linear transformation $T: R^2 \to R^2$ find all eigenvalues, eigen Vectors and Basis for eigen Space

$T(x,y) = (y,x)$

$T(e_1) = T(1,0) = (0,1) = 0.e_1 + e_2$

$T(e_2) = T(0,1) = (1,0) = 1.e_1 + 0.e_2$

$e_1 = (1,0)$, $e_2 = (0,1)$ standard Basis for R^2

Matrix of T is
i.e Matrix of linear Transformation $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A$

$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$ —— ①

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$

$\lambda^2 = 1 \Rightarrow \lambda = 1, -1$ Eigen Values.

For $\lambda = 1$

$(A - \lambda I)V = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Put $\lambda = 1$ in ①

$\Rightarrow -x + y = 0$
$x - y = 0$

$\Rightarrow y = x$ Let $y = a$, arbitrary value $\therefore x = a$

$\therefore V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigen Vector corresponding to $\lambda = 1$ is $[1 \ \ 1]^t$

Any linear combination of $[1 \ \ 1]^t$ is also the eigenvector for $\lambda = 1$

The Set $\{[1 \ \ 1]^t\}$ i.e set of linear combinations is a subspace of R^2 called Eigen Space having basis $\{[1 \ \ 1]^t\}$.

For $\lambda = -1$

$(A - \lambda I)V = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Put $\lambda = 1$ in ①

$\Rightarrow x + y = 0$
$x + y = 0$

$\Rightarrow x = -y$ Let $y = a$ arbitrary value $\therefore x = -a$

$\therefore V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -a \\ a \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigen Vector corresponding to $\lambda = -1$ is $[-1 \ \ 1]^t$

Any linear combination of $[-1 \ \ 1]^t$ is also eigen vector for $\lambda = -1$

The Set $\{[-1 \ \ 1]^t\}$ i.e set of linear combinations is a subspace of R^2 called Eigen Space having basis $\{[-1 \ \ 1]^t\}$

Q10 (iii) For the linear transformation $T: R^2 \to R^2$ find all eigen values and a basis for each eigen space.

7-3-17

$T(x,y) = (y,-x)$

$T(e_1) = T(1,0) = (0,-1) \;:=\; 0 e_1 + e_2$

$T(e_2) = T(0,1) = (1,0) \;:=\; 1\cdot e_1 + 0 e_2$

Matrix of linear transformation i.e. T $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \quad \text{——— (1)}$$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

$$\lambda = \pm i.$$

Eigen Values are not real i.e. $\pm i$

Q11 (i) For each of the following operators $T: R^3 \to R^3$, find all eigenvalues and a basis for each eigen space.

(i) $T(x,y,z) = (x+y+z,\ 2y+z,\ 2y+3z)$

Standard Basis for R^3
$e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

$T(1,0,0) = (1+0+0,\ 2(0)+0,\ 2(0)+3(0))$

$= (1,0,0) \;=\; 1\cdot e_1 + 0\cdot e_2 + 0\cdot e_3$

$T(0,1,0) = (1,2,2) \;=\; 1\cdot e_1 + 2 e_2 + 2 e_3$

$T(0,0,1) = (1,1,3) \;:=\; \cdot e_1 + 1\cdot e_2 + 3\cdot e_3$

$\Rightarrow$ Matrix of T $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} = A$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{bmatrix} \quad \text{—— (1)}$$

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$|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\{(2-\lambda)(3-\lambda)-2\} - 1(0-0) + 1(0-0) = 0$

$\Rightarrow (1-\lambda)(4-5\lambda+\lambda^2) = 0$

$\Rightarrow (1-\lambda)(\lambda-1)(\lambda-4) = 0$

Eigen Values are $\lambda = 1, 1, 4$

Let Eigen Vector 'V' for $\lambda = 1$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A-\lambda I)V = 0 \Rightarrow$

Put $\lambda=1$ in ① $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x + y + z = 0$ $\Rightarrow y = -z$ let $z = b$ & $x = a$

$0\cdot x + 1\cdot y + z = 0$ (arbitrary values)

$0\cdot x + 2y + 2z = 0$

$\therefore V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ -b \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Eigen Vectors corresponding to $\lambda=1$ are $[1\ 0\ 0]^t, [0\ -1\ 1]^t$

These eigen vectors generates the subspace of R^3 called Eigen Space have basis $\{[1\ 0\ 0]^t, [0\ -1\ 1]^t\}$

2nd Method to find Eigen Vector

$(A-\lambda I)V = 0$

$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$\sim_R \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Echelon form by $-R_1+R_2$, $-2R_1+R_3$

$y+z=0$

Let $z = b$

& $x = a$

then $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ -b \\ b \end{bmatrix}$

Now Eigen Vector 'V' for $\lambda = 4$ $(A-\lambda I)V = 0$

Put $\lambda=4$ in ① $\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x + y + z = 0$ $\Rightarrow 3x = y+z \Rightarrow \cancel{3}x = \cancel{3}y \Rightarrow x = y$

$-2y + z = 0$ $\Rightarrow z = 2y$

$2y - z = 0$ $\Rightarrow z = 2y$ Let

Let $x = a$ $\therefore V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ 2a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Eigen Vector corresponding to $\lambda = 4$ is $[1\ 1\ 2]^t$

Eigen Vector generates a subspace of R^3 called Eigen Space of R^3 corresponding to $\lambda = 4$ having basis $\{[1\ 1\ 2]^t\}$

2nd Method

$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ by R_2+R_3 Echelon form

$-3x+y+z=0$

$-2y+z=0$

$\Rightarrow z = 2y$

$\Rightarrow x = y$

Let $x = a$,

$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a \\ 2a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Exercise 8.1 - Chapter 08
Mathematical Methods
by S.M. Yusuf, A. Majeed and M. Amin
ILMI KITAB KHANA, LAHORE.

Infinite Series

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Sequence is a fn $f(n)$ whose domain is set of natural numbers and whose range is a subset of real numbers.

e.g $f(n) = \frac{1}{n+1}$ Domain $= \{1, 2, 3, \ldots\}$ Range $= \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

Generally seq is denoted by $\{a_n\}$, and a_n = n^{th} term of seq.

Infinite Sequence has infinite number of terms, eg $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \ldots$

$\frac{1}{n} = a_n = n^{th}$ term

Finite Sequence has finite number of terms

Convergence of a Sequence An infinite seq $\{a_n\}$ is said to be convergent if n^{th} term of seq tends to a definite real number 'l' as $n \to \infty$

i.e $\boxed{\lim_{n\to\infty} a_n = l}$

Divergence of a Sequence A seq $\{a_n\}$ is said to be Divergent if n^{th} term of seq tends to $\pm\infty$ as $n \to \infty$ i.e $\boxed{\lim_{n\to\infty} a_n = \pm\infty}$

Bounded Above If B is a fixed number such that $a_n \leq B$ for every +ive integral value of 'n', then we say $\{a_n\}$ seq is bounded above.

Bounded Below If B is a fixed number such that $a_n \geq B$ for every +ive integral value of 'n', then we say $\{a_n\}$ seq is bounded below.

Bounded Sequence is A seq which is bounded above & bounded below.

Monotonic Sequences A sequence $\{a_n\}$ is said to be

i) Non Decreasing if $a_{n+1} \geq a_n \quad \forall n$

ii) Non Increasing if $a_{n+1} \leq a_n \quad \forall n$

iii) strictly Increasing if $a_{n+1} > a_n \quad \forall n$

iv) strictly Decreasing if $a_{n+1} < a_n \quad \forall n$

Note

1) A convergent seq is Bounded but A Bounded seq need not be Convergent
2) A Bounded Monotonic Seq is Convergent
3) A Bounded above & Monotonic Increasing Seq is Convergent
4) A Bounded below & Monotonic Decreasing Seq is Convergent
5) An Unbounded Seq is Divergent

EXERCISE 6.1 N.M.

The nth term of a sequence is given. Determine whether the sequence converges or diverges. If it converges find its limit.

Q1 $\dfrac{2}{\sqrt{n^2+3}}$

SOL we have $a_n = \dfrac{2}{\sqrt{n^2+3}}$

then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n^2+3}}$

$= \lim\limits_{n\to\infty} \dfrac{2}{n\sqrt{1+3/n^2}} = \dfrac{2}{\infty\sqrt{1+\frac{3}{\infty}}}$

$\boxed{\lim\limits_{n\to\infty} a_n = 0}$ ∵ '0' is definite number. So the sequence $\{a_n\}$ is Convergent, having limit '0'

Q2 $\dfrac{(n-3)!}{(n-1)!}$

SOL. ∵ $a_n = \dfrac{(n-3)!}{(n-1)!}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{(n-3)!}{(n-1)!}$

$= \lim\limits_{n\to\infty} \dfrac{(n-3)!}{(n-1)(n-2)(n-3)!}$

$= \lim\limits_{n\to\infty} \dfrac{1}{(n-1)(n-2)} = \dfrac{1}{\infty}$

$\boxed{\lim\limits_{n\to\infty} a_n = 0}$

⇒ The Sequence $\{a_n\}$ Converges to 0

Q3 $1 + \dfrac{(-1)^n}{n}$

SOL. Here $a_n = 1 + \dfrac{(-1)^n}{n}$

then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left[1 + \dfrac{(-1)^n}{n}\right]$

⇒ $\lim\limits_{n\to\infty} a_n = 1 + 0 = 1$

⇒ $\boxed{\lim\limits_{n\to\infty} a_n = 1}$ ∴ $\{a_n\}$ is Convergent Seq.

Q.4 $a_n = \dfrac{\sqrt{n+1}}{n}$

SOL. ⇒ $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{\sqrt{n+1}}{n}$

$= \lim\limits_{n\to\infty} \dfrac{\sqrt{n}\sqrt{1+\frac{1}{n}}}{n}$

$= \lim\limits_{n\to\infty} \dfrac{\sqrt{1+\frac{1}{n}}}{\sqrt{n}} = 0$

$\boxed{\lim\limits_{n\to\infty} a_n = 0}$

∵ 0 is definite number

so Sequence $\{a_n\}$ Converges to '0'.

Q.5

SOL. $a_n = n^{\frac{1}{n}}$

∴ $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^{\frac{1}{n}}$ $\quad (\infty^0)$

So let $y = n^{\frac{1}{n}}$

⇒ $\ln y = \ln n^{\frac{1}{n}}$

$= \dfrac{1}{n}\ln n = \dfrac{\ln n}{n}$

⇒ $\lim\limits_{n\to\infty} \ln y = \lim\limits_{n\to\infty} \dfrac{\ln n}{n}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{n\to\infty} \dfrac{1/n}{1}$ (L'hospital Rule)

$= \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

⇒ $\lim\limits_{n\to\infty} \ln y = 0$

ALTERNATE:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^4+1}{4n^2-1}$$

$$= \lim_{n\to\infty} \frac{n^4\left(3+\frac{1}{n^4}\right)}{n^2\left(4-\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2\left(3+\frac{1}{n^4}\right)}{4-\frac{1}{n^2}}$$

$$= \frac{\infty(3+0)}{4-0} = \infty$$

$\Rightarrow$ Then $\{a_n\}$ is a Divergent Sequence

Q.8 $a_n = \frac{\ln(n+1)}{\sqrt{n}}$

Sol

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} \quad \left(\frac{\infty}{\infty}\right)$$

(by L, Hospital)

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} \quad \left(\frac{\infty}{\infty}\right)$$

(by L, Hospital)

$$= \lim_{n\to\infty} \frac{\cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{n}}}{1} = \lim_{n\to\infty} \frac{1}{\sqrt{n}}$$

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

Sequence $\{a_n\}$ Converges to 0

Q.9 $a_n = \frac{e^n}{n}$

Sol.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{e^n}{1}$$

$$\lim_{n\to\infty} a_n = \infty$$

So Seq $\{a_n\}$ is Divergent

Q.10 Easy $a_n = \ln n - \ln(n+1)$

Sol.

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\ln n - \ln(n+1)\right) \quad (\infty-\infty \text{ form})$$

$$= \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)$$

$$= \ln \lim_{n\to\infty}\left(\frac{n}{n+1}\right) \quad \left(\frac{\infty}{\infty}\right)$$

$$= \ln \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}}\left(\frac{1}{1+\frac{1}{n}}\right) = \ln 1 = 0$$

$$\boxed{\lim_{n\to\infty} a_n = 0}$$ $\therefore$ Seq $\{a_n\}$ converges to '0'

ALTERNATE Difficult

$\because a_n = \ln n - \ln(n+1)$ ($\infty - \infty$ form)

$$\Rightarrow a_n = \frac{\frac{1}{\ln(n+1)} - \frac{1}{\ln n}}{1/\ln n \cdot \ln(n+1)}$$

(4)

$\Rightarrow \lim_{n\to\infty} y = e^0 = 1$

or $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ Definite number

$$\boxed{\lim_{n\to\infty} a_n = 1}$$

$\Rightarrow$ The sequence $\{a_n\}$ converges to 1.

Easy **Q.6** $a_n = \frac{2^n}{(2n)!}$

Sol

$$\therefore = \frac{2^n}{(2n)(2n-1)(2n-2)(2n-3)(2n-4)\cdots 6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$$= \frac{2^n}{[(2n)(2n-2)(2n-4)(2n-6)\cdots 6\cdot4\cdot2][(2n-1)(2n-3)\cdots 5\cdot3\cdot1]}$$

$$= \frac{\cancel{2^n}}{[\cancel{2^n}(n)(n-1)(n-2)(n-3)\cdots 3\cdot2\cdot1][(2n-1)(2n-3)(2n-5)\cdots 5\cdot3\cdot1]}$$

$$a_n = \frac{1}{n!\,[(2n-1)(2n-3)(2n-5)\cdots 5\cdot3\cdot1]}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n!\,[(2n-1)(2n-3)\cdots 5\cdot3\cdot1]} = 0$$

$\Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}$ $\therefore$ Seq $\{a_n\}$ converges to 0

Difficult 2nd Method Q6

Sol Let $a_n = \frac{2^n}{\underline{|2n}}$ we prove that given seq $\{a_n\}$ is bounded & decreasing

$$a_{n+1} = \frac{2^{n+1}}{\underline{|2(n+1)}} = \frac{2^{n+1}}{\underline{|2n+2}}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{\underline{|2n+2}} \cdot \frac{\underline{|2n}}{2^n} = \frac{\cancel{2^n}\cdot 2 \cdot \cancel{\underline{|2n}}}{(2n+2)(2n+1)\cancel{\underline{|2n}}\,\cancel{2^n}}$$

$$\frac{a_{n+1}}{a_n} = \frac{2}{(2n+2)(2n+1)} = \frac{1}{(n+1)(2n+1)} < 1 \;\forall n$$

$\Rightarrow a_{n+1} < a_n \therefore$ given seq $\{a_n\}$ is decreasing

$$\text{Now } a_n = \frac{2^n}{\underline{|2n}} = \frac{\cancel{2^n}}{\cancel{2^n}\,\underline{|n}\,(2n-1)(2n-3)\cdots 3\cdot 1}$$

$$= \frac{1}{\underline{|n}\,(2n-1)(2n-3)\cdots 3\cdot1} \leq 1$$

$\therefore$ Seq is bounded

$\because$ Seq is bounded & Decreasing Hence Convergent.

Now $\lim_{n\to\infty} a_n$

$$= \frac{1}{\underline{|n}\,(2n-1)(2n-3)\cdots 3\cdot 1}$$

$= 0$

So limit of seq is 0.

Difficult 2nd Method

Q7 $a_n = \frac{3n^4+1}{4n^2-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^4+1}{4n^2-1} \left(\frac{\infty}{\infty}\right)$$

using L' Hospital Rule

$$= \lim_{n\to\infty} \frac{12n^3}{8n} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{24n^2}{8} = \infty$$

which is not a definite number

So Sequence $\{a_n\}$ is divergent sequence.

Q7 2° ALTERNATE - Easy

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^4+1}{4n^2-1}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cancel{n^4}\left[3+\frac{1}{n^4}\right]}{\cancel{n^4}\left[\frac{4}{n^2}-\frac{1}{n^4}\right]}$$

$$= \frac{3+0}{0} = \infty$$

$\Rightarrow \boxed{\lim_{n\to\infty} a_n = \infty}$ $\{a_n\}$ is a divergent Seq.

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{\ln(n+1)} - \frac{1}{\ln n}}{\frac{1}{(\ln n)(\ln(n+1))}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{(\cancel{n}+1) - \cancel{n}}{\frac{1}{\frac{1}{n}\ln(n+1) + \frac{1}{(n+1)}\ln n}} \quad (\text{by L'Hospital Rule})$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{\frac{\ln(n+1)}{n} + \frac{\ln n}{n+1}}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\frac{(n+1)\ln(n+1)+n\ln n}{n(n+1)}}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{n(n+1)}{(n+1)\ln(n+1)+n\ln n}} = \lim_{n\to\infty} \frac{(n+1)\ln(n+1)+n\ln n}{n^2+n}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot\frac{1}{n+1} + \ln(n+1) + n\cdot\frac{1}{n} + \ln n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{1 + \ln(n+1) + 1 + \ln n}{2n+1} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1} + \frac{1}{n}}{2} = \lim_{n\to\infty} \frac{n + n+1}{2n(n+1)} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{1+1}{4n+2} = 0$$

$\Rightarrow$ the sequence $\{a_n\}$ converges to 0.

Q.11 $a_n = \dfrac{\sin^2 n}{n}$

Sol

We know that

$$0 \leq \sin^2 n \leq 1$$

$\div$ by n

$$\Rightarrow \frac{0}{n} \leq \frac{\sin^2 n}{n} \leq \frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{0}{n} \leq \lim_{n\to\infty} \frac{\sin^2 n}{n} \leq \lim_{n\to\infty} \frac{1}{n}$$

$\frac{1}{\infty} = \frac{0}{0} = 0$

$$\Rightarrow 0 \leq \lim_{n\to\infty} \frac{\sin^2 n}{n} \leq 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin^2 n}{n} = 0$$

i.e. $\boxed{\lim_{n\to\infty} a_n = 0}$ $\therefore$ Seq $\{a_n\}$ converges to '0'

ALTERNATE Easy

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin^2 n}{n}$$

$\because$ value of Sin lies between $[-1, 1]$

& value of $\sin^2 n$ lies between $[0, 1]$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\text{any definite value in } [0,1]}{n}$$

$$= 0$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

$\Rightarrow$ Then sequence $\{a_n\}$ converges to '0'.

Q.12 $a_n = \frac{(2n)!}{(n!)^2}$

SOL.

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n)!}{(n!)^2}$

or $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)\cdots\cdots 6\cdot5\cdot4\cdot3\cdot2\cdot1}{(n!)(n!)}$

$= \lim_{n\to\infty} \frac{[(2n)(2n-2)(2n-4)\cdots 6\cdot4\cdot2][(2n-1)(2n-3)\cdots 5\cdot3\cdot1]}{(n!)(n!)}$

$= \lim_{n\to\infty} \frac{2^n[n(n-1)(n-2)\cdots 3\cdot2\cdot1][(2n-1)(2n-3)\cdots 5\cdot3\cdot1]}{(n!)(n!)}$

$= \lim_{n\to\infty} \frac{2^n \cancel{(n!)} (2n-1)(2n-3)(2n-5)\cdots 5\cdot3\cdot1}{\cancel{(n!)}[n(n-1)(n-2)\cdots 3\cdot2\cdot1]}$

$= \lim_{n\to\infty} \frac{2^n \cancel{n^n}\left[\left(2-\frac{1}{n}\right)\left(2-\frac{3}{n}\right)\left(2-\frac{5}{n}\right)\cdots \frac{3}{n}\cdot\frac{1}{n}\right]}{\cancel{n^n}\left[1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\frac{3}{n}\cdot\frac{2}{n}\cdot\frac{1}{n}\right]}$

$= \lim_{n\to\infty} \frac{2^n\left[\left(2-\frac{1}{n}\right)\left(2-\frac{3}{n}\right)\left(2-\frac{5}{n}\right)\cdots\cancel{\frac{1}{n}}\right]}{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)-\cancel{\frac{1}{n}}} = \infty$

$\boxed{\lim_{n\to\infty} a_n = \infty}$ $\Rightarrow$ $\{a_n\}$ seq Diverges.

Q.13 $a_n = \left(\frac{2-n^2}{3+n^2}\right)^n$

SOL.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{2-n^2}{3+n^2}\right)^n = \lim_{n\to\infty}\left(\frac{\cancel{n^2}\left(\frac{2}{n^2}-1\right)}{\cancel{n^2}\left(\frac{3}{n^2}+1\right)}\right)^n$

$= \lim_{n\to\infty}\left(\frac{0-1}{0+1}\right)^n = \lim_{n\to\infty}(-1)^n$

$\boxed{\lim_{n\to\infty} a_n = \pm 1}$ for odd & even 'n'

$\lim_{n\to\infty} a_n$ does not ~~not~~ exist because limit is not unique

So Sequence is divergent.

Q.14 $a_n = \frac{(\ln n)^2}{n}$

Sol, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(\ln n)^2}{n}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{n\to\infty} \frac{2\ln n \cdot \frac{1}{n}}{1}$ (using L'Hospital rule)

$= \lim_{n\to\infty} \frac{2\ln n}{n}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2}{n} = 0$

$\boxed{\lim_{n\to\infty} a_n = 0}$ $\therefore$ seq converges to '0'.

Q15 $a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$ Rationalize

Sol: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)\frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}$

$= \lim_{n\to\infty} \frac{\sqrt{n}\,(\cancel{n}+1-\cancel{n})}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\left\{\sqrt{\frac{n+1}{n}}+1\right\}\cancel{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n+1}{n}}+1}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$

$\boxed{\lim_{n\to\infty} a_n = \frac{1}{2}}$ $\therefore$ Seq Converges to $\frac{1}{2}$.

Q16 $a_n = \frac{5^n + (-1)^n}{5^{n+1} + (-1)^{n+1}} = \frac{5^n + (-1)^n}{5\cdot 5^n + (-1)^n(-1)}$

Sol

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n + (-1)^n}{5\cdot 5^n + (-1)^n(-1)} = \lim_{n\to\infty} \frac{\cancel{5^n}\left(1 + \frac{(-1)^n}{5^n}\right)}{\cancel{5^n}\left(5 - \frac{(-1)^n}{5^n}\right)}$

$= \lim_{n\to\infty} \frac{1 + \frac{(-1)^n}{5^n}}{5 - \frac{(-1)^n}{5^n}} = \frac{1}{5}$

$\because (-1)^n = \begin{cases} -1, & \text{for } n \text{ is odd} \\ 1, & \text{for } n \text{ is even} \end{cases}$

$\boxed{\lim_{n\to\infty} a_n = \frac{1}{5}}$ $\therefore$ Seq $\{a_n\}$ converges to $\frac{1}{5}$

Q. 17 $a_n = (c^n + d^n)^{\frac{1}{n}}$, $d > c > 0$

SOL

then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (d^n + c^n)^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} (d^n)^{\frac{1}{n}}\left[1+\left(\frac{c}{d}\right)^n\right]^{\frac{1}{n}}$ $\quad \frac{c}{d} < 1$

$= \lim\limits_{n\to\infty} d\left[1+\left(\frac{c}{d}\right)^n\right]^{\frac{1}{n}}$

$\therefore \lim\limits_{n\to\infty} a_n = d(1+0)^0 = d$

$\boxed{\lim\limits_{n\to\infty} a_n = d}$ $\therefore$ Seq $\{a_n\}$ is convergent

Q18 $a_n = \dfrac{5^n}{(n+1)^2}$

Sol $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{5^n}{(n+1)^2}$ $\left(\frac{\infty}{\infty}\right)$

L-Hospital Rule. $\frac{d}{dx}(a^x) = a^x \ln a$

$= \lim\limits_{n\to\infty} \dfrac{5^n (\ln 5)}{2(n+1)}$ $\left(\frac{\infty}{\infty}\right)$

L-Hospital Rule

$= \lim\limits_{n\to\infty} \dfrac{(5^n \ln 5).(\ln 5)}{2}$

$(\ln 5)$ is const.

$= \dfrac{(\ln 5)^2}{2} \lim\limits_{n\to\infty} 5^n$

$\boxed{\lim\limits_{n\to\infty} a_n = \infty}$

$\therefore \{a_n\}$ is Divergent

Easy

Q 19 $a_n = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n}$

We know $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots + \frac{x^n}{\lfloor n}$ as $n\to\infty$ (expansion up to n terms)

Put $x = 1$,

$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \cdots \frac{1}{\lfloor n}$ as $n\to\infty$ (up to n terms)

$e = 1 + a_n$

$e - 1 = a_n$

$\boxed{a_n = \lim\limits_{n\to\infty}(e-1) = [e-1]}$ Seq $\{a_n\}$ converges to $e-1$

Difficult

2nd Method

$a_n = (c^n + d^n)^{\frac{1}{n}}$

$= c\left(1+\left(\frac{d}{c}\right)^n\right)^{\frac{1}{n}}$ ∞^0 form

Let $y = \left[1+\left(\frac{d}{c}\right)^n\right]^{\frac{1}{n}}$

$\ln y = \ln\left[1+\left(\frac{d}{c}\right)^n\right]^{\frac{1}{n}}$

Solve yourself

Now $\lim\limits_{n\to\infty}\left(\frac{c}{d}\right)^n = 0$, $\because \frac{c}{d} < 1$

Q19 $a_n = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots \frac{1}{\lfloor n}$

$a_{n+1} = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots \frac{1}{\lfloor n} + \frac{1}{\lfloor n+1}$

$a_{n+1} - a_n = \frac{1}{\lfloor n+1}$

$a_{n+1} - a_n > 0$

$a_{n+1} > a_n$ $\therefore$ given Seq $\{a_n\}$ is Increasing

Now we prove $\{a_n\}$ is Bounded.

$a_n = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n}$

$= 1 + \frac{1}{1\cdot 2} + \frac{1}{3\cdot 2\cdot 1} + \cdots + \frac{1}{n\cdot(n-1)\cdots 3\cdot 2\cdot 1}$

$< 1 + \frac{1}{2} + \frac{1}{2\cdot 2} + \cdots + \frac{1}{2\cdot 2\cdot 2\cdots 2\cdot 2\cdot 1}$

$= 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^{n-1}}$

$= \dfrac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ G. Series $a = 1$, $r = \frac{1}{2}$, $\frac{a(1-r^n)}{1-r}$

$= 2\left(1 - \frac{1}{2^n}\right)$

$= 2 - \frac{1}{2^{n-1}}$

$a_n < 2$ Seq $\{a_n\}$ is bounded above and increasing so converges

Now $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots \frac{1}{\lfloor n}\right)$

$= \lim\limits_{n\to\infty}\left[\left(1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots \frac{1}{\lfloor n}\right) - 1\right]$

$\boxed{\lim\limits_{n\to\infty} a_n = e - 1}$

So the Seq $\{a_n\}$ has limit $e-1$ and it is convergent.

Q.20 $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$, $n \geq 1$

Sol Since $a_{n+1} = \sqrt{2+a_n}$, Since $a_1 = \sqrt{2}$
So $a_1 < 2$

for $n=1$ So $a_2 = \sqrt{2+a_1}$
$\because a_1 < 2$ $a_2 < \sqrt{2+2}$
$a_2 < \sqrt{4} \Rightarrow a_2 < 2$

Now for $n=2$ $a_3 = \sqrt{2+a_2}$
$\because a_2 < 2$ So $a_3 < \sqrt{2+2}$
$a_3 < \sqrt{4} \Rightarrow a_3 < 2$
and so on so $a_n < 2$
Similarly $a_{n+1} < 2$

$\Rightarrow$ Sequence is bounded above by 2.

Now since $a_{n+1} = \sqrt{2+a_n}$

$\therefore (a_{n+1})^2 = 2 + a_n$ subtracting $(a_n)^2$

$\Rightarrow (a_{n+1})^2 - (a_n)^2 = 2 + a_n - a_n^2$ $+a_n - a_n$

$= 2 + 2a_n - a_n - a_n^2$

$= 2(1+a_n) - a_n(1+a_n)$

$= (2-a_n)(1+a_n)$

> 0 $\because a_n < 2$ $\therefore 2 - a_n > 0$

$\Rightarrow (a_{n+1})^2 - (a_n)^2 > 0$

$\Rightarrow (a_{n+1})^2 > (a_n)^2 \Rightarrow a_{n+1} > a_n$

$\Rightarrow$ The Sequence is monotonically increasing

Since the sequence is bounded above and increasing, So Sequence is Convergent. So its limit will exist

Since the given sequence is convergent and let its limit is $a > 0$

i.e. $\lim_{n\to\infty} a_{n+1} = a$

$\lim_{n\to\infty} \sqrt{2+a_n} = a$

If $\lim_{n\to\infty} a_{n+1} = a$ then $\lim_{n\to\infty} a_n = a$ also

$\Rightarrow \sqrt{2+a} = a$

squaring $\Rightarrow 2 + a = a^2$

$\Rightarrow a^2 - a - 2 = 0$ $\Rightarrow a = \frac{+1 \pm \sqrt{1-4\cdot1\cdot(-2)}}{2} = \frac{1\pm\sqrt{9}}{2} = \frac{1\pm3}{2}$

$\Rightarrow a = -1, 2$

$\because a > 0$, $\therefore a = 2$, is required limit of Sequence.

Note A monotonic increasing seq which is bounded above is Convergent

ii) If $a_{n+1} \geq a_n$ then $\{a_n\}$ is Mon. Inc Seq.

iii) If $a_n < B$ where B is fixed number then $\{a_n\}$ is bounded above.

(20) Easy 2nd Method $a_{n+1} = \sqrt{2+a_n}$

Let $\lim_{n\to\infty} a_n = l$ —— (I)

then $\lim_{n\to\infty} a_{n+1} = l$ —— (II)

$\Rightarrow \lim_{n\to\infty} \sqrt{2+a_n} = l$ $\because$ (given)

squaring $\lim_{n\to\infty} (2+a_n) = l^2$

$2 + l = l^2$

$l^2 - l - 2 = 0$ Q Eq

$l^2 - 2l + l - 2 = 0$

$l(l-2) + 1(l-2) = 0$

$(l-2)(l+1) = 0$

$l = 2, -1$ $\therefore \boxed{l = 2}$

since the seq is a +ive terms seq so $l = -1$ is not possible

$\therefore \lim_{n\to\infty} a_n = l$

$\lim_{n\to\infty} a_n = 2.$

DEFINITION Let $\{a_n\} = a_1, a_2, a_3, a_4 \ldots\ldots$ be an Infinite Sequence
$\sum_1^\infty a_n = a_1 + a_2 + a_3 + \text{-----} + a_n + \text{--}$ be Infinite Series containing infinite number of terms.

Symbolically it is written as $\sum_{n=1}^\infty a_n$ or $\sum_1^\infty a_n$ or $\sum a_n$

where a_n is called n^{th} term of series

Sequence Of Partial Sum $\{S_n\}$

Let $\sum_{n=1}^\infty a_n = a_1 + a_2 + a_3 + \text{-----} + a_n + \text{-----}$ be an Infinite series.

Consider $S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$S_n = a_1 + a_2 + a_3 + \text{-----} a_n$

S_n is called nth partial sum of the series $\sum_{n=1}^\infty a_n$

$\{S_n\} = S_1, S_2, S_3, \text{-----} S_n \text{-----}$ is called Sequence of partial sums.

DEFINITION

If the sequence $\{S_n\}$ converges to limit S, then series $\sum_1^\infty a_n$ is said to converge to S.

$\Rightarrow \lim_{n\to\infty} S_n = S$

$\Rightarrow \sum_1^\infty a_n = S$

(S_n is n^{th} partial sum and S is sum of series)
If S_n cgs to S then $\{S_n\}$ seq converges to S.
If $\{S_n\}$ cgs to S then Series $\sum a_n$ cgs to S.

Also if $\{S_n\}$ diverges, then the series $\sum_1^\infty a_n$ is divergent

SOME WELL-KNOWN INFINITE SERIES

(1) INFINITE GEOMETRIC SERIES (IGS)

Consider the infinite geometric series

$$\sum_1^\infty a_n = \sum_1^\infty ak^{n-1} = a + ak + ak^2 + \ldots + ak^{n-1} + \ldots$$

We investigate the behaviour of this series for different values of r.

Here $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

or $S_n = \dfrac{a(1-r^n)}{1-r}$; $|r| < 1$

$= \dfrac{a(r^n - 1)}{r-1}$; $|r| > 1$

Case-1 if $|r| < 1$, then $r^n \longrightarrow 0$ as $n \longrightarrow \infty$

$\therefore \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{a(1-r^n)}{1-r} = \dfrac{a-0}{1-r} = \boxed{\dfrac{a}{1-r}}$ is a finite value

$\sum\limits_1^\infty a_n = \sum\limits_1^\infty ar^{n-1} = S_\infty = \lim\limits_{n\to\infty} S_n$ Converges to $\dfrac{a}{1-r}$

Case-2 If $|r| > 1$. Then $r^n \longrightarrow \infty$, as $n \longrightarrow \infty$

Then $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{a(r^n - 1)}{r-1} = \infty$

Thus $\{S_n\}$ diverge, so $\sum\limits_1^\infty a_n = \sum\limits_1^\infty ar^{n-1}$ diverges.

Case-3 If $r = 1$, Then

$S_\infty = \sum\limits_1^\infty ar^{n-1} = a + a + a + a + a + \cdots$

$\Rightarrow S_n = a + a + a + \cdots\cdots$ n times $= na$

So $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} na = \pm\infty$, according as

'a' is +ive or −ive $\Rightarrow \sum\limits_1^\infty ar^{n-1}$ diverges.

Case-4 if $r = -1$, Then

$\sum\limits_{n=1}^\infty ar^{n-1} = a - a + a - a + a - a + a - a + \cdots$

Then sequence of partial sums is

$\{S_n\} = S_1, S_2, S_3, S_4, S_5, \cdots = a, 0, a, 0, a, 0, a, \cdots$

$\because S_1 = a,\ S_2 = a - a = 0,\ S_3 = a - a + a = a, \cdots$

which diverges for $|r| = -1$ — limit not unique either 0 or 'a'

DEDUCTION An infinite G. Series $\begin{cases} \text{(i) Converges if } |r| < 1 \\ \text{(ii) Diverges if } |r| \geq 1 \ \& \ r = -1 \end{cases}$

EXAMPLE Find the sum of the series
$$\sum_1^\infty a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots\cdots$$

Sol Here $\sum_1^\infty a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$

Then nth partial sums is
$$S_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots\cdots + \frac{3}{10^n}$$

It is a G.S with $a = \frac{3}{10}$ and $r = \frac{1}{10} < 1$

using $S_n = \frac{a(1-r^n)}{1-r}$ put values
$$= \frac{\frac{3}{10}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}} = \frac{\frac{3}{10}\left(1-\frac{1}{10^n}\right)}{{}^{3}\!\!\not{9}/10} = \frac{1}{3}\left[1-\frac{1}{10^n}\right]$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(1-\frac{1}{10^n}\right) = \frac{1}{3}\left(1-\frac{1}{\infty}\right) = \frac{1}{3}$$

as $\lim_{n\to\infty} S_n = \frac{1}{3}$, So $\{S_n\}$ is convergent and so $\sum_1^\infty a_n$ converges and so $\frac{1}{3}$ is sum of the given series.

(Easy) **ALTERNATE** We are given that
$$\sum_1^\infty a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots\cdots$$

which is an infinite G. Series i.e
$$S_\infty = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots\cdots$$

Here $a = \frac{3}{10}$ and $\frac{1}{10} = r < 1$

So the given infinite G. Series is convergent

its sum can be found by the following formula
$$S_\infty = \frac{a}{1-r}$$ putting values of $a = \frac{3}{10}$ & $r = \frac{1}{10}$

$$\Rightarrow S_\infty = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{9/10} = \frac{1}{3}$$

$$\Rightarrow \sum_1^\infty a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots\cdots = \frac{1}{3}.$$

EXAMPLE Determine whether the following series converges or diverges

$$\sum_{1}^{\infty} \frac{1}{(n+2)(n+3)} = \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \frac{1}{5\cdot6} + \cdots\cdots$$

If it converges, find its sum.

SOL. The given series is

$$\frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \frac{1}{5\cdot6} + \cdots\cdots + \frac{1}{(n+2)(n+3)} + \cdots\cdots$$

then its nth partial sum of given series

$$S_n = \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \frac{1}{5\cdot6} + \cdots\cdots + \frac{1}{(n+2)(n+3)}$$

$$\Rightarrow S_n = \sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} \longrightarrow (1)$$

$$\because \frac{1}{(k+2)(k+3)} = \frac{(k+3)-(k+2)}{(k+2)(k+3)} = \frac{k+3}{(k+2)(k+3)} - \frac{k+2}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3}$$

$$= \sum_{k=1}^{n} \left[\frac{1}{k+2} - \frac{1}{k+3}\right]$$

$$= \left(\frac{1}{3} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n+2}} - \frac{1}{n+3}\right)$$

$$S_n = \frac{1}{3} - \frac{1}{n+3}$$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{1}{3} - \frac{1}{n+3}\right] = \frac{1}{3} - 0 = \frac{1}{3}$

Since $\lim_{n\to\infty} S_n = \frac{1}{3}$, so $\{S_n\}$ is convergent. Hence given series is convergent and its sum is $\frac{1}{3}$.

HARMONIC SERIES Test for convergence of the Harmonic series i.e

$$\sum_{1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots + \frac{1}{n} + \cdots\cdots$$

SOL: From given series:

$S_1 = 1$, $\quad S_2 = 1 + \frac{1}{2}$, $\quad S_3 = 1 + \frac{1}{2} + \frac{1}{3}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, $\cdots\cdots\cdots$

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n}$ $\quad \left(\because S_1 < S_2 < S_3 < S_4 < \cdots\right)$

It is obvious that $\{S_n\}$ is monotonically increasing sequence.

Now sequence $\{S_n\}$ is convergent if it bounded above but Here we shall prove that $\{S_n\}$ is unbounded and so it is divergent and diverges to ∞.

Now $1 > \frac{1}{2}$

$$\frac{1}{2}+\frac{1}{3} > \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$$

$$\frac{1}{8}+\frac{1}{9}+\cdots\cdots+\frac{1}{15} > \frac{1}{16}+\frac{1}{16}+\cdots\cdots+\frac{1}{16}\ (8\text{ times}) = \frac{1}{2}$$

So $1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\left(\frac{1}{8}+\frac{1}{9}+\cdots+\frac{1}{15}\right)+\frac{1}{16}+\cdots > \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$

Now consider the series

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$$

$\Rightarrow \quad S_n = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots \text{ n times} = \frac{n}{2}.$

$\Rightarrow \quad S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n}{2} = \frac{\infty}{2} = \infty$.

So the series $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$ is divergent

Consequently the series $1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\cdots$ whose sum is greater than the series

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$$

is also divergent. So

$$\sum_1^\infty a_n = \sum_1^\infty \frac{1}{n} \text{ is divergent.}$$

THE EULER'S SERIES

Investiagate the behaviour series i.e

$$\sum_1^\infty \frac{1}{n^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\cdots+\frac{1}{n^2}+\cdots$$

SOL: We have $S_n = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}$.

The Sequence is monotonically increasing, since

$S_1 = 1,\quad S_2 = 1+\frac{1}{4},\quad S_3 = 1+\frac{1}{4}+\frac{1}{9},\ \ldots$

and $S_1 < S_2 < S_3 < \cdots\cdots$

Now we check whether $\{S_n\}$ is bounded

$\because \; S_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots + \frac{1}{n^2}$

or $S_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} \cdots \frac{1}{4\cdot 4} + \cdots\cdots + \frac{1}{n\cdot n}$

or $S_n < 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots\cdots + \frac{1}{(n-1)n}$

$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots\cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$

$= 2 - \frac{1}{n}$

$\Rightarrow S_n < 2 - \frac{1}{n} \quad \forall\, n$

$\Rightarrow \lim_{n\to\infty} S_n < \lim_{n\to\infty}\left(2 - \frac{1}{n}\right) = 2$ i.e. $\lim_{n\to\infty} S_n < 2$

$\lim_{n\to\infty} S_n < 2 \quad \therefore \{S_n\}$ is bounded above by 2

$\because \{S_n\}$ is monotonically increasing and bounded above so $\{S_n\}$ is convergent sequence. Since the sequence of partial sums of the series is convergent, so the series $\sum_1^\infty \frac{1}{n^2}$ is convergent.

Theorem $\boxed{\text{If } \sum_1^\infty a_n \text{ converges then } \lim_{n\to\infty} a_n = 0}$

Proof Given series is (اگر کوئی series $\sum a_n$ cgs تو اس کی nth term کی $\lim_{n\to\infty}$ zero کے برابر ہوگی)
لیکن اگر nth term کی $\lim_{n\to\infty}$ zero کے برابر ہو تو لازمی نہیں ہے کہ series $\sum_1^\infty a_n$ cgs

See Note on Next Page

$\sum_1^\infty a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

then its nth partial sum is

(a_n is nth element of S_n)

$S_n = a_1 + a_2 + a_3 + \cdots\cdots + a_{n-1} + a_n$

$\Rightarrow S_{n-1} = a_1 + a_2 + a_3 + \cdots\cdots + a_{n-1}$

subtracting $\Rightarrow$ $S_n - S_{n-1} = a_n$

$\lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} \sum_1^\infty a_n$

Now since $\sum_1^\infty a_n$ converges (given)
So $\lim_{n\to\infty}(S_n - S_{n-1})$ converges
i.e. $\lim_{n\to\infty} S_n = \underset{(\text{say})}{S} = \lim_{n\to\infty} S_{n-1}$ as $n\to\infty$ so $n-1\to\infty$

Now since $n \to \infty$ so $n-1 \to \infty$

So $\lim_{n\to\infty} S_{n-1} = S$

Now $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$

$\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$

$= S - S = 0$

$\Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}$

NOTE: If $\lim_{n\to\infty} a_n = 0$, then $\sum_1^\infty a_n$ is not necessarily convergent

e.g. if $a_n = \frac{1}{n}$, Then $\lim_{n\to\infty}\frac{1}{n} = 0$, but $\sum_1^\infty a_n = \sum_1^\infty \frac{1}{n}$ is divergent. (see Harmonic Series)

DIVERGENT TEST (COROLLARY)

If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_1^\infty a_n$ diverges.

EXAMPLE Examine the series $\sum_1^\infty \frac{5n+2}{3n-1}$ for convergence

Sol.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{5n+2}{3n-1} = \lim_{n\to\infty}\frac{5+2/n}{3-\frac{1}{n}} = \frac{5}{3} \neq 0$$

So by divergent test the given series diverges.

THEOREMS: ① If $\sum_1^\infty a_n$ and, $\sum_1^\infty b_n$ are convergent series with sums S and T, Then Series $\sum_1^\infty (a_n + b_n)$ and $\sum_1^\infty (a_n - b_n)$ are convergent and sums of these series are

$$\sum_1^\infty (a_n + b_n) = \sum_1^\infty a_n + \sum_1^\infty b_n = S+T$$ (Sum of cgt series is cgt. (cgt + cgt = cgt))

and $$\sum_1^\infty (a_n - b_n) = \sum_1^\infty a_n - \sum_1^\infty b_n = S-T$$ (Diff of cgt series is cgt. (cgt − cgt = cgt))

(ii) If $\sum_1^\infty a_n$ converges and $\sum_1^\infty b_n$ diverges, then $\sum_1^\infty (a_n + b_n)$ diverges. (Sum of cgt + dgt series is dgt) (cgt + dgt = dgt)

(iii) If c is non-zero real number, then $\sum_1^\infty c a_n$ converges if $\sum_1^\infty a_n$ converges and $\sum_1^\infty a_n$ diverges if $\sum_1^\infty c a_n$ div.

(iv) '+' or '−' of a finite number of terms does not affect the convergence or divergence of an infinite series.

THE BASIC COMPARISON TEST (BCT)

let $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ be two positive terms series

then

(بڑی cvg تو چھوٹی cvg) (i) {if $a_n \le b_n$; $\forall\ n$, & if $\sum_1^\infty b_n$ converges} then $\sum_1^\infty a_n$ converges.

(چھوٹی dvg تو بڑی dvg) (ii) $\sum_1^\infty a_n$ diverges if $\{a_n \geq b_n$ and $\sum_1^\infty b_n$ diverges$\}$

EXAMPLE

Determine whether series $\sum_1^\infty \frac{1}{1+n^2}$ converges or diverges.

Soln. Given series is $\sum_1^\infty \frac{1}{n^2+1}$ $\Rightarrow a_n = \frac{1}{n^2+1} \rightarrow$ (i)

let $b_n = \frac{1}{n^2} \longrightarrow$ (ii)

from (i) and (ii)

$\frac{1}{n^2+1} < \frac{1}{n^2} \quad \Rightarrow a_n < b_n$

also from (ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2}$ is Convergent

So by BCT $\sum a_n$ is convergent.

HOW TO FIND b_n from a_n

Power of n in b_n

= Power of n in numerator of a_n − power of n in denominator of a_n

i.e $0-2=-2 \therefore b_n = n^{-2} = \frac{1}{n^2}$

Example Show that $\sum_1^\infty \frac{n+5}{n^2+4}$ diverges

$a_n = \frac{n+5}{n^2+4}$

$b_n = \frac{1}{n}$ $\left(\because \frac{1-2}{n} = \frac{-1}{n}\right)$

Now $\frac{n+5}{n^2+4} > \frac{1}{n}$

$a_n > b_n$

& $\because \sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ is divergent

So $\sum_1^\infty a_n = \sum_1^\infty \frac{n+5}{n^2+4}$ is divergent by BCT

چھوٹی dgt تو بڑی dgt

2nd Method

$a_n = \frac{n+5}{n^2+4}$, $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+5}{n^2+4}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \left(\frac{n+5}{n^2+4}\right) \cdot \frac{n}{1}$

$= \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{5}{n})}{\cancel{n^2}(1+\frac{4}{n^2})} \cdot \frac{\cancel{n}}{1}$

$= \frac{1+0}{1+0} = \boxed{1}$ definite

So both series behave alike

$\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ is dgt

So $\sum_1^\infty a_n = \sum_1^\infty \frac{n+5}{n^2+4}$ is also dgt.

Note

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+5}{n^2+4}$

$= \lim_{n\to\infty} \frac{n(1+\frac{5}{n})}{n^2(1+\frac{4}{n^2})}$

$= \lim_{n\to\infty} \frac{(1+\frac{5}{n})}{n(1+\frac{4}{n^2})}$

$= \frac{1}{\infty}$

$\boxed{\lim_{n\to\infty} a_n = 0}$

By Theorem it does not imply that series cgs or dgs. so use some other test.

THE LIMIT COMPARISON TEST.

Let $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ be two +ive terms series

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = l \neq 0$ then $\Rightarrow$ both series converges or both series diverges (depending on b_n.)

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and if $\sum_1^\infty b_n$ Converges then $\Longrightarrow \sum_1^\infty a_n$ also converges

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and if $\sum_1^\infty b_n$ diverges, then $\Longrightarrow \sum_1^\infty a_n$ also diverges.

EXAMPLE

use limit comparison test (L.C.T) determine each of following series converges or diverges

(i) $\sum_1^\infty \frac{n+1}{2n^2+1}$ (ii) $\frac{n-4}{n^3+n+3}$

SOL (i) $\because a_n \frac{n+1}{2n^2+1} \Rightarrow b_n = \frac{1}{n}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{2n^2+1} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n(n+1)}{2n^2+1}$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})}{\cancel{n^2}(2+\frac{1}{n})} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{2} = l \neq 0$$

$\Rightarrow$ both series behaves alike.

but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ diverges ($\because$ infinite H. Series)

So by Limit comparison test $\sum_1^\infty a_n = \sum_1^\infty \frac{n+1}{2n^2+1}$ is divergent.

(ii) $\sum_1^n \frac{n-4}{n^3+n+3}$

Here $a_n = \frac{n-4}{n^3+n+3} \Rightarrow b_n = \frac{1}{n^2}$

So $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n-4}{n^3+n+3} \times \frac{n^2}{1} = \lim_{n\to\infty} \frac{\cancel{n^3}(1-4/n)}{\cancel{n^3}(1+\frac{n}{n^3}+\frac{3}{n^3})}$

$$= \lim_{n\to\infty} \frac{1-\frac{4}{n}}{1+\frac{1}{n^2}+\frac{3}{n^3}} = 1 = l \neq 0$$

$\Rightarrow$ both series behaves alike.

but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n^2}$ is Convergent ($\because$ Euler Series)

So by Limit Comparison test $\sum_1^\infty a_n = \sum_1^\infty \frac{n-4}{n^3+n+3}$ Converges.

THE INTEGRAL TEST (or Cauchy's Integral Test)

Let $\sum_1^\infty a_n$ be +ive term series. If f is continuous and decreasing function on $[1,\infty[$, such that $f(n) = a_n$ for all +ive integers, then

(i) $\sum_1^\infty a_n$ Converges if $\int_1^\infty f(x)\,dx$ Converges.
(ii) $\sum_1^\infty a_n$ diverges if $\int_1^\infty f(x)\,dx$ diverges

EXAMPLE Investigate the behaviour of harmonic series $\sum_1^{\infty} \frac{1}{n}$ (by using integral test)

SOL we have $\sum_1^{\infty} \frac{1}{n}$

Here $a_n = \frac{1}{n} \Rightarrow f(x) = \frac{1}{x}$

then $\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \left|Ln\,x\right|_1^t$

$= \lim_{t\to\infty} (Ln\,t - Ln\,1) = \infty$ $\boxed{\because Ln\,1 = 0 \text{ \& } \lim_{n\to\infty} Ln\,t = Ln\,\infty = \infty}$

Since $\int_1^{\infty} f(x)\,dx$ diverges so $\sum_1^{\infty}\frac{1}{n}$ diverges.

P_SERIES (OR HYPERHORMONIC SERIES)

Test for convergence the P_series or Hyperharmonic Series i.e

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

SOL Here $a_n = \frac{1}{n^p} \Rightarrow f(x) = \frac{1}{x^p}$

Then $I = \int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \left|\frac{x^{-p+1}}{-p+1}\right|_1^{\infty}$

Case-1 when $p>1$ say $p = 1+q$, where q is +ive

then $I = \left|\frac{x^{-1-q+1}}{-1-q+1}\right| = \left|\frac{x^{-q}}{-q}\right|_1^{\infty} = \left|\frac{1}{-q x^q}\right|_1^{\infty} = 0 - \frac{1}{-q} = \frac{1}{q}$

$\Rightarrow \int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx$ Convergent. So $\sum_1^{\infty} \frac{1}{n^p}$ converges.

Case-2 when $p<1$, say $p = 1-q$, where q is +ive

then $I = \int^{\infty} \frac{1}{x^p}\,dx = \left|\frac{x^{-p+1}}{-p+1}\right|_1^{\infty} = \left|\frac{x^{-1+q+1}}{-1+q+1}\right|^{\infty}$

$$= \left| \frac{x^{q}}{q} \right|_1^\infty = \infty$$

$$\Rightarrow \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx \text{ is divergent}$$

So $\sum_1^n \frac{1}{n^p}$ is divergent for $p<1$

If $p=1$, Then p- Series becomes Harmonic which divergent.

DEDUCTION $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

(i) $\sum \frac{1}{n^p}$ is Convergent for $p>1$
(ii) $\sum \frac{1}{n^p}$ is divergent for $p \leq 1$

EXAMPLE Determine whether the series $\sum_1^\infty \frac{\tan^{-1} n}{1+n^2}$, converges or diverges.

SOL:- Given series is $\sum_1^\infty \frac{\tan^{-1} n}{1+n^2}$

$$\Rightarrow a_n = \frac{\tan^{-1} n}{1+n^2} \quad \Rightarrow f(x) = \frac{\tan^{-1} x}{1+x^2}$$

$$\Rightarrow f(x) > 0 \quad \text{for } x \geq 1$$

and $$f'(x) = \frac{(1+x^2)\cdot\frac{1}{1+x^2} - 2x\tan^{-1}x}{(1+x^2)^2} = \frac{1-2x\tan^{-1}x}{(1+x^2)^2}$$

Now $f'(x) < 0$ for $x \geq 1$

Hence $f(x)$ is decreasing function for $x \geq 1$ $\left(\because f'(x)<0 \text{ for } x \geq 1\right)$

Now $$\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{1+x^2}\,dx = \lim_{t\to\infty}\left|\frac{(\tan^{-1}x)^2}{2}\right|_1^t$$

$$= \frac{1}{2}\lim_{t\to\infty}\left\{\tan^{-1}t - \tan^{-1}1\right\} = \frac{1}{2}\left\{\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right\}$$

$$= \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right] = \frac{3\pi^2}{32}$$

So $\int_1^\infty f(x)\,dx$ Converges. So $\sum_1^\infty \frac{\tan^{-1} n}{1+n^2}$ is Convergent.

EXERCISE 8·2

Determine whether the given series converges or diverges if it converges find its sum. (Problems 1–5):

Q.1 $\sum_{1}^{\infty} \cos \pi n$

Sol. Here $a_n = \cos \pi n$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos \pi n = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$

Since $\lim_{n\to\infty} a_n \neq$ definite Real Value (limit is not unique So diverges)

So by divergence test

So $\sum_{1}^{\infty} \cos \pi n$ is divergent

Q.2 $\sum_{0}^{\infty} \frac{1}{(2+x)^n}$; $|x| < 1$

IGS is convergent if $r < 1$
IGS is divergent if $r \geq 1$ or $r = -1$

Here $a_n = \frac{1}{(2+x)^n}$

$\sum_{0}^{\infty} a_n = \sum_{0}^{\infty} \frac{1}{(2+x)^n} = 1 + \frac{1}{2+x} + \frac{1}{(2+x)^2} + \frac{1}{(2+x)^3} + \cdots$

Infinite G. Series

$a = 1$ & $r = \left|\frac{1}{2+x}\right| < 1$ $\quad \because |x| < 1$

since $|r| < 1$ $\therefore$ Series is Convergent.

$a_n = \frac{1}{(2+x)^n}$

$\lim_{n\to\infty} a_n = \frac{1}{\infty} = 0$ which does not imply that Series is definitely convergent, so check Infinite Geometric Series.

Its sums $S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{1}{1 - \frac{1}{2+x}}$

$\Rightarrow \quad S_\infty = \frac{1}{\frac{2+x-1}{2+x}} = \frac{2+x}{1+x}$

Q.3 $\sum_{0}^{\infty} \frac{2^{n/2}}{3^n}$

Sol we have $a_n = \frac{2^{\frac{n}{2}}}{3^n} = \left(\frac{\sqrt{2}}{3}\right)^n$

The Series is $1 + \frac{\sqrt{2}}{3} + \left(\frac{\sqrt{2}}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^3 + \cdots \infty$ I.G.S

$a = 1 \quad r = \frac{\sqrt{2}}{3}$

$r < 1$

So Series is cgt.

$\therefore S_\infty = \frac{a}{1-r}$

$= \frac{1}{1-\frac{\sqrt{2}}{3}}$

$= \frac{1}{\frac{3-\sqrt{2}}{3}} = \frac{3}{3-\sqrt{2}}$

Hence given Series is convergent and its sum is $\frac{3}{3-\sqrt{2}}$

2nd Method.

Apply Root test ....

$(a_n)^{\frac{1}{n}} = \left(\frac{2^{\frac{n}{2}}}{3^n}\right)^{\frac{1}{n}}$

$= \frac{2^{\frac{n}{2}\cdot\frac{1}{n}}}{3^{\frac{n}{n}}}$

$= \frac{\sqrt{2}}{3} = .47 < 1$

$(a_n)^{\frac{1}{n}} < 1$

So Converges.

Q.4 $\sum_1^\infty \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)$

Sol. Given series is $\sum_1^\infty \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)$

Here

$a_1 = \frac{1}{2^1} - \cancel{\frac{1}{2^2}}$

$a_2 = \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{2^3}}$

$a_3 = \cancel{\frac{1}{2^3}} - \cancel{\frac{1}{2^4}}$

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$a_{n-1} = \cancel{\frac{1}{2^{n-1}}} - \cancel{\frac{1}{2^n}}$

$a_n = \cancel{\frac{1}{2^n}} - \frac{1}{2^{n+1}}$

$S_n = a_1 + a_2 + \cdots + a_n = \frac{1}{2} - \frac{1}{2^{n+1}}$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}$ definite value

So the Series $\sum_1^\infty a_n = \sum_1^\infty \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)$ is convergent

2nd Method Easy.

$a_n = \frac{1}{2^n} - \frac{1}{2^{n+1}}$

$= \frac{1}{2^n}\left(1 - \frac{1}{2}\right)$

$= \frac{1}{2^n}\left(\frac{1}{2}\right) = \boxed{\frac{1}{2^{n+1}}}$

$\sum_1^\infty a_n = \sum_1^\infty \frac{1}{2^{n+1}} = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - - -$ IGS

Infinite G. Series with

$a = \frac{1}{4} \quad r = \frac{1}{2^3}\cdot\frac{2^2}{1} = \frac{1}{2} < 1$

So Series is Convergent

$S_\infty = \frac{a}{1-r}$

$= \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}}$

$= \frac{1}{4}\cdot\frac{2}{1} = \boxed{\frac{1}{2}}$

$\because$ If $\lim_{n\to\infty} S_n = S$, definite value then $\sum_1^\infty a_n$ converges

detail see page 7

Q.5 $\sum_{1}^{\infty} \frac{1}{9n^2+3n-2}$

Sol: we have $\sum_{1}^{\infty} \frac{1}{9n^2+3n-2} = \frac{1}{(3n+2)(3n-1)}$

$9n^2+6n-3n-2$
$= 3n(3n+2)-1(3n+2)$
$= (3n-1)(3n+2)$

By P. Fracti.

$$= \sum_{1}^{\infty} \frac{1}{(3n+2)(3n-1)} = \frac{A}{(3n-1)} + \frac{B}{(3n+2)}$$

$$\Rightarrow a_n = \frac{1}{(3n+2)(3n-1)} = \frac{1}{3}\left[\frac{1}{3n-1}\right] - \frac{1}{3}\left[\frac{1}{3n+2}\right]$$

$$\frac{1}{(3n+2)(3n-1)} = \frac{1}{3}\left(\frac{(3n+2)-(3n-1)}{(3n+2)(3n-1)}\right)$$

$$= \frac{1}{3}\left(\frac{3n+2}{(3n+2)(3n-1)} - \frac{3n-1}{(3n+2)(3n-1)}\right)$$

$$= \frac{1}{3}\left[\frac{1}{3n-1} - \frac{1}{3n+2}\right]$$

$$\Rightarrow a_1 = \frac{1}{3}\cdot\frac{1}{2} - \cancel{\frac{1}{3}\cdot\frac{1}{5}}$$

$$a_2 = \cancel{\frac{1}{3}\cdot\frac{1}{5}} - \cancel{\frac{1}{3}\cdot\frac{1}{8}}$$

$$a_3 = \cancel{\frac{1}{3}\cdot\frac{1}{8}} - \cancel{\frac{1}{3}\cdot\frac{1}{11}}$$

$$\cdots\cdots\cdots$$

$$a_{n-1} = \frac{1}{3}\left(\cancel{\frac{1}{3n-4}}\right) - \frac{1}{3}\left(\cancel{\frac{1}{3n-1}}\right)$$

$$a_n = \frac{1}{3}\left(\cancel{\frac{1}{3n-1}}\right) - \frac{1}{3}\left(\frac{1}{3n+2}\right)$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{1}{3}\cdot\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{3n+2} = \frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right]$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{6}$$

$\Rightarrow$ The Sequence $\{S_n\}$ is convergent. Hence the series $\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{1}{9n^2+3n-2}$ is Convergent.

Each of the following is the nth partial sums of an infinite series. Determine the series and check whether it converges (Problems 6 – 8).

$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$
$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$

$S_n - S_{n-1} = a_n$

(6) $S_n = \frac{3n}{4n+1}$

Sol $\because a_n = S_n - S_{n-1}$

$$= \frac{3n}{4n+1} - \frac{3(n-1)}{4(n-1)+1} = \frac{3n}{4n+1} - \frac{3n-3}{4n-3}$$

$$= \frac{\cancel{12n^2} - \cancel{9n} - \cancel{12n^2} + \cancel{9n} + 3}{(4n+1)(4n-3)} = \frac{3}{(4n+1)(4n-3)}$$

$$\Rightarrow a_n = \frac{3}{(4n+1)(4n-3)}$$

$\Rightarrow \therefore$ Required Infinite Series is

$$\text{so } \boxed{\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{3}{(4n+1)(4n-3)}}$$

$\therefore$ sum of I. Series is

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3n}{4n+1}$$

$$= \lim_{n\to\infty} \frac{3\cancel{n}}{\cancel{n}(4+\frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{3}{4+\frac{1}{n}}$$

$$\boxed{\lim_{n\to\infty} S_n = \frac{3}{4}} \text{ (definite value)}$$

$\Rightarrow$ The Sequence of partial sum i.e $\{S_n\}$ converges. Hence the series $\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{3}{(4n+1)(4n-3)}$ Converges and its sum is $= \frac{3}{4}$

Q.7 $S_n = \dfrac{n^2}{n+1}$

Sol: $\because a_n = S_n - S_{n-1}$

$$\Rightarrow a_n = \frac{n^2}{n+1} - \frac{(n-1)^2}{n}$$

$$= \frac{n^3 - (n+1)(n^2-2n+1)}{(n+1)(n)}$$

$$= \frac{\cancel{n^3} - \cancel{n^3} + 2n^2 + n - 1}{n(n+1)}$$

$$a_n = \frac{n^2+n-1}{n^2+n}$$

So required infinite series is

$$\boxed{\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{n^2+n-1}{n^2+n}}$$

$$\text{So } \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+n-1}{n^2+n}$$

$$= \lim_{n\to\infty} \frac{1+1/n-\frac{1}{n^2}}{1+\frac{1}{n}} = 1$$

$$\boxed{\lim_{n\to\infty} a_n \neq 0}$$

so by divergent test $\sum a_n$ is divergent

Difficult

ALTERNATE (For sum of infinite series)

$$\because a_n = \frac{3}{(4n+1)(4n-3)} = 3\left[\frac{(4n+1)-(4n-3)}{(4n+1)(4n-3)}\right]$$

$$a_n = \frac{3}{4}\left[\frac{1}{4n-3} - \frac{1}{4n+1}\right]$$

by partial fraction

$$\Rightarrow a_1 = \frac{3}{4}\left[\frac{1}{1} - \cancel{\frac{1}{5}}\right]$$

$$a_2 = \frac{3}{4}\left[\cancel{\frac{1}{5}} - \cancel{\frac{1}{9}}\right]$$

$$a_3 = \frac{3}{4}\left[\cancel{\frac{1}{9}} - \cancel{\frac{1}{13}}\right]$$

$$\cdots\cdots\cdots$$

$$a_{n-1} = \frac{3}{4}\left[\cancel{\frac{1}{4n-7}} - \cancel{\frac{1}{4n-3}}\right]$$

$$a_n = \frac{3}{4}\left[\cancel{\frac{1}{4n-3}} - \frac{1}{4n+1}\right]$$

$$S_n = a_1 + a_2 + \cdots + a_n = \frac{3}{4}\left(1 - \frac{1}{4n+1}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{4}\left(1 - \frac{1}{4n+1}\right) = \frac{3}{4}$$

* First partial sum ALTERNATE Easy.

$$S_n = \frac{n^2}{n+1}$$

$$\text{So } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2}{n+1}$$

$$= \lim_{n\to\infty} \frac{n^2}{n(1+\frac{1}{n})}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{1+\frac{1}{n}} = \infty \text{ not definite value}$$

$$\boxed{\lim_{n\to\infty} S_n = \infty}$$ So $\{S_n\}$ seq is Divergent

$\Rightarrow \therefore$ The series is divergent

Q.8 $S_n = \frac{1}{2^n}$

SOL $a_n = S_n - S_{n-1}$

$\therefore a_n = \frac{1}{2^n} - \frac{1}{2^{n-1}} = \frac{1}{2^n} - \frac{1}{\frac{2^n}{2}} = \frac{1}{2^n} - \frac{2}{2^n} = \frac{-1}{2^n}$

so $\boxed{\sum_1^\infty a_n = \sum_1^\infty \left(\frac{-1}{2^n}\right)}$ is required Infinite Series

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$ is the sum of required Infinite Series

$\boxed{\lim_{n\to\infty} S_n = 0}$ definite value

So Seq. $\{S_n\}$ is convergent and its sum $= 0$

$\therefore \sum_1^\infty a_n = \sum_1^\infty \left(\frac{-1}{2^n}\right)$ is convergent and its sum $= 0$

Q.9 Prove that if a positive term series $\sum_1^\infty a_n$ Converges, then the series $\sum_1^\infty \sqrt{a_n \cdot a_{n+1}}$ converges.

SOL Since a_n and a_{n+1} are both +ive terms $\forall$ all +ive integral values of n

$G.M = \sqrt{ab}$
$A.M = \frac{a+b}{2}$

G.Mean, $G = \sqrt{a_n \cdot a_{n+1}}$ and $A.M = \frac{a_n + a_{n+1}}{2}$

but $A.M. > G.M.$ (always)

$\boxed{\because A > G > H \text{ for } a \text{ and } b}$

$\Rightarrow \frac{a_n + a_{n+1}}{2} > \sqrt{a_n \cdot a_{n+1}}$

$\Rightarrow \sum_1^\infty \left(\frac{a_n + a_{n+1}}{2}\right) > \sum_1^\infty \sqrt{a_n \cdot a_{n+1}}$

$\Rightarrow \frac{1}{2} \sum_1^\infty (a_n + a_{n+1}) > \sum_1^\infty \sqrt{a_n \cdot a_{n+1}}$

Since $\sum a_n$ is convergent (given)

So $\sum a_{n+1}$ " also "

$\Rightarrow \sum a_n + \sum a_{n+1}$ is convergent

$\Rightarrow \frac{1}{2}(\sum a_n + \sum a_{n+1})$ " "

Then by comparison test

$\sum_1^\infty \sqrt{a_n \cdot a_{n+1}}$ is convergent

(بڑی کو تو چھوٹی کو)

Q.10 Give an example in which both $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ diverges but $\sum_1^\infty (a_n + b_n)$ converges

SOL: Let $\sum_1^\infty a_n = \sum_1^\infty \left(\frac{1}{n^2} - \frac{1}{n}\right)$ and $\sum_1^\infty b_n = \sum_1^\infty \left(\frac{1}{n^2} + \frac{1}{n}\right)$

both are divergent series ($\because$ Convergent $\pm$ divergent $=$ Divergent)

$$\text{but } \sum_1^\infty (a_n + b_n) = \sum_1^\infty a_n + \sum_1^\infty b_n$$
$$= \sum_1^\infty \left(\frac{1}{n^2} - \frac{1}{n}\right) + \sum_1^\infty \left(\frac{1}{n^2} + \frac{1}{n}\right)$$
$$= \sum_1^\infty \left(\frac{1}{n^2} - \cancel{\frac{1}{n}} + \frac{1}{n^2} + \cancel{\frac{1}{n}}\right) = \sum_1^\infty \frac{2}{n^2}$$
$$= 2\sum_1^\infty \frac{1}{n^2} \quad \text{(Euler's Series)}$$
$$= \text{Convergent Series}$$

Using Comparison Tests, investigate convergence or divergence of the series in Problems (11 —— 20)

BCT **Q.11** $\sum_1^\infty \frac{1}{n^2 + n^{3/2}}$

SOL: Here $a_n = \frac{1}{n^{\frac{1}{2}} + n^{\frac{3}{2}}}$

$\because 0 - \frac{3}{2} = -\frac{3}{2}$

$b_n = n^{-3/2}$

$b_n = \frac{1}{n^{3/2}}$

Now

$\because n^{\frac{1}{2}} + n^{\frac{3}{2}} > n^{\frac{3}{2}}$

$\Rightarrow \frac{1}{n^{\frac{1}{2}} + n^{\frac{3}{2}}} < \frac{1}{n^{\frac{3}{2}}}$

$\Rightarrow a_n < b_n$

but $\sum_1^\infty b_n = \sum \frac{1}{n^{\frac{3}{2}}}$

is convergent ($\because P = \frac{3}{2} > 1$)

So by comparison test (gt $\frac{1}{3}$, $p\frac{2}{3}$ gt $6\frac{2}{3}$)

$\sum_1^\infty a_n = \sum_1^\infty \frac{1}{n^{\frac{1}{2}} + n^{\frac{3}{2}}}$ is convergent.

B COMPARISON TEST

If Σa_n & Σb_n two +ive terms series, Then

(i) Σa_n converges iff $a_n < b_n$ and Σb_n converges.

(ii) Σa_n diverges iff $a_n > b_n$ and Σb_n is divergent.

P_ Series

$\sum_1^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

By L.C.T
ALTERNATE $a_n = \dfrac{1}{n^{\frac{1}{2}} \cdot n^{3/2}}$ take $b_n = \dfrac{1}{n^{3/2}}$

then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^{3/2}}{n^{3/2} + n^{1/2}} = \lim\limits_{n\to\infty} \dfrac{n^{3/2}}{n^{3/2}(1+\frac{1}{n})}$

$= \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{n}} = 1 \neq 0$ (definite number but not equal to zero)

$\Rightarrow$ both series behave alike.

but $\Sigma b_n = \Sigma \dfrac{1}{n^{3/2}}$ $\left(\because P = \frac{3}{2} > 1\right)$ is convergent

so by limit comparison test Σa_n is convergent

Wrong **Q.12** $\displaystyle\sum_{1}^{\infty} \frac{\sqrt{n}}{n+1}$

SOL:- Here $a_n = \dfrac{\sqrt{n}}{n+1} = \dfrac{1}{\frac{n+1}{\sqrt{n}}}$

$= \dfrac{1}{\sqrt{n} + \frac{1}{\sqrt{n}}}$

take $b_n = \dfrac{1}{n^{\frac{1}{2}}}$

$\because \sqrt{n} + \dfrac{1}{\sqrt{n}} > \sqrt{n}$

$\Rightarrow \dfrac{1}{\sqrt{n} + \frac{1}{\sqrt{n}}} < \dfrac{1}{\sqrt{n}}$

بڑی cgt تو چھوٹی cgt
چھوٹی dgt تو بڑی dgt

$\Rightarrow a_n < b_n$

Also $\Sigma b_n = \Sigma \dfrac{1}{\sqrt{n}} = \Sigma \dfrac{1}{n^{1/2}}$ $\left(\because P = \frac{1}{2} < 1\right)$ is Divergent

$\because$ Bigger term is dgt so we cannot say anything about smaller term

Q12 DO By L.C.T
ALTERNATE

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n+1} \times \dfrac{\sqrt{n}}{1}$

$= \lim\limits_{n\to\infty} \dfrac{n}{n+1}$

$= \lim\limits_{n\to\infty} \dfrac{n}{n(1+\frac{1}{n})}$

$= \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{n}} = 1 = l \neq 0$

$\Rightarrow$ both series Σa_n & Σb_n behave alike.

but $\Sigma b_n = \Sigma \dfrac{1}{\sqrt{n}}$ is div.

$\because P = \frac{1}{2} < 1$

so by Limit-C-Test Σa_n is also divergent

By L.C.T **Q.13** $\displaystyle\sum_{1}^{\infty} \frac{2}{\sqrt{n}+1}$

SOL. Here $a_n = \dfrac{2}{\sqrt{n}+1}$, take $b_n = \dfrac{1}{\sqrt{n}}$ $\left(\because 0 - \frac{1}{2} = -\frac{1}{2} \therefore b_n = n^{-\frac{1}{2}} = \frac{1}{\sqrt{n}}\right)$

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n}+1} \times \dfrac{\sqrt{n}}{1} = \lim\limits_{n\to\infty} \dfrac{2\sqrt{n}}{\sqrt{n}(1+\frac{1}{\sqrt{n}})} = \lim\limits_{n\to\infty} \dfrac{2}{1+\frac{1}{\sqrt{n}}} = 2 \neq 0$

$\Rightarrow$ both series behave alike

Now since $\Sigma b_n = \Sigma \dfrac{1}{n^{1/2}}$ $\left(\because P = \frac{1}{2} < 1\right)$ is divergent

So $\Sigma a_n = \Sigma \dfrac{2}{\sqrt{n}+1}$ is divergent

Q13 By B.C.T

ALTERNATE $a_n = \frac{2}{\sqrt{n}+1}$, take $b_n = \frac{1}{\sqrt{n}}$

Then $a_n - b_n = \frac{2}{\sqrt{n}+1} - \frac{1}{\sqrt{n}} \geq 0$ for $n = 1, 2, 3, \dots$

$\Rightarrow \quad a_n \geq b_n$

and $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{\sqrt{n}} = \sum_1^\infty \frac{1}{n^{1/2}}$ is divergent $(\because p = \frac{1}{2} < 1)$

So by B. comparison test $\sum_1^\infty a_n$ is divergent.

چھوٹی dgt تو بڑی dgt.

Q.14 $\sum_1^\infty \frac{1}{n^n}$

SOL here $a_n = \frac{1}{n^n}$ $\boxed{\text{WRONG}}$

$\Rightarrow \quad a_1 = \frac{1}{1} = 1$

$a_2 = \frac{1}{2^2} \nless \frac{1}{2^2} =$, $a_3 = \frac{1}{3^3} < \frac{1}{2^2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$a_4 = \frac{1}{4^4} < \frac{1}{2^3} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$a_5 = \frac{1}{5^5} < \frac{1}{2^4} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$

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$\Rightarrow \quad \sum_1^\infty \frac{1}{n^n} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \dots\dots$

$\Rightarrow \quad \sum_1^\infty \frac{1}{n^n} < 1 \cdot \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$

$=$ an infinite Geometric series with $r = \frac{1}{2} < 1$

$\therefore \Rightarrow \sum_1^\infty \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\dots$

is convergent series due to $r = \frac{1}{2} < 1$

also $\sum_1^\infty \frac{1}{n^n} < \sum_1^\infty \frac{1}{2^n}$

×————————×

By B.C.T

Q.14 $\sum_1^\infty \frac{1}{n^n}$

SOL Given series is $\sum_1^\infty a_n = \sum_1^\infty \frac{1}{n^n}$

Now $a_1 = \frac{1}{1^1} \geq \frac{1}{2^1}$

$a_2 = \frac{1}{2^2} = \frac{1}{2^2}$

$a_3 = \frac{1}{3^3} < \frac{1}{2^3}$

$a_4 = \frac{1}{4^4} < \frac{1}{2^4}$

$a_5 = \frac{1}{5^5} < \frac{1}{2^5}$

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$\Rightarrow a_n = \frac{1}{n^n} < \frac{1}{2^n}$ for $n \geq 3$

2nd Method Ratio Test

(14) $\sum \frac{1}{n^n}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n^{n+1}} \cdot \frac{n^n}{1}$

$= \lim_{n\to\infty} \frac{1}{\cancel{n^n} n} \cdot \frac{\cancel{n^n}}{1}$

$= \lim_{n\to\infty} \frac{1}{n}$

$= \frac{1}{\infty} = \boxed{0} < 1$

Hence series $\sum \frac{1}{n^n}$ Cgs

3rd Method

Q14 $\sum \frac{1}{n^n}$

We know $n^n > 2^n$ for $n \geq 3$

$\frac{1}{n^n} < \frac{1}{2^n}$

$\frac{1}{n^n} < \left(\frac{1}{2}\right)^n$

But $\sum_3^\infty \left(\frac{1}{2}\right)^n$ is IGS with $r = \frac{1}{2} < 1$

$\therefore$ It is Cgt.

So By B.CT $\sum \frac{1}{n^n}$ is Cgt.

but $\sum_1^\infty \frac{1}{2^n}$ is an infinite Geometric series with ratio $r = \frac{1}{2} < 1$. Hence $\sum_1^\infty \frac{1}{2^n} = \sum_1^\infty \left(\frac{1}{2}\right)^n$ is convergent.

So by B. Comparison test $\sum_1^\infty \frac{1}{n^n}$ is also convergent.

BCT **Q.15** $\sum_0^\infty \frac{1}{n!}$

SOL $\sum a_n = \sum \frac{1}{n!}$

Now $a_1 = \frac{1}{1!} = \frac{1}{2^0}$

$a_2 = \frac{1}{2!} = \frac{1}{2^1}$

$a_3 = \frac{1}{3!} < \frac{1}{2^2}$

$a_4 = \frac{1}{4!} < \frac{1}{2^3}$

$a_5 = \frac{1}{5!} < \frac{1}{2^4}$

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Q15 2nd Method Ratio Test

$\sum \frac{1}{n!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$

$= \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1}$

$= \lim_{n\to\infty} \frac{1}{(n+1)\, n!} \cdot \frac{n!}{1}$

$= \lim_{n\to\infty} \frac{1}{n+1}$

$= \frac{1}{\infty+1} = \boxed{0} < 1$

Hence Series Cgs

Easy by Q. Shah

Q16 $\sum \frac{e^{2n}+e^{-2n}}{e^n+e^{-n}}$

$a_n = \frac{e^{2n}+e^{-2n}}{e^n+e^{-n}}$

$= \frac{e^{2n}\left(1+\frac{1}{e^{4n}}\right)}{e^n\left(1+\frac{1}{e^{2n}}\right)}$

$= e^n \frac{\left(1+\frac{1}{e^{4n}}\right)}{\left(1+\frac{1}{e^{2n}}\right)}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^n \left(\frac{1+\frac{1}{e^{4n}}}{1+\frac{1}{e^{2n}}}\right)$

$= \infty \therefore$ Dgt.

$\Rightarrow a_n = \frac{1}{n!} < \frac{1}{2^{n-1}} = b_n$ (say)

but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{2^{n-1}}$ is an infinite Geometric series with

$r = \frac{1}{2} < 1$. Hence it is convergent. So by B.C.Test.

~~So~~ $\sum_1^\infty a_n$ and hence $\sum_0^\infty a_n$ is convergent. (cgt کو cgt ثابت کیا)

ByCT **Q.16** $\sum_1^\infty \frac{e^{2n}+e^{-2n}}{e^n+e^{-n}}$

SOL. Here $a_n = \frac{e^{2n}+e^{-2n}}{e^n+e^{-n}} = \frac{e^{2n}+\frac{1}{e^{2n}}}{e^n+\frac{1}{e^n}} = \frac{(e^{4n}+1)/e^{2n}}{(e^{2n}+1)/e^n}$

or $a_n = \frac{e^{4n}+1}{e^n(e^{2n}+1)} = \frac{e^{4n}+1}{e^{3n}+e^n}$

$e^{4n}-e^{3n} = e^n = b_n$

$e^{4n-3n}=e^{n}$

take $b_n = e^n$

$\Rightarrow \sum_1^\infty b_n = \sum_1^\infty e^n = e+e^2+e^3+e^4+\cdots$ an infinite G. Series

with $r=e=2.712818>1$

So $\sum_1^\infty b_n$ is divergent series

Also $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{e^{4n}+1}{e^{3n}+e^n}\Big/e^n$

$= \lim_{n\to\infty}\frac{e^{4n}+1}{e^{4n}+e^{2n}} = \lim_{n\to\infty}\frac{1+\frac{1}{e^{4n}}}{1+\frac{1}{e^{2n}}} = 1 \neq 0$

$\Rightarrow$ both series $\sum a_n$ & $\sum b_n$ behave alike. Now since $\sum b_n$ is divergent. So by Limit C. Test $\sum a_n$ also diverges.

LCT (M.T. 2014) **Q.17** $\sum_1^\infty \frac{\ln n}{n}$

SOL Here $a_n = \frac{\ln n}{n}$ $\quad\therefore b_n = \frac{1}{n}$

$\therefore b_n = \frac{0-1}{n} = \frac{-1}{n}$

Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\ln n}{n}\times\frac{n}{1} = \lim_{n\to\infty}\ln n$

$\Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = \infty$

$\because \sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ is divergent $(\because P=1)$ so by L.C.T.

$\sum_1^\infty a_n = \sum_1^\infty \frac{\ln n}{n}$ is divergent

ALTERNATE

$a_n = \frac{\ln n}{n} \Rightarrow f(x) = \frac{\ln x}{x}$

So $\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t \ln x\cdot\frac{1}{x}\,dx$

$= \lim_{t\to\infty}\left|\frac{(\ln x)^2}{2}\right|_1^t$

$= \lim_{t\to\infty}\left(\frac{(\ln t)^2}{2} - \frac{(\ln 1)^2}{2}\right)$

$= \infty$

$\Rightarrow$ According to integral test $\sum a_n$ is divergent

LCT **Q.18** $\sum_1^\infty \left(1+\frac{1}{n}\right)^n$

SOL $a_n = \left(1+\frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n = \frac{(n+1)^n}{n^n}$

take $b_n = n^0 = 1$

and $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{1} = e \neq 0 \Rightarrow$ both series behave alike

Now since $\sum_1^\infty b_n = \sum_1^\infty 1 = 1+1+1+1+\cdots = \infty$, $\Rightarrow \sum_1^\infty b_n$ is divergent

So by Limit comparison test $\sum_1^\infty a_n = \sum_1^\infty\left(1+\frac{1}{n}\right)^n$ is divergent

Q.18 $\sum_1^\infty\left(1+\frac{1}{n}\right)^n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0$

$\Rightarrow \sum a_n = \sum\left(1+\frac{1}{n}\right)^n$ is dgt.

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LCT Easy **Q.19** $\sum_{1}^{\infty} \sin\frac{\pi}{n}$

Sol $a_n = \sin\frac{\pi}{n}$

$b_n = \frac{\pi}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(\frac{\pi}{n})}{(\frac{\pi}{n})} = 1$$

$\left(\frac{\pi}{n} = x \;\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1 \right.$ as $n\to\infty$, $x \to 0$

so $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$ $\therefore$ both series behave alike

but $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \frac{\pi}{n} = \pi \sum_{1}^{\infty} \frac{1}{n}$

is divergent. so by Limit C.T.

$\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \sin\frac{\pi}{n}$ is divergent

ALTERNATE LCT

Here $a_n = \sin\frac{\pi}{n}$

take $b_n = \frac{1}{n}$ $\quad \because 0-1=-1$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{\pi}{n}}{1/n}$

$= \lim_{n\to\infty} \pi \frac{\sin\frac{\pi}{n}}{\pi/n}$

let $\frac{\pi}{n} = x$, so as $n\to\infty$, $x\to 0$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \pi \lim_{x\to 0} \frac{\sin x}{x} = \pi\cdot 1 = \pi \neq 0$

$\Rightarrow$ both series behave alike

but $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \frac{1}{n}$ is divergent

so by L.C.T. $\sum_{1}^{\infty} a_n$ is div.

BCT **Q.20** $\sum_{1}^{\infty} \frac{1}{e^{n^2}}$

Sol: $a_n = \frac{1}{e^{n^2}}$

take $b_n = \frac{1}{e^n}$ (say)

Now $\frac{1}{e^{n^2}} < \frac{1}{e^n}$ $\quad$ for $n \geq 1$ $\quad$ (Note $(e^{n})^2 = e^{2n}$)

$\Rightarrow a_n < b_n$.

but $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \frac{1}{e^n} = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots$ IGS

$\therefore r = \frac{1}{e} < 1$

so $\sum_{1}^{\infty} b_n$ is Convergent

Hence by B. Comparison test

$\sum a_n = \sum_{1}^{\infty} \frac{1}{e^{n^2}}$ is Convergent

Q20 Ratio Test

$\sum \frac{1}{e^{n^2}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$= \lim_{n\to\infty} \frac{1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{1}$

$= \frac{1}{e^{n^2+2n+1}} \cdot \frac{e^{n^2}}{1}$

$= \lim_{n\to\infty} \frac{1}{e^{2n+1}}$

$= \frac{1}{e^{\infty}}$

$= \boxed{0} < 1$

Hence $\sum \frac{1}{e^{n^2}}$ Converges.

Q20 Root Test

$\sum \frac{1}{e^{n^2}}$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}}$

$= \lim_{n\to\infty} \left(\frac{1}{e^{n^2}}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty} \left(e^{-n^2}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty} \left(e^{-n}\right)$

$= \lim_{n\to\infty} \left(\frac{1}{e^n}\right)$

$= \frac{1}{\infty}$

$= \boxed{0} < 1$

Hence series converges

In Problems (21 — 40), test each of following series converges or diverges.

Q21 $\sum_1^\infty n^2 e^{-n^3}$

Sol $a_n = n^2 e^{-n^3}$

$f(x) = x^2 e^{-x^3}$

$f'(x) = 2x e^{-x^3} + x^2 e^{-x^3}(-3x^2)$

$= x e^{-x^3}(2-3x^3) \quad < 0$

$\therefore f(x)$ is Montonically decreasing

Now $\int_1^\infty f(x)dx$

$= \int_1^\infty x^2 e^{-x^3} dx$

$= \lim_{t\to\infty} \left(-\frac{1}{3}\right)\int_1^t e^{-x^3}(-3x^2)dx$

$= \lim_{t\to\infty} \left(-\frac{1}{3}\right)\left| e^{-x^3} \right|_1^t$

$= \lim_{t\to\infty} \left(-\frac{1}{3}\right)\left[e^{-t^3} - e^{-1}\right]$

$= \left(-\frac{1}{3}\right)\left(\frac{1}{\infty} - \frac{1}{e}\right) = \boxed{\frac{1}{3e}}$ definite value

So $\int_1^\infty f(x)dx$ is cgt $\Rightarrow$ series $\sum_1^\infty n^2 e^{-n^3}$ also cgs

Note $\int x^2 e^{-x^3} dx$

$-\frac{1}{3}\int e^{-x^3}(-3x^2)\,dx$

Put $-x^3 = t$

$-3x^2 dx = dt$

$= -\frac{1}{3}\int e^t dt$

$= -\frac{1}{3} e^t$

$= -\frac{1}{3} e^{-x^3}$

LCT **Q22** $\sum_1^\infty \frac{1}{(2n-1)^{\frac{1}{3}}}$

Sol

$a_n = \frac{1}{(2n-1)^{\frac{1}{3}}}$, $b_n = \frac{1}{n^{\frac{1}{3}}}$ $\quad 0 - \frac{1}{3} = -\frac{1}{3}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\frac{1}{3}}}{(2n-1)^{\frac{1}{3}}}$

$= \lim_{n\to\infty} \left(\frac{n}{2n-1}\right)^{\frac{1}{3}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{2-\frac{1}{n}}\right)^{\frac{1}{3}} = \left(\frac{1}{2}\right)^{\frac{1}{3}} = l \neq 0$

both series behave alike

but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n^{\frac{1}{3}}}$ is divergent $\because P = \frac{1}{3} < 1$

So by L.C.T. $\sum_1^\infty \frac{1}{(2n-1)^{1/3}}$ is divergent

Q21 2nd Method by Rat

$\sum a_n = \sum n^2 e^{-n^3}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{e^{(n+1)^3}} \cdot \frac{e^{n^3}}{n^2}$

$= \left(\frac{n+1}{n}\right)^2 \frac{1}{e^{(n+1)^3 - n^3}}$

$= \cancel{\frac{n^2}{n^2}}\left(1+\frac{1}{n}\right)^2 \frac{1}{e^{n^3+3n^2+3n+1-n^3}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \frac{1}{e^{3n^2+3n+1}}$

$= \boxed{0<1}$

The series converges.

Integral Test **ALTERNATE**

$a_n = \frac{1}{(2n-1)^{1/3}}$

$\Rightarrow f(x) = \frac{1}{(2x-1)^{1/3}}$

So $\int_1^\infty f(x)dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x-1)^{1/3}} dx$

$= \lim_{t\to\infty}\int_1^t \frac{(2x-1)^{-\frac{1}{3}}(2)\,dx}{2}$

$= \lim_{t\to\infty} \frac{1}{2}\left|\frac{(2x-1)^{\frac{2}{3}}}{\frac{2}{3}}\right|_1^t$

$= \lim_{t\to\infty} \frac{3}{4}\left((2t-1)^{\frac{2}{3}} - 1\right)$

$= \frac{3}{4}(\infty - 1) = \infty$

$\Rightarrow \sum_1^\infty a_n = \sum_1^\infty \frac{1}{(2n-1)^{\frac{1}{3}}}$ is divergent

Q23 Q.3

CT **Q.23** $\sum_{1}^{\infty} \frac{2^n}{3^n+1}$

SOL. Here $a_n = \frac{2^n}{3^n+1}$

We know $\frac{2^n}{3^n+1} < \frac{2^n}{3^n}$

also $\frac{2^n}{3^n+1} < \left(\frac{2}{3}\right)^n$ $\forall$ values of n

$\Rightarrow a_n < b_n$

But $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \left(\frac{2}{3}\right)^n$

$= \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots$ IGS $r = \frac{2}{3} < 1$

So by Basic Comparison Test $\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{2^n}{3^n+1}$ is convergent

NOTE 2nd Method

Q23 by Root test ...

$\lim_{n\to\infty} (b_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left(\frac{2}{3}\right)^n\right]^{\frac{1}{n}} = \frac{2}{3} = l < 1$

$\Rightarrow \sum b_n$ is convergent

Q23' $\sum_{1}^{\infty} \frac{2^n}{3^n+1}$ see Ex 8.3 Q23

DT **Q.24** $\sum_{1}^{\infty} \frac{\ln n}{1+\ln n}$

SOL. Here $a_n = \frac{\ln n}{1+\ln n}$

Applying divergent test i.e

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{1+\ln n}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$ apply L'Hospital Rule

$= \lim_{n\to\infty} \frac{1/n}{1/n}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{n\to\infty} \frac{\cancel{n}\,1}{\cancel{n}\,1}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1) = 1 \neq 0$

$\therefore \sum \frac{\ln n}{1+\ln n}$ is divergent.

LCT **Q.25** $\sum_{1}^{\infty} \frac{\ln(n+1)}{n+2}$

SOL. Here $a_n = \frac{\ln(n+1)}{n+2} = \frac{n^0 \ln(n+1)}{n+2}$

So take $b_n = \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{n+2} \cdot \frac{n}{1}$

$= \lim_{n\to\infty} \frac{\cancel{n}\ln(n+1)}{\cancel{n}\left(1+\frac{2}{n}\right)} = \lim_{n\to\infty} \frac{\ln(n+1)}{1+\frac{2}{n}} = \infty$

NOTE: We can also take $b_n = \frac{1}{n+1}$ in Q25 but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ is divergent

so by limit comparison test $\sum_1^\infty a_n = \sum_1^\infty \frac{\ln(n+1)}{n+2}$ diverge.

NOTE We cannot take $b_n = \frac{1}{n^3}$ ∵ by L.C.T

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ & $\sum b_n$ dgs then $\sum a_n$ should diverge but here $\sum a_n$ is cgt so it is not satisfied

Q.26 $\sum_1^\infty \frac{\ln n}{n^3}$

Sol. Here $a_n = \frac{\ln n}{n^3} = \frac{n^0 \ln n}{n^3}$

so take $b_n = \frac{1}{n^3}$

(WRONG) See Note

then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^3} \times \frac{n^3}{1} = \lim_{n\to\infty} \ln n = \infty \neq 0$

But $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n^3}$ is Convergent (∵ $P = 3 > 1$)

So by L. C. test $\sum_1^\infty a_n = \sum_1^\infty \frac{\ln n}{n^3}$ is Convergent

✓ **ALTERNATE**

BCT **Q26** $a_n = \frac{\ln n}{n^3}$

We know $\ln n < n \quad \forall n \geq 1$

$\ln n < n$
$\ln 1 = 0 < 1$
$\ln 2 = .69 < 2$
$\ln 3 = 1.09 < 3$

$\Rightarrow \frac{\ln n}{n^3} < \frac{n}{n^3}$

$\Rightarrow \frac{\ln n}{n^3} < \frac{1}{n^2}$

∵ $\sum \frac{1}{n^2}$ is cgt ∴ $\sum \frac{\ln n}{n^3}$ is also cgt by BCT

cgt سے چھوٹا cgt ہے

BCT **Q.27** $\sum_1^\infty \frac{\sqrt{n}}{n^2 + \cos(2n-6)}$

Sol. Here $a_n = \frac{\sqrt{n}}{n^2 + \cos(2n-6)}$

Since $n^2 + \cos(2n-6) > n^2$

$\Rightarrow \frac{1}{n^2 + \cos(2n-6)} < \frac{1}{n^2}$

So $a_n = \frac{\sqrt{n}}{n^2 + \cos(2n-6)} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}} = b_n \quad \forall n \geq 1$

Now $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n^{3/2}}$ is convergent (∵ $P = \frac{3}{2} > 1$)

So by Basic C.T. $\sum_1^\infty a_n = \sum_1^\infty \frac{\sqrt{n}}{n^2 + \cos(2n-6)}$ is convergent

(28) $\sum_{1}^{\infty} \frac{1}{(n+1)\left[\ln(n+1)\right]^2}$ (31)

Integral Test

$a_n = \frac{1}{(n+1)\left[\ln(n+1)\right]^2}$

$f(x) = \frac{1}{(x+1)\left[\ln(x+1)\right]^2} = \left[\ln(x+1)\right]^{-2}(x+1)^{-1}$

$f'(x) = -2\left[\ln(x+1)\right]^{-3} \cdot \frac{1}{(x+1)} \cdot (x+1)^{-1} + \left[\ln(x+1)\right]^{-2}\left[-(x+1)^{-2}\right]$

$= \frac{-2}{\left[\ln(x+1)\right]^3 (x+1)^2} - \frac{1}{\left[\ln(x+1)\right]^2 (x+1)^2} < 0 \quad \forall x \in [1,\infty[$

So $f(x)$ is Monotonic Decreasing

Now $\int_1^{\infty} f(x)\,dx = \int \frac{1}{(x+1)\left[\ln(x+1)\right]^2}\,dx$

$= \lim_{t\to\infty} \int_1^t \ln(x+1)^{-2} \cdot \frac{1}{(x+1)}\,dx$

$= \lim_{t\to\infty} \left| \frac{\left[\ln(x+1)\right]^{-1}}{-1} \right|_1^t$

$= \lim_{t\to\infty} \left| \frac{-1}{\ln(x+1)} \right|_1^t$

$= \lim_{t\to\infty} \left(\frac{-1}{\ln(t+1)} + \frac{1}{\ln(1+1)} \right)$

$= \frac{-1}{\ln(\infty+1)} + \frac{1}{\ln 2}$

$\int_1^{\infty} f(x)\,dx = 0 + \frac{1}{\ln 2} \neq \infty$

Since $\int_1^{\infty} f(x)\,dx$ is convergent so

So $\sum_1^{\infty} \frac{1}{(n+1)\left[\ln(n+1)\right]^2}$ is Convergent.

Div. Test

(29) $\sum_1^{\infty} n^2 \sin^2 \frac{1}{n}$

$a_n = n^2 \sin^2 \frac{1}{n}$

$= \left(n \sin \frac{1}{n}\right)^2$

$a_n = \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}}\right)^2$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)^2$

$= \left(\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{(\frac{1}{n})}\right)^2$

$\lim_{n\to\infty} a_n = 1^2 = 1 \neq 0$

So By Divergence Test

$\sum n^2 \sin^2 \frac{1}{n}$ is Divergent.

Integral

Q30 $\sum_{2}^{\infty} \frac{1}{n(\ln n)^p}$

$$a_n = \frac{1}{n(\ln n)^p}$$

$$f(x) = \frac{1}{x(\ln x)^p} = x^{-1}(\ln x)^{-p}$$

$$f'(x) = \frac{(-1)}{x^2}\cdot(\ln x)^{-p} + \frac{(-p)}{(\ln x)^{p+1}}\left(\frac{1}{x}\right)\cdot x^{-1}$$

$$f'(x) = \frac{-1}{x^2(\ln x)^p} - \frac{p}{(\ln x)^{p+1}\,(x^2)} < 0 \quad \forall\, x \in [2,\infty[$$

So $f(x)$ is Monotonically decreasing.

Now $\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx$

$$= \lim_{t\to\infty}\int_2^{t}(\ln x)^{-p}\cdot\left(\frac{1}{x}\right)dx$$

$$= \lim_{t\to\infty}\left|\frac{(\ln x)^{-p+1}}{-p+1}\right|_2^{t}$$

$$= \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right]$$

$$= \frac{1}{(1-p)}\left[\frac{1}{(\ln \infty)^{p-1}} - \frac{1}{(\ln 2)^{p-1}}\right]$$

Now $\int_2^{\infty} f(x)dx = \begin{cases} \frac{1}{1-p}\left\{0 - \frac{1}{(\ln 2)^{p-1}}\right\} & \text{if } \underline{p>1} \\ \infty & \text{if } \underline{p\leq 1} \end{cases}$

So $\int_2^{\infty} f(x)dx$ <u>converges</u> if $p>1$ $\therefore \sum_{2}^{\infty}\frac{1}{n(\ln n)^p}$ <u>cgs</u>

& $\int_2^{\infty} f(x)dx$ <u>diverges</u> if $p\leq 1$ $\therefore \sum_{2}^{\infty}\frac{1}{n(\ln n)^p}$ <u>dgs</u>.

(38)

LCT **Q.31** $\sum_{1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$

SOL. Here $a_n = \frac{2n+1}{n(n+1)(n+2)}$

take $b_n = \frac{1}{n^2}$

So $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2n+1}{n(n+1)(n+2)} \times n^2 = \lim_{n\to\infty} \frac{\cancel{n^3}(2+\frac{1}{n})}{\cancel{n^3}(1+\frac{1}{n})(1+\frac{2}{n})}$

$= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{(1+\frac{1}{n})(1+\frac{2}{n})} = 2 \neq 0$

$\Rightarrow$ both series behave alike.

But $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \frac{1}{n^2}$ is convergent (Euler's Series or $p = 2 > 1$)

So by L.C.T, $\sum_{1}^{\infty} a_n$ is convergent.

BCT **Q.32** $\sum_{1}^{\infty} \frac{1}{n^{n-1}}$

SOL. Here $a_n = \frac{1}{n^{n-1}}$

$\Rightarrow a_1 = \frac{1}{1^0} = 1 = \frac{1}{2^0} = b_1$

$a_2 = \frac{1}{2^1} = \frac{1}{2} = b_2$

$a_3 = \frac{1}{3^2} < \frac{1}{2^2} = b_3$

$a_4 = \frac{1}{4^3} < \frac{1}{2^3} = b_4$

$a_5 = \frac{1}{5^4} < \frac{1}{2^4} = b_5$

$\vdots$

$a_n = \frac{1}{n^{n-1}} < \frac{1}{2^{n-1}} = b_n$ for $n \geq 3$

but $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \frac{1}{2^{n-1}}$ is IGS $r = \frac{1}{2} < 1$

Hence it is convergent.

Root Test **Q32** $\sum_{1}^{\infty} \frac{1}{n^{n-1}}$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n^{n-1}}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty} (n^{1-n})^{\frac{1}{n}}$

$= \lim_{n\to\infty} (n)^{\frac{1-n}{n}}$

$= \lim_{n\to\infty} n^{\frac{1}{n}} n^{-1}$

$= \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right) \lim_{n\to\infty} \frac{1}{n}$

$= 1 \cdot 0 = \boxed{0} < 1$

$\therefore$ Hence Cgt

Note first prove $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ using L'Hospital as in Q15 8.3

Q33 $\sum \frac{e^{\tan^{-1} n}}{1+n^2}$

Integral Test (33) $\sum_{1}^{\infty} \frac{e^{\tan^{-1} n}}{1+n^2}$

$$a_n = \frac{e^{\tan^{-1} n}}{1+n^2}$$

$$f(x) = \frac{e^{\tan^{-1} x}}{1+x^2}$$

$$f'(x) = \frac{(1+x^2)\, e^{\tan^{-1}x} \cdot \left(\frac{1}{1+x^2}\right) - e^{\tan^{-1}x} \cdot (2x)}{(1+x^2)^2}$$

$$f'(x) = e^{\tan^{-1}x} \frac{(1-2x)}{(1+x^2)^2} < 0 \quad \forall\, x \in [1, \infty[$$

So $f(x)$ is Monotonically decreasing

$$\text{Now} \int_1^\infty f(x)dx = \int_1^\infty \frac{e^{\tan^{-1}x}}{1+x^2}\, dx$$

$$= \lim_{t\to\infty} \int_1^t e^{\tan^{-1}x} \cdot \frac{dx}{1+x^2}$$

$$= \lim_{t\to\infty} \left| e^{\tan^{-1}x} \right|_1^t$$

$$= \lim_{t\to\infty} \left(e^{\tan^{-1}t} - e^{\tan^{-1}(1)} \right)$$

$$= \left(e^{\tan^{-1}\infty} - e^{\pi/4} \right)$$

$$\int_1^\infty f(x)dx = e^{\pi/2} - e^{\pi/4} \neq \infty$$

So $\int_1^\infty f(x)dx$ converges

$$\Rightarrow \sum_1^\infty \frac{e^{\tan^{-1} n}}{1+n^2} \text{ converges.}$$

LCT (34) $a_n = \frac{1}{n\sqrt{n^2-1}}$,

$$a_n = \frac{1}{n^2\sqrt{1-\frac{1}{n^2}}} \qquad b_n = \frac{1}{n^2} \quad \left(n^{0-2} = \frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{n^2\sqrt{1-\frac{1}{n^2}}} \times \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = \boxed{1}$$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ (definite). So both behaves alike.

$\sum b_n = \sum \frac{1}{n^2}$ is cgt. So $\sum a_n$ is cgt.

Integral Test (34) $\sum_{2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

$$a_n = \frac{1}{n\sqrt{n^2-1}}$$

$$f(x) = \frac{1}{x\sqrt{x^2-1}} = x^{-1}(x^2-1)^{-\frac{1}{2}}$$

$$f'(x) = \frac{-1}{x^2} \cdot (x^2-1)^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)(x^2-1)^{-\frac{3}{2}} \cdot 2x \cdot x^{-1}$$

$$f'(x) = \frac{-1}{x^2\sqrt{x^2-1}} - \frac{1}{(x^2-1)^{3/2}} \cancel{\frac{2x}{2x}}$$

$$f'(x) < 0 \quad \forall\, x \in [2, \infty[$$

So $f(x)$ is Monotonically Decreasing

$$\text{Now} \int_2^\infty f(x)dx = \int_2^\infty \frac{1}{x\sqrt{x^2-1}}\, dx$$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{x^2-1}}\, dx$$

$$= \lim_{t\to\infty} \left| \sec^{-1} x \right|_2^t$$

$$= \lim_{t\to\infty} \left[\sec^{-1} t - \sec^{-1} 2 \right]$$

$$= \left[\sec^{-1}\infty - \sec^{-1} 2 \right]$$

$$= \cos^{-1}\left(\frac{1}{\infty}\right) - \cos^{-1}\frac{1}{2}$$

$$= \cos^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)$$

$$\int_2^\infty f(x)dx = \frac{\pi}{2} - \frac{\pi}{3} \neq \infty$$

So $\int_2^\infty f(x)dx$ cgs

$$\Rightarrow \sum_2^\infty \frac{1}{n\sqrt{n^2-1}} \text{ cgs}$$

Note

$$\sec^{-1}\infty = y$$

$$\infty = \sec y$$

$$\infty = \frac{1}{\cos y}$$

$$\cos y = \frac{1}{\infty}$$

$$y = \cos^{-1}\left(\frac{1}{\infty}\right)$$

$$\sec^{-1}\infty = \cos^{-1}\left(\frac{1}{\infty}\right)$$

(40)

Integral test (35) $\sum_1^\infty \frac{n^2}{e^n}$ $\quad a_n = \frac{n^2}{e^n}$

$f(x) = \frac{x^2}{e^x} = x^2 e^{-x}$

$f'(x) = x^2(-e^{-x}) + 2x e^{-x} = x e^{-x}(2-x) < 0 \quad \forall\ x \in]2,\infty[$

So $f(x)$ is Monotonic Decreasing

Now $\int_1^\infty f(x)\,dx = \int_1^\infty \underset{I}{x^2}\,\underset{II}{e^{-x}}\,dx = -x^2 e^{-x} + 2\int x\, e^{-x}\,dx$

$= -x^2 e^{-x} + 2\left(-x e^{-x} + \int e^{-x}\,dx\right)$

$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x}$

$\Rightarrow \int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t f(x)\,dx = \lim_{t\to\infty}\left|-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right|_1^t$

$= \lim_{t\to\infty}\left(-\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + \frac{1}{e} + \frac{2}{e} + \frac{2}{e}\right)$

$= \lim_{t\to\infty}\left(\frac{-t^2-2t-1}{e^t} + \frac{5}{e}\right) = \lim_{t\to\infty}\left[\frac{5}{e} - \frac{(t+1)^2}{e^t}\right]$

$= \lim_{t\to\infty}\frac{5}{e} - \lim_{t\to\infty}\frac{(t+1)^2}{e^t} = \frac{5}{e} - \lim_{t\to\infty}\frac{(t+1)^2}{e^t}$

$\int_1^\infty f(x)\,dx = \frac{5}{e} - \lim_{t\to\infty}\frac{2(t+1)}{e^t} = \frac{5}{e} - \lim_{t\to\infty}\frac{2}{e^t} = \frac{5}{e} - 0 = \frac{5}{e} \neq \infty$

So by integral test $\sum_1^\infty \frac{n^2}{e^n}$ Cgs.

Q.36 $\sum_1^\infty \frac{2\cdot 4\cdot 6\cdots\cdots 2n}{1\cdot 3\cdot 5\cdots\cdots 2n-1}$

SOL. $a_n = \frac{2\cdot 4\cdot 6\cdots\cdots 2n}{1\cdot 3\cdot 5\cdots\cdots(2n-1)}$

$\Rightarrow a_n > \frac{2}{2}\cdot\frac{4}{4}\cdot\frac{6}{6}\cdots\cdots\frac{2n}{2n}$

$\Rightarrow a_n > 1\cdot 1\cdot 1\cdot 1\cdots\cdots 1$

$\Rightarrow a_n > 1 = b_n$ (say)

but $\sum_1^\infty b_n = \sum_1^\infty 1 = 1+1+1\cdots = \infty$

So $\sum b_n$ is divergent $\therefore \sum a_n$ is divergent by BCT

Q35 Root Test

$\sum_1^\infty \frac{n^2}{e^n}$

$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n^2}{e^n}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty}\frac{n^{2/n}}{e}$

$= \lim_{n\to\infty}\frac{(n^{\frac{1}{n}})^2}{e}$

$= \frac{(1)^2}{e} = \frac{1}{e} < 1$

Hence Series Cgs. $\quad \because \lim_{n\to\infty} n^{\frac{1}{n}} = 1$ proved as Q15 8.3

BCT **Q.37** $\sum_{1}^{\infty} \frac{1}{n-\sqrt{n}}$

SOL. Here $a_n = \frac{1}{n-\sqrt{n}}$, Let $b_n = \frac{1}{n}$

$$n-\sqrt{n} < n$$

$$\therefore \frac{1}{n-\sqrt{n}} > \frac{1}{n}$$

but $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \frac{1}{n}$ is divergent

(چھوٹی dgt تو بڑی dgt)

So by BCT $\sum_{1}^{\infty} \frac{1}{n-\sqrt{n}}$ is divergent.

Imp LCT 2000 **Q.38** $\sum_{0}^{\infty} \frac{5^n+n}{6^n+n}$

SOL. $a_n = \frac{5^n+n}{6^n+n} = \frac{5^n\left(1+\frac{n}{5^n}\right)}{6^n\left(1+\frac{n}{6^n}\right)}$. take $b_n = \left(\frac{5}{6}\right)^n$

$$= \left(\frac{5}{6}\right)^n \left[\frac{1+\frac{n}{5^n}}{1+\frac{n}{6^n}}\right]$$

and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{\left(\frac{5}{6}\right)^n}\left(\frac{1+\frac{n}{5^n}}{1+n/6^n}\right)}{\cancel{\left(5/6\right)^n}}$

$$= \lim_{n\to\infty}\left(\frac{1+\frac{n}{5^n}}{1+\frac{n}{6^n}}\right)$$

$\because \left(\frac{1+\frac{\infty}{\infty}}{1+\frac{\infty}{\infty}}\right)$ So first find $\frac{n}{5^n}$ & $\frac{n}{6^n}$

$\because$ By L Hospital Rule

$\frac{n}{5^n} = \lim_{n\to\infty} \frac{1}{5^n \ln 5} = \frac{1}{\infty} = 0$

Similarly $\frac{n}{6^n} \to 0$ as $n\to\infty$

$$= \frac{1+0}{1+0}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$$

$\Rightarrow$ both series behaves alike. but

$\sum_{0}^{\infty}\left(\frac{5}{6}\right)^n = \left(\frac{5}{6}\right)^0 + \left(\frac{5}{6}\right)^1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \cdots\cdots$ is an infinite G. Series

with $r<1$. Hence $\sum b_n$ is convergent $\Rightarrow \sum_{0}^{\infty} a_n$ is convergent.

LCT **Q.39** $\sum_{1}^{\infty} \frac{\ln(n+1)}{n^2}$

SOL: Here $a_n = \frac{\ln(n+1)}{n^2}$

Let $b_n = \frac{1}{n^{3/2}}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{n^2} \times n^{\frac{3}{2}}$

$= \lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{2}{2\sqrt{n}}}{1+0} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

But $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \frac{1}{n^{3/2}}$ is Cgt $\left(\because P = \frac{3}{2} > 1\right)$

So by L.C.T $\sum_{1}^{\infty} \frac{\ln(n+1)}{n^2}$ is Cgt.

NOTE We cannot take $b_n = \frac{1}{n^2}$, because

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{n^2}\cdot n^2 = \infty$

but $\sum b_n = \sum \frac{1}{n^2}$ is convergent. whereas according to L.C.T. when $\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = \infty}$ Then $\sum b_n$ should be div. (which is contradiction to the Th.)

Similarly we can not take $b_n = \frac{1}{n}$

$\because \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{n^2}\cdot n$

$= \lim_{n\to\infty} \frac{\ln(n+1)}{n} \quad \frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{1}{n+1}$ L.H. Rule

$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Now $\sum b_n$ should converge but $\sum b_n = \sum \frac{1}{n}$ diverges so again contradiction of Theorem.

So we take $b_n = \frac{1}{n^{3/2}}$

Integral Test **Q.40** $\sum_{10}^{\infty} \frac{1}{n(\ln n)\ln(\ln n)}$

SOL. Here $a_n = \frac{1}{n(\ln n)(\ln \ln n)}$

So $f(x) = \frac{1}{x\ln x(\ln \ln x)}$

$\Rightarrow f'(x) = \frac{0 - (1)\cdot\ln x\,\ln(\ln x) - \cancel{x}\cdot\frac{1}{\cancel{x}}\ln(\ln x) - \cancel{x}\,\cancel{\ln x}\cdot\frac{1}{\cancel{\ln x}}\cdot\frac{1}{\cancel{x}}}{\left[x\ln x\,(\ln(\ln x))\right]^2}$

$f'(x) = \frac{-\ln x\,\ln(\ln x) - \ln(\ln x) - 1}{\{x\ln x\,\ln(\ln x)\}^2} < 0 \quad \forall x \in [10,\infty[$

So $f(x)$ is Monotonically decreasing on $[10,\infty[$

Now $\int_{10}^{\infty} f(x) = \lim_{t\to\infty}\int_{10}^{t} \frac{1}{x\ln x(\ln \ln x)}dx = \lim_{t\to\infty}\int_{10}^{\infty} \frac{1}{\ln(\ln x)}\cdot\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)dx \quad \because \int\frac{dz}{z} = \ln z$

$= \lim_{t\to\infty} \left|\ln(\ln(\ln x))\right|_{10}^{t} = \lim_{t\to\infty}\left[\ln(\ln(\ln t)) - \ln(\ln(\ln 10))\right] = \infty$

so dgt

Since $\int_{10}^{\infty} f(x)\,dx$ is divergent

So $\sum_{10}^{\infty} a_n = \sum_{10}^{\infty} \frac{1}{\ln n\,(\ln(\ln n))}$ is divergent.

EXERCISE 8.3

Apply "**RATIO TEST**" to determine whether the given series converges or diverges (Problems 1—10)

RATIO TEST

Let $\sum_1^\infty a_n$ be a +ive term series and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, where l is non-negative real no. or ∞

(i) if $l<1$, series $\sum_1^\infty a_n$ converges

(ii) if $l>1$ or ∞, series $\sum_1^\infty a_n$ diverges

(iii) if $l=1$, test fails

Note: If Root test, Ratio test and Integral test do not hold then use B.C.T

Q.1 $\sum_0^\infty \frac{2^n}{(2n)!}$

Sol. $\Rightarrow a_n = \frac{2^n}{(2n)!}$

$\Rightarrow a_{n+1} = \frac{2^{n+1}}{(2n+2)!}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{2^n}$$

$$= \lim_{n\to\infty} \frac{2\cdot \cancel{2^n}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{2}{(2n+2)(2n+1)} = \boxed{0<1}$$

$\therefore$ Hence Convergent Series

Q.2 $\sum_1^\infty \frac{n!}{n^n}$

Sol. $a_n = \frac{n!}{n^n} \Rightarrow a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}\, n^n}{(n+1)^{n+1}\cdot \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\, n^n}{\cancel{(n+1)}(n+1)^n} = \lim_{n\to\infty} \frac{1}{(n+1)^n/n^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \boxed{\frac{1}{e}<1}$$

$\therefore$ Hence Convergent Series ($\because e = 2.71+$)

Q.3 $\sum_1^\infty 7^n / n(5^{n+1})$

Sol. $a_n = \frac{7^n}{n(5^{n+1})}$

$a_n = \frac{7^n}{5n(5^n)}$

$a_{n+1} = \frac{7^{n+1}}{5(n+1)5^{n+1}} = \frac{7\cdot 7^n}{25(n+1)5^n}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{7\cdot \cancel{7^n}}{\cancel{25}(n+1)\cancel{5^n}} \cdot \frac{\cancel{5}n(\cancel{5^n})}{\cancel{7^n}} = \lim_{n\to\infty} \frac{7n}{5(n+1)}$$

$$= \lim_{n\to\infty} \frac{7}{5(1+1/n)} = \boxed{\frac{7}{5}>1}$$

$\therefore \sum a_n$ diverges

Q.4 $\sum_{1}^{\infty} \frac{n}{n^2+1}$

$a_n = \lim_{n\to\infty} \frac{n}{n^2+1}$
$= \lim_{n\to\infty} \frac{n}{n^2(1+\frac{1}{n^2})}$
$= \lim_{n\to\infty} \frac{1}{n(1+\frac{1}{n^2})}$
$= \frac{1}{\infty}$
$a_n = 0$ So apply some other test.

SoL. $a_n = \frac{n}{n^2+1} \Rightarrow a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$

Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n^2+2n+2} \times \frac{n^2+1}{n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n^2}\left[1+\frac{2}{n}+\frac{2}{n^2}\right]} \times \frac{\cancel{n^2}(1+\frac{1}{n^2})}{\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{1}{n^2})}{(1+\frac{2}{n}+\frac{2}{n^2})} = 1$$ Test fails to determine the convergence or divergence

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

Note we can also solve it by integral test. $\because \int_1^{\infty} \frac{x}{x^2+1}\,dx$ can be integrated easily.

So we use L.C.T

let $\sum b_n = \sum \frac{1}{n}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left[\frac{n}{n^2+1} \times \frac{n}{1}\right] = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}}\left[\frac{1}{1+\frac{1}{n^2}}\right] = 1 \neq 0$

$\Rightarrow$ both Series behave alike. Since $\sum b_n = \sum \frac{1}{n}$ is divergent, So $\sum a_n$ is also <u>divergent</u>.

Q.5 $\sum_{1}^{\infty} \frac{(2n)!}{4^n}$

SoL $\because a_n = \frac{(2n)!}{4^n} \Rightarrow a_{n+1} = \frac{(2n+2)!}{4^{n+1}}$ $\quad \because 4^{n+1} = 4^n 4$

$= \frac{2n+2!}{4^n \cdot 4}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{4\cdot 4^n} \times \frac{4^n}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\,\cancel{2n!}}{4\cdot\cancel{4^n}} \times \frac{\cancel{4^n}}{\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{4} = \infty$$

Hence Divergent Series

Q.6 $\sum_{1}^{\infty} \frac{2^n}{n(n+2)}$

Sol. $\Rightarrow a_n = \frac{2^n}{n(n+2)}$, and $a_{n+1} = \frac{2^{n+1}}{(n+1)(n+3)}$

$$= \frac{2 \cdot 2^n}{(n+1)(n+3)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2\cdot \cancel{2^n}}{(n+1)(n+3)} \cdot \frac{n(n+2)}{\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{2\,\cancel{n^2}\left(1+\frac{2}{n}\right)}{\cancel{n^2}\left[1+\frac{1}{n}\right]\left[1+\frac{3}{n}\right]} = \lim_{n\to\infty} \frac{2\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}$$

$$= \boxed{2 > 1}$$ Hence divergent series

Q.7 $\sum_{1}^{\infty} \frac{(n+1)(n+2)}{n!}$

Sol. $\Rightarrow a_n = \frac{(n+1)(n+2)}{n!}$

$\Rightarrow a_{n+1} = \frac{(n+2)(n+3)}{(n+1)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(\cancel{n+2})(n+3)}{(n+1)!} \times \frac{n!}{(n+1)(\cancel{n+2})}$$

$$= \lim_{n\to\infty} \frac{(n+3)\,\cancel{n!}}{(n+1)(n+1)\,\cancel{n!}} = \lim_{n\to\infty} \frac{\left(1+\frac{3}{n}\right)}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$$

$$= \boxed{0 < 1}$$ Hence Convergent Series

Q.8 $n^3 e^{-n^4}$

Sol. $a_n = n^3 e^{-n^4}$

$a_{n+1} = (n+1)^3 e^{-(n+1)^4}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3 e^{-(n+1)^4}}{n^3 e^{-n^4}} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(1+\frac{1}{n}\right)^3 e^{-(n+1)^4}}{\cancel{n^3}\, e^{-n^4}}$$

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$$= \underset{n\to\infty}{Lt}\left(1+\frac{1}{n^3}\right)^3 \cdot \underset{n\to\infty}{Lt} \frac{e^{-(n+1)^4}}{e^{-n^4}}$$

$$= 1 \cdot \underset{n\to\infty}{Lt}\, e^{-(\cancel{n^4}+4n^3+6n^2+1-\cancel{n^4}+4n)}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{e^{4n^3+6n^2+1+4n}} = \boxed{0<1}$$

Hence Convergent Series

$$(1+n)^4 = 1+4n+\frac{4\cdot 3}{2\cdot 1}n^2 + \frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1}n^3 + n^4 = 1+4n+6n^2+4n^3+n^4$$

Q.9 $\sum_1^\infty \frac{(n!)^2}{(3n)!}$

SoL. $\Rightarrow a_n = \frac{(n!)^2}{(3n)!}$, $a_{n+1} = \frac{[(n+1)!]^2}{(3n+3)!}$

using ratio test. i.e.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{[(n+1)!]^2}{(3n+3)!}\times\frac{(3n)!}{(n!)^2}$$

$$= \lim_{n\to\infty}\frac{[(n+1)\cancel{n!}]^2}{(\cancel{n!})^2}\times\frac{\cancel{(3n)!}}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = \lim_{n\to\infty}\frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2}{n^{\cancel{3}}\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2}{n\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)} = \frac{1}{\infty} = \boxed{0<1}$$

Hence Convergent Series

Q.10 $\sum_1^\infty \frac{(n+2)!}{4!\, n!\, 2^n}$

SoL Here $a_n = \frac{(n+2)!}{4!\, n!\, 2^n}$, $a_{n+1} = \frac{(n+3)!}{4!\,(n+1)!\,2^{n+1}}$

$$\Rightarrow \underset{n\to\infty}{Lt}\frac{a_{n+1}}{a_n} = \underset{n\to\infty}{Lt}\frac{(n+3)!\quad 4!\, n!\, 2^n}{4!\,(n+1)!\,2^{n+1}\,(n+2)!}$$

$$= \underset{n\to\infty}{Lt}\frac{(n+3)\cancel{(n+2)!}\,\cancel{4!}\,n!\,\cancel{2^n}}{\cancel{4!}\,(n+1)!\,2\cdot\cancel{2^n}\,\cancel{(n+2)!}} = \underset{n\to\infty}{Lt}\frac{(n+3)\cancel{n!}}{2\cdot(n+1)\cancel{n!}}$$

$$= \underset{n\to\infty}{Lt}\frac{\cancel{n}\left(1+\frac{3}{n}\right)}{2\cancel{n}\left(1+\frac{1}{n}\right)} = \underset{n\to\infty}{Lt}\frac{1+\frac{3}{n}}{2\left(1+\frac{1}{n}\right)} = \boxed{\frac{1}{2}<1}$$

Hence Convergent Series

(15) Easy

B, Ratio Test 2001

$a_n = \frac{3^n}{n^3}$

$a_{n+1} = \frac{3^{n+1}}{(n+1)^3}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n}$

$= \lim_{n\to\infty} \frac{3\cancel{n^3}}{\cancel{n^3}(1+\frac{1}{n})^3}$

$= 3 > 1$

Dgt

Q.15 $\sum_{1}^{\infty} \frac{3^n}{n^3}$

SoL. Here $a_n = \frac{3^n}{n^3} \Rightarrow (a_n)^{\frac{1}{n}} = \left(\frac{3^n}{n^3}\right)^{\frac{1}{n}} = \frac{3}{n^{\frac{3}{n}}}$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{3}{(n^{\frac{1}{n}})^3}$$

$$= \frac{3}{(\lim_{n\to\infty} n^{\frac{1}{n}})^3} = \boxed{\frac{3}{1} > 1}$$ Hence Divergent

Proof $\lim_{n\to\infty} n^{\frac{1}{n}} = 1 \quad (\infty^0)$

Let $y = n^{\frac{1}{n}}$

$\ln y = \frac{1}{n} \ln n$

$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln n}{n} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{1/n}{1}$ L'Hopital

$\lim_{n\to\infty} \ln y = 0$

$\ln(\lim_{n\to\infty} y) = 0$

$\lim_{n\to\infty} y = e^0$ Antilog

$$\boxed{\lim_{n\to\infty} n^{\frac{1}{n}} = 1}$$

Q.16 $\sum_{1}^{\infty} n\left(\frac{\pi}{n}\right)^n$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \left[n\left(\frac{\pi}{n}\right)^n\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} n^{\frac{1}{n}}\left(\frac{\pi}{n}\right)$$

$$= \lim_{n\to\infty} n^{\frac{1}{n}} \cdot \lim_{n\to\infty} \frac{\pi}{n}$$

$$= 1 \cdot 0$$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = 0 < 1 \therefore \Sigma a_n$ Converges

In Problems 17–36, apply any appropriate test to determine the convergence or divergence of the series:

Root.T **Q.17** $\sum_{2}^{\infty} \frac{e^n}{(\ln n)^n}$

SoL. Here $a_n = \frac{e^n}{(\ln n)^n} \Rightarrow (a_n)^{\frac{1}{n}} = \left[\frac{e^n}{(\ln n)^n}\right]^{\frac{1}{n}} = \frac{e}{\ln n}$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{e}{\ln n} = \frac{e}{\infty} = 0 < 1 \Rightarrow \Sigma a_n \text{ is convergent}$$

2002A **Q.18** $\sum_{1}^{\infty} \frac{1^n + 2^n}{3^n}$

SoL $\Rightarrow a_n = \frac{1^n + 2^n}{3^n}$ $\quad a_{n+1} = \left(\frac{1^{n+1} + 2^{n+1}}{3^{n+1}}\right)$

$\because 1^n = 1^{n+1} = 1$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1^{n+1} + 2^{n+1}}{3^{n+1}} \times \frac{3^n}{1^n + 2^n} = \lim_{n\to\infty} \frac{1 + 2^{n+1}}{3(1 + 2^n)}$$

Apply Cauchy's Root test to determine whether the given series converges or diverges (Problems 11—16)

2002 A

Q.11 $\sum_1^{\infty} \left(\frac{3n+2}{2n-1}\right)^n$

SoL. $\Rightarrow a_n = \left(\frac{3n+2}{2n-1}\right)^n$

Then using Cauchy's Root-test i.e

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left(\frac{3n+2}{2n-1}\right)^{\cancel{n}}\right]^{\frac{1}{\cancel{n}}}$$

$$= \lim_{n\to\infty} \frac{3n+2}{2n-1} = \lim_{n\to\infty} \frac{\cancel{n}(3+\frac{2}{n})}{\cancel{n}(2-\frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{3+\frac{2}{n}}{2-\frac{1}{n}} = \boxed{\frac{3}{2} > 1} \Rightarrow \sum a_n \text{ diverges.}$$

ROOT TEST

Let $\sum_1^{\infty} a_n$ is +ve terms series and let $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = l$

l is nonnegative real no. or ∞

(i) if $l<1$, $\sum a_n$ converges

(ii) if $l>1$ or ∞, $\sum a_n$ diverges

(iii) if $l=1$ . . . Test fails

Q.12 $\sum_1^{\infty} \frac{1}{n^n}$

SoL. Here $a_n = \frac{1}{n^n}$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} (n^{-n})^{\frac{1}{n}} = \lim_{n\to\infty} n^{-1}$$

$$= \lim_{n\to\infty} \frac{1}{n} = \boxed{0 < 1} \Rightarrow \sum a_n \text{ Converges.}$$

Q.13 $\sum_1^{\infty} \left(\frac{n}{10}\right)^n$

SoL Here $a_n = \left(\frac{n}{10}\right)^n \Rightarrow (a_n)^{\frac{1}{n}} = \left[\left(\frac{n}{10}\right)^n\right]^{\frac{1}{n}} = \frac{n}{10}$

So by R. Test $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{10} = \boxed{\infty > 1}$ $\therefore \sum a_n$ Diverges

Q.14 $\sum_1^{\infty} \left(\frac{n}{1+n^3}\right)^n$

SoL. Here $a_n = \left(\frac{n}{1+n^3}\right)^n \Rightarrow (a_n)^{\frac{1}{n}} = \left[\left(\frac{n}{n^3+1}\right)^n\right]^{\frac{1}{n}} = \frac{n}{n^3+1}$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^3+1} = \lim_{n\to\infty} \frac{1}{n^2(1+\frac{1}{n^3})} = \boxed{0<1} \Rightarrow \sum a_n \text{ Converges.}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}\left(\frac{1}{2^n}+2\right)}{3\cdot\cancel{2^n}\left(\frac{1}{2^n}+1\right)}$$

$$= \frac{0+2}{3(0+1)} = \boxed{\frac{2}{3}<1}$$ Hence Convergent

2nd Method Q19

$a_n = \frac{1^n+2^n}{3^n}$

$a_n = \frac{1^n}{3^n} + \frac{2^n}{3^n}$

$a_n = b_n + c_n$

$\Sigma a_n = \Sigma b_n + \Sigma c_n$

$\Sigma b_n = \Sigma\left(\frac{1}{3}\right)^n$ is cgt $\because$ IGS $r=\frac{1}{3}<1$

$\Sigma c_n = \Sigma\left(\frac{2}{3}\right)^n$ is cgt $\because$ IGS $r=\frac{2}{3}<1$

$\therefore \Sigma a_n$ is convergent being sum of two cgt series.

Root **Q.19** $\sum_1^\infty \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

SoL

$$\Rightarrow a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$$

$$\Rightarrow \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left[\left(1+\frac{1}{\sqrt{n}}\right)^{n^{\frac{3}{2}}}\right]^{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\frac{n^{3/2}}{n}}}$$

Thus $$\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \boxed{\frac{1}{e}<1}$$

Q20 $\lim_{n\to\infty} a_n$

$= \lim_{n\to\infty} \frac{\ln n}{e^n}$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{e^n}$

$= \lim_{n\to\infty} \frac{1}{n e^n}$

$= 0$. Does not know cgs or dgs. So use some other Test

Ratio **Q-20** $\sum_1^\infty \frac{\ln n}{e^n}$

Sol

$a_n = \frac{\ln(n)}{e^n}$

$a_{n+1} = \frac{\ln(n+1)}{e^{n+1}}$

then $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{e^n\cdot e} \times \frac{\cancel{e^n}}{\ln n}$$

(L'Hospital Rule) $$= \frac{1}{e}\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{1}{e}\lim_{n\to\infty}\frac{n}{n+1}$$

$$= \frac{1}{e}\lim_{n\to\infty} \left(\frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}\right) = \frac{1}{e(1+0)}$$

$$= \boxed{\frac{1}{e}<1}$$ Hence Convergent

Root 2nd Method Q20

$\sum_1^\infty \frac{\ln n}{e^n}$, $\because \ln n < n$

$\therefore \Rightarrow \frac{\ln n}{e^n} < \frac{n}{e^n}$

Let $a_n = \frac{n}{e^n}$

$\Rightarrow \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{(e^n)^{\frac{1}{n}}}$ $\left(\because \lim_{n\to\infty} n^{\frac{1}{n}}=1\right)$

$= \frac{1}{e}<1$

$\Rightarrow \Sigma\frac{n}{e^n}$ is cgt

$\Rightarrow \Sigma\frac{\ln n}{e^n}$ is also cgt.

B. Compare Test

cgt (big) $\therefore$ cgt (s)

Ratio test

Q.21 $\sum_{1}^{\infty} \frac{(n!)^2 \, 2^n}{(2n+2)!}$

Sol

$\Rightarrow a_n = (n!)^2 \, 2^n / (2n+2)!$

$\Rightarrow a_{n+1} = \frac{[(n+1)!]^2 (2^{n+1})}{(2n+4)!}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2 (2\cdot 2^n)}{(2n+4)!} \times \frac{(2n+2)!}{(n!)^2 2^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cancel{(n!)^2} (2\cdot \cancel{2^n}) \times \cancel{(2n+2)!}}{(2n+4)(2n+3)\cancel{(2n+2)!}\cancel{(n!)^2}\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{2(n+1)^2}{(2n+4)(2n+3)} = \lim_{n\to\infty} \frac{2\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{4}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$= \frac{2}{4} = \boxed{\frac{1}{2} < 1}$$ Hence Convergent series

Q22 Can be solved Also by Integral Test

LCT **Q.22** $\sum_{1}^{\infty} \frac{\tan^{-1} n}{n^2}$

Sol Here $a_n = \frac{\tan^{-1} n}{n^2}$

let $b_n = \frac{1}{n^2}$ $\quad 0-2 = \frac{1}{n^2}$

using L. Comparison test

i.e $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\tan^{-1} n}{n^2}}{1/n^2}$

$$= \lim_{n\to\infty} \tan^{-1} n$$

$$= \tan^{-1} \infty = \frac{\pi}{2} = l \neq 0$$

($\frac{\pi}{2}$ is non ~~zero~~ finite)

by $\sum b_n = \sum \frac{1}{n^2}$

is convergent by Euler Series

so $\sum a_n$ is convergent

Root **Q.23** $\sum_{1}^{\infty}\left(\frac{5n}{2n+1}\right)^{3n}$,

SoL. Here $a_n = \left(\frac{5n}{2n+1}\right)^{3n}$,

apply root test i.e

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{5n}{2n+1}\right)^{3n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{5n}{2n+1}\right)^{3}$$

$$= \lim_{n\to\infty}\left(\frac{5\cancel{n}}{\cancel{n}(2+\frac{1}{n})}\right)^{3}$$

$$= \left(\frac{5}{2}\right)^{3} > 1$$ Hence divergent

Root **Q.24** $\sum_{1}^{\infty}\left(\frac{n!}{n^n}\right)^{n}$

SoL. Here $a_n = \left(\frac{n!}{n^n}\right)^{n}$

then by root test

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{n!}{n^n}\right)^{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n\cdot n\cdot n\cdots n\cdot n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{2}{n}\right)\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\cdots\left(\frac{3}{n}\right)\left(\frac{2}{n}\right)\left(\frac{1}{n}\right)$$

$$= 1\cdot 1\cdot 1\cdots 0\cdot 0\cdot 0 = \boxed{0<1}$$ Hence Convergent Series

LCT **Q.25** $\sum_{1}^{\infty}\frac{5\sqrt{n}+1}{\sqrt{n^3-2n^2+3}}$

SoL. Here $a_n = \frac{5\sqrt{n}+1}{\sqrt{n^3-2n^2+3}}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{5\sqrt{n}+1}{\sqrt{n^3-2n^2+3}}\cdot\frac{n}{1}$$

$\frac{1}{2}-\frac{3}{2}=-1$

$b_n = n^{-1} = \frac{1}{n}$

$$= \lim_{n\to\infty}\frac{n^{3/2}\left(5+\frac{1}{\sqrt{n}}\right)}{n^{3/2}\left(1-\frac{2}{n}+\frac{3}{n^3}\right)^{\frac{1}{2}}} = \frac{5}{1} \neq 0$$

By LCT

Σa_n and Σb_n behaves alike.

$\because \Sigma b_n = \Sigma\frac{1}{n}$ dgs so Σa_n dgs. (Harmonic series is dgt.)

$$= \lim_{n\to\infty} \frac{\sin \frac{1}{n^{1/2}}}{\left(1-\frac{2}{n}+\frac{3}{n^3}\right)^{1/2}} = 5 \neq 0$$

but Σb_n diverges. Hence by L. Comparison test (∵ Harmonic Series is dgt)

Q.26 $\sum_{0}^{\infty} \frac{2^n+n}{(n+1)!}$

Sol. $\Rightarrow a_n = \frac{2^n+n}{(n+1)!} \Rightarrow a_{n+1} = \frac{2^{n+1}+n+1}{(n+2)!}$

or $a_{n+1} = \frac{2\cdot 2^n+n+1}{(n+2)(n+1)!}$

By Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2\cdot 2^n+n+1)\cancel{(n+1)!}}{(n+2)\cancel{(n+1)!}\,(2^n+n)}$$

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$+n-n$

$$= \lim_{n\to\infty} \frac{2\cdot 2^n+2n+1-n}{(n+2)(2^n+n)}$$

$$= \lim_{n\to\infty} \left(\frac{2\cancel{(2^n+n)}}{(n+2)\cancel{(2^n+n)}} + \frac{1-n}{(n+2)(2^n+n)} \right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{n(1+\frac{2}{n})} + \frac{\cancel{n}(-1+\frac{1}{n})}{\cancel{n}(1+\frac{2}{n})(2^n+n)} \right)$$

$$= 0 - \frac{1}{\infty}$$

$$= \boxed{0<1}$$

Hence Σa_n Converges

ALTERNATE **NOTE:** Here we use the following Theorem i.e. If $\sum_{1}^{\infty} a_n$ and $\sum_{1}^{\infty} b_n$ are convergent series with sums S and T then $\sum_{1}^{\infty}(a_n+b_n)$ and $\sum_{1}^{\infty}(a_n-b_n)$ are convergent and sums of the series are

$$\sum_{1}^{\infty}(a_n+b_n) = \sum_{1}^{\infty} a_n + \sum_{1}^{\infty} b_n = S+T$$

$$\text{and } \sum_{1}^{\infty}(a_n-b_n) = \sum_{1}^{\infty} a_n - \sum_{1}^{\infty} b_n = S-T$$

(ii) "If $\sum_1^\infty a_n$ Converges and $\sum_1^\infty b_n$ diverges then $\sum_1^\infty (a_n+b_n)$ diverges"

Imp Q26 $\sum_0^\infty \frac{2^n+n}{(n+1)!}$

$a_n = \frac{2^n+n}{(n+1)!} \Rightarrow a_n = \frac{2^n}{(n+1)!} + \frac{n}{(n+1)!} \Rightarrow a_n = b_n + c_n$

where $b_n = \frac{2^n}{(n+1)!}$ & $c_n = \frac{n}{(n+1)!}$

$\Rightarrow b_{n+1} = \frac{2\cdot 2^n}{(n+2)!}$ $\quad c_{n+1} = \frac{n+1}{(n+2)!}$

$b_{n+1} = \frac{2\cdot 2^n}{(n+2)(n+1)!}$ $\quad c_{n+1} = \frac{(n+1)}{(n+2)(n+1)!}$

Then by ratio test

$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{2\cdot \cancel{2^n}}{(n+2)\cancel{(n+1)!}} \times \frac{\cancel{(n+1)!}}{\cancel{2^n}} = \lim_{n\to\infty} \frac{2}{n+2} = \boxed{0<1}$ $\therefore \sum b_n$ is Cgt

and $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \frac{\cancel{n+1}}{(n+2)\cancel{(n+1)!}} \times \frac{\cancel{(n+1)!}}{n}$

$= \lim_{n\to\infty} \frac{1}{n+2} = \boxed{0<1} \Rightarrow \therefore \sum c_n$ is convergent

$\because a_n = b_n + c_n \Rightarrow \sum_1^\infty a_n = \sum_1^\infty b_n + \sum_1^\infty c_n$

Since both $\sum_1^\infty b_n$ and $\sum_1^\infty c_n$ are convergent

So $\sum_1^\infty b_n + \sum_1^\infty c_n$ is also convergent

$\Rightarrow \sum_1^\infty a_n$ is convergent

Q.27 $\sum_{1}^{\infty} \frac{\sqrt{n}}{2^n} = \sum_{1}^{\infty} \frac{n^{\frac{1}{2}}}{2^n}$

Sol. $\Rightarrow a_n = \frac{n^{\frac{1}{2}}}{2^n}$

$\Rightarrow (a_n)^{\frac{1}{n}} = \left[\frac{(n)^{\frac{1}{2}}}{2^n}\right]^{\frac{1}{n}} = \frac{(n^{\frac{1}{2n}})}{2} = \frac{(n^{\frac{1}{n}})^{\frac{1}{2}}}{2}$

$\therefore \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(n^{\frac{1}{n}})^{\frac{1}{2}}}{2} = \frac{\sqrt{1}}{2}$

$= \boxed{\frac{1}{2} < 1} \quad \left(\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right)$

so by Root Test $\sum a_n$ Converges

Note $\boxed{\lim_{n\to\infty} n^{\frac{1}{2n}} = 1}$ (∞^0) $\boxed{(\infty)}$

let $y = n^{\frac{1}{2n}}$

$\Rightarrow \ln y = \frac{1}{2n} \ln n$

$\Rightarrow \lim_{n\to\infty} \ln y = \lim_{n\to\infty} \left(\frac{1}{2n}\right) \ln n \;\boxed{(0\cdot\infty)}$

$= \lim_{n\to\infty} \frac{\ln n}{2n}$ $\left(\frac{\infty}{\infty}\right)$ L'Hospital

$= \frac{1/n}{2}$

$= 0$

$\lim_{n\to\infty} y = e^0$

$= 1$

Q.28 $\sum_{1}^{\infty} \frac{n^n \cdot 2^n}{(n+2)!}$

Sol. Here $a_n = \frac{n^n \cdot 2^n}{(n+2)!}$

Formulae —

$\boxed{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e}$

$\Rightarrow a_{n+1} = \frac{(n+1)^{n+1} \cdot 2^{n+1}}{(n+3)!}$

$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} \cdot 2^{n+1}}{(n+3)!} \times \frac{(n+2)!}{n^n \cdot 2^n}$

$= \frac{(n+1)(n+1)^n \cdot 2 \cdot \cancel{2^n} \times \cancel{(n+2)!}}{(n+3)\cancel{(n+2)!}\; n^n \cancel{2^n}}$

$= \frac{2(n+1)(n+1)^n}{(n+3)\, n^n} = \frac{2(n+1)}{n+3} \cdot \left(\frac{n+1}{n}\right)^n$

$\frac{a_{n+1}}{a_n} = \frac{2\left(1+\frac{1}{n}\right)}{\left(1+\frac{3}{n}\right)} \cdot \left(1+\frac{1}{n}\right)^n$

$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2\left(1+\frac{1}{n}\right)}{\left(1+\frac{3}{n}\right)} \cdot \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^\infty$

$= \boxed{2 \cdot e > 1} \Rightarrow \sum a_n$ diverges.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(2+\frac{3}{n}\right)\left(3+\frac{1}{3^n}\right)\left(1+\frac{1}{4^n}\right)}{\left(4+\frac{1}{4^n}\right)\left(2+\frac{1}{n}\right)\left(1+\frac{1}{3^n}\right)}$$

$$= \frac{2\cdot 3\cdot 1}{4\cdot 2\cdot 1} = \frac{6}{8} = \boxed{\frac{3}{4} < 1}$$ Hence Convergent Series

Ratio **Q. 31** $\displaystyle\sum_{1}^{\infty} \frac{2^{2n-1}}{(2n-1)!}$

SoL. Here $a_n = \dfrac{2^{2n-1}}{(2n-1)!}$ & $a_{n+1} = \dfrac{2^{2n+1}}{(2n+1)!}$

Then $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{2n+1}}{(2n+1)!} \times \dfrac{(2n-1)!}{2^{2n-1}}$

$$= \frac{2\cdot \cancel{2^{2n}} \times \cancel{(2n-1)!}}{(2n+1)\,2n\,\cancel{(2n-1)!}\,2^{-1}\,\cancel{2^{2n}}} = \frac{\cancel{2}\cdot 2}{\cancel{2}n(2n+1)} = \frac{2}{n(2n+1)}$$

using ratio test,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{n(2n+1)} = \boxed{0<1}$$ Hence Convergent Series

Ratio **Q. 32** $\displaystyle\sum_{1}^{\infty} \frac{n!}{e^{n^2}}$

SoL. Here $a_n = \dfrac{n!}{e^{n^2}}$ & $a_{n+1} = \dfrac{(n+1)!}{e^{(n+1)^2}}$

Then $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{e^{(n+1)^2}} \times \dfrac{e^{n^2}}{n!} = \dfrac{(n+1)\cancel{n!}\cdot e^{n^2}}{e^{n^2+2n+1}\cdot\cancel{n!}}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\cancel{(e^{n^2})}}{\cancel{(e^{n^2})}e^{2n+1}}$$

using ratio test, we get

Ratio **Q.29** $\sum_{1}^{\infty} \frac{(2n+1)!}{n^2(n+1)!}$

Sol. Here $a_n = \frac{(2n+1)!}{n^2(n+1)!}$

$$\Rightarrow a_{n+1} = \frac{(2n+3)!}{(n+1)^2(n+2)!}$$

then $\frac{a_{n+1}}{a_n} = \frac{(2n+3)!}{(n+1)^2(n+2)!} \times \frac{n^2(n+1)!}{(2n+1)!}$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+3)(2n+2)\cancel{(2n+1)!}}{(n+1)^2(n+2)\cancel{(n+1)!}} \cdot \frac{n^2\cancel{(n+1)!}}{\cancel{(2n+1)!}}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^2(2n+3)(2n+2)}{(n+1)^2(n+2)} = \frac{n^4\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}{n^3\left(1+\frac{1}{n}\right)^2\left(1+\frac{2}{n}\right)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2\left(1+\frac{2}{n}\right)} = \boxed{\infty > 1}$$

Hence Divergent Series

Ratio **Q.30** $\sum_{1}^{\infty} \frac{(2n+1)(3^n+1)}{4^n+1}$

Sol. $a_n = \frac{(2n+1)(3^n+1)}{4^n+1}$

$$a_{n+1} = \frac{(2n+3)(3^{n+1}+1)}{(4^{n+1}+1)}$$

So $\frac{a_{n+1}}{a_n} = \frac{(2n+3)(3^{n+1}+1)}{(4^{n+1}+1)} \times \frac{(4^n+1)}{(2n+1)(3^n+1)}$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{n}\left(2+\frac{3}{n}\right)\cancel{3^n}\left(3+\frac{1}{3^n}\right)\cdot\cancel{4^n}\left(1+\frac{1}{4^n}\right)}{\cancel{4^n}\left(4+\frac{1}{4^n}\right)\cancel{n}\left(2+\frac{1}{n}\right)\cancel{3^n}\left(1+\frac{1}{3^n}\right)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2e^{2n+1}} = \frac{1}{\infty} = 0 < 1$$

$\Rightarrow \sum a_n$ is Convergent.

Integral **Q.33** $\sum_2^\infty \frac{1}{\sqrt{n}(\ln n)^3}$

Sol Here $a_n = \frac{1}{\sqrt{n}(\ln n)^3}$

Let $b_n = \frac{1}{n \ln n}$

Then $\frac{a_n}{b_n} = \frac{n \ln n}{\sqrt{n}(\ln n)^3} \Rightarrow$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{(\ln n)^2} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{2}{n}\ln n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{4\ln n} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{4\cdot\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{8}\sqrt{n} = \infty$$

but $\sum b_n = \sum \frac{1}{n\ln n}$ is divergent

$$\because \int_2^\infty f(x)\,dx = \lim_{n\to\infty} \int_2^n \frac{1}{x\ln x}\,dx$$

when $f(x) = \frac{1}{x\ln x}$

$$\int_2^\infty f(x)\,dx = $$

$$= \lim_{n\to\infty} \int_2^n \frac{\frac{1}{x}}{\ln x}\,dx$$

Q33 ALTERNATE LCT Easy

$a_n = \frac{1}{\sqrt{n}(\ln n)^3}$

Let $b_n = \frac{1}{n}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}(\ln n)^3}\cdot\frac{n}{1}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{(\ln n)^3} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{3(\ln n)^2}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{6(\ln n)^2} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{12\frac{\ln n}{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{24\ln n} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{24}{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{48} = \infty$$

but $\sum b_n = \sum \frac{1}{n}$ is div.

So by L.C.T. given series is div.

NOTE: We cannot take $b_n = \frac{1}{\sqrt{n}}$ because if so, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ So $\sum b_n$ should converge so that $\sum a_n$ cgs but $\sum b_n = \sum \frac{1}{\sqrt{n}}$ dgs $\because P<1$

$$= \lim_{n\to\infty} \left| \ln(\ln n) \right|_2^n = \infty$$

$$\Rightarrow \sum b_n = \int_2^\infty f(x)\,dx \text{ is divergent}$$

So $\sum a_n$ is divergent.

Ratio **Q.34** $1 + \frac{1\cdot2}{1\cdot3} + \frac{1\cdot2\cdot3}{1\cdot3\cdot5} + \frac{1\cdot2\cdot3\cdot4}{1\cdot3\cdot5\cdot7} + \cdots$

Sol. nth term $a_n = \frac{1\cdot2\cdot3\cdot4\cdots n}{1\cdot3\cdot5\cdot7\cdots(2n-1)}$

$a=1,\ d=2,\ n=n$
$a_n = ?$
$a_n = a+(n-1)d$
$= 1+(n-1)2$
$= 2n-1$

$$a_{n+1} = \frac{1\cdot2\cdot3\cdot4\cdots n\cdot(n+1)}{1\cdot3\cdot5\cdot7\cdots(2n-1)(2n+1)}$$

Then $\frac{a_{n+1}}{a_n} = \frac{1\cdot2\cdot3\cdot4\cdots n(n+1)}{1\cdot3\cdot5\cdot7\cdots(2n-1)(2n+1)} \times \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{1\cdot2\cdot3\cdot4\cdots n}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n+1} = \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(2+\frac{1}{n})}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right) = \boxed{\frac{1}{2} < 1}$$ Hence Convergent Series

Ratio **Q.35** $\frac{2}{5} + \frac{2\cdot4}{5\cdot8} + \frac{2\cdot4\cdot6}{5\cdot8\cdot11} + \cdots$

$a=5,\ d=3,\ n=n$
$a_n = ?$
$a_n = a+(n-1)d$
$= 5+(n-1)3$
$= 3n+2$

Sol. $\Rightarrow a_n = \frac{2\cdot4\cdot6\cdot8\cdots 2n}{5\cdot8\cdot11\cdots(3n+2)}$

$$\Rightarrow a_{n+1} = \frac{2\cdot4\cdot6\cdot8\cdots 2n\cdot(2n+2)}{5\cdot8\cdot11\cdots(3n+2)(3n+5)}$$

Then $\frac{a_{n+1}}{a_n} = \frac{2\cdot4\cdot6\cdot8\cdots 2n(2n+2)}{5\cdot8\cdot11\cdots(3n+2)(3n+5)} \times \frac{5\cdot8\cdot11\cdots(3n+2)}{2\cdot4\cdot6\cdot8\cdots 2n}$

$$= \frac{2n+2}{3n+5} = \frac{\cancel{n}(2+\frac{2}{n})}{\cancel{n}(3+\frac{5}{n})}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{2+\frac{2}{n}}{3+\frac{5}{n}}\right) = \boxed{\frac{2}{3}<1}$$ So by ratio test $\sum a_n$ is convergent

2nd

Q. 36 $\sum_{1}^{\infty} \frac{(a+1)(2a+1)(3a+1)\cdots\cdots(na+1)}{(b+1)(2b+1)(3b+1)\cdots\cdots(nb+1)}$, $a>0$, $b>0$

Sol. We have $a_n = \frac{(a+1)(2a+1)(3a+1)\cdots(na+1)}{(b+1)(2b+1)(3b+1)\cdots(nb+1)}$

$$a_{n+1} = \frac{(a+1)(2a+1)\cdots(na+1)\left[(n+1)a+1\right]}{(b+1)(2b+1)\cdots(nb+1)\left[(n+1)b+1\right]}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(a+1)(2a+1)\cdots\cdots(na+1)\left[(n+1)a+1\right]}{(b+1)(2b+1)\cdots\cdots(nb+1)\left[(n+1)b+1\right]} \times \frac{(b+1)(2b+1)\cdots\cdots(nb+1)}{(a+1)(2a+1)\cdots\cdots(na+1)}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)a+1}{(n+1)b+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{na+a+1}{nb+b+1} = \lim_{n\to\infty} \frac{\cancel{n}\left(a+\frac{a}{n}+\frac{1}{n}\right)}{\cancel{n}\left(b+\frac{b}{n}+\frac{1}{n}\right)}$$

$$= \frac{a+0+0}{b+0+0} = \frac{a}{b}$$

Then by ratio test

(i) $\sum a_n$ converges if $\frac{a}{b}<1$

(ii) $\sum a_n$ diverges if $\frac{a}{b}>1$

(iii) but if $\frac{a}{b}=1$ then $\Rightarrow a=b$

Then $a_n = \frac{(a+1)(2a+1)(3a+1)\cdots(na+1)}{(a+1)\;(2a+1)\;(3a+1)\cdots(na+1)}$

$\Rightarrow a_n = 1\cdot1\cdot1\cdot1\cdots\cdots n \text{ times} = 1$

So apply div. test i.e

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 = 1 \neq 0$$

Thus by divergent test for $a=b$

Given series is divergent.

Q. 37 If $x > 0$, Show that the series $\sum_{1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ Converges for $x < 4$.

SoL: Here $a_n = \frac{(n!)^2}{(2n)!} x^n$

and $a_{n+1} = \frac{[(n+1)!]^2}{(2n+2)!} x^{n+1}$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2 \cancel{x^{n}}x}{(2n+2)!} \times \frac{(2n)!}{(n!)^2 \cancel{x^n}} = \frac{((n+1))^2 (\cancel{n!})^2 \times \cancel{(2n)!}\, x}{(2n+2)(2n+1)\, \cancel{2n!}\, \cancel{(n!)^2}}$$

$$= \frac{(n+1)^2 x}{(2n+2)(2n+1)} = \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2 x}{\cancel{n^2}\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 x}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} = \boxed{\frac{x}{4}}$$

Now according to ratio test

Σa_n will converges only if $\frac{x}{4} < 1$. and it is only possible when $x < 4$

Q. 38 If $x > 0$, Prove that series $\sum_{1}^{\infty} \frac{1\cdot3\cdot5 \cdots (2n-1)}{1\cdot4\cdot7 \cdots (3n-2)} x^n$ converges for $x < \frac{3}{2}$

SoL. We have $a_n = \frac{1\cdot3\cdot5 \cdots (2n-1)}{1\cdot4\cdot7 \cdots (3n-2)} x^n$

$$\Rightarrow a_{n+1} = \frac{1\cdot3\cdot5 \cdots (2n-1)(2n+1)}{1\cdot4\cdot7 \cdots (3n-2)(3n+1)} x^{n+1}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{\cancel{1\cdot3\cdot5 \cdots (2n-1)}(2n+1)\, x \cdot \cancel{x^n}}{\cancel{1\cdot4\cdot7 \cdots (3n-2)}(3n+1)} \times \frac{\cancel{1\cdot4\cdot7 \cdots (3n-2)}}{\cancel{1\cdot3\cdot5 \cdots (2n-1)\, x^n}}$$

$$\therefore = \frac{2n+1}{3n+1} x = \frac{\cancel{n}\left(2+\frac{1}{n}\right)x}{\cancel{n}\left(3+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{\left(2+\frac{1}{n}\right)x}{\left(3+\frac{1}{n}\right)}\right) = \frac{2}{3} x$$

By Ratio test

Σa_n cgs if $\frac{2}{3}x < 1$

$\Rightarrow x < \frac{3}{2}$

Q.39 Show that $\sum_1^\infty \frac{x^n}{n!}$ converges $\forall$ +ive values of x.

Sol. $\because a_n = \frac{x^n}{n!}$ & $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{x \cdot \cancel{x^n}}{(n+1)!} \times \frac{n!}{\cancel{x^n}} = \frac{x \cdot \cancel{n!}}{(n+1)\,\cancel{n!}} = \frac{x}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1} = \boxed{0<1} \;\forall \text{ +ive values of } x.$$

$\Rightarrow$ by ratio test Σa_n convergent $\forall$ +ive values of x.

Q.40 Find those +ive values of x for which the series $1 + \sum_1^\infty \frac{x^{2n}}{2n}$ converges

Sol. It is obvious that Series will converges if $\sum_1^\infty \frac{x^{2n}}{2n}$ converges

$\because$ addition and substraction of finite number of terms does not effect the convergence or divergence of an infinite series.

So $a_n = \frac{x^{2n}}{2n}$

$$\Rightarrow a_{n+1} = \frac{x^{2n+2}}{2n+2} = \frac{x^2 \cdot x^{2n}}{2(n+1)}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{x^2 \cdot x^{2n}}{\cancel{2}(n+1)} \times \frac{\cancel{2}n}{\cancel{x^{2n}}} = \frac{x^2 \cdot n}{n(1+\frac{1}{n})} = \frac{x^2}{1+\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^2}{1+\frac{1}{n}} = x^2$$

Hence by ratio test the given series will converge if $x^2 < 1$ $\Rightarrow$ $x < 1$ (only +ive values of x)

$0 < x < 1$

ALTERNATING SERIES.

A Series in which terms are alternately +ive and negative (or negative and positive) is called an Alternating Series. Thus the series

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \cdots\cdots = \sum_1^{\infty} (-1)^{n-1} a_n$$

and

$$-a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - \cdots\cdots = \sum_1^{\infty} (-1)^n a_n$$

where 'a' is positive, are alternating Series.

ABSOLUTE CONVERGENCE OF ALTERNATING SERIES

Let $\sum_1^{\infty} (-1)^{n-1} a_n$ be an alternating Series. Then if the series $\sum_1^{\infty} \left|(-1)^{n-1} a_n\right|$ is convergent, we say that the alternating Series is <u>Absolutely Convergent</u>.

(Absolute ... Mod)
(... Mod)
(... Condition ...)

CONDITIONALLY CONVERGENT

If the Series $\sum_1^{\infty} \left|(-1)^{n-1} a_n\right|$ is divergent but $\sum_1^{\infty} (-1)^{n-1} a_n$ is convergent, we say that the alternating series is Conditionally Convergent.

EXAMPLE The Series $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{1}{n}\right|$ Converges conditionally but not absolutely.

<u>Sol</u> : $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{1}{n}\right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ as $\sum_1^{\infty} \frac{1}{n}$ is divergent

Where as $\sum_1^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent

Since here $a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

$\Rightarrow \quad n < n+1 \quad \forall$ +ive integral values of n

$\Rightarrow \quad \frac{1}{n} > \frac{1}{n+1} \quad \forall$ positive integral values of

$\Rightarrow \quad a_n > a_{n+1} \quad$ " " "

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

So both conditions for an alternating series to be convergent are satisfied. Hence $\sum_1^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent. Hence proved.

∵ an alternating Series
$a_1 - a_2 + a_3 - a_4 + \dots = \sum (-1)^{n-1} a_n$
such that $a_n > 0$, $\forall\, n = 1, 2, 3, \dots$
converge if
(i) $\lim_{n\to\infty} a_n = 0$
(ii) $a_n > a_{n+1}$ $\forall$ $n = 1, 2, 3, 4, \dots$

Theorem (Alternating Series test)

※ It is also called Leibniz test

Let $a_n > 0$, for $n = 1, 2, 3, 4, \dots$, then the series

$$a_1 - a_2 + a_3 - a_4 + a_5 + \dots + (-1)^{n-1} a_n + \dots = \sum_1^{\infty} (-1)^{n-1} a_n$$

Converges, if following two conditions are satisfied

(i) $\lim_{n\to\infty} a_n = 0$

(ii) $a_n > a_{n+1}$ for all +ive integral values of n i.e $\{a_n\}$ is non increasing.

(Also $\{a_n\}$ is non increasing sequence if $f'(x) < 0$ & $\lim_{n\to\infty} a_n = 0$ see Q 3, 5)

Note:- If any one condition of alternating series test for convergence is not satisfied. Then given alternating series will be divergent.

Proof Let S_n be the nth partial sum of the series $\sum (-1)^{n-1} a_n$ i.e $S_n = \sum (-1)^{n-1} a_n$

then $S_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - \dots - a_{2n-2} + a_{2n-1} - a_{2n}$

$$= a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

$$S_{2n} = a_1 - \{(a_2 - a_3) + (a_4 - a_5) + \dots + (a_{2n-2} - a_{2n-1})\} - a_{2n}$$

$\Rightarrow \quad S_{2n} < a_1 \quad (\because a_{n+1} < a_n$ and $a_n > 0\ \forall n)$

$\Rightarrow$ Sequence $\{S_{2n}\}$ is bounded above

Now $S_{2n+2} = (a_1 - a_2 + a_3 - a_4 + \dots - a_{2n}) + a_{2n+1} - a_{2n+2}$

$$S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2}$$

$\Rightarrow \quad S_{2n+2} - S_{2n} = a_{2n+1} - a_{2n+2}$

$\Rightarrow S_{2n+2} - S_{2n} > 0 \quad \forall n$ $\left(\begin{array}{l}\text{Since } a_n > 0\ \forall n \\ \Rightarrow a_{2n+1} > 0,\ a_{2n+2} > 0 \\ \text{and } a_{2n+2} < a_{2n+1}\end{array}\right)$

$\Rightarrow S_{2n+2} > S_{2n}.$

$\Rightarrow \{S_{2n}\}$ the Sequence, is monotonically increasing

a Sequence $\{a_n\}$ is monotonically increasing if $a_{n+1} > a_n\ \forall n$. i.e. $a_1 < a_2 < a_3 < a_4 \cdots\cdots$

$\Rightarrow$ The sequence $\{S_{2n}\}$ is Convergent

$\because$ a Sequence which is bounded above and monotonically increasing is Convergent

let $\lim\limits_{n\to\infty} S_{2n} = S$ (a finite real number the according to def. of convergent sequence)

Now $S_{2n+1} = (a_1 - a_2 + a_3 - a_4 + \cdots\cdots - a_{2n}) + a_{2n+1}$

or $S_{2n+1} = S_{2n} + a_{2n+1}$

$\because$ if finite number of terms in a infinite Series does not effect its behaviour

$\Rightarrow \lim\limits_{n\to\infty} S_{2n+1} = \lim\limits_{n\to\infty} S_{2n} + \lim\limits_{n\to\infty} a_{2n+1}$

$\because \lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow \lim\limits_{n\to\infty} a_{2n+1} = 0$

$= S + 0 = S$

$\Rightarrow$ The Sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ Converges to same real numbers. i.e.

$\lim\limits_{n\to\infty} S_n = S$ both for even and odd values of n. Hence the given Series is Convergent.

Theorem:- If the series $\sum\limits_1^\infty |a_n|$ Converges then so does the series i.e, if a series Converges absolutely then it Converges.

Proof:- Let $\sum\limits_1^\infty |a_n| = |a_1| + |a_2| + |a_3| + \cdots + |a_n| + \cdots$

be convergent.

Consider the series

$$\sum_1^\infty (a_n + |a_n|)$$

We have $a_n + |a_n| = \begin{cases} 0, & \text{if } a_n \text{ is negative} \\ 2|a_n|, & \text{if } a_n \text{ is positive.}\end{cases}$

Hence $0 \le a_n + |a_n| \le 2|a_n| \quad \longrightarrow (1)$

Theorem (Root Test for Absolute Convergent)

Let $\sum_{1}^{\infty} a_n$ be an infinite series and $\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} = l$, where l is non-negative number or ∞.

(i) if $l < 1$. The series is absolutely convergent.

(ii) if $l > 1$ or ∞, The series is divergent.

(iii) If $l = 1$, the test fails and the series may be absolutely convergent, conditionally convergent or divergent.

Exercise 8.4

Use the alternating series test to determine whether the given series converges (Problems 1–6):

ALTERNATING SERIES TEST (i.e. Leibniz Test)

The alternating series $\sum(-1)^{n-1} a_n$ is convergent if

(i) $a_n > a_{n+1}$ i.e. non increasing seq $\{a_n\}$

(ii) $\lim_{n\to\infty} a_n = 0$

(1) $\sum_{2}^{\infty} (-1)^n \frac{1}{n \ln n}$

Sol. Here $a_n = \frac{1}{n \ln n}$ (n^{th} term) $\Rightarrow a_{n+1} = \frac{1}{(n+1)\ln(n+1)}$

$\because n \ln n < (n+1)\ln(n+1)$

$\Rightarrow \frac{1}{n \ln n} > \frac{1}{(n+1)\ln(n+1)}$ $\qquad \{a_n\} = \left\{\frac{1}{n \ln n}\right\}$

$\Rightarrow a_n > a_{n+1}$ $\qquad$ So $\{a_n\}$ is nonincreasing seq.

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n \ln n} = 0$

So by Alternating Series test, $\sum_{2}^{\infty} (-1)^n \frac{1}{n \ln n}$ is convergent. (Since both conditions of convergence of an alternating series are satisfied)

$f(x) = \frac{1}{x \ln x} = (x \ln x)^{-1}$

$f'(x) = -(x \ln x)^{-2}\left\{x \cdot \frac{1}{x} + \ln x\right\} < 0$

$\therefore$ Seq $\{a_n\}$ is non increasing

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Q.2 $\sum_{1}^{\infty} \frac{\cos n\pi}{\sqrt{n\pi}}$

Sol. Here $\cos n\pi = \begin{cases} -1, & \text{when } n \text{ is odd} \\ 1, & \text{when } n \text{ is even} \end{cases}$

$\cos n\pi = -1+1-1+1-\cdots = (-1)^n$

$$\therefore \sum_{1}^{\infty} \frac{\cos n\pi}{\sqrt{n\pi}} = \sum_{1}^{\infty} \frac{(-1)^n}{\sqrt{n\pi}} \quad \text{(Alternating series)}$$

$$\Rightarrow a_n = \frac{1}{\sqrt{n\pi}} \quad \Rightarrow \quad a_{n+1} = \frac{1}{(n+1)\pi}$$

Since $n\pi < (n+1)\pi$

$$\Rightarrow \frac{1}{n\pi} > \frac{1}{(n+1)\pi} \quad \Rightarrow \quad \frac{1}{\sqrt{n\pi}} > \frac{1}{\sqrt{(n+1)\pi}}$$

$\Rightarrow a_n > a_{n+1}$ $\therefore \{a_n\}$ is non-increasing seq.

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n\pi}} = 0$

So by alternating series test, the given series is convergent.

2nd Method

$f(x) = \frac{1}{\sqrt{x\pi}} = (x\pi)^{-1/2}$

$f'(x) = -\frac{1}{2}(x\pi)^{-3/2}(\pi) = \frac{-\pi}{(x\pi)^{3/2}} < 0$

the seq $\{a_n\}$ is non-increasing

Q.3 $\sum_{1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1}$

Sol Here $a_n = \frac{n^2}{n^2+1}$, $\Rightarrow$

$$a_{n+1} = \frac{(n+1)^2}{(n+1)^2+1} \quad \text{(No need of it)}$$

Since second condition is not satisfied i.e. $\lim_{n\to\infty} a_n \neq 0$. So is not convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^2+1} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \neq 0$$ This cond is not satisfied. So

Series is not convergent. So is divergent.

2nd Method

$f(x) = \frac{x^2}{x^2+1}$

$f'(x) = \frac{(x^2+1)2x - x^2(2x)}{(x^2+1)^2}$

$f'(x) = \frac{2x}{(x^2+1)^2} > 0$

$\Rightarrow \{a_n\} = \left\{\frac{n^2}{n^2+1}\right\}$ is increasing, not non-increasing

and

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$

$\therefore$ Not Cgt

Q.4 $\sum_{1}^{\infty} (-1)^{n-1} \frac{1}{e^n}$

Sol. Here $a_n = \frac{1}{e^n}$ and $a_{n+1} = \frac{1}{e^{n+1}}$

$\because$ Since $e^n < e^{n+1}$

$\Rightarrow \frac{1}{e^n} > \frac{1}{e^{n+1}}$ i.e $a_n > a_{n+1}$.

also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{e^n} = 0$

$\therefore$ by alternating series test, $\sum_1^\infty (-1)^{n-1} \frac{1}{e^n}$ is Convergent.

2nd Method
$f(x) = \frac{1}{e^x} = e^{-x}$
$f'(x) = -\frac{1}{e^x} < 0$
Seq is non increasing

Q.5 $\sum_1^\infty (-1)^{n-1} \frac{n+4}{n^2+n}$

Sol: Here $a_n = \frac{n+4}{n^2+n}$ $\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+4}{n^2+n}$ $\left(\frac{\infty}{\infty}\right)$

or $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(1+\frac{4}{n})}{n^2(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{1+\frac{4}{n}}{n(1+\frac{1}{n})} = \boxed{0}$

$a_{n+1} = \frac{n+1+4}{(n+1)^2+(n+1)} = \frac{n+5}{(n+1)(n+1+1)} = \frac{n+5}{(n+1)(n+2)}$

$\therefore a_n - a_{n+1} = \frac{n+4}{n(n+1)} - \frac{n+5}{(n+1)(n+2)}$

$= \frac{(n+2)(n+4) - n(n+5)}{n(n+1)(n+2)}$

$a_n - a_{n+1} = \frac{n^2+6n+8-n^2-5n}{n(n+1)(n+2)} = \frac{n+8}{n(n+1)(n+2)} \rightarrow (1)$

2nd Method

$f(n) = \frac{n+4}{n^2+n}$

$f'(n) = \frac{(n^2+n)(1)-(n+4)(2n+1)}{(n^2+n)^2}$

$= \frac{n^2+n-2n^2-8n-n-4}{(n^2+n)^2}$

$= \frac{-n^2-8n-4}{(n^2+n)^2} < 0$

$\Rightarrow \{a_n\}$ is non-increasing

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+4}{n^2+n} = 0$

$\therefore$ Series is cgt.

Since (1) is a +ive quantity, because n is any +ive integer.

$\Rightarrow a_n - a_{n+1} > 0$ $\Rightarrow a_n > a_{n+1}$.

Since both conditions for an alternating series to be convergent. Thus given Series is convergent.

[V. Imp Note!]
If Mod i.e $\sum |a_n|$ cgs then Al. Series i.e $\sum a_n$ is Ab. Cgt
If Mod i.e $\sum |a_n|$ dgs but Al. Series i.e $\sum a_n$ Cgs then conditional Cgt (using AST)
If Mod dgs and Al. S. $\sum a_n$ also dgs then total Dgt

Q.7 $\displaystyle\sum_{1}^{\infty} \frac{(-1)^n \, n!}{2n!}$

SoL. $a_n = \dfrac{(-1)^n \, n!}{(2n)!}$

$$\Rightarrow |a_n| = \frac{n!}{2n!} \quad \text{and} \quad |a_{n+1}| = \frac{(n+1)!}{(2n+2)!}$$

using ratio test for absolute convergence test

$$\text{i.e } \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+2)!} \times \frac{2n!}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,\cancel{n!}}{(2n+2)(2n+1)\cancel{2n!}} \times \frac{\cancel{2n!}}{\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{n^2\left[2+\frac{2}{n}\right]\left[2+\frac{1}{n}\right]} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{n\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} = 0 < 1$$

So by Ratio test for (absolute convergence) the given Series is absolutely convergent.

Q.8 $\displaystyle\sum_{1}^{\infty} \frac{\sin\sqrt{n}}{\sqrt{n^3+1}}$

SoL. Here Mod $|a_n| = \left|\dfrac{\sin\sqrt{n}}{\sqrt{n^3+1}}\right|$.

Now

$\sin\sqrt{n} \leq 1$

$\dfrac{\sin\sqrt{n}}{\sqrt{n^3+1}} \leq \dfrac{1}{\sqrt{n^3+1}}$

$\dfrac{\sin\sqrt{n}}{\sqrt{n^3+1}} \leq \dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}}$

$\because$ values of $|\sin\sqrt{n}|$ lies between $[0, 1]$

but $\dfrac{1}{n^{3/2}}$ is cgt

$(\because p = 3/2 > 1)$ So

cgt ہو تو cgt ہے

So by B.C. Test $\sum a_n$ is absolutely convergent
(An absolute cgt series is convergent) ab. cgt تو cgt Mod

Q.6 $\sum_{1}^{\infty} (-1)^n \frac{\ln n}{n}$

Sol Here $a_n = \frac{\ln n}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$ (L'Hospital Rule)

$\lim_{n\to\infty} a_n = 0$

& $a_{n+1} = \frac{\ln(n+1)}{n+1}$

$\therefore a_n - a_{n+1} = \frac{\ln n}{n} - \frac{\ln(n+1)}{n+1}$

$= \frac{(n+1)\ln n - n\ln(n+1)}{n(n+1)}$

$= \frac{n\ln n + \ln n - n\ln(n+1)}{n(n+1)}$

$= \frac{n\ln n - n\ln(n+1) + \ln n}{n(n+1)}$

$= \frac{n(\ln n - \ln(n+1)) + \ln n}{n(n+1)}$

$a_n - a_{n+1} = \frac{n\ln\left(\frac{n}{n+1}\right) + \ln n}{n(n+1)} > 0$ $\left(\because n \text{ is any +ive integer}\right)$

$\Rightarrow a_n > a_{n+1}$. Thus by alternating series test

$\Rightarrow$ The given series is convergent.

2nd Method

$f(x) = \frac{\ln n}{n}$

$f'(x) = \frac{n\frac{1}{n} - \ln n}{n^2}$

$= \frac{1-\ln n}{n^2} < 0$ when $n>3$

$\Rightarrow \left\{\frac{\ln n}{n}\right\}$ is non-increasing

(ii) $\lim_{n\to\infty} \frac{\ln n}{n}$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$ (L Hospital Rule)

$= 0$

ALTERNATE:

To prove $a_n > a_{n+1}$

Consider $\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{n+1} \times \frac{n}{\ln n} < 1$

$\forall\ n \in Z^+$

(i.e +ive integral values of n)

$\Rightarrow \frac{a_{n+1}}{a_n} < 1 \Rightarrow a_{n+1} < a_n \quad \forall\ n \in Z^+$

Test the given series for

(i) absolute convergent (ii) Conditionally convergent

(iii) Divergent

(PROBLEMS 7—24)

cc

Q. 9 $\sum_{1}^{\infty} \frac{(-1)^n (n+2)}{n(n+1)}$ $\Rightarrow$ $a_n = \frac{(-1)^n (n+2)}{n(n+1)}$

step 1. **SOL.** from given Series $|a_n| = \frac{n+2}{n(n+1)} = \left|\frac{(-1)^n (n+2)}{n(n+1)}\right|$

$\Rightarrow |a_{n+1}| = \frac{n+3}{(n+1)(n+2)}$

using ratio test for absolutely convergent

i.e. $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n+3}{(\cancel{n+1})(n+2)} \times \frac{n(\cancel{n+1})}{(n+2)} = \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{3}{n})}{\cancel{n^2}(1+\frac{2}{n})(1+\frac{2}{n})} = 1$

$\Rightarrow$ Ratio test fails to determine that Alternating Series is Absolutely convergent or Conditionally convergent or divergent.

step 3. $\therefore$ Now using Alternating series test

$\because |a_n| - |a_{n+1}| = \frac{n+2}{n(n+1)} - \frac{n+3}{(n+1)(n+2)}$

$= \frac{\cancel{n^2}+4n+4-\cancel{n^2}-3n}{n(n+1)(n+2)}$

$= \frac{n+4}{n(n+1)(n+2)} > 0$ for $n = 1, 2, 3, \ldots$

ALTERNATING SERIES TEST The alternating Series $(-1)^{n-1} a_n$ is convergent if (i) $a_n > a_{n+1}$ $\forall n = 1, 2, 3, \ldots$ and (ii) $\lim_{n\to\infty} a_n = 0$

$\Rightarrow |a_n| - |a_{n+1}| > 0$ i.e. $|a_n| > |a_{n+1}|$

and $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n+2}{n(n+1)} = \lim_{n\to\infty} \frac{n(1+\frac{2}{n})}{n^2(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{(1+\frac{2}{n})}{n(1+\frac{1}{n})} = 0$

So by Alternating Series test $\sum_{1}^{\infty} (-1)^n \frac{n+2}{n(n+1)}$ is Convergent.

step 2. $\sum_{1}^{\infty} |a_n| = ?$ $\frac{(n+2)}{(n+1)} \cdot \frac{1}{n} > \frac{1}{n}$ $\frac{1}{n} = b_n$ (say)

then $|a_n| > |b_n|$ $\forall n = 1, 2, 3, \ldots$

& $\sum_{1}^{\infty} |b_n| = \sum_{1}^{\infty} \frac{1}{n}$ is divergent. ($\because$ Harmonic series)

$\therefore$ By C. Test $\Rightarrow \sum_{1}^{\infty} |a_n|$ is also Divergent. So check Alternating Series Test

$\therefore$ Alternating Series is Conditionally Convergent!

2002A

Q.10 $\sum_{1}^{\infty} (-1)^n \frac{n!}{1.3.5\ldots(2n-1)}$

SoL. $\Rightarrow a_n = \frac{(-1)^n n}{1.3.5\ldots(2n-1)}$

So $|a_n| = \frac{n!}{1.3.5\ldots(2n-1)} \Rightarrow |a_{n+1}| = \frac{(n+1)!}{1.3.5\ldots\ldots(2n+1)}$

using ratio test for absolute convergent i.e

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{1.3.5\ldots\cancel{(2n-1)}(2n+1)} \times \frac{\cancel{1.3.5\ldots(2n-1)}}{n!}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n}} = \boxed{\frac{1}{2}<1}$$

Hence Ab. Cgt Series

i.e $\sum a_n$ is Ab cgt (using Ratio test for Ab cgt.)

Q.11 $\sum_{1}^{\infty} (-1)^{n-1}\left(\frac{n+2}{3n-1}\right)^n$

SoL. Here $a_n = (-1)^{n-1}\left(\frac{n+2}{3n-1}\right)^n$

$\Rightarrow |a_n| = \frac{(n+2)^n}{(3n-1)^n}$

using root test for absolute convergent

i.e $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \left(\left(\frac{n+2}{3n-1}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{2}{n})}{\cancel{n}(3-\frac{1}{n})} = \boxed{\frac{1}{3}<1}$ Hence Ab. Cgt Series

Q.12 $\sum_{1}^{\infty} (-1)^n n \tan\frac{1}{n}$

SoL. $\Rightarrow a_n = (-1)^n n \tan\frac{1}{n}$

$\Rightarrow |a_n| = n \tan\left(\frac{1}{n}\right)$

Dgt Test
1. (If $\lim_{n\to\infty} a_n \neq 0$ then Σa_n is dgt)

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n \tan\frac{1}{n} = \lim_{n\to\infty} \frac{\tan(\frac{1}{n})}{1/n}$$

$$= \lim_{n\to\infty} \left(\left(\frac{\sin(\frac{1}{n})}{\frac{1}{n}}\right) \lim_{n\to\infty} \frac{1}{\cos(\frac{1}{n})} \right) = \boxed{1 \neq 0}$$ Hence $\sum_1^\infty |a_n|$ is dgt (1)

Now <u>Alternate Series Test</u> $\{a_n\} = \{n \tan\frac{1}{n}\}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \tan\frac{1}{n} = 1 \neq 0$ (as above)

By Al. Series Test, Σa_n diverges as one cond is not satisfied (2) of Al. Series Test

Therefore from (1) & (2) Σa_n is <u>Totally divergent</u>

<u>Note</u> If Mod of Al-Series is proved dgt by Dgt Test i.e $\lim_{n\to\infty} a_n \neq 0$ Then the series must be totally dgt. see Q12

IoT 2001 A <u>Q13</u> $(-1)^{n-1}\frac{n^2}{(n+2)!}$

Sol.

$$\Rightarrow a_n = \frac{(-1)^{n-1} n^2}{(n+2)!} \quad \text{So} \quad |a_n| = \frac{n^2}{(n+2)!}$$

Then $|a_{n+1}| = \frac{(n+1)^2}{(n+3)!}$

using ratio test

i.e $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+3)!} \times \frac{(n+2)!}{(n)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(n+3)\cancel{(n+2)!}} \times \frac{\cancel{(n+2)!}}{(n)^2}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{(n+3)} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{n+3}\right] = \boxed{0 < 1}$$

So by ratio test, the given series is **Absolutely convergent.**

<u>Q.14</u> $\sum_1^\infty (-1)^{n-1}\left[\sqrt{n+1} - \sqrt{n}\right]$

<u>Sol.</u> $\Rightarrow a_n = (-1)^{n-1}\left[\sqrt{n+1} - \sqrt{n}\right] \Rightarrow |a_n| = \sqrt{n+1} - \sqrt{n}$ (by Rationalising)

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\Rightarrow |a_{n+1}| = \frac{1}{\sqrt{n+2} + \sqrt{n+1}}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n+1}} = 1$$

اگر Rationalise نہ کریں تو آرام
صورت میں

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\infty - \infty}{\infty - \infty} = \frac{0}{0}$$

... L'Hospital ...

$\Rightarrow$ ratio test fails. So the given alternating Series is absolutely convergent or conditionally convergent or divergent.

Now Since $\sqrt{n} < \sqrt{n+1}$

$$\Rightarrow \sqrt{n+1} + \sqrt{n} < \sqrt{n+1} + \sqrt{n+1} = 2\sqrt{n+1}$$

$$\Rightarrow |a_n| = \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{2\sqrt{n+1}} = |b_n| \quad \forall n = 1,2,3,\dots$$

but $|b_n| = \dfrac{1}{2\sqrt{n+1}}$ is divergent.

because $\displaystyle\int_1^{\infty} \frac{1}{2(x+1)^{\frac{1}{2}}}\,dx = \frac{1}{2}\int_1^{\infty} (x+1)^{-\frac{1}{2}}\,dx = \frac{1}{2}\left|(x+1)^{\frac{1}{2}}\right|_1^{\infty} = \infty$.

So by Comparison test $\sum_1^{\infty} |a_n|$ is divergent. (dgt ...)

Now A.S.T. $\{a_n\} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}} < 1 \quad \forall n = 1,2,\dots$$

$\Rightarrow |a_{n+1}| < |a_n|$ $\therefore \{a_n\}$ is decreasing

And $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \boxed{0}$

$f(x) = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$

$f(x) = \sqrt{n+1} - \sqrt{n}$ after Rationalizing

$f'(x) = \frac{1}{2}(n+1)^{-\frac{1}{2}} - \frac{1}{2}(n)^{-\frac{1}{2}}$

$= \frac{1}{2}\left\{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right\} < 0$

$\Rightarrow f(n)$ is dec $\Rightarrow \{a_n\}$ is dec

$\Rightarrow \sum (-1)^{n-1}(\sqrt{n+1} - \sqrt{n})$ is cgt.

So the given series is convergent (by A. Series. Test)

but $\sum_1^{\infty} |a_n|$ is divergent. $\Rightarrow$ Given Series i.e.

$\Rightarrow \sum_1^{\infty} (-1)^{n-1}(\sqrt{n+1} - \sqrt{n})$ is conditionally convergent.

Q. 15 $\displaystyle\sum_2^{\infty} (-1)^n \frac{1}{\ln(\ln n)}$

Sol. $\Rightarrow a_n = (-1)^n \dfrac{1}{\ln(\ln n)}$ so $|a_n| = \dfrac{1}{\ln(\ln n)}$

Now Since $\ln n < n$

$\Rightarrow \ln(\ln n) < \ln n$

Combining both inequalities

$\Rightarrow \ln(\ln n) < \ln n < n$

$$\Rightarrow \frac{1}{\ln(\ln n)} > \frac{1}{n} = |b_n| \text{ (say)} \Rightarrow |a_n| > |b_n|$$

but $\sum_2^{\infty} |b_n| = \sum \frac{1}{n}$ is divergent Series

So $\sum_2^{\infty} |a_n|$ is Divergent $\longrightarrow$ ① چھوٹی dgt تو بڑی dgt

Now Al.S.T

say $\{a_n\} = \left\{\frac{1}{\ln \ln n}\right\}$ $\quad |a_n| = \frac{1}{\ln(\ln n)} \Rightarrow |a_{n+1}| = \frac{1}{\ln(\ln(n+1))}$

$$\Rightarrow \frac{|a_{n+1}|}{|a_n|} = \frac{\ln(\ln n)}{\ln(\ln(n+1))} < 1$$

$$\boxed{\begin{aligned} &\ln n < \ln(n+1) \\ &\Rightarrow \ln(\ln n) < \ln(\ln(n+1)) \\ &\Rightarrow \frac{\ln(\ln n)}{\ln(\ln(n+1))} < 1 \end{aligned}}$$

$\Rightarrow |a_{n+1}| < |a_n|$ $\therefore \{a_n\}$ is non increasing seq.

also $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\ln(\ln n)} = \boxed{0}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(\ln n)}$

$f(x) = \frac{1}{\ln(\ln x)} = (\ln(\ln x))^{-1}$

$f'(x) = -(\ln(\ln x))^{-2}\left(\frac{1}{x \ln x}\right) < 0$

$\Rightarrow \{a_n\}$ is decreasing

so by Al. Series Test $\sum_2^{\infty} (-1)^n \frac{1}{\ln \ln n}$ is cgt $\longrightarrow$ ②

combining ① & ② Given Series is Conditionally cgt.

Q.16 $\sum_0^{\infty} (-1)^{n+1} \text{arc} \tan n = \sum_0^{\infty} (-1)^{n+1} \tan^{-1} n$

Sol. Here $a_n = (-1)^{n+1} \tan^{-1} n$

$\Rightarrow |a_n| = \tan^{-1} n$

Since $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2} \boxed{\neq 0}$ so by divergent test

$\sum_1^{\infty} |a_n|$ is divergent $\longrightarrow$ ①

A.S.T

$\{a_n\} = \{\tan^{-1} n\}$

$\because \lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2} \neq 0$

$\therefore \Sigma a_n$ dgs

$$\int_1^\infty f(x)dx = \lim_{n\to\infty} \int_1^n 1\cdot\tan^{-1}x \, dx$$

$$= \lim_{n\to\infty} \left\{ \left| x\tan^{-1}x \right| - \int \frac{x}{1+x^2} dx \right\}$$

$\because \frac{1}{2}\int \frac{2x}{1+x^2}dx = \ln(1+x^2)$

$$= \lim_{n\to\infty} \left\{ \left| x\tan^{-1}x - \frac{1}{2} \ln(1+x^2) \right|_1^n \right\}$$

$$= \lim_{n\to\infty} \left(n\tan^{-1}n - \tan^{-1}1 - \frac{1}{2}\left(\ln(1+n^2)\right) + \frac{1}{2}\ln 2 \right) = \infty$$

So by integral test, $\sum_1^\infty |a_n|$ is divergent $\rightarrow$ (2)

Combining statements (1) and (2), the given series is **Divergent**

Q.17 (Integral) $\sum_2^\infty (-1)^{n-1} \frac{1}{n(\ln n)^2}$

$f(x) = \frac{1}{x(\ln x)^2} = x^{-1}(\ln x)^{-2}$

$f'(x) = -x^{-2}(\ln x)^{-2} - 2x^{-1}(\ln x)^{-3}\frac{1}{x}$

$= -\left[\frac{1}{x^2(\ln x)^2} + \frac{2}{x^2(\ln x)^3}\right]$

and $f'(x) < 0$.

f is non increasing for $x \geq 3$

Sol: $\Rightarrow |a_n| = \frac{1}{n(\ln n)^2}$

$\because f(x)$ is non increasing

So by integral test

$$\int_2^\infty \frac{1}{x(\ln x)^2} dx = \lim_{n\to\infty} \int_2^n (\ln x)^{-2}\left(\frac{1}{x}\right) dx$$

$$= \lim_{n\to\infty} \left| \frac{(\ln x)^{-1}}{-1} \right|_2^n = \lim_{n\to\infty} \left(\frac{-1}{\ln n} + \frac{1}{\ln 2} \right)$$

$$= \frac{1}{\ln 2}$$

$\Rightarrow \sum_2^\infty |a_n|$ is Convergent (by Integral test)

So by definition of absolute convergent of an alternating series. The given series is **Absolutely Convergent**.

Imp CC 2009 A **Q.18** $\sum_{1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$

Sol. Here $|a_n| = \frac{\sqrt{n}}{n+1}$, $b_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} \cdot \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})} = 1 \neq 0$$

$\therefore$ both behaves alike

$\frac{1}{2} - 1 = -\frac{1}{2}$

but $\sum_{1}^{\infty} b_n = \sum \frac{1}{\sqrt{n}}$ is divergent

So by L.C.T

$\sum_{1}^{\infty} |a_n|$ diverges ——— ①

Now A.S.T

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1}$$

$\{a_n\} = \{\frac{\sqrt{n}}{n+1}\}$

$$= \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}(\sqrt{n}+\frac{1}{\sqrt{n}})}$$

$$\boxed{\lim_{n\to\infty} a_n = \boxed{0}}$$

$\because$ A Sequence $\{a_n\}$ is said to be non-increasing if for $a_n = f(n)$ and $f'(n) < 0$
it means. if a sequence $\{a_n\}$ is non increasing i.e $a_n \geq a_{n+1}$ for $n = 1, 2, 3, \ldots$
Then it ... positive ...

$$\because f(n) = \frac{\sqrt{n}}{n+1}$$

$$f'(n) = \frac{\frac{1}{2\sqrt{n}}(n+1) - \sqrt{n}}{(n+1)^2}$$

$$= \frac{n+1-2n}{2\sqrt{n}(n+1)^2} = \frac{1-n}{2\sqrt{n}(n+1)^2} \boxed{<0} \text{ for } n \geq 2$$

$\Rightarrow \{a_n\}$ is non increasing Sequence $\therefore a_1 \geq a_2 \geq a_3 \geq a_4 \ldots$

So by alternating Series test when $\{a_n\}$ is non increasing sequence and $\lim_{n\to\infty} a_n = 0$. Then the alternating Series converges $\longrightarrow$ ②

So from statement ① & ②, We concluded that the given series Conditionally Convergent

NOTE a sequence $\{a_n\}$ is also **Non Increasing** if

$$\frac{|a_{n+1}|}{|a_n|} < 1 \quad \text{i.e } |a_{n+1}| < |a_n|$$
$$\text{or } |a_n| > |a_{n+1}|$$

e.g if $a_n = \frac{\sqrt{n}}{n+1} \Rightarrow a_{n+1} = \frac{\sqrt{n+1}}{n+2}$

then $\frac{a_{n+1}}{a_n} = \frac{\sqrt{n+1}}{n+2} \times \frac{n+1}{\sqrt{n}} = \sqrt{\frac{n+1}{n}} \cdot \frac{n+1}{n+2}$

$$= \sqrt{1+\frac{1}{n}} \times \frac{n+1}{n+2} < 1 \quad \text{for all } n = 1,2,3,\ldots$$

Q.19 $\sum_1^\infty \frac{(-1)^{n-1} n^2}{(2n+1)(n+5)}$

Sol. $\because |a_n| = \frac{n^2}{(2n+1)(n+5)}$ and $|a_{n+1}| = \frac{(n+1)^2}{(2n+3)(n+6)}$

using ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+3)(n+6)} \times \frac{(2n+1)(n+5)}{n^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})^2}{\cancel{n^2}\left[2+\frac{3}{n}\right]\left[1+\frac{6}{n}\right]} \times \frac{\cancel{n^2}\left[2+\frac{1}{n}\right]\left[1+\frac{5}{n}\right]}{\cancel{n^2}} = 1 .$$

i.e no conclusion can be drawn

Now divergent test

Divergent test
If $\lim_{n\to\infty} a_n \neq 0$ Then Σa_n is divergent

i.e $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{(2n+1)(n+5)} = \lim_{n\to\infty} \frac{1}{(2+\frac{1}{n})(1+\frac{5}{n})}$

$$= \boxed{\frac{1}{2} \neq 0} \Rightarrow \sum_1^\infty |a_n| \text{ is divergent}$$

A.S. Test $\{a_n\} = \left\{\frac{n^2}{(2n+1)(n+5)}\right\}$

$\lim_{n\to\infty} a_n = \lim \frac{n^2}{(2n+1)(n+1)} = \frac{1}{2} \neq 0$

$\therefore \Sigma a_n$ dgs.

Please write it

or $f'(n) = \dfrac{4n^3 + 22n^2 + 10n - 4n^3 - 11n^2}{4(2n^2+11n+5)^2}$

$= \dfrac{11n^2+10n}{(2n^2+11n+5)^2} > 0$ for all $n = 1, 2, 3, \dots$

$\Rightarrow f(n)$ is increasing i.e sequence

$\{a_n\}$ is increasing sequence.

So both conditions of alternating series test are not satisfied. Thus Given Series is not convergent. Implies that the given series is divergent.

Q. 20 $\sum_1^{\infty} (-1)^{n-1} \dfrac{2 \sinh n}{e^{2n}}$

Sol. Here $|a_n| = \dfrac{2 \sinh n}{e^{2n}} = \dfrac{\cancel{2}\left(\dfrac{e^n - e^{-n}}{\cancel{2}}\right)}{e^{2n}}$

$$|a_n| = \frac{e^n - \frac{1}{e^n}}{e^{2n}} = \frac{e^{2n}-1}{e^{3n}} < \frac{e^{2n}}{e^{3n}}$$

$$\therefore \Rightarrow |a_n| = \frac{e^{2n-1}}{e^{3n}} < \frac{1}{e^n}$$

Let $\Sigma b_n = \Sigma \frac{1}{e^n}$

$\therefore \sum_1^{\infty} b_n$ is cgt $\because$ I.G.S with $r < 1$

$\therefore \sum_1^{\infty} |a_n|$ is cgt by B.C.T (cgt

$\therefore \sum_1^{\infty} |a_n|$ is absolutely cgt.

Note A series is said to be absolutely convergent if the series $\sum_1^{\infty} |a_n|$ is convergent

DEFINITION A Series $\sum_1^{\infty} a_n$ is said to be absolutely convergent if the Series $\sum_1^{\infty} |a_n|$ is convergent

Q.21 $\sum_{1}^{\infty} \frac{\cos n\pi}{n+3} = \sum_{1}^{\infty} \frac{(-1)^n}{n+3}$

Sol. $a_n = \frac{(-1)^n}{n+3} \Rightarrow |a_n| = \frac{1}{n+3}$

Let $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{1}{n+3}\cdot\frac{n}{1}$$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \frac{\cancel{n}}{\cancel{n}(1+\frac{3}{n})} = \boxed{1 \neq 0}$$ ∴ both behave alike

$\because \sum b_n = \sum \frac{1}{n}$ is dgt.

So by L comparison test $\sum a_n$ is divergent.

Al. Series Test

$$|a_n| - |a_{n+1}| = \frac{1}{n+3} - \frac{1}{n+4} = \frac{n+4-n-3}{(n+3)(n+4)}$$

$$= \frac{1}{(n+3)(n+4)} > 0 \quad \forall \text{ +ive values of } n \text{ i.e. } n = 1,2,3,\dots$$

2nd Method

$f(x) = \frac{1}{x+3}$

$f'(x) = \frac{-1}{(x+3)^2} < 0$

Seq. is non increasing

$\Rightarrow |a_n| - |a_{n+1}| > 0$

$\Rightarrow |a_n| > |a_{n+1}|$ Hence non-increasing

ALTERNATING SERIES TEST

If $a_n > 0 \ \forall \ n = 1,2,3,\dots$

Then A-Series is convergent if

(i) $a_n > a_{n+1}$

(ii) $\lim_{n\to\infty} a_n = 0$

Also $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n+3} = \boxed{0}$

Since both conditions for an alternating series test are satisfied so given Alternating Series is Convergent.

∴ Hence $\sum a_n$ is conditionally convergent.

Q.22 $\sum_{1}^{\infty} \frac{n!}{(-2)^n}$

Sol. $\Rightarrow a_n = \frac{n!}{(-2)^n} \Rightarrow |a_n| = \frac{n!}{2^n}$

$$|a_{n+1}| = \frac{(n+1)!}{2^{n+1}} = \frac{(n+1)n!}{2\cdot 2^n}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)n!}{2\cdot 2^n}\cdot\frac{2^n}{n!} = \lim_{n\to\infty} \frac{n+1}{2} = \infty$$

∴ Divergent

So by Ratio test Given Alternating Series test is dgt.

Note . If Dgt Test, Ratio or Root test is applied to $\Sigma|a_n|$ and it is a dgt series. then Al. Series is totally dgt. No need of A.S.T.)

However if $\Sigma|a_n|$ dgs by B.C.T or L.C.T or Integral then we further apply Al.S.T for cond cgt or T dgt.

Q.23 B.2

Q.23

$$\sum_{0}^{\infty}\frac{(-2)^n}{3^n+1}$$

Sol

Here $a_n=\frac{(-2)^n}{3^n+1}$

$$|a_n|=\frac{2^n}{3^n+1}$$

$$|a_n|=\frac{2^n}{3^n+1}<\frac{2^n}{3^n}=|b_n| \text{ (say)}$$

$$\Sigma|b_n|=\Sigma\left(\frac{2}{3}\right)^n$$

$$=\left(\frac{2}{3}\right)^1+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots$$

infinite G. Series with $r=\frac{2}{3}<1$

So $\Sigma\left(\frac{2}{3}\right)^n$ is convergent Series. Thus by comparison test. given series is absolutely convergent

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\lim_{n\to\infty}\frac{2^{n+1}}{3^{n+1}+1}\cdot\frac{3^n+1}{2^n}$$

$$=\lim_{n\to\infty}\frac{\cancel{2^n}\cdot 2}{3^n\cdot 3+1}\cdot\frac{3^n+1}{\cancel{2^n}}$$

$$=\lim_{n\to\infty}\frac{2\cdot\cancel{3^n}(1+\frac{1}{3^n})}{\cancel{3^n}(3+\frac{1}{3^n})}$$

$$=\frac{2(1+0)}{3(3+0)}$$

$$=\frac{2}{3}<1$$

Hence Ab Cgt.

Q.25

$$\sum_{1}^{\infty}(-1)^{n+1}\frac{3n^2-3n+5}{n^3+n^2+n+1}$$

Sol $\Rightarrow |a_n|=\frac{3n^2-3n+5}{n^3+n^2+n+1}$

Step 1 Let $\Sigma b_n=\sum_{1}^{\infty}\frac{1}{n}$

$$\frac{|a_n|}{b_n}=\lim_{n\to\infty}\frac{(3n^2-3n+5)}{n^3+n^2+n+1}\times\frac{n}{1}$$

$$=\lim_{n\to\infty}\frac{\cancel{n^3}(3-3/n+5/n^2)}{\cancel{n^3}(1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3})}$$

$$=\boxed{3}\neq 0$$

so both series behave alike. as Σb_n is dgt so Σa_n is dgt.

Step 2 using Alternating Series test

$$\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\frac{3n^2-3n+5}{n^3+n^2+n+1}$$

$$\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\frac{\cancel{n^2}(3-\frac{3}{n}+\frac{5}{n^2})}{n^{\cancel{3}}(1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3})}$$

$$\lim_{n\to\infty}|a_n|=\boxed{0}$$

Now Since $f(n)=\frac{3n^2-3n+5}{n^3+n^2+n+1}$

$$\Rightarrow f'(n)=\frac{(n^3+n^2+n+1)(6n-3)-(3n^2-3n+5)(3n^2+2n+1)}{(n^3+n^2+n+1)^2}$$

$$f'(n) = \frac{6n^4+6n^3+6n^2+6n-3n^3-3n^2-3n-3-9n^4+9n^3-15n^2-6n^3+6n^2 - 10n-3n^2+3n-5}{(n^3+n^2+n+1)^2}$$

$$f'(n) = \frac{-3n^4+6n^3-9n^2-4n-8}{(n^3+n^2+n+1)^2} \boxed{<0} \quad \forall \text{ +ve values of } n$$

i.e. $n = 1, 2, 3 \ldots$

is decreasing sequence, so AEST is cgt.

$\Rightarrow$ So given series is conditionally cgt.

step 1 $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ — by stirling formula $\lfloor n = \sqrt{2\pi n}\, \frac{n^n}{e^n}$

$$\Rightarrow \lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{(3n^2-3n+5)}{n^3+n^2+n+1} \times \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^2\left(3-\frac{3}{n}+\frac{5}{n^2}\right)n}{n^3\left(1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}\right)} = \lim_{n\to\infty} \frac{3-\frac{3}{n}+\frac{5}{n^2}}{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}$$

$$= 3 = l \neq 0$$

$\Rightarrow$ both series behaves alike. but $\sum_1^\infty b_n$ is divergent implies that the series Hence $\sum_1^\infty |a_n|$ is also divergent-

Given series is conditionally convergent-

Q.26 $\sum_1^\infty (-1)^n \frac{n^n}{n!}$

SoL Here $a_n = (-1)^n \frac{n^n}{n!} \Rightarrow |a_n| = \left|\frac{(-1)^n n^n}{n!}\right| = \frac{n^n}{n!}$

$$|a_{n+1}| = \frac{(n+1)^{n+1}}{(n+1)!}$$

then using ~~absolute~~ ratio test for absolute convergent i.e.

Q.24 $\sum_{0}^{\infty}(-1)^n\left[\frac{\pi}{2} - \text{arc}\tan n\right]$

Sol. Here $a_n = (-1)^n\left(\frac{\pi}{2} - \tan^{-1} n\right)$

$\Rightarrow |a_n| = \frac{\pi}{2} - \tan^{-1} n$ so $f(x) = \frac{\pi}{2} - \tan^{-1} x$

$f'(x) = 0 - \frac{1}{1+x^2} < 0 \quad , x \geq 0$

$f(x)$ is non increasing on $[0, \infty[$

$\int_0^\infty f(x)dx$

$$= \lim_{x\to\infty}\int_0^x\left(\frac{\pi}{2} - \tan^{-1} x\right)dx = \lim_{x\to\infty}\int_0^x \frac{\pi}{2}dx - \lim_{x\to\infty}\int_0^x \tan^{-1} x\,dx$$

$$= \lim_{x\to\infty}\left|\frac{\pi}{2}x\right|_0^x - \lim_{x\to\infty}\int_0^x 1\cdot\tan^{-1} x\,dx$$

$$= \frac{\pi}{2}(\infty) - 0 - \lim_{x\to\infty}\left(\left|x\tan^{-1} x\right|_0^x - \frac{1}{2}\int_0^x \frac{2x}{1+x^2}dx\right)$$

$$= \infty - \lim_{x\to\infty}\left[x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]_0^x$$

$$= \infty - \infty + \infty = \infty$$

$\Rightarrow \sum|a_n|$ is divergent

So we check it that alternating series is convergent or divergent

Thus $|a_n| - |a_{n+1}|$

$= \left(\frac{\pi}{2} - \tan^{-1} n\right) - \left(\frac{\pi}{2} - \tan^{-1}(n+1)\right)$

$= \tan^{-1}(n+1) - \tan^{-1} n > 0$

$\Rightarrow |a_n| - |a_{n+1}| > 0$ or $\boxed{|a_n| > |a_{n+1}|}$ $\{a_n\}$ non increasing seq.

2nd Method

$f(x) = \frac{\pi}{2} - \tan^{-1} x$

$f'(x) = -\frac{1}{1+x^2} \boxed{<0}$

$\therefore \{a_n\}$ is non-increasing

& $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|\frac{\pi}{2} - \tan^{-1} n\right| = \frac{\pi}{2} - \frac{\pi}{2} = \boxed{0}$

Since both conditions of an alternating series are satisfied but $\sum|a_n|$ is divergent So given alternating series is convergent. Hence given series

Converges Conditionally

ALTERNATE Here $|a_n| = \frac{\pi}{2} - \tan^{-1} n$

$= \cot^{-1} n$

put $\frac{\pi}{2} - \tan^{-1} n = \alpha$

$\Rightarrow \tan^{-1} n = \frac{\pi}{2} - \alpha$

$n = \tan\left(\frac{\pi}{2} - \alpha\right)$

$n = \cot\alpha$

$\Rightarrow \cot^{-1} n = \alpha$

So

$\frac{\pi}{2} - \tan^{-1} n = \cot^{-1} n$

$f(x) = \cot^{-1} x$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}$$

$$= \lim_{n\to\infty}\frac{(n+1)^n\cdot(n+1)}{(n+1)\,n!}\times\frac{n!}{n^n} = \lim_{n\to\infty}\frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$$

So by ratio test for absolute convergent, Given Series is divergent.

Q.27 $\sum_1^\infty (-1)^{n+1}\frac{e^n}{n}$

Sol Here $a_n = (-1)^{n+1}\frac{e^n}{n} \Rightarrow |a_n| = \frac{e^n}{n}$

using root test for absolute convergent series

i.e. $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{e^n}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{e}{n^{\frac{1}{n}}}$ $\boxed{\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1 \text{ already proved}}$

$$= \frac{e}{1} = e > 1$$

So by root test for absolute convergent series, the given series is divergent.

$\because \left(3^{n^2}\right)^{\frac{1}{n}} = (3^n)$

Q.28 $\sum_1^\infty (-1)^{n+1}\frac{n^n}{3^{n^2}}$

Sol Here $|a_n| = n^n/3^{n^2}$. using root test for absolute convergent series i.e.

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n^n}{3^{n^2}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{3^n}\quad\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty}\frac{1}{3^n \ln 3}\quad \text{(using L'Hospital Rule)}\quad \left(\because \frac{d}{dx}a^x = a^x \ln a\right)$$

$= \boxed{0<1}$ So by Root T. for A. Convergent given Series is absolute Convergent.

Q.29 $\sum_1^\infty (-1)^n \frac{n^n}{a^{nb+c}}$ where $a>1$, b and c are real

Sol Here $|a_n| = \frac{n^n}{a^{nb+c}}$ using Root Test for absolute convergent i.e.

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n^n}{a^{nb+c}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{a^{b+c/n}} = \frac{\infty}{a^b} = \infty$$

So root test for absolute convergent, the given series is divergent.

using integral test

$$\int_0^\infty \cot^{-1}x\,dx = \lim_{t\to\infty}\int_0^t 1\cdot\cot^{-1}x\,dx$$

$$= \lim_{t\to\infty}\left[\left|x\cdot\cot^{-1}x\right|_0^t - \int_0^t \frac{-x}{1+x^2}\,dx\right]$$

$$= \lim_{t\to\infty}\left[t\cot^{-1}t - \cot^{-1}0 + \frac{1}{2}\left|\ln(1+x^2)\right|_0^t\right]$$

$$= \lim_{t\to\infty}\left[t\cot^{-1}t - \cot^{-1}0 + \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln 2\right] = \infty$$

Hence $\sum_0^\infty |a_n|$ is divergent, by integral test.

Now as $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\cot^{-1}n = \cot^{-1}\infty = 0$

$(\because \cot 0^\circ = \infty)$

and $|a_n| - |a_{n+1}| > 0$ (already proved)

Hence given series is convergent by A. S. test. but $\sum_0^\infty |a_n|$ is divergent. So given series is conditionally convergent.

Q. 30 — 35. find value of x for which the given series

(i) absolutely convergent (ii) Conditionally convergent

(iii) Divergent.

Q. 30 $\sum_1^\infty \frac{nx^n}{3^n}$

SOL. Here $|a_n| = \left|\frac{nx^n}{3^n}\right|$

$$|a_{n+1}| = \frac{|n+1|\,|x^{n+1}|}{|3^{n+1}|}$$

Root Test 2nd Method

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{nx^n}{3^n}\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{n^{\frac{1}{n}}|x|}{3}$$

$$= 1\cdot\frac{|x|}{3}$$

$$= \frac{|x|}{3}$$

using Ratio Test for absolutely convergent

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|n+1|\,|x^n\cdot x|}{3\cdot 3^n}\times\frac{|3^n|}{|n|\,|x^n|} = \lim_{n\to\infty}\frac{|n+1|\,|x|}{|3|\cdot|n|}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left|\frac{x}{3}\right| = \lim_{n\to\infty}\frac{n}{n}\left(1+\frac{1}{n}\right)\frac{|x|}{3} = \frac{|x|}{3}$$

Thus given series absolutely converges for $\frac{|x|}{3} < 1$ i.e. for $|x| < 3$ by ratio test

diverges for $\frac{|x|}{3} > 1$ i.e. for $|x| > 3$

Also if $\frac{|x|}{3} = 1 \Rightarrow |x| = 3 \Rightarrow x = \pm 3$

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(\pm 3)^n}{3^n} = \mp\infty \neq 0$

For $x = 3$: $\lim_{n\to\infty} \frac{n(3^n)}{(3^n)} = \infty$

For $x = -3$: $\lim_{n\to\infty} \frac{n(-3)^n}{(3^n)} = \lim_{n\to\infty} \frac{n(-1)^n 3^n}{(3^n)} = \infty$

so by divergent test the given series diverges for $x = \pm 3$ ... (ii)

combining statement (i) and (ii), we concluded that the given series is divergent for $|x| \geq 3$.

Q.31 $\sum_{1}^{\infty} \frac{x^n}{\sqrt{n}}$

Sol Here $|a_n| = \left|\frac{x^n}{\sqrt{n}}\right| \Rightarrow |a_n| = \frac{|x^n|}{|\sqrt{n}|} = \frac{|x^n|}{\sqrt{n}}$.

and $|a_{n+1}| = \frac{|x^{n+1}|}{\sqrt{n+1}} = \frac{|x||x^n|}{\sqrt{n+1}}$

2nd Method (By Root test)

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x|}{\sqrt{n^{\frac{1}{n}}}} = |x|$.

using Ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|\cancel{|x^n|}}{\sqrt{n+1}} \times \frac{\sqrt{n}}{\cancel{|x^n|}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}|x|$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n+1}{n}}}|x| = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}}|x|$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \longrightarrow (1)$$

if $|x| < 1$ Series is Ab. Cgt

if $|x| > 1$ divergent Series

if $|x| = 1$ Test Fail

$|x| = 1 \Rightarrow x = \pm 1$, For $x = 1$: $\sum_{1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{1}^{\infty} \frac{1}{\sqrt{n}} \Rightarrow a_n = \frac{1}{\sqrt{n}}$

Which is divergent. Thus the Series is also divergent for $x = 1$. and

For $x=-1$ $\sum_1^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is an alternating series. So using alternating series test

$f(x) = \frac{1}{\sqrt{x}}$
$f'(x) = -\frac{1}{2} x^{-3/2}$
$f'(x) < 0$
$\therefore$ non increasing

i.e. $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \quad \left(\because a_n = \frac{(-1)^n}{\sqrt{n}}\right)$

and $|a_n| = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = |a_{n+1}|$ for all $n \geq 1$.

So by alternating series test, Series $\sum_1^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Converges

but $\sum_1^{\infty} \frac{1}{\sqrt{n}} = \sum_1^{\infty} |a_n|$ is divergent, So by the definition of conditionally convergent, the series conditionally converges for $x=-1$.

We can use Ratio also
Root Test

Q. 32 $\sum_0^{\infty} \frac{4^n}{x^n}$

SOL. Here $a_n = \frac{4^n}{x^n} \Rightarrow |a_n| = \left|\frac{4^n}{x^n}\right| = \frac{4^n}{|x^n|}$

$= \left|\frac{4}{x}\right|^n = \left(\frac{4}{|x|}\right)^n$ Easy

I.G.S cgs if $r<1$

$\Sigma a_n = \sum^{\infty} \frac{4^n}{x^n}$

$|a_n|^{\frac{1}{n}} = \left|\frac{4}{x}\right|^{n\cdot\frac{1}{n}}$

$= \left|\frac{4}{x}\right|$

Series is cgt if $\left|\frac{4}{x}\right|<1$
dgt if $\left|\frac{4}{x}\right|>1$
Fails for $\left|\frac{4}{x}\right|=1$
i.e. $|x|=4$
$\Rightarrow x=\pm 4$

For $x=4$

For $x=-4$

$\therefore$ I.G.S Converges for $r = \frac{4}{x} < 1 \Rightarrow |x|>4$

I.G.S Diverges for $\frac{4}{|x|} > 1 \Rightarrow |x|<4$

I.G.S Diverges for $\frac{4}{|x|} = 1 \Rightarrow |x| = 4 \Rightarrow x = \pm 4$

Also When $x=0$, the series is not defined. Therefore it is divergent for $0<|x|\leq 4$

Q. 33 $\sum_0^{\infty} \frac{(-1)^n x^n}{3^n (n+2)!}$

Sol $a_n = \frac{(-1)^n x^n}{3^n (n+2)!} \Rightarrow |a_n| = \left|\frac{(-1)^n x^n}{3^n (n+2)!}\right| = \frac{|x|^n}{3^n (n+2)!}$

So $|a_{n+1}| = \frac{|x|^{n+1}}{3^{n+1}(n+3)!}$

using ratio test for absolute convergent i.e.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x||x|^n}{3\cdot 3^n (n+3)!} \times \frac{3^n (n+2)!}{|x|^n} = \lim_{n\to\infty} \frac{|x|(n+2)!}{3(n+3)(n+2)!}$$

$$= \underset{n\to\infty}{Lt} \frac{|x|}{3(n+3)} = 0 < 1$$

So by ratio-test for absolute convergence, the given alternating series is absolutely convergent for all values of x.

x ———— x

Q.34 $\sum_{1}^{\infty} (-1)^{n-1} \frac{x^n}{n(n+1)}$

SOL $\Rightarrow |a_n| = \left|\frac{x^n}{n(n+1)}\right| = \frac{|x^n|}{n(n+1)} = \frac{|x|^n}{n(n+1)}$

and $|a_{n+1}| = \frac{|x|^{n+1}}{(n+1)(n+2)}$

using ratio test for absolute convergent i.e

$$\underset{n\to\infty}{Lt} \frac{|a_{n+1}|}{|a_n|} = \underset{n\to\infty}{Lt} \frac{|x|^{n+1}}{(\cancel{n+1})(n+2)} \times \frac{n(\cancel{n+1})}{|x|^n} = \underset{n\to\infty}{Lt} \frac{n|x|}{n+2}$$

$$= \underset{n\to\infty}{Lt} \frac{|x|}{\frac{n+2}{n}} = \underset{n\to\infty}{Lt} \frac{|x|}{1+\frac{2}{n}} = \frac{|x|}{1+0} = |x|$$

The series converges absolutely for $|x|<1$ and diverges for $|x|>1$.

Also if $|x| = 1$ i.e $x = 1, -1$.

when $\underline{x=1}$ then given series will be $\sum_{1}^{\infty} (-1)^{n-1} \frac{1}{n(n+1)}$

for $\underline{x=-1}$ " " $\sum_{1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n(n+1)} = \sum_{1}^{\infty} (-1)^{2n-1} \frac{1}{n(n+1)}$

In both cases

$\frac{1}{n(n+1)} = \frac{1}{n^2+n} < \frac{1}{n^2} = b_n$ which is cgt so for $x=\pm 1$ given series is Absolutely Cgt

Thus the series is absolutely convergent for $|x| \le 1$ and divergent for $|x|>1$.

x ———— x

Q.35 $\sum_{0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

SOL Here $|a_n| = \left|\frac{x^{2n}}{2^{2n}(n!)^2}\right| = \frac{|x|^{2n}}{2^{2n}(n!)^2}$

$\Rightarrow |a_{n+1}| = \frac{|x|^{2n+2}}{2^{2n+2}[(n+1)!]^2}$

Using ratio test for absolute convergent series e...

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{2n+2}}{2^{2n+2}\left[(n+1)!\right]^2} \times \frac{2^{2n}(n!)^2}{|x|^{2n}}$$

$$= \lim_{n\to\infty} \frac{|x|^2 \cdot 2^{2n}(n!)^2}{2^2\, 2^{2n}\,(n+1)^2(n!)^2}$$

$$= \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 < 1$$

$\Rightarrow$ the given series is absolutely convergent for all values of x.

Power Series

An infinite series of the form

$$\sum_{0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots c_n x^n + \cdots \quad (I)$$

or $$\sum_{0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots c_n(x-a)^n + \cdots \quad (II)$$

is called Power Series in x or $(x-a)$, $c_n \in R$

Series (II) can be reduced to series (I) by putting $(x-a) = Y$

Convergence of Power Series (I) i.e $\sum_{0}^{\infty} c_n x^n$

We use the Ratio Test for Absolute Convergence or Root Test for Absolute convergence to find the values of x for which Power Series (I) converges.

Example. Find the values of x for which the power series $\sum_{0}^{\infty} n^{2n} x^{2n}$ converges.

Sol $\sum a_n = \sum n^{2n} x^{2n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{2(n+1)}\, x^{2(n+1)}}{n^{2n}\, x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^{2n}(n+1)^2 x^{2n} x^2}{n^{2n} x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^{2n} (n+1)^2 x^2\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^n (n+1)\, x\right|^2$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \quad \text{, when } x=0 \quad \therefore \text{ Series Converges for } x=0$$

$$= \infty \quad \text{, when } x \neq 0 \quad \therefore \text{ Series diverges for } x \neq 0$$

Easy 2nd Method

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left|n^{2n}x^{2n}\right|^{\frac{1}{n}}$$

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left|n^2 x^2\right|^{n\cdot\frac{1}{n}}$$

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty} n^2|x^2|$$

$$= 0 < 1 \quad \text{when } x=0$$

and $$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \infty \quad \text{when } x \neq 0$$

So series converges for $x=0$ & diverges for $x \neq 0$

Example Find values of x for which Power Series $\sum_{0}^{\infty} \frac{x^n}{2n!}$ converges.

Sol $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{2(n+1)!}}{\frac{x^n}{2n!}}\right| = \left|\frac{x^n \cdot x}{(2n+2)!} \times \frac{2n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{(2n+2)(2n+1)\cancel{2n!}} \times \cancel{2n!}\right| = \lim_{n\to\infty}\left|\frac{x}{(2n+2)(2n+1)}\right| = 0 < 1$$

$\therefore$ given series cgs for all values of x

Example Determine the values of x for which the power series $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$ converges absolutely, converges conditionally & diverges.

Sol: Ratio Test for Ab. Cgce.

$$|a_n| = \left|\frac{x^n}{\ln n}\right|$$

$$|a_{n+1}| = \left|\frac{x^{n+1}}{\ln(n+1)}\right|$$

$$|\ln n| = \ln n$$

$$|\ln(n+1)| = \ln(n+1)$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{x^{n+1}}{\ln(n+1)}\right| \times \left|\frac{\ln n}{x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^n \cdot x \cdot \ln n}{x^n \cdot \ln(n+1)}\right| \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} |x| \cdot \frac{\frac{1}{n}}{\frac{1}{n+1}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{n\to\infty} |x| \cdot \frac{1}{n} \cdot \frac{n+1}{1} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} |x| \cdot \frac{1}{1}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \quad \text{———} \; (1)$$

Now power series Converges Absolutely if $|x| < 1$

power series Diverges if $|x| > 1$

Test fails to determine, if $|x| = 1$

Now $|x| = 1 \Rightarrow x = \pm 1$

<u>For $x = -1$</u>

$$\sum a_n = \sum \frac{x^n}{\ln n} = \sum \frac{(-1)^n}{\ln n} \quad \text{(Alternating Series)}$$

Non-Mod

$$\sum |a_n| = \sum \left|\frac{1}{\ln n}\right| < \sum \frac{1}{n} \quad \text{which is dgt}$$

line $\sum |a_n|$ is dgt by BCT

So apply ALST.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln n} = \frac{1}{\infty} = \boxed{0}$$

Note $f(x) = \frac{1}{\ln x} = (\ln x)^{-1}$

$f'(x) = (-1)(\ln x)^{-2}\frac{1}{x} < 0$

$f'(x) < 0$ Non Increasing

Also $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$

$\Rightarrow \boxed{a_{n+1} < a_n}$ Hence non increasing seq

Therefore Alt Series is convergent and

Since $\sum |a_n|$ is dgt So Power Series is <u>Cond Cgt.</u>

<u>For $x = 1$</u>

$$\sum a_n = \sum \frac{x^n}{\ln n} = \sum \frac{1}{\ln n} > \sum \frac{1}{n} \quad \text{which is dgt.}$$

So $\sum a_n$ is dgt By BCT.

$\Rightarrow$ Hence Power Series

Converges absolutely for $|x| < 1$

Diverges for $|x| > 1$

Diverges for $x = 1$

Converges Conditionally for $x = -1$

$x \longrightarrow x$

Example Find interval of Convergence of p-series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Sol $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\lfloor n+1 \, x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{\lfloor n \, x^n}\right|$

$$= \left|\frac{\cancel{(n+1)}\cancel{\lfloor n}\,\cancel{x^n}\cdot x \cdot n^n}{(n+1)^n \cancel{(n+1)}\,\cancel{\lfloor n}\,\cancel{x^n}}\right|$$

$$= \left|\frac{x}{\frac{(n+1)^n}{n^n}}\right| = \left|\frac{x}{\left(\frac{n+1}{n}\right)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x}{\frac{\cancel{n}}{\cancel{n}}\left(1+\frac{1}{n}\right)^n}\right| = \left|\frac{x}{e}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{e}$$

Now

If $\frac{|x|}{e} < 1$: P.Series Cgs Abs.

$\frac{|x|}{e} < 1 \Rightarrow |x| < e$

$\Rightarrow -e < x < e$

Radius of Convergence $= e$

If $\frac{|x|}{e} > 1$ Power Series diverges

If $\frac{|x|}{e} = 1$ Test fails to determine

$\frac{|x|}{e} = 1 \Rightarrow |x| = e \Rightarrow x = \pm e$

2nd Method for term "$\frac{\lfloor n \, e^n}{n^n}$"

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\lfloor n+1 \, e^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{\lfloor n \, e^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(n+1)}\cancel{\lfloor n}\,\cancel{e^n}\cdot e \; n^n}{(n+1)^n\cdot\cancel{(n+1)}\,\cancel{\lfloor n}\,\cancel{e^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^n\cdot e}{(n+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^n\cdot e\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^{-n}\cdot e\right|$$

$$= \lim_{n\to\infty}\left|\left[\left(1+\frac{1}{n}\right)^n\right]^{-1}\cdot e\right|$$

$$= \lim_{n\to\infty}\left|e^{-1}\cdot e\right| = \boxed{1}$$

Test fails

Now for $x = e$, $\sum_{n=1}^{\infty}\frac{n!\,x^n}{n^n} = \sum_{n=1}^{\infty}\frac{n!}{n^n}e^n$ — (I)

for $x = -e$, $\sum_{n=1}^{\infty}\frac{n!\,x^n}{n^n} = \sum_{n=1}^{\infty}\frac{n!(-1)^n e^n}{n^n}$ — (II)

Consider the term $\frac{n!\,e^n}{n^n}$ (which is common in (I) & (II) i.e for $x = e$ or $x = -e$)

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\lfloor n \, e^n}{n^n}$$

If n is very large then by stirling Formula $\lfloor n = \sqrt{2\pi n}\,\frac{n^n}{e^n}$

$$= \lim_{n\to\infty}\left(\sqrt{2\pi n}\,\frac{\cancel{n^n}}{\cancel{e^n}}\right)\cdot\frac{\cancel{e^n}}{\cancel{n^n}}$$

$\lim_{n\to\infty} a_n = \infty \neq 0$ so both (I) & (II) diverges

1^{st} Cond. of AEST is not satisfied Hence dgt.

Therefore Interval of Convergence is $]-e, e[$

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Imp Note

1) If power series $\sum C_n x^n$ converges for $x = x_1$
then power series is converges for $|x| < |x_1|$

اگر P.S x_1 کے لیے cgs ہے تو x_1 سے چھوٹی values کے لیے بھی cgs ہوگی

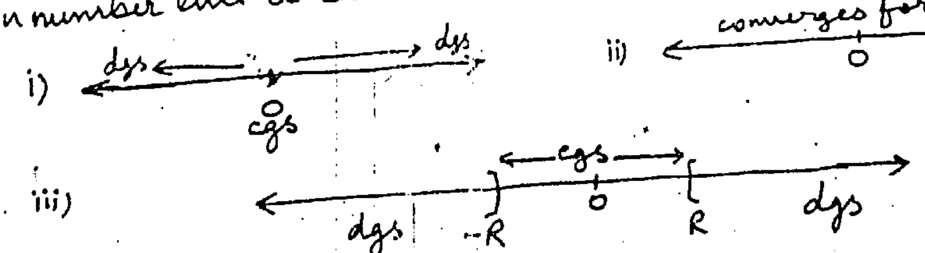
2) If power series $\sum C_n x^n$ diverges for $x = x_2$
then power series diverges for $|x| > |x_2|$

اگر سیریز P.S x_2 کے لیے dgs ہے تو x_2 سے بڑی تمام values کے لیے dgs ہے۔

3) For the power series $\sum\limits_0^{\infty} C_n x^n$ exactly one of the following conditions hold.

i) The series converges only for $x = 0$
ii) The series converges absolutely for all values of x
iii) The series converges absolutely for all values of x s.t $|x| < R$
The series diverges for all values of x s.t $|x| > R$ where R is +ive number

On number line it can be illustrated as.



i.e cgs for $|x| < R$
$-R < x < R$
dgs for $|x| > R$
$-R > x > R$ satisfy

Interval of Convergence & Radius of Convergence

The set of all values of x for which the power series converges is called Interval of convergence of the power series.

If power series converges absolutely for x s.t $|x| < R$
then R is called Radius of convergence where R is +ive number.

i) If P.Series cgs for $x=0$, then Its Interval of convergence $= 0$ & Radius of Convergence $= 0$
ii) If P.Series cgs for all x, then Its Interval of Convergence is $]-\infty, \infty[$ & Radius of convergence $= \infty$
iii) If P.Series cgs for $|x| < R$ and dgs for $|x| > R$ then Its Interval of Convergence is one of $[-R, R]$; $]-R, R[$; $]-R, R]$; $[-R, R[$ & Radius of Convergence is R
v) If P.Series $\sum\limits_0^{\infty} C_n (x-a)^n$ cgs for all x then Interval of Cgce $=]-\infty, \infty[$ & Radius of Cgce $= \infty$
vi) If P.Series $\sum\limits_0^{\infty} C_n (x-a)^n$ cgs only for $x = a$ then Interval of Cgce $= a$ + Radius of Cgce $= 0$
vii) If P.Series $\sum\limits_0^{\infty} C_n (x-a)^n$ cgs for $|x-a| < R$ then Interval of Cgce is one of $]a-R, a+R[$, $[a-R, a+R]$, $[a-R, a+R[$, $]a-R, a+R]$ + Radius of Cgce $= R$

Ex. 8.5

In each of the following, find the radius of convergence and interval of convergence (Problems 1–24):

> **Note:** In all questions n is +ive integer, but x has any real value

Q.1 $\sum_{n=0}^{\infty} \frac{x^n}{2n!}$

SOL. Here $|a_n| = \frac{|x^n|}{2n!}$ and $|a_{n+1}| = \frac{|x|^{n+1}}{(2n+2)!}$

$$\therefore \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(2n+2)!} \times \frac{2n!}{|x|^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{2n!}\,|x|}{(2n+2)(2n+1)\,\cancel{2n!}} = \lim_{n\to\infty} \frac{|x|}{(2n+2)(2n+1)}$$

$$= 0 < 1 \quad \text{for all values of } x$$

So by R. test for absolute convergence, the P Series converges absolutely for all values of x. Thus interval of convergence is $]-\infty, \infty[$ and radius of convergence 'R' is ∞.

Q.2 $\sum_{0}^{\infty} \frac{2^n x^n}{Ln(n+2)} \longrightarrow ①$

SOL. using ratio teste for A. Convergent when

$$a_n = \frac{2^n x^n}{Ln(n+2)} \quad \text{i.e.}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{2^{n+1} \cdot x^{n+1}}{Ln(n+3)} \times \frac{Ln(n+2)}{2^n \cdot x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2\, Ln(n+2)}{Ln(n+3)} \cdot x \right| = \lim_{n\to\infty} 2|x| \frac{Ln(n+2)}{Ln(n+3)} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} 2|x| \cdot \frac{\frac{1}{n+2}}{\frac{1}{n+3}} \left(\frac{0}{0}\right) = \lim_{n\to\infty} 2|x| \frac{n+3}{n+2} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} 2|x| \frac{\cancel{n}(1+\frac{3}{n})}{\cancel{n}(1+\frac{2}{n})} = 2|x| \cdot 1 \longrightarrow ①$$

① implies that the power series converges for $2|x| < 1$

$|x| < \frac{1}{2} \rightarrow (i)$

and power series diverges for $\because 2|x| > 1$ i.e for $|x| > \frac{1}{2} \rightarrow (ii)$

and when $2|x| = 1$ i.e $2x = \pm 1 \Rightarrow x = \pm\frac{1}{2}$

for $x = \frac{1}{2}$ Put in (i) the Series becomes $\sum_{n=0}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{\ln(n+2)}$ $\boxed{2^n \cdot \left(\frac{1}{2}\right)^n = 1}$

$$= \sum_{n=0}^{\infty} \frac{1}{\ln(n+2)} = \sum a_n$$

and since $\ln(n+2) < n+2 < 3n$

$$\Rightarrow \frac{1}{\ln(n+2)} > \frac{1}{3n} \Rightarrow a_n > b_n$$

i.e $\sum a_n = \sum \frac{1}{\ln(n+2)} > \sum \frac{1}{3n} = \sum b_n$

but $\sum b_n = \frac{1}{3}\sum \frac{1}{n}$ is divergent so by C.T. $\sum a_n = \sum_{n=0}^{\infty} \frac{1}{\ln(n+2)}$ is divergent

for $x = -\frac{1}{2}$ Put in (i) Series becomes $\sum_{n=0}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{\ln(n+2)} = \sum_{n=0}^{\infty} (-1)^n / \ln(n+2)$

which is an Alternating Series

ALST

$$|a_n| = \frac{1}{\ln(n+2)}$$

$$\Rightarrow \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\ln(n+2)} = 0$$

Now $f(x) = \frac{1}{\ln(x+2)} = [\ln(x+2)]^{-1}$

$f'(x) = \frac{-1}{[\ln(x+2)]^2} \cdot \frac{1}{(x+2)} < 0$ Non Increasing Seq.

Hence At $x = -\frac{1}{2}$, Al. Series is Cgt

So. series cgs for $|x| < \frac{1}{2}$ & $x = -\frac{1}{2}$ So

So Interval of Convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right[$

Radius of Convergence $= \frac{1}{2}$

Q.3 $\sum_{n=2}^{\infty} \frac{(x-5)^n \ln n}{n+1}$ ——— ①

SOL. $\Rightarrow \quad a_n = \frac{(x-5)^n \ln n}{n+1}$

using ratio test for absolute convergence i.e

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{\cancel{n}+1}\ln(n+1)}{n+2}\times\frac{n+1}{\cancel{(x-5)^n}\ln n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\ln(n+1)\cdot(x-5)}{(n+2)\ln n}\right|$$

$$= \lim_{n\to\infty}|x-5|\left(\frac{n+1}{n+2}\cdot\frac{\ln(n+1)}{\ln n}\right) = |x-5|\lim_{n\to\infty}\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(1+\frac{2}{n})}\cdot\frac{\ln(n+1)}{\ln n}$$

$\frac{\infty}{\infty}$

$$= |x-5|\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = |x-5|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$= |x-5|\cdot\lim_{n\to\infty}\cancel{n}\left(\frac{\cancel{n}}{1+\frac{1}{n}}\right) = |x-5|\cdot 1 = \boxed{|x-5|}$$

$\Rightarrow$ Power series converges if $|x-5|<1$ and

——— diverges if $|x-5|>1$

and when $|x-5|=1$

$x-5=\pm 1$

$x-5=1$, $x-5=-1$

$x=6$, $x=4$

for $x=4$, Put in ① $\quad \Sigma a_n = \sum_{n=2}^{\infty}\frac{(-1)^n \ln n}{n+1}$, which is an Alt. Series

where $|a_n| = \frac{\ln n}{n+1}$, Let $|b_n| = \frac{1}{n}$

$$\because \lim_{n\to\infty}\frac{|a_n|}{|b_n|} = \lim_{n\to\infty}\frac{\ln n}{n+1}\times n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\ln n = \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}.\ln n$$

$$= \ln\infty = \infty$$

Since $\Sigma|b_n| = \Sigma\frac{1}{n}$ is divergent

$\therefore$ $\Sigma|a_n|$ is divergent (by L.C. test) ①

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ALTERNA

Note using A Series Test

$$\underset{n\to\infty}{Lt}\ |a_n| = \frac{Ln\,n}{n+1}\left(\frac{\infty}{\infty}\right) = \underset{n\to\infty}{Lt}\ \frac{1/n}{1} = 0$$

and

$$|a_n| - |a_{n+1}| = \frac{Ln(n)}{n+1} - \frac{Ln(n+1)}{(n+2)}$$

$$= \frac{(n+2)\,ln(n) - (n+1)\,Ln(n+1)}{(n+1)(n+2)} > 0 \text{ for } n \geq 2$$

$$\Rightarrow |a_n| - |a_{n+1}| > 0 \Rightarrow |a_n| > |a_{n+1}|$$

Note: we can also prove that $|a_n| > |a_{n+1}|$ by taking $f(x) = \frac{\ln x}{x+1}$ and proving

$$\Rightarrow f'(x) = \frac{1}{(x+1)^2}\left[(x+1)\frac{1}{x} - \ln x\right]$$

$$= \frac{(x+1) - x\ln x}{x(x+1)^2} < 0$$

$\because x\ln x > x+1$ for $x \geq 4$.

Since both conditions of an A. Series Test are satisfied

So A- Series is convergent $\longrightarrow$ (2)

Combining (1) & (2) for $x = 4$, A-Series is conditionally Convergent.

For $x = 6$, the power series becomes $\sum_{n=2}^{\infty} \frac{Ln\,n}{n+1} = \Sigma a_n$

Let $\Sigma b_n = \Sigma \frac{1}{n}$

$$\underset{n\to\infty}{Lt}\ \frac{a_n}{b_n} = \underset{n\to\infty}{Lt}\left(\frac{Ln\,n}{n+1}\right)\cdot\frac{n}{1} = \underset{n\to\infty}{Lt}(\ln n)\left(\frac{n}{n+1}\right) = \underset{n\to\infty}{Lt}\ln n\left(\frac{1}{1+\frac{1}{n}}\right) = \underset{n\to\infty}{Lt}\ 1\cdot\ln n = \infty$$

By L. Comparision test $\Sigma a_n = \sum_{n=2}^{\infty}\frac{\ln n}{n+1}$ diverges ($\because \Sigma b_n = \Sigma\frac{1}{n}$ dgs)

So Series cgs for $|x-5| < 1$, $-1 < x-5 \leq 1$, $-1+5 < x < 1+5$, $4 < x < 6$

For $x=4$ cgs
For $x=6$ dgs

Therefore Interval of Convergence $[4, 6[$
Radius of Convergence $= 1$

(If P. Series converges for $|x-a| < R$ and diverges for $|x-a| > R$ Then Radius of Convergence $= R$)

Q.4 $\sum_{1}^{\infty} \frac{\sin n\pi x}{n^2} = \Sigma a_n$

Sol. $\Rightarrow \Sigma|a_n| = \sum_{1}^{\infty}\left|\frac{\sin n\pi x}{n^2}\right| = \frac{|\sin n\pi x|}{n^2}$ $0 \leq |\sin n\pi x| \leq 1$

$$= \frac{|\sin n\pi x|}{n^2} \leq \frac{1}{n^2}\ (\text{cgs})$$

$$|a_n| \leq b_n$$

$\because \Sigma b_n = \Sigma\frac{1}{n^2}$ is a cgt series

So by comparision test $\sum_{1}^{\infty}|a_n| = \sum_{1}^{\infty}\left|\frac{\sin n\pi x}{n^2}\right|$ is convergent. So P. Series (given) is absolutely convergent by A. cgt. Test, $\forall$ values of x. Thus its

"cgt" mean Convergent

Interval of convergence is $]-\infty, \infty[$

and radius of convergence is ∞

Q.5 $\sum_{n=1}^{\infty} n^2(x-2)^n$ ——— ①

SOL: Here $a_n = n^2(x-2)^n$

using R-T. for absolute cgt.

i.e $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 (x-2)^{\cancel{n}+1}}{n^2 \cancel{(x-2)^n}}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2}|x-2|$

$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})^2}{\cancel{n^2}} |x-2| = |x-2|$

$\Rightarrow$ P-Series is cgt for $|x-2|<1$, Equivalent to

------ dgt for $||x-2|>1$

$|x-2| = 1 \Rightarrow x-2=\pm 1$ or $x = 2\pm 1 = 3, 1$

For $x=1$ Put in ① Power Series becomes $\sum n^2(-1)^n$ (i.e A. Series)

$\Rightarrow \sum|a_n| = \sum n^2$ $\because \lim_{n\to\infty}|a_n| = \lim_{n\to\infty} n^2 = \infty \neq 0$

$\lim_{n\to\infty} a_n \neq 0$ So by divergent test $\therefore \sum |a_n|$ diverges.

Now for Al.S.T $f(x) = x^2 \Rightarrow f'(x) = 2x \not< 0$ $\therefore$ Not Non-Increasing

Hence Al. Series is divergent.

For $x=3$ Put in ① Power Series becomes $\sum n^2$ which is divergent (already proved)

Thus interval of convergence is $]1, 3[$

and radius of convergence is 1.

Q.6 $\sum_{n=1}^{\infty} \frac{n!\, x^n}{2n!}$

SOL. Here $a_n = \frac{n!\, x^n}{2n!}$

using R. Test. for A. cgt. i.e

$$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{Lt}\left|\frac{(n+1)!\,x^{n+1}}{(2n+2)!} \times \frac{2n!}{n!\,x^n}\right|$$

$$= \underset{n\to\infty}{Lt}\left|\frac{(n+1)\cancel{n!}\,x\cdot\cancel{x^n}}{(2n+2)(2n+1)\cancel{2n!}} \times \frac{\cancel{2n!}}{\cancel{n!}\,\cancel{x^n}}\right|$$

$$= \underset{n\to\infty}{Lt}\frac{1+\frac{1}{n}}{n\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}|x| = 0 < 1$$

So by ratio test for A-Convergent, Power Series Converges for all x

So Radius of Convergence is $=\infty$

and interval of convergence $]-\infty, \infty[$

Q.7 $\displaystyle\sum_{n=1}^{\infty} \frac{n\cdot 2^n (x-1)^n}{n+1}$ ——— (1)

Sol: $a_n = \dfrac{n\cdot 2^n (x-1)^n}{n+1}$

using Ratio test for absolute convergent i.e

$$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{Lt}\left|\frac{(n+1)\,2^{n+1}\,(x-1)^{n+1}}{n+2} \times \frac{n+1}{n\cdot\cancel{2^n}\,\cancel{(x-1)^n}}\right|$$

$$= \underset{n\to\infty}{Lt}\frac{(n+1)^2}{n(n+2)}\cdot 2|x-1| = \underset{n\to\infty}{Lt}\frac{\cancel{n}\left(1+\frac{1}{n}\right)^2}{\cancel{n}\left(1+\frac{2}{n}\right)}\,2|x-1|$$

$$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| = 2|x-1|$$

By Ratio Test Power Series Converges for $2|x-1|<1 \Rightarrow |x-1|<\frac{1}{2}$

Power Series diverges for $2|x-1|>1 \Rightarrow |x-1|>\frac{1}{2}$

$2|x-1|=1 \Rightarrow |x-1|=\frac{1}{2}$

$x-1 = \pm\frac{1}{2}$

$x = 1\pm\frac{1}{2}$

$= \frac{3}{2}, \frac{1}{2}$

For $x=\frac{3}{2}$ Put in (1): $\sum \dfrac{n\,2^n\left(\frac{3}{2}-1\right)^n}{n+1} = \sum \dfrac{n\,\cancel{2^n}\cancel{\left(\frac{1}{2}\right)^n}}{n+1} = \sum\frac{n}{n+1} = 1$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\dfrac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)} = 1 \neq 0$ so dgs by Divergent Test.

For $x=\frac{1}{2}$ Put in (1): Power series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{n\cdot 2^n\left(-\frac{1}{2}\right)^n}{n+1}$

or $= \sum_{n=1}^{\infty} \frac{n \cancel{2^n} (-1)^n \frac{1}{\cancel{2^n}}}{n+1} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ (Al. Series)

Mod

$\Rightarrow |a_n| = \frac{n}{n+1}$

then $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})} = 1 \neq 0$

So by Divergent test $\Sigma |a_n|$ diverges i.e $\lim_{n\to\infty} a_n \neq 0$

Now by ALST

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})} = 1 \neq 0,$

Ist Condition of ALST is not satisfied Hence given series is Dgt by DLST

Hence given series is Divergent for $x = \frac{1}{2}$

Therefore Power Series cgs for $|x-1| < \frac{1}{2}$

$-\frac{1}{2} < x-1 < \frac{1}{2}$

$1-\frac{1}{2} < x < \frac{1}{2}+1$

$\frac{1}{2} < x < \frac{3}{2}$

$\therefore$ Interval of Convergence is $]\frac{1}{2}, \frac{3}{2}[$

Radius of Convergence is $R = \frac{1}{2}$.

Q.8 $\sum_{0}^{\infty} \frac{(x-2)^n}{2^n \sqrt{n+1}}$ ——— ①

Sol. Here $|a_n| = \left| \frac{(x-2)^n}{2^n \sqrt{n+1}} \right| = \frac{|x-2|^n}{2^n \sqrt{n+1}}$

using ratio test for absolute convergence i.e

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x-2|^{n+1}}{2^{n+1} \sqrt{n+2}} \times \frac{\cancel{2^n} \sqrt{n+1}}{\cancel{|x-2|^n}}$

$= \lim_{n\to\infty} \frac{|x-2| \sqrt{n+1}}{2 \sqrt{n+2}} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{2}{n}}} \left(\frac{|x-2|}{2} \right)$

$= \frac{|x-2|}{2}$

$\Rightarrow$ Power series converges if $\frac{|x-2|}{2} < 1 \Rightarrow |x-2| < 2$

and Power series diverges if $\frac{|x-2|}{2} > 1 \Rightarrow |x-2| > 2$

$\Rightarrow$ $\left|\frac{x-2}{2}\right| = 1 \Rightarrow |x-2| = 2$

$x - 2 = \pm 2$

$x = 2 \pm 2$

$x = 4, 0$

For $x=4$ Put in ① $\sum_0^\infty a_n = \sum \frac{(4-2)^n}{2^n\sqrt{n+1}} = \sum \frac{\cancel{2^n}}{\cancel{2^n}\sqrt{n+1}} = \sum \frac{1}{\sqrt{n+1}}$

BCT $\because \sqrt{n+1} \leq n$

$\therefore \frac{1}{\sqrt{n+1}} \geq \frac{1}{n}$ So $\sum \frac{1}{b_n} = \sum \frac{1}{n}$ is Dgt we know already

Hence by BCT $\sum \frac{1}{\sqrt{n+1}}$ is Dgt.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ so may be cgt may be dgt.

For $x=0$ Put in ① Power Series becomes $\sum \frac{(-2)^n}{2^n\sqrt{n+1}} = \sum \frac{(-1)^n 2^n}{2^n\sqrt{n+1}} = \sum \frac{(-1)^n}{\sqrt{n+1}}$

which is an alternating Series.

where $\sum a_n = \sum \frac{(-1)^n}{\sqrt{n+1}}$

$\Rightarrow |a_n| = \frac{1}{\sqrt{n+1}}$

$\because \sqrt{n+1} \leq n$ for $n = 0, 1, 2, \ldots \infty$

$\Rightarrow \frac{1}{\sqrt{n+1}} \geq \frac{1}{n}$ So let $\sum b_n = \sum \frac{1}{n}$ which is divergent

also $|a_n| > b_n$ and $\sum b_n$ is divergent. So by B.C.T. $\sum |a_n|$ is divergent

using A. Series test:

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$

also $f(x) = \frac{1}{\sqrt{x+1}} = (x+1)^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}(x+1)^{-3/2} < 0$. $\therefore$ Nonincreasing

So by alternating Series (A.S.) test

A.S. is conditionally cgt for $x = 0$.

LCT
① ALTERNATE.

$\because |a_n| = \frac{1}{\sqrt{n+1}}$

let $|b_n| = \frac{1}{n}$

Then $\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = \lim_{n\to\infty} \frac{n}{\sqrt{n+1}}$

by L.C. Test

$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\sqrt{1+\frac{1}{n}}} = 1 \neq 0$

② Alternate

$\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{1}$

$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\sqrt{1+\frac{1}{n}}} = 1 \neq 0$

so depending on b_n

$b_n = \frac{1}{\sqrt{n}}$ is Dgt

$\therefore \sum a_n$ is Dgt

$\therefore$ Series is cgt for $|x-2| < 2$ and $x = 0$

$-2 < x-2 < 2$ and $x = 0$

$2-2 < x < 2+2$ and $x = 0$ So $[0, 4[$

$0 < x < 4$

Thus interval of convergence is

$$[0, 4[$$ & Radius of Cgce = 2.

Q.9 $\sum_{1}^{\infty} x^{2n}$

Sol. Here $a_n = x^{2n}$

using R.T. for A. convergent i.e $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{x^{2n}}\right| = |x^2| = |x|^2$

$\Rightarrow$ P. Series converge if $|x|^2 < 1$ i.e $|x| <$

and Power series diverges if $|x| > 1$.

So Radius of convergence is 1.

also $|x| < 1 \Rightarrow -1 < x < 1$

for $|x|^2 = 1 \Rightarrow |x| = 1 \Rightarrow x = \pm 1$

So for $x = \pm 1$ Power series becomes $\sum_{1}^{\infty} (\pm 1)^{2n} = \sum_{1}^{\infty} 1$ which is +ve term series, so is divergent. for

$x = \pm 1$.

Thus interval of convergence is $]-1, 1[$

Q.10 $\sum x^n/(\ln n)^n$

Sol. Here $|a_n|^{\frac{1}{n}} = \left|\frac{x^n}{(\ln n)^n}\right|^{\frac{1}{n}} = \left|\frac{x}{\ln n}\right| = \frac{|x|}{\ln n}$

using root test for absolute convergent

i.e $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x|}{\ln n} = 0 < 1$

So by Root test $\sum x^n/(\ln n)^n$ is absolutely convergent for all values of x.

Thus Interval of convergence $]-\infty, \infty[$,

Radius of convergence $= \infty$

Q. 11 $\sum_{1}^{\infty} n^n (x+1)^n$

SOL Here $|a_n| = n^n (x+1)^n$

using Root test for absolute convergence i.e

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|n^n (x+1)^n\right|^{1/n} = \lim_{n\to\infty} n|x+1|$$

or $\lim_{n\to\infty} |a_n|^{1/n} = 0$ for $x=-1$

$= \infty$ $\forall x$ (other than $x=-1$)

So by root test given power series converges absolutely for $x=-1$, and diverges for $\forall x$ (other than $x=-1$)

Since given power series converges only for $x=-1$ therefore its interval of convergence is a point $=-1$ or $[-1,-1]$ and its radius of convergence is $=$ '0'.

If Power Series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges at a pt. $x=a$ Then interval of convergence is $=a$ i.e one point and radius of convergence is zero.

X———————X

Q. 12 $\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n^2}$

SOL. Here $|a_n| = \left|\frac{2^n (x-3)^n}{n^2}\right| = \frac{2^n |(x-3)|^n}{n^2}$

using root test for absolute convergent i.e

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{2^n |(x-3)|^n}{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2|x-3|}{(n^2)^{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{2|x-3|}{\left(n^{\frac{1}{n}}\right)^2} = 2|x-3|$$

$\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1$ already proved

implies that power series converges if $2|x-3| < 1$ i.e $|x-3| < \frac{1}{2}$

and Power series diverges if $|x-3| > \frac{1}{2}$

$\Rightarrow$ radius of convergence 'R' $= \frac{1}{2}$

also $|x-3| < \frac{1}{2}$ implies that, $-\frac{1}{2} < x-3 < \frac{1}{2}$

or $-\frac{1}{2}+3 < x < \frac{1}{2}+3$

or $\frac{5}{2} < x < \frac{7}{2}$

For $|x-3| = \frac{1}{2} \Rightarrow x-3 = \pm\frac{1}{2}$

or $x = 3+\frac{1}{2},\ 3-\frac{1}{2} = \frac{7}{2}, \frac{5}{2}$

So for $x = \frac{5}{2}$, given power series becomes $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{5}{2}-3\right)^n}{n^2}$

$$= \sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which is an alternating series

where $\sum |a_n| = \sum \frac{1}{n^2}$ which is convergent, since $p = 2 > 1$

So power series is absolutely convergent for $x = \frac{5}{2}$

Now for $x = \frac{7}{2}$, power series becomes $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{7}{2}-3\right)^n}{n^2}$

$$= \sum_{n=1}^{\infty} \frac{2^n \cdot \left(\frac{1}{2}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{a +ive term series}$$

which is convergent

Thus interval of convergence is $\left[\frac{5}{2} : \frac{7}{2}\right]$

Q. NO. 13 $\sum_{1}^{\infty} \frac{1+(-1)^n}{n} x^n$

SOL: Given power series can be written as

$$\sum_{1}^{\infty} \frac{x^n}{n} + \sum_{1}^{\infty} \frac{(-1)^n x^n}{n}$$

$$= \sum a_n + \sum b_n \quad (\text{Say})$$

then by ratio test for absolute convergent

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}x\right|$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}|x| = |x|$$

$\Rightarrow$ The power series $\sum_{1}^{\infty} \frac{x^n}{n}$ converges for $|x| < 1$

and " " " " diverges for $|x| > 1$

also since $\sum b_n = \sum \frac{(-1)^n x^n}{n}$

then $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{n+1}}{n+1} \times \frac{n}{(-1)^n x^n}\right|$

$= \lim_{n\to\infty}\left|\frac{-n\, x}{n+1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)|x|$

$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)|x| = |x|$

$\Rightarrow$ The Power series $\sum_1^\infty \frac{(-1)^n x^n}{n}$ is **convergent if $|x|<1$**

and " " " " " **divergent if $|x|>1$**

$\Rightarrow$ both series converges for $|x|<1$

and " " diverges " $|x|>1$

Thus radius of convergence "R" $= 1$

In both series if $|x| = 1$ i.e $x = \pm 1$

So **for $x=1$**, $\sum a_n + \sum b_n = \sum \frac{1}{n} + \sum \frac{(-1)^n}{n} \longrightarrow ①$

and **for $x=-1$**, $\sum a_n + \sum b_n = \sum \frac{(-1)^n}{n} + \sum \frac{(-1)^n(-1)^n}{n}$

$= \sum \frac{(-1)^n}{n} + \sum \frac{(-1)^{2n}}{n}$

$= \sum \frac{(-1)^n}{n} + \sum \frac{1}{n} \longrightarrow ②$

from ① and ② we see that for $x = \pm 1$,

both power series $\therefore \sum a_n + \sum b_n$ becomes

$= \sum_1^\infty \frac{1}{n} + \sum_1^\infty \frac{(-1)^n}{n}$ = +ive Series + Al Series

= Dgt Series + [Dgt or Cgt or Cond Cgt]

$\because \sum \frac{1}{n}$ is dgt series, = Dgt. Series (By Theorem Page 262)

$\therefore$ Interval of Convergence is $]-1, 1[$

Radius of Convergence $= 1$

No Need 2 all aired:

also $\sum_1^\infty |b_n| = \sum_1^\infty \left|\frac{(-1)^n}{n}\right| = \sum_1^\infty \frac{1}{n}$ is divergent Series $\longrightarrow ③$

c.f. $\frac{|b_{n+1}|}{|b_n|} = \frac{1}{n+1} \times \frac{n}{1} = \frac{n}{n+1} < 1 \Rightarrow \frac{|b_{n+1}|}{|b_n|} < 1$

$\Rightarrow |b_{n+1}| < |b_n|$ or $|b_n| > |b_{n+1}|$

$\Rightarrow$ radius of convergence is $= 3$

if $|x| = 3 \Rightarrow x = \pm 3$

Then either $x = 3$ or $x = -3$. power series will become

$$\sum_{n=0}^{\infty}\left(\frac{9+3}{12}\right)^n = \sum_{n=0}^{\infty}(1)^n = \sum_{n=0}^{\infty} 1$$

which is a divergent series

$\because \sum_{}^{\infty} 1 = 1+1+1+\cdots\infty = \infty$

$\therefore \sum 1$ is divergent

therefore interval of convergence is $]-3, 3[$

x———————x

Q. 15 $\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(\ln n)^2} \Rightarrow |a_n| = \frac{|x|^n}{n(\ln n)^2}$

SOL. using ratio test for absolute convergent i.e

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)(\ln(n+1))^2}\times\frac{n(\ln n)^2}{|x|^n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\frac{\ln n}{\ln(n+1)}\right)^2|x|$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)\frac{\ln n}{\ln(n+1)}|x|$$

$$= \lim_{n\to\infty} 1 \cdot 1 \cdot |x| = |x|$$

$\because \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$

and $\lim_{n\to\infty}\frac{\ln n}{\ln n+1}\ \frac{\infty}{\infty}$

$= \lim_{n\to\infty}\frac{1/n}{1/n+1} = \lim_{n\to\infty}\frac{n+1}{n}$

$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$

So by ratio test power series converges if $|x| < 1$

and power series diverges if $|x| > 1$

$\therefore$ radius of convergence 'R' $= 1$

if $|x| = 1$ i.e $x = \pm 1$

So for $x = 1$, Power Series becomes $\sum_{2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ which is an alternating series $\Rightarrow |a_n| = \frac{1}{n(\ln n)^2}$

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n(\ln n)^2} = 0$$

and $|a_n| = \frac{1}{n(\ln n)^2} > \frac{1}{(n+1)\ln(n+1)} = |a_{n+1}|$

So by Alternating series test $\sum_{2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ converges for $x = 1$

For $x = -1$, Power series becomes $\sum \frac{(-1)^n(-1)^n}{n(\ln n)^2}$

$$= \sum \frac{(-1)^{2n}}{n(\ln n)^2} = \sum_{2}^{\infty} \frac{1}{n(\ln n)^2}$$

which is +ive term series and is convergent as already proved. Thus

interval of convergence is $[-1, 1]$.

x ——— x

Q.16 $\sum_{n=0}^{\infty} \frac{n x^n}{(n+1)(n+2)2^n}$

SOL. using ratio test for absolute convergent i.e

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{(n+2)(n+3)2^{n+1}} \times \frac{(n+1)(n+2)2^n}{n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2 x}{2n(n+3)}\right| = \lim_{n\to\infty}\left|\frac{(1+\frac{1}{n})^2 x}{2(1+\frac{3}{n})}\right|$$

$$= \lim_{n\to\infty} \frac{1}{2}|x| = \frac{|x|}{2}$$

$\Rightarrow$ Power series converges for $\frac{|x|}{2} < 1$ i.e $|x| < 2$

and " " diverges " $\frac{|x|}{2} > 1$ i.e $|x| > 2$

so radius of convergence is 'R' = 2.

Now when $|x| = 2$. i.e $x = \pm 2$. Then

for $x = -2$, Power series becomes

$$\sum_{0}^{\infty} \frac{n(-2)^n}{(n+1)(n+2)2^n} = \sum_{0}^{\infty} \frac{(-1)^n n \cancel{2^n}}{(n+1)(n+2)\cancel{2^n}} = \sum_{0}^{\infty} \frac{(-1)^n n}{(n+1)(n+2)}$$

which is an alternating series

with $|a_n| = \frac{n}{(n+1)(n+2)}$ i.e

using alternating series-test

(i) $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} \frac{n}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{1}{n(1+\frac{1}{n})(1+\frac{2}{n})} = 0$

and $|a_n| - |a_{n+1}| = \frac{n}{(n+1)(n+2)} - \frac{n+1}{(n+2)(n+3)}$

$$= \frac{n(n+3) - (n+1)^2}{(n+1)(n+2)(n+3)} = \frac{n-1}{(n+1)(n+2)(n+2)}$$

$\because \dfrac{n-1}{(n+1)(n+2)(n+3)} \geq 0$ for all $n = 0, 1, 2, 3, \ldots$

$\Rightarrow |a_n| - |a_{n+1}| \geq 0$ or $|a_n| > |a_{n+1}|$

So by alternating Series test $\sum a_n$ Converges. $\rightarrow$ (1)

also let $b_n = \frac{1}{n}$ $\Rightarrow \sum b_n = \sum \frac{1}{n}$ which is divergent

also $\lim\limits_{n\to\infty} \dfrac{|a_n|}{b_n} = \lim\limits_{n\to\infty} \dfrac{n}{(n+1)(n+2)} \times \dfrac{n}{1} = \lim\limits_{n\to\infty} \dfrac{n^2}{(n+1)(n+2)}$

$= \lim\limits_{n\to\infty} \dfrac{1}{(1+\frac{1}{n})(1+\frac{2}{n})} = 1 \neq 0$

$\Rightarrow \sum |a_n|$ and $\sum b_n$ behaves alike but $\sum b_n$ is divergent. So by L.C. test $\sum |a_n|$ diverges $\rightarrow$ (2)

Combining (1) & (2), implies the power series for $x = 2$, the series is conditionally convergent. and

for $x = 2$, Power series becomes

$$= \sum_0^{\infty} \frac{n \cdot 2^n}{(n+1)(n+2)2^n} = \sum_0^{\infty} \frac{n}{(n+1)(n+2)} = \sum a_n$$

which is divergent ($\because \sum b_n = \sum \frac{1}{n}$ is divergent)

So interval of convergence is $]-2, 2[$

Q.17 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n \sin^n x}{1}$ such that $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Sol. $\Rightarrow |a_n| = 2^n \sin^n x$

So using ratio test for absolute convergent i.e

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{2^{n+1}(\sin x)^{n+1}}{2^n (\sin x)^n}\right| = \lim\limits_{n\to\infty} 2|\sin x| = 2|\sin x|$

$\Rightarrow$ The power series converges if $2|\sin x| < 1 \Rightarrow |\sin x| < \frac{1}{2}$ i.e. $|x| < \frac{\pi}{6}$

and " " " diverges " $2|\sin x| > 1 \Rightarrow |\sin x| > \frac{1}{2}$ i.e $|x| > \frac{\pi}{6}$

So radius of convergence R is $= \frac{\pi}{6}$

Now when $|\sin x| = \frac{1}{2} \Rightarrow |x| = \frac{\pi}{6} \Rightarrow x = \pm\frac{\pi}{6}$

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For $x = \frac{\pi}{6}$, Power series becomes $\sum_0^\infty (-1)^n 2^n \left(\sin\frac{\pi}{6}\right)^n$

$= \sum_0^\infty (-1)^n 2^n \left(\frac{1}{2}\right)^n = \sum_0^\infty (-1)^n$

$\boxed{\because \sum_0^\infty (-1)^n = 1-1+1-1+\cdots \text{ is divergent. See Exm-3 Page 258}}$

which is an alternating series and is divergent.

For $x = -\frac{\pi}{6}$, Power series becomes $\sum_0^\infty (-1)^n 2^n \left[\sin\left(\frac{-\pi}{6}\right)\right]^n = \sum_0^\infty (-1)^n 2^n \left(\frac{-1}{2}\right)^n$

$= \sum_0^\infty (+1)^n$

$= 1+1+1+1 \cdots \infty$

which is divergent. $\boxed{\text{See Exm-3. Page 258 in which } \sum a_n = a + ax + ax^2 + \cdots \text{ is divergent for } |x| = 1}$

Q.18 $\sum_{n=2}^\infty \frac{(x-e)^n \ln n}{e^n}$

Sol. Here $|a_n| = \frac{(x-e)^n \ln n}{e^n}$

applying ratio test for absolute convergent

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-e)^{n+1} \ln(n+1)}{e\, e^n} \cdot \frac{e^n}{(x-e)^n \ln n}\right|$$

$$= \lim_{n\to\infty} \frac{|x-e|}{e} \frac{\ln(n+1)}{\ln n}$$

$$= \frac{|x-e|}{e} \cdot 1 = \left|\frac{x-e}{e}\right|$$

$\because \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}}$ (L'Hospital Rule)

$= \lim_{n\to\infty} \frac{n}{1+n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$

$= 1$

$\Rightarrow$ Power Series become convergent for $\left|\frac{x-e}{e}\right| < 1$ or $|x-e| < e$

" " " divergent for ~~pos~~ $\left|\frac{x-e}{e}\right| > 1$ or $|x-e| > e$

So radius of convergent 'R' is $= e$

For $\left|\frac{x-e}{e}\right| = 1$ i.e. $x - e = \pm e$ ~~or~~ or $x = 0, 2e$

So for $x = 0$, Power series becomes $\sum_2^\infty \frac{(-e)^n \ln n}{e^n} = \sum_2^\infty (-1)^n \ln n$

which is an alternating Series and since

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \ln n = \infty \neq 0 \Rightarrow$ } A. Series diverges

and $|a_n| \neq |a_{n+1}|$ ($\because \ln n \neq \ln(n+1)$)

Now for $x = 2e$, Power Series becomes $\sum_2^\infty \frac{e^n \ln n}{e^n} = \sum_2^\infty \ln n$

which is a +ve term series and is divergent (Already proved)

Since $|x - e| < e$ is equivalent to $-e < x - e < e$

or $0 < x < 2e$

Thus ~~req~~ interval of convergence is $]0, 2e[$

Q-19 $\sum_0^\infty \frac{(ax+b)^n}{c^n}$, $a>0$, $c>0$

SOL. using root test for absolute convergent i.e

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{ax+b}{c}\right| = \left|\frac{ax+b}{c}\right|$$

$\Rightarrow$ Power series converges if $\left|\frac{ax+b}{c}\right| < 1$ i.e $|ax+b| < c$

or $|a(x+\frac{b}{a})| < c$ or $a|x+\frac{b}{a}| < c$ or $|x+\frac{b}{a}| < \frac{c}{a}$

and diverges if $|x+\frac{b}{a}| > \frac{c}{a}$

$\Rightarrow$ radius of convergence 'R' is $= \frac{c}{a}$

if $|x+\frac{b}{a}| = \frac{c}{a}$ $\Rightarrow$ $x+\frac{b}{a} = \pm\frac{c}{a}$ or $x = \pm\frac{c}{a} - \frac{b}{a}$

$\Rightarrow$ $x = \frac{c-b}{a}$, $-\frac{c+b}{a}$

For $x = \frac{c-b}{a}$ $\Rightarrow$ $ax+b = c$ or $\frac{ax+b}{c} = 1$

So Power Series becomes $\sum_0^\infty (1)^n = \sum_0^\infty 1$ which divergent

For $x = -\frac{c+b}{a}$ $\Rightarrow$ $ax = -b-c$ or ~~ax+b~~ $ax+b = -c$

or $\frac{ax+b}{c} = -1$ so power series becomes

$\sum_2^\infty (-1)^n$ i.e an alternating series and is divergent.

Now Since $|x+\frac{b}{a}| < \frac{c}{a}$ is equivalent a

$-\frac{c}{a} < x+\frac{b}{a} < \frac{c}{a}$ or $-\frac{c}{a} - \frac{b}{a} < x < \frac{c}{a} - \frac{b}{a}$

or $-\left(\frac{c+b}{a}\right) < x < \frac{c-b}{a}$

$\Rightarrow$ Interval of convergence is $\left]-\left(\frac{b+c}{a}\right), \frac{c-b}{a}\right[$

Q.20 $\sum_1^\infty \frac{n(x-1)^n}{2^n(3n-1)}$

SOL. $\Rightarrow$ $|a_n| = \left|\frac{n(x-1)^n}{2^n(3n-1)}\right|$

using ratio test for absolute convergent i.e

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-1)^{\cancel{n}+1}}{2^{\cancel{n}+1}(3n+2)}\cdot\frac{\cancel{2^n}(3n-1)}{n\,\cancel{(x-1)^n}}\right|$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)(3n-1)}{2n(3n+2)}\right)|x-1| = \lim_{n\to\infty}\frac{(1+\frac{1}{n})(3-\frac{1}{n})}{2(3+\frac{2}{n})}|x-1| = \frac{|x-1|}{2}$$

$\Rightarrow$ Power series converges if $\frac{|x-1|}{2} < 1$ or $|x-1| < 2$

and " " diverges if $|x-1| > 2$

So radius of convergence R is $= 2$

When $|x-1| = 2$ $\Rightarrow$ $x - 1 = \pm 2$, $x = 3, -1$

For $x = -1$, Power Series becomes

$$\sum_{1}^{\infty} \frac{n(-2)^n}{2^n(3n-1)} = \sum_{1}^{\infty} \frac{(-1)^n \cancel{2^n} n}{\cancel{2^n}(3n-1)} = \sum_{1}^{\infty} (-1)^n \frac{n}{3n-1}.$$

~~Alternated~~ an alternating Series.

So by divergent Series test

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3} \neq 0,$$

$\Rightarrow \sum |a_n|$ is divergent. So $\sum a_n$ is divergent

because $|a_n| = \frac{n}{3n-1}$ and $|a_{n+1}| = \frac{n+1}{3n+2}$

and $|a_n| - |a_{n+1}| = \frac{n}{3n-1} - \frac{n+1}{3n+2} = \frac{3n^2+2n-3n^2-2n+1}{(3n-1)(3n+2)}$

$$= \frac{1}{(3n-1)(3n+2)} > 0$$

$\Rightarrow |a_n| - |a_{n+1}| > 0$ or $|a_n| > |a_{n+1}|$

but first condition of alternating is not satisfied

(i.e $\lim_{n\to\infty} |a_n| = 0$)

So alternating Series $\sum_{1}^{\infty} (-1)^n \frac{n}{3n-1}$ is not convergent

Hence is divergent for $x = -1$

For $x = 3$, Power series becomes

$$\sum_{1}^{\infty} \frac{n\, 2^n}{2^n(3n-1)} = \sum_{1}^{\infty} \frac{n}{3n-1} = \sum_{1}^{\infty} a_n$$

$\because \lim_{n\to\infty} a_n \neq 0$ So Power Series is divergent for $x = 3$

$|x-1| < 2$ is equivalent to $-2 < x-1 < 2$

$-1 < x < 3$

$\Rightarrow$ Interval of convergence is $]-1, 3[$

Q.21 $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$

SOL. Here $|a_n| = \frac{n^n x^n}{n!}$

and $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} \times \frac{n!}{n^n x^n}\right|$

$$= \left|\frac{\cancel{(n+1)}(n+1)^n \cancel{x^n} x \cdot \cancel{n!}}{\cancel{(n+1)} \cancel{n!}\; n^n \cancel{x^n}}\right| = \left(\frac{n+1}{n}\right)^n |x|$$

$$= \left(1+\frac{1}{n}\right)^n |x|$$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n |x| = e|x|$

$\boxed{\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$

So by ratio test for absolute convergent if

$e|x| < 1, \Rightarrow |x| < \frac{1}{e}$

and diverges if $|x| > \frac{1}{e}$

Thus radius of convergence is $R' = e$

if $e|x| = 1, \quad |x| = \frac{1}{e}, \quad x = \pm\frac{1}{e}$

for $x = -\frac{1}{e}$, Power series becomes $= \sum_1^\infty \frac{n^n}{n!}\left(-\frac{1}{e}\right)^n$

We know that for n is very large $= \sum_1^\infty (-1)^n \frac{n^n}{n!} \cdot \frac{1}{e^n}$ (An alternating Series)

$n! = \sqrt{2\pi n}\, \frac{n^n}{e^n}$

$$\Rightarrow \sum_1^\infty \frac{(-1)^n n^n}{n!\, e^n} = \sum_1^\infty \frac{(-1)^n \cancel{n^n}}{\cancel{e^n}} \cdot \frac{\cancel{e^n}}{\sqrt{2\pi n}\, \cancel{n^n}} = \sum_1 \frac{(-1)^n}{\sqrt{2\pi n}}$$

$\sum |a_n| = \sum \frac{1}{\sqrt{2\pi n}} = \frac{1}{\sqrt{2\pi}} \sum_1^\infty \frac{1}{\sqrt{n}}$ which is divergent $\boxed{\because p = \frac{1}{2} < 1 \text{ by p. Series}}$

Now using alternating Series test:

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi n}} = 0$

$|a_n| = \frac{1}{\sqrt{2\pi n}} > \frac{1}{\sqrt{2(n+1)\pi}} = |a_{n+1}|$

$\Rightarrow$ A. Series converges but $\sum |a_n|$ diverges.

$\Rightarrow$ Given Series is conditionally convergent.

for $x = \frac{1}{e}$, P. Series becomes $\sum_1^\infty \frac{n^n}{n!\, e^n} = \sum_1^\infty \frac{1}{\sqrt{2\pi n}}$ +ive term Series and is divergent

So interval of convergence is $\left[-\frac{1}{e}, \frac{1}{e}\right[$ Ans.

Q.22 $\sum_{1}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6\cdot 8 \cdots 2n} x^{2n+1}$

SOL. Here $|a_n| = \left| \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} x^{2n+1} \right|$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)}{2\cdot 4\cdot 6 \cdots 2n\,(2n+2)} \cdot \frac{x^{2n+3}}{1} \times \frac{2\cdot 4\cdot 6 \cdots 2n}{1\cdot 3\cdot 5 \cdots (2n-1)} \cdot \frac{1}{x^{2n+1}} \right|$$

$$= \left| \frac{2n+1}{2n+2} \cdot \frac{x^{2n+1+2}}{x^{2n+1}} \right| = \frac{2n+1}{2n+2} |x^2|$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{2+1/n}{2+\frac{2}{n}} \right) |x^2| = |x^2|$$

$\Rightarrow$ The Power series will converge if $|x^2|<1$ or $|x|^2<1$ or $|x|<1$

and " " diverge if $|x|>1$

So Radius of convergence 'R' is $= 1$.

For $|x| = 1 \Rightarrow x = \pm 1$.

when $x = 1$, then power series will become

$\sum_{1}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}$ which is an alternating Series

so applying divergent test i.e

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots \cancel{n}(2-\frac{1}{n})}{2\cdot 4\cdot 6 \cdots 2\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots (2-\frac{1}{n})}{2\cdot 4\cdot 6 \cdots 2} \neq 0 \longrightarrow (1)$$

Similarly for $x = -1$, Power Series becomes

$$\sum_{1}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n-1)\,(-1)^{2n+1}}{2\cdot 4\cdot 6 \cdots 2n} = \sum_{1}^{\infty} (-1)^{3n+1} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}$$

which is alternating series and also

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} \neq 0$ (already proved) $\longrightarrow (2)$

So from (1) & (2) by divergent test. power Series diverges in both cases and by A. Series test Since 1st condition is not satisfied. Hence is not convergent. Thus in result P. Series diverges for $x = \pm 1$. So interval of Convergence is $]-1, 1[$

Q.23 $\sum_{1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)x^n$

Sol. Here $|a_n| = \left|\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right)x^n\right|$

and $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(1+\frac{1}{2}+\frac{1}{3}\cdots\cdots+\frac{1}{n+1}\right)x^{n+1}}{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots\frac{1}{n}\right)\cdot x^n}\right|$

$\boxed{\because \left|\frac{x^{n+1}}{x^n}\right| = |x|}$

$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots-\frac{1}{n+1}}{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}}\right)|x|$

$= \lim_{n\to\infty} |x|$

$$\boxed{\begin{aligned}&\lim_{n\to\infty} 1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots\tfrac{1}{n+1}\\ &= \lim_{n\to\infty} 1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots\tfrac{1}{n}\\ &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots\end{aligned}}$$

$\Rightarrow$ Power series converges for $|x|<1$

and " " diverges " $|x|>1$

So radius of convergence is $=1$

for $|x| = 1 \Rightarrow x = \pm 1$.

if $x=1$, then power Series becomes

$\sum_{1}^{\infty} 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}(+1)^n = \sum_{1}^{\infty} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum_{1}^{\infty}\frac{1}{n}$

which is divergent.

and if $x=-1$, then power Series becomes

$\sum_{1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right)(-1)^n$ an alternating Series

$\Rightarrow \sum_{1}^{\infty}|a_n| = \sum_{1}^{\infty} 1+\frac{1}{2}+\frac{1}{n}+\cdots = \sum_{1}^{\infty}\frac{1}{n}$ which is divergent $\to$ (1)

~~but~~ also by alternating Series test

$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} \neq 0$

and $|a_n| = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\frac{1}{n+1}$

$= 1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n+1} = |a_{n+1}|$

So by alternating Series test Power Series is not convergent $\to$ (2)

So from (1) and (2), we concluded that power Series diverges for $x=-1$. So

interval of convergence is $]-1, 1[$.

Q.24 $\sum_{1}^{\infty} \frac{n!\,x^n}{1\cdot4\cdot7\cdots(3n-2)}$

SOL So $\frac{a_{n+1}}{a_n} = \frac{(n+1)!\,x^{n+1}}{1\cdot4\cdot7\cdots(3n-2)(3n+1)} \times \frac{1\cdot4\cdot7\cdots(3n-2)}{n!\,x^n}$

$$= \frac{(n+1)!\,x}{(3n+1)\,n!} = \frac{(n+1)\,n!\,x}{(3n+1)\,n!} = \frac{(n+1)\,x}{(3n+1)}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,x}{(3n+1)}\right| = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{3+\frac{1}{n}}\right)|x| = \frac{|x|}{3}$$

$\Rightarrow$ Power Series converges for $\frac{|x|}{3} < 1$ or $|x| < 3$

and " " diverges for $|x| > 3$

So radius of convergence 'R' = 3

for $|x| = 3$ i.e $x = \pm 3$

if $x = 3$, power Series becomes $\sum_{1}^{\infty} \frac{3^n\,n!}{1\cdot4\cdot7\cdots 3n-2}$

which +ve term Series

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n\,n!}{1\cdot4\cdot7\cdots(3n-2)} = \lim_{n\to\infty} \frac{3^n\,1\times2\times3\times4\cdots\cdots n}{1\times4\times7\cdots\cdots n(3-\frac{2}{n})}$

$$= \lim_{n\to\infty} 3^n \frac{1\cdot2\cdot3\times4\times\cdots}{1\cdot4\times7\times\cdots(3-\frac{2}{n})} = \infty \neq 0$$

So by divergent test, the series diverges for $x = 3$ $\rightarrow$ (1)

when $x = -3$, power series becomes $\sum_{1}^{\infty} \frac{(-1)^n\,3^n\,n!}{1\cdot4\cdot7\cdots 3n-2}$

which is an alternating Series

$\because |a_n| = \frac{3^n\,n!}{1\cdot4\cdot7\cdots 3n-2}$ and $|a_{n+1}| = \frac{3^{n+1}(n+1)!}{1\cdot4\cdot7\cdots 3n+1}$

and $\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}(n+1)\,n!}{1\cdot4\cdot7\cdots(3n-2)(3n+1)} \times \frac{1\cdot4\cdot7\cdots(3n-2)}{3^n\,n!}$

$$= \frac{3(n+1)}{3n+1} = \frac{3n+3}{3n+1} > 1$$

$\Rightarrow \frac{|a_{n+1}|}{|a_n|} > 1$ or $|a_{n+1}| > |a_n|$ So by A. Series test

Series is not convergent $\longrightarrow$ ①

and by divergent test

$\lim_{n\to\infty} |a_n| \neq 0$ (already proved)

The Series is divergent $\longrightarrow$ ②

form ① and ② we concluded that

Series diverges for $x = -3$

Thus interval of convergence is $]-3, 3[$

Q.25 Obtain a power Series representation of $\frac{x}{(1+x^2)^2}$ if $|x|<1$

SoL: $\because \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots x^n + \cdots$

replacing x by x^2, we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots x^{2n} + \cdots$$

$$\Rightarrow \frac{1}{(1+x^2)^2} = (1 - x^2 + x^4 - x^6 + x^8 + \cdots x^{2n} + \cdots)(1 - x^2 + x^4 - x^6 + x^8 + \cdots x^{2n} + \cdots)$$

$$= (1 - x^2 + x^4 - x^6 + x^8 + \cdots) + (-x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots) + (x^4 - x^6 + x^8 - x^{10} + x^{12} + \cdots)$$
$$+ (-x^6 + x^8 - x^{10} + x^{12} - x^{14} + \cdots) + (x^8 - x^{10} + x^{12} - x^{14} + x^{16} + \cdots)$$

$$= 1 - 2x^2 + 3x^4 - 4x^6 + 5x^8 - 6x^{10} + \cdots$$

$$\Rightarrow \frac{x}{(1+x^2)^2} = x - 2x^3 + 3x^5 - 4x^7 + 5x^9 - 6x^{11} + \cdots$$

$$= \sum_{0}^{\infty} (-1)^n (n+1) x^{2n+1}$$

is required power series

$1, 3, 5, 7, 9, \ldots$ n terms

$a_n = 1 + (n-1)2 = 2n-1$

FREQUENTLY USED MACLAURIN SERIES

① $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{0}^{\infty} x^n$; $|x|<1$

② $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots + (-x)^n + \cdots = \sum_{0}^{\infty} (-1)^n x^n$; $|x|<1$

③ $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$; $|x|\le 1$

④ $\ln\left(\frac{1+x}{1-x}\right) = 2\tanh^{-1}x = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) = 2\sum_{0}^{\infty} \frac{x^{2n+1}}{2n+1}$; $|x|<1$

⑤ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \sum_{1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$; $-1 < x \le 1$

⑥ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{0}^{\infty} \frac{x^n}{n!}$; $|x| < \infty$

Q 26 Find the sum of the series

$2 + 6x + 12x^2 + 20x^3 + \dots$

Sol $\because$ Maclaurins Series for $\frac{1}{1-x}$ is ∞ I.G.S

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\Rightarrow \frac{1}{(1-x)^2} = (1 + x + x^2 + x^3 + \dots)(1 + x + x^2 + x^3 + \dots)$$

$$= (1 + x + x^2 + x^3 + x^4 + \dots) + (x + x^2 + x^3 + x^4 + x^5 + \dots)$$

$$= (1 + x + x^2 + x^3 + x^4 + \dots) + (x^3 + x^4 + x^5 + x^6 + \dots)$$
$$+ (x^2 + x^3 + x^4 + x^5 + x^6 + \dots)$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$\Rightarrow \frac{1}{(1-x)^3} = (1 + 2x + 3x^2 + 4x^3 + \dots)(1 + x + x^2 + x^3 + \dots)$$

$$= (1 + x + x^2 + x^3 + \dots) + (2x + 2x^2 + 2x^3 + 2x^4 + \dots)$$
$$+ (3x^2 + 3x^3 + 3x^4 + 3x^5 + \dots) + (4x^3 + 4x^4 + 4x^5 + 4x^6 + \dots)$$

$$= 1 + 3x + 6x^2 + 10x^3 + \dots$$

$$\Rightarrow \frac{2}{(1-x)^3} = 2(1 + 3x + 6x^2 + 10x^3 + \dots)$$

$$= 2 + 6x + 12x^2 + 20x^3 + \dots$$

Thus sum of Series is $\frac{2}{(1-x)^3}$ ans.

$2 + 6x + 12x^2 + 20x^3 + \dots$

Q.27 using power series representation of $\frac{e^x - 1}{x}$, show that $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$

Sol: Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$\Rightarrow \frac{e^x - 1}{x} = \frac{1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots - 1}{x}$$

$$= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots = \sum_{0}^{\infty} \frac{x^n}{(n+1)!}$$

then $\dfrac{d}{dx}\sum_{0}^{n}\dfrac{x^n}{(n+1)!} = \sum_{0}^{\infty}\dfrac{d}{dx}\left(\dfrac{x^n}{(n+1)!}\right)$

$$= \sum_{1}^{\infty}\frac{n\,x^{n-1}}{(n+1)!} \longrightarrow (1)$$

$$\left(\frac{d}{dx}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right) = \frac{1}{2!}+\frac{2x}{3!}+\frac{3x^2}{4!}+\cdots = \sum_{1}^{\infty}\frac{n\,x^{n-1}}{(n+1)!}\right)$$

and $\dfrac{d}{dx}\left(1+\dfrac{x}{2!}+\dfrac{x^2}{3!}+\cdots\right) = \dfrac{d}{dx}\left(\dfrac{e^x-1}{x}\right)$

$$= \frac{x(e^x)-(e^x-1)\cdot 1}{x^2}$$

$$= \frac{xe^x-e^x+1}{x^2} = \frac{e^x(x-1)+1}{x^2} \longrightarrow (2)$$

Combining (1) and (2), we get

$$\frac{e^x(x-1)+1}{x^2} = \sum_{1}^{\infty}\frac{n\,x^{n-1}}{(n+1)!}$$

putting $x = 1$, both sides we get

$$\frac{e^1(1-1)+1}{1^2} = \sum_{1}^{\infty}\frac{n\,1^{n-1}}{(n+1)!}$$

or $$1 = \sum_{1}^{\infty}\frac{n}{(n+1)!}.$$ Proved.

Q.28 Find a Series of powers of x, that converges to $\tan x$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Sol We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\cdots = \sum_{0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

both series converges for all x.
So interval of convergence is $]-\infty, \infty[$

Now $\dfrac{1}{\cos x}$ exist if $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$\Rightarrow$ Common interval of convergence is $]-\frac{\pi}{2}, \frac{\pi}{2}[$

So

$$\sin x \cdot \frac{1}{\cos x} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\left(\frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}\right)$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} - \cdots \quad , \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

NOTE: The ans. given in the book is wrong.

$$\begin{array}{l|l}
 & x + x^3/3 + \frac{2x^5}{15} \\
\hline
1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots & x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
 & x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots \\
 & \frac{x^3}{3} + \frac{x^5}{30} \\
 & \frac{x^3}{3} - \frac{x^5}{6} \\
 & \frac{2x^5}{15} - \cdots \\
 & \frac{2x^5}{15} - \frac{x^7}{30}
\end{array}$$

$$\frac{x^5}{120} - \frac{x^5}{24} = \frac{x^5 - 5x^5}{120} = -\frac{4x^5}{120} = -\frac{x^5}{30}$$

$$-\frac{x^5}{30} + \frac{x^5}{6} = \frac{-x^5 + 5x^5}{30} = \frac{4x^5}{30} = \frac{2x^5}{15}$$

$$-\frac{x^3}{3!} + \frac{x^3}{2!} = -\frac{x^3}{6} + \frac{x^3}{2} = \frac{-x^3 + 3x^3}{6} = \frac{2x^3}{6} = \frac{x^3}{3}$$

Q-29 use power series to find the value of $\int_0^{1/2} \frac{dx}{1+x^4}$, to three ~~deci~~ places of decimal.

Sol. We know that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{①}$$

replace x by x^4 in ① we get

$$\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + \cdots$$

$$\Rightarrow \int_0^{1/2} \frac{1}{1+x^4}\,dx = \int_0^{1/2} (1 - x^4 + x^8 - x^{12} + \cdots)\,dx$$

$$= \left| x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots \right|_0^{1/2} = \left(\frac{1}{2} - \frac{(1/2)^5}{5} + \frac{(\frac{1}{2})^9}{9} - \frac{(\frac{1}{2})^{13}}{13} + \cdots\right) - 0$$

$$= \frac{1}{2} - \frac{1}{32(5)} + \frac{1}{512(9)} - \cdots = 0.493967 = 0.494 \text{ ans.}$$

Q-30 Estimate $\int_0^1 x^2 e^{-x^2}dx$ to three places of decimals.

Sol $\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \Rightarrow e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$

$$\Rightarrow x^2 e^{-x^2} = x^2\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) = x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \cdots$$

$$\Rightarrow \int_0^1 x^2 e^{-x^2}dx = \int_0^1 \left(x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \cdots\right)dx = \left|\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{14} - \frac{x^9}{54}\right|_0^1 = \frac{1}{3} - \frac{1}{5} + \frac{1}{14} - \frac{1}{54} + \cdots = \frac{352}{1890}$$

$$\approx 0.187 \text{ ans.}$$

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Chapter 9

Differential Equation:- An eq involving independent and dependent variables and the derivatives of the dependent variable with respect to one or more independent variables is called a Diff Eq. In $\frac{dy}{dx}$, $\frac{d^2z}{du^2}$
y, z are dependent variables & x, u are independent variables.

Ordinary Diff Eq:- An eq involving only derivatives of one or more dependent variables, with respect to a single independent variable is called ordinary diff eq. e.g $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$ is O.D.Eq, but $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dt}\right)^2 = 0$ is not O.D.Eq

Partial Diff Eq:- An eq involving partial derivatives of one or more dependent variables with respect to two or more independent variables is called partial diff eq.

Order of Diff Eq:- The order of a diff eq is the order of the highest derivative that occurs in the eq.

Degree of Diff Eq:- The degree of a diff eq is the power of highest order derivative (free of fractions) involved in a diff eq.

(i) $\frac{dy}{dx} + y\cos x = \sin x$	Ordinary Diff Eq	order 1	Degree 1
(ii) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$	"	order 2	Degree 1
(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$	"	order 2	Degree 2 ☆
(iv) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$	Partial Diff Eq	order 1	Degree 1
(v) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	"	order 2	Degree 1
(vi) $\frac{d^2v}{dx^2} + \frac{dv}{dx} = e^x$	Ordinary Diff Eq	order 2	Degree 1

☆ $\left[\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}\right]^2 = \left(\frac{d^2y}{dx^2}\right)^2 \Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$

Let $(y''')^{2/3} = 4 + y'$ — Order 3, degree 2.

cubing both sides $(y''')^2 = (4+y')^3$

$y''' = \sqrt{2x+3y}$ — Order 3, degree 1

$6x^2 \frac{d^3y}{dx^3} + \sin x \frac{d^2y}{dx^2} - \cos xy$ — Degree is undefined ∵ the unknown fn 'y' is argument of transcendental cosine fn and therefore can not be written as a polynomial in y and its derivatives.

Similarly $y'' + (y')^2 = \log y''$

& $\sin\left(\frac{dy}{dx}\right) = \frac{dy}{dx} + 3x + 2$

Linear Diff Eq.

A diff eq is said to be linear if

i) the dependent variable y and its derivatives are all of degree 'one' only.

ii) No products of 'y' and its derivatives are present.

iii) No transcendental fn of y or its derivatives are present.

e.g. $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$

$\frac{dy}{dx} - x^2y = \cos x.$

A diff eq which is not linear is called Non-Linear Diff Eq:

eg

i) $\frac{d^3y}{dx^3} + 4y^2 = 0$ (∵ power of $y \neq 1$)

ii) $\frac{d^2y}{dx^2} + 7y\frac{dy}{dx} + 12y = 0$ (∵ $7y\frac{dy}{dx}$ involves product of 'y' dependent variable & derivati

iii) $\frac{d^2y}{dx^2} + \sin xy = 0$ (involves transcendental fns of dependent variable.

iv) $5\left(\frac{dy}{dx}\right)^3 + 2\frac{d^2y}{dx^2} + 3y = 0$ (∵ degree of $\frac{dy}{dx}$ is not 1)

Exercise 9.1

① Classify each of the following eqs as ordinary or partial diff eq state the order and degree of each eq and determine whether the eq is linear or non-linear.

(i) $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = \cos x$

Ordinary Diff Eq, order 3, degree 1,
It is Linear Diff Eq.

(ii) $x^2 dy + y^2 dx = 0 \Rightarrow \frac{dy}{dx} + \frac{y^2}{x^2} = 0$

Ordinary Diff Eq, order 1, degree 1
It is non linear eq $\because$ power of $y \neq 1$

(iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Partial Diff Eq, order 2, degree 1
It is Linear Diff Eq.

iv) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} + u = 0$

It is Partial Diff Eq. order 2, degree 1
Non-linear Diff Eq $\because u\frac{\partial u}{\partial x}, u\frac{\partial u}{\partial y}$ i.e Product.

v) $\left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2} + y\right)^{3/2}$

Ordinary Diff Eq, order 2, Degree 3

$$\because \left[\left(\frac{dy}{dx}\right)^2\right]^2 = \left[\left(\frac{d^2y}{dx^2} + y\right)^{3/2}\right]^2$$

$$= \left(\frac{d^2y}{dx^2} + y\right)^3$$

Non linear Diff Eq $\because$ Degree $\neq 1$

General Solution or (Integral) or (Complete Primitive) :-

A sol of a diff eq which contains the number of arbitrary constants equal to the order of the diff eq is called General Sol.

Particular Solution :-

A sol obtained from the general sol by giving particular values to the constants is called a particular sol or integral.

Example: The general Sol of diff eq $\frac{d^2y}{dx^2}=0$ is $y=mx+c$. order 2 const 2, i,e m, c

whereas $y=3x+5$ is obtained by taking particular values $m=3$ & $c=5$.

Singular Sol :- (S.S)

A sol of a diff eq which cannot be obtained from the general sol by any choice of independent arbitrary const is called singular sol.

e.g the general sol of $y'=\sqrt{y}$ is $2\sqrt{y}=x+c$ and S.S is $y=0$

Note The arbitrary constants appearing in the general sol of a diff eq must be independent and to check this we show that they cannot be replaced by or reduced to a smaller number of const.

e.g $y=l\sin(x+\alpha)+m\cos x$ is the sol of $\frac{d^2y}{dx^2}+y=0$

it seems to contain three const l, m, α. But they are not independent as they can be reduced to 'two' only.

$$y = l\sin(x+\alpha)+m\cos x$$
$$= l\sin x\cos\alpha + l\cos x\sin\alpha + m\cos x$$
$$= (l\cos\alpha)\sin x + (m+l\sin\alpha)\cos x$$

or $y = A\sin x + B\cos x$ so two Arbitrary independent Consts A & B.

Initial Value Condition is a condition on the sol of a diff eq at one pt. i,e x_0 e.g $y(x_0)=a$, $y'(x_0)=b$ i,e at $x=\boxed{x_0}$ $y=a$ & $y'=b$

Boundary Value Condition is a cond on the sol of a diff eq at more than one pt. i,e x_0, x_1 $y(x_0)=a$, $y(x_1)=b$

Formation of a differential Eq. A diff eq is formed by the elimination of arbitrary constants from a relation of the form $f(x,y)=0$.

Since to eliminate one const we need two eqs, and to eliminate two constants we need three eqs and so on. N

Now we shall be given one eq of the form $f(x,y)=0$ and the remaining required number of eqs will be formed by differentiating given eq the required number of times.

This also shows that the order of the required diff eq can not exceed the number of constants to be eliminated.

Thus we shall not diff the given eq more than the number of consts. eg to form diff eq from

$$y^2 = cx \quad \text{——— (1)}$$

diff $$2y\frac{dy}{dx} = c \quad \text{——— (2)}$$

Required diff eq will be obtained by eliminating 'c' between (1) & (2)

So Put (2) in (1) $$y^2 = x\left(2y\frac{dy}{dx}\right)$$

Note As there is just one const, so the required diff eq is to be of order 'one'. i.e we should not diff (2) again to eliminate c as

Diff (1) $$2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\cdot\frac{dy}{dx} = 0$$ c eliminated

(6)

Ex 9.1

Q2 (i)

Form the diff eq of which the given fn is a sol.

(i) $y = x + 3e^{-x}$

Diff $y' = 1 - 3e^{-x}$

$= 1 - (y - x)$ $\because y = x + 3e^{-x}$ eliminating e

$y' + y = x + 1$

(ii) $y = (x^3 + c)e^{-3x}$, c being arbitrary const.

diff $y' = 3x^2 e^{-3x} + (x^3+c)(-3e^{-3x})$

$= 3x^2 e^{-3x} + (-3)y$ $\because y = (x^3+c)e^{-3x}$ c eliminated

$y' + 3y = 3x^2 e^{-3x}$

(iii) $ax + \ln|y| = y + b$

Diff $a + \frac{1}{y}y' = y'$

Diff $-y^{-2}y'y' + \frac{1}{y}y'' = y''$

$-\frac{(y')^2}{y^2} + \frac{y''}{y} = y''$

$\frac{-(y')^2 + yy''}{y^2} = y''$

$-(y')^2 + yy'' = y^2 y''$

$-(y')^2 + y''(y - y^2) = 0$

(iv) $y = ae^x + b\ln x + cx + d$ four const so diff four times

Diff $y' = ae^x + \frac{b}{x} + c$ —— (i)

Diff $y'' = ae^x - \frac{b}{x^2}$ —— (ii)

Diff $y''' = ae^x + \frac{2b}{x^3}$ —— (iii)

Diff $y^{iv} = ae^x - \frac{6b}{x^4}$ —— (iv)

Eliminating a & b from (ii) (iii) (iv)

$$e^x x^2 \begin{vmatrix} y'' & 1 & -1 \\ y''' & 1 & \frac{2}{x} \\ y^{iv} & 1 & \frac{-6}{x^2} \end{vmatrix} = 0$$

(v) $x^2+y^2+2gx+2fy+c=0$ three const g, f, c. So diff thrice

Diff $2x+2yy'+2g+2fy'=0$

$x+yy'+g+fy'=0$

$(x+g)+(y+f)y'=0$ — (1)

Diff $1+(y+f)y''+y'y'=0$ — (2)

Diff $(y+f)y'''+y'y''+[y'y''+y''y']=0$

$(y+f)y'''+3y'y''=0$ — (3)

$(y+f)=\dfrac{-3y'y''}{y'''}$ Put in (2)

from (2) $1+\left(\dfrac{-3y'y''}{y'''}\right)y''+y'^2=0$

$(1+y'^2)=\dfrac{3y'(y'')^2}{y'''}$

$(1+y'^2)y'''=3y'(y'')^2$

$3y'(y'')^2-(1+y'^2)y'''=0$

x ——————— x

(vi) $u=f(x,y,z)=(x^2+y^2+z^2)^{-\frac{1}{2}}$ Diff thrice partially w.r.t x, y, z.

$\dfrac{\partial u}{\partial x}=-\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$

$\dfrac{\partial u}{\partial x}=-\dfrac{x}{(x^2+y^2+z^2)^{3/2}}$

$\dfrac{\partial^2 u}{\partial x^2}=-\left(\dfrac{(x^2+y^2+z^2)^{3/2}-x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}2x}{(x^2+y^2+z^2)^3}\right)$

$=-\left(\dfrac{(x^2+y^2+z^2)^{\frac{3}{2}}-3x^2(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}\right)$

$=-\dfrac{(x^2+y^2+z^2)^{\frac{1}{2}}\{x^2+y^2+z^2-3x^2\}}{(x^2+y^2+z^2)^3}$

$\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{(-2x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}=\dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ — (1)

Similarly

$\dfrac{\partial^2 u}{\partial y^2}=\dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ — (ii)

& $\dfrac{\partial^2 u}{\partial z^2}=\dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$ — (iii)

Adding $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$

x ——————— x

vii) $u = f(x-ay) + g(x+ay)$ f & g are twice diff^{l} fns.

$$\frac{\partial u}{\partial x} = f'(x-ay) + g'(x+ay)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ay) + g''(x+ay) \quad \text{——} (1)$$

$$\frac{\partial u}{\partial y} = f'(x-ay)(-a) + g'(x+ay)(a)$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x-ay)(-a)(-a) + g''(x+ay)(a)(a)$$

$$= a^2\left[f''(x-ay) + g''(x+ay)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{using } (1)$$

③ MT (i) Find the differential eq of all circles of radius 'a'. (a is fixed)

Eq of circles of radius 'a': $(x-h)^2 + (y-k)^2 = a^2$

Diff $\quad 2(x-h) + 2(y-k)y' = 0$ (a is fixed given; h, k two arbitrary const so differentiate twice)

$$(x-h) + (y-k)y' = 0 \quad \text{——} (1)$$

Diff $\quad 1 + (y-k)y'' + y'y' = 0$

$$(y-k)y'' = -1 - y'^2$$

$$(y-k) = \frac{-(1+y'^2)}{y''} \quad \text{——} (2)$$

Put (2) in (1) $\quad (x-h) - \frac{(1+y'^2)}{y''}y' = 0$

$$(x-h) = \frac{(1+y'^2)}{y''}y' \quad \text{——} (3)$$

Squaring & Adding (2) & (3) to eliminate consts.

$$(x-h)^2 + (y-k)^2 = \left(\frac{1+y'^2}{y''}\right)^2 y'^2 + \left(\frac{1+y'^2}{y''}\right)^2$$

$$a^2 = \left(\frac{1+y'^2}{y''}\right)^2\left[y'^2 + 1\right]$$

$$a^2(y'')^2 = (1+y'^2)^2(y'^2+1)$$

$$a^2(y'')^2 = (1+y'^2)^3$$

⑨

(ii) Find the diff eq of circles that pass through origin.

Eq of all circles passing through origin is

$$x^2+y^2+2gx+2fy = 0 \quad \text{—— ①}$$ Two Const f, g So diff twice

Diff $2x+2yy'+2g+2fy' = 0$

$$x+yy'+g+fy' = 0$$

$$(x+g)+y'(y+f) = 0 \quad \text{—— ②}$$

Diff $1+(y+f)y''+y'y' = 0$

$$(y+f) = -\frac{(1+y'^2)}{y''} \quad \text{—— ③}$$

Put ③ in ② $(x+g)+y'\left(-\frac{(1+y'^2)}{y''}\right) = 0$

$$(x+g) = y'\left(\frac{1+y'^2}{y''}\right) \quad \text{—— ④}$$

Multiply ④ by x & ③ by y and adding

$$x(x+g)+y(y+f) = xy'\left(\frac{1+y'^2}{y''}\right)-y\frac{(1+y'^2)}{y''}$$

$$x^2+gx+y^2+fy = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$$

$$x^2+y^2+(gx+fy) = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$$

using ①

$$x^2+y^2+\left(\frac{x^2+y^2}{-2}\right) = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$$

$$\frac{x^2+y^2}{2} = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$$

$$(x^2+y^2)y'' = 2(xy'-y)(1+y'^2)$$

(iii) Find the diff eq of ellipses in standard form.

Ellipses in standard form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —— ①

Diff twice because two const a, b.

$$\frac{2x}{a^2}+\frac{2y}{b^2}y'=0$$

$$\frac{x}{a^2}+\frac{yy'}{b^2}=0 \quad —— ②$$

Diff $\frac{1}{a^2}+\frac{yy''+y'^2}{b^2}=0$

x by x $\Rightarrow$ $\frac{x}{a^2}+\frac{x(yy''+y'^2)}{b^2}=0$

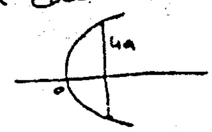
$$\Rightarrow \frac{x}{a^2}=-\frac{x(yy''+y'^2)}{b^2} \quad —— ③$$

Put ③ in ② $\Rightarrow$ $-\frac{x(yy''+y'^2)}{b^2}+\frac{yy'}{b^2}=0$

$$\Rightarrow -xyy''-xy'^2+yy'=0$$

$$xy'^2+xyy''-yy'=0$$

(iv) Find the diff eq of Parabolas each of which has a latus rectum 4a and whose axes are ∥ to x-axis.



$y^2=4ax$

$(y-k)^2=4a(x-h)$

∴ axis ∥ to x-axis.

Eq of given parabola is $(y-k)^2=4a(x-h)$ —— ①

Diff twice because two const h, k.

$$2(y-k)y'=4a$$

$$(y-k)y'=2a \quad —— ②$$

$$y'y'+(y-k)y''=0$$

$$y'^2+(y-k)y''=0$$

$$(y-k)=-\frac{y'^2}{y''} \quad —— ③$$

Put ③ in ②

$$\left[-\frac{y'^2}{y''}\right]y'=2a$$

$$-y'^3=2ay''$$

$$0=2ay''+y'^3$$

(11)

(v) Find diff eq of Hyperbolas in standard form.

standard eq of Hyperbolas is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ —— ①

Diff twice ∵ two const a, b

Diff $\frac{2x}{a^2} - \frac{2y}{b^2}y' = 0$

$\frac{x}{a^2} - \frac{yy'}{b^2} = 0$ —— ②

Diff $\frac{1}{a^2} - \frac{(yy'' + y'^2)}{b^2} = 0$

× by $x \Rightarrow \frac{x}{a^2} - \frac{x(yy'' + y'^2)}{b^2} = 0$ —

$\Rightarrow \frac{x}{a^2} = \frac{x(yy'' + y'^2)}{b^2}$ —— ③

Put ③ in ②

$\frac{x(yy'' + y'^2)}{b^2} - \frac{yy'}{b^2} = 0$

$xyy'' + xy'^2 - yy' = 0$

vi) Find diff eq of conics which coincide with the axes of coordinates.
Find diff eq of conics whose axes coincide with axes of coord.

$ax^2 + by^2 = 1$ —— ① is Eq of conics whose axes coincide with axes of coord.

Diff twice because two const a, b

Diff $2ax + 2byy' = 0$

$ax + byy' = 0$ —— ②

Diff $a + b(yy'' + y'^2) = 0$ —— ③

Eliminating a, b from ①, ②, ③

$$\begin{vmatrix} x^2 & y^2 & 1 \\ x & yy' & 0 \\ 1 & yy'' + y'^2 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & yy' \\ 1 & yy'' + y'^2 \end{vmatrix} = 0$$

$\Rightarrow x(yy'' + y'^2) - yy' = 0$

(4) Solve the following initial value problems. (at one value of x)

(i) $\frac{dy}{dx} = -\frac{x}{y}$, $Y(3)=4$

G. Sol is $x^2+y^2=c^2$

$3^2+4^2 = c^2 \Rightarrow \boxed{c=5}$

$\therefore x^2+y^2=25$ is req sol.

(ii) $\frac{dy}{dx}+y = 2xe^{x}$, $Y(-1) = e+3$

G. Sol is $y=(x^2+c)e^{-x}$

$e+3 = (1+c)e^{-(-1)}$ $\quad \because Y(-1)=e+3$

$\cancel{e}+3 = \cancel{e} + ce \Rightarrow \boxed{c=\frac{3}{e}}$

$\therefore Y = (x^2+\frac{3}{e})e^{-x}$ is Particular Sol

(iii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12 = 0$, $Y(0)=-2$, $Y'(0)=6$

G. Sol is $Y = Ae^{4x}+Be^{-3x}$ —— (1)

$\therefore (-2) = Ae^{0}+Be^{0}$ $\quad \because Y(0)=-2$

$-2 = A+B$ —— (II)

diff (1) $Y' = 4Ae^{4x} - 3Be^{-3x}$

$6 = 4Ae^{0} - 3Be^{0}$ $\quad \because Y'(0)=6$

$6 = 4A - 3B$ —— (III)

solving (II) & (III)

× (II) by 4. $\quad -8 = \cancel{4A}+4B$

$\quad 6 = \cancel{4A} - 3B$

$-14 = 7B$ $\quad \boxed{B=-2}$

using (II) $\boxed{0 = A}$. $\therefore Y = -2e^{-3x}$ is P. Sol.

(iv) $x\frac{dy}{dx}+2y = 4x^2$ $\quad Y(1)=2$

G. Sol is $y = x^2+\frac{c}{x^2}$

$2 = 1^2+\frac{c}{1^2}$ $\quad \because Y(1)=2$

$\boxed{1=c}$

$\therefore$ P. Sol is $y = x^2+\frac{1}{x^2}$

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(v) $x^3\dfrac{d^3y}{dx^3} - 3x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} - 6y = 0$, $y(2)=0$, $y'(2)=2$, $y''(2)=6$

G.Sol is $y = c_1x + c_2x^2 + c_3x^3$ —(I)

$y' = c_1 + c_2(2x) + c_3(3x^2)$ —(II)

$y'' = 2c_2 + c_3 6x$ —(III)

from (I) $0 = 2c_1 + 4c_2 + 8c_3$ (IV) $\because y(2)=0$

from (II) $2 = c_1 + 4c_2 + 12c_3$ (V) $\because y'(2)=2$

from (III) $6 = 2c_2 + 12c_3$ (VI) $\because y''(2)=6$

from (IV) $0 = c_1 + 2c_2 + 4c_3$

from (V) $2 = c_1 + 4c_2 + 12c_3$ subtracting

$-2 = -2c_2 - 8c_3$

from (VI) $6 = 2c_2 + 12c_3$ Adding

$4 = 0 + 4c_3$

$\boxed{1 = c_3}$

from (VI) $\boxed{-3 = c_2}$

from (V) $\boxed{2 = c_1}$

$\therefore$ P.Sol is $y = 2x - 3x^2 + x^3$

5 (i) Solve boundary value Problem (values of x are more than one)

$\dfrac{d^2y}{dx^2} + y = 0$, $y(0) = 1$ $y'(\frac{\pi}{2}) = -1$

G.Sol is $y = c_1\sin x + c_2\cos x$ —(I)

$y' = c_1\cos x - c_2\sin x$ —(II)

from (I) $\boxed{1 = c_2}$ $\because y(0)=1$

from (II) $\boxed{-1 = -c_2}$ $\because y'(\frac{\pi}{2}) = -1$

$y = c_1\sin x + \cos x$ is P.Sol.

(ii) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 2y = 0$, $y(0) = 0$ $y(1) = 1$

$y = c_1 e^x + c_2 e^{3x}$ is the G. Sol — (I)

$0 = c_1 e^0 + c_2 e^0$ $\because y(0) = 0$

$0 = c_1 + c_2$ — (II) $\Rightarrow c_1 = -c_2$

$1 = c_1 e + c_2 e^3$ $\because y(1) = 1$

$\dfrac{1}{e} = c_1 + c_2 e^2$ — (III) $\div$ by e

Subtract (III) from (II)

$$\begin{array}{l} c_1 + c_2 = 0 \\ c_1 + c_2 e^2 = \frac{1}{e} \\ \hline c_2 - c_2 e^2 = -\frac{1}{e} \end{array}$$

$c_2(1-e^2) = -\dfrac{1}{e}$ $\Rightarrow c_2 = \dfrac{-1/e}{1-e^2}$

$\Rightarrow c_2 = \dfrac{-1}{e(1-e^2)}$

using (II) $\therefore c_1 = \dfrac{1}{e(1-e^2)}$

$y = \dfrac{1}{e(1-e^2)} e^x + \left(\dfrac{-1}{e(1-e^2)}\right) e^{3x}$

$= \dfrac{1}{e(1-e^2)} \left[e^x - e^{3x} \right]$ Ans.

If c_1 or c_2 has two different values as $c_1 = -1$ & $c_1 = 2$ then we cannot determine c_2, hence No Solution exist.

See Example 7.

9.2-01

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Differential Eqs of First order and First Degree..

An ordinary differential eq of first order and first degree can be expressed as $\frac{dy}{dx} = f(x,y)$

or $M(x,y)\,dx + N(x,y)\,dy = 0$

The General sol of such an eq will contain only one arbitrary const.

We discuss these types of diffl eqs of first degree & first order.

1) Separable Variables Eqs
2) Homogeneous Eqs
3) Non Homogenous Eqs.
4) Exact Eqs
5) Non Exact. Eqs
6) Linear Eqs
7) Bernoulli Eqs

Type 1 Separable Eq. is a diffl eq of the form $M(x)dx + N(y)dy = 0$

where $M(x)$ is fn of x alone and $N(y)$ is fn of y alone.

To Solve we separate variables & integrate.

Note Constant of integration 'c' can be replaced by $\log c$, e^c, $\tan^{-1}c$ etc whichever is suitable for simplification.

Note Since '$\ln x$' when x is negative is not defined so it is better to write $\ln|x|$ i.e modulus when there is possibility of a -ive number.

Ex 9.2

Solve

(1) $\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$

$y\,dy = \frac{x^2}{1+x^3}\,dx$

$\int y\,dy = \frac{1}{3}\int \frac{3x^2}{1+x^3}\,dx$

$\frac{y^2}{2} = \frac{1}{3}\ln(1+x^3) + c$

$\frac{3y^2}{2} = \ln(1+x^3) + 3c$

$3y^2 = 2\ln(1+x^3) + 6c$

$3y^2 = 2\ln(1+x^3) + c'$

(2) $\frac{dy}{dx} + y^2 \sin x = 0$

$\frac{dy}{dx} = -y^2 \sin x$

$\int \frac{dy}{y^2} = -\int \sin x\,dx$

$\frac{y^{-1}}{-1} = -(-\cos x) + c$

$-\frac{1}{y} = \cos x + c$

(3) $\frac{dy}{dx} = 1 + x + y^2 + xy^2$

$\frac{dy}{dx} = (1+x) + y^2(1+x)$

$\frac{dy}{dx} = (1+x)(1+y^2)$

$\int \frac{dy}{1+y^2} = \int (1+x)\,dx$

$\tan^{-1} y = x + \frac{x^2}{2} + c$

$2\tan^{-1} y = 2x + x^2 + c'$

(5) $\frac{dy}{dx} = 2x^2 + y - x^2 y + xy - 2x - 2$

$= 2x^2 - 2x - 2 + y - x^2 y + xy$

$= 2(x^2 - x - 1) - y(-1 + x^2 - x)$

$\frac{dy}{dx} = (x^2 - x - 1)(2 - y)$

$\int \frac{dy}{2-y} = \int (x^2 - x - 1)\,dx$

$-\int \frac{-dy}{2-y} = \int (x^2 - x - 1)\,dx$

$-\ln|2-y| = \frac{x^3}{3} - \frac{x^2}{2} - x + c$

$-\ln|2-y| = \frac{2x^3 - 3x^2 - 6x + 6c}{6}$

$-6\ln|2-y| = 2x^3 - 3x^2 - 6x + 6c$

$\ln|2-y|^{-6} = (2x^3 - 3x^2 - 6x + 6c)\ln e$

$\ln|2-y|^{-6} = \ln e^{2x^3 - 3x^2 - 6x + 6c}$

$|2-y|^{-6} = e^{2x^3 - 3x^2 - 6x} \cdot e^{6c}$

$|2-y|^{-6} = c_1 e^{2x^3 - 3x^2 - 6x}$

(4) $(xy + 2x + y + 2)\,dx + (x^2 + 2x)\,dy = 0$

$[x(y+2) + (y+2)]\,dx + x(x+2)\,dy = 0$

$[(y+2)(x+1)]\,dx + x(x+2)\,dy = 0$

$\div$ by $x(x+2)(y+2)$

$\frac{x+1}{x(x+2)}\,dx + \frac{1}{y+2}\,dy = 0$

$\int \frac{x+1}{x^2+2x}\,dx + \int \frac{dy}{y+2} = 0$

$\frac{1}{2}\int \frac{2x+2}{x^2+2x}\,dx + \int \frac{dy}{y+2} = 0$

$\ln(y+2) = -\frac{1}{2}\ln(x^2+2x) + \ln c$

$y+2 = \frac{c}{\sqrt{x^2+2x}}$

⑥ $\operatorname{Cosec} y\, dx + \sec x\, dy = 0$

$\div$ by $\operatorname{Cosec} y \sec x$

$\Rightarrow \frac{1}{\sec x} dx + \frac{dy}{\operatorname{Cosec} y} = 0$

$\Rightarrow \int \cos x\, dx + \int \sin y\, dy = \int 0\, dx$

$\Rightarrow \sin x - \cos y = c$ general Sol

⑦ $y(1+x)dx + x(1+y)dy = 0$

$\div$ by xy

$\Rightarrow \frac{(1+x)}{x} dx + \frac{(1+y)}{y} dy = 0$

$\Rightarrow \int(\frac{1}{x}+1)dx + \int(\frac{1}{y}+1)dy = \int 0\, dx$

$\Rightarrow \ln x + x + \ln y + y = C$

$\Rightarrow x + y + \ln(xy) = C$

⑨ $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

$\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$ if $|x|<1$, $|y|<1$

or $\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$

$\sin^{-1} y = -\sin^{-1} x + c$

$y = \sin(c - \sin^{-1} x)$ is G.Sol.

$\frac{dy}{dx} + \sqrt{\frac{y^2-1}{x^2-1}} = 0$ if $|x|>1$, $|y|>1$

$\frac{dy}{dx} = -\sqrt{\frac{y^2-1}{x^2-1}}$

$\int \frac{dy}{\sqrt{y^2-1}} = -\int \frac{dx}{\sqrt{x^2-1}}$

$\cosh^{-1} y = -\cosh^{-1} x + c$

$y = \cosh(c - \cosh^{-1} x)$

⑧ $y\sqrt{1+x^2}\, dx + x\sqrt{1+y^2}\, dy = 0$

$\div$ by xy

$\Rightarrow \int \frac{\sqrt{1+x^2}}{x} dx + \int \frac{\sqrt{1+y^2}}{y} dy = \int 0\, dx$

Put $\sqrt{1+x^2} = t$

$1+x^2 = t^2$

$2x\,dx = 2t\,dt$

$x\,dx = t\,dt$

Put $\sqrt{1+y^2} = z$

$1+y^2 = z^2$

$2y\,dy = 2z\,dz$

$y\,dy = z\,dz$

Therefore

$\Rightarrow \int \frac{\sqrt{1+x^2}}{x^2} x\,dx + \int \frac{\sqrt{1+y^2}}{y^2} y\,dy = \int 0\, dx$

$\Rightarrow \int \frac{t \cdot t\,dt}{t^2-1} + \int \frac{z \cdot z\,dz}{z^2-1} = c$

$\Rightarrow \int\left(\frac{t^2-1+1}{t^2-1}\right)dt + \int \frac{z^2-1+1}{z^2-1} dz = c$

$\Rightarrow \int\left(1 + \frac{1}{t^2-1}\right)dt + \int\left(1 + \frac{1}{z^2-1}\right)dz = c$

$\Rightarrow t + \frac{1}{2}\ln\left(\frac{t-1}{t+1}\right) + z + \frac{1}{2}\ln\left(\frac{z-1}{z+1}\right) = c$

$\Rightarrow \sqrt{1+x^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) + \sqrt{1+y^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+y^2}-1}{\sqrt{1+y^2}+1}\right) = c$

⑩ $(e^x+1)y\,dy = (y+1)e^x\,dx$

$\div$ by $(e^x+1)(y+1)$

$\Rightarrow \int \frac{y\,dy}{y+1} = \int \frac{e^x\,dx}{e^x+1}$

$\Rightarrow \int\left(\frac{y+1-1}{y+1}\right)dy = \int \frac{e^x}{e^x+1} dx$

$\Rightarrow \int\left(1 - \frac{1}{y+1}\right)dy = \int \frac{e^x\,dx}{e^x+1}$

$\Rightarrow y - \ln(y+1) = \ln(e^x+1) + \ln c$

$\Rightarrow y = \ln(y+1) + \ln(e^x+1) + \ln c$

$\Rightarrow y = \ln\{(y+1)(e^x+1)c\}$

$\Rightarrow e^y = c(y+1)(e^x+1)$

(11) $\dfrac{dy}{dx} = \dfrac{y^3+2y}{x^2+3x}$

$\dfrac{dy}{y^3+2y} = \dfrac{dx}{x^2+3x}$

By Partial Fractions from (I) & (II)

$\left[\dfrac{1}{2y} - \dfrac{y}{2(y^2+2)}\right]dy = \left[\dfrac{1}{3x} - \dfrac{1}{3(x+3)}\right]dx$

$\int \dfrac{dy}{2y} - \int \dfrac{(2y)\,dy}{4(y^2+2)} = \int \dfrac{dx}{3x} - \int \dfrac{dx}{3(x+3)}$

$\dfrac{1}{2}\ln y - \dfrac{1}{4}\ln(y^2+2) = \dfrac{1}{3}\ln x - \dfrac{1}{3}\ln(x+3) + \ln c$

$\ln y^{\frac{1}{2}} - \ln(y^2+2)^{\frac{1}{4}} = \ln x^{\frac{1}{3}} - \ln(x+3)^{\frac{1}{3}} + \ln c^{\frac{1}{3}}$

$\ln\left[\dfrac{y^{\frac{1}{2}}}{(y^2+2)^{\frac{1}{4}}}\right] = \ln\left[\dfrac{cx}{(x+3)}\right]^{\frac{1}{3}}$

$\dfrac{y^{\frac{1}{2}}}{(y^2+2)^{\frac{1}{4}}} = \left(\dfrac{cx}{x+3}\right)^{\frac{1}{3}}$ is G. Sol.

Partial Fractions

$\dfrac{1}{y(y^2+2)} = \dfrac{A}{y} + \dfrac{By+C}{y^2+2}$

$1 = A(y^2+2) + (By+C)y$

Put $y=0 \Rightarrow 1 = A(2) \Rightarrow \boxed{A=\frac{1}{2}}$

comparing coeff of $y^2 \rightarrow 0 = A+B$

$0 = \frac{1}{2} + B \quad \boxed{B=-\frac{1}{2}}$

comparing coeff of $y \Rightarrow \boxed{C=0}$

Thus $\dfrac{1}{y(y^2+2)} = \dfrac{1}{2y} - \dfrac{1(y)+0}{2(y^2+2)}$

$\dfrac{1}{y(y^2+2)} = \dfrac{1}{2y} - \dfrac{y}{2(y^2+2)}$ —— (I)

Now $\dfrac{1}{x(x+3)} = \dfrac{A}{x} + \dfrac{B}{x+3}$

$1 = A(x+3) + Bx$

Put $x=0 \Rightarrow \boxed{A=\frac{1}{3}}$

Put $x+3=0 \Rightarrow \boxed{B=-\frac{1}{3}}$

$\therefore \dfrac{1}{x(x+3)} = \dfrac{1}{3x} - \dfrac{1}{3(x+3)}$ —— (II)

(12) $(\sin x + \cos x)dy + (\cos x - \sin x)dx = 0$

$\div$ by $(\sin x + \cos x)$

$\int dy + \int \dfrac{(\cos x - \sin x)}{(\sin x + \cos x)}dx = \int 0\,dx$

$y + \ln(\sin x + \cos x) = c$

$y\ln e + \ln(\sin x + \cos x) = c\ln e$

$\ln e^y + \ln(\sin x + \cos x) = \ln e^c$

$\ln\left[e^y(\sin x + \cos x)\right] = \ln e^c$

$e^y(\sin x + \cos x) = c'$

$e^y = \dfrac{c'}{\sin x + \cos x}$ G. Sol.

(13) $(2x\cos y)dx + x^2(\sec y - \sin y)dy = 0$

$\div$ by $x^2\cos y$

$\dfrac{2\cancel{x}\cancel{\cos y}}{x^{\cancel{2}}\cancel{\cos y}}dx + \dfrac{\cancel{x^2}(\sec y - \sin y)}{\cancel{x^2}\cos y}dy = 0$

$\int \dfrac{2dx}{x} + \int\left(\dfrac{\sec y}{\cos y} - \dfrac{\sin y}{\cos y}\right)dy = \int 0$

$2\int \dfrac{dx}{x} + \int(\sec^2 y - \tan y)dy = \int 0\,dx$

$2\ln x + \tan y - \ln\sec y = c$

$2\ln x = \ln\sec y - \tan c + c$ G. Sol

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(13) $e^{x}\left(1+\frac{dy}{dx}\right) = xe^{-y}$

$$1+\frac{dy}{dx} = xe^{-y}e^{-x}$$

$$1+\frac{dy}{dx} = xe^{-(x+y)}$$

Not separable

So Put $z = x+y$

$$\frac{dz}{dx} = 1+\frac{dy}{dx}$$

$\therefore$

$$\frac{dz}{dx} = xe^{-z}$$

$$\int e^{z}dz = \int x\,dx$$

$$e^{z} = \frac{x^2}{2}+c$$

$$e^{x+y} = \frac{x^2}{2}+c$$

$$\ln e^{x+y} = \ln\left(\frac{x^2}{2}+c\right)$$

$$(x+y)\ln e = \ln\left(\frac{x^2}{2}+c\right)$$

$$(x+y)\,1 = \ln\left(\frac{x^2}{2}+c\right)$$

$$y = \ln\left(\frac{x^2}{2}+c\right)-x \quad \text{G.Sol}$$

(14) $xe^{x^2+y}dx = y\,dy$

$$xe^{x^2}e^{y}dx = y\,dy$$

$$xe^{x^2}dx = ye^{-y}dy$$

$$\frac{1}{2}\int e^{x^2}(2x)\,dx = \int ye^{-y}dy \quad \text{IBP}$$

$\because \begin{cases} e^{x^2}=t & e^{x^2}2x\,dx=dt \\ \int e^{x^2}2x\,dx=\int dt=t=e^{x^2} \end{cases}$

$$\text{So } \frac{1}{2}e^{x^2} = y\frac{e^{-y}}{-1}-\int 1\cdot\frac{e^{-y}}{-1}dy$$

$$= -ye^{-y}+\int e^{-y}dy$$

$$= -ye^{-y}+\frac{e^{-y}}{-1}+c$$

$$\frac{1}{2}e^{x^2} = -ye^{-y}-e^{-y}+c$$

$$e^{x^2} = -2e^{-y}(y+1)+2c$$

$$e^{x^2} = -2e^{-y}(y+1)+c' \quad \text{G.Sol}$$

(16) Solve the initial value problems.

$$2(y-1)dy = (3x^2+4x+2)dx \qquad y(0)=-1$$

$$2\int(y-1)\,dy = \int(3x^2+4x+2)\,dx$$

$$2\left(\frac{y^2}{2}-y\right) = 3\frac{x^3}{3}+4\frac{x^2}{2}+2x+c$$

$$\cancel{2}\left(\frac{y^2-2y}{\cancel{2}}\right) = x^3+2x^2+2x+c$$

$\because y(0)=-1$

$$\therefore (-1)^2-2(-1) = 0+0+0+c$$

$$\boxed{3 = c}$$

$$\text{Hence } y^2-2y = x^3+2x^2+2x+3$$

$$+1-1 \quad y^2-2y+1 = x^3+2x^2+2x+3+1$$

$$(y-1)^2 = x^3+2x^2+2x+4$$

$$y-1 = \pm\sqrt{x^3+2x^2+2x+4}$$

$$y = 1\pm\sqrt{x^3+2x^2+2x+4}$$

$y(0)=-1$ does not satisfy $y=1+\sqrt{x^3+2x^2+2x+4}$

$$\because -1 = 1+\sqrt{0+0+0+4}$$

$$-1 = 1+2 \quad \text{Impossible}$$

$y(0)=-1$ satisfy $y=1-\sqrt{x^3+2x^2+2x+4}$

$$-1 = 1-\sqrt{0+0+0+4}$$

$$-1 = 1-2 \quad \text{true}$$

So $y = 1-\sqrt{x^3+2x^2+2x+4}$ is Sol.

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Q17 $(3x+8)(y^2+4)\,dx - 4y(x^2+5x+6)\,dy = 0$, $y(1) = 2$

$\div$ by $(y^2+4)(x^2+5x+6)$

$$\Rightarrow \frac{3x+8}{x^2+5x+6}\,dx - \frac{4y}{y^2+4}\,dy = 0$$

By Partial Fractions

$$\frac{3x+8}{x^2+5x+6}$$

$$\therefore \frac{3x+8}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{x+3} \quad \text{--- (I)}$$

$$3x+8 = A(x+3) + B(x+2) \quad \text{--- (II)}$$

Put $x+3=0 \Rightarrow x=-3$
$\Rightarrow \boxed{B=1}$

Put $x+2=0 \Rightarrow x=-2$
$\Rightarrow \boxed{A=2}$

$$\therefore \frac{3x+8}{(x+2)(x+3)} = \frac{2}{(x+2)} + \frac{1}{x+3}$$

$$\Rightarrow \left(\frac{2}{x+2} + \frac{1}{x+3}\right)dx - \frac{4y}{y^2+4}\,dy = 0$$

$$\Rightarrow \int\frac{2}{x+2}\,dx + \int\frac{dx}{x+3} - 2\int\frac{2y}{y^2+4}\,dy = \int 0\,dx$$

$$\Rightarrow 2\ln(x+2) + \ln(x+3) - 2\ln(y^2+4) = \ln c$$

$$\ln\left[\frac{(x+2)^2(x+3)}{(y^2+4)^2}\right] = \ln c$$

antilog $$\frac{(x+2)^2(x+3)}{(y^2+4)^2} = c$$

$\because y(1)=2 \Rightarrow \left.\begin{matrix} y=2 \\ x=1 \end{matrix}\right) \Rightarrow c = \dfrac{(1+2)^2(1+3)}{(2^2+4)^2}$

$$c = \frac{36}{64} = \frac{9}{16}$$

$$\therefore \frac{(x+2)^2(x+3)}{(y^2+4)^2} = \frac{9}{16}$$

$$16(x+2)^2(x+3) = 9(y^2+4)^2 \quad \text{Ans.}$$

(18) $(1+2y^2)dy = y\cos x\,dx$, $y(0)=1$

$\div$ by y.

$\Rightarrow \left(\frac{1+2y^2}{y}\right)dy = \cos x\,dx$

$\Rightarrow \int\left(\frac{1}{y}+2y\right)dy = \int \cos x\,dx$

$\Rightarrow \ln y + \cancel{2}\frac{y^2}{\cancel{2}} = \sin x + C$

$\because y(0)=1$

$\therefore \ln 1 + 1 = 0 + C$

$\boxed{1 = C}$

$\Rightarrow \therefore \ln y + y^2 = \sin x + 1$

(19) $8\cos^2 y\,dx + \operatorname{cosec}^2 x\,dy = 0$, $y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

$\div$ by $\cos^2 y \operatorname{cosec}^2 x$

$\Rightarrow \frac{8}{\operatorname{cosec}^2 x}dx + \frac{1}{\cos^2 y}dy = 0$

$\Rightarrow \int 8\sin^2 x\,dx + \int \sec^2 y\,dy = \int 0\,dx$

$\Rightarrow 4\int 2\sin^2 x\,dx + \tan y = C$

$\Rightarrow 4\int (1-\cos 2x)\,dx + \tan y = C$

$\Rightarrow 4\left(x - \frac{\sin 2x}{2}\right) + \tan y = C$

$\Rightarrow 4x - 2\sin 2x + \tan y = C$

$\Rightarrow \tan y = -4x + 2\sin 2x + C$

$\because y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

$\therefore \tan\left(\frac{\pi}{4}\right) = -4\left(\frac{\pi}{12}\right) + 2\sin 2\left(\frac{\pi}{12}\right) + C$

$1 = -\frac{\pi}{3} + 2\sin\frac{\pi}{6} + C$

$1 = -\frac{\pi}{3} + 2\left(\frac{1}{2}\right) + C$

$\cancel{1} - \cancel{1} + \frac{\pi}{3} = C \Rightarrow \boxed{C = \frac{\pi}{3}}$

$\Rightarrow \therefore \tan y = -4x + 2\sin 2x + \frac{\pi}{3}$ Ans

(20) $\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}$, $y(0) = -\frac{1}{\sqrt{2}}$

$\Rightarrow 4y^3\,dy = x(x^2+1)\,dx$

$\Rightarrow 4\int y^3\,dy = \int (x^3+x)\,dx$

$\Rightarrow \cancel{4}\frac{y^4}{\cancel{4}} = \frac{x^4}{4} + \frac{x^2}{2} + C$

$\because y(0) = -\frac{1}{\sqrt{2}}$

$\therefore \left(-\frac{1}{\sqrt{2}}\right)^4 = 0 + 0 + C$

$\boxed{\frac{1}{4} = C}$

$\Rightarrow \therefore y^4 = \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4}$

$\Rightarrow 4y^4 = x^4 + 2x^2 + 1$

$\Rightarrow 4y^4 = (x^2+1)^2$

$\Rightarrow 2y^2 = x^2+1$

$\Rightarrow y^2 = \frac{x^2+1}{2}$

$\Rightarrow y = \pm\sqrt{\frac{x^2+1}{2}}$

$\Rightarrow y = -\sqrt{\frac{x^2+1}{2}}$ Ans.

Note: $y = +\sqrt{\frac{x^2+1}{2}}$ is not satisfied by $y(0) = -\frac{1}{\sqrt{2}}$. So leave it.

Homogeneous Diff Eq. (H.DE)

A differential eq. of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be homogeneous diff eq if both fns $f(x,y)$ & $g(x,y)$ are homogeneous of same degree.

Homogeneous Fn:-

A function $f(x,y)$ is said to be homogeneous of degree 'n', if it can be written as $f(tx, ty) = t^n f(x,y)$

e.g. $f(x,y) = \sqrt{xy}$
$f(tx,ty) = \sqrt{tx\,ty} = t\sqrt{xy}$
$f(x,y)$ is homogeneous fn of degree '1'

To Solve Put $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

then by method of separable variable we solve.

Ex 9.3

① $(x-y)dx + (x+y)dy = 0$

$x\,dx - y\,dx + x\,dy + y\,dy = 0$
Not separable.

$(x+y)\,dy = -(x-y)\,dx$

$\frac{dy}{dx} = \frac{y-x}{x+y}$ H.DE (i)

Put $y = vx$ (ii)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ (iii)

using (ii) (iii) in (i)

$v + x\frac{dv}{dx} = \frac{vx - x}{x + vx}$

$x\frac{dv}{dx} = \frac{\cancel{x}(v-1)}{\cancel{x}(1+v)} - v$

$= \frac{\cancel{v} - 1 - \cancel{v} - v^2}{1+v}$

$x\frac{dv}{dx} = -\frac{(v^2+1)}{1+v}$

$\int \frac{v+1}{v^2+1}\,dv = -\int \frac{dx}{x}$

$\frac{1}{2}\int \frac{2v\,dv}{v^2+1} + \int \frac{dv}{v^2+1} = -\int \frac{dx}{x}$

$\frac{1}{2}\ln(v^2+1) + \tan^{-1}v = -\ln x + c$

$\rightarrow \ln(v^2+1)^{\frac{1}{2}} + \tan^{-1}v + \ln x = c$

$\ln\sqrt{\frac{y^2}{x^2}+1} + \tan^{-1}\left(\frac{y}{x}\right) + \ln x = c$

$\ln\sqrt{y^2+x^2} - \cancel{\ln\sqrt{x^2}} + \tan^{-1}\left(\frac{y}{x}\right) + \cancel{\ln x} = c$

$\ln\sqrt{y^2+x^2} + \tan^{-1}\left(\frac{y}{x}\right) = c$

9.3 – 1

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Homogeneous Diff Eq (H.DE)

A differential eq of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$ is said to be homogeneous diff eq if both fns $f(x,y)$ & $g(x,y)$ are homogeneous of same degree.

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$\dfrac{dy}{dx} = \dfrac{y-x}{x+y}$ H.DE ——— (i)

Put $y = vx$ ——— (ii)

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ ——— (iii)

using (ii) (iii) in (i)

$v + x\dfrac{dv}{dx} = \dfrac{vx - x}{x + vx}$

$x\dfrac{dv}{dx} = \dfrac{\cancel{x}(v-1)}{\cancel{x}(1+v)} - v$

$= \dfrac{\cancel{v} - 1 - \cancel{v} - v^2}{1+v}$

$x\dfrac{dv}{dx} = -\dfrac{(v^2+1)}{1+v}$

$\displaystyle\int \frac{v+1}{v^2+1}\,dv = -\int \frac{dx}{x}$

$\displaystyle\frac{1}{2}\int \frac{2v\,dv}{v^2+1} + \int \frac{dv}{v^2+1} = -\int \frac{dx}{x}$

$\frac{1}{2}\ln(v^2+1) + \tan^{-1}v = -\ln x + c$

$\ln(v^2+1)^{\frac{1}{2}} + \tan^{-1}v + \ln x = c$

$\ln\sqrt{\dfrac{y^2}{x^2}+1} + \tan^{-1}\left(\dfrac{y}{x}\right) + \ln x = c$

$\ln\sqrt{y^2+x^2} - \cancel{\ln\sqrt{x^2}} + \tan^{-1}\left(\dfrac{y}{x}\right) + \cancel{\ln x} = c$

$\ln\sqrt{y^2+x^2} + \tan^{-1}\left(\dfrac{y}{x}\right) = c$

② $(y^2+2xy)dx + x^2dy = 0$

$x^2dy = -(y^2+2xy)dx$

$\frac{dy}{dx} = -\frac{(y^2+2xy)}{x^2}$ H.D.E ——①

Put $y = vx$ ——⑪

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ——⑪⑪

using ⑪ & ⑪⑪ in ①

$v + x\frac{dv}{dx} = -\frac{(v^2x^2+2xvx)}{x^2}$

$x\frac{dv}{dx} = -\frac{\cancel{x^2}(v^2+2v)}{\cancel{x^2}} - v$

$x\frac{dv}{dx} = -(v^2+3v)$

$\int\frac{dv}{v^2+3v} = -\int\frac{dx}{x}$ separating variables

$\int\frac{1}{v(v+3)}dv = -\int\frac{dx}{x}$

$\frac{1}{3}\int\frac{3}{v(v+3)}dv = -\int\frac{dx}{x}$

$\frac{1}{3}\int\frac{v+3-v}{v(v+3)}dv = -\int\frac{dx}{x}$

$\frac{1}{3}\int\left(\frac{1}{v} - \frac{1}{v+3}\right)dv = -\int\frac{dx}{x}$

$\frac{1}{3}\ln v - \frac{1}{3}\ln(v+3) = -\ln x + \ln c$

$\ln\left[\frac{v^{\frac{1}{3}}}{(v+3)^{\frac{1}{3}}}\right] = \ln\frac{c}{x}$

Antilog $\frac{v^{\frac{1}{3}}}{(v+3)^{\frac{1}{3}}} = \frac{c}{x}$

$x\cdot v^{\frac{1}{3}} = c(v+3)^{\frac{1}{3}}$

$x\left(\frac{y}{x}\right)^{\frac{1}{3}} = c\left(\frac{y}{x}+3\right)^{\frac{1}{3}}$

$x\frac{y^{\frac{1}{3}}}{\cancel{x^{\frac{1}{3}}}}\cdot\cancel{x^{\frac{1}{3}}} = c(y+3x)^{\frac{1}{3}}$

$xy^{\frac{1}{3}} = c(y+3x)^{\frac{1}{3}}$

$x^3y = c'(y+3x)$

Q3 $(x^2-3y^2)dx + 2xydy = 0$

$2xydy = -(x^2-3y^2)dx$

$\frac{dy}{dx} = \frac{3y^2-x^2}{2xy}$ H.D.E ——①

Put $y = vx$ ——⑪

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ——⑪⑪

using ⑪⑪⑪ in ①

$v + x\frac{dv}{dx} = \frac{3v^2x^2-x^2}{2xvx}$

$x\frac{dv}{dx} = \frac{(3v^2-1)\cancel{x^2}}{2v\cancel{x^2}} - v$

$x\frac{dv}{dx} = \frac{3v^2-1-2v^2}{2v}$

$x\frac{dv}{dx} = \frac{v^2-1}{2v}$

$\int\frac{2v}{v^2-1}dv = \int\frac{dx}{x}$ separating variables

$\ln(v^2-1) = \ln x + \ln c$

$\ln\left(\frac{y^2}{x^2}-1\right) = \ln cx$

$\frac{y^2-x^2}{x^2} = cx$

$y^2-x^2 = (cx)x^2$

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④ $(x^2+3y^2)dx - 2xy\,dy = 0$

$(x^2+3y^2)\,dx = 2xy\,dy$

$\frac{x^2+3y^2}{2xy} = \frac{dy}{dx}$ —— HDE ①

Put $y = vx$ —— ⑪

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— ⑪⑪

using ⑪⑪⑪ in ①

$v + x\frac{dv}{dx} = \frac{x^2+3v^2x^2}{2xvx}$

$x\frac{dv}{dx} = \frac{\cancel{x^2}(1+3v^2)}{\cancel{x^2}2v} - v$

$x\frac{dv}{dx} = \frac{1+3v^2-2v^2}{2v}$

$x\frac{dv}{dx} = \frac{1+v^2}{2v}$

$\int\frac{2v}{1+v^2}dv = \int\frac{dx}{x}$ separating variables

$\ln(1+v^2) = \ln x + \ln c$

$\ln(1+v^2) = \ln cx$

$(1+\frac{y^2}{x^2}) = cx$

$\frac{x^2+y^2}{x^2} = cx$

$x^2+y^2 = (cx)\,x^2$

⑤ $(x^2+xy+y^2)dx - x^2dy = 0$

$(x^2+xy+y^2)dx = x^2dy$

$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$ HDE ①

Put $y = vx$ —— ⑪

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— ⑪⑪

using ⑪⑪⑪ in ①

$v + x\frac{dv}{dx} = \frac{x^2+x(vx)+v^2x^2}{x^2}$

$x\frac{dv}{dx} = \frac{(1+v+v^2)\cancel{x^2}}{\cancel{x^2}} - v$

$\int\frac{dv}{1+v^2} = \int\frac{dx}{x}$ separating variables

$\tan^{-1}v = \ln x + c$

$\tan^{-1}(\frac{y}{x}) = \ln x + c$

⑥ $(x^2+3xy+y^2)dx - x^2dy = 0$

$(x^2+3xy+y^2)dx = x^2dy$

$\frac{dy}{dx} = \frac{x^2+3xy+y^2}{x^2}$ —— HDE ①

Put $y = vx$ —— ⑪

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— ⑪⑪

using ⑪⑪⑪ in ①

$v + x\frac{dv}{dx} = \frac{x^2+3xvx+v^2x^2}{x^2}$

$x\frac{dv}{dx} = \frac{\cancel{x^2}(1+3v+v^2)}{\cancel{x^2}} - v$

$x\frac{dv}{dx} = 1+2v+v^2$

$\int\frac{dv}{(1+v)^2} = \int\frac{dx}{x}$ separating variables

$-\frac{1}{(v+1)} = \ln x + c$

$\frac{-1}{(\frac{y}{x}+1)} = \ln x + c$

$\frac{-1}{\frac{y+x}{x}} = \ln x + c$

$\frac{-x}{(x+y)} = \ln x + c$

(7) $\dfrac{dy}{dx} = \dfrac{4y-3x}{2x-y}$ HDE —— (i)

Put $y = vx$ —— (ii)

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ —— (iii)

using (ii) (iii) in (i)

$$v + x\frac{dv}{dx} = \frac{4vx-3x}{2x-vx}$$

$$x\frac{dv}{dx} = \frac{\cancel{x}(4v-3)}{\cancel{x}(2-v)} - v$$

$$x\frac{dv}{dx} = \frac{4v-3-2v+v^2}{2-v}$$

$$\int \frac{2-v}{v^2+2v-3}\,dv = \int \frac{dx}{x} \quad \text{—— (iv)}$$ separating variables

By Partial Fractions

$$\frac{2-v}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}$$

$$2-v = A(v-1) + B(v+3)$$

Put $v+3=0 \Rightarrow +5 = -4A \Rightarrow \boxed{A = \frac{-5}{4}}$

Put $v-1=0 \Rightarrow 1 = 4B \Rightarrow \boxed{B = \frac{1}{4}}$

$$\therefore \frac{2-v}{(v+3)(v-1)} = \frac{-5}{4(v+3)} + \frac{1}{4(v-1)}$$

from (iv)

$$\frac{-5}{4}\int\frac{dv}{v+3} + \frac{1}{4}\int\frac{dv}{v-1} = \int\frac{dx}{x}$$

$$\frac{-5}{4}\ln(v+3) + \frac{1}{4}\ln(v-1) = \ln x + \ln c$$

$$-\ln(v+3)^5 + \ln(v-1) = 4\ln cx$$

$$\ln\frac{(v-1)}{(v+3)^5} = \ln c^4x^4$$

Antilog

$$\frac{\left(\frac{y}{x}-1\right)}{\left(\frac{y}{x}+3\right)^5} = c^4x^4$$

$$\frac{(y-x)x^5}{x(y+3x)^5x^4} = c'$$

$$\frac{(y-x)}{(y+3x)^5} = c' \quad \text{Ans.}$$

(8) $x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\frac{y}{x} - x\right)dx$

$\dfrac{dy}{dx} = \dfrac{y\sin\frac{y}{x} - x}{x\sin\left(\frac{y}{x}\right)}$ HDE —— (i)

Put $y = vx$ —— (ii)

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ —— (iii)

using (ii) (iii) in (i)

$$v + x\frac{dv}{dx} = \frac{vx\sin\frac{vx}{x} - x}{x\sin\left(\frac{vx}{x}\right)}$$

$$x\frac{dv}{dx} = \frac{\cancel{x}(v\sin v - 1)}{\cancel{x}\sin v} - v$$

$$x\frac{dv}{dx} = \frac{v\cancel{\sin v} - 1 - v\cancel{\sin v}}{\sin v}$$

$$\int \sin v\,dv = \int -\frac{dx}{x}$$ separating variables

$$-\cos v = -\ln x + c$$

$$\cos v = \ln x - c$$

$$\cos\frac{y}{x} = \ln x - c$$

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(9) $\left(x^3 + y^2\sqrt{x^2+y^2}\right)dx - xy\sqrt{x^2+y^2}\,dy = 0$

$$x^3 + y^2\sqrt{x^2+y^2}\,dx = xy\sqrt{x^2+y^2}\,dy$$

$$\frac{dy}{dx} = \frac{x^3 + y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}} \quad \text{HDE} \text{ ——— (i)}$$

$$\left\{\frac{dy}{dx} = \frac{t^3x^3 + t^2y^2\sqrt{t^2x^2+t^2y^2}}{tx\,ty\sqrt{t^2x^2+t^2y^2}} = \frac{t^3\,x^3 + y^2t^3\sqrt{x^2+y^2})}{t^2xy\,t\sqrt{x^2+y^2}} = \frac{t^3\left(x^3+y^2\sqrt{x^2+y^2}\right)}{t^3\left(xy\sqrt{x^2+y^2}\right)}\right.$$

Degree of N & D is same '3' So Homogeneous.

Put $y = vx$ ——— (ii)

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \text{ ——— (iii)}$$

using (ii) & (iii) in (i)

$$v + x\frac{dv}{dx} = \frac{x^3 + v^2x^2\sqrt{x^2+v^2x^2}}{xvx\sqrt{x^2+v^2x^2}}$$

$$= \frac{\cancel{x^3}\left(1+v^2\sqrt{1+v^2}\right)}{\cancel{x^3}\,v\sqrt{1+v^2}} - v$$

$$x\frac{dv}{dx} = \frac{1+v^2\cancel{\sqrt{1+v^2}} - v^2\cancel{\sqrt{1+v^2}}}{v\sqrt{1+v^2}}$$

$$x\frac{dv}{dx} = \frac{1}{v\sqrt{1+v^2}}$$

$$\int v\sqrt{1+v^2}\,dv = \int\frac{dx}{x} \qquad \text{separating variables}$$

$$\frac{1}{2}\int\sqrt{1+v^2}\,(2v)\,dv = \int\frac{dx}{x}$$

$$\frac{1}{2}\,\frac{(1+v^2)^{3/2}}{3/2} = \ln x + c$$

$$\frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}(1+v^2)^{3/2}$$

$$\left(1+\frac{y^2}{x^2}\right)^{3/2} = 3\ln x + 3c$$

$$\left(\frac{x^2+y^2}{x^2}\right)^{3/2} = \ln x^3 + c'$$

$$\frac{(x^2+y^2)^{3/2}}{x^3} = \ln x^3 + c'$$

$$(x^2+y^2)^{3/2} = x^3\ln x^3 + c'x^3$$

Imp

(10) $\left(\sqrt{x+y}+\sqrt{x-y}\right)dx-\left(\sqrt{x+y}-\sqrt{x-y}\right)dy=0$

$$\left(\sqrt{x+y}+\sqrt{x-y}\right)dx=\left(\sqrt{x+y}-\sqrt{x-y}\right)dy$$

$$\frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}=\frac{dy}{dx} \quad \text{HDE} \quad \text{——— (i)}$$

Put $y = vx$ ——— (ii)

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \quad \text{——— (iii)}$$

using (ii) (iii) in (i)

$$v+x\frac{dv}{dx}=\frac{\sqrt{x+vx}+\sqrt{x-vx}}{\sqrt{x+vx}-\sqrt{x-vx}}$$

$$v+x\frac{dv}{dx}=\frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}\left(\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}\right)$$

Rationalize

$$v+x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}\times\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}+\sqrt{1-v}}$$

$$=\frac{(1+v)+(1-v)+2\sqrt{1+v}\sqrt{1-v}}{(1+v)-(1-v)}$$

$$=\frac{2+2\sqrt{1-v^2}}{2v}$$

$$x\frac{dv}{dx}=\frac{1+\sqrt{1-v^2}}{v}-v$$

$$x\frac{dv}{dx}=\frac{1+\sqrt{1-v^2}-v^2}{v}$$

$$\int\frac{v\,dv}{(1-v^2)+\sqrt{1-v^2}}=\int\frac{dx}{x} \quad \text{——— (iv)}$$

separate variables

$$\int\frac{v\,dv}{(1-v^2)+\sqrt{1-v^2}}$$

Put $v=\sin x$
$dv=\cos x\,dx$

$$\int\frac{\sin x\cos x\,dx}{(1-\sin^2x)+\sqrt{1-\sin^2x}}$$

$$\int\frac{\sin x\cos x\,dx}{\cos^2x+\cos x}$$

$$=\int\frac{\sin x\cancel{\cos x}\,dx}{\cancel{\cos x}(\cos x+1)}$$

$$=-\int\frac{-\sin x\,dx}{(\cos x+1)}$$

$$=-\ln(\cos x+1) \quad \text{Put in (iv)}$$

$$-\ln(\cos x+1)=\ln x+\ln c$$

$$-\ln\left(\sqrt{1-\sin^2x}+1\right)=\ln cx$$

$$-\ln\left(\sqrt{1-v^2}+1\right)=\ln cx$$

$$-\ln\left(\sqrt{1-\frac{y^2}{x^2}}+1\right)=\ln cx$$

$$-\ln\left(\frac{\sqrt{x^2-y^2}}{x}+1\right)=\ln(cx)$$

$$\ln\left(\frac{\sqrt{x^2-y^2}+x}{x}\right)^{-1}=\ln(cx)$$

Antilog

$$\frac{\cancel{x}}{\sqrt{x^2-y^2}+x}=c\cancel{x}$$

$$c'x=\sqrt{x^2-y^2}+x$$

where $c'=\frac{1}{c}$

(11) $\frac{dy}{dx} = \frac{x+y}{x}$ ——— (i) , $y(1)=1$

Put $y = vx$ ——— (ii)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ——— (iii)

using (ii) (iii) in (i)

$v + x\frac{dv}{dx} = \frac{x+vx}{x}$

$x\frac{dv}{dx} = \frac{\cancel{x}(1+v)}{\cancel{x}} - v$

$x\frac{dv}{dx} = 1$

$\int dv = \int \frac{dx}{x}$ separating variables

$v = \ln x + c$

$\frac{y}{x} = \ln x + c$

$\because y(1)=1$

$\therefore \frac{1}{1} = \ln 1 + c$

$\boxed{1 = 0 + c}$

So $\frac{y}{x} = \ln x + 1$

$y = x\ln x + x = x[\ln x + 1]$ Ans

(12) $(y+\sqrt{x^2+y^2})dx - xdy = 0$, $y(1)=0$

$\because$ $(y+\sqrt{x^2+y^2})dx = xdy$

$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ HDE ——— (i)

Put $y = vx$ ——— (ii)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ——— (iii)

using (ii) (iii) in (i)

$v + x\frac{dv}{dx} = \frac{vx+\sqrt{x^2+v^2x^2}}{x}$

$= \frac{x(v+\sqrt{1+v^2})}{x}$

$x\frac{dv}{dx} = \cancel{v} + \sqrt{1+v^2} - \cancel{v}$

$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$ separating variables

$\sinh^{-1} v = \ln x + c$

$\sinh^{-1}\left(\frac{y}{x}\right) = \ln x + c$

$\because y(1)=0$

$\therefore \sinh^{-1}\left(\frac{0}{1}\right) = \ln 1 + c$

$\boxed{0 = c}$

$\therefore \sinh^{-1}\left(\frac{y}{x}\right) = \ln x$

$\because \sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$

$\Rightarrow \ln\left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right) = \ln x$

Antilog

$\Rightarrow \frac{y+\sqrt{x^2+y^2}}{x} = x$

$\Rightarrow y + \sqrt{x^2+y^2} = x^2$

$y = x^2 - \sqrt{x^2+y^2}$

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(13) $(2x-5y)dx + (4x-y)dy = 0 \quad , \quad y(1) = 4$

$(4x-y)dy = -(2x-5y)dx$

$\frac{dy}{dx} = \frac{5y-2x}{4x-y}$ —— HDE (i)

Put $y = vx$ —— (ii)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— (iii)

using (ii) (iii) in (i)

$v + x\frac{dv}{dx} = \frac{5vx-2x}{4x-vx}$

$= \frac{\cancel{x}(5v-2)}{\cancel{x}(4-v)} - v$

$= \frac{5v-2-4v+v^2}{4-v}$

$x\frac{dv}{dx} = \frac{v^2+v-2}{4-v}$

$\int \frac{4-v}{v^2+v-2}dv = \int \frac{dx}{x}$ separating variables.

By Partial Fractions

$\int \frac{(4-v)dv}{(v-1)(v+2)} = \int \frac{dx}{x}$

$\int \left[\frac{dv}{v-1} - \frac{2}{(v+2)}\right]dv = \int \frac{dx}{x}$

$\Rightarrow \ln(v-1) - 2\ln(v+2) = \ln x + \ln c$

$\Rightarrow \ln\left[\frac{(v-1)}{(v+2)^2}\right] = \ln cx$

Antilog

$\Rightarrow \frac{(\frac{y}{x}-1)}{(\frac{y}{x}+2)^2} = cx$

$\Rightarrow \frac{y-x}{\cancel{x}\frac{(y+2x)^2}{\cancel{x^2}}} \cdot \frac{1}{\cancel{x}} = c$

Partial Fractions.

$\frac{4-v}{(v-1)(v+2)} = \frac{A}{v-1} + \frac{B}{v+2}$

$4-v = A(v+2) + B(v-1)$

$v+2=0 \Rightarrow v=-2$

$\Rightarrow \boxed{B=-2}$

$v-1=0 \Rightarrow v=1$

$\boxed{A=1}$

So $\frac{4-v}{(v-1)(v+2)} = \frac{1}{v-1} - \frac{2}{v+2}$

$\because y(1) = 4$

$\therefore \frac{4-1}{(4+2)^2} = c$

$c = \frac{3}{36} = \frac{1}{12}$

$\Rightarrow \frac{y-x}{(y+2x)^2} = \frac{1}{12} \Rightarrow 12(y-x) = (y+2x)^2$

(14) $(3x^2+9xy+5y^2)dx-(6x^2+4xy)dy=0 \quad , \quad y(2)=-6$

$$(6x^2+4xy)dy=(3x^2+9xy+5y^2)dx$$

$$\frac{dy}{dx}=\frac{3x^2+9xy+5y^2}{6x^2+4xy} \quad \text{HDE} \text{ --- (i)}$$

Put $y=vx$ ——— (ii)

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \text{ ——— (iii)}$$

using (ii) (iii) in (i)

$$v+x\frac{dv}{dx}=\frac{3x^2+9xvx+5v^2x^2}{6x^2+4xvx}$$

$$x\frac{dv}{dx}=\frac{\cancel{x^2}(3+9v+5v^2)}{\cancel{x^2}(6+4v)}-v$$

$$x\frac{dv}{dx}=\frac{3+9v+5v^2-6v-4v^2}{6+4v}$$

$$x\frac{dv}{dx}=\frac{v^2+3v+3}{6+4v}$$

$$\int\frac{(4v+6)dv}{v^2+3v+3}=\int\frac{dx}{x} \quad \text{separating variables}$$

$$2\int\frac{(2v+3)dv}{v^2+3v+3}=\int\frac{dx}{x}$$

$$2\ln(v^2+3v+3)=\ln x+\ln c$$

$$\ln(v^2+3v+3)^2=\ln cx$$

$$(v^2+3v+3)^2=cx$$

$$\left(\frac{y^2}{x^2}+3\frac{y}{x}+3\right)^2=cx$$

$$\left(\frac{y^2+3yx+3x^2}{x^2}\right)^2=cx$$

$$\frac{y^2+3yx+3x^2}{x^2}=\sqrt{cx}$$

$$y^2+3yx+3x^2=x^2\sqrt{cx}$$

$\because y(2)=-6$

$$\therefore 36-36+12=4\sqrt{2c}$$

$$\frac{12}{4}=\sqrt{2c}$$

$$(3)^2=2c \Rightarrow \boxed{c=\frac{9}{2}}$$

Therefore

$$3x^2+3xy+y^2=x^2\sqrt{\frac{9}{2}x}$$

$$3x^2+3xy+y^2=\frac{3}{\sqrt{2}}x^{2+\frac{1}{2}}$$

$$\sqrt{2}(3x^2+3xy+y^2)=3x^{5/2}$$

$$2(3x^2+3xy+y^2)^2=9x^5$$

Non Homogeneous Diff Eq:- is of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

Non Homogeneous Diff Eq are of two types.

Type 1 if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ **To solve** Put $\left.\begin{array}{l} x = X+h \\ y = Y+K \end{array}\right] \Rightarrow \begin{array}{l} dx = dX \\ dy = dY \end{array}$ so $\dfrac{dy}{dx} = \dfrac{dY}{dX}$

find values of h & K.

Type 2 if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ **To Solve** Put $z = a_1x+b_1y$ and solve by separable variable in x & z

(15) $\dfrac{dy}{dx} = \dfrac{x+3y-5}{x-y-1}$ ...... NHDEq $\quad \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \because \dfrac{1}{1} \neq \dfrac{3}{-1}$

Put $\left.\begin{array}{l} x = X+h \\ y = Y+K \end{array}\right] \Rightarrow \dfrac{dy}{dx} = \dfrac{dY}{dX}$

$$\frac{dY}{dX} = \frac{X+h+3(Y+K)-5}{X+h-(Y+K)-1}$$

where $h+3K-5=0$, & $h-K-1=0$

Subtract

$$\begin{array}{l} h+3K-5=0 \\ h-K-1=0 \\ \hline 4K-4=0 \\ \boxed{K=1} \end{array}$$

Add

$$\begin{array}{l} h+3K-5=0 \\ 3h-3K-3=0 \\ \hline 4h-8=0 \\ \boxed{h=2} \end{array}$$

$x = X+2$
$y = Y+1$

$$\frac{dY}{dX} = \frac{X+3Y}{X-Y} \quad \text{HDE} \text{ — (i)}$$

Put $Y = VX$ — (ii)

$$\frac{dY}{dX} = V + X\frac{dV}{dX} \text{ — (iii)}$$

using (ii) (iii) in (i)

$$V + X\frac{dV}{dX} = \frac{X+3VX}{X-VX}$$

$$X\frac{dV}{dX} = \frac{\cancel{X}(1+3V)}{\cancel{X}(1-V)} - V$$

$$X\frac{dV}{dX} = \frac{1+3V-V+V^2}{1-V}$$

$$X\frac{dV}{dX} = \frac{(1+V)^2}{1-V}$$

$$\int \frac{1-V}{(1+V)^2}\,dV = \int \frac{dX}{X} \quad \text{separating variables}$$

$$\int \frac{1\,dV}{(1+V)^2} - \int \frac{V\,dV}{(1+V)^2} = \int \frac{dX}{X}$$

$$\int (1+V)^{-2}dV - \int \frac{1+V-1}{(1+V)^2}dV = \int \frac{dX}{X}$$

$$\Rightarrow \int (1+V)^{-2}dV - \int \frac{1}{1+V}dV + \int \frac{1\,dV}{(1+V)^2} = \int \frac{dX}{X}$$

$$\Rightarrow \frac{-1}{(1+V)} - \ln(1+V) - \frac{1}{(1+V)} = \ln X + \ln c$$

$$\Rightarrow \frac{-2}{1+V} = \ln(1+V) + \ln X + \ln c$$

$$\Rightarrow \frac{-2}{1+V} = \ln c\,X(1+V)$$

$$\Rightarrow \frac{-2}{1+\frac{Y}{X}} = \ln c\,X\left(1+\frac{Y}{X}\right)$$

$$\Rightarrow \frac{-2X}{X+Y} = \ln c\,(X+Y)$$

$$\Rightarrow \frac{-2(x-2)}{x-2+y-1} = \ln c\,(x-2+y-1)$$

$$\Rightarrow \frac{-2x+4}{x+y-3} = \ln c\,(x+y-3) \quad \text{Ans.}$$

(16) $\dfrac{dy}{dx} = -\dfrac{(4x+3y+15)}{2x+y+7}$ NHDEq $\qquad \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \therefore \dfrac{4}{2} \neq \dfrac{3}{1}$

Put $\left.\begin{array}{l} x = X+h \\ y = Y+K \end{array}\right] \Rightarrow \dfrac{dy}{dx} = \dfrac{dY}{dX}$

$\therefore \dfrac{dY}{dX} = -\dfrac{(4X+4h+3Y+3K+15)}{2X+2h+Y+K+7}$

where $-4h-3K-15=0$
& $2h+K+7=0$

Add
$-4h-3K-15=0$
$4h+2K+14=0$
$-K-1=0$
$\boxed{-1=K}$

$-4h-3(-1)-15=0$
$-12=4h$
$\boxed{-3=h}$

$x = X-3$
$y = Y-1$

$\dfrac{dY}{dX} = -\dfrac{4X-3Y}{2X+Y}$ HDE (i)

Put $Y = VX$ (ii)

$\dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$ (iii)

using (ii) (iii) in (i)

$V + X\dfrac{dV}{dX} = -\dfrac{4X-3VX}{2X+VX}$

$X\dfrac{dV}{dX} = -\dfrac{\cancel{X}(4+3V)}{\cancel{X}(2+V)} - V$

$= -\dfrac{4-3V-2V-V^2}{2+V}$

$X\dfrac{dV}{dX} = -\dfrac{(V^2+5V+4)}{2+V}$

$\displaystyle\int \frac{(V+2)dV}{V^2+5V+4} = -\int\frac{dX}{X}$ separating variables.

Partial Fractions

$\dfrac{V+2}{V^2+5V+4} = \dfrac{V+2}{(V+1)(V+4)} = \dfrac{A}{V+1} + \dfrac{B}{V+4}$

$V+2 = A(V+4) + B(V+1)$

Put $V+4=0 \quad V=-4 \Rightarrow \boxed{B=\frac{2}{3}}$

$V+1=0 \quad V=-1 \Rightarrow \boxed{A=\frac{1}{3}}$

$\therefore \dfrac{V+2}{(V+1)(V+4)} = \dfrac{1}{3(V+1)} + \dfrac{2}{3(V+4)}$

By P.F

$\dfrac{1}{3}\displaystyle\int\frac{dV}{(V+1)} + \frac{2}{3}\int\frac{dV}{V+4} = -\int\frac{dX}{X}$

$\dfrac{1}{3}\ln(V+1) + \dfrac{2}{3}\ln(V+4) = -\ln X + \ln c$

$\dfrac{1}{3}\ln\left[(V+1)(V+4)^2\right] = \ln\left(\dfrac{c}{X}\right)$

$\ln\left[\left(\dfrac{Y}{X}+1\right)\left(\dfrac{Y}{X}+4\right)^2\right]^{\frac{1}{3}} = \ln\left(\dfrac{c}{X}\right)$

Antilog cube $\left(\dfrac{Y+X}{X}\right)\left(\dfrac{Y+4X}{X}\right)^2 = \dfrac{c^3}{X^3}$

$(Y+X)(Y+4X)^2 \dfrac{\cancel{X^3}}{\cancel{X^3}} = c'$

$(y+1+x+3)(y+1+4x+12) = c' \Longrightarrow (x+y+4)(4x+y+13) = c'$ Ans

(17) $(3y-7x-3)dx + (7y-3x-7)dy = 0$

$(7y-3x-7)dy = -(3y-7x-3)dx$

$\dfrac{dy}{dx} = \dfrac{-3y+7x+3}{7y-3x-7}$ NHDEq

$\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \because \dfrac{7}{-3} \neq \dfrac{-3}{7}$

Put $\left.\begin{array}{l} x = X+h \\ y = Y+K \end{array}\right\} \Rightarrow \dfrac{dy}{dx} = \dfrac{dY}{dX}$

where $-3K + 7h + 3 = 0$
& $7K - 3h - 7 = 0$

Add $\cancel{-21K} + 49h + 21 = 0$
$\cancel{21K} - 9h - 21 = 0$

$40h = 0$
$\boxed{h = 0}$

$-3K + 3 = 0$
$\boxed{K = \frac{-3}{-3} = 1}$

$x = X + 0$
$y = Y + 1$

$\therefore \dfrac{dY}{dX} = \dfrac{-3(Y+K)+7(X+h)+3}{7(Y+K)-3(X+h)-7}$

$\dfrac{dY}{dX} = \dfrac{-3Y+7X}{7Y-3X}$ HDE (i)

Put $Y = VX$ (ii)

$\dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$ (iii)

using (ii) & (iii) in (i)

$V + X\dfrac{dV}{dX} = \dfrac{-3VX+7X}{7VX-3X}$

$X\dfrac{dV}{dX} = \dfrac{\cancel{X}(-3V+7)}{\cancel{X}(7V-3)} - V$

$X\dfrac{dV}{dX} = \dfrac{-\cancel{3V}+7-7V^2+\cancel{3V}}{7V-3}$

$\displaystyle\int \frac{7V-3}{7(1-V^2)}dV = \int\frac{dX}{X}$ separating variables

Partial Fractions

$\dfrac{7V-3}{7(1-V^2)} = \dfrac{V-\frac{3}{7}}{(1-V)(1+V)} = \dfrac{A}{1-V} + \dfrac{B}{1+V}$

$V - \frac{3}{7} = A(1+V) + B(1-V)$

Put $1+V=0 \Rightarrow -1-\frac{3}{7} = 2B$ $\boxed{B = -\frac{5}{7}}$

Put $1-V=0 \Rightarrow 1-\frac{3}{7} = 2A$ $\boxed{A = \frac{2}{7}}$

By PF $\displaystyle\frac{2}{7}\int\frac{dV}{1-V} - \frac{5}{7}\int\frac{dV}{1+V} = \int\frac{dX}{X}$

$-\frac{2}{7}\ln(1-V) - \frac{5}{7}\ln(1+V) = \ln X + \ln c$

$\frac{1}{7}\ln\left[(1-V)^{-2}(1+V)^{-5}\right] = \ln cX$

$\ln\left[\left(1-\frac{Y}{X}\right)^{-2}\left(1+\frac{Y}{X}\right)^{-5}\right] = 7\ln cX$

$\ln\left[\left(\frac{X-Y}{X}\right)^{-2}\left(\frac{X+Y}{X}\right)^{-5}\right] = \ln c^7X^7$

Antilog

$\dfrac{X^2}{(X-Y)^2}\dfrac{X^5}{(X+Y)^5} = c^7X^7$

$\dfrac{\cancel{X^7}}{\cancel{X^7}c^7} = (X-Y)^2(X+Y)^5$

$\to c' = (X-Y)^2(X+Y)^5$

$\therefore c' = (x-(y-1))^2(x+y-1)^5$

$c' = (x-y+1)^2(x+y-1)^5$

(20) $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$ NHDEq $\qquad \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow \frac{1}{2} \neq \frac{-2}{1}$

Put $\left.\begin{array}{l} x = X+h \\ y = Y+k \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$

$\therefore \frac{dY}{dX} = \frac{X+h-2Y-2k+5}{2X+2h+Y+k-1}$.

where $h-2k+5=0$
& $2h+k-1=0$

Add
$2h-4k+10=0$
$\cancel{2h}+k-1=0$
$-5k+11=0$
$\boxed{k=+\frac{11}{5}}$

$h-2k+5=0$
$+4h+2k-2=0$
$5h+3=0$
$\boxed{h=\frac{-3}{5}}$

$x = X-\frac{3}{5}$ $\qquad y=Y+\frac{11}{5}$

$\frac{dY}{dX} = \frac{X-2Y}{2X+Y}$ HDE ——— (I)

Put $Y = VX$ ——— (II)

$\frac{dY}{dX} = V+X\frac{dV}{dX}$ ——— (III)

using (II) (III) in (I)

$V+X\frac{dV}{dX} = \frac{X-2VX}{2X+VX}$

$X\frac{dV}{dX} = \frac{\cancel{X}(1-2V)}{\cancel{X}(2+V)} - V$

$X\frac{dV}{dX} = \frac{1-2V-2V-V^2}{2+V}$

$\frac{2+V}{1-4V-V^2}\,dV = \frac{dX}{X}$

$\int \frac{2+V}{V^2+4V-1}\,dV = -\int \frac{dX}{X}$

$\frac{1}{2}\left[\ln(V^2+4V-1)\right] = -\ln X + \ln c$

$\ln\sqrt{V^2+4V-1} = \ln\frac{c}{X}$

Antilog $\sqrt{\frac{Y^2}{X^2}+\frac{4Y}{X}-1} = \frac{c}{X}$

$\frac{Y^2+4XY-X^2}{X^2} = \frac{c^2}{X^2}$

$Y^2+4XY-X^2 = c^2$

$\left(y-\frac{11}{5}\right)^2 + 4\left(x+\frac{3}{5}\right)\left(y-\frac{11}{5}\right) - \left(x+\frac{3}{5}\right)^2 = c^2$

Type 2 $\quad \frac{a_1}{a_2} = \frac{b_1}{b_2} \quad \because \frac{3}{3} = \frac{-4}{-4}$, $\; 1 = 1$

(18) $\frac{dy}{dx} = \frac{3x-4y-2}{3x-4y-3}$ NHDE —— (i)

Put $3x-4y = z$ —— (ii)

$3 - 4\frac{dy}{dx} = \frac{dz}{dx}$

$3 - \frac{dz}{dx} = 4\frac{dy}{dx}$

$\frac{1}{4}\left(3 - \frac{dz}{dx}\right) = \frac{dy}{dx}$ —— (iii)

using (ii) & (iii) in (i)

$\frac{1}{4}\left(3 - \frac{dz}{dx}\right) = \frac{z-2}{z-3}$

$3 - \frac{dz}{dx} = \frac{4z-8}{z-3}$

$3 - \frac{(4z-8)}{z-3} = \frac{dz}{dx}$

$\frac{3z-9-4z+8}{z-3} = \frac{dz}{dx}$

$-\left(\frac{1+z}{z-3}\right) = \frac{dz}{dx}$

$-dx = \frac{(z-3)}{1+z}dz$ separately variables

$\frac{z-3-1+1}{1+z} = -dx$

$\int \frac{(z+1)-4}{1+z}dz = -\int dx$

$\int \left(1 - \frac{4}{1+z}\right)dz = -\int dx$

$z - 4\ln(1+z) = -x + c$

$(3x-4y) - 4\ln(1+3x-4y) = -x + c$

$4x - 4y - 4\ln(1+3x-4y) = c$

$x - y - \ln(1+3x-4y) = \frac{c}{4}$

$x - y - \ln(1+3x-4y) = c'$

(19) $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$ —— (i) $\quad$ Type 2 $\frac{1}{1} = \frac{-1}{-1}$

Put $y - x = z$ —— (ii)

$\frac{dy}{dx} - 1 = \frac{dz}{dx}$

$\frac{dy}{dx} = 1 + \frac{dz}{dx}$ —— (iii)

using (ii) & (iii) in (i)

$1 + \frac{dz}{dx} = \frac{z+1}{z+5}$

$\frac{dz}{dx} = \frac{z+1}{z+5} - 1$

$= \frac{z+1-z-5}{z+5}$

$\frac{dz}{dx} = \frac{-4}{z+5}$

$\int (z+5)dz = -4\int dx$ separately variables

$\frac{z^2}{2} + 5z = -4x + c$

$\frac{z^2+10z}{2} = -4x + c$

$z^2 + 10z = -8x + 2c$

$(y-x)^2 + 10(y-x) = -8x + c'$

$(y-x)^2 + 10(y-x) + 8x = c'$

$(y-x)^2 + 10y - 10x + 8x = c'$

$(y-x)^2 + 10y - 2x = c'$

(36)

Exact Diff Eq. (EDE)

A diff eq of the form $M(x,y)dx + N(x,y)dy = 0$

is said to be an Exact diff^{l} eq if it is expressible in total diff^{l} i.e. $d[f(x,y)]$

$\left[d\{f(x,y)\} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \right.$

Condition for an Exact Diff Eq.

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

M, N have 1st order continuous partial derivatives.

$M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$

$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ $\because$ M,N have 1st order cont partial derivatives.

To Solve

1) Integrate M w.r.t 'x' keeping 'y' const

2) Add the integral w.r.t 'y' of the terms of N free from 'x'.

3) & Equate to arbitrary const.

i.e $\int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

Ex 9.4

① Solve $(3x^2+4xy)dx + (2x^2+2y)dy = 0$

$M = 3x^2+4xy$, $N = 2x^2+2y$

$\frac{\partial M}{\partial y} = 0+4x$ $\quad \frac{\partial N}{\partial x} = 4x+0$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ So given Diff Eq is Exact.

Now $\int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = C$

$\therefore \int(3x^2+4xy)\,dx + \int 2y\,dy = C$

$3\frac{x^3}{3} + \frac{4x^2y}{2} + \frac{2y^2}{2} = c$

$x^3 + 2x^2y + y^2 = c$

② $(2xy+y-\tan y)dx+(x^2-x\tan^2 y+\sec^2 y)dy=0$

$\therefore M=2xy+y-\tan y \quad , \quad N=x^2-x\tan^2 y+\sec^2 y$

$\frac{\partial M}{\partial y}=2x+1-\sec^2 y \quad , \quad \frac{\partial N}{\partial x}=2x-\tan^2 y+0$

$=2x-\tan^2 y$

$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ So given diff eq is Exact

$\int M dx+\int(\text{terms of } N \text{ free from } x)dy=c$

$\int(2xy+y-\tan y)dx+\int \sec^2 y\, dy=c$

$\frac{2x^2y}{2}+xy-x\tan y+\tan y=c$

$x^2y+xy-x\tan y+\tan y=c$

③ $\left(\frac{x+y}{y-1}\right)dx-\frac{1}{2}\left(\frac{x+1}{y-1}\right)^2dy=0$

$M=\frac{x+y}{y-1} \qquad N=-\frac{1}{2}\left(\frac{x+1}{y-1}\right)^2$

$N=-\frac{1}{2}\frac{(x^2+2x+1)}{(y-1)^2}$

$\frac{\partial M}{\partial y}=\frac{(y-1)(0+1)-(x+y)(1)}{(y-1)^2} \qquad \frac{\partial N}{\partial x}=\frac{-(2x+2)}{2(y-1)^2}$

$=\frac{\cancel{y}-1-x-\cancel{y}}{(y-1)^2} \qquad =\frac{-x-1}{(y-1)^2}$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \quad \therefore$ given diff eq is Exact.

$\int M dx+\int(\text{terms of } N \text{ free from } x)dy=c$

$\int\left(\frac{x+y}{y-1}\right)dx+\int -\frac{1}{2(y-1)^2}dy=c$

$\left(\frac{1}{y-1}\right)\int(x+y)dx+\left(-\frac{1}{2}\right)\int(y-1)^{-2}dy=c$

$\left(\frac{1}{y-1}\right)\left(\frac{x^2}{2}+xy\right)+\left(-\frac{1}{2}\right)\left(\frac{-1}{y-1}\right)=c$

$\frac{x^2+2xy}{2(y-1)}+\frac{1}{2(y-1)}=c$

$x^2+2xy+1=c'(y-1)$ Ans

④ $\frac{dy}{dx}=-\frac{(ax+hy)}{hx+by}$

$(hx+by)dy=-(ax+hy)dx$

$(ax+hy)dx+(hx+by)dy=0$

$M=ax+hy \qquad N=hx+by$

$\frac{\partial M}{\partial y}=0+h \qquad \frac{\partial N}{\partial x}=h$

$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ Hence Exact Diff Eq

$\int M dx+\int(\text{terms of } N \text{ free from } x)dy=c$

$\int(ax+hy)dx+\int by\, dy=c$

$a\frac{x^2}{2}+hxy+b\frac{y^2}{2}=c$

$ax^2+2hxy+by^2=c'$

⑤ $(1+\ln xy)dx+\left(1+\frac{x}{y}\right)dy=0$

$M=1+\ln xy \qquad N=1+\frac{x}{y}$

$\frac{\partial M}{\partial y}=0+\frac{1}{xy}\cdot x \qquad \frac{\partial N}{\partial x}=0+\frac{1}{y}$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M dx+\int(\text{terms of } N \text{ free from } x)dy=c$

$\int(1+\ln xy)dx+\int 1\cdot dy=c$

$\int dx+\int \ln xy\, dx+\int dy=c$

$x+\left[\ln xy\cdot(x)-\int\frac{1}{xy}\cdot y\cdot x\, dx\right]+y=c$

$x+x\ln xy-\int dx+y=c$

$\cancel{x}+x\ln xy-\cancel{x}+y=c$

$x\ln xy+y=c$

(6) $\dfrac{y\,dx + x\,dy}{1-x^2y^2} + x\,dx = 0$

$$\frac{y\,dx}{1-x^2y^2} + \frac{x\,dy}{1-x^2y^2} + x\,dx = 0$$

$$\left(x+\frac{y}{1-x^2y^2}\right)dx + \frac{x\,dy}{1-x^2y^2} = 0$$

$$M = x+\frac{y}{1-x^2y^2}$$

$$\frac{\partial M}{\partial y} = 0+\frac{(1-x^2y^2)\cdot 1 - y(-2x^2y)}{(1-x^2y^2)^2}$$

$$= \frac{1-x^2y^2+2x^2y^2}{(1-x^2y^2)^2} = \frac{1+x^2y^2}{(1-x^2y^2)^2}$$

$$N = \frac{x}{1-x^2y^2}$$

$$\frac{\partial N}{\partial x} = \frac{(1-x^2y^2)1 - x(-2xy^2)}{(1-x^2y^2)^2}$$

$$= \frac{1-x^2y^2+2x^2y^2}{(1-x^2y^2)^2} = \frac{1+x^2y^2}{(1-x^2y^2)^2}$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ Exact Diff Eq.

$\int M\,dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$$\int\left(x+\frac{y}{1-x^2y^2}\right)dx + \text{Nil} = c$$

$$\int x\,dx + \int \frac{y\,dx}{1-x^2y^2} = c$$

$$\frac{x^2}{2} + \int \frac{\frac{y}{y^2}}{\frac{1}{y^2}-\frac{x^2y^2}{y^2}}\,dx = c$$

$$\frac{x^2}{2} + \frac{1}{y}\int\frac{dx}{\left(\frac{1}{y}\right)^2 - x^2} = c$$

$$\frac{x^2}{2} + \frac{1}{y}\left\{\frac{1}{2\left(\frac{1}{y}\right)} \ln\left|\frac{\frac{1}{y}+x}{\frac{1}{y}-x}\right|\right\} = c$$

$$\frac{x^2}{2} + \frac{1}{2}\ln\left|\frac{1+xy}{1-xy}\right| = c$$

$$x^2 + \ln\left|\frac{1+xy}{1-xy}\right| = \bar{c} \quad \text{Ans.}$$

(7) $(6xy+2y^2-5)dx + (3x^2+4xy-6)dy$

$M = 6xy+2y^2-5$, $N = 3x^2+4xy-6$

$\dfrac{\partial M}{\partial y} = 6x+4y$ $\qquad \dfrac{\partial N}{\partial x} = 6x+4y$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M\,dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$$\int(6xy+2y^2-5)dx + \int -6\,dy = c$$

$$\frac{6x^2}{2}y + 2xy^2 - 5x - 6y = c$$

$$3x^2y + 2xy^2 - 5x - 6y = c$$

(8) $(y\sec^2 x + \sec x\tan x)dx + (\tan x + 2y)dy = 0$

$M = y\sec^2 x + \sec x \tan x$, $N = \tan x + 2y$

$\dfrac{\partial M}{\partial y} = \sec^2 x$ $\qquad \dfrac{\partial N}{\partial x} = \sec^2 x$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M\,dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$$\int(y\sec^2 x + \sec x\tan x)dx + \int 2y\,dy = c$$

$$y\tan x + \sec x + y^2 = c$$

(9) $(y\cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0$

$M = y\cos x + 2xe^y$, $N = \sin x + x^2e^y - 1$

$\dfrac{\partial M}{\partial y} = \cos x + 2xe^y$ $\qquad \dfrac{\partial N}{\partial x} = \cos x + 2xe^y$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Hence Exact diff eq.

$\int M\,dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$$\int(y\cos x + 2xe^y)dx + \int -1\,dy = c$$

$$y\sin x + \cancel{2}\frac{x^2}{\cancel{2}}e^y - y = c$$

$$y\sin x + x^2e^y - y = c$$

(10) $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + (xe^{xy}\cos 2x - 3)dy = 0$

Sol $M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x$ $\qquad N = xe^{xy}\cos 2x - 3$

$\frac{\partial M}{\partial y} = \{ye^{xy}(x) + e^{xy}\}\cos 2x - 2\sin 2x\, e^{xy}(x)$

$= e^{xy}\{yx\cos 2x + \cos 2x - 2\sin 2x(x)\}$

$\frac{\partial N}{\partial x} = 1.e^{xy}\cos 2x + xe^{xy}(y)\cos 2x + xe^{xy}(-2\sin 2x) - 0$

$= e^{xy}(\cos 2x + xy\cos 2x - x2\sin 2x)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + \int -3dy = c$

$y\int e^{xy}\cos 2x\,dx - 2\int e^{xy}\sin 2x + 2\int x\,dx - 3\int dy = c$

$y\left[\cos 2x\frac{e^{xy}}{y} - \int -\sin 2x(2)\frac{e^{xy}}{y}dx\right] - 2\int e^{xy}\sin 2x + \cancel{2}\frac{x^2}{\cancel{2}} - 3y = c$

$\cos 2x\, e^{xy} + 2\int \cancel{\sin 2x}\, e^{xy}dx - 2\int e^{xy}\cancel{\sin 2x} + x^2 - 3y = c$

$\cos 2x\, e^{xy} + x^2 - 3y = c$

(11) Solve the initial value problems.

$(2xy - 3)dx + (x^2 + 4y)dy = 0, \quad y(1) = 2$

$M = 2xy - 3 \qquad N = x^2 + 4y$

$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(2xy - 3)dx + \int 4y\,dy = c$

$\cancel{2}\frac{x^2}{\cancel{2}}y - 3x + 4\frac{y^2}{2} = c$

$x^2y - 3x + 2y^2 = c$

$\because y(1) = 2$ (given)

$\therefore 2 - 3 + 8 = c$

$\boxed{7 = c}$

Hence $x^2y - 3x + 2y^2 = 7$

$\longrightarrow$ P. Sol.

(12) $(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0$ $\quad y(0) = 2$

$M = 2x\cos y + 3x^2y \qquad N = x^3 - x^2\sin y - y$

$\frac{\partial M}{\partial y} = 2x(-\sin y) + 3x^2 \qquad \frac{\partial N}{\partial x} = 3x^2 - 2x\sin y - 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(2x\cos y + 3x^2y)dx + \int -y\,dy = c$

$2\frac{x^2}{2}\cos y + 3\frac{x^3}{3}y - \frac{y^2}{2} = c$

$x^2\cos y + x^3y - \frac{y^2}{2} = c$

$\because y(0) = 2$

$0 + 0 - \frac{4}{2} = c$

$\boxed{-2 = c}$

$x^2\cos y + x^3y - \frac{y^2}{2} = -2$

(41)

(13) $(3x^2y^2 - y^3 + 2x)dx + (2x^3y - 3xy^2 + 1)dy = 0 \quad , \; y(-2) = 1$

$M = 3x^2y^2 - y^3 + 2x \qquad N = 2x^3y - 3xy^2 + 1$

$\frac{\partial M}{\partial y} = 6x^2y - 3y^2 \qquad \frac{\partial N}{\partial x} = 6x^2y - 3y^2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq

$\int M dx + \int (\text{terms of } N \text{ free from } x) dy = c$

$\int (3x^2y^2 - y^3 + 2x)dx + \int 1\, dy = c$

$3\frac{x^3}{3}y^2 - xy^3 + 2\frac{x^2}{2} + y = c$

$x^3y^2 - xy^3 + x^2 + y = c$

$\because y(-2) = 1$

$\therefore -8 + 2 + 4 + 1 = c$

$\boxed{-1 = c}$

Hence $x^3y^2 - xy^3 + x^2 + y + 1 = 0$

(14) $\left(\frac{3-y}{x^2}\right)dx + \left(\frac{y^2 - 2x}{xy^2}\right)dy = 0$

$M = \frac{3-y}{x^2} \qquad N = \frac{y^2 - 2x}{xy^2}$

$\frac{\partial M}{\partial y} = \frac{-1}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{y^2}\left[\frac{x(0-2) - (y^2 - 2x)1}{x^2}\right]$

$= \frac{(-2x - y^2 + 2x)}{y^2x^2}$

$\frac{\partial N}{\partial x} = \frac{-y^2}{y^2x^2} = \frac{-1}{x^2}$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore$ Exact Diff Eq.

$\int M dx + \int (\text{terms of } N \text{ free from } x) dy = c$

$\int \left(\frac{3-y}{x^2}\right)dx + \int \frac{-2}{y^2} dy = c$

$(3-y)\int \frac{dx}{x^2} - 2\int \frac{dy}{y^2} = c$

$(3-y)\left(\frac{-1}{x}\right) - 2\left(-\frac{1}{y}\right) = c$

$\frac{-y(3-y) + 2x}{xy} = c$

$\rightarrow -y(3-y) + 2x = cxy$

$\because y(-1) = 2$

$\therefore -2 + 2(-1) = c(-1)(2)$

$-4 = c(-2)$

$\boxed{+2 = c}$

$\therefore -y(3-y) + 2x = 2xy$

$y^2 - 3y + 2x - 2xy = 0$

(15) $(4x^3e^{x+y} + x^4e^{x+y} + 2x)dx + (x^4e^{x+y} + 2y)dy$

$M = 4x^3e^{x+y} + x^4e^{x+y} + 2x$

$\frac{\partial M}{\partial y} = 4x^3e^{x+y}(0+1) + x^4e^{x+y}(0+1) + 0$

$\frac{\partial M}{\partial y} = (4x^3 + x^4)e^{x+y}$

$N = x^4e^{x+y} + 2y$

$\frac{\partial N}{\partial x} = 4x^3e^{x+y} + x^4e^{x+y}(1+0) + 0$

$\frac{\partial N}{\partial x} = (4x^3 + x^4)e^{x+y}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq

$\int M dx + \int (\text{terms of } N \text{ free from } x) dy = c.$

$\int (4x^3e^{x+y} + x^4e^{x+y} + 2x)dx + \int 2y\, dy = c$

$\int 4x^3e^{x+y}dx + \int \underset{I}{x^4}\underset{II}{e^{x+y}}dx + \int 2x\,dx + \int 2y\,dy = c$

$\cancel{\int 4x^3e^{x+y}dx} + \left\{x^4e^{x+y} - \cancel{\int 4x^3e^{x+y}dx}\right\} + x^2 + y^2 = c$

$x^4e^{x+y} + x^2 + y^2 = c$

$\because y(0) = 1 \implies e^{0+1}(0)^4 + 0^2 + 1^2 = c$

$\boxed{1 = c}$

$\therefore x^4e^{x+y} + x^2 + y^2 = 1$

Ans.

Non Exact Diff Eq

A diff eq of the form $Mdx+Ndy=0$ is said to be non-exact iff $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now if this diff eq is multiplied by a suitable function; then the resulting eq is Exact Diff^l Eq. This suitable fn is called Integrating Factor (I.F)

Note The number of Integrating Factors may be infinite.

Some Rules to Find Integrating Factors

If $Mdx+Ndy=0$ is not exact then find Integrating Factor using

1) $\frac{M_y-N_x}{N}=P$ then $I.F=e^{\int Pdx}$ where P is fn of x alone.

2) $\frac{N_x-M_y}{M}=Q$ then $I.F=e^{\int Qdy}$ where Q is fn of y alone

3) If $Mdx+Ndy=0$ is Homogeneous then $I.F=\frac{1}{xM+yN}$ where $xM+yN\neq 0$

4) If diff eq is the form $y.f(xy)dx+x.g(xy)dy=0$ then $I.F=\frac{1}{xM-yN}$ where $xM-yN\neq 0$

Note In some cases I.F can be found only after properly regrouping the terms of a diff eq and then recognising each group as an Exact differential of known function.

1) $xdy+ydx=d(xy)$

2) $\frac{xdy-ydx}{x^2}=d\left(\frac{y}{x}\right)$

3) $\frac{ydx-xdy}{y^2}=d\left(\frac{x}{y}\right)$

4) $xdx+ydy=d\left(\frac{x^2+y^2}{2}\right)$

5) $\frac{xdy+ydx}{xy}=d(\log(xy))$

6) $\frac{xdy-ydx}{x^2+y^2}=d\left(\tan^{-1}\frac{y}{x}\right)$

7) $\frac{ydx-xdy}{x^2+y^2}=d\left(\tan^{-1}\frac{x}{y}\right)$

8) $\frac{xdy+ydx}{x^2y^2}=d\left(-\frac{1}{xy}\right)$

(42)

Ex 9.5

Solve by finding an I.F

① $(xy^2+y)dx - x\,dy = 0$ — ①

$M = xy^2+y$ $\quad N = -x$

$M_y = 2xy+1$ $\quad N_x = -1$

$\because M_y \neq N_x \therefore$ Non Exact

$\dfrac{M_y - N_x}{N} = \dfrac{2xy+1+1}{-x}$ Not fn of x alone.

$\dfrac{N_x - M_y}{M} = \dfrac{-1-2xy-1}{xy^2+y}$

$= \dfrac{-2\cancel{(1+xy)}}{y\cancel{(xy+1)}} = -\dfrac{2}{y}$ Fn of 'y' alone

$\therefore I.F = e^{-\int \frac{2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \dfrac{1}{y^2}$

Multiply both sides of eq① by $IF = \dfrac{1}{y^2}$

$\dfrac{1}{y^2}(xy^2+y)dx - \dfrac{x}{y^2}dy = 0$

$\left(x + \dfrac{1}{y}\right)dx - \dfrac{x}{y^2}dy = 0$ — ⑪

Now $M = x + \dfrac{1}{y}$ $\quad N = -\dfrac{x}{y^2}$

$M_y = -\dfrac{1}{y^2}$ $\quad N_x = -\dfrac{1}{y^2}$

$\because M_y = N_x \therefore$ ⑪ is Exact Diff Eq

So $\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(x+\dfrac{1}{y}\right)dx + \text{Nil} = c$

$\dfrac{x^2}{2} + \dfrac{x}{y} = c$

③ $(x^2+x-y)dx + x\,dy = 0$ — ①

$M = x^2+x-y$ $\quad N = x$

$M_y = -1$ $\quad N_x = 1$

$M_y \neq N_x \therefore$ Non Exact Diff Eq

$\dfrac{M_y - N_x}{N} = \dfrac{-1-1}{x} = -\dfrac{2}{x}$ fn of x alone.

$\therefore I.F = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \dfrac{1}{x^2}$

Multiply both sides of eq① by $I.F = \dfrac{1}{x^2}$

$\dfrac{1}{x^2}(x^2+x-y)dx + \dfrac{x}{x^2}dy = 0$

$\left(1 + \dfrac{1}{x} - \dfrac{y}{x^2}\right)dx + \dfrac{1}{x}dy = 0$ — ⑪

Now $M = 1 + \dfrac{1}{x} - \dfrac{y}{x^2}$ $\quad N = \dfrac{1}{x}$

$M_y = -\dfrac{1}{x^2}$ $\quad N_x = -\dfrac{1}{x^2}$

$M_y = N_x \therefore$ ⑪ is Exact Diff Eq.

So $\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(1+\dfrac{1}{x}-\dfrac{y}{x^2}\right)dx + \text{Nil} = c$

$x + \ln x + \dfrac{y}{x} = c$

9 - [illegible]

(43)

④ $dy + \left(\dfrac{y - \sin x}{x}\right)dx = 0$ ——— ①

$M = \dfrac{y - \sin x}{x}$ $\quad N = 1$

$M_y = \dfrac{1}{x} - 0$ $\quad N_x = 0$

$M_y \neq N_x \therefore$ ① is Non Exact Diff Eq

Now $\dfrac{M_y - N_x}{N} = \dfrac{\frac{1}{x} - 0}{1} = \dfrac{1}{x}$ fn of x alone

$I.F = e^{\int \frac{1}{x}dx} = e^{\ln x} = \boxed{x}$

Multiplying both sides of eq ① by I.F $= x$

$x\,dy + \cancel{x}\left(\dfrac{y - \sin x}{\cancel{x}}\right)dx = 0$ ——— ⑪

$M = y - \sin x$ $\quad N = x$

$M_y = 1$ $\quad N_x = 1$

$M_y = N_x \therefore$ ⑪ is Exact Diff Eq.

$\int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\int (y - \sin x)\,dx = c$

$xy + \cos x = c$

⑤ $y(2xy + e^x)dx - e^x dy = 0$

$(2xy^2 + e^x y)dx - e^x dy = 0$ ——— ①

$M = 2xy^2 + e^x y$ $\quad N = -e^x$

$M_y = 4xy + e^x$ $\quad N_x = -e^x$

$M_y \neq N_x \therefore$ ① is Non Exact Diff Eq.

$\dfrac{M_y - N_x}{N} = \dfrac{4xy + e^x + e^x}{-e^x}$ Not fn of x alone

$\dfrac{N_x - M_y}{M} = \dfrac{-e^x - 4xy - e^x}{2xy^2 + ye^x} = \dfrac{-2e^x - 4xy}{y(2xy + e^x)}$

$= \dfrac{-2(\cancel{e^x + 2xy})}{y(\cancel{2xy + e^x})} = \dfrac{-2}{y}$ fn of y alone

$I.F = e^{\int -\frac{2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \boxed{\dfrac{1}{y^2}}$

Multiply both sides of ① by I.F $= \dfrac{1}{y^2}$

$\dfrac{1}{y^2}(2xy^2 + e^x y)dx - \dfrac{1}{y^2}e^x dy = 0$

$\left(2x + \dfrac{e^x}{y}\right)dx - \dfrac{e^x}{y^2}dy = 0$ ——— ⑪

$M = 2x + \dfrac{e^x}{y}$ $\quad N = -\dfrac{e^x}{y^2}$

$M_y = 0 + \left(-\dfrac{e^x}{y^2}\right)$ $\quad N_x = -\dfrac{e^x}{y^2}$

$M_y = N_x \therefore$ ⑪ is Exact Diff Eq.

$\int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\int \left(2x + \dfrac{e^x}{y}\right)dx + \text{Nil} = c$

$x^2 + \dfrac{e^x}{y} = c$ Ans.

⑥ $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$ — ①

$M = y^4 + 2y$ $\quad N = xy^3 + 2y^4 - 4x$

$M_y = 4y^3 + 2$ $\quad N_x = y^3 - 4$

$M_y \neq N_x \therefore$ ① is Non Exact Diff Eq.

$\dfrac{N_x - M_y}{M} = \dfrac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \dfrac{-3y^3 - 6}{y(y^3 + 2)} = \dfrac{-3}{y}$

$\therefore I.F = e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = e^{\ln y^{-3}} = y^{-3} = \boxed{\dfrac{1}{y^3}}$

× by $\frac{1}{y^3}$

$\dfrac{1}{y^3}(y^4 + 2y)dx + \dfrac{1}{y^3}(xy^3 + 2y^4 - 4x)dy = 0$

$\left(y + \dfrac{2}{y^2}\right)dx + \left(x + 2y - \dfrac{4x}{y^3}\right)dy = 0$ ——— ⑪

Now $M = y + \dfrac{2}{y^2}$ $\quad N = x + 2y - \dfrac{4x}{y^3}$

$M_y = 1 - \dfrac{4}{y^3}$ $\quad N_x = 1 + 0 - \dfrac{4}{y^3}$

$\because M_y = N_x \therefore$ ⑪ is Exact Diff Eq.

$\int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\int \left(y + \dfrac{2}{y^2}\right)dx + \int 2y\,dy = c$

$xy + \dfrac{2x}{y^2} + \dfrac{2y^2}{2} = c$

$xy + \dfrac{2x}{y^2} + y^2 = c$

(4)

(7) $(x^2+y^2+2x)\,dx + 2y\,dy = 0$ —(1)

$M = x^2+y^2+2x \qquad N = 2y$

$M_y = 2y \qquad N_x = 0$

$M_y \neq N_x$ ∴ (1) is Non Exact Diff Eq

$\dfrac{N_x - M_y}{M} = \dfrac{0-2y}{x^2+y^2+2x}$ Not fn of y only

$\dfrac{M_y - N_x}{N} = \dfrac{2y-0}{2y} = 1 = x^0$ fn of x only

$I.F = e^{\int 1.dx} = e^x$

Multiply both sides of eq (1) by $I.F = \boxed{e^x}$

$e^x(x^2+y^2+2x)\,dx + e^x(2y)\,dy = 0$ —(11)

$M = e^x(x^2+y^2+2x) \qquad N = e^x 2y$

$M_y = e^x 2y \qquad N_x = e^x 2y$

$M_y = N_x$ ∴ (11) is Exact Diff Eq.

∴ $\int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\int e^x(x^2+y^2+2x)\,dx + \text{Nil} = c$

$\int e^x x^2\,dx + \int e^x y^2\,dx + \int e^x 2x\,dx = c$

$x^2 e^x - \cancel{\int 2x e^x\,dx} + e^x y^2 + \cancel{\int e^x 2x\,dx} = c$

$(x^2+y^2)e^x = c$

(10) $(4x+3y^2)\,dx + 2xy\,dy = 0$ —(1)

$M = 4x+3y^2 \qquad N = 2xy$

$M_y = 0+6y \qquad N_x = 2y$

$M_y \neq N_x$ ∴ (1) is Non Exact Diff Eq

$\dfrac{N_x - M_y}{M} = \dfrac{2y-6y}{4x+3y^2}$ not fn of y alone

$\dfrac{M_y - N_x}{N} = \dfrac{6y-2y}{2xy} = \dfrac{4y}{2xy} = \dfrac{2}{x}$ fn of x alone.

i.e. Easy 2nd Method on Page (ii)

(8) $(x^2+y^2)\,dx - 2xy\,dy = 0$ —(1)

$M = x^2+y^2 \qquad N = -2xy$

$M_y = 2y \qquad N_x = -2y$

$M_y \neq N_x$ ∴ (1) is Non Exact Diff Eq

$\dfrac{N_x - M_y}{M} = \dfrac{-2y-2y}{x^2+y^2}$ Not fn of y only

$\dfrac{M_y - N_x}{N} = \dfrac{2y+2y}{-2xy} = \dfrac{4y}{-2xy} = \dfrac{-2}{x}$ fn of x only

$I.F = e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \boxed{\dfrac{1}{x^2}}$

Multiply both sides of eq (1) by $I.F = \dfrac{1}{x^2}$

$\dfrac{1}{x^2}(x^2+y^2)\,dx - \dfrac{1}{x^2}(2xy)\,dy = 0$

$\left(1+\dfrac{y^2}{x^2}\right)dx - \dfrac{2y}{x}\,dy = 0$ —(11)

$M = 1+\dfrac{y^2}{x^2} \qquad N = -\dfrac{2y}{x}$

$M_y = \dfrac{2y}{x^2} \qquad N_x = +\dfrac{2y}{x^2}$

$M_y = N_x$ ∴ (11) is Exact Diff Eq.

∴ $\int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\int\left(1+\dfrac{y^2}{x^2}\right)dx + \text{Nil} = c$

$x - \dfrac{y^2}{x} = c$ Ans.

Note (8) can be done by $I.F = \dfrac{1}{xM+yN}$ i.e. Homogeneous Method.

∴ $I.F = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Multiply both sides of (1) by $I.F = \boxed{x^2}$

$(4x^3+3y^2x^2)\,dx + (2x^3y)\,dy = 0$ —(11)

$M_y = 6yx^2 \qquad N_x = 6x^2y$

$M_y = N_x$ ∴ (11) is Exact Diff Eq.

∴ $\int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\int(4x^3+3y^2x^2)\,dx + \text{Nil} = c$

$\cancel{4}\dfrac{x^4}{\cancel{4}} + \cancel{3}y^2\dfrac{x^3}{\cancel{3}} = c$

$x^4 + y^2x^3 = c$ Ans

(45)

(12) $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$ —(1)

$M = 3x^2y^4 + 2xy \qquad N = 2x^3y^3 - x^2$

$M_y = 12x^2y^3 + 2x \qquad N_x = 6x^2y^3 - 2x$

$M_y \neq N_x$ ∴ (1) is Non Exact Diff Eq

$$\frac{M_y - N_x}{N} = \frac{12x^2y^3 + \cancel{2x} - 6x^2y^3 - \cancel{2x}}{2x^3y^3 - x^2} = \frac{6\cancel{x^2}y^3}{\cancel{x^2}(2xy^3 - 1)}$$

$$\frac{N_x - M_y}{M} = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{3x^2y^4 + 2xy} = \frac{-6x^2y^3 - 4x}{xy(3xy^3 + 2)}$$

$$= \frac{-2x\cancel{(3xy^3+2)}}{xy\cancel{(3xy^3+2)}} = -\frac{2}{y} \text{ fn of } y \text{ alone.}$$

So $I.F = e^{\int -\frac{2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \boxed{\frac{1}{y^2}}$

Multiply both sides of eq (1) by $I.F = \frac{1}{y^2}$

$$\frac{1}{y^2}(3x^2y^4 + 2xy)dx + \frac{1}{y^2}(2x^3y^3 - x^2)dy = 0$$

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0 \quad \text{—(11)}$$

$M = 3x^2y^2 + \frac{2x}{y} \qquad N = 2x^3y - \frac{x^2}{y^2}$

$M_y = 6x^2y - \frac{2x}{y^2} \qquad N_x = 6x^2y - \frac{2x}{y^2}$

$M_y = N_x$ ∴ (11) is Exact Diffl Eq.

∴ $\int M dx + \int(\text{terms of } N \text{ free from } x)dy = C$

$$\int\left(3x^2y^2 + \frac{2x}{y}\right)dx + \text{Nil} = C$$

$$\cancel{3}\frac{x^3}{\cancel{3}}y^2 + \cancel{2}\frac{x^2}{\cancel{2}y} = C$$

$$x^3y^2 + \frac{x^2}{y} = C \quad \text{Ans}$$

(14) $\frac{dy}{dx} = e^{2x} + y - 1$

$dy = (e^{2x} + y - 1)dx$

$(e^{2x} + y - 1)dx - dy = 0$ —(1)

$M = e^{2x} + y - 1 \qquad N = -1$

$M_y = 1 \qquad N_x = 0$

$M_y \neq N_x$ ∴ (1) is Non Exact Diff Eq

$\frac{N_x - M_y}{M} = \frac{0-1}{e^{2x}+y-1}$ not fn of y alone

$\frac{M_y - N_x}{N} = \frac{1-0}{-1} = -1 = -x^0$ fn of x alone.

∴ $I.F = e^{\int -1 dx} = \boxed{e^{-x}}$

Multiply both sides of eq (1) by e^{-x}

$e^{-x}(e^{2x} + y - 1)dx - e^{-x}dy = 0$

$(e^{x} + e^{-x}y - e^{-x})dx - e^{-x}dy = 0$ —(11)

$M = e^{x} + e^{-x}y - e^{-x} \qquad N = -e^{-x}$

$M_y = e^{-x} \qquad N_x = +e^{-x}$

$M_y = N_x$ ∴ (11) is Exact Diffl Eq

So $\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(e^{x} + e^{-x}y - e^{-x})dx + \text{Nil} = c$

$e^{x} - e^{-x}y + e^{-x} = c$

2nd Method of Q15 see on Page 46 & 3rd Method on 49

Q15 $(y^2 + xy)dx - x^2dy = 0$

$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}$ HDE —(1)

Put $y = vx$ —(ii)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —(iii)

$v + x\frac{dv}{dx} = \frac{v^2x^2 + xvx}{x^2}$

$v + x\frac{dv}{dx} = \frac{x^2(v^2+v)}{x^2}$

$x\frac{dv}{dx} = v^2 + \cancel{v} - \cancel{v}$

$\frac{dv}{v^2} = \frac{dx}{x}$ separable

$\int \frac{dv}{v^2} = \int \frac{dx}{x}$

$-\frac{1}{v} = \ln x + c$

$0 = \ln x + \frac{1}{v} + c$

$= \ln x + \frac{x}{y} + c$ ∵ $y = vx$, $v = \frac{y}{x}$

(46)

Easy Method on Page ...

(15) $(y^2+xy)dx - x^2dy = 0$ —— ①

$M = y^2+xy \qquad N = -x^2$

$M_y = 2y+x \qquad N_x = -2x$

$M_y \neq N_x \quad \therefore$ ① is Non Exact Diffl Eq

$\dfrac{M_y - N_x}{N} = \dfrac{2y+x+2x}{-x^2}$ Not fn of x only

$\dfrac{N_x - M_y}{M} = \dfrac{-2x-2y-x}{y^2+xy}$ Not fn of y only

① is Homogeneous diff eq of degree 2.

$\therefore xM + yN = xy^2 + \cancel{x^2y} + (-\cancel{yx^2})$

$I.F = \dfrac{1}{xM+yN} = \boxed{\dfrac{1}{xy^2}}$

Multiply both sides of ① by $I.F = \dfrac{1}{xy^2}$

$\dfrac{1}{xy^2}(y^2+xy)dx - \dfrac{1}{xy^2}x^2dy = 0$

$\left(\dfrac{1}{x} + \dfrac{1}{y}\right)dx - \dfrac{x}{y^2}dy = 0$ —— ⑪

$M = \dfrac{1}{x} + \dfrac{1}{y} \qquad N = -\dfrac{x}{y^2}$

$M_y = -\dfrac{1}{y^2} \qquad N_x = -\dfrac{1}{y^2}$

$M_y = N_x \quad \therefore$ ⑪ is Exact Diffl Eq

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(\dfrac{1}{x} + \dfrac{1}{y}\right)dx + \text{Nil} = c$

$\ln x + \dfrac{x}{y} = c$ Ans.

(16) $(3xy+y^2)dx + (x^2+xy)dy = 0$ —— ①

$M = 3xy+y^2 \qquad N = x^2+xy$

$M_y = 3x+2y \qquad N_x = 2x+y$

$M_y \neq N_x \quad \therefore$ ① is Non Exact Diffl Eq

$\dfrac{N_x - M_y}{M} = \dfrac{2x+y-3x+2y}{3xy+y^2} = -\dfrac{x+3y}{y(3x+y)}$ Not fn of y only

$\dfrac{M_y - N_x}{N} = \dfrac{3x+2y-2x-y}{x^2+xy} = \dfrac{(x+y)}{x(x+y)} = \dfrac{1}{x}$ fn of x only

$I.F = e^{\int \frac{1}{x}dx} = e^{\ln x} = \boxed{x}$

Multiply both sides of eq ① by $I.F = x$

$(3x^2y + xy^2)dx + (x^3+x^2y)dy = 0$ —— ⑪

$M = 3x^2y+xy^2 \qquad N = x^3+x^2y$

$M_y = 3x^2+2xy \qquad N_x = 3x^2+2xy$

$M_y = N_x \quad \therefore$ ⑪ is Exact Diffl Eq.

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(3x^2y+xy^2)dx + \text{Nil} = c$

$3\dfrac{x^3}{3}y + \dfrac{x^2y^2}{2} = c$

$x^3y + \dfrac{x^2y^2}{2} = c$ Ans.

(23) $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$

$y - x = \dfrac{dy}{dx}(x+y)$

$\dfrac{dy}{dx} = \dfrac{y-x}{x+y}$ H.D.Eq —— ①

$y = vx$ —— ⑪

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ —— ⑪⑪

$v + x\dfrac{dv}{dx} = \dfrac{vx-x}{x+vx}$

$x\dfrac{dv}{dx} = \dfrac{\cancel{x}(v-1)}{\cancel{x}(1+v)} - v$

$x\dfrac{dv}{dx} = \dfrac{v-1-v(1+v)}{1+v}$

$x\dfrac{dv}{dx} = \dfrac{\cancel{v}-1-\cancel{v}-v^2}{1+v}$

$\int\dfrac{(1+v)}{1+v^2}dv = -\int\dfrac{dx}{x}$

$\int\dfrac{dv}{1+v^2} + \int\dfrac{v\,dv}{1+v^2} = -\int\dfrac{dx}{x}$

$\tan^{-1}v + \dfrac{1}{2}\int\dfrac{2v\,dv}{1+v^2} = -\ln x$

$\tan^{-1}v + \dfrac{1}{2}\ln(1+v^2) + \ln x = c$

$\tan^{-1}v + \ln(1+v^2)^{\frac{1}{2}} + \ln x = c$

$\tan^{-1}\dfrac{y}{x} + \ln\left(1+\dfrac{y^2}{x^2}\right)^{\frac{1}{2}} + \ln x = c$

$\tan^{-1}\dfrac{y}{x} + \ln\left(\dfrac{\sqrt{x^2+y^2}}{\cancel{x}}\right)(\cancel{x}) = c$

$\tan^{-1}\dfrac{y}{x} + \ln\sqrt{x^2+y^2} = c$ Ans

(47)

7) $(3x^2y + 2xy + y^3)dx + (x^2+y^2)dy = 0$ — ①

$M = 3x^2y + 2xy + y^3$ $\quad N = x^2+y^2$

$M_y = 3x^2 + 2x + 3y^2$ $\quad N_x = 2x$

$M_y \neq N_x \quad \therefore$ ① is Non Exact Diff Eq.

$\frac{M_y - N_x}{N} = \frac{3x^2 + \cancel{2x} + 3y^2 - \cancel{2x}}{x^2+y^2}$

$= \frac{3(x^2+y^2)}{(x^2+y^2)} = 3 = 3x^0$ fn of x only.

$I.F = e^{\int 3\,dx} = \boxed{e^{3x}}$

Multiply both sides of eq ① by $I.F = e^{3x}$

$(3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x})dx + (e^{3x}x^2 + e^{3x}y^2)dy = 0$ — ⑪

$M = 3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x}$ $\quad N = e^{3x}x^2 + e^{3x}y^2$

$M_y = 3x^2e^{3x} + 2xe^{3x} + 3y^2e^{3x}$ $\quad N_x = 3e^{3x}x^2 + e^{3x}2x + e^{3x}3y^2$

$M_y = N_x \quad \therefore$ ⑪ is Exact Diffl Eq.

$\int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\Rightarrow \int(3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x})dx + \text{Nil} = c$

$\Rightarrow 3y\int x^2e^{3x}dx + 2y\int xe^{3x}dx + y^3\int e^{3x}dx = c$

$\Rightarrow 3y\left[x^2\frac{e^{3x}}{3} - \int 2x\frac{e^{3x}}{3}dx\right] + 2y\int xe^{3x} + y^3\frac{e^{3x}}{3} = c$

$\Rightarrow x^2ye^{3x} - \cancel{2y\int xe^{3x}dx} + \cancel{2y\int xe^{3x}} + y^3\frac{e^{3x}}{3} = c$

$\Rightarrow x^2ye^{3x} + y^3\frac{e^{3x}}{3} = c$

(18) $y\,dx + (2xy - e^{-2y})dy = 0$ — ①

$M = y$ $\quad N = 2xy - e^{-2y}$

$M_y = 1$ $\quad N_x = 2y$

$M_y \neq N_x$ ① is Non Exact Diff Eq

$\frac{M_y - N_x}{N} = \frac{1-2y}{2xy - e^{-2y}}$ Not fn of x alone

$\frac{N_x - M_y}{M} = \frac{2y-1}{y} = 2 - \frac{1}{y}$ fn of y alone

$I.F = e^{\int(2-\frac{1}{y})dy} = e^{2y - \ln y}$

$= e^{2y + \ln y^{-1}} = e^{2y}e^{\ln(\frac{1}{y})} = \boxed{e^{2y}\cdot\frac{1}{y}}$

Multiply ① by $I.F = e^{2y}\cdot\frac{1}{y}$

$e^{2y}\frac{1}{y}y\,dx + e^{2y}\cdot\frac{1}{y}(2xy - e^{-2y})dy = 0$

$e^{2y}dx + (e^{2y}2x - \frac{1}{y})dy = 0$ — ⑪

$M = e^{2y}$ $\quad N = e^{2y}2x - \frac{1}{y}$

$M_y = 2e^{2y}$ $\quad N_x = 2e^{2y} - 0$

$M_y = N_x \quad \therefore$ ⑪ is Exact Diffl Eq

$\therefore \int M\,dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\Rightarrow \int e^{2y}dx + \int -\frac{1}{y}dy = c$

$\Rightarrow xe^{2y} - \ln y = c$

⑧ $(x^2+y^2)dx - 2xy\,dy = 0$

$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ — (i)

Put $y = vx$ — (ii)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ — (iii)

Put (ii) & (iii) in (i)

$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2xvx}$

$v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{x^2\,2v}$

$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$

$= \frac{1+v^2-2v^2}{2v}$

$x\frac{dv}{dx} = \frac{1-v^2}{2v}$

$\int\frac{2v}{1-v^2}dv = \int\frac{dx}{x}$

$-\int\frac{-2v}{1-v^2}dv = \int\frac{dx}{x}$

$\ln(1-v^2) = -\ln x + \ln c$

$\ln(1-v^2) = \ln\left(\frac{c}{x}\right)$

$1 - v^2 = \frac{c}{x}$

$1 - \frac{y^2}{x^2} = \frac{c}{x}$

$\frac{x^2-y^2}{x^2} = \frac{c}{x}$

$x^2 - y^2 = \frac{c}{x}x^2$

$x^2 - y^2 = cx$

(48)

(19) $e^x dx + (e^x \cot y + 2y \operatorname{cosec} y)dy = 0$ —(1)

$M = e^x \qquad N = e^x \cot y + 2y \operatorname{cosec} y$

$M_y = 0 \qquad N_x = e^x \cot y$

$M_y \neq N_x \therefore$ (1) is Non Exact Diff Eq

$\dfrac{M_y - N_x}{N} = \dfrac{0 - e^x \cot y}{e^x \cot y + 2y \operatorname{cosec} y}$ Not fn of x alone

$\dfrac{N_x - M_y}{M} = \dfrac{e^x \cot y - 0}{e^x} = \cot y$ fn of y alone

$\therefore I.F = e^{\int \cot y\, dy} = e^{\ln \sin y} = \boxed{\sin y}$

Multiply both sides of (1) by $I.F = \sin y$

$\sin y\, e^x dx + (\sin y\, e^x \cot y + 2y \sin y \operatorname{cosec} y) dy = 0$

$\sin y\, e^x dx + (e^x \cos y + 2y)dy = 0$ (11)

$M = \sin y\, e^x \qquad N = e^x \cos y + 2y$

$M_y = \cos y\, e^x \qquad N_x = e^x \cos y + 0$

$M_y = N_x \therefore$ (11) is Exact Diff Eq

$\therefore \int M dx + \int (\text{terms of } N \text{ free from } x) dy = c$

$\int e^x \sin y\, dx + \int 2y\, dy = c$

$e^x \sin y + \dfrac{2 \cdot y^2}{2} = c$

$e^x \sin y + y^2 = c$

(20) $(x+2)\sin y\, dx + x \cos y\, dy = 0$ (1)

$M = (x+2)\sin y \qquad N = x \cos y$

$M_y = (x+2)\cos y \qquad N_x = \cos y$

$M_y \neq N_x \therefore$ (1) is Non Exact

$\dfrac{N_x - M_y}{M} = \dfrac{\cos y - (x+2)\cos y}{(x+2)\sin y}$ Not fn of y

$\dfrac{M_y - N_x}{N} = \dfrac{(x+2)\cos y - \cos y}{x \cos y}$

$= \dfrac{((x+2) - 1)\cancel{\cos y}}{x \cancel{\cos y}} = \dfrac{x+1}{x}$

$\dfrac{M_y - N_x}{N} = 1 + \dfrac{1}{x}$ fn of x alone

$I.F = e^{\int (1+\frac{1}{x}) dx} = e^{x + \ln x} = e^x e^{\ln x}$

$= \boxed{e^x x}$

Multiply by $x e^x$ on both sides of (1)

$x e^x (x+2)\sin y\, dx + x e^x x \cos y\, dy = 0$ (11)

$M = x e^x (x+2)\sin y \quad , N = x^2 e^x \cos y$

$M = (x^2 e^x + 2x e^x)\sin y , N_x = (2x e^x + x^2 e^x)\cos y$

$M_y = (x^2 e^x + 2x e^x)\cos y$

$M_y = N_x \therefore$ (11) is Exact Diff Eq.

$\therefore \int M dx + \int (\text{terms of } N \text{ free from } x) dy = c$

$\int (\underset{I}{x^2} \underset{II}{e^x} \sin y + 2x e^x \sin y) dx + \text{Nil} = c$

$\int \underset{I}{x^2} \underset{II}{e^x} \sin y\, dx + \int 2x e^x \sin y\, dx = c$

$x^2 e^x \sin y - \cancel{\int 2x e^x \sin y\, dx} + \cancel{\int 2x e^x \sin y\, dx} = c$

$x^2 e^x \sin y = c$

Easy Method on Page 46.

3) $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$

$x - y + (x+y)\frac{dy}{dx} = 0$

$(x-y)dx + (x+y)dy = 0$ —— (1)

$M = x - y \qquad N = x+y$

$M_y = 0 - 1 \qquad N_x = 1 + 0$

$\frac{M_y - N_x}{N} = \frac{-1-1}{x+y}$ Not fn of x alone

$\frac{N_x - M_y}{M} = \frac{1+1}{x-y}$ Not fn of y alone

∴ (1) is Homogeneous So $I.F = \frac{1}{xM + yN}$

$I.F = \frac{1}{x(x-y) + y(x+y)} = \frac{1}{x^2 - \cancel{xy} + \cancel{xy} + y^2}$

$= \frac{1}{x^2+y^2}$

Multiply (1) by $I.F = \frac{1}{x^2+y^2}$

$\frac{(x-y)}{x^2+y^2}dx + \frac{(x+y)}{x^2+y^2}dy$ —— (11)

$M_y = \frac{(x^2+y^2)(-1) - (x-y)(2y)}{(x^2+y^2)^2}, \quad N_x = \frac{(x^2+y^2).1 - (x+y)2x}{(x^2+y^2)^2}$

$M_y = \frac{-[x^2+y^2 - 2xy + 2y^2]}{(x^2+y^2)^2}, \quad N_x = \frac{x^2+y^2 - 2x^2 - 2xy}{(x^2+y^2)^2}$

$M_y = \frac{+y^2 - x^2 - 2xy}{(x^2+y^2)^2} \qquad N_x = \frac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}$

$M_y = N_x$ ∴ (11) is Exact Diffl Eq.

∴ $\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(\frac{x-y}{x^2+y^2}\right)dx + \text{Nil} = c$

$\int \frac{x\,dx}{x^2+y^2} - \int \frac{y\,dx}{x^2+y^2} = c$

$\frac{1}{2}\int \frac{2x\,dx}{x^2+y^2} - \int \frac{d}{dx}\tan^{-1}\frac{x}{y} = c$ $\quad \left(\because \frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2}\right.$

$\frac{1}{2}\ln(x^2+y^2) - \tan^{-1}\frac{x}{y} = c$ $\quad \left.\therefore \int \frac{a\,dx}{a^2+x^2} = \tan^{-1}\frac{x}{a}\right)$

(49) (16) $(3xy + y^2)dx + (x^2 + xy)dy = 0$

$\frac{dy}{dx} = -\frac{(3xy + y^2)}{x^2 + xy}$ —— (1)

Put $y = vx$ —— (11)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— (111)

Put (11) & (111) in (1)

$v + x\frac{dv}{dx} = -\frac{(3x^2v + v^2x^2)}{x^2 + x^2v}$

$x\frac{dv}{dx} = -\frac{\cancel{x^2}(3v + v^2)}{\cancel{x^2}(1+v)} - v$

$= \frac{-3v - v^2 - v(1+v)}{1+v}$ LCM

$= \frac{-3v - v^2 - v - v^2}{1+v}$

$x\frac{dv}{dx} = -\frac{4v - 2v^2}{1+v}$

$x\frac{dv}{dx} = -2\frac{(2v + v^2)}{1+v}$

$\int \frac{1+v}{2v+v^2}dv = -2\int\frac{dx}{x}$ separating Variables

$\frac{1}{2}\int\frac{(2+2v)}{2v+v^2}dv = -2\int\frac{dx}{x}$

$\frac{1}{2}\ln(2v + v^2) = -2\ln x + \ln c$

$\ln(2v+v^2)^{\frac{1}{2}} = \ln x^{-2} + \ln c$

$\ln\sqrt{2v+v^2} = \ln(c x^{-2})$

$\sqrt{2v+v^2} = \frac{c}{x^2}$

Sqring $2v + v^2 = \frac{c^2}{x^4}$

$v(2+v) = \frac{c^2}{x^4}$

$\frac{y}{x}\left(2 + \frac{y}{x}\right) = \frac{c^2}{x^4}$

$\frac{y}{x}\left(\frac{2x+y}{x}\right) = \frac{c^2}{x^4}$

$x^4\frac{y(2x+y)}{x^2} = c^2$

$x^2(2xy + y^2) = c^2$

$2x^3y + x^2y^2 = c^2$ Ans

(49)

Easy Method and Lengthy

(3) $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$

$x - y + (x+y)\dfrac{dy}{dx} = 0$

$(x-y)dx + (x+y)dy = 0$ —— (I)

$M = x - y \qquad N = x + y$

$M_y = 0 - 1 \qquad N_x = 1 + 0$

$\dfrac{M_y - N_x}{N} = \dfrac{-1-1}{x+y}$ Not fn of x alone

$\dfrac{N_x - M_y}{M} = \dfrac{1+1}{x-y}$ Not fn of y alone

$\because$ (I) is Homogeneous So $I.F = \dfrac{1}{xM + yN}$

$I.F = \dfrac{1}{x(x-y)+y(x+y)} = \dfrac{1}{x^2 - \cancel{xy} + \cancel{xy} + y^2}$
$= \dfrac{1}{x^2+y^2}$

Multiply (I) by $I.F = \dfrac{1}{x^2+y^2}$

$\dfrac{(x-y)}{x^2+y^2}dx + \dfrac{(x+y)}{x^2+y^2}dy$ —— (II)

$M_y = \dfrac{(x^2+y^2)(-1) - (x-y)(2y)}{(x^2+y^2)^2}, \quad N_x = \dfrac{(x^2+y^2).1 - (x+y)2x}{(x^2+y^2)^2}$

$M_y = -\dfrac{[x^2+y^2 - 2xy + 2y^2]}{(x^2+y^2)^2}, \quad N_x = \dfrac{x^2+y^2 - 2x^2 - 2xy}{(x^2+y^2)^2}$

$M_y = +\dfrac{y^2 - x^2 - 2xy}{(x^2+y^2)^2} \qquad N_x = \dfrac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}$

$M_y = N_x \therefore$ (II) is Exact Diff Eq.

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(\dfrac{x-y}{x^2+y^2}\right)dx + Nil = c$

$\int \dfrac{x\,dx}{x^2+y^2} - \int \dfrac{y\,dx}{x^2+y^2} = c$

$\dfrac{1}{2}\int \dfrac{2x\,dx}{x^2+y^2} - \int \dfrac{d}{dx}\tan^{-1}\dfrac{x}{y} = c \qquad \left(\because \dfrac{d}{dx}\tan^{-1}\dfrac{x}{a} = \dfrac{a}{a^2+x^2}\right)$

$\dfrac{1}{2}\ln(x^2+y^2) - \tan^{-1}\dfrac{x}{y} = c$

(15) $(y^2 + xy)dx - x^2 dy = 0$ —— (I)

$M = y^2 + xy \qquad N = -x^2$

$M_y = 2y + x \qquad N_x = -2x$

$M_y \neq N_x \therefore$ (I) is not Exact Diff Eq.

$\dfrac{M_y - N_x}{N} = \dfrac{2y + x + 2x}{-x^2}$ Not fn of x alone

$\dfrac{N_x - M_y}{M} = \dfrac{-2x - 2y - x}{y^2 + xy}$ Not fn of y alone.

(I) is Homogeneous diff eq of degree (2)

$\therefore I.F = \dfrac{1}{xM + yN} = \dfrac{1}{xy^2 + \cancel{x^2y} + y(\cancel{-x^2})} = \boxed{\dfrac{1}{xy^2}}$

Multiply both sides of (I) by $I.F = \dfrac{1}{xy^2}$

$\dfrac{1}{xy^2}(y^2 + xy)dx - \dfrac{1}{xy^2}x^2 dy = 0$

$\left(\dfrac{1}{x} + \dfrac{1}{y}\right)dx - \dfrac{x}{y^2}dy = 0$ —— (II)

$M = \dfrac{1}{x} + \dfrac{1}{y} \qquad N = -\dfrac{x}{y^2}$

$M_y = -\dfrac{1}{y^2} \qquad N_x = -\dfrac{1}{y^2}$

$\because M_y = N_x \therefore$ (II) is Exact Diff Eq.

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(\dfrac{1}{x} + \dfrac{1}{y}\right)dx + Nil = c$

$\ln x + \dfrac{x}{y} = c$

(50)

Q9 $(3y+4xy^2)dx+(2x+3x^2y)dy=0$ ——(i)

$M_y=3+8xy \qquad N_x=2+6xy$

$M_y\neq N_x$ ∴ (i) is Non Exact.

$$\frac{M_y-N_x}{N}=\frac{3+8xy-2-6xy}{2x+3x^2y}=\frac{1+2xy}{x(2+3xy)}$$

$$\frac{N_x-M_y}{M}=\frac{2+6xy-3-8xy}{3y+4xy^2}=\frac{-1-2xy}{y(3+4xy)}$$

Eq (i) is not Homogeneous

Eq (i) is of the form $y f(xy)dx+x g(xy)dy=0$

∴ (i) is $y(3+4xy)dx+x(2+3xy)dy=0$

$$\text{So I.F}=\frac{1}{xM-yN}=\frac{1}{3xy+4x^2y^2-2xy-3x^2y^2}=\frac{1}{xy+x^2y^2}$$

Multiply both sides of eq (i) by I.F $=\frac{1}{xy+x^2y^2}$

$$\therefore \frac{(3y+4xy^2)}{(xy+x^2y^2)}dx+\frac{(2x+3x^2y)}{(xy+x^2y^2)}dy=0$$

$$\frac{\cancel{y}}{\cancel{y}}\left[\frac{3+4xy}{x+x^2y}\right]dx+\frac{\cancel{x}}{\cancel{x}}\left[\frac{2+3xy}{y+xy^2}\right]dy=0 \quad\text{——(ii)}$$

$$M_y=\frac{(x+x^2y)(4x)-(3+4xy)(x^2)}{(x+x^2y)^2}$$

$$=\frac{4x^2+\cancel{4x^3y}-3x^2-\cancel{4x^3y}}{(x+x^2y)^2}=\frac{\cancel{x^2}}{\cancel{x^2}(1+xy)^2}$$

$$=\frac{1}{(1+xy)^2}$$

$$N_x=\frac{(y+xy^2)(3y)-(2+3xy)(y^2)}{(y+xy^2)^2}$$

$$=\frac{3y^2+\cancel{3xy^3}-2y^2-\cancel{3xy^3}}{(y+xy^2)^2}=\frac{\cancel{y^2}}{\cancel{y^2}(1+xy)^2}$$

$$=\frac{1}{(1+xy)^2}$$

$M_y=N_x$ ∴ (ii) is Exact Diff. Eq.

See above

$$\therefore \int M dx+\int(\text{terms of } N \text{ free from } x)dy=c$$

$$\int\left(\frac{3+4xy}{x+x^2y}\right)dx+\int\frac{2}{y}dy=c$$

$$\int\frac{3+3xy+xy}{x(1+xy)}dx+\int\frac{2}{y}dy=c$$

$$3\int\frac{\cancel{(1+xy)}}{x\cancel{(1+xy)}}dx+\int\frac{\cancel{x}y}{\cancel{x}(1+xy)}dx+\int\frac{2}{y}dy=c$$

$$3\int\frac{dx}{x}+\int\frac{y}{1+xy}dx+\int\frac{2}{y}dy=c$$

$$3\ln x+\ln(1+xy)+2\ln y=c$$

$$3\ln x+\ln(1+xy)+\ln y^2=c\ln e$$

$$\ln x^3(1+xy)y^2=\ln e^c$$

$$\ln x^3(1+xy)y^2=\ln e^c$$

Antilog $x^3(1+xy)y^2=e^c$

$$x^3(1+xy)y^2=c'$$

$$\because N=\frac{2+3xy}{y(1+xy)}$$

$$=\frac{2+2xy+xy}{y(1+xy)}$$

$$=\frac{2\cancel{(1+xy)}}{y\cancel{(1+xy)}}+\frac{x\cancel{y}}{\cancel{y}(1+xy)}$$

$$=\frac{2}{y}+\frac{x}{1+xy}$$

$\frac{2}{y}$ free from x.

51

(11) $(y-xy^2)\,dx+(x+x^2y^2)\,dy=0$

$M=y-xy^2 \qquad N=x+x^2y^2$

$M_y=1-2xy \qquad N_x=1+2xy^2$

$\frac{M_y-N_x}{N}=\frac{1-2xy-1-2xy^2}{x+x^2y^2}=\frac{-2x(y+y^2)}{x(1+xy^2)}$ Not fn of x

$\frac{N_x-M_y}{M}=\frac{1+2xy^2-1+2xy}{y-xy^2}=\frac{2xy(y+1)}{y(1-xy)}$ Not fn of y

Rearranging $y\,dx-xy^2dx+x\,dy+x^2y^2dy=0$

$y\,dx+x\,dy-xy^2dx+x^2y^2dy=0$

x & ÷ by x $\quad y\,dx+x\,dy-x^2y^2\left(\frac{dx}{x}\right)+x^2y^2dy=0$

$y\,dx+x\,dy-x^2y^2\left(\frac{dx}{x}-dy\right)=0$

÷ by x^2y^2 on both sides $\quad \frac{y\,dx+x\,dy}{x^2y^2}-\frac{\cancel{x^2y^2}}{\cancel{x^2y^2}}\left(\frac{dx}{x}-dy\right)=0$

$d\left(-\frac{1}{xy}\right)-\frac{dx}{x}+dy=0$

Integrating $\quad -\frac{1}{xy}-\ln|x|+y=c$

(2) $x\,dy-y\,dx=(x^2+y^2)\,dx$ —— (1)

$(x^2+y^2+y)\,dx-x\,dy=0$

$M_y=2y+1 \qquad N_x=-1$

$\because M_y\neq N_x$ Hence (1) is Non Exact

$\frac{N_x-M_y}{M}=\frac{-1-2y-1}{x^2+y^2+y}$ Not fn of y

$\frac{M_y-N_x}{N}=\frac{2y+1+1}{-x}$ Not fn of x

from (1) $\quad x\,dy-y\,dx=(x^2+y^2)\,dx$

$\int\frac{x\,dy-y\,dx}{x^2+y^2}=\int dx$

$\tan^{-1}\left(\frac{y}{x}\right)=x+c$

$\left(\frac{y}{x}\right)=\tan(x+c)$

$y=x\tan(x+c)$ Ans

Linear Diff Eqs

A diff eq of the form $\boxed{\frac{dy}{dx} + P_{(x)}y = Q_{(x)}}$ —— ① (P & Q are fn of x only) is called a linear diff eq, because it is linear in 'y' and $\frac{dy}{dx}$.

To Solve: Multiply both sides of eq ① by I.F $e^{\int P dx}$ then L.H.S of ① becomes exact diff of y & $e^{\int P dx}$ i.e $d(y e^{\int P dx})$ and then Integrating both sides

∴ Solution is given by $\boxed{\int d(y \times I.F) = \int Q \times I.F\, dx + c}$

Similarly

A diff eq of the form $\boxed{\frac{dx}{dy} + P_{(y)}x = Q_{(y)}}$ —— ⑪ (P, Q are fns of y only) is called a linear diff eq, because it is linear in 'x' & $\frac{dx}{dy}$.

To solve: Multiply both sides of eq ⑪ by IF $e^{\int P dy}$, then LHS of ⑪ becomes exact diff of x & $e^{\int P dy}$ i.e $d(x e^{\int P dy})$ and then Integrating both sides.

∴ solution is given by $\boxed{\int d(x \times I.F) = \int Q \times I.F\, dy + c}$

Ex 9.6

① $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$ LDE in y

$I.F = e^{\int P dx} = e^{\int \frac{2x+1}{x} dx} = e^{\int (2+\frac{1}{x}) dx}$

$= e^{2x + \ln x} = e^{2x} e^{\ln x} = e^{2x} x$

∴ Sol is given by $\int d(y \times I.F) = \int Q \times I.F dx + c$

$\Rightarrow \int d(y e^{2x} x) = \int e^{-2x} \cdot e^{2x} x\, dx + c$

$\Rightarrow y e^{2x} x = \int x\, dx + c$

$\Rightarrow x y e^{2x} = \frac{x^2}{2} + c$

② $\frac{dy}{dx} + \frac{3}{x} y = 6x^2$

$I.F = e^{\int P dx} = e^{\int \frac{3}{x} dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

∴ Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y x^3) = \int 6x^2 \cdot x^3\, dx + c$

$\Rightarrow y x^3 = \int 6x^5\, dx + c$

$\Rightarrow y x^3 = 6\frac{x^6}{6} + c$

$\Rightarrow x^3 y = x^6 + c$

(3) $\dfrac{dy}{dx} + \dfrac{y}{x\ln x} = \dfrac{3x^2}{\ln x}$ (LDE in y)

$I.F = e^{\int P dx} = e^{\int \frac{1}{x\ln x}dx} = e^{\int \frac{dx/x}{\ln x}}$

$I.F = e^{\ln(\ln x)} = \boxed{\ln x}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y\ln x) = \int \dfrac{3x^2}{\cancel{\ln x}} \cancel{\ln x}\, dx + c$

$y\ln x = \cancel{3}\dfrac{x^3}{\cancel{3}} + c$

$\Rightarrow y = \dfrac{x^3+c}{\ln x}$

(4) $\dfrac{dy}{dx} + 3y = 3x^2 e^{-3x}$ (LDE in y)

$I.F = e^{\int P dx} = e^{\int 3dx} = \boxed{e^{3x}}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y e^{3x}) = \int 3x^2 \cancel{e^{-3x}}\, \cancel{e^{3x}}\, dx + c$

$y e^{3x} = x^3 + c$

$y = e^{-3x}(x^3 + c)$

(7) $(x+1)\dfrac{dy}{dx} - ny = e^x (x+1)^{n+1}$

$\dfrac{dy}{dx} - \dfrac{n}{(x+1)}y = e^x (x+1)^n$ (LDE in y)

$I.F = e^{\int P dx} = e^{-\int \frac{n}{x+1}dx} = e^{-n\ln(x+1)} = e^{\ln(x+1)^{-n}}$

$I.F = (x+1)^{-n} = \boxed{\dfrac{1}{(x+1)^n}}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d\left(y \cdot \dfrac{1}{(x+1)^n}\right) = \int e^x \cancel{(x+1)^n} \cdot \dfrac{1}{\cancel{(x+1)^n}}\, dx + c$

$\dfrac{y}{(x+1)^n} = e^x + c$

$y = (e^x + c)(x+1)^n$ Ans.

(5) $\cos^3 x \dfrac{dy}{dx} + y\cos x = \sin x$

$\dfrac{dy}{dx} + \dfrac{y\cos x}{\cos^3 x} = \dfrac{\sin x}{\cos^3 x}$

$\dfrac{dy}{dx} + \sec^2 x\, y = \sec^2 x \tan x$ (LDE in y)

$I.F = e^{\int P dx} = e^{\int \sec^2 x\, dx} = \boxed{e^{\tan x}}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y \times e^{\tan x}) = \int \sec^2 x \tan x\, e^{\tan x}\, dx + c$ $\quad$ $\tan x = t$, $\sec^2 x\, dx = dt$

$\Rightarrow y e^{\tan x} = \int \underset{II}{e^t}\, \underset{I}{t}\, dt + c$

$= t e^t - \int 1 \cdot e^t dt + c$

$= t e^t - e^t + c$

$y e^{\tan x} = e^t (t-1) + c$

$y e^{\tan x} = e^{\tan x}(\tan x - 1) + c$

$y = (\tan x - 1) + c e^{-\tan x}$

(6) $x\dfrac{dy}{dx} + (1 + x\cot x)y = x$

$\dfrac{dy}{dx} + \left(\dfrac{1}{x} + \cot x\right)y = 1$ (LDE in y)

$I.F = e^{\int P dx} = e^{\int (\frac{1}{x} + \cot x)dx} = e^{\ln x + \ln \sin x}$

$I.F = e^{\ln(x\sin x)} = \boxed{x\sin x}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y x\sin x) = \int \underset{I}{x} \underset{II}{\sin x}\, dx + c$

$y x\sin x = x(-\cos x) - \int 1 \cdot (-\cos x)\, dx$

$= x(-\cos x) + \int \cos x\, dx$

$y x\sin x = -x\cos x + \sin x + c$

$y = -\cot x + \dfrac{1}{x} + \dfrac{c}{x}\operatorname{cosec} x$

(54)

(8) $(x^2+1)\frac{dy}{dx}+2xy = 4x^2$

$\frac{dy}{dx}+\left(\frac{2x}{x^2+1}\right)y = \frac{4x^2}{x^2+1}$ (LDE in y)

$I.F = e^{\int\left(\frac{2x}{x^2+1}\right)dx} = e^{\ln(x^2+1)} = \boxed{x^2+1}$

Sol is given by $\int d(y.I.F) = \int Q.I.F\, dx + c$

$\Rightarrow \int d(y(x^2+1)) = \int \frac{4x^2}{\cancel{(x^2+1)}}\cancel{(x^2+1)}\,dx + c$

$y(x^2+1) = \frac{4x^3}{3} + c$

$3y(x^2+1) = 4x^3 + c'$

(9) $x\frac{dy}{dx}+2y = \sin x$

$\frac{dy}{dx}+\frac{2}{x}y = \frac{\sin x}{x}$ (LDE in y)

$IF = e^{\int P dx} = e^{\int\frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = \boxed{x^2}$

Sol is given by $\int d(y.I.F) = \int Q.I.F\, dx + C$

$\Rightarrow \int d(yx^2) = \int \frac{\sin x}{x}x^2 dx + C$

$yx^2 = \int \underset{I}{x}\,\underset{II}{\sin x}\, dx + c$

$yx^2 = x(-\cos x) - \int 1.(-\cos x)dx + c$

$y = \frac{1}{x^2}\left(-x\cos x + \sin x + c\right)$

(10) $(1+x^2)\frac{dy}{dx} + 4xy = \frac{1}{(1+x^2)^2}$

$\frac{dy}{dx}+\left(\frac{4x}{1+x^2}\right)y = \frac{1}{(1+x^2)^3}$ (LDE in y)

$I.F = e^{\int\frac{4x}{1+x^2}dx} = e^{2\ln(1+x^2)} = e^{\ln(1+x^2)^2} = \boxed{(1+x^2)^2}$

Sol is given by $\int d(y.I.F) = \int Q.I.F\,dx + c$

$\Rightarrow \int d\left(y(1+x^2)^2\right) = \int \frac{1}{(1+x^2)^3}(1+x^2)^2 dx + c$

$y(1+x^2)^2 = \int \frac{dx}{1+x^2} + c \quad \Rightarrow y = \frac{1}{(1+x^2)^2}\left[\tan^{-1}x + c\right]$ Ans.

(11) $\frac{dy}{dx} = \frac{1}{e^y - x}$

Reciprocal

$\frac{dx}{dy} = e^y - x$

$\frac{dx}{dy} + x = e^y$ (LDE in x)

$I.F = e^{\int 1.dy} = \boxed{e^y}$

Sol is given by $\int d(x.I.F) = \int Q.I.F\,dy + C$

$\Rightarrow \int d(xe^y) = \int e^y e^y dy + C$

$\Rightarrow xe^y = \int e^{2y}dy + c$

$x = \frac{1}{e^y}\left(\frac{e^{2y}}{2} + c\right)$

$x = \frac{e^y}{2} + ce^{-y}$

(12) $(x+2y^3)\frac{dy}{dx} = y$

$\left(\frac{1}{x+2y^3}\right)\frac{dx}{dy} = \frac{1}{y}$

$\frac{dx}{dy} = \frac{x+2y^3}{y}$

$\frac{dx}{dy} = \frac{x}{y} + 2y^2$

$\frac{dx}{dy} - \left(\frac{1}{y}\right)x = 2y^2$ (LDE in x)

$I.F = e^{\int-\left(\frac{1}{y}\right)dy} = e^{-\ln y} = e^{\ln y^{-1}} = y^{-1} = \boxed{\frac{1}{y}}$

Sol is given by $\int d(x.I.F) = \int Q.I.F\,dy + C$

$\Rightarrow \int d\left(x.\frac{1}{y}\right) = \int 2y^2.\frac{1}{y}\,dy + C$

$\Rightarrow \frac{x}{y} = \int 2y\,dy + c$

$\Rightarrow x = y\left(2\frac{y^2}{2} + c\right)$

$\Rightarrow x = y^3 + cy$

(55)

Bernoulli Eq is the diff eq of the form $\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n}$ ——— ①

To Solve ① Divide the eq ① by y^n $\Rightarrow y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$

② Multiply both sides by $(1-n)$ $\Rightarrow (1-n)y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n}(1-n) = (1-n)Q(x)$

③ Put $y^{1-n} = v$ $\Rightarrow$ $\frac{dv}{dx} + P(x)\,v\,(1-n) = (1-n)Q(x)$ which is L.D.Eq in v

& Diff $(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$ Now solve it easily as before.

Note The diff eq of the form $\boxed{\frac{dx}{dy} + P(y)\,x = Q(y)\,x^n}$ is also called Bernoulli Eq

(13) $x\frac{dy}{dx} + y = y^2 \ln x$

$\div$ by x

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{\ln x}{x}\,y^2 \quad \text{Bernoulli Eq.}$$

$\div$ by y^2

$$y^{-2}\frac{dy}{dx} + \left(\frac{1}{x}\right)y^{-1} = \frac{\ln x}{x}$$

$\times$ by -1

$$-y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -\frac{\ln x}{x}$$

Put $y^{-1} = v$

$$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} + \left(\frac{-1}{x}\right)v = -\frac{\ln x}{x} \quad \text{(LDE in } v\text{)}$$

$$\text{I.F} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \boxed{x^{-1}}$$

Sol is given by $\int d(v\,x^{-1}) = \int -\frac{\ln x}{x}\,x^{-1}dx + c$

$$\Rightarrow \frac{v}{x} = -\int \underset{\text{I}}{\ln x}\,.\,\underset{\text{II}}{x^{-2}}\,dx + c$$

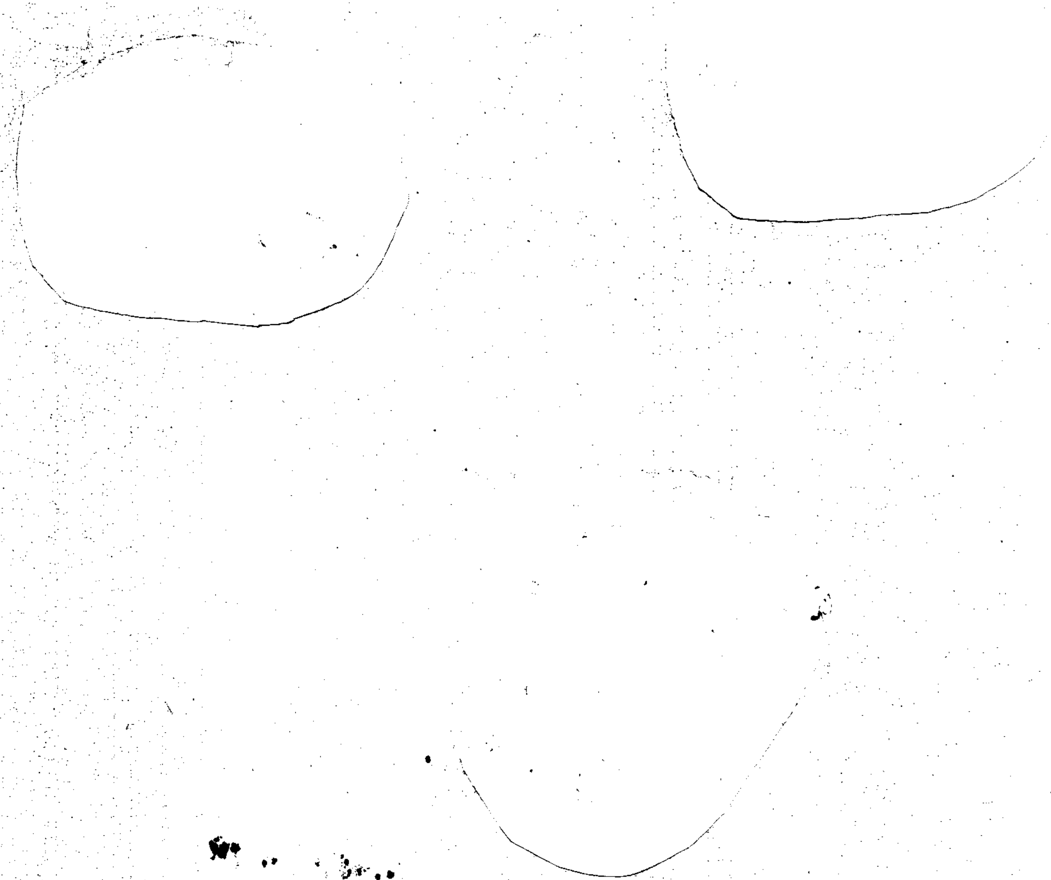
$$\frac{v}{x} = -\left(\ln x\,\frac{x^{-1}}{-1} - \int \frac{1}{x}\,\frac{x^{-1}}{-1}\,dx\right) + c$$

$$\frac{y^{-1}}{x} = \frac{1}{x}\ln x - \int x^{-2}\,dx + c$$

$$\frac{1}{xy} = \frac{1}{x}\ln x - \frac{x^{-1}}{-1} + c$$

$$\frac{1}{y} = \frac{x\ln x}{x} + x\,x^{-1} + cx$$

$$\frac{1}{y} = \ln x + 1 + cx$$



(14) $\frac{dy}{dx} + y = xy^3$ Bernoulli Eq

by y^3: $y^{-3}\frac{dy}{dx} + y^{-2} = x$

by (-2): $-2y^{-3}\frac{dy}{dx} + (-2)y^{-2} = -2x$

Put $y^{-2} = v$

$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + (-2)v = -2x$ (LDE in V)

$I.F = e^{\int -2dx} = e^{-2x}$

Sol is given by $\int d(v e^{-2x}) = \int (-2x)e^{-2x} dx + c$

Put $-2x = t$, $-2dx = dt$, $dx = \frac{dt}{-2}$

$\Rightarrow ve^{-2x} = \int t e^{t} \frac{dt}{-2} + c$

$\Rightarrow ve^{-2x} = -\frac{1}{2}\int \underset{I}{t}\, \underset{II}{e^{t}} dt + c$

$\Rightarrow ve^{-2x} = -\frac{1}{2}\left(t e^{t} - \int 1.e^{t}dt\right] + c$

$\Rightarrow ve^{-2x} = -\frac{1}{2}\left(te^{t} - e^{t}\right) + c$

$\Rightarrow y^{-2}e^{-2x} = -\frac{1}{2}e^{t}(t-1) + c$

$\Rightarrow \frac{1}{y^2}e^{-2x} = -\frac{1}{2}e^{-2x}(-2x-1) + c$

$\Rightarrow \frac{1}{y^2} = -\frac{1}{2}\frac{e^{-2x}}{e^{-2x}}(-2x-1) + \frac{c}{e^{-2x}}$

$\Rightarrow \frac{1}{y^2} = -\frac{1}{2}(-2x-1) + ce^{2x}$

(15) $x\frac{dy}{dx} - 2x^2y = y\ln y$

$\frac{dy}{dx} - 2xy = \frac{y\ln y}{x}$

$\frac{1}{y}\frac{dy}{dx} - 2x = \frac{\ln y}{x}$

Put $\ln y = v$

$\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} - 2x = \frac{v}{x}$

$\frac{dv}{dx} - \frac{v}{x} = 2x$ (LDE in V)

$I.F = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \boxed{x^{-1}}$

Sol is given by $\int d(vx^{-1}) = \int 2x\, x^{-1}dx + c$

$\Rightarrow vx^{-1} = \int 2dx + c$

$\Rightarrow \frac{v}{x} = 2x + c$

$\Rightarrow \frac{\ln y}{x} = 2x + c$

$\Rightarrow \ln y = 2x^2 + cx$

Available at www.mathcity.org

(57)

(16) $(x^2+1)\dfrac{dy}{dx} + 4xy = x \quad , \quad y(2)=1$

$$\frac{dy}{dx} + \left[\frac{4x}{x^2+1}\right]y = \frac{x}{(x^2+1)} \qquad \text{(LDE in } y)$$

$$\text{I.F} = e^{\int \frac{4x}{(x^2+1)}dx} = e^{2\ln(x^2+1)} = e^{\ln(x^2+1)^2} = \boxed{(x^2+1)^2}$$

Sol is given by $\int d\left(y(x^2+1)^2\right) = \int \dfrac{x}{(x^2+1)}(x^2+1)^2\,dx + c$

$$\Rightarrow y(x^2+1)^2 = \int x(x^2+1)\,dx + c$$

$$\Rightarrow y(x^2+1)^2 = \int (x^3+x)\,dx + c$$

$$\Rightarrow y(x^2+1)^2 = \frac{x^4}{4} + \frac{x^2}{2} + c$$

$\because y(2)=1$

$\therefore 1(25) = 6+c$

$\boxed{19 = c}$

$$\therefore y(x^2+1)^2 = \frac{x^4}{4} + \frac{x^2}{2} + 19$$

(19) $x(2+x)\dfrac{dy}{dx} + 2(1+x)y = 1+3x^2, \quad y(-1)=1$

$$\frac{dy}{dx} + \left[\frac{2(1+x)}{x(2+x)}\right]y = \frac{1+3x^2}{x(2+x)} \qquad \text{(LDE in } y)$$

$$\text{I.F} = e^{\int \frac{2+2x}{2x+x^2}dx} = e^{\ln(2x+x^2)} = \boxed{2x+x^2}$$

Sol is given by $\int d\left(y(2x+x^2)\right) = \int \dfrac{1+3x^2}{x(2+x)}(2x+x^2)\,dx + c$

$$\Rightarrow y(2x+x^2) = x + \cancel{3}\frac{x^3}{\cancel{3}} + c$$

$$\Rightarrow y = \frac{x+x^3+c}{2x+x^2}$$

$\because y(-1) = 1$

$\therefore 1 = \dfrac{-1-1+c}{-2+1}$

$-1 = -2+c$

$\boxed{1 = c}$

$$\therefore y = \frac{x+x^3+1}{2x+x^2} \quad \text{Ans.}$$

(17) $e^x\left(y-3(e^x+1)^2\right)dx + (e^x+1)dy = 0, \quad y(0)=4$

$$(e^x+1)dy = -e^x\left(y-3(e^x+1)^2\right)dx$$

$$\frac{dy}{dx} = \frac{-e^x}{e^x+1}\left[y-3(e^x+1)^2\right]$$

$$\frac{dy}{dx} = -\left(\frac{e^x}{e^x+1}\right)y + \frac{3e^x(e^x+1)^2}{(e^x+1)}$$

$$\frac{dy}{dx} + \left(\frac{e^x}{e^x+1}\right)y = 3e^x(e^x+1) \qquad \text{(LDE in } y)$$

$$\text{I.F} = e^{\int \frac{e^x}{e^x+1}dx} = e^{\ln(e^x+1)} = \boxed{e^x+1}$$

Sol is given by $\int d\left[y(e^x+1)\right] = \int 3e^x(e^x+1)^2\,dx + c$

$$\Rightarrow y(e^x+1) = \cancel{3}\frac{(e^x+1)^3}{\cancel{3}} + c$$

$$\Rightarrow y = (e^x+1)^2 + c$$

$\because y(0) = 4$

$4 = (1+1)^2 + c$

$\boxed{0 = c}$

$$\Rightarrow y = (e^x+1)^2 \quad \text{Ans}$$

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(18) $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$, $y(1)=2$

$\frac{dy}{dx} + \frac{1}{2x}y = xy^{-3}$ Bernoulli Eq.

$\div$ by y^{-3} $\quad y^3\frac{dy}{dx} + \frac{1}{2x}y^4 = x$

$\times$ by 4 $\quad 4y^3\frac{dy}{dx} + \frac{2}{x}y^4 = 4x$

Put $y^4 = v$

$4y^3\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + \frac{2}{x}v = 4x$ (LDE in v)

$I.F = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = \boxed{x^2}$

Sol is given by $\int d(vx^2) = \int (4x)x^2\,dx + c$

$\Rightarrow vx^2 = \int 4x^3\,dx + c$

$y^4x^2 = \cancel{4}\frac{x^4}{\cancel{4}} + c$

$\because y(1)=2$

$(2)^4 1^2 = 1^4 + c$

$16 - 1 = c$

$\boxed{15 = c}$

$y^4x^2 = x^4 + 15$

(20) $\frac{dy}{dx} + \frac{3y}{x} = x^2y^2$, $y(1)=2$ Bernoulli Eq

$\div$ by y^2 $\quad y^{-2}\frac{dy}{dx} + \frac{3}{x}y^{-1} = x^2$

$\times$ by (-1) $\quad -y^{-2}\frac{dy}{dx} - \frac{3}{x}y^{-1} = -x^2$

Put $y^{-1} = v$

$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} - \frac{3}{x}v = -x^2$ (LDE in v)

$I.F = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = \boxed{x^{-3}}$

Sol is given by $\int d(vx^{-3}) = \int (-x^2)x^{-3}\,dx + c$

$\Rightarrow vx^{-3} = -\int x^{-1}\,dx + c$

$\Rightarrow \frac{v}{x^3} = -\int \frac{dx}{x} + c$

$\Rightarrow \frac{1}{yx^3} = -\ln x + c$

$\Rightarrow \frac{1}{y} = x^3(\ln x^{-1} + c)$

$\because y(1)=2$

$\therefore \frac{1}{2} = 1(\ln 1^{-1} + c)$

$\boxed{\frac{1}{2} = c}$

$\therefore \frac{1}{y} = x^3(\ln x^{-1} + \frac{1}{2})$ Ans

Orthogonal Trajectory is a curve that intersects every curve of another family at right angle.

If all the curves of a family of curves intersect orthogonally to all the curves of another family of curves then the two families are called Orthogonal Trajectories.

To Solve
1) Differentiate
2) Eliminate const using eq ①
3) Find $\frac{dy}{dx}$ or $r\frac{d\theta}{dr}$ for given curve.
4) Find $\frac{dy}{dx}$ or $r\frac{d\theta}{dr}$ for family of O.Ts
5) Solve by previous methods according i.e separating variables or Homogeneous or LDE or Exact.

Ex 9.7

Find the orthogonal trajectories of each of the following curves.

① $x^2 - y^2 = c$ one const.

Diff $2x - 2y\frac{dy}{dx} = 0$ const eliminated.

$x - y\frac{dy}{dx} = 0$

$x = y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{x}{y}$ D Eq of given family

$\frac{dy}{dx} = -\frac{y}{x}$ D Eq of OTs

$\int\frac{dy}{y} = -\int\frac{dx}{x}$ separating variables

$\ln y = -\ln x + \ln c$

$\ln y = \ln\frac{c}{x}$

$y = \frac{c}{x}$

$xy = c$ is Required family of orthogonal trajectories.

② $x = cy^2$ —①

Diff $1 = c\,2y\frac{dy}{dx}$

$\frac{1}{2cy} = \frac{dy}{dx}$

$\frac{1}{2\left(\frac{x}{y\cancel{y}}\right)\cancel{y}} = \frac{dy}{dx}$ using ① const eliminated

$\frac{dy}{dx} = \frac{y}{2x}$ DEq of given family

$\frac{dy}{dx} = -\frac{2x}{y}$ DEq of OTs

$\int y\,dy = -\int 2x\,dx$

$\frac{y^2}{\cancel{2}} = -\frac{\cancel{2}x^2}{\cancel{2}} + c$

$y^2 = -2x^2 + 2c$

$y^2 = -2x^2 + K$

Required family of orthogonal trajectories

③ $x^2+y^2=cx$ ——— one const. ①

diff $2x+2y\frac{dy}{dx}=c$ — const. eliminated using ①

$\therefore\ x^2+y^2=\left(2x+2y\frac{dy}{dx}\right)x$

$x^2+y^2=2x^2+2xy\frac{dy}{dx}$

$y^2-x^2=2xy\frac{dy}{dx}$

$\frac{y^2-x^2}{2xy}=\frac{dy}{dx}$ diff eq for given family

$-\frac{2xy}{y^2-x^2}=\frac{dy}{dx}$ diff eq for orthogonal trajectories

$\frac{2xy}{x^2-y^2}=\frac{dy}{dx}$ Homogeneous Eq.

Put $y=vx$

$\frac{dy}{dx}=v+x\frac{dv}{dx}$

$v+x\frac{dv}{dx}=\frac{2xvx}{x^2-v^2x^2}$

$=\frac{2v\cancel{x^2}}{(1-v^2)\cancel{x^2}}$

$x\frac{dv}{dx}=\frac{2v}{1-v^2}-v$

$x\frac{dv}{dx}=\frac{2v-v+v^3}{1-v^2}$

$x\frac{dv}{dx}=\frac{v+v^3}{1-v^2}$

$\int\frac{1-v^2}{v+v^3}dv=\int\frac{dx}{x}$ separating variables.

$\int\frac{1-v^2}{v(1+v^2)}dv=\int\frac{dx}{x}$ ——— ⑪

By Partial Fractions

$\frac{1-v^2}{v(1+v^2)}=\frac{A}{v}+\frac{Bv+C}{1+v^2}$

$1-v^2=A(1+v^2)+(Bv+C)v$

Put $v=0$ $\boxed{1=A}$

Comparing coeffts of v^2; $-1=A+B$

$-1=1+B$

$\boxed{-2=B}$

Comparing coeffts of v, $\boxed{0=C}$

$\therefore\ \frac{1-v^2}{v(1+v^2)}=\frac{1}{v}+\frac{(-2v)}{1+v^2}$

So ⑪ becomes

$\int\left(\frac{1}{v}-\frac{2v}{1+v^2}\right)dv=\int\frac{1}{x}dx$

$\ln v-\ln(1+v^2)=\ln x+\ln K$

$\ln\left(\frac{v}{1+v^2}\right)=\ln Kx$

$\frac{v}{1+v^2}=Kx$

$\frac{\frac{y}{x}}{1+\frac{y^2}{x^2}}=Kx$

$\frac{\frac{y}{x}}{\frac{x^2+y^2}{x^2}}=Kx$

$\frac{y\cancel{x}}{x^2+y^2}=K\cancel{x}\Rightarrow\boxed{y=K(x^2+y^2)}$ Required family of O. Trajectories

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(4) $y = e^{cx}$ one const. (1)

Diff $\frac{dy}{dx} = ce^{cx}$

$\frac{dy}{dx} = cy$

$\frac{1}{y}\frac{dy}{dx} = c$

$y = e^{\frac{x}{y}\frac{dy}{dx}}$ using (1) const. eliminated

$\ln y = \ln e^{\frac{x}{y}\frac{dy}{dx}}$

$\ln y = \frac{x}{y}\frac{dy}{dx}.\ln e$

$\frac{y}{x}\ln y = \frac{dy}{dx}\cdot 1$, diff eq of given family

$-\frac{x}{y\ln y} = \frac{dy}{dx}$, diff eq of family of O.Ts

$\int -x\,dx = \int \underset{II}{y}\, \underset{I}{\ln y}\, dy$ separating variables.

$\Rightarrow \ln y\,\frac{y^2}{2} - \int \frac{1}{y}\frac{y^2}{2}\,dy = -\frac{x^2}{2} + K$

$\Rightarrow \ln y\cdot\frac{y^2}{2} - \frac{1}{2}\left(\frac{y^2}{2}\right) = -\frac{x^2}{2} + K$

$\Rightarrow \frac{y^2}{2}\ln y - \frac{y^2}{4} = -\frac{x^2}{2} + K$

$\Rightarrow \frac{2y^2\ln y - y^2}{4} = -\frac{x^2}{2} + K$

$\Rightarrow y^2\left(\frac{2\ln y - 1}{4}\right) = -\frac{x^2}{2} + K$

$\Rightarrow y^2(\ln y^2 - 1) = 4\left(-\frac{x^2}{2} + K\right)$

$\Rightarrow y^2(\ln y^2 - 1) = -2x^2 + 4K$

$\Rightarrow y^2(\ln y^2 - 1) = 2(K' - x^2)$

where $K' = 2K$.

(5) $y = x - 1 + ce^{-x}$ one const (1)

Diff $\frac{dy}{dx} = 1 - 0 - ce^{-x}$

$= 1 - (y - x + 1)$ const eliminated using (1)

$\frac{dy}{dx} = \cancel{1} - y + x - \cancel{1}$

$\frac{dy}{dx} = x - y$ is diff eq of given family

$\frac{dy}{dx} = \frac{-1}{x-y}$ diff eq of family of O.Ts.

$\frac{dy}{dx} = \frac{1}{y-x}$

$\frac{dx}{dy} = y - x$ taking Reciprocal

$\frac{dx}{dy} + x = y$ LDE in x

I.F $= e^{\int 1\,dy} = e^{y}$

Sol is given by $\int d(xe^{y}) = \int \underset{I}{y}\,\underset{II}{e^{y}}\,dy + K$

$\Rightarrow xe^{y} = ye^{y} - \int 1.e^{y}\,dy + K$

$xe^{y} = ye^{y} - e^{y} + K$

$xe^{y} = e^{y}(y-1) + K$

$x = \frac{e^{y}(y-1)}{e^{y}} + \frac{K}{e^{y}}$

$x = (y-1) + K.e^{-y}$

is required family of O.Ts.

(6) $xy = c$

Diff $x\frac{dy}{dx} + y = 0$ const eliminated

$\frac{dy}{dx} = -\frac{y}{x}$ diff eq of given family

$\frac{dy}{dx} = \frac{x}{y}$ diff eq of family of O.Ts

$\int y\,dy = \int x\,dx$ separating variables

$\frac{y^2}{2} = \frac{x^2}{2} + K$

$y^2 = x^2 + 2K$

$y^2 = x^2 + K'$

(7) $x = \frac{y^2}{4} + \frac{c}{y^2}$ —— (1)

Diff $1 = \frac{2y}{4}\frac{dy}{dx} - \frac{2c}{y^3}\frac{dy}{dx}$

$1 = \frac{dy}{dx}\left(\frac{y}{2} - \frac{2c}{y^3}\right)$

$\frac{dy}{dx} = \frac{1}{\frac{y}{2} - \frac{2c}{y^3}}$

$= \frac{1}{\frac{y}{2} - \frac{2}{y^3}\left(y^2\left(x - \frac{y^2}{4}\right)\right)}$ ∵ using (1) Const eliminated

$= \frac{1}{\frac{y}{2} - \frac{2xy^2}{y^3} + \frac{2y^4}{4y^3}}$

$= \frac{1}{\frac{y}{2} - \frac{2x}{y} + \frac{y}{2}}$

$= \frac{1}{\frac{y^2 - 4x + y^2}{2y}}$

$= \frac{2y}{2y^2 - 4x}$

$\frac{dy}{dx} = \frac{y}{y^2 - 2x}$ diff eq of given family

$\frac{dy}{dx} = -\frac{(y^2 - 2x)}{y}$

$\frac{dy}{dx} = \frac{2x - y^2}{y}$

$\frac{dy}{dx} = \frac{2x}{y} - y$

$\frac{dy}{dx} + y = 2xy^{-1}$ Bernoulli Eq

÷ by y^{-1} $\quad y\frac{dy}{dx} + y^2 = 2x$

× by 2 $\quad 2y\frac{dy}{dx} + 2y^2 = 4x$

Put $y^2 = V$

$2y\frac{dy}{dx} = \frac{dV}{dx}$ $\quad$ ∴ $\frac{dV}{dx} + 2V = 4x$ LDE in V

see above

I.F $= e^{\int 2\,dx} = e^{2x}$

sol is given by $\int d(Ve^{2x}) = \int 4xe^{2x}\,dx + C$

$\Rightarrow Ve^{2x} = \int e^{2x}(2x)\,2dx + C$ $\quad$ Put $2x = t$, $2dx = dt$

$= \int \underset{II}{e^t}\,\underset{I}{t}\,dt + C$

$= te^t - \int 1\,e^t + C$

$\cancel{te^t - e^t + c}$

$Ve^{2x} = e^t(t-1) + C$

$y^2e^{2x} = e^{2x}(2x-1) + C$

$y^2 = \frac{\cancel{e^{2x}}(2x-1)}{\cancel{e^{2x}}} + \frac{C}{e^{2x}}$

$y^2 = (2x-1) + Ce^{-2x}$

is required family of O.Ts.

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$$y = (x-c)^2 \quad \text{---}(1)$$

$$\frac{dy}{dx} = 2(x-c)$$

$$\frac{dy}{dx} = 2(\pm\sqrt{y}) \quad \text{const eliminated using (1) D.Eq of given eq}$$

$$\frac{dy}{dx} = \frac{-1}{\pm 2\sqrt{y}} \quad \text{D.Eq of O.Ts}$$

$$\pm\sqrt{y}\,dy = -\frac{dx}{2} \quad \text{separating variables}$$

$$\int\sqrt{y}\,dy = -\int\frac{dx}{2}$$

$$\pm\frac{2}{3}y^{3/2} = -\frac{1}{2}x + K$$

$$\pm\frac{2}{3}y^{3/2} = \frac{-x+2K}{2}$$

$$\pm\frac{4}{3}y^{3/2} = -x+2K$$

$$\text{sq. } \frac{16}{9}y^3 = (-x+K')^2$$

$$16y^3 = 9(-x+K')^2 \quad \text{Required family of O.Ts.}$$

D 2nd Method $\frac{dy}{dx} = \frac{-2xy}{y^2+x^2}$

$$(y^2+x^2)dy = -2xy\,dx$$

$$(y^2+x^2)dy + 2xy\,dx = 0$$

$$M_y = 2x \qquad N_x = 2x$$

$\because M_y = N_x \quad \therefore$ Exact Eq

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)\,dy = C$

$$\int 2xy\,dx + \int y^2 dy = C$$

$$\cancel{2}\frac{x^2}{\cancel{2}}y + \frac{y^3}{3} = C$$

$$\frac{3x^2y + y^3}{3} = C$$

$$y^3 + 3x^2y = 3C$$

$$y^3 + 3x^2y = C'$$

(9) $y^2 = x^2 + cx$ ——— (1)

$$2y\frac{dy}{dx} = 2x + c$$

$$2y\frac{dy}{dx} - 2x = c$$

$$y^2 = x^2 + \left(2y\frac{dy}{dx} - 2x\right)x \quad \text{const eliminated using (1)}$$

$$= x^2 + 2xy\frac{dy}{dx} - 2x^2$$

$$y^2 + x^2 = 2xy\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^2+x^2}{2xy} \quad \text{diff Eq of given eq.}$$

$$\frac{dy}{dx} = \frac{-2xy}{y^2+x^2} \quad \text{D.Eq of O.Ts.}$$

Homogeneous Eq.

So Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{-2xvx}{v^2x^2+x^2}$$

$$x\frac{dv}{dx} = \frac{-2v\cancel{x^2}}{(v^2+1)\cancel{x^2}} - v$$

$$x\frac{dv}{dx} = \frac{-(2v + v(v^2+1))}{(v^2+1)}$$

$$x\frac{dv}{dx} = \frac{-(3v+v^3)}{v^2+1}$$

$$\int\frac{(v^2+1)\,dv}{3v+v^3} = \int -\frac{dx}{x}$$

$$\frac{1}{3}\int\frac{(3v^2+3)\,dv}{v^3+3v} = -\int\frac{dx}{x}$$

$$\frac{1}{3}\ln(v^3+3v) = -\ln x + \ln c$$

$$\ln(v^3+3v)^{\frac{1}{3}} = \ln\frac{c}{x}$$

Antilog $\left(\frac{y^3}{x^3} + 3\frac{y}{x}\right)^{\frac{1}{3}} = \frac{c}{x}$

Cubing $\left(\frac{y^3+3yx^2}{x^3}\right) = \left(\frac{c}{x}\right)^3$

$$y^3 + 3x^2y = \frac{c^3}{\cancel{x^3}}\cancel{x^3}$$

$$y^3 + 3x^2y = K \qquad K = c^3$$

(10) $x^2+y^2 = 1+2cy$ ——— (1)

Diff $2x+2y\frac{dy}{dx} = 2c\frac{dy}{dx}$

$2\left(x+y\frac{dy}{dx}\right) = 2c\frac{dy}{dx}$

$\frac{2}{2\frac{dy}{dx}}\left(x+y\frac{dy}{dx}\right) = c$

$x\cdot\frac{dx}{dy}+y\,\cancel{\frac{dy}{dx}}\cdot\cancel{\frac{dx}{dy}} = c$

$x^2+y^2 = 1+2\left[x\frac{dx}{dy}+y\right]y$ Const eliminated using (1)

$x^2+y^2 = 1+2xy\frac{dx}{dy}+2y^2$

$x^2-y^2-1 = 2xy\frac{dx}{dy}$

$\frac{x^2-y^2-1}{2xy} = \frac{dx}{dy}$

$\frac{dy}{dx} = \frac{2xy}{x^2-y^2-1}$ D.Eq of given family

$\frac{dy}{dx} = \frac{-(x^2-y^2-1)}{2xy}$ D.Eq of O.Ts

$\frac{dy}{dx} = \frac{1-x^2}{2xy}+\frac{y^2}{2xy}$

$\frac{dy}{dx}-\frac{y}{2x} = \frac{1-x^2}{2x}\cdot y^{-1}$ Bernoulli Eq

$\div$ by y^{-1} $\quad y\frac{dy}{dx}-\frac{y^2}{2x} = \frac{1-x^2}{2x}$

$\times$ by 2 $\quad 2y\frac{dy}{dx}-\cancel{2}\frac{y^2}{\cancel{2}x} = \cancel{2}\frac{(1-x^2)}{\cancel{2}x}$

Put $y^2=v$

$2y\frac{dy}{dx}=\frac{dv}{dx}$ $\quad\therefore \frac{dv}{dx}+\left(-\frac{1}{x}\right)v = \frac{1-x^2}{x}$ (LDE in v)

$I.F = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$ see above

(from below)

Sol is given by $\int d\left(v\frac{1}{x}\right) = \int \frac{1-x^2}{x}\cdot\frac{1}{x}dx + K$

$\Rightarrow \frac{v}{x} = \int \frac{1-x^2}{x^2}dx + K$

$= \int \frac{dx}{x^2} - \int dx + K$

$\frac{v}{x} = \frac{x^{-1}}{-1} - x + K$

$\frac{v}{x} = -\frac{1}{x} - x + K$

$\because v=y^2$ $\quad \frac{y^2}{x} = \frac{-1-x^2+Kx}{x}$

$y^2 = -1-x^2+Kx$ Ans.

$y^2-Kx+x^2 = -1$

$y^2-\frac{2}{2}Kx+x^2 = -1$

$y^2-2K'x+x^2=-1$ $\quad K'=\frac{K}{2}$

$x^2-2K'x+K'^2-K'^2+y^2=-1$

$(x-K')^2+y^2=-1+K'^2$

$(x-K')^2+y^2 = K'^2-1$

$K'^2>1$

Eq of Required O.Ts.

(11) $r = a(1+\sin\theta)$ —— (1)

Diff $\frac{dr}{d\theta} = a\cos\theta$

$\frac{dr}{d\theta} = \left(\frac{r}{1+\sin\theta}\right)\cos\theta$ using (1) const eliminated

$\frac{1+\sin\theta}{\cos\theta} = r\frac{d\theta}{dr}$, D.Eq of given family

$-\frac{\cos\theta}{1+\sin\theta} = r\frac{d\theta}{dr}$, D.Eq of family of OTs

$\int\frac{dr}{r} = -\int\frac{1+\sin\theta}{\cos\theta}\,d\theta$

$\int\frac{dr}{r} = -\int\frac{1}{\cos\theta}\,d\theta - \int\frac{\sin\theta\,d\theta}{\cos\theta}$

$\int\frac{dr}{r} = -\int\sec\theta\,d\theta - \int\tan\theta\,d\theta$

$\ln r = -\ln(\sec\theta+\tan\theta) - (-\ln\cos\theta) + \ln c$

$\ln r = \ln\left(\frac{c\cos\theta}{\sec\theta+\tan\theta}\right)$

$r = \frac{c\cos\theta}{\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}$

$r = \frac{c\cos^2\theta}{1+\sin\theta}$

$= \frac{c(1-\sin^2\theta)}{}$

$= c\,\frac{(1-\sin\theta)\cancel{(1+\sin\theta)}}{\cancel{(1+\sin\theta)}}$

$r = c(1-\sin\theta)$

is required family of O.Ts.

(12) $r^2 = a\sin 2\theta$ —— (1)

Diff $2r\frac{dr}{d\theta} = 2a\cos 2\theta$

$2r\frac{dr}{d\theta} = 2\left(\frac{r^2}{\sin 2\theta}\right)\cos 2\theta$ using (1) const eliminated

$\frac{\cancel{2}\cancel{r}\frac{dr}{d\theta}}{\cancel{2}r^{\cancel{2}}} = \frac{\cos 2\theta}{\sin 2\theta}$

$\frac{r}{dr/d\theta} = \frac{\sin 2\theta}{\cos 2\theta}$ Reciprocal

$\frac{r\,d\theta}{dr} = \frac{\sin 2\theta}{\cos 2\theta}$ D.Eq of given family

$\frac{r\,d\theta}{dr} = -\frac{\cos 2\theta}{\sin 2\theta}$ D.Eq of family of OTs

(Reciprocal) separating variables

$\int\frac{dr}{r} = -\int\frac{\sin 2\theta}{\cos 2\theta}\,d\theta$

$\int\frac{dr}{r} = \frac{1}{2}\int\frac{-\sin 2\theta(2)\,d\theta}{\cos 2\theta}$

$\ln r = \frac{1}{2}\ln\cos 2\theta + \ln c$

$\ln r = \ln(\cos 2\theta)^{\frac{1}{2}} + \ln c$

$\ln r = \ln c\sqrt{\cos 2\theta}$

Antilog $r = c\sqrt{\cos 2\theta}$

$r^2 = c^2\cos 2\theta$

$r^2 = c'\cos 2\theta$. $c^2 = c'$

is required family of O.Ts.

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(13) $r^n = a^n \cos n\theta$ ——— (1)

Diff $n r^{n-1}\frac{dr}{d\theta} = -a^n \sin n\theta\,(n)$

$r^{n-1}\frac{dr}{d\theta} = -\left(\frac{r^n}{\cos n\theta}\right)\sin n\theta$ using (1) const eliminated

$\frac{r^{n-1}}{r^n}\frac{dr}{d\theta} = -\frac{\sin n\theta}{\cos n\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$

(Reciprocal)

$r\frac{d\theta}{dr} = -\cot n\theta$ D.Eq of given family

$r\frac{d\theta}{dr} = -(-\tan n\theta)$ D.Eq of family of O.Ts.

$\frac{d\theta}{\tan n\theta} = \frac{dr}{r}$ separating variables

$\int\frac{dr}{r} = \int \cot n\theta\, d\theta$

$\int\frac{dr}{r} = \frac{1}{n}\int \cot n\theta\; n d\theta$

$\ln r = \frac{1}{n}\ln\sin(n\theta) + \ln c$

$\ln r = \ln(\sin n\theta)^{\frac{1}{n}} + \ln c$

$\ln r = \ln c(\sin n\theta)^{\frac{1}{n}}$

$r = c(\sin n\theta)^{\frac{1}{n}}$

$r^n = c^n \sin n\theta$

(14) $r = \frac{a}{2+\cos\theta}$ ——— (1)

Diff $\frac{dr}{d\theta} = -a(2+\cos\theta)^{-2}(-\sin\theta)$

$\frac{dr}{d\theta} = \frac{a\sin\theta}{(2+\cos\theta)^2}$

$\frac{dr}{d\theta} = \frac{[r(2+\cos\theta)]\sin\theta}{(2+\cos\theta)^2}$ using (1) const eliminated

$\frac{dr}{d\theta} = \frac{r\sin\theta}{2+\cos\theta}$

$\frac{d\theta}{dr} = \frac{2+\cos\theta}{r\sin\theta}$

$r\frac{d\theta}{dr} = \frac{2+\cos\theta}{\sin\theta}$ D.Eq of given family

$r\frac{d\theta}{dr} = \frac{-\sin\theta}{2+\cos\theta}$ D.Eq of family of O.Ts

$\int\frac{dr}{r} = \int\frac{2+\cos\theta}{-\sin\theta}d\theta$ separating variables

$\int\frac{dr}{r} = -\int 2\csc\theta\, d\theta - \int\cot\theta\, d\theta$

$\ln r = -2\ln(\csc\theta - \cot\theta) - \ln\sin\theta + \ln c$

$\ln r = \ln(\csc\theta-\cot\theta)^{-2} + \ln\sin\theta^{-1} + \ln c$

$\ln r = \ln\frac{c}{\sin\theta(\csc\theta-\cot\theta)^2}$

$r = \frac{c}{\sin\theta\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}\right)^2}$

$r = \frac{c\,\sin^{\cancel{2}}\theta}{\cancel{\sin}\theta(1-\cos\theta)^2}$

$r\frac{\sin^3\theta}{\sin^3\theta} = \frac{c\sin\theta}{(1-\cos\theta)^2}$

$r\sin^3\theta = \frac{c\sin^4\theta}{(1-\cos\theta)^2}$

$r\sin^3\theta = \frac{c(\sin^2\theta)^2}{(1-\cos\theta)^2}$

$r\sin^3\theta = \frac{c(1-\cos^2\theta)^2}{(1-\cos\theta)^2}$

$r\sin^3\theta = \frac{c\left((1-\cos\theta)(1+\cos\theta)\right)^2}{(1-\cos\theta)^2}$

$r\sin^3\theta = c(1+\cos\theta)^2$ Ans

(xii)

(17) $y^2 = 4cx + 4c^2$ ——— (1)

Diff $2y\frac{dy}{dx} = 4c$

$\frac{2y}{4}\frac{dy}{dx} = c$

$y^2 = 4\left(\frac{y}{2}\frac{dy}{dx}\right)x + 4\left(\frac{y}{2}\frac{dy}{dx}\right)^2$ using (1) Const eliminated

$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$

$y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$ ——— (2) D.Eq of given family

Put $\frac{dy}{dx} = \frac{-1}{dy/dx}$ For D.Eq of family of O.Ts.

$\therefore y = 2x\left(\frac{-1}{dy/dx}\right) + y\left(\frac{-1}{dy/dx}\right)^2$

$y = -2x\frac{1}{(dy/dx)} + y\frac{1}{(dy/dx)^2}$

LCM $y\left(\frac{dy}{dx}\right)^2 = -2x\left(\frac{dy}{dx}\right) + y$

$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) = y$ same as (2)

Hence $y^2 = 4cx + 4c^2$ is self orthogonal

(18) $\frac{x^2}{c^2} + \frac{y^2}{c^2-1} = 1$ ——— (1)

Diff $\frac{2x}{c^2} + \frac{2y}{c^2-1}\left(\frac{dy}{dx}\right) = 0$

$\frac{2y}{c^2-1}\cdot\frac{dy}{dx} = -\frac{2x}{c^2}$

$\frac{dy}{dx} = -\frac{\cancel{2}x}{c^2}\cdot\frac{(c^2-1)}{\cancel{2}y}$

$y' = -\frac{x}{y}\frac{(c^2-1)}{c^2}$

$= -\frac{x}{y}\left(\frac{c^2}{c^2} - \frac{1}{c^2}\right)$

$y' = -\frac{x}{y}\left(1 - \frac{1}{c^2}\right)$

$\frac{yy'}{x} = -1 + \frac{1}{c^2}$

$\frac{1+yy'}{x} = \frac{1}{c^2}$

$\frac{x+yy'}{x} = \frac{1}{c^2}$

$c^2 = \frac{x}{x+yy'}$, Put in (1) to eliminate const.

$\frac{x^2}{\frac{x}{x+yy'}} + \frac{y^2}{\frac{x}{x+yy'} - 1} = 1$ using (1) Const eliminated

$x^2\left(\frac{x+yy'}{x}\right) + \left(\frac{y^2}{\frac{\cancel{x}-\cancel{x}-yy'}{x+yy'}}\right) = 1$

$x^2 + xyy' + y^2\frac{(x+yy')}{-yy'} = 1$

$\frac{x^2y' + xyy'^2 - xy - y^2y'}{y'} = 1$

$x^2y' + xyy'^2 - xy - y^2y' = y'$

$x^2y' + xyy'^2 - xy - y^2y' - y' = 0$

$y'(x^2 + xyy' - y^2 - 1) - xy = 0$ ——— (2) D.Eq of given family

Put $y' = -\frac{1}{y'}$ For D.Eq of family of O.Ts

$\therefore -\frac{1}{y'}\left[x^2 + xy\left(-\frac{1}{y'}\right) - y^2 - 1\right] - xy = 0$

LCM $\frac{-\left(x^2 - \frac{xy}{y'} - y^2 - 1\right) - xyy'}{y'} = 0$

LCM $\frac{-x^2y' + xy + y^2y' + y' - xyy'^2}{y'^2} = 0$

$-x^2y' + xy + y^2y' + y' - xyy'^2 = 0$

$x^2y' - xy - y^2y' - y' + xyy'^2 = 0$

$y'(x^2 + xyy' - y^2 - 1) - xy = 0$ same as (2) Hence (1) is self orthogonal.

8 Non Linear Diff. Equation of order One

Non Linear Diff. Eq. of order one

An eq. which is not linear, is called Non-Linear. (see ch: 10)

Consider the non-linear diff. eq. of first order

$$x^2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y^2 - y = 0$$

or $\quad x^2P^2 + xP - y^2 - y = 0 \qquad$ where $\; P = \frac{dy}{dx}$

or $\quad f(x, y, P) = 0$

Thus, we usually, represents the Non-Linear diff. eq. of the first order by $f(x, y, P) = 0$ where $P = \frac{dy}{dx}$

We shall discuss the four techniques to solve the eq. $f(x,y,P) = 0$

1. Solvable for P
2. Solvable for y
3. Solvable for x
4. Clairaut's eq.

$xy^2\left(\frac{-dx}{dy}\right)^2 + (x^2y - y^2 - y)\frac{dx}{dy} = xy^2$

by $x\left(\frac{dy}{dx}\right)^2 xy^2 + (x^2y + y^3 - y)\left(\frac{dx}{dy}\right)\left(\frac{dy}{dx}\right)^2 = xy^2\left(\frac{dy}{dx}\right)^2$

$xy^2 - (x^2y - y^2 - y)\frac{dy}{dx} = xy^2\left(\frac{dy}{dx}\right)^2$

$xy^2\left(\frac{dy}{dx}\right)^2 + (x^2y - y^3 - y)\frac{dy}{dx} = xy^2$ which is same $\frac{x^2}{c^2} + \frac{y^2}{c-1} = 1$

Solvable for P

The diff. eq. $f(x,y,P) = 0$ is said to be solvable for P if it can be reduced into Linear factors.

Example $x^2P^2 + xP - y^2 - y = 0$.

Sol:-

$$x^2P^2 - y^2 + xP - y = 0$$

$$\Rightarrow (xP+y)(xP-y) + (xP-y) = 0$$

$$\Rightarrow (xP-y)[xP+y+1] = 0$$

$$\Rightarrow xP-y = 0 \quad \text{or} \quad xP+y+1 = 0$$

$xP - y$

$$\Rightarrow x\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \int\frac{dy}{y} = \int\frac{dx}{x}$$

$$\Rightarrow \ln y = \ln x + \ln c$$

$$\Rightarrow \ln y = \ln cx$$

$$\Rightarrow y = cx$$

$$\Rightarrow y - cx = 0$$

$xP + y + 1 = 0$

$$\Rightarrow x\frac{dy}{dx} = -(y+1)$$

$$\Rightarrow \frac{dy}{y+1} = -\frac{dx}{x}$$

$$\Rightarrow \int\frac{dy}{y+1} = -\int\frac{dx}{x}$$

$$\Rightarrow \ln(y+1) = -\ln x + \ln c$$

$$\Rightarrow \ln(y+1) = \ln cx^{-1}$$

$$\Rightarrow y+1 = cx^{-1}$$

$$\Rightarrow x(y+1) - c = 0$$

Hence the general sol. is $(y-cx)(xy+x-c) = 0$

Example $xP^3 - (x^2+x+y)P^2 + (x^2+xy+y)P - xy = 0$

Sol:-

Since the given eq. is satisfied by $P = 1$

$$\therefore\ (P-1)[xP^2-(x^2+y)P+xy]=0$$
$$\Rightarrow (P-1)(xP^2-x^2P-yP+xy)=0$$
$$\Rightarrow (P-1)[xP(P-x)-y(P-x)]=0$$
$$\Rightarrow (P-1)(P-x)(xP-y)=0$$
$$\Rightarrow P-1=0 \quad \text{or} \quad P-x=0 \quad \text{or} \quad xP-y=0$$

$$\begin{array}{c|cccc} 1 & x & -x^2-x-y & x^2+xy+y & -xy \\ & & x & -x^2-y & xy \\ \hline & x & -x^2-y & xy & 0 \end{array}$$

$P-1=0$

$\Rightarrow \dfrac{dy}{dx}=1$

$\Rightarrow dy=dx$

$\Rightarrow \int dy=\int dx$

$\Rightarrow y=x+c$

$\Rightarrow y-x-c=0$

$P-x=0$

$\dfrac{dy}{dx}=x$

$\Rightarrow dy=x\,dx$

$\Rightarrow \int dy=\int x\,dx$

$\Rightarrow y=x^2/2+c$

$\Rightarrow y-x^2/2-c=0$

$xP-y=0$

$x\dfrac{dy}{dx}=y$

$\Rightarrow \dfrac{dy}{y}=\dfrac{dx}{x}$

$\Rightarrow \int\dfrac{dy}{y}=\int\dfrac{dx}{x}$

$\Rightarrow \ln y=\ln x+\ln c$

$\Rightarrow y=cx$

$\Rightarrow y-cx=0$

Hence the general sol. is $(y-x-c)(y-x^2/2-c)(y-cx)=0$

Solvable for Y

The diff. eq. $f(x,y,P)=0$ is said to be solvable for y if it cannot be factorised and can be put in the form

$$y=F(x,P)$$

Example $\quad y+Px=P^2x^4$

Sol:- $\quad y=P^2x^4-Px$ ——— ①

Diff. ① w.r.t. x, we get

$$\frac{dy}{dx}=4x^3P^2+2x^4P\frac{dP}{dx}-P-x\frac{dP}{dx}$$
$$\Rightarrow P=4x^3P^2+2x^4P\frac{dP}{dx}-P-x\frac{dP}{dx}$$
$$\Rightarrow 2P-4x^3P^2=x(2x^3P-1)\frac{dP}{dx}$$
$$\Rightarrow 2P(1-2x^3P)-x(1-2x^3P)\frac{dP}{dx}=0$$

Example $\quad y=P^2x+P$ ——— ①

Sol:-

Diff. eq. ① w.r.t x we get

$$\frac{dy}{dx}=P^2+2xP\frac{dP}{dx}+\frac{dP}{dx}$$
$$\Rightarrow P=P^2+(2xP+1)\frac{dP}{dx}$$
$$\Rightarrow (2xP+1)\frac{dP}{dx}+P^2-P=0$$
$$\Rightarrow \frac{dP}{dx}=\frac{P-P^2}{2xP+1}$$

$$\Rightarrow (1-2Px^3)(2P+x\frac{dP}{dx}) = 0$$

$$\Rightarrow 1-2Px^3 = 0 \quad \text{or} \quad 2P+x\frac{dP}{dx} = 0$$

Consider,

$$2P+x\frac{dP}{dx} = 0$$

$$\Rightarrow x\frac{dP}{dx} = -2P$$

$$\Rightarrow \frac{dP}{P} = -2\frac{dx}{x}$$

$$\Rightarrow \int\frac{dP}{P} = -2\int\frac{dx}{x}$$

$$\Rightarrow \ln P = -2\ln x + \ln c$$

$$= \ln x^{-2} + \ln c$$

$$= \ln cx^{-2}$$

$$\Rightarrow P = c/x^2 \quad \text{———} \quad (2)$$

Eliminating P from (1), (2)

we get, $y = c^2 - c/x$

$$\Rightarrow xy = c^2x - c$$

$$\Rightarrow xy - c^2x + c = 0$$

$$\Rightarrow \frac{dx}{dP} = \frac{2Px+1}{P(1-P)}$$

$$\Rightarrow \frac{dx}{dP} = \frac{2x}{1-P} + \frac{1}{P(1-P)}$$

$$\Rightarrow \frac{dx}{dP} + \left(\frac{2}{P-1}\right)x = \frac{-1}{P(P-1)} \quad \text{———} \quad (2)$$

It is linear in x, $f(P) =$

$$\therefore I.F = e^{\int f(P)dP} = e^{\int\frac{2dP}{P-1}} = e^{2\ln(P-1)} = (P-1)^2$$

Multiplying (2) by its I.F, we get

$$(P-1)^2\frac{dx}{dP} + 2(P-1)x = -\frac{P-1}{P}$$

$$\Rightarrow (P-1)^2dx + 2(P-1)x\,dP = -\left(\frac{P-1}{P}\right)dP$$

$$\Rightarrow d[x(P-1)^2] = (-1+1/P)\,dP$$

$$\Rightarrow \int d[x(P-1)^2] = \int(1/P - 1)\,dP$$

$$\Rightarrow x(P-1)^2 = \ln P - P + c$$

$$\Rightarrow x = \frac{c-P+\ln P}{(P-1)^2} \quad \text{———} \quad (3)$$

Putting value of in eq. (1), we get

$$y = P^2\left(\frac{c-P+\ln P}{(P-1)^2}\right) + P \quad \text{———} \quad (4)$$

(3), (4) give the parametric sol. of (1)

Solvable for X

The diff. eq. $f(x,y,P) = 0$ is said to be solvable for x if it cannot be factorized and can be put in the form

$$x = F(y,P)$$

Example

$$xP = 1 + P^2$$

Sol:-

$$x = \frac{1}{P} + P \quad \text{———} \quad (1)$$

Differentiating eq. (1) w.r.t y, we get.

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$$\frac{dx}{dy} = 1 - \frac{1}{P^2}\frac{dP}{dy} + \frac{dP}{dy}$$

$$\Rightarrow \quad \frac{1}{P} = \left(1 - \frac{1}{P^2}\right)\frac{dP}{dy}$$

$$\Rightarrow \quad dy = \left(P - \frac{1}{P}\right)dP$$

$$\Rightarrow \quad \int dy = \int \left(P - \frac{1}{P}\right)dP$$

$$\Rightarrow \quad y = \frac{P^2}{2} - \ln P + c \quad \text{——— (2)}$$

Thus (1), (2) give the general sol. of the given eq. in paramt. form

Clairaut's Eq.

An eq. of the type $y = xP + f(P)$ where $P = \frac{dy}{dx}$ is called Clairaut's Equation

Theorem

General solution of the eq. $y = xP + f(P)$ is $y = cx + f(c)$.

Proof

$$y = xP + f(P) \quad \text{——— (1)}$$

Differentiating (1) w.r.t x, we get

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + f'(P)\frac{dP}{dx}$$

$$\cancel{\ } + \left(x + f'(P)\right)\frac{dP}{dx}$$

Remark

In the above theorem, if we consider $x+f'(P)=0$

if we consider $x+f'(p)=0$

or $x=-f'(p)$ putting in eq. ① of the above theorem

we get $y=-Pf'(P)+f(P)$

The parametric Eqs.

$$x=-f'(P)$$
$$y=f(P)-Pf'(P)$$

represent the singular sol. of $y=xP+f(P)$

($\because$ This sol. involves no arbitrary constant. called singular sol.)

Example

Find the general sol. and singular sol. of $y=xP+\frac{1}{4}P^4$ —— ①

Sol:-

It is clairaut's eq.

General sol:-

$$y=cx+\frac{1}{4}c^4$$

Singular sol:-

know that,

the clairaut's eq.

form is

Example

Find the general sol. and singular sol. of $x^2(y-Px)=yP^2$ —— ①

Sol:- $yP^2+Px^3-x^2y=0$

It is not solvable for P, y, x

we can convert ① into clairaut's eq. as

Let $u=x^2$, $v=y^2$

$\therefore$ $du=2x\,dx$, $dv=2y\,dy$

Now $\dfrac{2y\,dy}{2x\,dx}=\dfrac{dv}{du}$

$\rightarrow$ $\dfrac{dy}{dx}=\dfrac{x\,dv}{y\,du}$

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We can eliminat P from ③, as

Since $P = (-x)^{1/3}$

$\therefore \quad y = -\frac{3}{4}(-x)^{4/3}$

$= -\frac{3}{4} x^{4/3}$

$\Rightarrow \quad 4y = -3x^{4/3}$

$\Rightarrow \quad 64y^3 = -3x^4$

$\Rightarrow \quad 64y^3 + 3x^4 = 0$ req. singul. sol.

It is clairaut's eq.

$\therefore$ its general sol. is

$v = cu + c^2$

$\Rightarrow \quad y^2 = cx^2 + c^2$ req. general sol.

Singular sol.

since $\quad v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$

$\Rightarrow \quad v = uq + q^2 \quad , \quad q = \frac{dv}{du}$

$\therefore$ singular sol of above eq. is

$$\left.\begin{aligned} u &= -f'(q) \\ v &= f(q) - q f'(q) \end{aligned}\right] \text{———} ②$$

where

$f(q) = q^2 \quad \therefore \quad f'(q) = 2q$

Hence ② becomes,

$$\left.\begin{aligned} u &= -2q \\ v &= q^2 - 2q^2 = -q^2 \end{aligned}\right] \text{———} ③$$

we can eliminate q from ③

since $\quad q = -u/2$

$\therefore \quad v = -(-u/2)^2$

$\Rightarrow \quad v = -u^2/4$

$\Rightarrow \quad y^2 = -\frac{1}{4}x^4$

$\Rightarrow \quad 4y^2 + x^4 = 0$ req. sing. sol. of ①

EXERCISE 9.8

(1) $P^2 + P - 6 = 0$

Sol:-

$$P^2 - 2P + 3P - 6 = 0$$

$$\Rightarrow P(P-2) + 3(P-2) = 0$$

$$\Rightarrow (P-2)(P+3) = 0$$

$$\Rightarrow P-2 = 0 \quad \text{or} \quad P+3 = 0$$

Now

$$P - 2 = 0$$

$$\Rightarrow \frac{dy}{dx} = 2$$

$$\Rightarrow dy = 2dx$$

$$\Rightarrow \int dy = 2\int dx$$

$$\Rightarrow y = 2x + c$$

$$\Rightarrow y - 2x - c = 0$$

$$P + 3 = 0$$

$$\Rightarrow \frac{dy}{dx} + 3 = 0$$

$$\Rightarrow dy = -3dx$$

$$\Rightarrow \int dy = -3\int dx$$

$$\Rightarrow y = -3x + c$$

$$\Rightarrow 3x + y - c = 0$$

Hence the general sol. is $(y-2x-c)(3x+y-c) = 0$

(2) $x^2P^2 + xyP - 6y^2 = 0$

Sol:-

$$x^2P^2 - 2xyP + 3xyP - 6y^2 = 0$$

$$\Rightarrow xP(xP-2y) + 3y(xP-2y) = 0$$

$$\Rightarrow (xP-2y)(xP+3y) = 0$$

$$\Rightarrow (xP-2y) = 0 \quad \text{or} \quad (xP+3y) = 0$$

Now $xP - 2y = 0$

$$\Rightarrow P = 2y/x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$$

$$\Rightarrow \ln y = 2\ln x + \ln c$$

$$\Rightarrow \ln y = \ln x^2 + \ln c$$

$$xP + 3y = 0$$

$$\Rightarrow P = -3y/x$$

$$\Rightarrow \frac{dy}{dx} = -3y/x$$

$$\Rightarrow \frac{dy}{y} = -\frac{3dx}{x}$$

$$\Rightarrow \int \frac{dy}{y} = -3\int \frac{dx}{x}$$

$\Rightarrow \ln y = \ln cx^2$

$\Rightarrow y = cx^2$

$\Rightarrow y - cx^2 = 0$

$\Rightarrow \ln y = -3\ln x + \ln c$

$\Rightarrow \ln y = \ln x^{-3} + \ln c$

$\Rightarrow \ln y = \ln cx^{-3}$

$\Rightarrow y = cx^{-3}$

$\Rightarrow y - c/x^3 = 0$

Hence the generol sol. is $(y-cx^2)(y-c/x^3) = 0$

3 $P^2y + (x-y)P - x = 0$

Sol:-

$P^2y + xP - yP - x = 0$

$\Rightarrow P(Py+x) - (Py+x) = 0$

$\Rightarrow (Py+x)(P-1) = 0$

$\Rightarrow Py + x = 0$ or $P - 1 = 0$

$Py + x = 0$

$\Rightarrow P = -x/y$

$\Rightarrow \dfrac{dy}{dx} = -x/y$

$\Rightarrow y\,dy = -x\,dx$

$\Rightarrow \int y\,dy = -\int x\,dx$

$\Rightarrow y^2/2 = -x^2/2 + c_1$

$\Rightarrow x^2 + y^2 = c$

$\Rightarrow x^2 + y^2 - c = 0$

$P - 1 = 0$

$\Rightarrow \dfrac{dy}{dx} = 1$

$\Rightarrow dy = dx$

$\Rightarrow \int dy = \int dx$

$\Rightarrow y = x + c$

$\Rightarrow y - x - c = 0$

Hence the req. sol. is $(x^2+y^2-c)(y-x-c) = 0$

4 $P^3 - (x^2+xy+y^2)P + x^2y + xy^2 = 0$

Sol:-

Since the given eq. is satisfied by $P = x$

x	1	0	$-x^2-xy-y^2$	x^2y+xy^2
		x	x^2	$-x^2y-xy^2$
	1	x	$-xy-y^2$	0

$\therefore (P-x)(P^2+xP-xy-y^2) = 0$

$\Rightarrow (P-x)[P^2-y^2+x(P-y)] = 0$

$\Rightarrow (P-x)[(P+y)(P-y)+x(P-y)] = 0$

$\Rightarrow (P-x)(P-y)(P+y+x) = 0$

$\Rightarrow P-x=0$ or $P-y=0$ or $P+x+y=0$

$P-x=0$

$\Rightarrow \dfrac{dy}{dx} = x$

$\Rightarrow dy = x\,dx$

$\Rightarrow \int dy = \int x\,dx$

$\Rightarrow y = x^2/2 + c_1$

$\Rightarrow 2y - x^2 = 2c_1$

$\Rightarrow 2y - x^2 - c = 0$

$P-y=0$

$\Rightarrow \dfrac{dy}{dx} = y$

$\Rightarrow \dfrac{dy}{y} = dx$

$\Rightarrow \int \dfrac{dy}{y} = \int dx$

$\Rightarrow \ln y = x + c_2$

$\Rightarrow \ln y = \ln e^x + \ln c$

$\Rightarrow \ln y = \ln ce^x$

$\Rightarrow y = ce^x$

$\Rightarrow y - ce^x = 0$

$P+x+y=0$

$\Rightarrow \dfrac{dy}{dx} + y = -x$ (linear in y)

$I.F = e^{\int 1 dx} = e^x$

Multiplying the above eq. by I.F, we get

$e^x \dfrac{dy}{dx} + ye^x = -xe^x$

$\Rightarrow e^x dy + ye^x dx = -xe^x dx$

$\Rightarrow d(ye^x) = -xe^x dx$

$\Rightarrow \int d(ye^x) = -\int xe^x dx$

$\Rightarrow ye^x = -[x\cdot e^x - \int e^x dx]$ by Parts

$= -xe^x + e^x + c$

$\Rightarrow y = -x + 1 + ce^{-x}$

$\Rightarrow x + y - 1 - ce^{-x} = 0$

Hence the req. sol. is $(2y - x^2 - c)(x + y - 1 - ce^{-x})(y - ce^x) = 0$

5 $xP^2 + (y-1-x^2)P - x(y-1) = 0$

Sol:-

Since the given eq. is satisfied by $P = x$

x	x	$y-1-x^2$	$-xy+x$
		x^2	$xy-x$
	x	$y-1$	0

$\therefore (P-x)(xP + y - 1) = 0$

$\Rightarrow P - x = 0$, or $xP + y - 1 = 0$

$P - x = 0$

$\Rightarrow \dfrac{dy}{dx} = x$

$\Rightarrow dy = x\,dx$

$\Rightarrow \int dy = \int x\,dx$

$\Rightarrow y = x^2/2 + c_1$

$\Rightarrow 2y - x^2 = 2c_1$

$\Rightarrow 2y - x^2 - c' = 0$

$xP + y - 1 = 0$

$x\dfrac{dy}{dx} + y = 1$

$\Rightarrow x\,dy + y\,dx = dx$

$\Rightarrow d(xy) = dx$

$\Rightarrow \int d(xy) = \int dx$

$\Rightarrow xy = x + c_2$

$\Rightarrow xy - x - c_2 = 0$

Hence the req. sol. is $(2y - x^2 - c)(xy - x - c_2) = 0$

6. $xyP^2 + (x+y)P + 1 = 0$

Sol:-

$xyP^2 + xP + yP + 1 = 0$

$\Rightarrow xP(yP+1) + (yP+1) = 0$

$\Rightarrow (yP+1)(xP+1) = 0$

$\Rightarrow yP+1 = 0$ or $xP+1 = 0$

$yP + 1 = 0$

$\Rightarrow y\dfrac{dy}{dx} + 1 = 0$

$\Rightarrow y\,dy = -dx$

$\Rightarrow \int y\,dy = -\int dx$

$\Rightarrow y^2/2 = -x + c_1$

$\Rightarrow y^2 = -2x + 2c_1$

$\Rightarrow y^2 + 2x - c = 0$

$xP + 1 = 0$

$\Rightarrow x\dfrac{dy}{dx} = -1$

$\Rightarrow dy = -\dfrac{dx}{x}$

$\Rightarrow \int dy = -\int \dfrac{dx}{x}$

$\Rightarrow y = -\ln x + \ln c$

$\Rightarrow y = \ln \dfrac{c}{x}$

$\Rightarrow y - \ln\dfrac{c}{x} = 0$

Hence the req. sol. is $(y^2+2x-c)(y - \ln\frac{c}{x}) = 0$

7. $P^2 - (x^2y+3)P + 3x^2y = 0$

Sol:-

$P^2 - x^2yP - 3P + 3x^2y = 0$

$\Rightarrow P(P-x^2y) - 3(P-x^2y) = 0$

$\Rightarrow (P-x^2y)(P-3) = 0$

$\Rightarrow P-x^2y = 0$ or $P-3 = 0$

$P - x^2y = 0$

$\Rightarrow \dfrac{dy}{dx} = x^2y$

$\Rightarrow \dfrac{dy}{y} = x^2dx$

$\Rightarrow \int\dfrac{dy}{y} = \int x^2dx$

$\Rightarrow \ln y = x^3/3 + \ln c_1$

$\Rightarrow \ln y + \ln c = x^3/3$

$\Rightarrow \ln cy - x^3/3 = 0$

$\Rightarrow 3\ln cy - x^3 = 0$

$P - 3 = 0$

$\Rightarrow \dfrac{dy}{dx} = 3$

$\Rightarrow dy = 3dx$

$\Rightarrow \int dy = 3\int dx$

$\Rightarrow y = 3x + c$

$\Rightarrow y - 3x - c = 0$

Hence the req. sol. is $(3\ln cy - x^3)(y-3x-c) = 0$

8 $yP^2 + (x - y^2)P - xy = 0$

Sol:-

$$yP^2 + xP - y^2P - xy = 0$$
$$\Rightarrow P(yP + x) - y(yP + x) = 0$$
$$\Rightarrow (yP + x)(P - y) = 0$$
$$\Rightarrow yP + x = 0 \quad \text{or} \quad P - y = 0$$

$$yP + x = 0$$
$$\Rightarrow y\frac{dy}{dx} = -x$$
$$\Rightarrow y\,dy = -x\,dx$$
$$\Rightarrow \int y\,dy = -\int x\,dx$$
$$\Rightarrow y^2/2 = -x^2/2 + c_1$$
$$\Rightarrow y^2 = -x^2 + 2c_1$$
$$\Rightarrow y^2 + x^2 - c = 0$$

$$P - y = 0$$
$$\Rightarrow \frac{dy}{dx} = y$$
$$\Rightarrow \frac{dy}{y} = dx$$
$$\Rightarrow \int \frac{dy}{y} = \int dx$$
$$\Rightarrow \ln y = x + \ln c$$
$$\Rightarrow \ln y = \ln e^x + \ln c$$
$$\Rightarrow \ln y = \ln ce^x$$
$$\Rightarrow y = ce^x \Rightarrow y - ce^x = 0$$

Hence the req. sol. is $(y^2 + x^2 - c)(y - ce^x) = 0$

9 $(y+x)^2P^2 + (2y^2 + xy - x^2)P + y(y - x) = 0$

Sol:-

$$\text{Since } 2y^2 + xy - x^2 = 2y^2 + 2xy - xy - x^2$$
$$= 2y(y+x) - x(y+x)$$
$$= (x+y)(2y - x)$$

$$\therefore (y+x)^2P^2 + (y+x)(2y - x)P + y(y - x) = 0$$
$$\Rightarrow (y+x)^2P^2 + (y+x)(y + y - x)P + y(y - x) = 0$$
$$\Rightarrow (y+x)^2P^2 + (y+x)yP + (y+x)(y-x)P + y(y-x) = 0$$
$$\Rightarrow (y+x)P[(y+x)P + y] + (y-x)[(y+x)P + y] = 0$$
$$\Rightarrow [(y+x)P + y][(y+x)P + y - x] = 0$$
$$\Rightarrow (y+x)P + y = 0 \quad \text{or} \quad (y+x)P + y - x = 0$$

$(y+x)P + y = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-y}{y+x}$

$\Rightarrow (y+x)\,dy = -y\,dx$

$\Rightarrow y\,dy + x\,dy = -y\,dx$

$\Rightarrow x\,dy + y\,dx = -y\,dy$

$\Rightarrow \int d(xy) = -\int y\,dy$

$\Rightarrow xy = -\dfrac{y^2}{2} + c$

$\Rightarrow xy + \dfrac{y^2}{2} - c = 0$

$(y+x)P + y - x = 0$

$\Rightarrow (y+x)\dfrac{dy}{dx} = x-y$

$\Rightarrow (y+x)\,dy = (x-y)\,dx$

$\Rightarrow y\,dy + x\,dy = x\,dx - y\,dx$

$\Rightarrow y\,dy + x\,dy + y\,dx = x\,dx$

$\Rightarrow \int y\,dy + \int d(xy) = \int x\,dx$

$\Rightarrow \dfrac{y^2}{2} + xy = \dfrac{x^2}{2} + c$

$\Rightarrow xy - \dfrac{x^2}{2} - \dfrac{y^2}{2} - c = 0$

Hence the req. sol. is $\left(xy + \dfrac{y^2}{2} - c\right)\left(xy - \dfrac{x^2}{2} - \dfrac{y^2}{2} - c\right) = 0$

Q10 $xy(x^2+y^2)(P^2-1) = P(x^4+x^2y^2+y^4)$

Sol:-

$xy(x^2+y^2)P^2 - xy(x^2+y^2) - P[(x^2+y^2)^2 - x^2y^2] = 0$

$\Rightarrow xy(x^2+y^2)P^2 - xy(x^2+y^2) - P(x^2+y^2)^2 + Px^2y^2 = 0$

$\Rightarrow P(x^2+y^2)[xyP - (x^2+y^2)] + xy[Pxy - (x^2+y^2)] = 0$

$\Rightarrow [xyP - (x^2+y^2)][P(x^2+y^2) + xy] = 0$

$\Rightarrow xyP - (x^2+y^2) = 0$ or $P(x^2+y^2) + xy = 0$

$xyP - (x^2+y^2) = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$

$\Rightarrow (x^2+y^2)\,dx - xy\,dy = 0$ ——— ①

$(M\,dx + N\,dy = 0)$

Let $M = x^2+y^2$; $N = -xy$

$\therefore \dfrac{\partial M}{\partial y} = 2y$; $\dfrac{\partial N}{\partial x} = -y$

① is not exact, we find I·F of ①

$\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \dfrac{2y+y}{-xy} = \dfrac{3y}{-xy} = -\dfrac{3}{x} = P(x)$

$P(x^2+y^2) + xy = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-xy}{x^2+y^2}$

$\Rightarrow xy\,dx + (x^2+y^2)\,dy = 0$ ——— ③

$(M\,dx + N\,dy = 0)$

Let $M = xy$; $N = x^2+y^2$

$\therefore \dfrac{\partial M}{\partial y} = x$; $\dfrac{\partial N}{\partial x} = 2x$

③ is not exact, we find I·F,

$\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \dfrac{x-2x}{x^2+y^2} = \dfrac{-x}{x^2+y^2} \neq P(x)$

$$I\cdot F = e^{\int -3/x\,dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$$

Multiplying ① by its I·F, we get

$$(x^{-1} + x^{-3}y^2)\,dx - x^{-2}y\,dy = 0 \quad \text{——②}$$

② is exact and here

$$M = x^{-1} + x^{-3}y^2, \quad N = -x^{-2}y$$

$$\int M\,dx = \int (x^{-1} + x^{-3}y^2)\,dx \quad (y \text{ is const.})$$

$$= \int \frac{dx}{x} + y^2\int x^{-3}\,dx$$

$$= \ln x + y^2\, x^{-2}/-2$$

Hence sol. of ② is,

$$\ln x - \frac{y^2}{2x^2} = c$$

$$\Rightarrow 2x^2\ln x - y^2 = 2cx^2$$

$$\Rightarrow 2x^2\ln x - y^2 - 2cx^2 = 0$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2x - x}{xy} = \frac{x}{xy} = \frac{1}{y} = P(y)$$

$$I\cdot F = e^{\int 1/y\,dy} = e^{\ln y} = y$$

Multiplying ③ by its I·F, we get.

$$xy^2\,dx + (x^2y + y^3)\,dy = 0 \quad \text{——④}$$

④ is exact, and here

$$M = xy^2 \quad ; \quad N = x^2y + y^3$$

$$\int M\,dx = \int xy^2\,dx \quad , \ (y \text{ is constant})$$

$$= x^2y^2/2$$

$$\int y^3\,dy = y^4/4$$

Hence sol. of ④ is,

$$x^2y^2/2 + y^4/4 = c_1$$

$$\text{or} \quad 2x^2y^2 + y^4 = 4c_1$$

$$\text{or} \quad 2x^2y^2 + y^4 - c = 0$$

Hence req. sol. is $(2x^2\ln x - y^2 - 2cx^2)(2x^2y^2 + y^4 - c) = 0$

11 $\quad xP^2 - 3yP + 9x^2 = 0$

Sol:-

$$3yP = xP^2 + 9x^2$$

$$\Rightarrow \quad y = \tfrac{1}{3}xP + 3x^2P^{-1} \quad \text{———①}$$

Diff. the above eq. w.r.t x

$$\frac{dy}{dx} = \tfrac{1}{3}\left(x\frac{dP}{dx} + P\right) + 3\left(-x^2P^{-2}\frac{dP}{dx} + 2xP^{-1}\right)$$

$$\Rightarrow P = \tfrac{1}{3}x\frac{dP}{dx} + \frac{P}{3} - 3x^2P^{-2}\frac{dP}{dx} + 6xP^{-1}$$

$$\Rightarrow P = x/3\,(1 - 9xP^{-2})\frac{dP}{dx} + \frac{P}{3} + 6xP^{-1}$$

$$\Rightarrow \tfrac{2}{3}P - 6xP^{-1} = \tfrac{x}{3}(1 - 9xP^{-2})\frac{dP}{dx}$$

$$\Rightarrow \tfrac{2}{3}P(1 - 9xP^{-2}) - \tfrac{x}{3}(1 - 9xP^{-2})\frac{dP}{dx} = 0$$

$$\Rightarrow \tfrac{1}{3}(1 - 9xP^{-2})\left(2P - x\frac{dP}{dx}\right) = 0$$

$$2P = x\frac{dP}{dx}$$

12 $\quad P^2 + x^3P - 2x^2y = 0$

Sol:-

$$2x^2y = P^2 + x^3P$$

$$\Rightarrow \quad y = \tfrac{1}{2}x^{-2}P^2 + \frac{xP}{2} \quad \text{———①}$$

Diff. the above eq. w.r.t. x, we get

$$\frac{dy}{dx} = \tfrac{1}{2}\left(x^{-2}\,2P\frac{dP}{dx} - 2x^{-3}P^2\right) + \tfrac{1}{2}\left(x\frac{dP}{dx} + P\right)$$

$$\Rightarrow P = \frac{P}{x^2}\frac{dP}{dx} - \frac{P^2}{x^3} + \tfrac{1}{2}x\frac{dP}{dx} + \frac{P}{2}$$

$$\Rightarrow \left(P/x^2 + x/2\right)\frac{dP}{dx} - P^2/x^3 - P/2 = 0$$

$$\Rightarrow x\left(P/x^3 + 1/2\right)\frac{dP}{dx} - P\left(P/x^3 + \tfrac{1}{2}\right) = 0$$

$$\Rightarrow \left(P/x^3 + 1/2\right)\left(x\frac{dP}{dx} - P\right) = 0$$

$$\Rightarrow P/x^3 + 1/2 = 0 \quad \text{or} \quad x\frac{dP}{dx} - P = 0$$

$$\Rightarrow 1-9xp^{-2}=0 \quad \text{or} \quad 2p-x\frac{dp}{dx}=0$$

Consider;

$$2p-x\frac{dp}{dx}=0$$

$$\Rightarrow x\frac{dp}{dx}=2p$$

$$\Rightarrow \frac{dp}{dx}=\frac{2p}{x}$$

$$\Rightarrow \int\frac{dp}{p}=\int 2\frac{dx}{x}$$

$$\Rightarrow \ln p = 2\ln x+\ln c$$

$$\Rightarrow \ln p = \ln x^2+\ln c$$

$$\Rightarrow \ln p = \ln cx^2$$

$$\Rightarrow p = cx^2$$

Put this value of p in eq. ①

we get, $y=\frac{1}{3}x\cdot cx^2+\frac{3x^2}{cx^2}$

$$\Rightarrow cy = \frac{1}{3}c^2x^3+3$$

$$\Rightarrow 3cy-c^2x^3-9=0$$

$$\Rightarrow c^2x^3-3cy+9=0$$

13 $p^2+4x^5p-12x^4y=0$

Sol:-

$$12x^4y = p^2+4x^5p$$

$$\Rightarrow y = \frac{p^2}{12x^4}+\frac{xp}{3}$$

$$\Rightarrow y = \tfrac{1}{12}x^{-4}p^2+\tfrac{1}{3}xp \quad \text{——} ①$$

Diff. ① w.r.t x, we get.

$$\frac{dy}{dx}=\frac{1}{12}\left(x^{-4}\cdot 2p\frac{dp}{dx}-4p^2x^{-5}\right)+\frac{1}{3}\left(x\frac{dp}{dx}+p\right)$$

$$\Rightarrow p = \tfrac{1}{6}x^{-4}p\frac{dp}{dx}-\tfrac{1}{3}p^2x^{-5}+\tfrac{1}{3}x\frac{dp}{dx}+\tfrac{1}{3}p$$

$$\Rightarrow \frac{2}{3}p = \frac{1}{6x^4}p\frac{dp}{dx}+\frac{1}{3}x\frac{dp}{dx}-\frac{1}{3x^5}p^2$$

Consider

$$x\frac{dp}{dx}=p$$

$$\Rightarrow \int\frac{dp}{p}=\int\frac{dx}{x}$$

$$\Rightarrow \ln p = \ln x+\ln c$$

$$\Rightarrow \ln p = \ln cx$$

$$\Rightarrow p = cx$$

To eliminate p from ① put this value of p in ①, we get

$$y=\frac{1}{2}\frac{c^2x^2}{x^2}+\frac{x\cdot cx}{2}$$

$$\Rightarrow y = c^2/2+cx^2/2$$

$$\Rightarrow 2y = c^2+cx^2$$

14 $x^8p^2+3xp+9y=0$

Sol:-

$$y=-\tfrac{1}{9}(x^8p^2+3xp) \quad \text{——} ①$$

Diff. ① w.r.t x, we get.

$$\frac{dy}{dx}=-\tfrac{1}{9}\left(8x^7p^2+2x^8p\frac{dp}{dx}\right)-\tfrac{1}{3}\left(x\frac{dp}{dx}+p\right)$$

$$\Rightarrow p = -\tfrac{8}{9}p^2x^7-\tfrac{2}{9}px^8\frac{dp}{dx}-\tfrac{x}{3}\frac{dp}{dx}-\tfrac{p}{3}$$

$$\Rightarrow \frac{4p}{3}+\frac{8p^2}{9}x^7=-\tfrac{x}{3}\left(1+\tfrac{2}{3}x^7p\right)\frac{dp}{dx}$$

$$\Rightarrow \tfrac{4}{3}p\left(1+\tfrac{2}{3}px^7\right)+\tfrac{x}{3}\left(1+\tfrac{2}{3}x^7p\right)\frac{dp}{dx}=0$$

$$\Rightarrow \left(1+\tfrac{2}{3}px^7\right)\left(\tfrac{4}{3}p+\tfrac{x}{3}\frac{dp}{dx}\right)=0$$

$$\Rightarrow 1+\tfrac{2}{3}px^7=0 \quad \text{or} \quad \tfrac{4}{3}p+\tfrac{x}{3}\frac{dp}{dx}=0$$

Consider,

$$\tfrac{4}{3}p+\tfrac{x}{3}\frac{dp}{dx}=0$$

$$\Rightarrow \tfrac{x}{3}\frac{dp}{dx}=-\tfrac{4}{3}p$$

$$\Rightarrow \frac{2}{3}P + \frac{1}{3x^5}P^2 = \frac{1}{3}\left(\frac{1}{2x^4}P + x\right)\frac{dP}{dx}$$

$$\Rightarrow \frac{2P}{3}\left(1+\frac{P}{2x^5}\right) = \frac{x}{3}\left(1+\frac{P}{2x^5}\right)\frac{dP}{dx}$$

$$\Rightarrow \frac{2P}{3}\left(1+\frac{P}{2x^5}\right) - \frac{x}{3}\left(1+\frac{P}{2x^5}\right)\frac{dP}{dx} = 0$$

$$\Rightarrow \left(1+\frac{P}{2x^5}\right)\left(\frac{2P}{3} - \frac{x}{3}\frac{dP}{dx}\right) = 0$$

$$\Rightarrow 1+\frac{P}{2x^5} = 0 \quad \text{or} \quad \frac{2P}{3} - \frac{x}{3}\frac{dP}{dx} = 0$$

consider

$$\frac{2P}{3} - \frac{x}{3}\frac{dP}{dx} = 0$$

$$\Rightarrow \frac{x}{3}\frac{dP}{dx} = \frac{2}{3}P$$

$$\Rightarrow \int\frac{dP}{P} = 2\int\frac{dx}{x}$$

$$\Rightarrow \ln P = 2\ln x + \ln c$$

$$\Rightarrow \ln P = \ln x^2 + \ln c$$

$$\Rightarrow \ln P = \ln cx^2$$

$$\Rightarrow P = cx^2$$

Put above value of P in ①
we get,

$$y = \frac{1}{12x^4}\cdot c^2x^4 + \frac{1}{3}x\cdot cx^2$$

$$\Rightarrow y = c^2/12 + cx^3/3$$

$$\Rightarrow 12y = c^2 + 4cx^3$$

$$\Rightarrow 12y = c(c+4x^3)$$

(15) $P^2 + 3xP - y = 0$

Sol:-

$$y = 3xP + P^2 \quad \text{———} ①$$

Diff. ① w.r.t x, we get

$$\frac{dy}{dx} = 3\left(x\frac{dP}{dx} + P\right) + 2P\frac{dP}{dx}$$

$$\Rightarrow P = 3x\frac{dP}{dx} + 3P + 2P\frac{dP}{dx}$$

$$\Rightarrow \int\frac{dP}{P} = -4\int\frac{dx}{x}$$

$$\Rightarrow \ln P = -4\ln x + \ln c$$

$$\Rightarrow \ln P = \ln x^{-4} + \ln c$$

$$\Rightarrow \ln P = \ln cx^{-4}$$

$$\Rightarrow P = cx^{-4}$$

Put above value of P, in ①,
we get,

$$y = -\frac{1}{9}\left(x^8\cdot c^2x^{-8} + 3x\cdot cx^{-4}\right)$$

$$= -1/9\left(c^2 + 3cx^{-3}\right)$$

$$\Rightarrow -9y = c^2 + \frac{3c}{x^3}$$

$$\Rightarrow -9x^3y = c^2x^3 + 3c$$

(16) $y = Px + x^3P^2$ ——— ①

Sol:-

Diff. eq. ①, w.r.t x, we get

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + 2x^3P\frac{dP}{dx} + 3x^2P^2$$

$$\Rightarrow \cancel{P} = \cancel{P} + x(1+2x^2P)\frac{dP}{dx} + 3x^2P^2$$

$$\Rightarrow (1+2x^2P)\frac{dP}{dx} = -3xP^2$$

$$\Rightarrow \frac{dP}{dx} = \frac{-3xP^2}{1+2x^2P}$$

$$\Rightarrow \frac{dx}{dP} = \frac{1+2x^2P}{-3xP^2}$$

$$\Rightarrow \frac{dx}{dP} = -\frac{1}{3xP^2} - \frac{2x}{3P}$$

$$\Rightarrow \frac{dx}{dP} + \frac{2}{3P}x = -\frac{1}{3P^2}x^{-1}$$

(It is Bernoulli eq.)

Multiplying the above eq. by x

i-e $$x\frac{dx}{dP} + \frac{2}{3P}x^2 = -\frac{1}{3P^2}$$

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$\Rightarrow -2P = (3x+2P)\frac{dP}{dx}$

$\Rightarrow (3x+2P)\frac{dP}{dx} = -2P$

$\Rightarrow \frac{dP}{dx} = \frac{-2P}{3x+2P}$

$\Rightarrow \frac{dx}{dP} = \frac{3x+2P}{-2P}$

$\Rightarrow \frac{dx}{dP} = -\frac{3x}{2P} - 1$

$\Rightarrow \frac{dx}{dP} + \frac{3}{2P}x = -1$

(It is linear in x)

$$I.F = e^{\int \frac{3}{2P}dP} = e^{\frac{3}{2}\ln P} = e^{\ln P^{3/2}} = P^{3/2}$$

Multiplying the above eq. by I.F

$P^{3/2}\frac{dx}{dP} + \frac{3x}{2P}P^{3/2} = -P^{3/2}$

$\Rightarrow P^{3/2}dx + \frac{3}{2}xP^{1/2}dP = -P^{3/2}dP$

$\Rightarrow d(xP^{3/2}) = -P^{3/2}dP$

$\Rightarrow \int d(xP^{3/2}) = -\int P^{3/2}dP$

$\Rightarrow xP^{3/2} = -\frac{P^{5/2}}{5/2} + c.$

$\Rightarrow x = -\frac{2}{5}P + cP^{-3/2}$ ——— ②

It is difficult to find value of P.

So putting this value of x in eq. ①

we get,

$y = 3P(-2P/5 + cP^{-3/2}) + P^2$ ——— ③

Thus ②, ③, is a parametric sol. of the given eq.

Let $v = x^2 \therefore \frac{dv}{dP} = 2x\frac{dx}{dP}$

$\Rightarrow \frac{1}{2}\frac{dv}{dP} = x\frac{dx}{dP}$

Hence the above eq. become:

$\frac{1}{2}\frac{dv}{dP} + \frac{2v}{3P} = -\frac{1}{3P^2}$

$\Rightarrow \frac{dv}{dP} + \frac{4v}{3P} = -\frac{2}{3P^2}$

(It is linear in v)

$$I.F = e^{\int \frac{4}{3P}dP} = e^{\frac{4}{3}\ln P} = e^{\ln P^{4/3}} = P^{4/3}$$

Multiplying the above eq. by I.F we get,

$P^{4/3}\frac{dv}{dP} + \frac{4vP^{1/3}}{3} = -\frac{2}{3}P^{-2/3}$

$\Rightarrow P^{4/3}dv + \frac{4}{3}vP^{1/3}dP = -\frac{2}{3}P^{-2/3}dP$

$\Rightarrow d(vP^{4/3}) = -2/3\, P^{-2/3}dP$

$\Rightarrow \int d(vP^{4/3}) = -2/3\int P^{-2/3}dP$

$\Rightarrow vP^{4/3} = -\frac{2}{3}\frac{P^{1/3}}{1/3} + c$

$\Rightarrow vP^{4/3} = -2P^{1/3} + c$

$\Rightarrow x^2P^{4/3} = -2P^{1/3} + c \quad \because v = x^2$

$\Rightarrow x^2 = -2/P + c/P^{4/3}$ ——— ②

It is difficult to find the value of P.

So putting this value of x in ①

we get

$y = P(-\frac{2}{P} + cP^{-4/3})^{1/2} + P^2(-2/P + cP^{-4/3})^{3/2}$ ——— ③

Hence ②, ③ give the parametric sol. of the given eq.

17 $xP^2 - 2yP + ax = 0$

Sol:–

$$2yP = ax + xP^2$$

$$\Rightarrow y = \frac{a}{2}xP^{-1} + \frac{1}{2}xP \quad \text{——} ①$$

Diff. ①, w.r.t x, we get,

$$\frac{dy}{dx} = \frac{a}{2}\left(-xP^{-2}\frac{dP}{dx} + P^{-1}\right) + \frac{1}{2}\left(x\frac{dP}{dx} + P\right)$$

$$\Rightarrow P = -\tfrac{1}{2}axP^{-2}\frac{dP}{dx} + \tfrac{1}{2}aP^{-1} + \tfrac{1}{2}x\frac{dP}{dx} + \tfrac{P}{2}$$

$$\Rightarrow \tfrac{P}{2} = \tfrac{1}{2}x(1 - aP^{-2})\frac{dP}{dx} + \tfrac{1}{2}aP^{-1}$$

$$\Rightarrow \tfrac{1}{2}x(1 - aP^{-2})\frac{dP}{dx} + \tfrac{1}{2}aP^{-1} - \tfrac{1}{2}P = 0$$

$$\Rightarrow \tfrac{1}{2}x(1 - aP^{-2})\frac{dP}{dx} - \tfrac{1}{2}P(1 - aP^{-2}) = 0$$

$$\Rightarrow \tfrac{1}{2}(1 - aP^{-2})\left(x\frac{dP}{dx} - P\right) = 0$$

$$\Rightarrow 1 - aP^{-2} = 0 \quad \text{or} \quad x\frac{dP}{dx} - P = 0$$

Consider,

$$x\frac{dP}{dx} - P = 0$$

$$\Rightarrow x\frac{dP}{dx} = P$$

$$\Rightarrow \int\frac{dP}{P} = \int\frac{dx}{x}$$

$$\Rightarrow \ln P = \ln x + \ln c$$

$$\Rightarrow \ln P = \ln cx$$

$$\Rightarrow P = cx$$

Put this value of P in ①

We get $y = \frac{ax}{2}(cx)^{-1} + \frac{1}{2}x \cdot cx$

$$\Rightarrow y = \frac{ac^{-1}}{2} + \frac{cx^2}{2}$$

$$\Rightarrow 2cy = a + c^2x^2$$

$$\Rightarrow c^2x^2 - 2cy + a = 0$$

18 $P = \tan\left(x - \frac{P}{1+P^2}\right)$

Sol:–

$$\tan^{-1}P = x - \frac{P}{1+P^2}$$

$$\Rightarrow x = \tan^{-1}P + \frac{P}{1+P^2} \quad \text{——} ①$$

Diff. ① w.r.t y we get

$$\frac{dx}{dy} = \frac{1}{1+P^2}\frac{dP}{dy} + \frac{(1+P^2)\frac{dP}{dy} - P\cdot 2P\frac{dP}{dy}}{(1+P^2)^2}$$

$$\Rightarrow \frac{1}{P} = \frac{1}{1+P^2}\frac{dP}{dy} + \left[\frac{1-P^2}{(1+P^2)^2}\right]\frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = \left[\frac{1}{1+P^2} + \frac{1-P^2}{(1+P^2)^2}\right]\frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = \left(\frac{1+P^2+1-P^2}{(1+P^2)^2}\right)\frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = \frac{2}{(1+P^2)^2}\frac{dP}{dy}$$

$$\Rightarrow dy = \frac{2P}{(1+P^2)^2}dP$$

$$\Rightarrow \int dy = \int(1+P^2)^{-2}\cdot 2P\,dP$$

$$\Rightarrow y = -(1+P^2)^{-1} + c$$

$$\Rightarrow y = -\frac{1}{1+P^2} + c \quad \text{——} ②$$

①, ② give the parametric sol. of the given eq.

20 $aP^2 + Py - x = 0$

Sol:–

$$x = Py + aP^2 \quad \text{——} ①$$

Diff. ① w.r.t. y, we get.

$$\frac{dx}{dy} = P + y\frac{dP}{dy} + 2aP\frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = P + (y + 2aP)\frac{dP}{dy}$$

19 $p^3 - 4xyp + 8y^2 = 0$

Sol:

$4xyp = p^3 + 8y^2$

$\Rightarrow x = \frac{1}{4}p^2y^{-1} + 2p^{-1}y$ —— ①

Diff. ① w.r.t y, we get

$$\frac{dx}{dy} = \frac{1}{4}\left(-p^2y^{-2} + 2y^{-1}p\frac{dp}{dy}\right) + 2\left(-yp^{-2}\frac{dp}{dy} + p^{-1}\right)$$

$$\Rightarrow \frac{1}{p} = -\frac{p^2}{4y^2} + \frac{p}{2y}\frac{dp}{dy} - \frac{2y}{p^2}\frac{dp}{dy} + \frac{2}{p}$$

$$\Rightarrow 4py^2 = -p^4 + 2yp^3\frac{dp}{dy} - 8y^3\frac{dp}{dy} + 8y^2p$$

$$\Rightarrow -4py^2 + p^4 = 2yp^3\frac{dp}{dy} - 8y^3\frac{dp}{dy}$$

$$\Rightarrow p(p^3 - 4y^2) = 2y(p^3 - 4y^2)\frac{dp}{dy}$$

$$\Rightarrow 2y(p^3 - 4y^2)\frac{dp}{dy} - p(p^3 - 4y^2) = 0$$

$$\Rightarrow (p^3 - 4y^2)\left(2y\frac{dp}{dy} - p\right) = 0$$

$$\Rightarrow p^3 - 4y^2 = 0 \quad \text{or} \quad 2y\frac{dp}{dy} - p = 0$$

Consider,

$$2y\frac{dp}{dy} - p = 0$$

$$\Rightarrow 2y\frac{dp}{dy} = p$$

$$\Rightarrow 2\int\frac{dp}{p} = \int\frac{dy}{y}$$

$$\Rightarrow 2\ln p = \ln y + \ln c_1$$

$$\Rightarrow \ln p^2 = \ln c_1 y$$

$$\Rightarrow p^2 = c_1 y$$

$\Rightarrow y = p^2/c_1$ or $y = cp^2$ —— ②

Put this value of y in ①, we get

$$x = \frac{1}{4}p^2 \cdot c^{-1}p^{-2} + 2p^{-1}\cdot cp^2$$

$$\Rightarrow x = \frac{1}{4c} + 2cp$$

$\Rightarrow x = (1 + 8c^2p)/4c$ —— ③

②, ③ give param. sol. of given eq.

$$\Rightarrow (y + 2ap)\frac{dp}{dy} = \frac{1}{p} - p$$

$$= \frac{1 - p^2}{p}$$

$$\Rightarrow \frac{dp}{dy} = \frac{1 - p^2}{p(y + 2ap)}$$

$$\Rightarrow \frac{dy}{dp} = \frac{py + 2ap^2}{1 - p^2}$$

$$\Rightarrow \frac{dy}{dp} + \frac{p}{p^2 - 1}y = \frac{-2ap^2}{p^2 - 1}$$

(It is linear in y)

$$\text{I.F.} = e^{\int\frac{p}{p^2-1}dp} = e^{\frac{1}{2}\int\frac{2p\,dp}{p^2-1}} = e^{\frac{1}{2}\ln(p^2-1)} = e^{\ln(p^2-1)^{1/2}} = \sqrt{p^2-1}$$

Multiplying the above eq. by I.F.

$$\sqrt{p^2-1}\,\frac{dy}{dp} + \frac{p}{\sqrt{p^2-1}}y = \frac{-2ap^2}{\sqrt{p^2-1}}$$

$$\Rightarrow \sqrt{p^2-1}\,dy + \frac{p}{\sqrt{p^2-1}}y\,dp = \frac{-2ap^2}{\sqrt{p^2-1}}dp$$

$$\Rightarrow \int d\left(y\sqrt{p^2-1}\right) = \int\frac{-2ap^2}{\sqrt{p^2-1}}dp$$

$$\Rightarrow y\sqrt{p^2-1} = -2a\int\frac{p^2}{\sqrt{p^2-1}}dp$$

$$= -2a\int\frac{(p^2-1)+1}{\sqrt{p^2-1}}dp$$

$$= -2a\int\sqrt{p^2-1}\,dp - 2a\int\frac{dp}{\sqrt{p^2-1}}$$

$$= -2a\left[\frac{p\sqrt{p^2-1}}{2} - \frac{1}{2}\cosh^{-1}p\right] - 2a\cosh^{-1}p$$

$$= -ap\sqrt{p^2-1} + a\cosh^{-1}p - 2a\cosh^{-1}p$$

$$= -ap\sqrt{p^2-1} - a\cosh^{-1}p + c$$

$$\Rightarrow y = -ap + \frac{c - a\cosh^{-1}p}{\sqrt{p^2-1}}$$

21 $e^{4x}(P-1) + e^{2y}P^2 = 0$ —— ①

Sol:-

① is not solvable for P, for y so, we convert ①, in clairot's eq. as,

Let $u = e^{2x}$, $v = e^{2y}$

∴ $du = 2e^{2x}dx$, $dv = 2e^{2y}dy$

Now $\frac{2e^{2y}dy}{2e^{2x}dx} = \frac{dv}{du}$

⇒ $\frac{v\,dy}{u\,dx} = \frac{dv}{du}$

⇒ $\frac{dy}{dx} = \frac{u}{v}\frac{dv}{du}$

⇒ $P = \frac{u}{v}\frac{dv}{du}$ ∵ $P = \frac{dy}{dx}$

Hence eq. ① becomes,

$u^2\left(\frac{u}{v}\frac{dv}{du} - 1\right) + v\left(\frac{u}{v}\frac{dv}{du}\right)^2 = 0$

⇒ $\frac{u}{v}\frac{dv}{du} - 1 + \frac{u}{v}\left(\frac{dv}{du}\right)^2 = 0$

⇒ $u\frac{dv}{du} - v + \left(\frac{dv}{du}\right)^2 = 0$

⇒ $v = u\left(\frac{dv}{du}\right) + \left(\frac{dv}{du}\right)^2$

which is clairaut's eq.

Hence its general sol. is,

$v = uc + c^2$

⇒ $e^{2y} = e^{2x}c + c^2$

23

$P^2\cos y + P\sin x\cos x\cos y - \sin y\cos^2 x = 0$ —— ①

It is not solvable for P, y, x

22 $yP^2 - 2xP + y = 0$

Sol:-

It is solvable for x, so we take

$2xP = yP^2 + y$

⇒ $x = \frac{1}{2}yP + \frac{1}{2}yP^{-1}$ —— ①

Diff. ① w.r.t y, we get

$\frac{dx}{dy} = \frac{1}{2}\left(y\frac{dP}{dy} + P\right) + \frac{1}{2}\left(-yP^{-2}\frac{dP}{dy} + P^{-1}\right)$

⇒ $2\cdot\frac{1}{P} = y\frac{dP}{dy} + P - \frac{y}{P^2}\frac{dP}{dy} + \frac{1}{P}$

⇒ $\frac{2}{P} - \frac{1}{P} - P = y\left(1 - \frac{1}{P^2}\right)\frac{dP}{dy}$

⇒ $\frac{1}{P} - P = y\left(1 - \frac{1}{P^2}\right)\frac{dP}{dy}$

⇒ $-P\left(1 - \frac{1}{P^2}\right) - y\left(1 - \frac{1}{P^2}\right)\frac{dP}{dy} = 0$

⇒ $P\left(1 - \frac{1}{P^2}\right) + y\left(1 - \frac{1}{P^2}\right)\frac{dP}{dy} = 0$

⇒ $\left(1 - \frac{1}{P^2}\right)\left(P + y\frac{dP}{dy}\right) = 0$

⇒ $1 - \frac{1}{P^2} = 0$ or $P + y\frac{dP}{dy} = 0$

Consider,

$P + y\frac{dP}{dy} = 0$

⇒ $y\frac{dP}{dy} = -P$

⇒ $\frac{dP}{P} = -\frac{dy}{y}$

⇒ $\int\frac{dP}{P} = -\int\frac{dy}{y}$

⇒ $\ln P = -\ln y + \ln c$

⇒ $\ln P = \ln cy^{-1}$

⇒ $P = cy^{-1}$ put in ①

We get $x = \frac{1}{2}c + \frac{1}{2c}y^2$

⇒ $2cx = c^2 + y^2 = 0$

So, we converts it into clairau's form as,

Let $u = \sin x$, $v = \sin y$

$\therefore du = \cos x\,dx$, $dv = \cos y\,dy$

Now,

$$\frac{\cos y\,dy}{\cos x\,dx} = \frac{dv}{du}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{\cos y}\frac{dv}{du}$$

$$\Rightarrow P = \frac{\cos x}{\cos y}\frac{dv}{du}$$

Hence ① becomes,

$$\cancel{\cos^2 y}\cdot\frac{\cos^2 x}{\cancel{\cos^2 y}}\left(\frac{dv}{du}\right)^2 + \frac{\cos x}{\cancel{\cos y}}\frac{dv}{du}\cdot u\cos x\,\cancel{\cos y} - v\cos^2 x = 0$$

$$\Rightarrow \cos^2 x\left(\frac{dv}{du}\right)^2 + u\cos^2 x\frac{dv}{du} - v\cos^2 x = 0$$

$$\Rightarrow \left(\frac{dv}{du}\right)^2 + u\frac{dv}{du} - v = 0$$

$$\Rightarrow v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2 = 0$$

It is clairauts eq. so its general sol. is,

$$v = uc + c^2$$

$$\Rightarrow \sin y = c\sin x + c^2$$

25 $y^2(y - xP) = x^4P^2$ ——— ①

Sol:-

$$y^3 - xy^2P - x^4P^2 = 0$$

It is not solvable for P, x, y

So, we convert it into Clairaut's form as,

24 $(Px - y)(Py + x) = 2P$.

Sol:-

$$P^2xy + Px^2 - Py^2 - xy - 2P = 0 \text{ ——— ①}$$

It is not solvable for P, x, y

So, we convert it into clairaut's eq. as,

Let $u = x^2$, $v = y^2$

$\therefore du = 2x\,dx$, $dv = 2y\,dy$

Now

$$\frac{2y\,dy}{2x\,dx} = \frac{dv}{du}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du}$$

$$\Rightarrow P = \frac{x}{y}\frac{dv}{du}$$

Putting in the given eq., we get

$$\left(\frac{x}{y}\frac{dv}{du}\cdot x - y\right)\left(\frac{x}{\cancel{y}}\frac{dv}{du}\cdot\cancel{y} + x\right) = 2\frac{x}{y}\frac{dv}{du}$$

$$\Rightarrow \left(\frac{x^2}{y}\frac{dv}{du} - y\right)\left(x\frac{dv}{du} + x\right) = 2\frac{x}{y}\frac{dv}{du}$$

$$\Rightarrow \cancel{x}\left(x^2\frac{dv}{du} - y^2\right)\left(\frac{dv}{du} + 1\right) = 2\cancel{x}\frac{dv}{du}$$

$$\Rightarrow \left(u\frac{dv}{du} - v\right)\left(\frac{dv}{du} + 1\right) = 2\frac{dv}{du}$$

$$\Rightarrow u\frac{dv}{du} - v = \frac{2\frac{dv}{du}}{\frac{dv}{du} + 1}$$

$$\Rightarrow v = u\frac{dv}{du} - \frac{2\frac{dv}{du}}{1 + \frac{dv}{du}}$$

which is clairaut's form and its sol. is,

$$v = u\cdot c - \frac{2c}{1+c}$$

$$\Rightarrow y^2 = cx^2 - \frac{2c}{1+c}$$

Let $u = 1/x$, $v = 1/y$

$\therefore$ $du = -\frac{dx}{x^2}$, $dv = -\frac{dy}{y^2}$

Now $\frac{dy}{y^2} \Big/ \frac{dx}{x^2} = dv/du$

$\Rightarrow \frac{dy}{y^2} \cdot \frac{x^2}{dx} = \frac{dv}{du}$

$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x^2} \frac{dv}{du}$

$\Rightarrow P = \frac{u^2}{v^2} \frac{dv}{du}$

Hence ① becomes as,

$$\frac{1}{v^2}\left(\frac{1}{v} - \frac{1}{u} \cdot \frac{u^2}{v^2} \frac{dv}{du}\right) = \frac{1}{u^4} \cdot \frac{u^4}{v^4}\left(\frac{dv}{du}\right)^2$$

$$\Rightarrow \frac{1}{v^2}\left(\frac{1}{v} - \frac{u}{v^2} \frac{dv}{du}\right) = \frac{1}{v^4}\left(\frac{dv}{du}\right)^2$$

$$\Rightarrow \frac{1}{v^4}\left(v - u\frac{dv}{du}\right) = \frac{1}{v^4}\left(\frac{dv}{du}\right)^2$$

$$\Rightarrow v - u\frac{dv}{du} = \left(\frac{dv}{du}\right)^2$$

$$\Rightarrow v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$$

which is clairaut's form,
Hence its general sol. is,

$$v = u \cdot c + c^2$$

$$\Rightarrow 1/y = c/x + c^2$$

$$\Rightarrow x = cy + c^2 xy$$

27 $y = xP - e^P$ ——— ①

It is clairaut's eq.

General sol. of ①

$$y = cx - e^c$$

Singular sol. of ①

26 Find the general sol. and singular sol. of the diff. eqs. from 26 to 30

$y = xP + \ln P$ ——— ①

Sol:-

It is Clairaut's eq.

General sol. of ①

$$y = cx + \ln c$$

Singular sol. of ①

We know that,
the singular sol. of the Clairaut's eq. $y = xP + f(P)$ in parametric form, is

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - Pf'(P) \end{aligned}\right\} \text{——— ②}$$

Where.

$f(P) = -\ln P$ $\therefore f'(P) = -\frac{1}{P}$

Hence ② becomes, as

$$\left.\begin{aligned} x &= 1/P \\ y &= -\ln P - P \cdot -1/P = -\ln P + 1 \end{aligned}\right\} \text{——— ③}$$

We can eliminate P in eqs. ③, as since $P = 1/x$

$$\therefore y = -\ln(1/x) + 1$$

$$= -\ln x^{-1} + 1$$

$$= \ln x + 1$$ req. s. sol. of ①

28 $y = xP + a\sqrt{1+P^2}$ ——— ①

Sol:- It is clairaut's eq.

General sol:-

$$y = cx + a\sqrt{1+c^2}$$

We know that,

singular sol. of the clairaut's eq. $y = xP + f(P)$ in param. is,

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - Pf'(P) \end{aligned}\right] \text{——} ②$$

Where,

$$f(P) = -e^{P} \quad \therefore \quad f'(P) = -e^{P}$$

Hence ② becomes, as.

$$\left.\begin{aligned} x &= e^{P} \\ y &= -e^{P} - P(-e^{P}) = -e^{P} + Pe^{P} \end{aligned}\right] ③$$

we can eliminat P from ③

Since $x = e^{P}$ or $\ln x = P$

$$\therefore y = -x + \ln x \cdot x$$

$$= x(\ln x - 1) \quad \text{req. sol. of ①}$$

29 $y = xP - \sqrt{P}$ ——①

Sol:- It is clairaut's eq.

General sol:-

$$y = cx - \sqrt{c}$$

Singular sol:-

We know that,

singular sol. of the clairaut's eq. $y = xP + f(P)$ in param. is

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - Pf'(P) \end{aligned}\right] ②$$

Where

$$f(P) = -\sqrt{P} \quad \therefore \quad f'(P) = -\frac{1}{2\sqrt{P}}$$

Hence ② becomes,

$$\left.\begin{aligned} x &= \frac{1}{2\sqrt{P}} \\ y &= -\sqrt{P} + P \cdot \frac{1}{2\sqrt{P}} = -\sqrt{P} + \frac{\sqrt{P}}{2} \end{aligned}\right] ③$$

Singular sol:-

we know that,

singular sol. of the clairaut's eq. $y = xP + f(P)$ in paramet. is,

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - Pf'(P) \end{aligned}\right] \text{——} ②$$

where

$$f(P) = a\sqrt{1+P^2} \quad \therefore \quad f'(P) = \frac{aP}{\sqrt{1+P^2}}$$

Hence ② becomes, as

$$\left.\begin{aligned} x &= -\frac{aP}{\sqrt{1+P^2}} \\ y &= a\sqrt{1+P^2} - \frac{aP^2}{\sqrt{1+P^2}} = \frac{a}{\sqrt{1+P^2}} \end{aligned}\right] ③$$

We can eliminat P from ③, as squaring and adding two eqs, we get,

$$x^2 + y^2 = \frac{a^2P^2}{1+P^2} + \frac{a^2}{1+P^2}$$

$$= \frac{a^2P^2 + a^2}{1+P^2}$$

$$= \frac{a^2(1+\cancel{P^2})}{(\cancel{1}+P^2)}$$

$$= a^2 \quad \text{req. s. sol. of ①}$$

30 $y = xP + P^3$ ——①

Sol:- It is clairaut's eq.

General sol:-

$$y = cx + c^3$$

Singular sol:-

We know that,

Singular sol. of the clairaut's eq. $y = xP + f(P)$ in paramet. is we can eliminat P from ③

Since $\sqrt{P} = \frac{1}{2x}$

$\therefore \quad y = -\frac{1}{2x} + \frac{1}{2x}\cdot\frac{1}{2}$

$= -\frac{1}{2x} + \frac{1}{4x}$

$= -\frac{1}{4x}$ req. s.sol. of ①

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - P f'(P) \end{aligned}\right\} \text{—— ②}$$

Where

$f(P) = P^3 \quad \therefore \quad f'(P) = 3P^2$

Hence ② becomes, as

$$\left.\begin{aligned} x &= -3P^2 \\ y &= P^3 - 3P^3 = -2P^3 \end{aligned}\right\} \text{—— ③}$$

we can eliminate P from ③ as,

Since $x = -3P^2$

or $P = \pm\sqrt{\frac{-x}{3}}$

$\therefore \quad y = -2\left(\pm\sqrt{\frac{-x}{3}}\right)^3$

$= -2\left(\pm\sqrt{\frac{-x}{3}}\right)^2\left(\pm\sqrt{\frac{-x}{3}}\right)$

$= -2\left(\frac{-x}{3}\right)\left(\pm\sqrt{\frac{-x}{3}}\right)$

$= \frac{2x}{3}\left(\pm\sqrt{\frac{-x}{3}}\right)$

$\therefore \quad y^2 = \frac{4x^2}{9}\left(-\frac{x}{3}\right)$

$\Rightarrow 27y^2 = -4x^3$ req. s.sol. of ①

Exercise # 9.9
Mathematical Methods

9 The Singular Solutions

Singular Solution

A diff. eq. $f(x,y,P)=0$ may possess a solution which does not involve any arbitrary constant and, in general, is not obtained from the general solution by giving any particular value to the arbitrary constants, is called Singular Solution.

P-discriminant

Consider the non-linear diff. eq. of first order $f(x,y,P)=0$ ------ ①

Differentiating ① partially w.r.t P, i-e $\dfrac{\partial f(x,y,P)}{\partial P}=0$ ------ ②

If we eliminate P from ① and ② then the eliminant eq. (resulting eq.), is called P-discriminant for eq. ①

Remark

If the eq. ① is quadratic in P i-e is of the shape $AP^2+BP+C=0$ then the P-disc. is given by $B^2-4AC=0$

C-discriminant

Consider the non-linear diff. eq. of first order, $f(x,y,P)=0$ ------ ①

Let the general solution of ①, be $\phi(x,y,c)=0$ ------ ②

Different. ② partially w.r.t. c, i-e $\dfrac{\partial \phi(x,y,c)}{\partial c}=0$ ------ ③

If, we eliminat c from ② and ③, we get the eliminant eq. (resulting eq.) called c-disc. for ②

Remark

If eq. ② is quadratic in c, i-e of the shape $Ac^2+Bc+C=0$ then the c-disc. is given by $B^2-4AC=0$

Determination of Sing. Sol.

Suppose, we want to find the singular sol. of diff. eq. $f(x,y,p)=0$ ——— ①

1. Find the general solution of ①
2. Find the c-discriminant
3. Find the P-discriminant
4. The common part in both the discriminants, that satisfies the diff. eq ①, is the singular solution of ①

The P disc. Method

We can obtain the singular solution of the diff. eq. $f(x,y,P)=0$ —① directly from eq. ①, as

Find P-discriminant for ①'. The part of this relation that satisfies the diff. eq. ①, is the singular sol. of ①

Example

Solve and find singular sol. of

$$P^2-xP+y=0 \quad \text{———} ①$$

Sol:- $y = xP-P^2$

It is clairaut's eq. so its general sol is $y = cx-c^2$

or $c^2-cx+y=0$ ——②

Singular sol.

we find the singular sol. of ①, as

Example

Solve and find singular sol. of

$$xP^2-2yP+4x=0 \quad \text{———} ①$$

Sol:-

$$2yP = xP^2+4x$$

$$\Rightarrow 2y = xP+4xP^{-1}$$

Differentiating w.r.t x.

$$2\frac{dy}{dx} = x\frac{dP}{dx}+P+4\left(-xP^{-2}\frac{dP}{dx}+P^{-1}\right)$$

Example

Solve and find singular sol. of

$$(x^2-1)P^2 - 2xyP - x^2 = 0 \quad \text{——— (1)}$$

Sol:-

$$2xyP = (x^2-1)P^2 - x^2$$

$$\Rightarrow 2xyP = x^2P^2 - P^2 - x^2$$

$$\Rightarrow 2y = xP - x^{-1}P - xP^{-1}$$

Diff. w.r.t x, we get

$$2\frac{dy}{dx} = x\frac{dP}{dx} + P - \left(x^{-1}\frac{dP}{dx} - x^{-2}P\right) - \left(-xP^{-2}\frac{dP}{dx} + P^{-1}\right)$$

$$\Rightarrow 2P = x\frac{dP}{dx} + P - \frac{1}{x}\frac{dP}{dx} + \frac{P}{x^2} + \frac{x}{P^2}\frac{dP}{dx} - \frac{1}{P}$$

$$\Rightarrow P + \frac{1}{P} - \frac{P}{x^2} = \left(x - \frac{1}{x} + \frac{x}{P^2}\right)\frac{dP}{dx}$$

$$\Rightarrow P\left(1 + \frac{1}{P^2} - \frac{1}{x^2}\right) - x\left(1 - \frac{1}{x^2} + \frac{1}{P^2}\right)\frac{dP}{dx} = 0$$

$$\Rightarrow \left(1 + \frac{1}{P^2} - \frac{1}{x^2}\right)\left(P - x\frac{dP}{dx}\right) = 0$$

$$\Rightarrow 1 + \frac{1}{P^2} - \frac{1}{x^2} = 0 \quad \text{or} \quad P - x\frac{dP}{dx} = 0$$

Consider,

$$P - x\frac{dP}{dx} = 0$$

$$\Rightarrow x\frac{dP}{dx} = P$$

$$\Rightarrow \int\frac{dP}{P} = \int\frac{dx}{x}$$

$$\Rightarrow \ln P = \ln x + \ln c$$

$$\Rightarrow P = cx$$

putting in eq. (1), we get

$$(x^2-1)c^2x^2 - 2cx^2y - x^2 = 0 \quad \text{——— (2)}$$

(req. general sol. of (1))

$$(-2y)^2 - 4x\cdot 4x = 0$$

$$\Rightarrow y^2 = 4x$$

Since c-disc. and P-disc. are same

$\therefore$ $y^2 = 4x$ is the singular sol. of (1)

Example

By finding the P-disc. find the singular sol. of

$$x^3P^2 + x^2yP + a^3 = 0 \quad \text{——— (1)}$$

Sol:-

P-discriminant

P-disc. of (1), is given as

$$B^2 - 4AC = 0$$

$$\Rightarrow x^4y^2 - 4a^3x^3 = 0$$

$$\Rightarrow x^3(xy^2 - 4a^3) = 0$$

$$\Rightarrow x^3 = 0 \quad \text{or} \quad xy^2 - 4a^3 = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad xy^2 - 4a^3 = 0$$

Take $x = 0$

Since $x = 0$ $\therefore$ $\frac{dx}{dy} = 0$

or $\frac{1}{P} = 0$

we, first write (1), as

$$x^3 + x^2y\cdot\frac{1}{P} + a^3\left(\frac{1}{P}\right)^2 = 0 \quad \text{——— (2)}$$

L.H.S. of eq. (2)

$$= x^3 + x^2y\cdot\frac{1}{P} + a^3\left(\frac{1}{P}\right)^2$$

$$= 0 + 0 + 0 + 0$$

$$= 0 = \text{R.H.S. of eq. 2}$$

$\therefore$ $x = 0$ is a singular sol. of (1)

C-discriminant

c-disc for ②, is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-x)^2 - 4y = 0$$

$$\Rightarrow x^2 = 4y$$

P-discriminant

P-disc. for ①, is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-x)^2 - 4y = 0$$

$$\Rightarrow x^2 = 4y$$

Since c-disc and p-disc. are same

$\therefore$ $x^2 = 4y$ is singular sol. of ①

Example

Solve $x^2p^2 + yp(2x+y) + y^2 = 0$ —— ①

by making the substitutions

$y = u$, $xy = v$

and find the singular sol.

Sol:-

Here $y = u$, $x = \frac{v}{u}$

$\therefore dy = du$ $\quad dx = \frac{udv - vdu}{u^2}$

Now ! $\frac{dy}{dx} = \frac{u^2 du}{udv - vdu}$

$$\Rightarrow \quad P = \frac{u^2 du}{udv - vdu}$$

Hence eq. ① becomes, as

$$x^2\left(\frac{u^2du}{udv-vdu}\right)^2 + y\left(\frac{u^2du}{udv-vdu}\right) + y^2 = 0$$

$$\Rightarrow 2P = x\frac{dP}{dx} + P - \frac{4x}{P^2}\frac{dP}{dx} + \frac{4}{P}$$

$$\Rightarrow P - \frac{4}{P} = x\left(1 - \frac{4}{P^2}\right)\frac{dP}{dx}$$

$$\Rightarrow P\left(1 - \frac{4}{P^2}\right) - x\left(1 - \frac{4}{P^2}\right)\frac{dP}{dx} = 0$$

$$\Rightarrow \left(1 - \frac{4}{P^2}\right)\left(P - x\frac{dP}{dx}\right) = 0$$

$$\Rightarrow 1 - \frac{4}{P^2} = 0 \quad \text{or} \quad P - x\frac{dP}{dx} = 0$$

Consider,

$$P - x\frac{dP}{dx} = 0$$

$$\Rightarrow x\frac{dP}{dx} = P$$

$$\Rightarrow \int\frac{dP}{P} = \int\frac{dx}{x}$$

$$\Rightarrow \ln P = \ln x + \ln c$$

$$\Rightarrow P = cx$$

putting in ①, we get

$$x \cdot c^2x^2 - 2y \cdot cx + 4x = 0$$

$$\Rightarrow c^2x^2 - 2yc + 4 = 0 \quad \text{---} \quad (2)$$

(general sol. of ①)

Singular Sol:-

we find the singular sol of ① as

C-discriminant

c-disc. of ② is given as

$$B^2 - 4AC = 0$$

$$\Rightarrow (-2y)^2 - 4x^2 \cdot 4 = 0$$

$$\Rightarrow y^2 = 4x^2$$

P-discriminant

P-disc. of ①, is given by

$$B^2 - 4AC = 0$$

Singular sol:

we find singular sol. of ① as,

C-discriminant

c-disc. of ②, is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-2x^2y)^2 - 4x^2(x^2-1)(-x^2) = 0$$

$$\Rightarrow y^2 + x^2 - 1 = 0$$

$$\Rightarrow x^2 + y^2 = 1$$

P-discriminant

P-disc. of ① is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-2xy)^2 - 4(x^2-1)(-x^2) = 0$$

$$\Rightarrow y^2 + x^2 - 1 = 0$$

$$\Rightarrow x^2 + y^2 = 1$$

Since c-disc. and p-disc. are same

$\therefore x^2 + y^2 = 1$ is a singular sol. of ①

Take $y^2x - 4a^3 = 0$

$$\Rightarrow x = 4a^3y^{-2}$$

Diff. w.r.t y, we get

$$\frac{dx}{dy} = -8a^3y^{-3}$$

$$\Rightarrow \frac{1}{p} = -\frac{8a^3}{y^3}$$

Now

L.H.S. of eq. ②

$$= x^3 + x^2y\cdot\frac{1}{p} + a^3\left(\frac{1}{p}\right)^2$$

$$= \left(\frac{4a^3}{y^2}\right)^3 + \left(\frac{4a^3}{y^2}\right)^2\cdot y\cdot\left(-\frac{8a^3}{y^3}\right) + a^3\left(-\frac{8a^3}{y^3}\right)^2$$

$$= \frac{64a^9}{y^6} - \frac{128a^9}{y^6} + \frac{64a^9}{y^6}$$

$$= \frac{64a^9 - 128a^9 + 64a^9}{y^6}$$

$$= 0$$

$$= \text{R.H.S. of eq. ②}$$

i-e eq. ② is satisfied by $y^2x - 4a^3 = 0$

$\therefore y^2x - 4a^3 = 0$ is a singular sol. of ①

EXERCISE 9.9

(1) Solve and find singular sol. of $y = px + p^n$ ——— ①

Sol:- It is clairaut's eq.

so its general sol. is

$$y = cx + c^n \quad \text{———} ②$$

Singular sol

We find singular sol. of ①, as

C-discriminant

Diff. ② partially w.r.t c, we get

$$0 = x + nc^{n-1}$$

$$\Rightarrow c = \left(-\frac{x}{n}\right)^{\frac{1}{n-1}}$$

putting value of c in ②, we get

$$y = \left(-\frac{x}{n}\right)^{\frac{1}{n-1}} x + \left(-\frac{x}{n}\right)^{\frac{n}{n-1}}$$

P-discriminant

Diff. ① partially w.r.t. P, we get

$$0 = x + np^{n-1}$$

$$\Rightarrow p = \left(-\frac{x}{n}\right)^{\frac{1}{n-1}}$$

putting value of P in ①, we get

$$y = \left(-\frac{x}{n}\right)^{\frac{1}{n-1}} x + \left(-\frac{x}{n}\right)^{\frac{n}{n-1}}$$

Since c-disc and P-disc are same

$$\therefore \quad y = \left(-\frac{x}{n}\right)^{\frac{1}{n-1}} x + \left(-\frac{x}{n}\right)^{\frac{n}{n-1}}$$

is a singular sol. of ①

(2) Solve and find sing sol of

$$p^2(x^2-a^2) - 2pxy + y^2 - b^2 = 0 \quad \text{———} ①$$

Sol:-

$$p^2x^2 - a^2p^2 - 2pxy + y^2 - b^2 = 0$$

It is not solvable for P, x, y

So, we write it as

$$p^2x^2 - 2pxy + y^2 - p^2a^2 - b^2 = 0$$

$$\Rightarrow p^2x^2 - 2pxy + y^2 = a^2p^2 + b^2$$

$$\Rightarrow (px-y)^2 = a^2p^2 + b^2$$

$$\Rightarrow px - y = \pm\sqrt{a^2p^2+b^2}$$

$$\Rightarrow y = xp \pm \sqrt{a^2p^2+b^2}$$

It is clairaut's eq.

so its general sol. is,

$$y = cx \pm \sqrt{a^2c^2+b^2}$$

$$\Rightarrow y - cx = \pm\sqrt{a^2c^2+b^2}$$

$$\Rightarrow (y-cx)^2 = a^2c^2+b^2$$

$$\Rightarrow y^2 + c^2x^2 - 2cxy = a^2c^2 + b^2$$

$$\Rightarrow c^2x^2 - a^2c^2 - 2cxy + y^2 - b^2 = 0.$$

$$\Rightarrow \underline{(x^2-a^2)}c^2 - \underline{2xy}c + \underline{(y^2-b^2)} = 0 \quad \text{———} ②$$

Singular sol.

We now find the s.sol. of ① as,

C-discriminant.

c-disc. of eq. ② is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-2xy)^2 - 4(x^2-a^2)(y^2-b^2) = 0$$

$$\Rightarrow 4x^2y^2 - 4(x^2y^2 - b^2x^2 - a^2y^2 + a^2b^2) = 0$$

3

$$xP^2 + (x-y)P + 1 - y = 0 \quad \text{---} ①$$

Sol:-

$$xP^2 + xP - yP + 1 - y = 0$$

It is not solvable for x, y, P

so, we write it as,

$$xP(P+1) - y(P+1) + 1 = 0$$

$$\Rightarrow xP - y + \frac{1}{P+1} = 0$$

$$\Rightarrow y = xP + \frac{1}{P+1}$$

It is clairaut's eq. so, its general sol. is $y = cx + \frac{1}{c+1}$

$$\Rightarrow y(c+1) = c(c+1)x + 1$$

$$\Rightarrow cy + y = c^2x + cx + 1$$

$$\Rightarrow c^2x + cx - cy + 1 - y = 0$$

$$\Rightarrow c^2x + c(x-y) + (1-y) = 0 \quad \text{---} ②$$

Singular sol:-

We find singular sol of ①, as

C-discriminant

c-disc. of ② is, given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (x-y)^2 - 4x(1-y) = 0$$

$$\Rightarrow x^2 + y^2 - 2xy - 4x + 4xy = 0$$

$$\Rightarrow x^2 + y^2 + 2xy - 4x = 0$$

$$\Rightarrow (x+y)^2 - 4x = 0$$

P-discriminant

P-disc. of ①, is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (x-y)^2 - 4x(1-y) = 0$$

$$\Rightarrow 4x^2y^2 - 4x^2y^2 + 4b^2x^2 + 4a^2y^2 - 4a^2b^2 = 0$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

P-discriminant

P-disc. of eq. ① is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-2xy)^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$$\Rightarrow 4x^2y^2 - 4(x^2y^2 - b^2x^2 - a^2y^2 + a^2b^2) = 0$$

$$\Rightarrow \cancel{x^2y^2} - \cancel{x^2y^2} + b^2x^2 + a^2y^2 - a^2b^2 = 0$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

Since c-disc. and P-disc are same

$\therefore b^2x^2 + a^2y^2 = a^2b^2$ is the singular sol. of eq. ①

4

$$4P^2 = 9x \quad \text{---} ①$$

Sol:-

$$x = \frac{4}{9}P^2$$

Diff. it w.r.t y, we get

$$\frac{dx}{dy} = \frac{4}{9} \cdot 2P \frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = \frac{8}{9}P \frac{dP}{dy}$$

$$\Rightarrow 9dy = 8P^2 dP$$

$$\Rightarrow \int 8P^2 dP = \int 9 dy$$

$$\Rightarrow \frac{8}{3}P^3 = 9y + c_1$$

$$\Rightarrow \frac{8}{27}P^3 = y + c_1/9$$

$$\Rightarrow \left(\frac{2}{3}P\right)^3 = y + c$$

$$\Rightarrow \frac{2}{3}P = (y+c)^{1/3}$$

$$\Rightarrow P = \frac{3}{2}(y+c)^{1/3}$$

$\Rightarrow x^2+y^2-2xy-4x+4xy=0$

$\Rightarrow x^2+y^2+2xy-4x=0$

$\Rightarrow (x+y)^2-4x=0$

Since c-disc. and P-disc. are same. $\therefore$ singular sol. of ① is $(x+y)^2-4x=0$

5 $4xP^2=(3x-1)^2$

Sol:-

$$4xP^2=(3x-1)^2 \quad \text{---①}$$

$$P^2=\frac{(3x-1)^2}{4x}$$

$$\sqrt{P^2}=\sqrt{\frac{(3x-1)^2}{4x}}$$

$$P=\pm\frac{(3x-1)}{2\sqrt{x}}$$

$$dy/dx=\pm\left[\frac{3x}{2\sqrt{x}}-\frac{1}{2\sqrt{x}}\right] \quad \because \frac{1}{P}=\frac{dx}{dy}$$

$$dy=\pm\left[\tfrac{3}{2}x^{1/2}-\tfrac{1}{2}x^{-1/2}\right]dx$$

$$\int dy=\pm\int\left[\tfrac{3}{2}x^{1/2}-\tfrac{1}{2}x^{-1/2}\right]dx$$

$$\Rightarrow y=\pm\left[\int\tfrac{3}{2}x^{1/2}dx-\int\tfrac{1}{2}x^{-1/2}dx\right]$$

$$\Rightarrow y=\pm\left[x^{3/2}-x^{1/2}+c\right]$$

$$\Rightarrow y=\pm\left[x^{3/2}-x^{1/2}\right]-c$$

$$y+c=\pm\left[x^{3/2}-x^{1/2}\right]$$

$$(y+c)^2=\left[x^{3/2}-x^{1/2}\right]^2$$

$$(y+c)^2=x^3+x-2x^2$$

$$(y+c)^2=x(x^2+1-2x)$$

$$y^2+c^2+2cy=x(x-1)^2$$

$$c^2+2cy+y^2-x(x-1)^2=0$$

$$c^2+2yc+[y^2-x(x-1)^2]=0 \quad \text{---②}$$

It is quad. in "c"

Singular Solution: We now find the singular sol. as,

Putting in ①, we get

$$4\cdot\left[\tfrac{3}{2}(y+c)^{1/3}\right]^2=9x$$

$$\Rightarrow 4\cdot\tfrac{9}{4}(y+c)^{2/3}=9x$$

$$\Rightarrow (y+c)^{2/3}=x$$

$$\Rightarrow (y+c)^2=x^3$$

$$\Rightarrow y^2+c^2+2cy=x^3$$

$$\Rightarrow c^2+2yc+y^2-x^3=0 \quad \text{---②}$$

Singular sol.

We now find the singular sol. as

C-discriminant

C-disc. of ②, is given by

$$B^2-4AC=0$$

$$\Rightarrow (2y)^2-4(y^2-x^3)=0$$

$$\Rightarrow 4y^2-4(y^2-x^3)=0$$

$$\Rightarrow x^3=0$$

P-discriminant

P-disc. of ①, is given by

$$B^2-4AC=0$$

$$\Rightarrow 0-4(4)(-9x)=0$$

$$\Rightarrow x=0$$

Common in c-disc., P-disc is $x=0$

But ① is not satisfied by $x=0$

$\therefore$ no singular sol. of ① exists

6 $P^2+2Px^3-4x^2y=0$

Sol:-

$$4x^2y=P^2+2Px^3$$

C - Discriminant:

C-discriminant of ② is given by,

$B^2 - 4AC = 0$

$\Rightarrow (2y)^2 - 4(1)[y^2 - x(x-1)^2] = 0$

$\Rightarrow 4y^2 - 4y^2 + 4x(x-1)^2 = 0$

$\Rightarrow 4x(x-1)^2 = 0$

$\Rightarrow x(x-1)^2 = 0$

$\Rightarrow x = 0$ or $(x-1)^2 = 0$

P- Discriminant:

P-discriminant of ① is given by,

$B^2 - 4AC = 0$

$\Rightarrow 0 - 4(4x)(3x-1)^2 = 0$

$\Rightarrow x(3x-1)^2 = 0$

$\Rightarrow x = 0$ or $(3x-1)^2 = 0$

Since $x=0$ is common in both P-disc. and C-disc.

$\therefore$ $x=0$ is not singular solution of eq ①.

$\because$ It does not satisfy eq (I).

(7) $6P^2y^2 + 3PX - y = 0$

Sol:-

$3PX = y - 6P^2y^2$

$\Rightarrow 3x = yP^{-1} - 6Py^2$

Diff. w.r.t y, we get,

$$3\frac{dx}{dy} = -yP^{-2}\frac{dP}{dy} + P^{-1} - 6\left(2Py + y^2\frac{dP}{dy}\right)$$

$$\Rightarrow 3\cdot\frac{1}{P} = -\frac{y}{P^2}\frac{dP}{dy} + \frac{1}{P} - 12yP - 6y^2\frac{dP}{dy}$$

$4y = P^2x^{-2} + 2PX$

Differentiating w.r.t x, we get

$$4\frac{dy}{dx} = -2x^{-3}P^2 + 2x^{-2}P\frac{dP}{dx} + 2\left(P + x\frac{dP}{dx}\right)$$

$$\Rightarrow 2P = -x^{-3}P^2 + P + (x^{-2}P + x)\frac{dP}{dx}$$

$$\Rightarrow \left(P + \frac{P^2}{x^3}\right) - \left(\frac{P}{x^2} + x\right)\frac{dP}{dx} = 0$$

$$\Rightarrow P\left(1 + \frac{P}{x^3}\right) - x\left(\frac{P}{x^3} + 1\right)\frac{dP}{dx} = 0$$

$$\Rightarrow \left(1 + \frac{P}{x^3}\right)\left(P - x\frac{dP}{dx}\right) = 0$$

$$\Rightarrow 1 + \frac{P}{x^3} = 0 \quad \text{or} \quad P - x\frac{dP}{dx} = 0$$

Consider,

$$P - x\frac{dP}{dx} = 0$$

$$\Rightarrow x\frac{dP}{dx} = P$$

$$\Rightarrow \int\frac{dP}{P} = \int\frac{dx}{x}$$

$$\Rightarrow \ln P = \ln x + \ln c$$

$$\Rightarrow P = cx$$

Putting in eq. ①, we get

$$c^2x^2 + 2cx^4 - 4x^2y = 0 \quad \text{——} ②$$

Singular sol.

We now find the s.sol. of ①, as

C- discriminant

C-disc. of ② is given by

$B^2 - 4AC = 0$

$\Rightarrow (2x^4)^2 - 4x^2(-4x^2y) = 0$

$\Rightarrow 4x^8 + 16x^4y = 0$

$\Rightarrow x^4 + 4y = 0$

P- discriminant

P-disc. of ① is given by

$$\frac{2}{P}+12yP = -y\left(\frac{1}{P^2}+6y\right)\frac{dP}{dy}$$

$$\Rightarrow 2P\left(\frac{1}{P^2}+6y\right)+y\left(\frac{1}{P^2}+6y\right)\frac{dP}{dy}=0$$

$$\Rightarrow \left(\frac{1}{P^2}+6y\right)\left(2P+\frac{dP}{dy}\right)=0$$

$$\Rightarrow \frac{1}{P^2}+6y=0 \quad \text{or} \quad 2P+y\frac{dP}{dy}=0$$

Consider,

$$2P+y\frac{dP}{dy}=0$$

$$\Rightarrow y\frac{dP}{dy}=-2P$$

$$\Rightarrow \int\frac{dP}{P}=-2\int\frac{dy}{y}$$

$$\Rightarrow \ln P=-2\ln y+\ln c$$

$$\Rightarrow P=c/y^2$$

putting in ①, we get

$$6\cdot\frac{c^2}{y^4}\cdot y^2+3\cdot\frac{c}{y^2}\cdot x-y=0$$

$$\Rightarrow 6c^2+3cx-y^3=0 \quad \text{——②}$$

Singular Sol:-

we now find the s.sol. of ①, as

C-discriminant

C-disc. of ②, is given by

$$B^2-4AC=0$$

$$\Rightarrow 9x^2-4(6)(-y^3)=0$$

$$\Rightarrow 9x^2+24y^3=0$$

$$\Rightarrow 3x^2+8y^3=0$$

P-discriminant

P-disc. of ①, is given by

$$B^2-4AC=0$$

$$\Rightarrow (2x^3)^2-4(-4x^2y)=0$$

$$\Rightarrow 4x^6+16x^2y=0$$

$$\Rightarrow x^4+4y=0$$

Since c-disc. and P-disc. are same.

$\therefore$ $x^4+4y=0$ is s.sol. of ①

8 $x^3P^2+x^2yP+1=0$ ...... ①

Sol:-

$$x^2yP=-x^3P^2-1$$

$$\Rightarrow y=-xP-x^{-2}P^{-1}$$

Diff. w.r.t x, we get

$$\frac{dy}{dx}=-\left(x\frac{dP}{dx}+P\right)-\left(-x^{-2}P^{-2}\frac{dP}{dx}-2x^{-3}P^{-1}\right)$$

$$\Rightarrow P=-x\frac{dP}{dx}-P+\frac{1}{x^2P^2}\frac{dP}{dx}+\frac{2}{x^3P}$$

$$\Rightarrow 2P-\frac{2}{x^3P}=-\left(x-\frac{1}{x^2P^2}\right)\frac{dP}{dx}$$

$$\Rightarrow 2P\left(1-\frac{1}{x^3P^2}\right)+x\left(1-\frac{1}{x^3P^2}\right)\frac{dP}{dx}=0$$

$$\Rightarrow \left(1-\frac{1}{x^3P^2}\right)\left(2P+x\frac{dP}{dx}\right)=0$$

$$\Rightarrow 1-\frac{1}{x^3P^2}=0 \quad \text{or} \quad 2P+x\frac{dP}{dx}=0$$

Consider,

$$2P+x\frac{dP}{dx}=0$$

$$\Rightarrow x\frac{dP}{dx}=-2P$$

$$\Rightarrow \int\frac{dP}{P}=-2\int\frac{dx}{x}$$

$$\Rightarrow \ln P=-2\ln x+\ln c$$

$$B^2-4AC=0$$

$$\Rightarrow \quad 9x^2-4\,6y^2(-y)=0$$

$$\Rightarrow \quad 9x^2+24y^3=0$$

$$\Rightarrow \quad 3x^2+8y^3=0$$

Since C-disc. is equal to P-disc. $\therefore$ $3x^2+8y^3=0$ is a singular sol. of ①

Q9 $xP^4-2yP^3+12x^3=0$ ——— ①

Sol:-

$$2yP^3 = xP^4+12x^3$$

$$\Rightarrow 2y = xP+12x^3P^{-3}$$

Diff. it w.r.t x, we get

$$2\frac{dy}{dx} = x\frac{dP}{dx}+P+12\left(-3x^3P^{-4}\frac{dP}{dx}+3x^2P^{-3}\right)$$

$$\Rightarrow 2P = x\frac{dP}{dx}+P-36x^3P^{-4}\frac{dP}{dx}+36x^2P^{-3}$$

$$\Rightarrow P-\frac{36x^2}{P^3} = x\frac{dP}{dx}-\frac{36x^3}{P^4}\frac{dP}{dx}$$

$$\Rightarrow P-\frac{36x^2}{P^3} = x\left(1-\frac{36x^2}{P^4}\right)\frac{dP}{dx}$$

$$\Rightarrow P\left(1-\frac{36x^2}{P^4}\right)-x\left(1-\frac{36x^2}{P^4}\right)\frac{dP}{dx}=0$$

$$\Rightarrow \left(1-\frac{36x^2}{P^4}\right)\left(P-x\frac{dP}{dx}\right)=0$$

$$\Rightarrow 1-\frac{36x^2}{P^4}=0 \quad \text{or} \quad P-x\frac{dP}{dx}=0$$

Consider

$$P-x\frac{dP}{dx}=0$$

$$\Rightarrow x\frac{dP}{dx} = P$$

$$\Rightarrow \int\frac{dP}{P} = \int\frac{dx}{x}$$

$$\Rightarrow \ln P = \ln x^{-2}+\ln c$$

$$\Rightarrow P = c/x^2$$

putting in eq. ①, we get,

$$x^3\cdot\frac{c^2}{x^4}+x^2y\cdot\frac{c}{x^2}+1=0$$

$$\Rightarrow c^2x+cxy+x=0 \text{ ——— ②}$$

Singular sol.

we now find the s.sol. of ① as

C-discriminant

c-disc. of ② is given by

$$B^2-4AC=0$$

$$\Rightarrow \quad x^2y^2-4x=0$$

$$\Rightarrow \quad x(x.y^2-4)=0$$

$$\Rightarrow \quad x=0 \;,\; xy^2-4=0$$

P-discriminant

p-disc. of ①, is given by

$$B^2-4AC=0$$

$$\Rightarrow \quad x^4y^2-4x^3=0$$

$$\Rightarrow \quad x^3(xy^2-4)=0$$

$$\Rightarrow \quad x.x^2(xy^2-4)=0$$

$$\Rightarrow \quad x=0,\; x^2=0,\; xy^2-4=0$$

Since $x=0$, $xy^2-4=0$ are common in both c-disc. and P-disc.

$\therefore$ $x=0$, $xy^2-4=0$ are the singular sols. of ①

$\Rightarrow \ln P = \ln x + \ln c$

$\Rightarrow P = cx$

putting in eq. ①, we get,

$c^4x^5 - 2c^3x^3y + 12x^3 = 0$

$\Rightarrow c^4x^2 - 2c^3y + 12 = 0$ ——②

Singular sol:-

we now, find singular sol. as

C-discriminant

Since ② is not quadratic

∴ c-disc. of ② is found as,

Diff. ②, partially w.r.t c, we get

$4c^3x^5 - 6c^2x^3y = 0$

$\Rightarrow 2cx^2 - 3y = 0$

$\Rightarrow c = \frac{3y}{2x^2}$ put in ②, we get

$\left(\frac{3y}{2x^2}\right)^4 x^2 - 2\left(\frac{3y}{2x^2}\right)^3 y + 12 = 0$

$\Rightarrow \frac{81y^4}{16x^6} - \frac{27y^4}{4x^6} + 12 = 0$

$\Rightarrow 81y^4 - 108y^4 + 192x^6 = 0$

$\Rightarrow -27y^4 + 192x^6 = 0$

$\Rightarrow -9y^4 + 64x^6 = 0$

$\Rightarrow 9y^4 = 64x^6$

P-discriminant

Diff. ①, partially w.r.t P, we get

$4xP^3 - 6yP^2 = 0$

$\Rightarrow 2xP - 3y = 0$

$\Rightarrow P = \frac{3y}{2x}$ putting in ①

10 $8P^3x - 12P^2y - 27x = 0$ ——①

Sol:

$12P^2y = -27x + 8P^3x$

$\Rightarrow 12y = 8Px - 27P^{-2}x$

Diff. w.r.t x, we get

$12\frac{dy}{dx} = 8\left(P + x\frac{dP}{dx}\right) - 27\left(P^{-2} - 2xP^{-3}\frac{dP}{dx}\right)$

$\Rightarrow 12P = 8P + 8x\frac{dP}{dx} - \frac{27}{P^2} + \frac{54x}{P^3}\frac{dP}{dx}$

$\Rightarrow 4P + \frac{27}{P^2} = 2x\left(4 + \frac{27}{P^3}\right)\frac{dP}{dx}$

$\Rightarrow P\left(4 + \frac{27}{P^3}\right) - 2x\left(4 + \frac{27}{P^3}\right)\frac{dP}{dx} = 0$

$\Rightarrow \left(4 + \frac{27}{P^3}\right)\left(P - 2x\frac{dP}{dx}\right) = 0$

$\Rightarrow 4 + \frac{27}{P^3} = 0$ or $P - 2x\frac{dP}{dx} = 0$

Consider,

$P - 2x\frac{dP}{dx} = 0$

$\Rightarrow 2x\frac{dP}{dx} = P$

$\Rightarrow \int\frac{dP}{P} = \frac{1}{2}\int\frac{dx}{x}$

$\Rightarrow \ln P = \frac{1}{2}\ln x + \ln c$

$\Rightarrow P = c\sqrt{x}$

putting in ①, we get

$8(c\sqrt{x})^3x - 12(c\sqrt{x})^2y - 27x = 0$

$\Rightarrow 8c^3x^{3/2}x - 12c^2xy - 27x = 0$

$\Rightarrow 8c^3x^{3/2} - 12c^2y - 27 = 0$ —— ②

Singular sol:-

we now find the s.sol. of ① as,

$$x\left(\frac{3y}{2x}\right)^4 - 2y\left(\frac{3y}{2x}\right)^3 + 12x^3 = 0$$

$$\Rightarrow \frac{81y^4}{16x^3} - \frac{27y^4}{4x^3} + 12x^3 = 0$$

$$\Rightarrow 81y^4 - 108y^4 + 192x^6 = 0$$

$$\Rightarrow -27y^4 + 192x^6$$

$$\Rightarrow -9y^4 + 64x^6$$

$$\Rightarrow 9y^4 = 64x^6$$

Since C-disc. and P-disc. are same

$\therefore$ $9y^4 = 64x^6$ is the S.sol.

(11) Investigate for singular sol. by finding P-disc.

$$e^P - P + xy - x - 1 = 0 \quad \text{——①}$$

Sol:-

P-discriminant

Diff. ①, partially w.r.t P, we get.

$$e^P - 1 = 0$$

$$\Rightarrow e^P = 1$$

$$\Rightarrow \ln e^P = \ln 1$$

$$\Rightarrow P \ln e = \ln 1$$

$$\Rightarrow P = 0 \quad (\because \ln e = 1, \ \ln 1 = 0)$$

putting in eq. ①, we get

$$xy - x = 0$$

$$\Rightarrow x(y-1) = 0$$

$$\Rightarrow x = 0, \quad y - 1 = 0$$

Take $x = 0$

Since $x = 0$ $\therefore$ $\frac{dx}{dy} = 0$

or $\frac{1}{P} = 0$

Since eq. ① is not satisfied

C-discriminant

Diff. ② partially w.r.t C, we get

$$24c^2x^{3/2} - 24cy = 0$$

$$\Rightarrow cx^{3/2} - y = 0$$

$$\Rightarrow c = \frac{y}{x^{3/2}} \quad \text{putting in ②, we get}$$

$$8\left(\frac{y}{x^{3/2}}\right)^3 x^{3/2} - 12\left(\frac{y}{x^{3/2}}\right)^2 y - 27 = 0$$

$$\Rightarrow 8\frac{y^3}{x^3} - 12\frac{y^3}{x^3} - 27 = 0$$

$$\Rightarrow 8y^3 - 12y^3 - 27x^3 = 0$$

$$\Rightarrow -4y^3 - 27x^3 = 0$$

$$\Rightarrow 27x^3 + 4y^3 = 0$$

P-discriminant

Diff. ① partially w.r.t P, we get

$$24P^2x - 24Py = 0$$

$$\Rightarrow Px = y$$

$$\Rightarrow P = y/x \quad \text{putting in ① we get}$$

$$8\frac{y^3}{x^3}\cdot x - 12\frac{y^2}{x^2}\cdot y - 27x = 0$$

$$\Rightarrow 8\frac{y^3}{x^2} - 12\frac{y^3}{x^2} - 27x = 0$$

$$\Rightarrow 8y^3 - 12y^3 - 27x^3 = 0$$

$$\Rightarrow -4y^3 - 27x^3 = 0$$

$$\Rightarrow 27x^3 + 4y^3 = 0$$

Since c-disc. and P-disc. are same

$\therefore$ $27x^3 + 4y^3 = 0$ is the S.sol. of ①

(12) Investigate for singular sol. by finding P-disc.

$$4x(x-1)(x-2)P^2 - (3x^2 - 6x + 2)^2 = 0$$

by $x = 0$ and $1/p = 0$

$\therefore$ $x = 0$ is not singul. sol. of ①

Take $y-1=0$

$y = 1$ and $\frac{dy}{dx} = 0$

or $P = 0$

Since ① is satisfied by $y=1$ and $P = 0$

$\therefore$ $y-1 = 0$ is a S. sol. of ①

13 $P^4 - 2P^2 + 1 - y^4 = 0$ —— ①

Sol:-

P-discriminant

Diff. ① partially w.r.t P, we get

$4P^3 - 4P = 0$

$\Rightarrow$ $4P(P^2-1) = 0$

$\Rightarrow$ $P(P-1)(P+1) = 0$

$\Rightarrow$ $P = 0, P = 1, P = -1$

Putting in eq. ①, we get

$1 - y^4 = 0$, $y^4 = 0$

or $(1-y^2)(1+y^2) = 0$, $y = 0$

or $(1-y)(1+y)(1+y^2) = 0$, $y = 0$

o $1-y = 0$, $1+y = 0$, $y = 0$

($1+y^2 = 0$ is rejected $\because$ y is imaginary)

Take $1-y=0$

$y = 1$ $\therefore \frac{dy}{dx} = 0$ i-e $P = 0$

Since ① is satisfied by $y=1$ and $\frac{dy}{dx} = 0$

Sol:-

P-discriminant

P-disc. of ① is given by

$B^2 - 4AC = 0$

$\Rightarrow$ $0 - 4.4x(x-1)(x-2)[-(3x^2-6x+2)^2] = 0$

$\Rightarrow$ $x(x-1)(x-2)(3x^2-6x+2)^2 = 0$

$\Rightarrow$ $x = 0$, $x-1 = 0$, $x-2 = 0$

$3x^2 - 6x + 2 = 0$

First, we write ①, as

$4x(x-1)(x-2) = (3x^2-6x+2)^2 \cdot \frac{1}{P^2}$ —— ②

Take $x=0$

$x = 0$ $\therefore \frac{dx}{dy} = 0$ i-e $\frac{1}{P} = 0$

Since ② is satisfied by $x = 0$ and $\frac{dx}{dy} = 0$

$\therefore$ $x = 0$ is a S. sol. of ①

Take $x-1=0$

$x = 1$ $\therefore \frac{dx}{dy} = 0$ i-e $\frac{1}{P} = 0$

Since ② is satisfied by $x = 1$ and $\frac{dx}{dy} = 0$

$\therefore$ $x-1 = 0$ is a S. sol. of ①

Take $x-2=0$

$x = 2$ $\therefore \frac{dx}{dy} = 0$ i-e $\frac{1}{P} = 0$

Since ② is satisfied by $x = 2$ and $\frac{dx}{dy} = 0$

$\therefore$ $x-2 = 0$ is a S. sol. of ①

$\therefore\ y-1=0$ is s.sol. of ①

Take $1+y=0$

$y=-1 \quad \therefore \frac{dy}{dx}=0$ i.e $p=0$

Since ① is satisfied by

$y=-1$ and $\frac{dy}{dx}=0$

$\therefore\ y+1=0$ is s.sol. of ①

Take $y=0$

$y=0 \quad \therefore \frac{dy}{dx}=0$ i.e $P=0$

Since ① is not satisfied by

$y=0$. and $\frac{dy}{dx}=0$

$\therefore\ y=0$ is not a s.sol. of ①

15 $\quad 4P^3+3xP-y=0$ ——①

Sol:-

P-discriminant

Diff. ①, partially w.r.t P, we get

$12P^2+3x=0$

$\Rightarrow 4P^2=-x$

$\Rightarrow P^2=-x/4$

$\Rightarrow P=\pm\sqrt{-x/4}$

putting in eq. ①, we get

$4\left(\pm(-x/4)^{3/2}\right)+3x\left(\pm(-x/4)^{1/2}\right)-y=0$

$\Rightarrow \pm(-x/4)^{1/2}\left(4(-x/4)+3x\right)-y=0$

$\Rightarrow \pm(-x/4)^{1/2}(-2x)-y=0$

$\Rightarrow y=\pm 2(-x/4)^{1/2}x$

$\Rightarrow y^2=4(-x/4)\cdot x^2$

$\Rightarrow y^2=-x^3$

$\Rightarrow x^3+y^2=0$

Take $3x^2-6x+2=0$

$x=\frac{6\pm\sqrt{36-24}}{6}=\frac{6\pm\sqrt{12}}{6}=\frac{3\pm\sqrt{3}}{3}$

Thus $x=1\pm\frac{1}{\sqrt{3}} \quad \therefore \frac{dx}{dy}=0$ i.e $1/p=0$

Since ② is not satisfied

by $x=1\pm\frac{1}{\sqrt{3}}$ and $\frac{dx}{dy}=0$

$\therefore\ 3x^2-6x+2=0$ is not s.sol. of ①

14 $\quad P^3-4xyP+8y^2=0$ ——①

Sol:-

P-discriminant

Diff. ① partially w.r.t. P, we get

$3P^2-4xy=0$

$\Rightarrow 3P^2=4xy$

$\Rightarrow P^2=\frac{4}{3}xy$

$\Rightarrow P=\pm\left(\frac{4}{3}xy\right)^{1/2}$

putting in eq. ①, we get,

$\pm\left(\frac{4}{3}xy\right)^{3/2}-4xy\left[\pm\left(\frac{4}{3}xy\right)^{1/2}\right]+8y^2=0$

$\Rightarrow \pm\left(\frac{4}{3}xy\right)^{1/2}\cdot\left[\frac{4}{3}xy-4xy\right]+8y^2=0$

$\Rightarrow \pm\left(\frac{4}{3}xy\right)^{1/2}\cdot\left(-\frac{8}{3}xy\right)+8y^2=0$

$\Rightarrow -2\left[\pm\left(\frac{4}{3}xy\right)^{1/2}\cdot\left(\frac{4}{3}xy\right)\right]+8y^2=0$

$\Rightarrow \pm\left(\frac{4}{3}xy\right)^{3/2}-4y^2=0$

$\Rightarrow \pm\left(\frac{4}{3}xy\right)^{3/2}=4y^2$

$\Rightarrow \left(\frac{4}{3}xy\right)^3=16y^4$

$\Rightarrow \frac{64}{27}x^3y^3-16y^4=0$

$\Rightarrow \frac{4}{27}x^3y^3-y^4=0$

Take $x^3+y^2=0$

$$y^2=-x^3$$

$$\therefore\ 2y\frac{dy}{dx}=-3x^2$$

$$\Rightarrow\ 2yP=-3x^2$$

$$\Rightarrow\ P=-\frac{3x^2}{2y}$$

Putting in L.H.S of eq. ①, we get

L.H.S of eq. ①

$$=4P^3+3xP-y$$

$$=4\left(-\frac{3x^2}{2y}\right)^3+3x\left(-\frac{3x^2}{2y}\right)-y$$

$$=4\left(-\frac{27x^6}{8y^3}\right)-\frac{9x^3}{2y}-y$$

$$=-\frac{27x^6}{2y^3}-\frac{9x^3}{2y}-y$$

$$=\frac{27(x^3)^2}{2y^3}-\frac{9x^3}{2y}-y$$

$$=\frac{27(-y^2)^2}{2y^3}-\frac{9(-y^2)}{2y}-y \quad \because x^3=-y^2$$

$$=\frac{27y^4}{2y^3}+\frac{9y^2}{2y}-y$$

$$=\frac{27y}{2}+\frac{9y}{2}-y$$

$$=\frac{27y+9y-2y}{2}=17y\neq 0=\text{R.H.S}$$

16 $P^3-2x^2P-4xy=0$ ——— ①

Sol:-

P-discriminant

Diff. ① partially w.r.t P, we get

$$3P^2-2x^2=0$$

$$\Rightarrow\ 3P^2=2x^2$$

$$\Rightarrow\ P=\pm\left(\tfrac{2}{3}\right)^{1/2}x$$

$$4x^3y^3-27y^4=0$$

$$\Rightarrow\ y^3(4x^3-27y)=0$$

$$\Rightarrow\ y^3=0\ ,\ 4x^3-27y=0$$

$$\Rightarrow\ y=0\ ,\ 4x^3-27y=0$$

Take $y=0$

$y=0\ \therefore\ \frac{dy}{dx}=0$ i.e $P=0$

Since ① is satisfied by $y=0$ and $\frac{dy}{dx}=0$

$\therefore\ y=0$ is a s.sol. of ①

Take $4x^3-27y=0$

$$4x^3-27y=0$$

$$\Rightarrow\ 27y=4x^3$$

$$\Rightarrow\ y=\frac{4}{27}x^3\quad \therefore\ \frac{dy}{dx}=\frac{4}{9}x^2$$

or $P=4/9\,x^2$

Putting in L.H.S of eq. ①

L.H.S. of eq. ①

$$=P^3-4xyP+8y^2$$

$$=\left(\tfrac{4}{9}x^2\right)^3-4x\left(\tfrac{4}{27}x^3\right)\left(\tfrac{4}{9}x^2\right)+8\left(\tfrac{4}{27}x^3\right)^2$$

$$=\frac{64}{729}x^6-\frac{64}{243}x^6+\frac{128}{729}x^6$$

$$=\frac{64x^6-192x^6+128x^6}{729}=0=\text{R.H.S}$$

Since eq. ① is satisfied by $y=\frac{4}{27}x^3$ and $\frac{dy}{dx}=\frac{4}{9}x^2$

$\therefore\ 4x^3-27y=0$ is a s.sol. of ① putting in eq. ①, we get

$$\left(\pm\left(\tfrac{2}{3}\right)^{1/2}x\right)^3-2x^2\left(\pm\left(\tfrac{2}{3}\right)^{1/2}x\right)-4xy=0$$

$$\Rightarrow \pm\left(\tfrac{2}{3}\right)^{3/2}x^3-2\left(\pm(2/3)^{1/2}x^3\right)-4xy=0$$

$$\Rightarrow \pm\left(\tfrac{2}{3}\right)^{1/2}x^3\left(\tfrac{2}{3}-2\right)-4xy=0$$

$$\Rightarrow \pm\left(\tfrac{2}{3}\right)^{1/2}x^3\left(-4/3\right)=4xy$$

$$\Rightarrow \pm\left(\tfrac{2}{3}\right)^{1/2}x^3 = -3xy$$

$$\Rightarrow \tfrac{2}{3}x^6 = 9x^2y^2$$

$$\Rightarrow 2x^6-27x^2y^2=0$$

$$\Rightarrow x^2(2x^4-27y^2)=0$$

$$\Rightarrow x^2=0,\quad 2x^4-27y^2=0$$

$$\Rightarrow x=0 \quad 2x^4-27y^2=0$$

Take $x=0$

$x=0 \;\therefore\; \frac{dx}{dy}=0$ i.e $\frac{1}{P}=0$

first, we write ①, as,

$$1-2x^2\cdot\left(\tfrac{1}{P}\right)^2-4xy\cdot\left(\tfrac{1}{P}\right)^3=0 \quad \text{—②}$$

since ②, is not satisfied by $x=0$ and $\frac{dx}{dy}=0$

$\therefore$ $x=0$ is not a s. sol. of ①

Take $2x^4-27y^2=0$

$$2x^4-27y^2=0$$

$$\Rightarrow 27y^2=2x^4$$

$$\Rightarrow y^2=\tfrac{2}{27}x^4$$

$$\Rightarrow y=\pm\left(\tfrac{2}{27}\right)^{1/2}x^2$$

$$\therefore \frac{dy}{dx}=\pm2\left(\tfrac{2}{27}\right)^{1/2}x$$

$$\Rightarrow P=\pm2\left(\tfrac{2}{27}\right)^{1/2}x$$

Putting in L.H.S of eq. ①

L.H.S of eq. ①

$$=P^3-2x^2P-4xy$$

$$=\left[\pm2\left(\tfrac{2}{27}\right)^{1/2}x\right]^3-2x^2\left[\pm2\left(\tfrac{2}{27}\right)^{1/2}x\right]-4x\left[\pm\left(\tfrac{2}{27}\right)^{1/2}x^2\right]$$

$$=\pm8\left(\tfrac{2}{27}\right)^{3/2}x^3-\left[\pm4\left(\tfrac{2}{27}\right)^{1/2}x^3\right]-\left[\pm4\left(\tfrac{2}{27}\right)^{1/2}x^3\right]$$

$$=\pm4x^3\left(\tfrac{2}{27}\right)^{1/2}\left[2\cdot\left(\tfrac{2}{27}\right)-1-1\right]$$

THE END.

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CH#10

Higher Order Linear Differential Eqs

Diff eqs of Ist Degree, but not of Ist order. OR

Linear Diff eqs with constant coefficients.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = F(x)$$

If $F(x)=0$, then Homogeneous Linear Diff. Eq.
If $F(x)\neq 0$, then NonHomogeneous Linear Diff Eq.

1) **Real and Distinct Roots** If $m_1, m_2, \ldots m_n$ are distinct real roots then General Sol is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

2) **Real and Equal Roots** If $m_1 = m_2$ are roots then G. Sol is $y = (c_1 + c_2 x) e^{m_1 x}$
If $m_1 = m_2 = m_3$ are real roots then G. Sol is $y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x}$

3) **Imaginary & Distinct Roots** If Roots are $\alpha \pm i\beta$ then G. Sol is $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

4) **Imaginary & Repeated Roots** If Roots are $\alpha \pm i\beta$, & $\alpha \pm i\beta$ then G. Sol $y = e^{\alpha x}\{(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\}$

Ex 10.1

Solve

① $(9D^2 - 12D + 4)y = 0$

$9D^2 - 12D + 4 = 0$ Characteristic Eq or Auxiliary eq.

$D = \frac{12 \pm \sqrt{144 - 4.9.4}}{18} = \frac{12 \pm \sqrt{144-144}}{18}$

$D = \frac{12}{18} + 0,\ \frac{12}{18} - 0 = \boxed{\frac{2}{3}, \frac{2}{3}}$ (case II)

$\therefore$ G. Sol is $y = (c_1 + c_2 x) e^{\frac{2}{3}x}$

② $(75D^2 + 50D + 12)y = 0$

$75D^2 + 50D + 12 = 0$ characteristic Eq

$D = \frac{-50 \pm \sqrt{2500 - }}{2(75)} = \frac{-50 \pm \sqrt{2500 - 3600}}{150}$

$= \frac{-50 \pm \sqrt{-1100}}{150} = \frac{-50 \pm \sqrt{(-1)(100)}}{150} = \frac{-50 \pm 10\sqrt{-11}}{150}$

$= -10\left(\frac{-5 \pm \sqrt{-11}}{150}\right) = \frac{-5 \pm i\sqrt{11}}{15} = \boxed{-\frac{1}{3} \pm \frac{i\sqrt{11}}{15}}$

G. Sol $y = e^{-\frac{1}{3}x}\left(c_1 \cos\frac{\sqrt{11}}{15}x + c_2 \sin\frac{\sqrt{11}}{15}x\right)$

③ $(D^3 - 4D^2 + D + 6)y = 0$

$D^3 - 4D^2 + D + 6 = 0$ Characteristic Eq.

-1	1	-4	1	6
	$\downarrow$	-1	5	-6
	1	-5	6	$\boxed{0}$

$D^2 - 5D + 6 = 0$ Depressed Eq

$(D-2)(D-3) = 0$

$\boxed{D = 2, 3 \& -1}$

G. Sol is $y = c_1 e^{-1x} + c_2 e^{2x} + c_3 e^{3x}$

④ $(D^3 + D^2 + D + 1)y = 0$

$D^3 + D^2 + D + 1 = 0$ characteristic Eq

-1	1	1	1	1
	$\downarrow$	-1	0	-1
	1	0	1	$\boxed{0}$

$D^2 + 0D + 1 = 0$

$D^2 = -1$

$D = \pm i$

G. Sol. $y =$ $\therefore D = -1, \pm i$

G. Sol is $y = c_1 e^{-1x} + e^{0x}(c_2 \cos x + c_3 \sin x)$

$y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x$

Solution of Non Homogeneous Linear Diff Eq of order 'n'.

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) Y = F(x)$$

The solution consist of two parts

(i) Complementary Function (C.F) :- It is sol of Homogeneous L. Diff Eq i.e $(a_0 D^n + a_1 D^{n-1} + \cdots + a_n) Y = 0$. It is denoted by Y_c

(ii) Particular Integral (P.I) :- It is sol of $\dfrac{1}{a_0 D^n + a_1 D^{n-1} + \cdots a_n} F(x)$. It is denoted by Y_p

$\therefore$ General Sol $Y = Y_c + Y_p$

Properties of Differential Operator 'D' $= \frac{d}{dx}$

1) $D^n(ae^{bx}) = a(b)^n e^{bx}$ 'D is replaced by b'

ii) $F(D)(ae^{bx}) = aF(b)e^{bx}$ 'D is replaced by b'

iii) $D^n(e^{bx})U = e^{bx}(D+b)^n U$ b is added in D in D

iv) $F(D)e^{bx}U = e^{bx}F(D+b)U$ b is added in D

v) $\dfrac{1}{F(D)}ae^{bx} = \dfrac{ae^{bx}}{F(b)}$ 'D is replaced by b'

vi) $\dfrac{1}{F(D)}e^{bx}.U = e^{bx}\dfrac{1}{F(D+b)}.U$, b is added in D

vii) $\dfrac{1}{F(D^2)}\sin(ax) = \dfrac{\sin ax}{F(-a^2)}$ D^2 is replaced by $(-a^2)$

viii) $\dfrac{1}{F(D^2)}\cos(ax) = \dfrac{\cos ax}{F(-a^2)}$ } only for D^2

(ix) $\dfrac{1}{F(D)}\sin bx = \text{Im}\dfrac{1}{F(D)}e^{ibx} = \text{Im}\dfrac{e^{ibx}}{F(ib)}$

x) $\dfrac{1}{F(D)}\cos bx = \text{Re}\dfrac{1}{F(D)}e^{ibx} = \text{Re}\dfrac{e^{ibx}}{F(ib)}$

xi) $D^2\cos bx = (-b^2)\cos bx$

xii) $D^2\sin bx = (-b^2)\sin bx$ } only for D^2

xiii) $\dfrac{1}{D^2}\cos bx = \dfrac{1}{-b^2}\cos bx$

xiv) $\dfrac{1}{D^2}\sin bx = \dfrac{1}{-b^2}\sin bx$ } only for D^2

Imp Note

1) when, $\dfrac{1}{F(D)}(ae^{bx}) = \dfrac{ae^{bx}}{F(b)}$ if $F(b)=0$

then $\dfrac{1}{F(D)}ae^{bx} = \dfrac{x}{F'(D)}(ae^{bx})$

$= \dfrac{x}{F'(b)}ae^{bx}$ if $F'(b)=0$

then $\dfrac{x}{F'(D)}ae^{bx} = \dfrac{x^2}{F''(D)}ae^{bx}$

$= \dfrac{x^2}{F''(b)}ae^{bx}$ & so on. if $F''(b)\neq 0$

$$D(\sin x) = \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{1}{D}(\sin x) = \int \sin x\,dx = -\cos x$$

B. Series. If n is -ve or fraction.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \cdots$$

We apply B. Series when $\frac{1}{}F(x)$ is other than Sin, Cos or e^{bx} see Q4, 5, 9,

Solution of Non Homogeneous Linear Diff Eq of order 'n'.

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n)Y = F(x)$$

The solution consist of two parts

(i) Complementary Function (C.F) :- It is sol of Homogeneous L. Diff Eq i.e $(a_0 D^n + a_1 D^{n-1} + \cdots + a_n)Y = 0$. It is denoted by Y_c

(ii) Particular Integral (P.I) :- It is sol of $\frac{1}{a_0 D^n + a_1 D^{n-1} + \cdots a_n} F(x)$. It is denoted by Y_p

$\therefore$ General Sol $Y = Y_c + Y_p$

Properties of Differential Operator 'D' $= \frac{d}{dx}$

i) $D^n(ae^{bx}) = a(b)^n e^{bx}$ 'D is replaced by b'

ii) $F(D)(ae^{bx}) = aF(b)e^{bx}$ 'D is replaced by b'

iii) $D^n(e^{bx})U = e^{bx}(D+b)^n U$ b is added in D in D

iv) $F(D)e^{bx}U = e^{bx}F(D+b)U$ b is added in D

v) $\frac{1}{F(D)} ae^{bx} = \frac{ae^{bx}}{F(b)}$ 'D is replaced by b'

vi) $\frac{1}{F(D)} e^{bx}.U = e^{bx}\frac{1}{F(D+b)}.U$, b is added in D

vii) $\frac{1}{F(D^2)} \sin(ax) = \frac{\sin ax}{F(-a^2)}$ D^2 is replaced by $(-a^2)$

viii) $\frac{1}{F(D^2)} \cos(ax) = \frac{\cos ax}{F(-a^2)}$ } only for D^2

(ix) $\frac{1}{F(D)} \sin bx = \text{Im} \frac{1}{F(D)} e^{ibx}$ where $\frac{e^{ibx}}{F(ib)}$

x) $\frac{1}{F(D)} \cos bx = \text{Re} \frac{1}{F(D)} e^{ibx} = \text{Re} \frac{e^{ibx}}{F(ib)}$

xi) $D^2 \cos bx = (-b^2)\cos bx$

xii) $D^2 \sin bx = (-b^2)\sin bx$ } only for D^2

xiii) $\frac{1}{D^2} \cos bx = \frac{1}{-b^2} \cos bx$

xiv) $\frac{1}{D^2} \sin bx = \frac{1}{-b^2} \sin bx$ } only for D^2

Imp Note

1) when, $\frac{1}{F(D)}(ae^{bx}) = \frac{ae^{bx}}{F(b)}$ if $F(b)=0$

then $\frac{1}{F(D)} ae^{bx} = \frac{x}{F'(D)}(ae^{bx})$

$= \frac{x}{F'(b)} ae^{bx}$ if $F'(b)=0$

then $\frac{x}{F'(D)} ae^{bx} = \frac{x^2}{F''(D)} ae^{bx}$

$= \frac{x^2}{F''(b)} ae^{bx}$ & so on. if $F''(b) \neq 0$

$$D(\sin x) = \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{1}{D}(\sin x) = \int \sin x\, dx = -\cos x$$

B. Series. If n is -ve or fraction.

$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \cdots$

We apply B. Series when $F(x)$ is other than Sin, Cos or e^{bx} see Q4, 5, 9,

Ex 10.2

Sol

$$\text{①}\ (D^2+3D-4)y = 15e^{x}$$

For C.F

$D^2+3D-4 = 0$ Characteristic Eq

$$D = \frac{-3 \pm \sqrt{9+4.1.4}}{2} = \frac{-3\pm\sqrt{25}}{2}$$

$$= \frac{-3\pm 5}{2} = 1, -4$$

$$y_c = c_1e^{x} + c_2e^{-4x}$$

For P.I

$$y_p = \frac{1}{D^2+3D-4}15e^{x}$$

(Failure Case) $\because \frac{15e^{x}}{1+3-4} = \infty$

$$= \frac{x}{2D+3}(15e^{x})$$

$$= \frac{x15e^{x}}{2(1)+3} = \frac{15xe^{x}}{5} = 3xe^{x}$$

$$y = y_c + y_p = c_1e^{x} + c_2e^{-4x} + 3xe^{x}$$

Sol

$$\text{②}\ (D^2-3D+2)y = e^{x}+e^{2x}$$

For C.F

$D^2-3D+2 = 0$

$$D = \frac{3\pm\sqrt{9-4.1.2}}{2} = \frac{3\pm\sqrt{1}}{2}$$

$$= \frac{3\pm1}{2} = 2, 1$$

$$y_c = c_1e^{2x} + c_2e^{x}$$

For P.I

$$y_p = \frac{1}{D^2-3D+2}(e^{x}+e^{2x})$$

$$= \frac{1}{D^2-3D+2}e^{x} + \frac{1}{D^2-3D+2}e^{2x}$$

$\because$ Failure Case

$$= \frac{x}{2D-3}(e^{x}) + \frac{x}{2D-3}(e^{2x})$$

$$= \frac{xe^{x}}{2(1)-3} + \frac{xe^{2x}}{2(2)-3}$$

$$y_p = -xe^{x} + xe^{2x}$$

$$y = y_c + y_p$$

$$= c_1e^{x} + c_2e^{2x} + xe^{2x} - xe^{x}.$$

Sol

$$\text{③}\ (D^2-2D-3)y = 2e^{x} - 10\sin x$$

For C.F

$D^2-2D-3 = 0$ Characteristic Eq

$$D = \frac{2\pm\sqrt{4+4.1.3}}{2} = \frac{2\pm\sqrt{16}}{2} = \frac{2\pm4}{2} = 3, -1$$

$$y_c = c_1e^{3x} + c_2e^{-x}$$

For P.I

$$y_p = \frac{1}{D^2-2D-3}(2e^{x} - 10\sin x)$$

$$= \frac{1}{D^2-2D-3}2e^{x} - \frac{1}{D^2-2D-3}(10\sin x)$$

$$= \frac{2e^{x}}{1^2-2(1)-3} - \frac{10\sin x}{-1^2-2D-3}$$

D^2 is replaced by $-a^2$, $D^2 = -1^2$

$$= \frac{2e^{x}}{-4} - \frac{10\sin x}{-4-2D}$$

$$= -\frac{e^{x}}{2} + \frac{5\sin x}{2+D}$$

$$= -\frac{e^{x}}{2} + \frac{5(2-D)\sin x}{(2-D)(2+D)}$$

$\times$ & $\div$ by $(2-$

$$= -\frac{e^{x}}{2} + \frac{5(2-D)\sin x}{4-D^2}$$

D^2 by $-$
D^2 by $-$

$$= -\frac{e^{x}}{2} + \frac{5(2-D)\sin x}{4-(-1)}$$

$$= -\frac{e^{x}}{2} + \frac{\cancel{5}}{\cancel{5}}(2-D)\sin x$$

$$= -\frac{e^{x}}{2} + 2\sin x - D(\sin x)$$

$$y_p = -\frac{e^{x}}{2} + 2\sin x - \cos x$$

$$y = y_c + y_p$$

$$= c_1e^{3x} + c_2e^{-x} - \frac{e^{x}}{2} + 2\sin x - \cos$$

$\boxed{10.2.21}$

Q2 $y_p = \dfrac{e^x + e^{2x}}{D^2 - 3D + 2}$

$= \dfrac{e^x}{(D-1)(D-2)} + \dfrac{1}{(D-1)(D-2)} e^{2x}$

$= \dfrac{e^x}{(D-1)(1-2)} + \dfrac{e^{2x}}{(2-1)(D-2)}$

$= \dfrac{-e^x}{D-1} + \dfrac{e^{2x}}{(D-2)}$

Failure case $= \dfrac{-x e^x}{1-0} + \dfrac{x e^{2x}}{1}$

Q1 $y_p = \dfrac{15 e^x}{D^2 + 3D - 4}$

$= \dfrac{15 e^x}{(D+4)(D-1)}$

$= \dfrac{15 e^x}{(1+4)(D-1)}$

$= \dfrac{15 e^x}{5(D-1)}$

$= \dfrac{3 e^x}{D-1}$

$= \dfrac{3 x e^x}{1}$ Failure case

Q3 $\dfrac{2e^x - 10\sin x}{D^2 - 2D - 3}$

$= \dfrac{2 e^x}{(D-3)(D+1)} - \dfrac{10 \sin x}{(D^2 - 2D - 3)}$

$= \dfrac{2 e^x}{(1-3)(1+1)} - \dfrac{10 \sin x}{(-1^2) - 2D - 3}$

$= \dfrac{\cancel{2} e^x}{\cancel{-2}(2)} - \dfrac{10 \sin x}{-4 - 2D}$

$= \dfrac{e^x}{-2} + \dfrac{5 \sin x}{2 + D}$

$+ \dfrac{5(2-D)\sin x}{(2^2 - D^2)}$

$+ \dfrac{\cancel{5}(2\sin x - \cos x)}{2^2 - (-1^2)}$

Q3 $\dfrac{1}{D^2 - 2D - 3} \sin x = \text{Im} \dfrac{1}{D^2 - 2D - 3} e^{ix}$

D replaced by i

$= \text{Im} \dfrac{1}{i^2 - 2i - 3} e^{ix}$

$= \text{Im} \dfrac{1}{-4 - 2i} e^{ix}$ ∴ i in Denominator

$= \text{Im} \dfrac{1}{-(4+2i)(4-2i)} (4-2i) e^{ix}$

$= \text{Im} \dfrac{(4-2i)}{-(16+4)} [\cos x + i \sin x]$

$= \text{Im} \dfrac{-2(2-i)}{20} (\cos x + i \sin x)$

$= \text{Im} \dfrac{(2-i)}{10} (\cos x + i \sin x)$

$= \dfrac{-\text{Im}}{10} \left(2\cos x + 2i\sin x - i(\cos x + i \sin x) \right]$

$\dfrac{-1}{10} (2 \sin x - \cos x)$

10.2-3

(5)

Q4 $(D^4 - 2D^3 + D)y = x^4 + 3x + 1$

$D(D^3 - 2D^2 + 1)y = 0$

Either $D = 0$, OR $D^3 - 2D^2 + 0D + 1 = 0$

$$\begin{array}{c|cccc} & 1 & -2 & 0 & 1 \\ 1 & \downarrow & 1 & -1 & -1 \\ \hline & 1 & -1 & -1 & \boxed{0} \end{array}$$

$D^2 - D - 1 = 0$

$D = \frac{1 \pm \sqrt{1+4\cdot1\cdot1}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$\therefore D = 0, 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

So $y_c = c_1 e^{0x} + c_2 e^{1\cdot x} + c_3 e^{\left(\frac{1+\sqrt{5}}{2}\right)x} + c_4 e^{\left(\frac{1-\sqrt{5}}{2}\right)x}$

For P.I $y_p = \frac{1}{D^4 - 2D^3 + D}(x^4 + 3x + 1)$

$= \frac{1}{D[D^3 - 2D^2 + 1]}(x^4 + 3x + 1)$

$y_p = \frac{1}{D}\left[1 + (D^3 - 2D^2)\right]^{-1}\{x^4 + 3x + 1\}$

$\because (1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \cdots$

$y_p = \frac{1}{D}\left[1 + (-1)(D^3 - 2D^2) + \frac{(-1)(-1-1)}{\underline{|2}}(D^3 - 2D^2)^2 + \cdots\right]\{x^4 + 3x + 1\}$

$= \frac{1}{D}\left[1 - D^3 + 2D^2 + \frac{\cancel{2}}{\cancel{\underline{|2}}}D^6 + 4D^4 - 4D^5 + \cdots\right]\{x^4 + 3x + 1\}$

$= \frac{1}{D}\left[1 + 2D^2 - D^3 + 4D^4 - 4D^5 + D^6\right]\{x^4 + 3x + 1\}$

$= \frac{1}{D}\left[x^4 + 3x + 1 + 2D^2(x^4 + 3x + 1) - D^3(x^4 + 3x + 1) + 4D^4(x^4 + 3x + 1) - 4D^5(x^4 + 3x + 1) + D^6(x^4 + 3x + 1)\right]$

$= \frac{1}{D}\left[x^4 + 3x + 1 + 2(12x^2) - 24x + 4(24) - 0 + 0\right]$

$= \frac{x^5}{5} + \frac{3x^2}{2} + x + 24\left(\frac{x^3}{3}\right) - 24\frac{x^2}{2} + 96x$

$y_p = \frac{x^5}{5} + 8x^3 - 21\frac{x^2}{2} + 97x$

$y = y_c + y_p$

$= c_1 + c_2 e^x + c_3 e^{\left(\frac{1+\sqrt{5}}{2}\right)x} + c_4 e^{\left(\frac{1-\sqrt{5}}{2}\right)x} + \frac{x^5}{5} + 8x^3 - 21\frac{x^2}{2} + 97x$

(5) $(D^3 - D^2 + D - 1)y = 4\sin x$

$D^3 - D^2 + D - 1 = 0$

$$\begin{array}{c|cccc} & 1 & -1 & 1 & -1 \\ 1 & \downarrow & 1 & 0 & 1 \\ \hline & 1 & 0 & 1 & \boxed{0} \end{array}$$

$D^2 + 1 = 0$

$D^2 = -1$

$D = \pm i$

$\therefore$ Roots are $D = 1, \pm i$

So $y_c = c_1 e^x + e^{0x}(c_2 \cos x + c_3 \sin x)$

$y_p = \frac{1}{D^3 - D^2 + D - 1}(4\sin x)$

$= \frac{1}{D(D^2) - D^2 + D - 1} 4\sin x = \frac{1}{D(-1^2) - (-1^2) + D - 1} 4\sin x$

$y_p = \frac{1}{-D + 1 + D - 1} = \frac{1}{0} = \infty$ failure

$\therefore y_p = \frac{x}{3D^2 - 2D + 1} 4\sin x$

$= \frac{x}{3(-1^2) - 2D + 1} 4\sin x = \frac{4x\sin x}{-2 - 2D}$

$= \frac{4x \sin x}{-2(1 + D)} = \frac{-2x\sin x}{1 + D}$

$= \frac{-2x(1 - D)\sin x}{(1+D)(1-D)} = \frac{-2x(1-D)\sin x}{1 - D^2}$

$= \frac{-2x(1-D)\sin x}{1 - (-1^2)} = \frac{-2x\sin x + 2x\cos x}{2}$

$y_p = \cancel{2}\left(\frac{-x\sin x + x\cos x}{\cancel{2}}\right)$

G.Sol $y = y_c + y_p$

$= c_1 e^x + c_2 \cos x + c_3 \sin x + x\cos x$

$\boxed{10.2\text{-}24}$

(6) $(D^3-2D^2-3D+10)y = 40\cos x$

$D^3-2D^2-3D+10=0$

$$\begin{array}{r|rrrr} & 1 & -2 & -3 & 10 \\ -2 & \downarrow & -2 & 8 & -10 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

$D^2-4D+5=0$

$D = \frac{4\pm\sqrt{16-4.1.5}}{2} = \frac{4\pm\sqrt{-4}}{2}$

$D = \frac{4\pm i2}{2} = \frac{\cancel{2}(2\pm i)}{\cancel{2}}$

$D = -2,\ 2\pm i$

$y_c = c_1e^{-2x} + e^{2x}(c_2\cos x + c_3\sin x)$

$y_p = \frac{40\cos x}{D^3-2D^2-3D+10}$

$= \frac{40\cos x}{D(D^2)-2D^2-3D+10}$

$= \frac{40\cos x}{D(-1^2)-2(-1^2)-3D+10}$

$= \frac{40\cos x}{-D+2-3D+10}$

$= \frac{40\cos x}{-4D+12}$

$= \frac{40\cos x}{-4(D-3)}$

$= -10\frac{(D+3)\cos x}{(D-3)(D+3)}$

$= -10\frac{(D+3)\cos x}{D^2-9}$

$= \frac{-10[D(\cos x)+3\cos x]}{(-1^2)-9}$

$y_p = \frac{\cancel{10}}{\cancel{10}}[-\sin x+3\cos x]$

$y = y_c+y_p = c_1e^{-2x} + e^{2x}(c_2\cos x + c_3\sin x) - \sin x + 3\cos x$

(7) (6) $(D^2+4)y = 4\sin^2 x$

$D^2+4=0 \Rightarrow D^2=-4 \Rightarrow D=\pm 2i$

$y_c = e^{0x}(c_1\cos 2x + c_2\sin 2x)$

$y_p = \frac{1}{D^2+4}(4\sin^2 x) = 2\frac{(2\sin^2 x)}{D^2+4}$

$= 2\frac{(1-\cos 2x)}{D^2+4} = \frac{2(1)}{D^2+4} - \frac{2\cos 2x}{D^2+4}$ — (1)

$= \frac{2e^{0x}}{D^2+4} - \cancel{2}\frac{x(\cos 2x)}{\cancel{2}D+0}$ failure case $1=e^{0x}$

$= \frac{2}{0^2+4} - \frac{x\cos 2x}{D}$

$y_p = \frac12 - x\frac{\sin 2x}{2}$

$y = y_c+y_p$

$y = (c_1\cos 2x + c_2\sin 2x) + \frac12 - \frac{x\sin 2x}{2})$

2nd Method from (1): $\frac{2}{D^2+4}(1) = 2(D^2+4)^{-1}(1) = 2\cdot 4^{-1}(1+\frac{D^2}{4})^{-1}(1) = \frac{2}{4}[1+(-1)(\frac{D^2}{4})+\dots]$

$= \frac12[1+0] = \frac12$

(8) $(D^3+D)y = 2x^2+3\sin x$

$D^3+D=0 \Rightarrow D(D^2+1)=0 \Rightarrow D=0,\ D^2=-1$

$D^2 = i^2$

$D = \pm i$

$\therefore D = 0, \pm i$

$y_c = c_1e^{0x} + e^{0x}(c_2\cos x + c_3\sin x)$

$y_c = c_1 + c_2\cos x + c_3\sin x$

$y_p = \frac{1}{(D^3+D)}[2x^2+3\sin x]$

$= \frac{1}{D^3+D}2x^2 + \frac{1}{D^3+D}3\sin x = \frac{1}{D(D^2+1)}2x^2 + \frac{1}{D(D^2+1)}3\sin x$

$= \frac{1}{(D^2+1)}\frac{2x^3}{3} + \frac{1}{(D^2+1)}(-3\cos x) = (1+D^2)^{-1}(\frac23 x^3) + \frac{x(-3)}{2D}$ ∵ failure

$= [1+(-1)(D^2) + \frac{(-1)(-1-1)}{2!}(D^2)^2 + \dots](\frac23 x^3) + [x(-\frac32)\frac{\cos x}{D}]$

$= (1-D^2+D^4+\dots)(\frac23 x^3) + [-(\frac{3x}{2})(\sin x)]$

$= \frac23 x^3 - \frac{2}{\cancel{3}}(\cancel{3})(2x) + 0 - \frac{3x}{2}\sin x$

$= \frac23 x^3 - 4x - \frac{3x}{2}\sin x$

$y = y_c+y_p = c_1 + c_2\cos x + c_3\sin x + \frac23 x^3 - 4x - \frac{3x}{2}\sin x$ Ans

10.2-5

⑨ $(D^4+D^2)y = 3x^2+6\sin x-2\cos x$

$D^4+D^2=0$

$D^2(D^2+1)=0$

$D^2=0,\ D^2+1=0$

$D=0,0,\ D^2=-1$

$D=\pm i$

$\therefore D=0,0,\pm i$

$y_c=(c_1+c_2x)e^{0x}+e^{0x}(c_3\cos x+c_4\sin x)$

$y_c=c_1+c_2x+c_3\cos x+c_4\sin x$

$y_p=\dfrac{1}{D^4+D^2}(3x^2+6\sin x-2\cos x)$

$=\dfrac{1}{D^4+D^2}(3x^2)+\dfrac{1}{D^4+D^2}6\sin x-\dfrac{1}{D^4+D^2}2\cos x$

$=\dfrac{1}{D^2(D^2+1)}3x^2+\dfrac{1}{D^2(D^2+1)}6\sin x-\dfrac{1}{D^2(D^2+1)}2\cos x$

$=\dfrac{1}{(D^2+1)}3\left(\dfrac{x^4}{12}\right)+\dfrac{6}{(D^2+1)}(-\sin x)-\dfrac{2}{(D^2+1)}(-\cos x)$

failure case $=(1+D^2)^{-1}\left(\dfrac{x^4}{4}\right)-\dfrac{6x\sin x}{2D}+\dfrac{\cancel{2}x\cos x}{\cancel{2}D}$

$=(1+D^2)^{-1}\left(\dfrac{x^4}{4}\right)-\dfrac{(3x)}{D}\sin x+\dfrac{x}{D}(\cos x)$

B.S $=\left[1+(-1)(D^2)+\dfrac{(-1)(-1-1)}{\underline{2}}(D^2)^2+\cdots\right]\dfrac{x^4}{4}-3x(-\cos x)+x\sin x$

$=\left(1-D^2+\dfrac{2}{2}D^4+\cdots\right)\dfrac{x^4}{4}+3x\cos x+x\sin x$

$=\left(\dfrac{x^4}{4}-\dfrac{12x^2}{4}+\dfrac{24}{4}+\cdots\right)+3x\cos x+x\sin x$

$y_p=\dfrac{x^4}{4}-3x^2+6+3x\cos x+x\sin x$

$y=y_c+y_p$

$y=c_1+c_2x+c_3\cos x+c_4\sin x+\dfrac{x^4}{4}-3x^2+6+3x\cos x+x\sin x$

⑪ ⑦ $(D^3-D^2+3D+5)y=e^x\sin 2x$

$D^3-D^2+3D+5=0$

$$\begin{array}{r|rrrr} & 1 & -1 & 3 & 5 \\ -1 & \downarrow & -1 & 2 & -5 \\ \hline & 1 & -2 & 5 & 0 \end{array}$$

$D^2-2D+5=0$

$D=\dfrac{2\pm\sqrt{4-4\cdot1\cdot5}}{2}$

$=\dfrac{2\pm\sqrt{-16}}{2}=\dfrac{2\pm4i}{2}$

$=\dfrac{\cancel{2}(1\pm2i)}{\cancel{2}}=1\pm2i$

$\therefore D=-1,\ 1\pm2i$

$y_c=c_1e^{-x}+e^x(c_2\cos 2x+c_3\sin 2x)$

$y_p=\dfrac{1}{D^3-D^2+3D+5}e^x\sin 2x$

$=\dfrac{e^x}{(D+1)^3-(D+1)^2+3(D+1)+5}\sin 2x$

$=\dfrac{e^x}{[D^3+1+3D^2+3D]-[D^2+1+2D]+3D+3+5}\sin 2x$

$=\dfrac{e^x}{D^3+2D^2+4D+8}\sin 2x$ ——①

$=\dfrac{e^x}{D(D^2)+2D^2+4D+8}\sin 2x$

D is replaced $-a^2$

$=\dfrac{e^x}{D(-2^2)+2(-2^2)+4D+8}\sin 2x$

$=\dfrac{e^x}{D(-4)-8+4D+8}$

failure case

$\therefore y_p=\dfrac{xe^x}{3D^2+4D+4}\sin 2x$

$=\dfrac{xe^x}{3(-2^2)+4D+4}\sin 2x$

$=\dfrac{xe^x}{4D-8}\sin 2x=\dfrac{xe^x}{4}\dfrac{1}{(D-2)}\sin 2x$

$=\dfrac{xe^x}{4}\dfrac{(D+2)\sin 2x}{D^2-4}=\dfrac{xe^x}{4}\dfrac{(D+2)}{(-2^2-4)}$

$=\dfrac{xe^x}{4(-8)}[2\cos 2x+2\sin 2x]$

$y_p=-\dfrac{xe^x}{\cancel{32}_{16}}\cancel{2}(\cos 2x+\sin 2x)$

$y=y_c+y_p$

$=c_1e^{-x}+e^x(c_2\cos 2x+c_3\sin 2x)-\dfrac{xe^x}{16}(\cos 2x+\sin 2x)$

Ex. 2.6 (8)

(13) $(D^2-7D+12)y = e^{2x}(x^3-5x^2)$

Characteristic Eq. $D^2-7D+12=0$

$$D = \frac{+7\pm\sqrt{49-4\cdot1\cdot12}}{2} = \frac{7\pm\sqrt{1}}{2}$$

$$= \frac{7+1}{2},\ \frac{7-1}{2} = 4, 3$$

$$y_c = c_1e^{3x}+c_2e^{4x}$$

$$y_p = \frac{e^{2x}(x^3-5x^2)}{D^2-7D+12}$$

$$= e^{2x}\frac{1}{(D+2)^2-7(D+2)+12}(x^3-5x^2)$$

$$= e^{2x}\frac{1}{D^2+4+4D-7D-14+12}(x^3-5x^2)$$

$$= e^{2x}\frac{1}{D^2-3D+2}(x^3-5x^2)$$

$$= \frac{e^{2x}}{2}\left[\frac{1}{1+\left(\frac{D^2-3D}{2}\right)}\right](x^3-5x^2)$$

$$= \frac{e^{2x}}{2}\left[1+\left(\frac{D^2-3D}{2}\right)\right]^{-1}(x^3-5x^2)$$

$$= \frac{e^{2x}}{2}\left[1+(-1)\left(\frac{D^2-3D}{2}\right)+\frac{(-1)(-1-1)}{2}\left(\frac{D^2-3D}{2}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(\frac{D^2-3D}{2}\right)^3\right](x^3-5x^2)$$

$$= \frac{e^{2x}}{2}\left[1+\frac{3D-D^2}{2}+\frac{D^4+9D^2-6D^3}{4}+\frac{27D^3}{8}\right](x^3-5x^2)$$

D^4 & higher neglected

$$= \frac{e^{2x}}{2}\left[1+\frac{3}{2}D-\frac{D^2}{2}+\frac{9D^2}{4}-\frac{6D^3}{4}+\frac{27D^3}{8}\right](x^3-5x^2)$$

$$= \frac{e^{2x}}{2}\left[1+\frac{3}{2}D+\frac{7D^2}{4}+\frac{15}{8}D^3\right](x^3-5x^2)$$

$$= \frac{e^{2x}}{2}\left[x^3-5x^2+\frac{3}{2}(3x^2-10x)+\frac{7}{4}(6x-10)+\frac{15}{8}(6)\right]$$

$$= \frac{e^{2x}}{2}\left[x^3-5x^2+\frac{9x^2}{2}-15x+\frac{21x}{2}-\frac{35}{2}+\frac{45}{4}\right]$$

$$y_p = \frac{e^{2x}}{2}\left[x^3-\frac{x^2}{2}-\frac{9x}{2}-\frac{25}{4}\right] = \frac{e^{2x}}{8}\left[4x^3-2x^2-18x-25\right]$$

$$y = c_1e^{3x}+c_2e^{4x}+\frac{e^{2x}}{8}(4x^3-2x^2-18x-25)$$

(10) $(D^2-2D+4)y = e^{x}\cos x$

$D^2-2D+4=0$ Characteristic Eq

$$D = \frac{2\pm\sqrt{4-4\cdot1\cdot4}}{2} = \frac{2\pm\sqrt{4-16}}{2}$$

$$= \frac{2\pm\sqrt{-12}}{2} = \frac{2\pm i2\sqrt{3}}{2} = \frac{\cancel{2}(1\pm i\sqrt{3})}{\cancel{2}}$$

$$y_c = e^{1x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x)$$

$$y_p = \frac{1}{D^2-2D+4}e^{x}\cos x$$

$$= e^{x}\frac{1}{(D+1)^2-2(D+1)+4}\cos x$$

$$= e^{x}\frac{1}{D^2+2D+1-2D-2+4}\cos x$$

$$= e^{x}\frac{1}{D^2+3}\cos x$$

$$y_p = \frac{e^{x}\cos x}{-1^2+3} = \frac{e^{x}\cos x}{2}$$

$$y = y_c+y_p$$

$$= e^{x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x)+\frac{e^{x}\cos x}{2}$$

10.2-7

(9)

DS (14) $(D^4+8D^2-9)y = 9x^3+5\cos 2x$

Characteristic eq

$D^4+8D^2-9=0$

$$\begin{array}{r|rrrrr} & 1 & 0 & 8 & 0 & -9 \\ +1 & \downarrow & +1 & +1 & 9 & 9 \\ \hline & 1 & +1 & 9 & 9 & 0 \\ -1 & \downarrow & -1 & 0 & -9 & \\ \hline & 1 & 0 & 9 & 0 & \end{array}$$

$D^2+9=0 \quad D^2=-9$

$D=1,-1,\pm 3i$

$y_c = c_1e^{x}+c_2e^{-x}+e^{0x}(c_3\cos 3x+c_4\sin 3x)$

$y_p = \dfrac{9x^3+5\cos 2x}{D^4+8D^2-9}$

$= \dfrac{9x^3}{D^4+8D^2-9} + \dfrac{5\cos 2x}{D^4+8D^2-9}$

$= \dfrac{1}{(-9)\left[1-\left(\frac{D^4+8D^2}{9}\right)\right]}9x^3 + \dfrac{5\cos 2x}{(-2^2)^2+8(-2^2)-9}$

$= -\left[1-\frac{(D^4+8D^2)}{9}\right]^{-1}x^3 + \dfrac{5\cos 2x}{16-32-9}$

$= -\left[1-(-1)\left(\frac{D^4+8D^2}{9}\right)\right]x^3 + \dfrac{5\cos 2x}{-25}$

$= -\left[x^3+0+\frac{8}{9}(6x)\right] - \dfrac{\cos 2x}{5}$

$y_p = -x^3-\frac{16}{3}x-\frac{\cos 2x}{5}$

$y = c_1e^{x}+c_2e^{-x}+e^{0x}(c_3\cos 3x+c_4\sin 3x$

(12) $(D^3-7D-6)y = e^{2x}+xe^{2x}$

$D^3-7D-6=0$ Characteristic Eq

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & -6 \\ -1 & \downarrow & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$D^2-D-6=0$

$D=\dfrac{1\pm\sqrt{1+4\cdot1\cdot6}}{2} = \dfrac{1\pm5}{2} = 3,-2$

$\therefore D=-1,-2,3$

So $y_c = c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}$

$y_p = \dfrac{1}{D^3-7D-6}(e^{2x}+xe^{2x})$

$= \dfrac{1}{D^3-7D-6}(1+x)e^{2x}$

$= e^{2x}\dfrac{1}{(D+2)^3-7(D+2)-6}(1+x)$

$= \dfrac{e^{2x}}{D^3+2^3+3\cdot2\cdot D(D+2)-7D-14-6}(1+x)$

$= \dfrac{e^{2x}}{D^3-12+6D^2+12D-7D}(1+x)$

$= \dfrac{e^{2x}}{D^3+6D^2+5D-12}(1+x)$

$= \dfrac{e^{2x}}{-12}\left[\dfrac{1}{1+\left(\frac{D^3+6D^2+5D}{-12}\right)}\right](1+x)$

$= \dfrac{e^{2x}}{-12}\left[1-\left(\frac{D^3+6D^2+5D}{12}\right)\right]^{-1}(1+x)$ Apply B.Se

$= \dfrac{e^{2x}}{-12}\left[1-(-1)\left(\frac{D^3+6D^2+5D}{12}\right)\right](1+x$

$= \dfrac{e^{2x}}{-12}\left[(1+x)+\left(\frac{D^3+6D^2+5D}{12}\right)(1+x)\right]$

$= \dfrac{e^{2x}}{-12}\left[(1+x)+\frac{1}{12}\left[D^3(1+x)+6D^2(1+x)\right.\right.$

$= \dfrac{e^{2x}}{-12}\left[(1+x)+\frac{1}{12}(0+0+5(0+1))\right]$

$= \dfrac{e^{2x}}{-12}\left(1+x+\frac{5}{12}\right)$

$y_p = \dfrac{e^{2x}}{-12}\left[\frac{12+5}{12}+x\right] = \dfrac{e^{2x}}{-12}\left[\right.$

$y = y_c+y_p$

$y = c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}+\dfrac{e^{2x}}{-12}\left(\frac{17}{12}+x\right)$

Ans

$\boxed{Ex\,10.2 - 8}$

15. $(D^4+3D^2-4)y = \sinh x - \cos^2 x$

$D^4+3D^2-4=0$

	1	0	3	0	−4
1	↓	1	1	4	4
	1	1	4	4	$\underline{0}$
−1	↓	−1	0	−4	
	1	0	4	$\underline{0}$	

$D^2+4=0$

$D^2=-4 \qquad D=\pm 2i$

$\therefore D = 1, -1, \pm 2i$

$y_c = c_1 e^{x} + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x$

$$y_p = \frac{\sinh x - \cos^2 x}{D^4+3D^2-4}$$

$$= \frac{\sinh x}{D^4+3D^2-4} - \frac{\cos^2 x}{D^4+3D^2-4}$$

$$= \frac{\frac{e^{x}-e^{-x}}{2}}{(D+1)(D-1)(D^2+4)} - \frac{\left(\frac{1+\cos 2x}{2}\right)}{D^4+3D^2-4}$$

$$= \left[\frac{1}{2(D+1)(D-1)(D^2+4)}e^{x} - \frac{1}{2(D+1)(D-1)(D^2+4)}e^{-x}\right] - \left[\frac{1}{2(D^4+3D^2-4)} + \frac{\cos 2x}{2(D^4+3D^2-4)}\right]$$

$$= \left[\frac{e^{x}}{2(1+1)(D-1)(1+4)} - \frac{e^{-x}}{2(D+1)(-1-1)\{(-1)^2+4\}}\right] - \left[\frac{1}{2(-4)\{1-(D^4+3D^2)\}} + \frac{\cos 2x}{2(D^2-1)(D^2+4)}\right]$$

$$= \left[\frac{e^{x}}{20(D-1)} + \frac{e^{-x}}{20(D+1)}\right] - \left[\frac{1}{-8}\{1-(D^4+3D^2)\}^{-1} + \frac{\cos 2x}{2(-2^2-1)(D^2+4)}\right]$$

$$= \left[\frac{xe^{x}}{20} + \frac{xe^{-x}}{20}\right] - \left[-\frac{1}{8} + \frac{x}{(-10)(2D)}\cos 2x\right]$$

$$= \left[\frac{x}{20}(e^{x}+e^{-x})\right] - \left[-\frac{1}{8} + \frac{x}{-20}\,\frac{\sin 2x}{2}\right]$$

$$= \left[\frac{x\cosh x}{10} + \frac{1}{8} + \frac{x\sin 2x}{40}\right]$$

$\because \frac{e^{x}+e^{-x}}{2} = \cos$…

$\frac{e^{x}-e^{-x}}{2} = $…

$y = y_c + y_p = c_1 e^{x} + c_2 e^{-x} + (c_3\cos 2x + c_4 \sin 2x) + \frac{x\cosh x}{10} + \frac{1}{8} + \frac{x\sin 2x}{40}$

(2c) $y''' + 3y'' + 7y' + 5y = 16e^{-x}\cos 2x,$ $y(0) = 2$, $y'(0) = -4$, $y''(0) = -2$

D Characteristic Eq is

$D^3 + 3D^2 + 7D + 5 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 7 & 5 \\ & \downarrow & -1 & -2 & -5 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

$\therefore D = -1$

$\&$ $D^2 + 2D + 5 = 0$

$D = \dfrac{-2 \pm \sqrt{4 - 4\cdot1\cdot5}}{2} = \dfrac{-2 \pm \sqrt{-16}}{2}$

$D = \dfrac{-2 \pm i4}{2} = -1 \pm 2i$

$y_c = c_1 e^{-x} + e^{-x}(c_2 \cos 2x + c_3 \sin 2x)$

$y_p = \dfrac{16e^{-x}\cos 2x}{D^3 + 3D^2 + 7D + 5}$

$= 16e^{-x} \dfrac{1}{(D-1)^3 + 3(D-1)^2 + 7(D-1) + 5} \cos 2x$

$= 16e^{-x} \dfrac{\cos 2x}{[D^3 - 1 - 3D(D-1)] + [3D^2 + 3 - 6D] + 7D - 7 + 5}$

$= \dfrac{16e^{-x}\cos 2x}{D^3 + 4D}$

$= \dfrac{16e^{-x}\cos 2x}{D(D^2+4)} = \dfrac{16e^{-x}\cos 2x}{D(-2^2+4)}$ Failure Case

$= \dfrac{16e^{-x}\, x\cos 2x}{3D^2 + 4}$ x by N by x & take Derivative of D

$= \dfrac{+16e^{-x}\, x\cos 2x}{3(-2^2) + 4}$

$= \dfrac{+16}{8} e^{-x}\, x\cos 2x$

$y_p = -2e^{-x}\, x\cos 2x$

$y = c_1 e^{-x} + e^{-x}(c_2 \cos 2x + c_3 \sin 2x) - 2e^{-x}\, x\cos 2x$

Ex 0.2.7-10

$$y = c_1 e^{-x} + e^{-x}(c_2 \cos 2x + c_3 \sin 2x) - 2e^{-x} x\cos 2x$$

$$y' = -c_1 e^{-x} - e^{-x}(c_2\cos 2x + c_3 \sin 2x) + e^{-x}(c_2 \sin 2x (2) + 2c_3 \cos 2x) - 2(-1)e^{-x}x\cos 2x - 2e^{-x}\cos 2x - 2e^{-x}x(-2\sin 2x)$$

$$= -c_1 e^{-x} - e^{-x}(c_2\cos 2x + c_3 \sin 2x) + e^{-x}(-2c_2 \sin 2x + 2c_3\cos 2x) + 2e^{-x}x\cos 2x - 2e^{-x}\cos 2x + 4xe^{-x}\sin 2x$$

$$y'' = c_1 e^{-x} + e^{-x}(c_2\cos 2x + c_3 \sin 2x) - e^{-x}(c_2(-2)\sin 2x + 2c_3\cos 2x) - e^{-x}(-2c_2\sin 2x + 2c_3 \ldots) + e^{-x}(-4c_2\cos 2x - 4c_3 \sin 2x) + 2e^{-x}\cos 2x - 2e^{-x}(-2\sin 2x) + 2e^{-x}x(-2\sin 2x)$$
$$+ 2(-1)e^{-x}x\cos 2x + 2e^{-x}\cos 2x + 2e^{-x}x(-2\sin 2x) + 4xe^{-x}(\cos 2x)2.$$
$$+ 4e^{-x}\sin 2x - 4xe^{-x}\sin 2x + 4xe^{-x}(\cos 2x)2.$$

Apply $y(0) = 2 \quad \Rightarrow \quad 2 = c_1 + c_2$ —— ①

$y'(0) = -4 \quad \Rightarrow \quad -4 = -c_1 - c_2 + 2c_3 + 0 - 2$

$\cancel{+}2 = \cancel{+}(c_1 + c_2 - 2c_3)$ —— ⑪

$$-2 = c_1 + c_2 - 2c_3 - 2c_3 - 4c_2 + 2 + 0 + 2 + 0 + 0 - 0 + 0$$

$y''(0) = -2 \quad \Rightarrow \quad -2 = c_1 + c_2 - 4c_3 - 4c_2 + 4$

$-2 = c_1 + c_2 - 4c_3 - 4c_2$ —— ⑪⑪

$-6 = c_1 - 3c_2 - 4c_3$

Put ① in ⑪ $\quad 2 = 2 - 2c_3 \quad \Rightarrow \quad 0 = -2c_3 \quad \Rightarrow \quad \boxed{c_3 = 0}$

Put $c_3 = 0$ in ⑪ $\quad 2 = c_1 + c_2 - 0$

Put $c_3 = 0$ in ⑪⑪ $\quad -6 = c_1 - 3c_2 - 0$

$8 = 4c_2 \quad \Rightarrow \quad c_2 = \frac{8}{4} = \boxed{2}$

Put $c_2 = 2$ in ① $\quad 2 = c_1 + 2 \quad \Rightarrow \quad \boxed{c_1 = 0}$

The required sol is $y = 0 + e^{-x}(2\cos 2x + 0) - 2e^{-x}x\cos 2x$

$$y = 2e^{-x}\cos 2x - 2xe^{-x}\cos 2x$$

10.2-11

(11)

(16) $(D^2-8D+15)y = 9xe^{2x}$ $\quad y(0)=5, \; y'(0)=10$

$D^2-8D+15 = 0$ ∴ Characteristic Eq

$D = \frac{8 \pm \sqrt{64-4.1.15}}{2} = \frac{8 \pm \sqrt{64-60}}{2}$

$= \frac{8 \pm \sqrt{4}}{2} = \frac{8 \pm 2}{2} = 5, 3$

$y_c = c_1 e^{3x} + c_2 e^{5x}$

$y_p = \frac{1}{D^2-8D+15} 9xe^{2x}$

$= e^{2x} \frac{1}{(D+2)^2-8(D+2)+15} 9x$

$= e^{2x} \frac{1}{D^2+4+4D-8D-16+15} 9x$

$= e^{2x} \frac{1}{D^2-4D+3} 9x$

$= \frac{e^{2x}}{\cancel{3}} \frac{1}{\left(\frac{D^2-4D}{3}+1\right)} \cancel{9}^{3}x$

$= e^{2x}\left[1+\left(\frac{D^2-4D}{3}\right)\right]^{-1} 3x$ Apply B.Series

$= e^{2x}\left[1+(-1)\left(\frac{D^2-4D}{3}\right)+\cdots\right] 3x$

$= e^{2x}\left[3x - \frac{1}{\cancel{3}}(D^2-4D)\cancel{3}x\right]$

$= e^{2x}\left[3x-(0-4)\right]$

$y_p = 3xe^{2x}+4e^{2x}$

$y = y_c + y_p = c_1e^{3x}+c_2e^{5x}+3xe^{2x}+4e^{2x}$ — (I)

Diff (I) $y' = 3c_1e^{3x}+5c_2e^{5x}+3e^{2x}+6xe^{2x}+8e^{2x}$ — (II)

$y(0)=5 \Rightarrow$ Put $x=0, y=5$ in (I) $\quad 5 = c_1e^0+c_2+0+4e^0$

$5 = c_1+c_2+4$

$1 = c_1+c_2$ — (III)

$y'(0)=10 \Rightarrow$ Put $x=0, y'=10$ in (II) $\quad 10 = 3c_1+5c_2+11$

$-1 = 3c_1+5c_2$ — (IV)

Solve (III) & (IV)

$3 = 3c_1+3c_2$
$-1 = 3c_1+5c_2$
$4 = -2c_2$

$5 = 5c_1+5c_2$
$-1 = 3c_1+5c_2$
$6 = 2c_1$

Hence from (I) $c_1=3, c_2=-2$

(17) $y''-4y'+13y = 8\sin 3x$ $\quad y(0)=1, \; y'(0)=2$

$(D^2-4D+13)y = 8\sin 3x$

$D^2-4D+13 = 0$ Characteristic Eq

$D = \frac{4 \pm \sqrt{16-4.1.13}}{2} = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2}$

$D = \frac{4 \pm 6i}{2} = \cancel{2}\frac{(2 \pm 3i)}{\cancel{2}} = 2 \pm 3i$

$y_c = e^{2x}(c_1\cos 3x + c_2 \sin 3x)$

$y_p = \frac{1}{D^2-4D+13} 8\sin 3x$

$= 8\frac{(1)}{-3^2-4D+13}\sin 3x$ $\quad$ D^2 is replaced by $-a^2$ i.e -3^2

$= 8 \frac{1}{4-4D}$

$= \frac{8}{4}\frac{1}{1-D}\sin 3x$ $\quad$ × & ÷ by $1+D$

$= 2\frac{(1+D)}{1-D^2}\sin 3x$

$= 2 \frac{(1+D)\sin 3x}{1-(-3^2)}$

$= \frac{2}{10}\left[\sin 3x + \cos 3x\,(3)\right]$

$y_p = \frac{1}{5}\left[\sin 3x + 3\cos 3x\right]$

$y = y_c + y_p$

$y = e^{2x}(c_1\cos 3x + c_2 \sin 3x) + \frac{1}{5}[\sin 3x + 3\cos$

$y' = 2e^{2x}(c_1\cos 3x + c_2\sin 3x) + e^{2x}(-3c_1\sin 3x + 3c$

$+\frac{1}{5}[\cos 3x\,(3) + 9\quad(-\sin 3x)]$ —

$y(0)=1$ Put $x=0, y=1$ in (I) $\Rightarrow 1 = c_1 + \frac{3}{5} \Rightarrow c_1 = 1-\frac{3}{5}$

$c_1 = \frac{5-3}{5}$

$y'(0)=2$ Put $x=0, y'=2$ in (II) $\Rightarrow 2 = 2c_1+3c_2+\frac{3}{5}$

$2 = 2\left(\frac{2}{5}\right)+3c_2+\frac{3}{5}$

$2-\frac{4}{5}-\frac{3}{5} = 3c_2$

$\frac{10-7}{5} = 3c_2 \Rightarrow c_2 = \frac{3}{5}\cdot\frac{1}{3}$

So $y = e^{2x}\left[\frac{2}{5}\cos 3x + \frac{1}{5}\sin 3x\right] + \frac{1}{5}[\sin 3x +$

$= \frac{1}{5}\left[e^{2x}(\cos 3x + \sin 3x) + \sin 3x + 3\cos\right.$

10.2-12

12

(18) $y'' - 4y = 2 - 8x \quad y(0)=0,\; y'(0)=5$

$(D^2-4)y = 2-8x$

$D^2 - 4 = 0$ Characteristic Eq.

$D^2 = 4$

$D = \pm 2$

$\therefore y_c = c_1 e^{2x} + c_2 e^{-2x}$

$y_p = \frac{1}{D^2-4}(2-8x)$

$= \frac{1}{-4\left(1+\frac{D^2}{-4}\right)}(2-8x)$

$= \frac{1}{-4}\left(1-\frac{D^2}{4}\right)^{-1}(2-8x)$ Apply B. Series

$= -\frac{1}{4}\left[1-(-1)\frac{D^2}{4}+\cdots\right](2-8x)$

$= -\frac{1}{4}\left[1+\frac{D^2}{4}\right](2-8x)$

$= -\frac{1}{4}\left[2-8x+\frac{D^2}{4}(2-8x)\right]$

$= -\frac{1}{4}(2-8x)+\frac{1}{4}(0-0)$

$= -\frac{2}{4}+\frac{8x}{4}$

$y_p = -\frac{1}{2}+2x$

$y = y_c + y_p$

$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{2} + 2x$ — (I)

$y' = 2c_1 e^{2x} + (-2)c_2 e^{-2x} - 0 + 2$ — (II)

$y(0)=0 \Rightarrow 0 = c_1 + c_2 - \frac{1}{2}$

Put $x=0, y=0$ in (I)

$\frac{1}{2} = c_1 + c_2$ — (III)

$y'(0)=5 \Rightarrow 5 = 2c_1 - 2c_2 + 2$

Put $x=0, y'=5$ in (II)

$3 = 2c_1 - 2c_2$ — (IV)

Solving (III) & (IV)

$\frac{1}{\cancel{2}}(\cancel{2}) = 2c_1 + \cancel{2c_2}$

$3 = 2c_1 - \cancel{2c_2}$

$4 = 4c_1$

$\boxed{1 = c_1}$

$\frac{1}{2}(2) = 2c_1 + 2c_2$

$3 = 2c_1 - 2c_2$ (subtracting: $-$, $+$)

$-2 = 4c_2$

$-\frac{2}{4} = c_2$

$\boxed{-\frac{1}{2} = c_2}$

from (I)

$y = e^{2x} - \frac{1}{2}e^{-2x} - \frac{1}{2} + 2x$ Ans.

(19) $y'' + y = x\sin x \quad y(0)=1,\; y'(0)=2$

$(D^2+1)y = x\sin x$

$D^2+1 = 0$ Characteristic Eq

$D^2 = -1$

$D = \pm i$

$\therefore y_c = c_1\cos x + c_2\sin x$

$y_p = \frac{1}{D^2+1}x\sin x$

$= \frac{1}{-1^2+1}x\sin x$ failure case

$\therefore y_p = \frac{x}{2D}(x\sin x)$

$= \frac{x}{2}\cdot\frac{1}{D}(\underset{I}{x}\,\underset{II}{\sin x})$ I.B.P

$= \frac{x}{2}\left[x\cdot(-\cos x) - \int 1\cdot(-\cos x)\,dx\right]$

$= \frac{x}{2}\left[-x\cos x + \int\cos x\,dx\right]$

$= \frac{x}{2}\left[-x\cos x + \sin x\right]$

$y_p = -\frac{x^2\cos x}{2} + \frac{x\sin x}{2}$

$y = y_c + y_p$

$y = c_1\cos x + c_2\sin x - \frac{x^2\cos x}{2} + \frac{x\sin x}{2}$ — (I)

$y' = -c_1\sin x + c_2\cos x - \frac{1}{2}\left[2x\cos x - x^2\sin x\right] + \frac{1}{2}\left[1.\sin x + x\cos x\right]$ — (II)

$y(0)=1 \Rightarrow \boxed{1 = c_1}$

$x=0, y=1$ in (I)

$y'(0)=2 \Rightarrow \boxed{2 = c_2}$

$x=0, y'=2$ in (II)

$\therefore y = 1\cos x + 2\sin x - \frac{x^2\cos x}{2} + \frac{x\sin x}{2}$

Ans.

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Method of Undetermined Coefficients

This is an alternate method for finding particular integral i,e Y_p, when $F(x)$ (i,e RHS of question) has terms like e^{ax}, $\sin ax$, $\cos ax$, x^m, and product of these terms. In this method we first find possible terms of 'Y_p' by writing all the derivatives of the terms in $F(x)$.

The Particular Integral 'Y_p' is constructed according to the following table.

Table

If $F(x)$ is of the form	then take Y_p as
i) a	$A x^K$
ii) ax^n (n +ive integer)	$x^K(A_0x^n + A_1x^{n-1} + \dots A_{n-1}x + A_n)$
iii) $ax^n e^{rx}$ (n is +ive integer)	$x^K(A_0x^n + A_1x^{n-1} + \dots A_{n-1}x + A_n)e^{rx}$
iv) $a\cos ax$ or $a\sin ax$ (for both)	$x^K(A\cos ax + B\sin ax)$
v) $ax^n e^{rx}\cos ax$ or $ax^n e^{rx}\sin ax$ (for both)	$x^K[(A_0x^n + A_1x^{n-1} + \dots A_{n-1}x + A_n)e^{rx}\cos ax + (B_0x^n + B_1x^{n-1} + \dots B_{n-1}x + B_n)e^{rx}\sin ax]$

Note

i) In x^K, K is the smallest non-negative integer i,e 0,1,2,3... which will ensure that no term in Y_p is already in C.F. i.e 'Y_c'

ii) If $F(x)$ is sum of several terms write Y_p for each term individually and then add up all of them.

iii) Y_p is written on the basis of type of fn mentioned in $F(x)$

iv) From Y_p fns are compared with Y_c by giving values to $K = 0, 1, 2 \dots$

v) If no term of Y_c is in Y_p, then put $K = 0$

vi) Start giving values to 'K' $= 0, 1, 2 \dots$ in Y_p, and leave those values of 'K' which make Y_p similar to terms in Y_c.

vii) The Y_p and its derivatives i,e $Y, Y', Y'' \dots$ will be substituted in the eq. ① and coeffts. of like terms on the LHS & RHS will be equated to determine the U.C $A, B, \dots$

10.3-1

(27)

EXERCISE 10.3

Solve by the method of U.C. (Problem 1-9):

Q.1 $y'' - 4y' + 4y = 1 \cdot e^{2x} \longrightarrow (1)$ $\quad F(x) = e^{2x}$

Sol. $\Rightarrow$ Auxilliary Eqn. is $D^2 - 4D + 4 = 0 \Rightarrow D = +2, +2$

$\therefore y_c = (c_1 + c_2 x) e^{2x} = c_1 e^{2x} + c_2 x' e^{2x}$

let us suppose that y_p of (1) is

$y_p = x^K A e^{2x}$ $\quad \because$ from (1) $F(x) = e^{2x}$ (see (iii) of table)

If we put $K=0$ in y_p we get Ae^{2x} similar to $c_1 e^{2x}$ in y_c, so leave $K=0$

Now put $K=1$ in y_p we get Axe^{2x} similar to $c_2 x e^{2x}$ in y_c, so leave $K=1$. So put $K=2$

$\therefore y_p = A x^2 e^{2x} \longrightarrow (2) \Rightarrow y'_p = 2Axe^{2x} + 2Ax^2 e^{2x}$

and $y''_p = 2Ae^{2x} + 4Axe^{2x} + 4Axe^{2x} + 4Ax^2 e^{2x}$

or $y''_p = 2Ae^{2x} + 8Axe^{2x} + 4Ax^2 e^{2x}$

put values of y_p, y'_p, y''_p in (1) and simplify

becomes $2Ae^{2x} + \cancel{8Axe^{2x}} + \cancel{4Ax^2 e^{2x}} - \cancel{8Axe^{2x}} - \cancel{8Ax^2 e^{2x}} + \cancel{4Ax^2 e^{2x}} = e^{2x} \longrightarrow (3)$

Comparing coeff. of like powers. We get

$2Ae^{2x} = 1 \cdot e^{2x} \Rightarrow 2A = 1 \Rightarrow A = 1/2$. put in (2)

from (2)

$\therefore y_p = \frac{1}{2} x^2 e^{2x}$ $\qquad y = y_c + y_p$

... Thus general sol. is

or $y = (c_1 + c_2 x) e^{2x} + \frac{1}{2} x^2 e^{2x}$ ans.

Q.2 $y'' + 2y' + 5y = 6 \sin 2x + 7 \cos 2x \longrightarrow (1)$ $\quad F(x) = 6\sin 2x + 7\cos 2x$

Sol A. Eqn. is $D^2 + 2D + 5 = 0$

$\Rightarrow D = \frac{-2 \pm \sqrt{4 - 20}}{2 \cdot 1} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$\Rightarrow y_c = e^{-x}[c_1 \cos 2x + c_2 \sin 2x] = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$

or $y_c = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$

Now let $y_p = x^K [A \sin 2x + B \cos 2x]$ (see (iv) of table)

Since no term of y_c is in y_p (due to absence of e^{-x} in y_p) thus put $K = 0$ in (2)

we have $y_p = A \sin 2x + B \cos 2x$

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$\Rightarrow y_p' = 2A\cos 2x - 2B\sin 2x$

and $y_p'' = -4A\sin 2x - 4B\cos 2x$

put values of y_p, y_p' and y_p'' in (1) we get

$-4A\sin 2x - 4B\cos 2x - 4A\cos 2x - 4B\sin 2x + 5A\sin 2x + 5B\cos 2x = 6\sin 2x + 7\cos 2x$

or $(A-4B)\sin 2x + (4A+B)\cos 2x = 6\sin 2x + 7\cos 2x$ — (3)

Comparing coeff of $\sin 2x$ & $\cos 2x$ we get

from (3) $A - 4B = 6$ & $4A + B = 7$ from (3)

or $4A - 16B = 24$
$4A + B = 7$
$\Rightarrow -17B = +17 \Rightarrow B = -1$
Thus $A = 2$

So $y_p = 2\sin 2x - \cos 2x$

Thus G. Sol is $y = e^{-x}(c_1\cos 2x + c_2\sin 2x) + 2\sin 2x - \cos 2x$ Ans

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Q.3 $2y'' + 3y' + y - x^2 + 3\sin x = 0$ $F(x) = x^2 + 3\sin x$

Sol A. Eq is $2D^2 + 3D + 1 = 0$

$\Rightarrow D = \frac{-3 \pm \sqrt{9-8}}{2 \cdot 2} = \frac{-3 \pm 1}{4} = -\frac{1}{2}, -1$

$\Rightarrow y_c = c_1 e^{-x} + c_2 e^{-\frac{1}{2}x}$

For P.I of Eqn (1) we find P.I of

$2y'' + 3y' + y = x^2$ → (2)

and $2y'' + 3y' + y = -3\sin x$ → (3)

and then (+) both results we get actual P.I of (1)

Now let y_p of (2) is

(from 1st of table) $y_p = x^k\{Ax^2 + Bx + C\}$ → (4) (due to absence of e^x in y_p) $k = 0$

since no term of y_c is in (4) so put $k = 0$

$y_{p_1} = Ax^2 + Bx + C$

$\Rightarrow y_{p_1}' = 2Ax + B$ and $y_{p_1}'' = 2A$ put value of y_{p_1}, y_{p_1}'

and y_{p_1}'' in (2) we get

$2 \cdot 2A + 6Ax + 3B + Ax^2 + Bx + C = 1x^2 + 0x + 0$

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$\Rightarrow Ax^2 + (6A+B)x + (4A+3B+C) = x^2$ — (5)

Comparing coeff. we get

from (5) $A = 1$, $6A + B = 0 \Rightarrow 6 + B = 0 \Rightarrow B = -6$

and $4\cdot 1 + 3(-6) + C = 0 \Rightarrow 4 - 18 + C = 0$ or $C = 14$.

So y_p of (2) is

$y_p = x^2 - 6x + 14$

Now let y_p of (3) is

(from table) $y_p = x^k [C_1 \cos x + D \sin x] \rightarrow$ (6)

Since no term of y_c is in (6). Thus put $k=0$ in (6)

$\therefore y_p = C_1 \cos x + D \sin x \Rightarrow y_p' = -C_1 \sin x + D \cos x$

and $y_p'' = -C_1 \cos x - D \sin x$ put values in (3), we get

$-C_1 \cos x - D\sin x - 3C_1 \sin x + 3D \cos x + C_1 \cos x + D \sin x = 3 \sin x$

$-2C_1 \cos x - 2D \sin x - 3C_1 \sin x + 3D\cos x$ ... $= 3 \sin x + 0 \cos x$

or $(-3C_1 - D)\sin x + (-C_1 + 3D)\cos x = 3 \sin x + 0\cos x$

Comparing coeff. we get

$-3C_1 - D = 3$ and $-C_1 + 3D = 0$

or $-9C_1 - 3D = 9$

and $-C_1 + 3D = 0$

$-10C_1 = 9 \Rightarrow C_1 = -9/10$. So $-D = 3 + 3C_1 = 3 - \frac{27}{10}$

or $-D = \frac{3}{10} \Rightarrow D = -\frac{3}{10}$

So y_p of Eq. (3) is

$y_p = -\frac{9}{10}\cos x - \frac{3}{10}\sin x$

Thus y_p of Eq. (1) is $= x^2 - 6x + 14 - \frac{9}{10}\cos x - \frac{3}{10}\sin x$

$\therefore$ G. Sol. of (1) is

$y = C_1 e^{-x} + C_2 e^{-\frac{1}{2}x} + x^2 - 6x + 14 - \frac{9}{10}\cos x - \frac{3}{10}\sin x$ Ans

x ———————— x

Q.4 $y'' + 2y' + y = e^{x}\cos x \rightarrow$ (1) $F(x) = e^{x}\cos x$

Sol A. Eqn of (1) is

$D^2 + 2D + 1 = 0 \Rightarrow D = -1, -1$

$\Rightarrow y_c = (c_1 + c_2 x)e^{-x} = c_1 e^{-x} + c_2 x e^{-x}$

Now let y_p of ① is

(from table) $y_p = x^k [A\cos x + B\sin x] e^x$

Since no term of y_c is in y_p so put $k = 0$, in y_p

$\Rightarrow y_p = (A\cos x + B\sin x)e^x$

$\Rightarrow y_p' = (A\cos x + B\sin x)e^x + [-A\sin x + B\cos x]e^x$

$y_p' = (A+B)\cos x\, e^x + (B-A)\sin x \cdot e^x$

and $y_p'' = (A+B)\cos x \cdot e^x - (A+B)\sin x \cdot e^x + (B-A)\cos x \cdot e^x$
$+ (B-A)\sin x \cdot e^x$

or $y_p'' = 2B\cos x \cdot e^x - 2A\sin x \cdot e^x$

put values in ① we get

$2B\cos x\, e^x - 2A\sin x \cdot e^x + 2A\cos x\, e^x + 2B\cos x\, e^x + 2B\sin x \cdot e^x - 2A\sin x \cdot e^x$
$+ A\cos x\, e^x + B\sin x\, e^x = e^x\cos x$

or $(3A + 4B)\cos x \cdot e^x + (-4A + 3B)\sin x \cdot e^x = e^x\cos x$

comparing coeff we get

$3A + 4B = 1$ and $-4A + 3B = 0$

$\Rightarrow 12A + 16B = 4$ & $-12A + 9B = 0$

or $-12A + 9B = 0$

$25B = 4 \Rightarrow B = \frac{4}{25}$, $-4A = -3\left(\frac{4}{25}\right)$

$\Rightarrow A = \frac{3}{25}$

$\therefore y_p = \left(\frac{3}{25}\cos x + \frac{4}{25}\sin x\right)e^x$

$\therefore$ G. Sol. is

$y = (c_1 + c_2 x)e^{-x} + \left(\frac{3}{25}\cos x + \frac{4}{25}\sin x\right)e^x$ ans

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Q.5 $y'' + y = 12\cos^2 x \to ①$

Sol A. Eq. of ① is $D^2 + 1 = 0 \Rightarrow D = 0 \pm i$

$\Rightarrow y_c = e^{0x}[c_1\cos x + c_2\sin x] = c_1\cos x + c_2\sin x$

Now Since $\cos^2 x = \frac{1+\cos 2x}{2}$ So ① will be

So (1) can be written as

$y'' + y = 12\left(\frac{1+\cos 2x}{2}\right)$

or $y'' + y = 6 + 6\cos 2x \rightarrow (2)$

Now for finding y_p of (2). We find saperately y_p's of

$y'' + y = 6 \rightarrow (3)$ and $y'' + y = 6\cos 2x \rightarrow (4)$

let y_p of (3) is $\quad F(x) = g(3) = 6 \quad$ Since no term of (3) is in y_c. So

(from using table) $y_p = x^K A$ $\qquad y_p = x^0 A = 1 \cdot A$

put $K=0$ in y_p we get

or $y_p = A \Rightarrow y_p' = 0, \; y_p'' = 0$ put in (3), we get

$0 + A = 6 \Rightarrow A = 6$

Thus $y_p = 6$ (i.e P.I. of (3))

Again let y_p of (4) is $y_p = x^K[B\cos 2x + D\sin 2x]$

Since no term of y_c is in y_p. So put $K=0$ in y_p, we get

$y_p = B\cos 2x + D\sin 2x \Rightarrow y_p' = -2B\sin 2x + 2D\cos 2x$

and $y_p'' = -4B\cos 2x - 4D\sin 2x$ put values of y_p'', y_p at place of y'' and y in (4), we get

$-4B\cos 2x - 4D\sin 2x + B\cos 2x + D\sin 2x = 6\cos 2x$

$\Rightarrow -3B\cos 2x - 3D\sin 2x = 6\cos 2x$ (by comparing coeff.)

$\Rightarrow -3B = 6$ and $-3D = 0$

$\Rightarrow B = -2$ and $D = 0$

$\therefore y_p = -2\cos 2x + 0\sin 2x$

or $y_p = -2\cos 2x \Rightarrow y_p$ of (1) $= 6 - 2\cos 2x$

Thus G. Sol. of eq. (1) is

$y = c_1\cos x + c_2\sin x + 6 - 2\cos 2x$ Ans

Q.6 $y'' - 3y' + 2y = \cancel{x}\; e^x + 2xe^x \rightarrow (1)$

Sol. A. Eqn of (1) is $D^2 - 3D + 2 = 0$

$(D-1)(D-2) = 0 \Rightarrow D = 1, 2$

$\boxed{10\text{-}3\text{-}6}$ (32)

$\Rightarrow y_c = c_1 e^{x} + c_2 e^{2x}$

For P.I. of (1), we find saperately P.I of

$y'' - 3y' + 2y = 2x^2 \to (2)$ & $y'' - 3y' + 2y = 2xe^{x} \to (3)$

$F(x) = 2x^2$ $\quad F(x) = 2xe^{x}$

Let y_p of (2) is $= x^{K}[Ax^2 + Bx + C]$ P.I of (2) is

Since no term of y_c is in y_p of (2) $\therefore K=0$, So

$y_p = Ax^2 + Bx + C \Rightarrow y_p' = 2Ax + B$

and $y_p'' = 2A$ put values of y_p, y_p', y_p'' at the place of y, y', y'' respectively, we get

$2A - 6Ax - 3B + 2Ax^2 + 2Bx + 2C = 2x^2$

or $2Ax^2 + (-6A + 2B)x + (2A - 3B + 2C) = 2x^2$

Comparing Coeff of x^2, x & x^0 we get

$2A = 2 \Rightarrow \boxed{A=1}$, $-6A + 2B = 0$ or $-6 + 2B = 0 \Rightarrow \boxed{B=3}$

and $2A - 3B + 2C = 0$ or $2 - 3\times 3 + 2C = 0 \Rightarrow C = \frac{7}{2}$

So P.I. of (2) is $y_p = x^2 + 3x + \frac{7}{2}$

Now let P.I of (3) is

Common factor e^{x} of y_p is al...

(from table) $y_p = x^{K}[Dx + E]e^{x}$

Since if $K=0$ then Ee^{x} of y_p is similar to c_1e^{x} of y_c

So leave $K=0$ & Put $K=1$

$\therefore y_p = x^1[Dx + E]e^{x} = [Dx^2 + Ex]e^{x} = Dx^2e^{x} + (E+2D)xe^{x} + Ee^{x}$

or $y_p' = \{Dx^2 + Ex\}e^{x} + \{2Dx + E\}e^{x}$

$y_p' = \{Dx^2 + (2D+E)x + E\}e^{x}$

$\Rightarrow y_p'' = \{Dx^2 + (2D+E)x + E\}e^{x} + \{2xD + 2D + E\}e^{x}$

or $y_p'' = \{Dx^2 + (4D+E)x + (2D+2E)\}e^{x}$

put values of y_p, y_p' & y_p'' in place of y, y', y'' in eqn (3), we get

$\{Dx^2 + (4D+E)x + (2D+2E)\}e^{x} - 3\{Dx^2 + (2D+E)x + E\}e^{x} + 2(Dx^2 + Ex)e^{x} = 2xe^{x}$

Comparing coeff of like powers

$3D - 3D = 0$, and $4D + E - 6D - 3E + 2E = 2$

$-2D = 2 \Rightarrow D = -1$

$2D+2E-3E=0$ or $-2-E=0 \Rightarrow E=-2$

$\therefore$ P.I. of (3) is

$y_p = (-x^2-2x)e^x$

Thus P.I. of (1) is

$y_p = x^2+3x+\frac{7}{2}-(x^2+2x)e^x$

So G. sol is

$y = c_1e^x + c_2e^{2x} + x^2 + 3x + \frac{7}{2} - (x^2+2x)e^x$ is

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Q.7 $y''' + y' = 2x^2 + 4\sin x \rightarrow (1)$

Sol A. Eqn is $D^3+D=0$, $D(D^2+1)=0 \Rightarrow D=0, \pm i$

$\Rightarrow y_c = c_1e^{0x} + c_2\cos x + c_3\sin x$

$y_c = c_1 + c_2\cos x + c_3\sin x$

Now for finding P.I. of (1). We find saperately P.I. of

$y''' + y' = 2x^2 \rightarrow (2)$ And $y''' + y' = 4\sin x \rightarrow (3)$

$\because F(x)$ of (2) $= 2x^2$

let P.I. of (2) is $y_p = x^k[Ax^2+Bx+C]$

Since if we put $k=0$ then 'C' of y_p is similar to 'c_1' of y_c, being const so put $k=1$ in y_p.

$\therefore$ we get $y_p = x^1[Ax^2+Bx+C]$

or $y_p = Ax^3+Bx^2+Cx$

$\Rightarrow y_p' = 3x^2A+2xB+C$, $y_p'' = 6xA+2B$

and $y_p''' = 6A$ put values of y_p''' and y_p' in

(2) We get $6A+3x^2A+2xB+C = 2x^2$

$\Rightarrow 3A = 2 \Rightarrow \boxed{A=\frac{2}{3}}$, $2B=0 \Rightarrow \boxed{B=0}$

and $6A+C=0 \Rightarrow C=-6\times\frac{2}{3} \Rightarrow \boxed{C=-4}$

$\Rightarrow y_p = \frac{2}{3}x^3 + 0x^2 + (-4)x$

or $\boxed{y_p = \frac{2}{3}x^3 - 4x}$

$\because F(x)$ of (3) is $4\sin x$

Now let P.I. of (3) is $y_p = x^k(D\cos x + E\sin x)$

If we put $k=0$ in y_p then $D\cos x$ is similar to $c_2\cos x$ of y_c & $E\sin x$ of y_p is similar to $c_3\sin x$ of y_c so leave $k=0$ & Put $k=1$

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$y_p = x^1[D\cos x + E\sin x] = Dx\cos x + Ex\sin x$

$\Rightarrow y_p' = -Dx\sin x + D\cos x + E\sin x + Ex\cos x$

$y_p' = (D+Ex)\cos x + (E-Dx)\sin x$

$y_p'' = E\cos x - (D+Ex)\sin x - D\sin x + (E-Dx)\cos x$

$y_p'' = -(2D+Ex)\sin x + (2E-Dx)\cos x$

$y''' = -(2D+Ex)\cos x - E\sin x + (Dx-2E)\sin x - D\cos x$

$= -(3D+Ex)\cos x + (Dx-3E)\sin x$

put values of y_p''' and y_p' in (3) we get

$-(3D+Ex)\cos x + (Dx-3E)\sin x + (D+Ex)\cos x + (E-Dx)\sin x = 4\sin x$

$-3D\cos x - \cancel{Ex}\cos x + \cancel{Dx}\sin x - \underline{3E\sin x} + D\cos x + \cancel{Ex}\cos x$

$+\underline{E\sin x} - \cancel{Dx}\sin x = 4\sin x$

$-2D\cos x - 2E\sin x = 4\sin x$ comparing coeff of $\sin x$ and $\cos x$, we get

$\Rightarrow -2D = 0 \quad | \quad$ and $-2E = 4$

$\Rightarrow D = 0 \quad | \quad \Rightarrow E = -2$

So $y_p = 0x\cos x - 2x\sin x = -2x\sin x$

Thus G. Sol. of (1) is

$y_1 = c_1 + c_2\cos x + c_3\sin x + \frac{2}{3}x^3 - 4x - 2x\sin x$

x———x

Q8 $y''' + y'' + 3y' - 5y = 5\sin 2x + 10x^2 + 3x + 7$

Sol. A. Eqn is $D^3 + D^2 + 3D - 5 = 0$

1	1	3	-5
0	1	2	5
	2	5	0

$\Rightarrow (D-1)(D^2+2D+5) = 0$

$\Rightarrow D = 1$ and $D = \frac{-2 \pm \sqrt{4-20}}{2 \cdot 1} = -1 \pm 2i$

$\Rightarrow y_c = c_1e^x + e^{-x}(c_2\cos 2x + c_3\sin 2x)$

For P.I. of (1), we first find P.I. of

$y''' + y'' + 3y' - 5y = 5\sin 2x \longrightarrow (2)$

and $y''' + y'' + 3y' - 5y = 10x^2 + 3x + 7 \longrightarrow (3)$

let y_p of (2) is $y_p = x^k[A\cos 2x + B\sin 2x]$

Since no term of y_c is in y_p, so put $k=0$ in y_p

we get $y_p = A\cos 2x + B\sin 2x$

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$\Rightarrow y_p' = -2A\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$

$y_p''' = +8A\sin 2x - 8B\cos 2x$ put values of y_p', y_p'', y_p'''

and y_p in (2), we get

$8A\sin 2x - 8B\cos 2x - 4A\cos 2x - 4B\sin 2x - 6A\sin 2x + 6B\cos 2x$
$-5A\cos 2x - 5B\sin 2x = 5\sin 2x$

$2A\sin 2x - 9B\sin 2x - 9A\cos 2x - 2B\cos 2x = 5\sin 2x$

$\Rightarrow (2A - 9B)\sin 2x + (-9A - 2B)\cos 2x = 5\sin 2x$

$\Rightarrow 2A - 9B = 5$ and $-9A - 2B = 0$

or $18A - 81B = 45$ & $-18A - 4B = 0$

$-18A - 4B = 0$

$-85B = 45 \Rightarrow B = -\frac{45}{85} \Rightarrow \boxed{B = -\frac{9}{17}}$

$\Rightarrow -9A = +2\left(-\frac{9}{17}\right)$ or $9A = \frac{18}{17}$ or $\boxed{A = \frac{2}{17}}$

$\Rightarrow \boxed{y_p = \frac{2}{17}\cos 2x - \frac{9}{17}\sin 2x}$

Now let P.I. of (3) is

$y_p = (Cx^2 + Dx + E)x^k$

Since no term of y_c is in y_p, so put $k=0$ in y_p

$\Rightarrow y_p = Cx^2 + Dx + E \Rightarrow y_p' = 2Cx + D$

$y_p'' = 2C$, $y_p''' = 0$, put values in (3), we get

$0 + 2C + 6Cx + 3D - 5Cx^2 - 5Dx - 5E = 10x^2 + 3x + 7$

or $-5Cx^2 + (6C - 5D)x + (3D - 5E + 2C) = 10x^2 + 3x + 7$

$\Rightarrow -5C = 10 \Rightarrow \boxed{C = -2}$, $6C - 5D = 3$ or $-12 - 5D = 3 \Rightarrow \boxed{D =}$

and $3D - 5E + 2C = 7$ or $-9 - 5E - 4 = 7$ or $\boxed{E = 4}$

$\therefore \boxed{y_p = -2x^2 - 3x + 4}$ (P.I. of (3))

So G. Sol. of (1) is

$y = c_1 e^{x} + e^{-x}\left[c_2\cos 2x + c_3\sin 2x\right] + \frac{2}{17}\cos 2x - \frac{9}{17}\sin 2x - 2x^2 - 3x + 4$

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$$y_p = x^k\left[E_1 \cos x + F_1 \sin x\right]e^{-x}$$

If we put $K=0$ in y_p we get $[E_1\cos x + F_1 \sin x]e^{-x}$ similar to $(c_1\cos x + c_2 \sin x)e^{-x}$ of y_c So leave $K=0$ + Put $K=1$

we get $y_p = x\left(E_1 \cos x + F_1 \sin x\right)e^{-x}$

or $y_p = \left(E_1 x \cos x + F_1 x \sin x\right)e^{-x}$

Now as $C = F = E$ ($\because$ all's are constants)

Then $y_p = \left[(Cx)\cos x + (Cx)\sin x\right]e^{-x} \rightarrow (7)$

but we see that y_p of eqn (7) is already present y_p of eq (5)

Thus P.I. of (1) is only the sum of Eq's (5) & (6)

i.e. $y_p = e^{x}\left[(Ax^3 + Bx^2 + Cx)\cos x + (Ax^3 + Bx^2 + Cx)\sin x\right] + D_1 e^{-x}$ Ans

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Q 10(ii) $y'' + 3y' + 2y = e^{x}(x^2+1)\sin 2x + 4e^{x} + 3e^{-x}\cos x \longrightarrow (1)$

Sol: A. Eqn of (1) is $D^2 + 3D + 2 = 0 \Rightarrow D = -1, -2$

$\Rightarrow y_c = c_1 e^{-x} + c_2 e^{-2x}$

We first find P.I. of each term of right saperately and then their sum for finding P.I. of (1)

i.e. P.I. of $y'' + 3y' + 2y = e^{x}(x^2+1)\sin 2x \rightarrow (2)$

$y'' + 3y' + 2y = 4e^{x} \rightarrow (3)$

and $y'' + 3y' + 2y = 3e^{-x}\cos x \rightarrow (4)$

Let P.I. of (2) is $y_p = e^{x}\left\{[Ax^2 + Bx + C]\sin 2x + (Ax^2 + Bx + C)\cos 2x\right\}x^k \rightarrow (5)$ $\longrightarrow (6)$

P.I. of (3) is $y_p = x^k D e^{x}$

P.I. of (4) is $y_p = x^k\left[E \cos x + F \sin x\right]e^{-x} \longrightarrow (7)$

Since no term of y_c is in (5), (6) and (7) So put $K=0$ in (5), (6) and (7), and then adding, We get

$$y_p = e^{x}\left\{(Ax^2 + Bx + C)\sin 2x + (Ax^2 + Bx + C)\cos 2x\right\}x^0 + x^0 D e^{x} + x^0\left(E \cos x + F \sin x\right)e^{-x}$$

or $y_p = e^{x}\left\{(Ax^2 + Bx + C)\sin 2x + (Ax^2 + Bx + C)\cos 2x\right\} + D e^{x} + \left(E \cos x + F \sin x\right)e^{-x}$ Ans.

Q.9 $y^{iv} + 8y'' + 16y = \sin x \rightarrow ①$

Sol: A. Eqn. is $D^4 + 8D^2 + 16 = 0$ $(D^2+4)^2 = 0 \Rightarrow (D^2+4)(D^2+4) = 0$

The roots are complex can repeated

$\Rightarrow D^2 = \pm 2i, \pm 2i \Rightarrow$

$\Rightarrow y_c = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

let P.I of ① is $y_p = x^k[A\sin x + B\cos x]$ $\therefore k = 0$, in y_p.

Since no term of y_c is in y_p

$\therefore y_p = x^0[A\sin x + B\cos x] \Rightarrow y_p = A\sin x + B\cos x$

$\Rightarrow y_p' = A\cos x - B\sin x$, $y_p'' = -A\sin x - B\cos x$

$y_p''' = -A\cos x + B\sin x$, $y_p'''' = A\sin x + B\cos x$

put values of y_p'''', y_p'', y_p at place of y'''', y'' & y in ①

$A\sin x + B\cos x - 8A\sin x - 8B\cos x + 16A\sin x + 16B\cos x = \sin x$

$9A\sin x + 9B\cos x = 1 \cdot \sin x \Rightarrow 9A = 1$, $9B = 0$

$\Rightarrow A = \frac{1}{9}$, $B = 0$ $\Rightarrow y_p = \frac{1}{9}\sin x + 0\cos x$ or $y_p = \frac{1}{9}\sin x$

Thus G. Sol. of ① is $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x + \frac{1}{9}\sin x$ ans.

Q.10 Write the general term of P.I (without evaluating U.C)

(i) for $y'' + 2y' + 2y = 4e^{-x}x^2\sin x + 3e^{-x} + 2e^{-x}\cos x \rightarrow ①$

Sol A. Eqn. is $D^2 + 2D + 2 = 0 \Rightarrow D = \frac{-2 \pm \sqrt{4-8}}{2 \cdot 1} = -1 \pm i$

$\Rightarrow y_c = e^{-x}[c_1\cos x + c_2\sin x]$

For find P.I of ① we first find

P.I. of $y'' + 2y' + 2y = 4e^{-x}x^2\sin x \rightarrow ②$

$y'' + 2y' + 2y = 3e^{-x} \rightarrow ③$

$y'' + 2y' + 2y = 2e^{-x}\cos x \rightarrow ④$

> **NOTE:** In This Question To find undermined coefficients is not required. Here we have to find only general form of P.I

let P.I of ② is

$y_p = x^k\{e^{-x}[Ax^2 + Bx + C]\sin x + e^{-x}[Dx^2 + Ex + F]\cos x\}$ (from table)

which is similar to $e^{-x}C\cos x + e^{-x}C\sin x$; if we put $k=0$ in y_p we get $e^{-x}(c_1\cos x + c_2\sin x)$ of y_c. Put $\boxed{k=1}$ in y_p

$y_p = x\{e^{-x}(Ax^2 + Bx + C)\sin x + e^{-x}(Dx^2 + Ex + F)\cos x\}$

or $y_p = e^{-x}(Ax^3 + Bx^2 + Cx)\sin x + e^{-x}(Dx^3 + Ex^2 + Fx)\cos x \rightarrow ⑤$

let P.I of ③ is $y_p = x^k D_1 e^{-x}$ since no term of y_c is in y_p so $k=0$

$\Rightarrow y_p \Rightarrow y_p = D_1 e^{-x} \rightarrow ⑥$ $(\because x^0 = 1)$

also let P.I of ④ is

10.4-0

original (15)

The Cauchy Euler Diff Eq

A diff eq of the form $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots a_{n-1} x \frac{dy}{dx} + a_n y = F(x)$ —— ①

or $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \dots a_{n-1} x D + a_n) y = F(x)$ $(a_i \in R$

is called Cauchy Euler Diff Eq (variable coeffts).

This eq can be reduced to a linear diff eq with const coeffts as

Put $x = e^t \Rightarrow t = \ln x$

$x = e^t$
$\ln x = \ln e^t$
$\ln x = t \ln e$
$\ln x = t$
$\frac{1}{x} = \frac{dt}{dx}$

$\frac{dx}{dy} = e^t \frac{dt}{dy}$

$\frac{dy}{dx} = \frac{1}{e^t} \frac{dy}{dt}$

$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt}$

$x \frac{dy}{dx} = \frac{dy}{dt}$

$D = \frac{d}{dx}$
$\Delta = \frac{d}{dt}$

$xDy = \Delta y \Rightarrow \boxed{xD = \Delta}$

Again $\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt}$

Diff $\frac{d^2y}{dx^2} = \frac{1}{x} \frac{d^2y}{dt^2} \cdot \frac{dt}{dx} - \frac{1}{x^2} \frac{dy}{dt}$

$= \frac{1}{x} \frac{d^2y}{dt^2} \frac{1}{x} - \frac{1}{x^2} \frac{dy}{dt}$

$\frac{d^2y}{dx^2} = \frac{1}{x^2} \left(\frac{d^2y}{dt^2} - \frac{dy}{dt} \right)$

$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

$x^2 D^2 y = \Delta^2 y - \Delta y$

$(x^2 D^2) y = (\Delta^2 - \Delta) y$

$x^2 D^2 = \Delta^2 - \Delta$

$\boxed{x^2 D^2 = \Delta(\Delta - 1)}$

$\boxed{x^3 D^3 = \Delta(\Delta-1)(\Delta-2)}$

$\boxed{x^n D^n = \Delta(\Delta-1)(\Delta-2) \dots (\Delta-(n-1))}$

substituting values of $xD, x^2D^2, x^3D^3, \dots$ in ① we obtain an eq of nth order w[ith] const coeffts having 't' as independent variable. Now it can be solved

Ex No 10.4

① $(x^2D^2+7xD+5)Y = x^5$ ——— ①

$(\Delta(\Delta-1)+7\Delta+5)Y = e^{5t}$

$(\Delta^2-\Delta+7\Delta+5)Y = e^{5t}$

$(\Delta^2+6\Delta+5)Y = e^{5t}$

Put $x=e^t$
$\Rightarrow t = \ln x$
$xD=\Delta$
$x^2D^2=\Delta(\Delta-1)$

For Characteristic eq

$\Delta^2+6\Delta+5 = 0$

$\Delta^2+1\Delta+5\Delta+5 = 0$

$\Delta(\Delta+1)+5(\Delta+1) = 0$

$(\Delta+1)(\Delta+5) = 0$

$\Delta = -1, -5$

$Y_c = c_1e^{-t}+c_2e^{-5t}$

$= \frac{c_1}{e^t}+\frac{c_2}{e^{5t}}$

$Y_c = \frac{c_1}{x}+\frac{c_2}{x^5}$ ——— ②

For Particular Integral

$Y_p = \frac{1}{\Delta^2+6\Delta+5}e^{5t}$

$= \frac{e^{5t}}{5^2+6(5)+5}$

$= \frac{e^{5t}}{60}$

$Y_p = \frac{x^5}{60}$ ——— ③ $\because x = e^t$

So general sol is

$Y = Y_c+Y_p$

$= \frac{c_1}{x}+\frac{c_2}{x^5}+\frac{x^5}{60}$

② $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y = x^2\sin(\ln x)$

$(x^2D^2-3xD+5)Y = x^2\sin(\ln x)$

Put $x=e^t$, $t=\ln x$
$xD=\Delta$, $x^2D^2=\Delta(\Delta-1)$

$(\Delta(\Delta-1)-3\Delta+5)Y = e^{2t}\sin t$

$(\Delta^2-\Delta-3\Delta+5)Y = e^{2t}\sin t$

$(\Delta^2-4\Delta+5)Y = e^{2t}\sin t$

For characteristic Eq.

$\Delta^2-4\Delta+5 = 0$

$\Delta = \frac{4\pm\sqrt{16-20}}{2} = \frac{4\pm\sqrt{-4}}{2} = \frac{4\pm 2i}{2}$

$= 2\pm i$

$Y_c = e^{2t}(c_1\cos t+c_2\sin t)$

$Y_c = x^2(c_1\cos\ln x+c_2\sin\ln x)$

For Particular Integral

$Y_p = \frac{1}{\Delta^2-4\Delta+5}e^{2t}\sin t$

$= e^{2t}\frac{1}{(\Delta+2)^2-4(\Delta+2)+5}\sin t$

$= e^{2t}\frac{1}{\Delta^2+\cancel{4\Delta}+4-\cancel{4\Delta}-8+5}\sin t$

$= e^{2t}\frac{1}{(\Delta^2+1)}\sin t$ $\because$ (failure case $-1^2+1=0$)

$= e^{2t}\frac{t}{2\Delta}\sin t$

$= \frac{te^{2t}}{2}(-\cos t)$ $\because \frac{1}{\Delta}\sin t = -\cos t$

$Y_p = -\frac{1}{2}te^{2t}\cos t$

$Y_p = -\frac{1}{2}\ln x\, x^2\cos(\ln x)$

Hence General Sol

$Y = Y_c+Y_p$

$Y = x^2(c_1\cos\ln x+c_2\sin\ln x) - \frac{1}{2}\ln x\, x^2\cos(\ln x)$

16.4-2

3. $x^2\frac{d^2y}{dx^2} - (2m-1)x\frac{dy}{dx} + (m^2+n^2)y = n^2x^m\ln x$ —— (1)

$[x^2D^2 - (2m-1)xD + (m^2+n^2)]y = n^2x^m\ln x$

Put $x = e^t$
$t = \ln x$
$xD = \Delta$
$x^2D^2 = \Delta(\Delta-1) = \Delta^2-\Delta$

$\therefore [\Delta^2-\Delta-(2m-1)\Delta+(m^2+n^2)]y = n^2e^{mt}.t$

$(\Delta^2-\cancel{\Delta}-2m\Delta+\cancel{\Delta}+m^2+n^2)y = n^2e^{mt}.t$

$(\Delta^2-2m\Delta+m^2+n^2)y = n^2e^{mt}t$ —— (II)

(LDEq with const coeff)

Characteristic Eq of (II) is $\Delta^2-2m\Delta+m^2+n^2 = 0$

$\Delta = \frac{2m\pm\sqrt{4m^2-4m^2-4n^2}}{2} = \frac{2m\pm\sqrt{-4n^2}}{2}$

$\Delta = \frac{2m\pm 2in}{2} = \frac{\cancel{2}(m\pm in)}{\cancel{2}} = \boxed{m\pm in}$

$y_c = e^{mt}(c_1\cos nt + c_2\sin nt)$

$\because \left(\begin{matrix} x = e^t \\ x^m = e^{mt}\end{matrix}\right) \because \left\{\begin{matrix} t = \ln x \\ nt = n\ln x \\ nt = \ln x^n\end{matrix}\right.$

$y_c = x^m(c_1\cos\ln x^n + c_2\sin\ln x^n)$

Now $y_p = \frac{1}{\Delta^2-2m\Delta+m^2+n^2}(n^2e^{mt}.t)$

$= n^2e^{mt}\frac{1}{(\Delta+m)^2-2m(\Delta+m)+m^2+n^2}(t)$

$= n^2e^{mt}\frac{1}{\Delta^2+\cancel{2\Delta m}+\cancel{m^2}-\cancel{2\Delta m}-\cancel{2m^2}+\cancel{m^2}+n^2}(t)$

$= n^2e^{mt}\frac{1}{(\Delta^2+n^2)}(t)$

$= \frac{\cancel{n^2}e^{mt}}{\cancel{n^2}(\frac{\Delta^2}{n^2}+1)}(t) = e^{mt}\left(1+\frac{\Delta^2}{n^2}\right)^{-1}(t)$

$= e^{mt}\left(1-\frac{\Delta^2}{n^2}\right)t = e^{mt}(t) - \frac{e^{mt}}{n^2}(0)$

$= e^{mt}(t) = e^{m\ln x}(\ln x) = e^{\ln x^m}(\ln x)$

$y_p = x^m\ln x$

So G.Sol. $y = y_c + y_p$

$= x^m(c_1\cos\ln x^n + c_2\sin\ln x^n) + x^m\ln x.$

Ex 10.4-3 (18)

Q4 $(4x^2D^2-4xD+3)y=\sin\ln(-x)$ — ①

Cauchy-Euler Eq.

Put $-x=e^t \Rightarrow t=\ln(-x)$

$xD=\Delta$

$x^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$

Put in ①

$(4(\Delta^2-\Delta)-4\Delta+3)y=\sin t$

$(4\Delta^2-8\Delta+3)y=\sin t$ — ②

Characteristic Eq of ② is

$$4\Delta^2-8\Delta+3=0$$

$$\Delta=\frac{8\pm\sqrt{64-4\cdot4\cdot3}}{8}=\frac{8\pm\sqrt{64-48}}{8}$$

$$=\frac{8\pm4}{8}=\frac{3}{2},\frac{1}{2}$$

$$y_c=c_1e^{\frac{3}{2}t}+c_2e^{t/2}$$

$$y_p=\frac{\sin t}{4\Delta^2-8\Delta+3}$$

$$=\frac{\sin t}{4(-1^2)-8\Delta+3}=\frac{\sin t}{-(1+8\Delta)}$$

$$=\frac{-(1-8\Delta)\sin t}{(1+8\Delta)(1-8\Delta)}=\frac{-(1-8\Delta)\sin t}{1-64\Delta^2}$$

$$=-\frac{(1-8\Delta)\sin t}{1-64(-1^2)}=\frac{-\sin t+8\cos t}{65}\text{ Ans}$$

$\boxed{\text{2nd Method}}$ $y_p=\dfrac{\sin t}{4\Delta^2-8\Delta+3}=\operatorname{Im}\dfrac{e^{it}}{4\Delta^2-8\Delta+3}$

$$=\operatorname{Im}\frac{e^{it}}{4(i)^2-8(i)+3}=\operatorname{Im}\frac{(1-8i)}{-(1+8i)(1-8i)}e^{it}$$

$$=\operatorname{Im}\frac{-1}{65}(1-8i)e^{it}=\frac{-1}{65}\operatorname{Im}(1-8i)[\cos t+i\sin t]$$

$$y_p=-\frac{1}{65}\left[\sin t-8\cos t\right]=\frac{8\cos t-\sin t}{65}\text{ Ans}$$

$$y=c_1e^{3t/2}+c_2e^{t/2}+\frac{8}{65}\cos t-\frac{1}{65}\sin t$$

(5) $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y = 10x+\frac{10}{x}$

$(x^3D^3+2x^2D^2+2)y = 10x+\frac{10}{x}$ —— (I)

Put $x=e^t \Rightarrow t=\ln x$
$xD = \Delta$
$x^2D^2 = \Delta(\Delta-1)$
$x^3D^3 = \Delta(\Delta-1)(\Delta-2)$

$(\Delta(\Delta-1)(\Delta-2)+2\Delta(\Delta-1)+2)y = 10e^t+\frac{10}{e^t}$

$(\Delta^3-3\Delta^2+2\Delta+2(\Delta^2-\Delta)+2)y = 10e^t+10e^{-t}$

$(\Delta^3-3\Delta^2+\cancel{2\Delta}+2\Delta^2-\cancel{2\Delta}+2)y = 10e^t+10e^{-t}$

$(\Delta^3-\Delta^2+2)y = 10e^t+10e^{-t}$ —— (II)

Characteristic Eq

$\Delta^3-\Delta^2+2 = 0$

-1	1	-1	0	2
	↓	-1	2	-2
	1	-2	2	$\boxed{0}$

So $\Delta=-1$

Depressed Eq is

$\Delta^2-2\Delta+2 = 0$

$\Delta = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm 2i}{2} = 1\pm i$

$\therefore\ \Delta = -1,\ 1\pm i$

Hence $y_c = c_1e^{-t}+e^t(c_2\cos t+c_3\sin t)$

$y_c = \frac{c_1}{x}+x(c_2\cos(\ln x)+c_3\sin(\ln x))$

Now $y_p = \frac{1}{(\Delta^3-\Delta^2+2)}(10e^t+10e^{-t})$

$= \frac{1}{\Delta^3-\Delta^2+2}10e^t+\frac{1}{\Delta^3-\Delta^2+2}10e^{-t}$

failure case $\because \frac{1}{-1-(-1)^2+2}=\infty$

Multiply by t Take Derivative

$= \frac{10e^t}{1-1+2}+\frac{t}{3\Delta^2-2\Delta+0}(10e^{-t})$

$= \frac{10e^t}{2}+\frac{t\,10e^{-t}}{3(-1)^2-2(-1)}$

$= 5e^t+\frac{10}{5}te^{-t}$

$y_p = 5x+2\ln x\left(\frac{1}{x}\right)$

$y = y_c+y_p$

$= \frac{c_1}{x}+x(c_2\cos\ln x+c_3\sin\ln x)+5x+2\frac{\ln x}{x}$

10.4-5

(6) $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1$

Non Cauchy Euler Diff Eq.

$\div$ by x: $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \dfrac{1}{x}$

$(x^3D^3 + 2x^2D^2 - xD + 1)y = \dfrac{1}{x}$

Put $x = e^t \Rightarrow t = \ln x$

$xD = \Delta$

$x^2D^2 = \Delta(\Delta-1)$

$x^3D^3 = \Delta(\Delta-1)(\Delta-2)$

$(\Delta(\Delta-1)(\Delta-2) + 2\Delta(\Delta-1) - \Delta + 1)y = \dfrac{1}{e^t}$

$(\Delta^3 - 3\Delta^2 + \cancel{2\Delta} + 2\Delta^2 - \cancel{2\Delta} - \Delta + 1)y = e^{-t}$

$(\Delta^3 - \Delta^2 - \Delta + 1)y = e^{-t}$

(LDEq with const coeffs)

Characteristic Eq

$\Delta^3 - \Delta^2 - \Delta + 1 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -1 & -1 & 1 \\ & \downarrow & +1 & 0 & -1 \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

$\therefore \Delta = 1$

and Depressed Eq is

$\Delta^2 - 1 = 0$

$\Rightarrow \Delta^2 = 1 \Rightarrow \Delta = \pm 1$

$\therefore \Delta = 1, 1, -1$

$y_c = (c_1 + c_2t)e^t + c_3e^{-t}$

$y_c = (c_1 + c_2\ln x)x + \dfrac{c_3}{x}$

$y_p = \dfrac{1}{\Delta^3 - \Delta^2 - \Delta + 1}(e^{-t})$

$\dfrac{1}{(-1)^3 - (-1)^2 - (-1) + 1} = \dfrac{1}{-1-1+1+1} = \infty$

$= \dfrac{t}{3\Delta^2 - 2\Delta - 1}e^{-t}$

$= \dfrac{te^{-t}}{3(-1)^2 - 2(-1) - 1} = \dfrac{te^{-t}}{3+2-1}$

$y_p = \dfrac{te^{-t}}{4}$

$= \dfrac{\ln x}{4}\cdot\dfrac{1}{x}$

$\because t = \ln x$

$e^t = x$

$e^{-t} = \dfrac{1}{x}$

So $y = y_c + y_p$

$= (c_1 + c_2\ln x)x + \dfrac{c_3}{x} + \dfrac{\ln x}{4x}$ Ans.

10.3/7 6 (21) (7)

(7) $(x^3D^3+4x^2D^2-5xD-15)y=x^4$ Cauchy-Euler Eq.

Let $x=e^t \Rightarrow t=\ln x$

$xD=\Delta$

$x^2D^2=\Delta^2-\Delta$

$x^3D^3=\Delta^3-3\Delta^2+2\Delta$

Put in (1) $(\Delta^3-3\Delta^2+2\Delta+4\Delta^2-4\Delta-5\Delta-15)y=e^{4t}$

$(\Delta^3+\Delta^2-7\Delta-15)y=e^{4t}$

Characteristic Eq is $\Delta^3+\Delta^2-7\Delta-15=0$

$$\begin{array}{c|cccc} & 1 & 1 & -7 & -15 \\ 3 & \downarrow & 3 & 12 & 15 \\ \hline & 1 & 4 & 5 & 0 \end{array}$$

$\Delta^2+4\Delta+5=0$

$\Delta=\frac{-4\pm\sqrt{16-4\cdot1\cdot5}}{2}=\frac{-4\pm\sqrt{-4}}{2}$

$=\frac{-4\pm2i}{2}=-2\pm i$

$y_c=c_1e^{3t}+e^{-2t}(c_2\cos t+c_3\sin t)$

$y_p=\frac{1}{\Delta^3+\Delta^2-7\Delta-15}e^{4t}$

$=\frac{e^{4t}}{64+16-28-15}=\frac{e^{4t}}{37}$

$y=y_c+y_p=c_1e^{3t}+e^{-2t}(c_2\cos t+c_3\sin t)+\frac{e^{4t}}{37}$

Replacing t by ln x

$$\boxed{y=c_1x^3+x^{-2}[c_2\cos(\ln x)+c_3\sin(\ln x)]+\frac{x^4}{37}}$$

(8) $((x+1)^2D^2+(x+1)D+1)y=4[\cos\ln(x+1)]^2$

Let $x+1=e^t$

$\Rightarrow t=\ln(x+1)$

$(x+1)D=\Delta$

$(x+1)^2D^2=\Delta^2-\Delta$

Putting values

$(\Delta^2-\Delta+\Delta+1)y=4[\cos t]^2$

$\Delta^2+1=4\cos^2t$

$\Delta^2+1=2(1+\cos2t)$

Characteristic Eq is $\Delta^2+1=0$

$\Delta=\pm i$

$y_c=e^{0t}(c_1\cos t+c_2\sin t)$

$y_p=\frac{2(1+\cos2t)}{\Delta^2+1}$

$=\frac{2}{\Delta^2+1}+\frac{2\cos2t}{\Delta^2+1}$

$=(1+\Delta^2)^{-1}2+\frac{2\cos2t}{(-2^2)+1}$

$=2+\frac{2\cos2t}{-3}$

$y_p=2-\frac{2\cos2t}{3}$

$\boxed{y=c_1\cos t+c_2\sin t+2-\frac{2\cos2t}{3}}$

Replacing t by ln(x+1)

$$\boxed{y=c_1\cos(\ln(x+1))+c_2\sin(\ln(x+1))+2-\frac{2\cos2(\ln(x+1))}{3}}$$

$\boxed{10.4-7}$ (22)

(10) $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-6y=10x^2$ where $y(1)=1$, $y'(1)=-6$

$(x^2D^2+2xD-6)y=10x^2$ —— (I) Cauchy Euler Diff Eq

Put $x=e^t \Rightarrow t=\ln x$

$xD=\Delta$

$x^2D^2=\Delta(\Delta-1)$

$(\Delta(\Delta-1)+2\Delta-6)y=10e^{2t}$

$(\Delta^2-\Delta+2\Delta-6)y=10e^{2t}$

$(\Delta^2+\Delta-6)y=10e^{2t}$

$\Delta^2+\Delta-6=0$

$\Delta^2+3\Delta-2\Delta-6=0$

$\Delta(\Delta+3)-2(\Delta+3)=0$

$(\Delta+3)(\Delta-2)=0 \Rightarrow \boxed{\Delta=-3,+2}$

$y_c=c_1e^{2t}+c_2e^{-3t}$

$\boxed{y_c=c_1x^2+c_2\frac{1}{x^3}}$

$y_p=\frac{1}{\Delta^2+\Delta-6}10e^{2t}$ $\quad\because \frac{1}{2^2+2-6}=\infty$ Failure Case.

$=\frac{t}{2\Delta+1}10e^{2t}$

$y_p=\frac{10te^{2t}}{2(2)+1}=2te^{2t}=\boxed{2(\ln x)(x^2)}$

$y=y_c+y_p=c_1x^2+c_2\frac{1}{x^3}+2(\ln x)x^2$ —— (II)

$y'=2c_1x-3c_2x^{-4}+2\left[\frac{1}{x}\cdot x^2+\ln x(2x)\right]$

$y'=2c_1x-\frac{3c_2}{x^4}+2x+2\ln x(2x)$

$y'=2c_1x-\frac{3c_2}{x^4}+2x+4x\ln x$ —— (III) $\quad\because \ln 1=0$

$y(1)=1\Rightarrow$ from (II) $1=c_1+c_2+0$ —— (IV)

$y'(1)=-6\Rightarrow$ from (III) $-6=2c_1-3c_2+2+0$

$-8=2c_1-3c_2$ —— (V)

from (IV) & (V)

$2=2c_1+2c_2$

$-8=2c_1-3c_2$

$10=5c_2 \Rightarrow c_2=\frac{10}{5}=\boxed{2}$

Put c_2 in (IV) $1=c_1+2 \Rightarrow c_1=\boxed{-1}$

$\therefore$ (II) becomes $y=-1x^2+\frac{2}{x^3}+2\ln x(x^2)$ Ans.

$\boxed{\text{Ex } 10.4\text{-}8}$ 23

(11) $x^2y'' - 2xy' + 2y = x\ln x$; $y(1) = 1$, $y'(1) = 0$

$(x^2D^2 - 2xD + 2)y = x\ln x$

Let $x = e^t$ $\quad t = \ln x$

$xD = \Delta$

$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

Substituting we get.

$(\Delta^2 - \Delta - 2\Delta + 2)y = e^t \cdot t$

$(\Delta^2 - 3\Delta + 2)y = te^t \quad —(1)$

Characteristic Eq of (1) is

$\Delta^2 - 3\Delta + 2 = 0$

$\Delta^2 - \Delta - 2\Delta + 2 = 0$

$\Delta(\Delta - 1) - 2(\Delta - 1) = 0$

$(\Delta - 2)(\Delta - 1) = 0$

$\Delta = 1, 2.$

$y_c = c_1 e^t + c_2 e^{2t}$

$y_p = \dfrac{te^t}{\Delta^2 - 3\Delta + 2} = e^t \dfrac{t}{(\Delta+1)^2 - 3(\Delta+1) + 2}$ (Shift Theorem)

$= e^t \dfrac{t}{\Delta^2 + 2\Delta + 1 - 3\Delta - 3 + 2} = e^t \dfrac{t}{\Delta^2 - \Delta} = e^t \dfrac{t}{\Delta(\Delta - 1)}$

$= -e^t \dfrac{t}{\Delta(1 - \Delta)} = -e^t \dfrac{1}{\Delta}(1 - \Delta)^{-1} t$

$= -e^t \dfrac{1}{\Delta}(1 - (-1)\Delta)t = -e^t \dfrac{1}{\Delta}(1 + \Delta)t = -e^t \dfrac{1}{\Delta}(t + \Delta t)$

$= -e^t \int (t + 1)\, dt = -e^t\left(\dfrac{t^2}{2} + t\right) = -\dfrac{e^t}{2}(t^2 + 2t)$

General Sol is $y = c_1 e^t + c_2 e^{2t} - \dfrac{e^t}{2}(t^2 + 2t)$

Replace t by ln x $\quad y = c_1 e^{\ln x} + c_2 e^{2\ln x} - \dfrac{e^{\ln x}}{2}((\ln x)^2 + 2\ln x)$

$y = c_1 x + c_2 x^2 - \dfrac{1}{2} x[(\ln x)^2 + 2\ln x]$

$\ldots x\left[2\ln x \cdot \dfrac{1}{x} + 2 \cdot \dfrac{1}{x}\right)$ → see above *

* From below.

$y(1) = 1 \Rightarrow 1 = c_1 \cdot 1 + c_2 \cdot 1 - \frac{1}{2} \cdot 1(\ln 1 + 2\ln 1]$

$1 = c_1 + c_2 \qquad \because \ln 1 = 0$

$y'(1) = 0 \Rightarrow 0 = c_1 + 2c_2 - 1$

from $c_1 + c_2 = 1 \Rightarrow c_1 = 1 - c_2$

so $0 = 1 - c_2 + 2c_2 - 1 \Rightarrow \boxed{c_2 = 0}$

$\therefore 1 = c_1 + 0 \Rightarrow \boxed{c_1 = 1}$

Required Sol. $y = x - \frac{1}{2} x[(\ln x)^2 + 2\ln x]$

$= x - \frac{x}{2}(\ln x)^2 - x\ln x.$

Ans.

10.4 - 9 (24)

(12) $(x^3D^3 + 2x^2D^2 + xD - 1)y = 15\cos(2\ln x)$ $\quad y(1)=2$, $y'(1)=-3$, $y''(1)=0$

Let $x = e^t$ $\quad t = \ln x$.

$xD = \Delta$

$x^2D^2 = \Delta^2 - \Delta$

$x^3D^3 = \Delta^3 - 3\Delta^2 + 2\Delta$

Substituting we get.

$(\Delta^3 - 3\Delta^2 + 2\Delta + 2\Delta^2 - 2\Delta + \Delta - 1)y = 15\cos 2t$

$(\Delta^3 - \Delta^2 + \Delta - 1)y = 15\cos 2t$ ——— (1)

Characteristic Eq of (1)

$\Delta^3 - \Delta^2 + \Delta - 1 = 0$

$(\Delta - 1)(\Delta^2 + 1) = 0$

$$\begin{array}{c|cccc} & 1 & -1 & 1 & -1 \\ 1 & \downarrow & 1 & 0 & 1 \\ \hline & 1 & 0 & 1 & \boxed{0} \end{array}$$

$\Delta = 1, \pm i$ $\qquad \Delta^2 + 1 = 0$

$y_c = c_1 e^t + c_2\cos t + c_3\sin t$

$y_p = \dfrac{15\cos 2t}{\Delta^3 - \Delta^2 + \Delta - 1} = 15\dfrac{\cos 2t}{\Delta(\Delta^2) - \Delta^2 + \Delta - 1} = 15\dfrac{\cos 2t}{\Delta(-2^2) - (-2^2) + \Delta - 1} = 15\dfrac{\cos 2t}{-4\Delta + 4 + \Delta - 1}$

$= 15\dfrac{\cos 2t}{-3\Delta + 3} = \dfrac{15}{3}\dfrac{\cos 2t}{(-\Delta + 1)} = 5\dfrac{(1+\Delta)\cos 2t}{(1+\Delta)(1-\Delta)} = \dfrac{5(1+\Delta)\cos 2t}{1 - \Delta^2}$

$= \dfrac{5}{1-(-4)}(1+\Delta)\cos 2t = \dfrac{5}{5}(\cos 2t + 2(-\sin 2t)) = \cos 2t - 2\sin 2t.$

General Sol $\quad y = c_1 e^t + c_2\cos t + c_3\sin t + \cos 2t - 2\sin 2t$

Replace t by '$\ln x$' $\quad y = c_1 x + c_2\cos(\ln x) + c_3\sin(\ln x) + \cos 2(\ln x) - 2\sin(2\ln x)$

$y' = c_1 - c_2\dfrac{\sin(\ln x)}{x} + c_3\dfrac{\cos\ln x}{x} - 2\dfrac{\sin(2\ln x)}{x} - 4\dfrac{\cos(2\ln x)}{x}$

$y'' = -c_2\dfrac{\cos(\ln x)}{x^2} + c_2\dfrac{\sin\ln x}{x^2} - c_3\dfrac{\sin(\ln x)}{x^2} - c_3\dfrac{\cos\ln x}{x^2} - 4\dfrac{\cos(2\ln x)}{x^2}$

$\qquad + 2\dfrac{\sin(2\ln x)}{x^2} + 8\dfrac{\sin(2\ln x)}{x^2} + 4\dfrac{\cos(2\ln x)}{x^2}$

$y(1) = 2 \Rightarrow 2 = c_1 + c_2 + 1 \Rightarrow c_1 + c_2 = 1$ ——— (2)

$y'(1) = -3 \Rightarrow -3 = c_1 + c_3 - 4 \Rightarrow c_1 + c_3 = 1$ ——— (3)

$y''(1) = 0 \Rightarrow 0 = -c_2 - c_3 - 4 + 4 \Rightarrow c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$ ——— (4)

from (2) $\quad c_1 + (-c_3) = 1$

$\quad \& \quad c_1 + c_3 = 1$

$\quad 2c_1 = 2 \Rightarrow \boxed{c_1 = 1} \quad \therefore \boxed{c_2 = 0}$ $\quad \boxed{c_3 = 0}$

Hence $y = x + \cos 2(\ln x) - 2\sin(2\ln x)$

10-4-10

Eqs Reducible to Cauchy's form

$$a_0 (a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = \phi(a+bx)$$

such a diff eq is reducible to cauchy's form. In order to solve it We first reduce it to Cauchy's diff eq

Put $a+bx = z$

Diff w.r.t 'x'

$b = \frac{dz}{dx}$

$b = \frac{dz}{dy} \cdot \frac{dy}{dx}$

$$\boxed{b\frac{dy}{dz} = \frac{dy}{dx}}$$

Diff $\frac{d^2y}{dx^2} = b\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$

$= b\frac{d^2y}{dz^2}\cdot b \quad \because z=a+bx \Rightarrow \frac{dz}{dx}=b$

$$\boxed{\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dz^2}}$$

Similarly $\boxed{\frac{d^3y}{dx^3} = b^3\frac{d^3y}{dz^3}}$

So $\boxed{\frac{d^ny}{dx^n} = b^n\frac{d^ny}{dz^n}}$

So above eq becomes

$$a_0 z^n b^n \frac{d^n y}{dz^n} + a_1 z^{n-1} b^{n-1}\frac{d^{n-1}y}{dz^{n-1}} + \dots a_{n-1} z b\frac{dy}{dz} + a_n y = \phi(z)$$

which is Cauchy-Euler Eq

Note If we put $\boxed{a=0,\ b=1}$ in eq reducible to Cauchy it becomes Cauchy's Diff Eq.

Note 'b' is coeff of x in $(a+bx)$

(9) $(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

Put $2x+1 = z$

$\frac{dy}{dx} = 2\frac{dy}{dz}$

$\frac{d^2y}{dx^2} = 2^2\frac{d^2y}{dz^2}$

$\therefore z^2(2^2\frac{d^2y}{dz^2}) - 6z(2\frac{dy}{dz}) + 16y = 8z^2$

$4z^2\frac{d^2y}{dz^2} - 12z\frac{dy}{dz} + 16y = 8z^2$

$z^2\frac{d^2y}{dz^2} - 3z\frac{dy}{dz} + 4y = 2z^2$

$(z^2D^2 - 3zD + 4)y = 2z^2$ — Cauchy E.D.Eq (1)

$z = e^t \Rightarrow t = \ln z$

$zD = \Delta$

$z^2D^2 = \Delta(\Delta-1)$

$(\Delta(\Delta-1) - 3\Delta + 4)y = 2e^{2t}$

$(\Delta^2 - \Delta - 3\Delta + 4)y = 2e^{2t}$

$(\Delta^2 - 4\Delta + 4)y = 2e^{2t}$ — (11)

$\Delta^2 - 4\Delta + 4 = 0$

$(\Delta-2)^2 = 0 \Rightarrow \boxed{\Delta = 2,2}$

$\therefore y_c = (c_1 + c_2 t)e^{2t} \Rightarrow y_c = (c_1 + c_2\ln z)z^2$

$y_c = [c_1 + c_2\ln(2x+1)](2x+1)^2$

from (11)

$y_p = \frac{1}{\Delta^2 - 4\Delta + 4}(2e^{2t})$ $\quad \frac{1}{8-8} = \frac{1}{0}$ Failure case

$= \frac{t}{2\Delta - 4}\, 2e^{2t}$ $\quad \frac{t}{2(2)-4} = \infty$ Failure case

$= \frac{t^2}{2}\, 2e^{2t}$

$= t^2 e^{2t}$

$= z^2(\ln z)^2$

$y_p = (2x+1)^2[\ln(2x+1)]^2$

So $y = y_c + y_p$

$y = [c_1 + c_2\ln(2x+1)](2x+1)^2 + (2x+1)^2(\ln(2x+1))^2$

$= (2x+1)^2[c_1 + c_2\ln(2x+1) + \ln(2x+1) ...$

(1) Reduction of order. If one solution of the second order linear eq.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = 0 \quad \text{—— (1)}$$

(where P, q are functions of x or consts). is known. Then we can use it to find the general soln. of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = F(x) \quad \text{—— (2)}$$

This procedure is known as "method of reduction of order".

Suppose it is known that $y = y_1$ is a soln. of (1)

we assume that $y = vy_1$ —(3) is a soln. of (2), where v is some function of x.

from (3), we have

$$\frac{dy}{dx} = v\frac{dy_1}{dx} + y_1\frac{dv}{dx} \quad \text{—— (4)}$$

$$\frac{d^2y}{dx^2} = v\frac{d^2y_1}{dx^2} + \frac{dy_1}{dx}\frac{dv}{dx} + y_1\frac{d^2v}{dx^2} + \frac{dy_1}{dx}\frac{dv}{dx}$$

$$= v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2} \quad \text{—— (5)}$$

Put (3), (4), (5) in (2)

$$v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2} + Pv\frac{dy_1}{dx} + Py_1\frac{dv}{dx} + qvy_1 = F(x)$$

or

$$y_1\frac{d^2v}{dx^2} + \left(2\frac{dy_1}{dx} + Py_1\right)\frac{dv}{dx} + \left(\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + qy_1\right)v = F(x)$$

or

$$y_1\frac{d^2v}{dx^2} + \left(2\frac{dy_1}{dx} + Py_1\right)\frac{dv}{dx} = F(x) \qquad (\because y = y_1 \text{ is a soln. of (1)})$$

Put $\frac{dv}{dx} = u$

So $y_1\frac{du}{dx} + \left(2\frac{dy_1}{dx} + Py_1\right)u = F(x)$

It is a linear diff. eq. in u & can be

(2)

Solved for U

from $\frac{dv}{dx} = U$, we determine V & hence the Soln. $y_p = V y_1$. general soln. is $y = y_c + y_p$

Note: It is easy to see that

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + q\,y = 0$$

is satisfied by $y = e^x$ if $1+p+q=0$

& by $y = x$ if $p+qx = 0$

Exercise No: 10.5

Solve:

(1) $\frac{d^2y}{dx^2} + y = \sec^3 x$ — const coeffs So $y = y_c + y_p$

Sol. Given

$$\frac{d^2y}{dx^2} + y = \sec^3 x \quad \text{--- (1)}$$

Consider $\frac{d^2y}{dx^2} + y = 0$ — (2)

$(D^2+1)y = 0$ where $D = \frac{d}{dx}$

$F(D)y = 0$

where $F(D) = D^2+1$

Characteristic eq. is $F(m) = m^2+1 = 0$

$\Rightarrow m = \pm i$

So $\boxed{y_c = c_1\cos x + c_2\sin x}$

Put $c_1 = 1$ & $c_2 = 0$

So $y_1 = \cos x$ is a soln. of (2)

Suppose $y = Vy_1 = V\cos x$ is a soln. of (1)

10.5-3 $\Rightarrow \dfrac{dy}{dx} = -V\sin x + \dfrac{dv}{dx}\cos x$

$\dfrac{d^2y}{dx^2} = -V\cos x - \dfrac{dv}{dx}\sin x - \dfrac{dv}{dx}\sin x + \dfrac{d^2v}{dx^2}\cos x$

$= -V\cos x - 2\sin x\dfrac{dv}{dx} + \cos x\dfrac{d^2v}{dx^2}$

Put values in (1)

$-\cancel{V\cos x} - 2\sin x\dfrac{dv}{dx} + \cos x\dfrac{d^2v}{dx^2} + \cancel{V\cos x} = \sec^3 x$

$\cos x\dfrac{d^2v}{dx^2} - 2\sin x\dfrac{dv}{dx} = \sec^3 x$

or

$\dfrac{d^2v}{dx^2} - 2\tan x\dfrac{dv}{dx} = \sec^4 x$

Put $\dfrac{dv}{dx} = U$

So $\dfrac{du}{dx} + (-2\tan x)U = \sec^4 x$ ——— (3)

It is a linear diff. eq. in U

$\therefore$ I.F. $= e^{-2\int\tan x\,dx} = e^{-2\ln\sec x} = e^{\ln\sec^{-2}x} = \dfrac{1}{\sec^2 x} = \cos^2 x$

Multiplying both sides of (3) by I.F. $\cos^2 x$

$\therefore \int d(U\cos^2 x) = \int \sec^2 x\,dx$

$U\cos^2 x = \tan x$

(∵ No const. of integration since y_p does not contain const.)

Do 2nd Method

or $U = \tan x\sec^2 x$

$\dfrac{dv}{dx} = \tan x\sec^2 x$

$\Rightarrow$

$\int dv = \int \tan x\cdot\sec^2 x\,dx$

$V = \dfrac{\tan^2 x}{2}$

So $y_p = \dfrac{\tan^2 x}{2}\cos x$

Hence general soln. is

$y = y_c + y_p = C_1\cos x + C_2\sin x + \dfrac{1}{2}\tan^2 x\cos x$

2nd Method

$\therefore U\cos^2 x = \tan x$

$U = \dfrac{\sin x}{\cos x\cos^2 x}$

$\dfrac{dv}{dx} = \cos^{-3}x\sin x$

$V = -\int \cos^{-3}x(-\sin x)dx$

$= -\dfrac{\cos^{-2}x}{-2}$

$V = \dfrac{1}{2}\sec^2 x$

$\therefore y_p = V\cos x = \dfrac{1}{2}\sec^2 x\cos x$

$\boxed{y_p = \dfrac{1}{2}\sec x}$

$y = C_1\cos x + C_2\sin x + \dfrac{1}{2}\sec x$. Ans

$\boxed{10.5-4}$

(9)

(2) $\dfrac{d^2y}{dx^2} + 4y = 4\tan 2x$ (const coeffs so $y = y_c + y_p$)

Sol: Given

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x \quad \text{—— (1)}$$

For C.F.

Consider $\dfrac{d^2y}{dx^2} + 4y = 0$ —— (2)

$(D^2+4)y = 0$

$F(D)y = 0$

where $F(D) = D^2+4$

Characteristic eq. is

$F(m) = m^2+4 = 0$

$\Rightarrow m = \pm 2i$

So $\boxed{y_c = C_1\cos 2x + C_2\sin 2x}$

Put $C_1 = 1$ & $C_2 = 0$

So $y_1 = \cos 2x$ is also a soln of (2)

Suppose $y = vy_1 = v\cos 2x$ is a soln of (1)

$$\frac{dy}{dx} = -2v\sin 2x + \frac{dv}{dx}\cos 2x$$

$$\frac{d^2y}{dx^2} = -2\left[v(2\cos 2x) + \frac{dv}{dx}\sin 2x\right] + \frac{d^2v}{dx^2}\cos 2x + \frac{dv}{dx}(-2\sin 2x)$$

$$= -4v\cos 2x - 2\sin 2x\frac{dv}{dx} + \cos 2x\frac{d^2v}{dx^2} - 2\sin 2x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = -4v\cos 2x - 4\sin 2x\frac{dv}{dx} + \cos 2x\frac{d^2v}{dx^2}$$

Put values in (1)

$$\cancel{-4v\cos 2x} - 4\sin 2x\frac{dv}{dx} + \cos 2x\frac{d^2v}{dx^2} + \cancel{4v\cos 2x} = 4\tan 2x$$

$$\cos 2x\frac{d^2v}{dx^2} - 4\sin 2x\frac{dv}{dx} = 4\tan 2x$$

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$$\frac{d^2v}{dx^2} - 4\tan 2x \frac{dv}{dx} = \frac{4\tan 2x}{\cos 2x}$$

Put $\frac{dv}{dx} = U$

So $\frac{du}{dx} - (4\tan 2x) U = \frac{4\tan 2x}{\cos 2x}$ ——— (3)

It is a linear diff. eq. in U.

$$I.F. = e^{-4\int \tan 2x\, dx} = e^{-4\frac{\ln \sec 2x}{2}} = e^{-2\ln \sec 2x} = e^{\ln \sec^{-2} 2x} = \frac{1}{\sec^2 2x} = \cos^2 2x$$

$\therefore$ I.F. $= \cos^2 2x$

Multiplying both sides of eq (3) by I.F. $\cos^2 2x$

$$\int d(U\cos^2 2x) = \int \frac{4\tan 2x}{\cancel{\cos 2x}} \cdot \cos^{\cancel{2}} 2x\, dx$$

$$U\cos^2 2x = 4\int \sin 2x\, dx$$

$$= 4\left[-\frac{\cos 2x}{2}\right]$$

$$U\cos^2 2x = -2\cos 2x$$

or $U = -\frac{2}{\cos 2x}$

$$\frac{dv}{dx} = -2\sec 2x$$

$$\int dv = -2\int \sec 2x\, dx$$

$$V = -2\left[\frac{\ln(\sec 2x + \tan 2x)}{2}\right] = -\ln[\sec 2x + \tan 2x]$$

$$\text{or} \quad = -\ln\left[\frac{1+\sin 2x}{\cos 2x}\right] = -\ln\left[\frac{(\cos x + \sin x)^2}{\cos^2 x - \sin^2 x}\right]$$

$(\cos x + \sin x)^2 = \cos^2 x + \sin^2 x + \sin 2x = 1 + \sin 2x$

$$= -\ln\left[\frac{\cos x + \sin x}{\cos x - \sin x}\right]$$

So $V = -\ln\left[\frac{1+\tan x}{1-\tan x}\right] = -\ln\tan(\pi/4 + x)$

Hence $y_p = Vy_1 = -\ln\tan(\pi/4 + x)\cdot\cos 2x$

So general soln. is

$$y = y_c + y_p = C_1\cos 2x + C_2\sin 2x - \ln|\tan(\pi/4 + x)|\cos 2x$$

10.5-6

(6)

(i) $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$ (Legender eq. of order one) variable coeffts

Sol. Given

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \quad \text{——— (1)}$$

$\dfrac{d^2y}{dx^2} - \dfrac{2x}{1-x^2}\left(\dfrac{dy}{dx}\right) + \dfrac{2y}{1-x^2} = 0$

$P + Qx = 0$

$\dfrac{-2x}{1-x^2} + \left(\dfrac{2}{1-x^2}\right)x = 0$

$0 = 0$

Hence $y_c = x$

We note that $y = x$ is a soln. of (1)

So $\boxed{y_c = x}$ $\therefore y_1 = x$

Suppose $y = vy_1 = vx$ is also a soln. of (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

$$= 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Put in (1)

$$(1-x^2)\left[2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right] - 2x\left[v + x\frac{dv}{dx}\right] + 2vx = 0$$

or

$$2(1-x^2)\frac{dv}{dx} + x(1-x^2)\frac{d^2v}{dx^2} - \cancel{2xv} - 2x^2\frac{dv}{dx} + \cancel{2vx} = 0$$

$$x(1-x^2)\frac{d^2v}{dx^2} + 2(1-2x^2)\frac{dv}{dx} = 0$$

or

$$\frac{d^2v}{dx^2} + \frac{2(1-2x^2)}{x(1-x^2)}\frac{dv}{dx} = 0$$

Put $\dfrac{dv}{dx} = u$

So

$$\frac{du}{dx} + \frac{2(1-2x^2)}{x(1-x^2)}u = 0$$

or $$\frac{du}{dx} = \frac{2(2x^2-1)}{x(1-x^2)}u$$

or $$\int\frac{du}{u} = 2\int\frac{2x^2-1}{x(1-x)(1+x)}dx \quad \text{——— (A)}$$

10.5 Now $\frac{4x^2-2}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$ (7)

$\Rightarrow 4x^2-2 = A(1-x^2) + Bx(1+x) + Cx(1-x)$ —— I

Put $x = 0$ in I

$-2 = A \Rightarrow \boxed{A = -2}$

$1-x=0 \Rightarrow$ Put $x = 1$ in I

$2 = B(2) \Rightarrow \boxed{B = 1}$

$1+x=0 \Rightarrow$ Put $x = -1$ in I

$2 = C(-1)(+2)$

$\Rightarrow \boxed{C = -\frac{2}{2} = -1}$

So $\frac{4x^2-2}{x(1-x)(1+x)} = \frac{-2}{x} + \frac{1}{(1-x)} - \frac{1}{(1+x)}$

So eq. A becomes

$$\int \frac{du}{u} = \int \frac{-2dx}{x} - \int \frac{(-1)\,dx}{(1-x)} - \int \frac{1}{(1+x)} dx$$

$$\ln u = -2\ln x - \ln(1-x) - \ln(1+x) + \ln c_1$$

$$\ln u = \ln\left[\frac{c_1}{x^2(1-x)(1+x)}\right]$$

or $\ln u = \ln\left[\frac{c_1}{x^2(1-x^2)}\right]$

$$\Rightarrow u = \frac{c_1}{x^2(1-x^2)}$$

or

$$\frac{dv}{dx} = \frac{c_1}{x^2(1-x^2)}$$

$$\Rightarrow \int dv = \int \frac{c_1}{x^2(1-x^2)} dx$$

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or $V = \int \frac{C_1}{x^2(1-x^2)} dx$ —— (B)

Now

$$\frac{1}{x^2(1-x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{1+x}$$

$\Rightarrow 1 = Ax(1-x^2) + B(1-x^2) + Cx^2(1+x) + Dx^2(1-x)$ —— II

Put $x = 0$ $1 = B$ $\Rightarrow \boxed{B = 1}$

Put $x = 1$ $1 = C(2)$ $\Rightarrow \boxed{C = \frac{1}{2}}$

Put $x = -1$ $1 = D(2)$ $\Rightarrow \boxed{D = \frac{1}{2}}$

Comparing Coff of x^3 on both sides in II

$0 = -A + C - D$

or $0 = -A + \frac{1}{2} - \frac{1}{2}$ $\Rightarrow \boxed{A = 0}$

So eq. (B) is

$$V = C_1 \int \frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)} dx$$

$$= C_1\left[\frac{-1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)\right] + C_2$$

$$V = C_1\left[-\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + C_2$$

So

$$y = Vx = \left\{C_1\left[\frac{-1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + C_2\right\}x$$

So gen. soln. is

$$y = C_1\left[\frac{-x}{x} + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + C_2 x$$

or $y = C_1\left[-1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + C_2 x$

10.5-9 (4) $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 1$ Variable coeffs $\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{y}{x-1} = \frac{1}{x-1}$ (9)

Sol. Given

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 1 \quad \text{——— (1)}$$

$P + Qx = 0$

$\frac{-x}{x-1} + \frac{1 \cdot x}{x-1} = 0$

$0 = 0$

Hence $y = x$ is sol of (2)

For C.F.

Consider $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ ——— (2)

We note that $y = x$ is a soln of (2) $\therefore \boxed{y_1 = x}$

Suppose that $y = vy_1 = vx$ is a soln of (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{dv}{dx} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Put values in (1)

$$(x-1)\left[2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right] - x\left[v + x\frac{dv}{dx}\right] + vx = 1$$

$$2(x-1)\frac{dv}{dx} + x(x-1)\frac{d^2v}{dx^2} - \cancel{xv} - x^2\frac{dv}{dx} + \cancel{vx} = 1$$

or $x(x-1)\frac{d^2v}{dx^2} + (2x-2-x^2)\frac{dv}{dx} = 1$

or $\frac{d^2v}{dx^2} - \frac{x^2-2x+2}{x(x-1)}\frac{dv}{dx} = \frac{1}{x(x-1)}$

Put $\frac{dv}{dx} = u$

$\frac{x-2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$x-2 = A(x-1) + Bx$

$x=0 \Rightarrow -2 = -A$

$\boxed{2 = A}$

$x-1=0 \Rightarrow x=1$

$\Rightarrow 1-2 = 0+B$

$\boxed{-1 = B}$

$\therefore \frac{x-2}{x(x-1)} = \frac{2}{x} - \frac{1}{x-1}$

So

$$\frac{du}{dx} - \frac{x^2-2x+2}{x(x-1)}u = \frac{1}{x(x-1)} \quad \text{——— (3)}$$

It is a linear diff eq in u

$$\text{I.F.} = e^{-\int \frac{x^2-2x+2}{x^2-x}dx} = e^{-\int \frac{(x^2-x)+(-x+2)}{x^2-x}dx} = e^{-\int \left(1 - \frac{x-2}{x^2-x}\right)dx}$$

$$= e^{\int \left(\frac{x-2}{x^2-x} - 1\right)dx} = e^{\int \frac{x-2}{x(x-1)}dx} \cdot e^{-x} = e^{\int\left(\frac{2}{x} - \frac{1}{x-1}\right)dx} \cdot e^{-x}$$

See above detail

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10-5-11

(5) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 8x^3$ variable coeffts

Sol. Given

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 8x^3 \quad \text{--- (1)}$$

$P + Qx = 0$

$\frac{x}{x^2} + \frac{(-1)x}{x^2} = 0$

$0 = 0$

$\Rightarrow y = x$ is a sol.

For C.F.

Consider $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

$$(x^2D^2 + xD - 1)y = 0 \quad \text{--- (2)} \quad \text{where } D = \frac{d}{dx}$$

Put $x = e^t \Rightarrow t = \ln x$

Then $xD = \Delta$

$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

So eq. (2) becomes

$(\Delta^2 - \Delta + \Delta - 1)y = 0$

$(\Delta^2 - 1)y = 0$

$F(\Delta)y = 0$

where $F(\Delta) = \Delta^2 - 1$

Characteristic eq. is

$F(m) = m^2 - 1 = 0 \Rightarrow m = \pm 1.$

So

$y_c = c_1e^{t} + c_2e^{-t} = c_1x + \frac{c_2}{x}$

or $\boxed{y_c = c_1x + \frac{c_2}{x}}$

Put $c_1 = 1$ & $c_2 = 0$

No Need.

So $y = x$ is a soln of (2)

Suppose $y = vx$ is a soln. of (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

10.5-12

(12)

Put in (1)

$$x^2\left[2\frac{dv}{dx}+x\frac{d^2v}{dx^2}\right]+x\left[v+x\frac{dv}{dx}\right]-vx=8x^3$$

$$2x^2\frac{dv}{dx}+x^3\frac{d^2v}{dx^2}+\cancel{vx}+x^2\frac{dv}{dx}-\cancel{vx}=8x^3$$

$$x^3\frac{d^2v}{dx^2}+3x^2\frac{dv}{dx}=8x^3$$

$$\frac{d^2v}{dx^2}+\frac{3}{x}\frac{dv}{dx}=8$$

Put $\frac{dv}{dx}=U$

So $\frac{du}{dx}+\frac{3}{x}U=8$

It is a linear diff eq

$I.F=e^{3\int\frac{1}{x}dx}=e^{3\ln x}=e^{\ln x^3}=x^3$

Multiplying both sides by I.F. x^3

$$\int d(Ux^3)=\int 8x^3\,dx$$

$$Ux^3=\frac{8}{4}x^4$$

or

$$U=\frac{8}{4}x$$

or $U=2x$

$$\frac{dv}{dx}=2x$$

$$\int dv=\int 2x\,dx$$

$$V=x^2$$

So $y_p=vx=x^2(x)=x^3$

Note If using $P+Qx=0$, $y=x$ is sol direct

then $\int d(Ux^3)=\int 8x^3\,dx$

$Ux^3=2x^4+C$ ← Note

$U=2x+Cx^{-3}$

$\frac{dv}{dx}=2x+Cx^{-3}$

$\int dv=\int(2x+Cx^{-3})dx$

$V=x^2+\frac{Cx^{-2}}{-2}+D$

$\therefore Y=Vx$

$=\left[x^2-\frac{1}{2}Cx^{-2}+D\right]x$

$Y=x^3-\frac{1}{2}Cx^{-1}+Dx$ G.sol.

So general soln is $y=y_c+y_p=C_1x+\frac{C_2}{x}+x^3$ ✓

10.5-13

(13)

(6) $x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3e^x$ variable coeffts

Soln. Given

$$x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3e^x \quad \text{--- (1)}$$

$P+Qx=0$

$-\frac{(x^2+2x)}{x^2}+\frac{(x+2)}{x^2}x=0$

$-(x^2+2x)+x^2+2x=0$

$-\cancel{x^2}-\cancel{2x}+\cancel{x^2}+\cancel{2x}=0$

$0=0$

For C.F.

Consider $x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = 0$ --- (2)

We note that $y = x$ is a soln of (2)

Suppose $y = vx$ is a soln of (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} + 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Put in (1)

$$x^2\left[2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right] - (x^2+2x)\left[v + x\frac{dv}{dx}\right] + (x+2)vx = x^3e^x$$

$$\cancel{2x^2\frac{dv}{dx}} + x^3\frac{d^2v}{dx^2} - \cancel{x^2v} - \cancel{2xv} - x^3\frac{dv}{dx} - \cancel{2x^2\frac{dv}{dx}} + \cancel{vx^2} + \cancel{2vx} = x^3e^x$$

$$x^3\frac{d^2v}{dx^2} - x^3\frac{dv}{dx} = x^3e^x$$

or

$$\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x$$

Put $U = \frac{dv}{dx}$

So $\frac{dU}{dx} - U = e^x$ --- (3)

It is a linear eq in U

$$I.F. = e^{-\int 1dx} = e^{-x}$$

Multiplying both sides of (3) by I.F. e^{-x}

$$\int d(Ue^{-x}) = \int 1\,dx$$

$$Ue^{-x} = x + c$$

10-5-14

(4)

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or $U = xe^x + ce^x$

$$\frac{dv}{dx} = xe^x + ce^x$$

$$\int dv = \int xe^x\,dx + c\int e^x\,dx$$

$$V = xe^x - \int e^x \cdot 1\,dx + ce^x$$

$$= xe^x - e^x + ce^x + c_2$$

$$V = xe^x + (c-1)e^x + c_2$$

So

$y_p = Vx = x^2e^x + c_1xe^x + c_2x$ is req. gen. soln.

~~$(x\sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = 0$~~

~~Sol:- Given~~

~~$$(x\sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = 0 \quad (1)$$~~

~~We note that $y = x$ is a soln. of (1)~~

~~Also $y = \cos x$ is another soln. of (1)~~

~~Hence general soln. is~~

~~$y = c_1x + c_2\cos x$~~

---

(8) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{1}{(1+e^x)^2}$ const coeffs.

Sol:- Given

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{1}{(1+e^x)^2} \quad (1)$$

For C.F.

Consider $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \quad (2)$

$$(D^2 + 2D + 1)y = 0$$

$$F(D)y = 0$$

1003

10-5-15 where $F(D) = (D+1)^2$

Characteristic eq. is

$$F(m) = (m+1)^2 = 0 \Rightarrow \boxed{m = -1, -1}$$

So $\boxed{y_c = (C_1 + C_2 x)e^{-x}}$

Put $c_1 = 1$ & $c_2 = 0$

So $y = e^{-x}$ is a soln. of ②

Suppose $y = ve^{-x}$ is a soln. of ①

$$\frac{dy}{dx} = v(-e^{-x}) + \frac{dv}{dx}(e^{-x})$$

$$\frac{dy}{dx} = e^{-x}\left[\frac{dv}{dx} - v\right]$$

$$\frac{d^2y}{dx^2} = e^{-x}\left[\frac{d^2v}{dx^2} - \frac{dv}{dx}\right] + (-e^{-x})\left[\frac{dv}{dx} - v\right]$$

$$= e^{-x}\left[\frac{d^2v}{dx^2} - \frac{dv}{dx} - \frac{dv}{dx} + v\right] = e^{-x}\left[\frac{d^2v}{dx^2} - 2\frac{dv}{dx} + v\right]$$

Put in ①

$$e^{-x}\left[\frac{d^2v}{dx^2} - 2\frac{dv}{dx} + v\right] + 2e^{-x}\left[\frac{dv}{dx} - v\right] + ve^{-x} = \frac{1}{(1+e^x)^2}$$

$$e^{-x}\frac{d^2v}{dx^2} - \cancel{2e^{-x}\frac{dv}{dx}} + \cancel{ve^{-x}} + \cancel{2e^{-x}\frac{dv}{dx}} - \cancel{2ve^{-x}} + \cancel{ve^{-x}} = \frac{1}{(1+e^x)^2}$$

$$\frac{d^2v}{dx^2} = \frac{e^x}{(1+e^x)^2}$$

$$\int \frac{d^2v}{dx^2}dx = \int (1+e^x)^{-2} e^x \, dx$$

$$\frac{dv}{dx} = \frac{(1+e^x)^{-1}}{-1}$$

$$\frac{dv}{dx} = \frac{-1}{1+e^x}$$

10-5-16

(16)

100%

or $\int dv = -\int (1+e^{x})^{-1} dx$

$$V = -\int [e^{x}(e^{-x}+1)]^{-1} dx$$

$$= -\int (1+e^{-x})^{-1} e^{-x} dx$$

$$= \int \frac{-e^{-x}}{(1+e^{-x})} dx$$

$$V = \ln(1+e^{-x})$$

$$y_p = Ve^{-x} = \ln(1+e^{-x}).e^{-x}$$

So general soln. is

$$y = y_c + y_p$$

$$= (C_1+C_2x)e^{-x} + e^{-x}\ln(1+e^{-x})$$

$$y = e^{-x}[C_1+C_2x+\ln(1+e^{-x})]$$

(10) $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 3y = 2\sec x$

Sol. Given

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 3y = 2\sec x \quad \text{(1)}$$

F.S.C.F.

Consider $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 3y = 0$ —— (2)

We note that $y = \sin x$ is a soln of (2) (given)

Suppose $y = V\sin x$ is a soln. of (1)

$$\frac{dy}{dx} = V\cos x + \sin x \frac{dV}{dx}$$

$$\frac{d^2y}{dx^2} = \cancel{-V\sin x + \frac{dv}{dx}\cos x + \sin x\frac{d^2V}{dx^2}} + \cos x\frac{dv}{dx}$$

Put in (1)

$$\left[-V\sin x + 2\frac{dv}{dx}\cos x + \sin x\frac{d^2v}{dx^2}\right] - 2\tan x\left[V\cos x + \sin x\frac{dv}{dx}\right] + 3V\sin x = 2\sec x$$

$$-V\sin x + 2\frac{dv}{dx}\cos x + \sin x\frac{d^2v}{dx^2} - 2\tan x V\cos x - 2\tan\sin x\frac{dv}{dx} + 3V\sin x = 2\sec x$$

$$\sin x\frac{d^2v}{dx^2} + 2\left(\cos x - \frac{\sin^2 x}{\cos x}\right)\frac{dv}{dx} - \cancel{2V\sin x} + \cancel{2V\sin x} = 2\sec x$$

or $$\frac{d^2v}{dx^2} + 2\left[\frac{\cos^2 x - \sin^2 x}{\sin x\cos x}\right]\frac{dv}{dx} = \frac{2}{\sin x\cos x}$$

$$\frac{d^2v}{dx^2} + 4\frac{\cos 2x}{\sin 2x}\frac{dv}{dx} = \frac{4}{\sin 2x}$$

Put $\frac{dv}{dx} = U$

So

$$\frac{du}{dx} + 4\frac{\cos 2x}{\sin 2x}U = \frac{4}{\sin 2x} \quad \text{(3)}$$

It is a linear diff. eq. in U.

$$I.F. = e^{2\int\frac{2\cos 2x}{\sin 2x}dx} = e^{2\ln\sin 2x} = e^{\ln(\sin 2x)^2} = \sin^2 2x$$

Multiplying both sides of eq. (3) by I.F. $\sin^2 2x$

$$\int d(U\sin^2 2x) = 4\int\sin 2x\,dx$$

$$U\sin^2 2x = 4\left[-\frac{\cos 2x}{2}\right] + C_1$$

or

$$U = \frac{-2\cos 2x}{\sin^2 2x} + C_1\sin^{-2}2x$$

$$\frac{dv}{dx} = \frac{2\cos 2x}{\sin^2 2x} + C_1\sin^{-2}2x$$

$\boxed{10.5-18}$

(18)

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$\cancel{\int dV = \int \sin^{-2}2x\,(2\cos 2x)\,dx + C_1\int \operatorname{cosec}^2 2x\,dx}$

$$V = -\frac{(\sin 2x)^{-1}}{-1} - \frac{C_1}{2}\cot 2x + C_2$$

or $$V = \frac{1}{\sin 2x} - \frac{C_1}{2}\cot 2x + C_2$$

So general soln. is

$$y = V\sin x$$

$$= \left[\frac{1}{\sin 2x} - \frac{C_1}{2}\cot 2x + C_2\right]\sin x$$

$$= \frac{\cancel{\sin x}}{2\cancel{\sin x}\cos x} - \frac{C_1}{2}\,\frac{\cos 2x}{2\sin x\cos x}(\sin x) + C_2\sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{4}\,\frac{\cos 2x}{\cos x} + C_2\sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{4}\left[(2\cos^2 x - 1)\sec x\right] + C_2\sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{4}\left(2\cos x - \sec x\right) + C_2\sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{2}\cos x + \frac{C_1}{4}\sec x + C_2\sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{2}\left[\cos x - \frac{1}{2}\sec x\right] + C_2\sin x$$

$$y = \frac{1}{2}\sec x + C_1'\left(\cos x - \frac{1}{2}\sec x\right) + C_2\sin x$$

where $C_1' = -\frac{C_1}{2}$

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10.5.19

$$x\frac{d^2y}{dx^2}-(2x+1)\frac{dy}{dx}+(x+1)y$$

Sol. Given

$$x\frac{d^2y}{dx^2}-(2x+1)\frac{dy}{dx}+(x+1)y=(x^2+x-1)e^x \quad \text{---}(1)$$

For C.F

Consider $x\frac{d^2y}{dx^2}-(2x+1)\frac{dy}{dx}+(x+1)y=0 \quad \text{---}(2)$

We note that $y=e^x$ is a soln. of (2)

$\because P+Q x\neq 0$
where $P=\frac{-(2x+1)}{x}$ & $Q=\frac{x+1}{x}$
$\therefore 1+P+Q=1-\frac{(2x+1)}{x}+\frac{x+1}{x}$
$=\frac{\cancel{x}-2x-1+x+1}{x}$
$=0$
$\therefore y=e^x$ is sol of (2)

Suppose that $y=ve^x$ is a soln. of (1)

$$\frac{dy}{dx}=ve^x+\frac{dv}{dx}e^x$$

$$\frac{d^2y}{dx^2}=ve^x+\frac{dv}{dx}e^x+\frac{dv}{dx}e^x+\frac{d^2v}{dx^2}e^x$$

$$\frac{d^2y}{dx^2}=ve^x+2\frac{dv}{dx}e^x+\frac{d^2v}{dx^2}e^x$$

Put in (1)

$$x\left[ve^x+2\frac{dv}{dx}e^x+\frac{d^2v}{dx^2}e^x\right]-(2x+1)\left[ve^x+\frac{dv}{dx}e^x\right]+(x+1)ve^x=(x^2+x-1)e^x$$

$$vxe^x+2x\cancel{\frac{dv}{dx}}e^x+xe^x\frac{d^2v}{dx^2}-2vxe^x-2xe^x\cancel{\frac{dv}{dx}}-\cancel{v}e^x-\frac{dv}{dx}e^x+vxe^x+\cancel{v}e^x=(x^2+x-1)e^x$$

$$v\cancel{x}+x\frac{d^2v}{dx^2}-2\cancel{v}x-\frac{dv}{dx}+v\cancel{x}=(x^2+x-1)$$

or $x\frac{d^2v}{dx^2}-\frac{dv}{dx}=x^2+x-1$

or

$$\frac{d^2v}{dx^2}-\frac{1}{x}\frac{dv}{dx}=x+1-\frac{1}{x}$$

Put $\frac{dv}{dx}=U$

So $\frac{du}{dx}-\frac{1}{x}U=x-\frac{1}{x}+1 \quad \text{---}(3)$

It is a linear diff eq

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(20) I.F. $= e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$

Multiplying both sides of ③ by I.F. $\frac{1}{x}$

$$\int d\left(U\cdot\frac{1}{x}\right) = \int\left(1-\frac{1}{x^2}+\frac{1}{x}\right)dx$$

$$\frac{U}{x} = x+\frac{1}{x}+\ln x + c_1$$

or $U = x^2+1+x\ln x + c_1 x$

$$\frac{dv}{dx} = x^2+1+x\ln x + c_1 x$$

$$\int dV = \int x^2+1+x\ln x + c_1 x\, dx$$

$$V = \frac{x^3}{3}+x+\ln x\cdot\frac{x^2}{2}-\int\frac{x^2}{2}\cdot\frac{1}{x}dx + c_1\frac{x^2}{2}+C_2$$

$$= \frac{x^3}{3}+x+\frac{x^2\ln x}{2}-\left[\frac{x^2}{4}\right]+C_1\frac{x^2}{2}+C_2$$

or

$$V = \frac{x^3}{3}+\frac{x^2\ln x}{2}-\frac{x^2}{4}+\frac{x^2}{2}C_1+C_2$$

So general Soln. is

$$y = Ve^x$$

$$= e^x\left[\frac{x^3}{3}+\frac{x^2\ln x}{2}-\frac{x^2}{4}+\frac{C_1x^2}{2}+C_2\right]$$

---

(9) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y = (1+x+x^2+\dots+x^{25})e^{2x}$

Sol. Given

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y = (1+x+x^2+\dots+x^{25})e^{2x} \quad \text{—①}$$

For C.F.

Consider $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y = 0$ —②

$$(D^2-4D+4)y = 0$$

or $F(D)y = 0$

10.5.21 where $F(D) = D^2 - 4D + 4$

Characteristic eq. is

$$F(m) = m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0 \Rightarrow \boxed{m = 2, 2}$$

So $\boxed{y_c = (C_1 + C_2 x) e^{2x}}$

Put $C_1 = 1$ & $C_2 = 0$

So $y = e^{2x}$ is a soln of ②

Suppose that $y = V e^{2x}$ is a soln of ①

$$\frac{dy}{dx} = V(2e^{2x}) + \frac{dv}{dx} e^{2x}$$

$$= e^{2x}\left[2V + \frac{dv}{dx}\right]$$

$$\frac{d^2y}{dx^2} = e^{2x}\left[2\frac{dv}{dx} + \frac{d^2v}{dx^2}\right] + 2e^{2x}\left[2V + \frac{dv}{dx}\right]$$

$$= e^{2x}\left[2\frac{dv}{dx} + \frac{d^2v}{dx^2} + 4V + 2\frac{dv}{dx}\right]$$

or

$$\frac{d^2y}{dx^2} = e^{2x}\left[\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4V\right]$$

Put Values in ①

$$e^{2x}\left[\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4V\right] - e^{2x}4\left[2V + \frac{dv}{dx}\right] + 4Ve^{2x} = (1 + x + x^2 + x^{25})e^{2x}$$

$$\frac{d^2v}{dx^2} + \cancel{4\frac{dv}{dx}} + \cancel{4V} - \cancel{8V} - \cancel{4\frac{dv}{dx}} + \cancel{4V} = (1 + x + \cdots + x^{25})$$

$$\frac{d^2v}{dx^2} = (1 + x + \cdots + x^{25})$$

Integ. w.r.t. x

$$\int \frac{d^2v}{dx^2}\, dx = \int (1 + x + x^2 + \cdots + x^{25})\, dx$$

10-5-22

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$$\frac{dy}{dx} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{26}}{26}$$

$$V = \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots + \frac{x^{27}}{26\cdot 27}$$

So

$$y_p = Ve^{2x}$$

$$\therefore y_p = \left[\frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots + \frac{x^{27}}{26\cdot 27}\right]e^{2x}$$

So general soln. is

$$y = y_c + y_p$$

$$= (C_1 + C_2 x)e^{2x} + \left[\frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots + \frac{x^{27}}{26\cdot 27}\right]e^{2x}$$

or $y = e^{2x}\left[C_1 + C_2 x + \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots + \frac{x^{27}}{26\cdot 27}\right]$

Solved examples

Example (1) $\frac{d^2y}{dx^2} + y = \operatorname{Cosec}x$

Sol:- Given

$$\frac{d^2y}{dx^2} + y = \operatorname{Cosec}x \quad \text{——— (1)}$$

For C.F.

Consider $\frac{d^2y}{dx^2} + y = 0$ ——— (2)

$$(D^2+1)y = 0$$

$$F(D)y = 0$$

where $F(D) = D^2+1$

Characteristic eq. is

$$F(m) = m^2+1 = 0$$

$$\Rightarrow \boxed{m = \pm i}$$

So $\boxed{y_c = C_1\cos x + C_2\sin x}$

Put $C_1 = 1$ & $C_2 = 0$

In books, we used $C_1 = 0$ & $C_2 = 1$ then question will become more easy than this method

So $y = \cos x$ is a Soln of ②

$\boxed{\text{Suppose}}$ that $\boxed{y = V\cos x}$ is a Soln. of ①

$$\frac{dy}{dx} = -V\sin x + \cos x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = -V\cos x - \sin x\frac{dv}{dx} - \sin x\frac{dv}{dx} + \cos x\frac{d^2V}{dx^2}$$

$$\frac{d^2y}{dx^2} = -V\cos x - 2\sin x\frac{dv}{dx} + \cos x\frac{d^2v}{dx^2}$$

Put values in ①

$$-V\cancel{\cos x} - 2\sin x\frac{dv}{dx} + \cos x\frac{d^2v}{dx^2} + V\cancel{\cos x} = \operatorname{Cosec} x$$

$$\cos x\frac{d^2v}{dx^2} - 2\sin x\frac{dv}{dx} = \operatorname{Cosec} x$$

Put $\frac{dv}{dx} = U$

So $\frac{dU}{dx} - 2\tan x\, U = \frac{1}{\sin x\cos x}$ ——— ③

It is a linear diff eq in U

$$I.F. = e^{-2\int\tan x\,dx} = e^{-2\ln\sec x} = e^{\ln\sec^{-2}x} = \frac{1}{\sec^2 x} = \cos^2 x.$$

Multiplying both sides of ③ by $I.F. = \cos^2 x$

$$\int d(U\cos^2 x) = \int \cot x\, dx$$

$$U\cos^2 x = \ln\sin x$$

$$U = \ln\sin x\,.\sec^2 x$$

$$\frac{dv}{dx} = \ln\sin x\,.\sec^2 x$$

$$\int dv = \int \ln\sin x\,.\sec^2 x\,.dx$$

$$= \ln\sin x\,.\tan x - \int \tan x\,.\frac{\cos x}{\sin x}\,dx$$

$$V = \tan x\,.\ln\sin x - x$$

10.5-24

(22)

So $y_p = V\cos x$

$= \left[\tan x \ln \sin x - x\right]\cos x$

$y_p = \sin x \ln \sin x - x\cos x$

So general soln is

$y = y_c + y_p$

$= C_1\cos x + C_2 \sin x - x\cos x + \sin x \ln|\sin x|$

Example (18) $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$

Sol:- Given

$(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$ ——— (1)

For C.F.

Consider $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = 0$ ——— (2)

We note that $y = e^{2x}$ is a soln of (2)

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$Y = e^{ax}$ is a sol of

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + QY = 0$

if $a^2 + Pa + Q = 0$

$2^2 + \left(\frac{-2x-5}{x+2}\right)2 + \frac{2}{x+2} = 0$

$\frac{4(x+2) - 4x - 10 + 2}{x+2} = 0$

$4x + 8 - 4x - 10 + 2 = 0$

$0 = 0$

Suppose $y = Ve^{2x}$ is a soln of (1)

$\frac{dy}{dx} = V(2e^{2x}) + \frac{dv}{dx}e^{2x}$

$= e^{2x}\left[2V + \frac{dv}{dx}\right]$

$\frac{d^2y}{dx^2} = e^{2x}\left[2\frac{dv}{dx} + \frac{d^2v}{dx^2}\right] + (2e^{2x})\left[2V + \frac{dv}{dx}\right]$

$= e^{2x}\left[2\frac{dv}{dx} + \frac{d^2v}{dx^2} + 4V + 2\frac{dv}{dx}\right]$

$\frac{d^2y}{dx^2} = e^{2x}\left[\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4V\right]$

Put values in (1)

$(x+2)e^{2x}\left[\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4V\right] - (2x+5)e^{2x}\left[2V + \frac{dv}{dx}\right] + 2Ve^{2x} = (x+1)e^x$

10.5-25

$$(x+2)\left[\frac{d^2v}{dx^2}+4\frac{dv}{dx}+4v\right]-(2x+5)\left[2v+\frac{dv}{dx}\right]+2v=(x+1)e^{-x}$$

$$(x+2)\frac{d^2v}{dx^2}+\left[4(x+2)-(2x+5)\right]\frac{dv}{dx}+4v(x+2)-2v(2x+5)+2v=(x+1)e^{-x}$$

$$(x+2)\frac{d^2v}{dx^2}+(2x+3)\frac{dv}{dx}+(\cancel{4x}+\cancel{8}-\cancel{4x}-\cancel{16}+\cancel{2})v=(x+1)e^{-x}$$

$$(x+2)\frac{d^2v}{dx^2}+(2x+3)\frac{dv}{dx}=(x+1)e^{-x}$$

or

$$\frac{d^2v}{dx^2}+\left(\frac{2x+3}{x+2}\right)\frac{dv}{dx}=\frac{(x+1)e^{-x}}{x+2}$$

Put $\frac{dv}{dx}=U$

So

$$\frac{du}{dx}+\left(\frac{2x+3}{x+2}\right)U=\frac{(x+1)e^{-x}}{x+2} \quad \text{③}$$

It is a linear diff eq in U

$$\text{I.F.}=\exp\int\frac{2x+3}{x+2}dx=\exp\int 2-\frac{1}{(x+2)}dx=\exp\left[2x-\ln(x+2)\right]$$

$$=e^{2x}\cdot e^{-\ln(x+2)}=e^{2x}\cdot e^{\ln(x+2)^{-1}}=\frac{e^{2x}}{x+2}$$

Multiplying both sides of ③ by I.F. $\frac{e^{2x}}{x+2}$

$$\int d\left(U\cdot\frac{e^{2x}}{x+2}\right)=\int\frac{x+1}{(x+2)^2}e^{x}dx$$

$$\frac{Ue^{2x}}{x+2}=\int\frac{(x+2)-1}{(x+2)^2}e^{x}dx$$

$$=\int\frac{e^x}{x+2}dx-\int\frac{e^x}{(x+2)^2}dx$$

$$=\frac{1}{(x+2)}e^x-\int e^x\cdot\cancel{\frac{-1}{(x+2)^2}}dx-\int\cancel{\frac{e^x}{(x+2)^2}}dx$$

$$U\frac{e^{2x}}{x+2}=\frac{e^x}{x+2}+C_1$$

10.5-26

(26)

1014

$$V = e^{-x} + c_1(x+2)e^{-2x}$$

$$\frac{dv}{dx} = e^{-x} + c_1(x+2)e^{-2x}$$

$$\int dv = \int e^{-x}dx + c_1\int e^{-2x}(x+2)\,dx$$

$$V = -e^{-x} + c_1\left\{(x+2)\cdot\frac{e^{-2x}}{-2} - \int\frac{e^{-2x}}{-2}\cdot 1\,dx\right\} + c_2$$

$$= -e^{-x} - \frac{c_1(x+2)}{2}e^{-2x} + \frac{c_1}{2}\left[\frac{e^{-2x}}{-2}\right] + c_2$$

$$V = -e^{-x} - \frac{c_1(x+2)}{2}e^{-2x} - \frac{c_1}{4}\left[e^{-2x}\right] + c_2$$

$$= -e^{-x} - c_1e^{-2x}\left[\frac{x+2}{2} + \frac{1}{4}\right] + c_2$$

$$= -e^{-x} - c_1e^{-2x}\left[\frac{2x+4+1}{4}\right] + c_2$$

$$V = -e^{-x} - \frac{1}{4}c_1(2x+5)e^{-2x} + c_2$$

So general soln. is

$$y = Ve^{2x} = -e^{x} - \frac{1}{4}c_1(2x+5) + c_2e^{+2x}$$

☆ Method of Variation of Parameters ☆

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + Qy = F(x) \quad -(1)$$

Steps

1. $y_c = c_1y_1 + c_2y_2$
2. $y_p = u_1y_1 + u_2y_2$
3. $y_1 = ?$ $\quad y_2 = ?$
4. $F(x) = $ RHS of (1)
5. $W = y_1y_2' - y_1'y_2$
6. $u_1 = \int\frac{-y_2F(x)}{W}dx$
7. $u_2 = \int\frac{y_1F(x)}{}$

$\boxed{10\cdot6-1}$

Ex 10.6

Q1 $\dfrac{d^2y}{dx^2}+4y=\sec 2x$ —— ①

$(D^2+4)y=\sec 2x$

$D^2+4=0$

$D^2=-4$

$D^2=(i)^2 4$

$D=\pm 2i$

$\therefore y_c=c_1\cos 2x+c_2\sin 2x$ —— ②

Replace c_1 by u_1 & c_2 by u_2 in y_c we get

Assumed $y_p=u_1\cos 2x+u_2\sin 2x$ —— ③

from ② $y_1=\cos 2x$ & $y_2=\sin 2x$

$y_1'=-2\sin 2x$ & $y_2'=2\cos 2x$

$W=y_1y_2'-y_2y_1'$

$=(\cos 2x)(2\cos 2x)-(\sin 2x)(-2\sin 2x)$

$=2\cos^2 2x+2\sin^2 2x$

$W=2(\cos^2 2x+\sin^2 2x)=2(1)=2$

$F(x)=\sec 2x$

$u_1=\int -\dfrac{y_2F(x)}{W}dx$

$=\int -\dfrac{\sin 2x\ \sec 2x}{2}dx=\int -\dfrac{1}{2}\dfrac{\sin 2x}{\cos 2x}dx$

$u_1=\frac{+1}{4}\int \dfrac{-2\sin 2x}{\cos 2x}dx=\dfrac{1}{4}\ln|\cos 2x|$

$u_2=\int \dfrac{y_1F(x)}{W}dx$

$u_2=\int \dfrac{\cos 2x\ \sec 2x}{2}dx=\frac{1}{2}\int dx=\frac{1}{2}x$

Put values of u_1 & u_2 in ③ i.e Assumed y_p

we get $y_p=\frac{1}{4}\ln|\cos 2x|\cos 2x+\frac{x}{2}\sin 2x$

G.Sol $y=y_c+y_p$

$y=c_1\cos 2x+c_2\sin 2x+\frac{1}{4}(\ln|\cos 2x|)+\frac{x}{2}\sin 2x$ Ans.

Working Rule

For sol of $y''+Py'+Qy=F(x)$

① Find $y_c=c_1y_1+c_2y_2$

If $y_c=c_1\cos x+c_2\sin x$

then Replace c_1 by u_1 & c_2 by u_2

② to get assumed $y_p=u_1\cos x+u_2\sin x$

where u_1 & u_2 are fns of 'x'

Note on comparing y_c we get

③ $y_1=\cos x$ & $y_2=\sin x$

$W=y_1y_2'-y_2y_1'$

$F(x)=$ R.H.S of ①

④ For finding u_1 & u_2 use formulae

$u_1=\int -\dfrac{y_2F(x)}{W}dx$

$u_2=\int \dfrac{y_1F(x)}{W}dx$

⑤ Put values of u_1 & u_2 in Assumed y_p

⑥ For General Sol

$y=y_c+y_p$.

Q2 $\dfrac{d^2y}{dx^2} + y = \tan x \sec x$ — ①

$(D^2+1)y = \tan x \sec x$

$D^2 + 1 = 0$

$D = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$ — ②

$y_p = u_1 \cos x + u_2 \sin x$ (supposed) — ③

from ② $y_1 = \cos x$ & $y_2 = \sin x$

$W = y_1 y_2' - y_1' y_2$

$W = \cos x \cos x + \sin x \sin x$

$W = \cos^2 x + \sin^2 x = \boxed{1}$

$u_1 = \int \dfrac{-y_2 F(x)}{W} dx$

$= \int -\dfrac{\sin x \tan x \sec x}{1} dx$

$= \int -\sin x \dfrac{\sin x}{\cos x} \dfrac{1}{\cos x} dx$

$= \int -\tan^2 x \, dx$

$= \int (1 - \sec^2 x) dx$

$u_1 = x - \tan x$

$u_2 = \int \dfrac{y_1 F(x)}{W} dx$

$= \int \dfrac{\cancel{\cos x} \tan x \cancel{\sec x}}{1} dx$

$= -\ln|\cos x|$

$u_2 = \ln \sec x$. Put u_1 & u_2 in y_p ③

$\therefore y_p = (x - \tan x)\cos x + \ln \sec x \sin x$

G.Sol $y = y_c + y_p$

$y = c_1 \cos x + c_2 \sin x + (x - \tan x)\cos x + \ln \sec x \sin x$.

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④ $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = e^{-2x} \sec x$ — ①

$(D^2 + 4D + 5)y = e^{-2x} \sec x$

$D^2 + 4D + 5 = 0$

$D = \dfrac{-4 \pm \sqrt{16-20}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i$

$y_c = e^{-2x}(c_1 \cos x + c_2 \sin x)$ — ②

$y_p = e^{-2x}(u_1 \cos x + u_2 \sin x)$ (supposed) ③

from ② $y_1 = e^{-2x}\cos x$ & $y_2 = e^{-2x} \sin x$

$W = y_1 y_2' - y_2 y_1'$

$= e^{-2x}\cos x[\cos x e^{-2x} - 2e^{-2x}\sin x] - [e^{-2x}\sin x](-\sin x e^{-2x} - 2\cos x e^{-2x})$

$= e^{-4x}[\cos^2 x - 2\cos x \sin x] + e^{-4x}[\sin^2 x + 2\sin x \cos x]$

$= e^{-4x}[\cos^2 x \cancel{- 2\cos x \sin x} + \sin^2 x \cancel{+ 2\sin x \cos x}]$

$= e^{-4x}$

$u_1 = \int \dfrac{-y_2 F(x)}{W} dx$

$= \int \dfrac{-e^{-2x}\sin x \, e^{-2x}\sec x}{e^{-4x}} dx = \int -\dfrac{\sin x}{\cos x} dx$

$= \ln|\cos x|$

$u_2 = \int \dfrac{y_1 F(x)}{W} dx$

$u_2 = \int \dfrac{\cos x \cancel{e^{-2x}} \cancel{e^{-2x}} \sec x}{\cancel{e^{-4x}}} dx = \int dx = x$

Put u_1, u_2 in ③

$\therefore y_p = e^{-2x}(\ln|\cos x| \cos x + x \sin x)$

G.Sol $y = y_c + y_p$

$= e^{-2x}(c_1 \cos x + c_2 \sin x) + e^{-x}[\cos x \ln \cos x + x \sin x]$ Ans

③ $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = (1+e^{-x})^{-1}$$

$$(D^2 - 3D + 2)y = 0$$

$$D^2 - 3D + 2 = 0$$

$$D^2 - D - 2D + 2 = 0$$

$$D[D-1] - 2[D-1] = 0$$

$$(D-2)(D-1) = 0.$$

$$D = 1, 2.$$

$$y_c = c_1 e^{x} + c_2 e^{2x}$$

$$y_p = u_1 e^{x} + u_2 e^{2x}$$

$$F(x) = \frac{1}{1+e^{-x}} \qquad y_1 = e^{x},\ y_2 = e^{2x}$$

$$W = y_1 y_2' - y_2 y_1'$$

$$= e^{x}\,2e^{2x} - e^{2x}e^{x}$$

$$= 2e^{3x} - e^{3x} = \boxed{e^{3x}}$$

$$u_1 = \int -\frac{y_2 F(x)}{W}\,dx$$

$$= -\int \frac{e^{2x}\cdot 1}{e^{3x}(1+e^{-x})}\,dx$$

$$= -\int \frac{e^{-x}}{(1+e^{-x})}\,dx = \ln(1+e^{-x})$$

$$u_2 = \int \frac{y_1 F(x)}{W}\,dx$$

$$= \int \frac{e^{x}\cdot 1}{e^{3x}(1+e^{-x})}\,dx$$

$$= \int \frac{e^{-2x}}{1+e^{-x}}\,dx \qquad \text{Put } e^{-x} = z$$

$$-e^{-x}dx = dz$$

$$dx = \frac{dz}{-z}$$

$$= \int \frac{-z^2\cdot\frac{dz}{z}}{1+z}$$

$$= \int \frac{-z\,dz}{1+z} = \int \frac{(-z-1+1)dz}{1+z}$$

$$u_2 = -\int dz + \int \frac{dz}{1+z} = -z + \ln(z+1) = -e^{-x} + \ln(1+e^{-x})$$

$$y_p = \ln(1+e^{-x})e^{x} + (-e^{-x} + \ln(1+e^{-x}))e^{2x}$$

General Sol $y = y_c + y_p$

$y = c_1e^{x} + c_2e^{2x} + \ln(1+e^{-x})e^{x} - e^{x} + \ln(1+e^{-x})e^{2x}$

⑤ $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \frac{e^{2x}}{1+x}$$

$$D^2 - 4D + 4 = 0$$

$$(D-2)^2 = 0 \qquad D = 2, 2$$

$$y_c = (c_1 + c_2 x)e^{2x}$$

$$y_p = u_1 e^{2x} + u_2 x e^{2x}$$

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}, \quad F(x) = \frac{e^{2x}}{1+x}$$

$$W = y_1 y_2' - y_2 y_1' = e^{2x}(e^{2x} + 2xe^{2x}) - xe^{2x}\cdot 2e^{2x}$$

$$W = e^{4x}$$

$$u_1 = \int \frac{-y_2 F(x)\,dx}{W} = -\int xe^{2x}\frac{e^{2x}}{1+x}\frac{dx}{e^{4x}}$$

$$= -\int \frac{x\,dx}{1+x} = -\int \frac{(x+1-1)dx}{1+x}$$

$$= -x + \ln|1+x|$$

$$u_2 = \int \frac{y_1 F(x)\,dx}{W} = \int e^{2x}\frac{e^{2x}}{1+x}\frac{dx}{e^{4x}}$$

$$u_2 \Rightarrow \int \frac{dx}{1+x} = \ln|1+x|$$

$$y_p = [-x + \ln|1+x|]e^{2x} + \ln|1+x|\,xe^{2x}$$

General Sol $y = y_c + y_p$

$$y = (c_1 + c_2x)e^{2x} + [-x + \ln|1+x|]e^{2x} + \ln|1+x|\,xe^{2x}.$$

x———x

10.6

(8) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = e^{-x}\ln x$

$(D^2+2D+1)y=0$

$D^2+2D+1=0$

$(D+1)^2=0 \Rightarrow D=-1,-1$

$y_c=(c_1+c_2x)e^{-x}$

$y_p=(u_1+u_2x)e^{-x}=u_1e^{-x}+u_2xe^{-x}$

$y_1=e^{-x}, \quad y_2=xe^{-x}, \quad F(x)=e^{-x}\ln x$

$W=y_1y_2'-y_1'y_2 = e^{-x}(-xe^{-x}+e^{-x})-(-e^{-x})(xe^{-x})$

$=-\cancel{xe^{-2x}}+e^{-2x}+\cancel{xe^{-2x}}=\boxed{e^{-2x}}$

$u_1=\int -\frac{y_2F(x)}{W}dx=\int -\frac{xe^{-x}e^{-x}\ln x}{e^{-2x}}dx$

$=-\int \underset{II}{x}\,\underset{I}{\ln x}\,dx = -\left(\ln x\frac{x^2}{2}-\int\frac{1}{x}\cdot\frac{x^2}{2}dx\right)$

$=-\ln x\frac{x^2}{2}+\int\frac{x}{2}dx = -\ln x\frac{x^2}{2}+\frac{x^2}{4}$

$u_2=\int\frac{y_1F(x)}{W}dx=\int\frac{e^{-x}e^{-x}\ln x}{e^{-2x}}dx$

$=\int\ln x\,dx = \ln x\cdot x-\int\frac{1}{x}\cdot x\,dx$

$=x\ln x-\int dx = x\ln x - x$

$y_p=\left(-\ln x\frac{x^2}{2}+\frac{x^2}{4}\right)e^{-x}+(x\ln x-x)xe^{-x}$

$=\left[-\ln x\cdot\frac{x^2}{2}+\frac{x^2}{4}+x^2\ln x-x^2\right]e^{-x}$

$=\left[-2x^2\ln x+x^2+4x^2\ln x-4x^2\right]\frac{e^{-x}}{4}$

$y_p=(2x^2\ln x-3x^2)\frac{e^{-x}}{4}$

General Sol $y=y_c+y_p=(c_1+c_2x)e^{-x}+(2x^2\ln x-3x^2)\frac{e^{-x}}{4}$.

(10) $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=x^3e^x$

$\frac{d^2y}{dx^2}-\frac{2x}{x^2}\frac{dy}{dx}+\frac{2y}{x^2}=\frac{x^3e^x}{x^2}$

$\frac{d^2y}{dx^2}-\frac{2}{x}\frac{dy}{dx}+\frac{2y}{x^2}=xe^x$

$P=-\frac{2}{x} \quad Q=\frac{2}{x^2} \quad F(x)=xe^x$

$P+xQ=-\frac{2}{x}+x\cdot\frac{2}{x^2}=0$

∴ $y_1=x$ is another sol of associated homogeneous eq.

$y_2=x^2$ is given sol.

∴ $y_c=c_1x+c_2x^2$

$y_p=u_1x+u_2x^2 \quad y_1=x, y_2=x^2$

$W=y_1y_2'-y_1'y_2 \quad F(x)=xe^x$

$W=2x^2-x^2=\boxed{x^2}$

$u_1=\int -\frac{y_2F(x)}{W}dx$

$=\int -\frac{x^2\cdot xe^x}{x^2}dx = -\int \underset{I}{x}\,\underset{II}{e^x}\,dx$

$u_1=-xe^x+\int 1\cdot e^x dx = -xe^x+e^x$

$u_2=\int\frac{y_1F(x)}{W}dx=\int\frac{x\,xe^x}{x^2}$

$u_2=\int e^x = e^x$

$y_p=(-xe^x+e^x)x+e^xx^2$

$=-\cancel{x^2e^x}+xe^x+\cancel{e^xx^2}$

$y_p=xe^x$

General Sol $y=y_c+y_p$

$y=c_1x+c_2x^2+xe^x$

⑥ $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \sin^{-1}x$ ——①

$(D^2 - 2D + 1)y = e^x \sin^{-1}x$

$D^2 - 2D + 1 = 0$

$(D-1)^2 = 0$

$D = 1, 1$

$y_c = (c_1 + c_2 x)e^x$ ——②

$y_p = (u_1 + u_2 x)e^x$ (Supposed) ——③

from ② $y_1 = e^x$ & $y_2 = xe^x$

$W = y_1 y_2' - y_1' y_2$

$= e^x(1 \cdot e^x + xe^x) - e^x \, xe^x$

$= e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}}$

$W = e^{2x}$

$u_1 = \int \dfrac{-y_2 F(x)}{W} dx$

$= \int \dfrac{-xe^x e^x \sin^{-1}x}{e^{2x}} dx$

$= \int -x \sin^{-1}x \, dx$ IBP

$= -\dfrac{x^2}{2}\sin^{-1}x + \int \dfrac{x^2}{2} \dfrac{dx}{\sqrt{1-x^2}}$

$= -\dfrac{x^2}{2}\sin^{-1}x - \dfrac{1}{2}\int \dfrac{1-x^2-1}{\sqrt{1-x^2}} dx$

$= -\dfrac{x^2}{2}\sin^{-1}x - \dfrac{1}{2}\int \sqrt{1-x^2}\,dx + \dfrac{1}{2}\int \dfrac{dx}{\sqrt{1-x^2}}$

$= -\dfrac{x^2}{2}\sin^{-1}x - \dfrac{1}{2}\left[\dfrac{x\sqrt{1-x^2}}{2} + \dfrac{1}{2}\sin^{-1}x\right] + \dfrac{\sin^{-1}x}{2}$

$= -\dfrac{x^2}{2}\sin^{-1}x - \dfrac{x\sqrt{1-x^2}}{4} - \dfrac{\sin^{-1}x}{4} + \dfrac{\sin^{-1}x}{2}$

$u_1 = -\dfrac{x^2}{2}\sin^{-1}x - \dfrac{x\sqrt{1-x^2}}{4} + \dfrac{\sin^{-1}x}{4}$

$u_2 = \int \dfrac{y_1 F(x)}{W} dx =$

$= \int \dfrac{\cancel{e^x}\,\cancel{e^x}\sin^{-1}x}{\cancel{e^{2x}}} dx = \int 1 \cdot \sin^{-1}x \, dx$ IBP

$u_2 = x\sin^{-1}x - \int x \cdot \dfrac{1}{\sqrt{1-x^2}} dx$

$u_2 = x\sin^{-1}x + \dfrac{1}{2}\int (1-x^2)^{-\frac{1}{2}}(-2x)\,dx$

$= x\sin^{-1}x + \dfrac{1}{2}\dfrac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$

$u_2 = x\sin^{-1}x + \dfrac{\cancel{1}}{\cancel{2}}\dfrac{\sqrt{1-x^2}}{\cancel{\frac{1}{2}}}$

Put u_1, u_2 in ③

$y_p = e^x\left[\left(-\dfrac{x^2}{2}\sin^{-1}x - \dfrac{x\sqrt{1-x^2} + \sin^{-1}x}{4}\right) + \left(x\sin^{-1}x + \sqrt{1-x^2}\right)x\right]$

$y_p = e^x\left[-\dfrac{x^2}{2}\sin^{-1}x - \dfrac{x\sqrt{1-x^2}}{4} + \dfrac{\sin^{-1}x}{4} + x^2\sin^{-1}x + x\sqrt{1-x^2}\right]$

G.Sol $y = y_c + y_p$

$= (c_1 + c_2 x)e^x + e^x\left(-\dfrac{x^2}{2}\sin^{-1}x - \dfrac{x\sqrt{1-x^2}}{4} + \sin^{-1}x + x^2\sin^{-1}x + x\sqrt{1-x^2}\right)$

10.6-8

Q7 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^x \tan 2x$ —①

$(D^2-2D+5)y = e^x \tan 2x$

$D^2-2D+5 = 0$

$D = \frac{2\pm\sqrt{4-20}}{2} = \frac{2\pm i4}{2} = 1\pm i2$

$y_c = e^x(c_1\cos 2x + c_2 \sin 2x)$ —②

$y_p = e^x(u_1\cos 2x + u_2\sin 2x)$ supposed —③

for ② $y_1 = e^x\cos 2x$ & $y_2 = e^x\sin 2x$

$y_1' = e^x\cos 2x + e^x(-2\sin 2x)$

$y_2' = e^x\sin 2x + e^x\cos 2x(2)$

$W = y_1 y_2' - y_1' y_2$

$= (e^x\cos 2x)\, e^x(\sin 2x + 2\cos 2x)$
$\quad - e^x(\cos 2x - 2\sin 2x)\, e^x \sin 2x$

$= e^{2x}(\cos 2x\cancel{\sin 2x} + 2\cos^2 2x - \cancel{\sin 2x\cos 2x} + 2\sin^2 2x)$

$W = e^{2x}\, 2(\cos^2 2x + \sin^2 2x) = \boxed{2e^{2x}}$

$u_1 = \int -\frac{y_2 F(x)}{W}dx = \int -\frac{\cancel{e^x}\sin 2x\, \cancel{e^x}\tan 2x\, dx}{2\cancel{e^{2x}}}$

$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}dx$

Let $\sin 2x = t$
$2\cos 2x\, dx = dt$
$dx = \frac{dt}{2\cos 2x}$

$= -\frac{1}{2}\int \frac{t^2}{\cos 2x}\,\frac{dt}{2\cos 2x}$

$= -\frac{1}{4}\int \frac{t^2\, dt}{1-t^2}$ $\quad \because \cos^2 2x = 1-\sin^2 2x = 1-t^2$

$= \frac{1}{4}\int\left[\frac{(1-t^2)}{(1-t^2)} - 1\right]dt$

$= \frac{1}{4}\int dt - \frac{1}{4}\int\frac{dt}{1-t^2}$

$= \frac{1}{4}t - \frac{1}{4}\left[\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)\right]$ $\quad \because \int\frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left[\frac{a+x}{a-x}\right]$

$u_1 = \frac{\sin 2x}{4} - \frac{1}{8}\ln\left(\frac{1+\sin 2x}{1-\sin 2x}\right)$

$u_1 = \frac{\sin 2x}{4} - \frac{1}{8}\ln\left(\frac{\sin^2 x + \cos^2 x + 2\sin x\cos x}{\sin^2 x + \cos^2 x - 2\sin x\cos x}\right)$

$= \frac{\sin 2x}{4} - \frac{1}{8}\ln\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)^2$

$= \frac{\sin 2x}{4} - \frac{1}{8}\ln\left(\frac{1+\tan x}{1-\tan x}\right)^2$

$= \frac{\sin 2x}{4} - \frac{2}{8}\ln\left(\tan\left(\frac{\pi}{4}+x\right)\right)$

$\because \tan\frac{\pi}{4} = 1$

$\because \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$u_1 = \frac{\sin 2x}{4} - \frac{1}{4}\ln\left(\tan\left(\frac{\pi}{4}+x\right)\right)$

$u_2 = \int \frac{y_1 F(x)}{W}dx$

$= \int \frac{\cancel{e^x}\cos 2x \cdot \cancel{e^x}\tan 2x}{2\cancel{e^{2x}}}dx$

$= \frac{1}{2}\int \cancel{\cos 2x}\,\frac{\sin 2x}{\cancel{\cos 2x}}dx$

$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)$

$= -\frac{1}{4}\cos 2x$ $\quad$ Put u_1, u_2 in y_p

$\therefore y_p = \frac{e^x}{4}\left[\sin 2x - \ln\left(\tan\left(\frac{\pi}{4}+x\right)\right)\cos 2x - \left(\frac{\cos 2x}{4}\right)\sin 2x\right]$

G.Sol $y = y_c + y_p$

$y = e^x(c_1\cos 2x + c_2\sin 2x) + \frac{e^x}{4}\left[\sin 2x - \ln\left(\tan\left(\frac{\pi}{4}+x\right)\right)\cos 2x - \left(\frac{\cos 2x}{4}\right)\sin 2x\right]$

Ans.

⑨ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2e^{-x}\tan^2 x$

$(D^2+2D+2)y = 0$

$D^2+2D+2=0$

$D = \frac{-2\pm\sqrt{4-4\cdot1\cdot2}}{2}$

$D = \frac{-2\pm 2i\sqrt{-4}}{2} = -1\pm i$

$y_c = e^{-x}(c_1\cos x + c_2\sin x)$

$y_p = e^{-x}(u_1\cos x + u_2\sin x)$

$y_p = u_1 e^{-x}\cos x + u_2 e^{-x}\sin x$

$u_1 = \int \frac{-y_2 F(x)}{W}dx$

$= \int \frac{-e^{-x}\sin x\,(2e^{-x}\tan^2 x)}{e^{-2x}}dx$

$= -2\int \sin x\tan^2 x\,dx$

$= -2\int \frac{\sin^3 x}{\cos^2 x}dx$ Put $\cos x = z$, $-\sin x\,dx = dz$

$= 2\int \frac{\sin^2 x}{\cos^2 x}(-\sin x\,dx)$ Also $1-\cos^2 x = \sin^2 x$, $1-z^2 = \sin^2 x$.

$= 2\int \frac{(1-z^2)dz}{z^2}$

$= 2\int \frac{dz}{z^2} - 2\int dz$

$= -\frac{2}{z} - 2z = -2\left[\frac{1}{\cos x} + \cos x\right]$

$-2(\cos^2 x + 1)$

$y_1 = e^{-x}\cos x$, $y_2 = e^{-x}\sin x$, $F(x) = 2e^{-x}\tan^2 x$

$W = y_1y_2' - y_1'y_2$

$= e^{-x}\cos x(-e^{-x}\sin x + e^{-x}\cos x) - (-e^{-x}\cos x - e^{-x}\sin x)(e^{-x}\sin x)$

$= -\cancel{e^{-2x}\cos x\sin x} + e^{-2x}\cos^2 x + \cancel{e^{-2x}\cos x\sin x} + e^{-2x}\sin^2 x$

$= e^{-2x}(\cos^2 x + \sin^2 x)$

$\boxed{W = e^{-2x}}$

$u_2 = \int \frac{y_1 F(x)dx}{W} = \int \frac{e^{-x}\cos x\; 2e^{-x}\tan^2 x\,dx}{e^{-2x}}$

$u_2 = 2\int \cos x\tan^2 x\,dx$

$= 2\int \frac{\sin^2 x}{\cos x}dx$ Put $\sin x = z$, $\cos x\,dx = dz$

$= 2\int \frac{z^2\cos x\,dx}{\cos^2 x}$

$= 2\int \frac{z^2\,dz}{1-z^2} = -2\int \frac{-z^2}{1-z^2}dz$

$= -2\int \frac{1-z^2-1}{1-z^2}dz$

$= -2\int\left(1 - \frac{1}{1-z^2}\right)dz$

$= -2\int dz + 2\int \frac{dz}{1-z^2}$

$= -2z + 2\cdot\frac{1}{2}\ln\frac{(1+z)}{1-z}$

$= -2\sin x + \ln\left|\frac{1+\sin x}{1-\sin x}\right|$

$u_2 = -2\sin x + \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$

$y_p = -2(\cos^2 x+1)e^{-x}\cos x + \left[-2\sin x + \ln\left(\tan\frac{x}{2}+\frac{\pi}{4}\right)\right](e^{-x}\sin x)$

General Sol $y = y_c + y_p$

$= e^{-x}(c_1\cos x + c_2\sin x) + \left[-2(\cos^2 x+1)e^{-x}\cos x + \left(-2\sin x + \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right)e^{-x}\sin x\right]$ Ans.

$\boxed{10.6}$ $\boxed{10.6-8}$

(11) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{1}{1+x}$ $y_1 = \frac{1}{x}$ is a sol (given)

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x^2(1+x)} \quad \text{—(1)}$$

$P = \frac{1}{x}$ $Q = \frac{-1}{x^2}$

$P + Qx = \frac{1}{x} + \left(\frac{-1}{x^2}\right)x = 0$

$\therefore y_2 = x$ is another sol of $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0$.

$y_c = c_1\frac{1}{x} + c_2 x$

$y_p = u_1 \cdot \frac{1}{x} + u_2 \cdot x$

$y_1 = \frac{1}{x}$, $y_2 = x$, $F(x) = \frac{1}{x^2(1+x)}$

$W = y_1 y_2' - y_1' y_2 = \frac{1}{x} \cdot 1 - \left(-\frac{1}{x^2}\right)x$

$W = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$

$u_1 = \int \frac{-y_2 F(x)}{W}dx = -\int x \cdot \frac{1}{x^2(1+x)} \cdot \frac{x}{2}dx$

$u_1 = -\frac{1}{2}\int\frac{dx}{1+x} = -\frac{1}{2}\ln(1+x)$

$u_2 = \int \frac{y_1 F(x)}{W}dx$

$= \int \frac{1}{x} \cdot \frac{1}{x^2(1+x)} \cdot \frac{x}{2}dx$

$= \frac{1}{2}\int\frac{dx}{x^2(1+x)}$ —— (A)

$\frac{1}{x^2(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1+x}$

$1 = Ax(1+x) + B(1+x) + Cx^2$

$x = 0 \Rightarrow \boxed{1 = B}$

$x = -1 \Rightarrow \boxed{1 = C}$

Comparing coeff of x^2

$0 = A + C$

$0 = A + 1 \Rightarrow \boxed{A = -1}$

$\therefore \frac{1}{x^2(1+x)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{1+x}$

$\therefore$ (A) becomes $\frac{1}{2}\int\left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{1+x}\right)dx$

$u_2 = -\frac{1}{2}\ln x - \frac{1}{2x} + \frac{1}{2}\ln|1+x|$

$y_p = \left[\frac{-1}{2}\ln(1+x) \cdot \frac{1}{x}\right] + \left[-\frac{1}{2}\ln x - \frac{1}{2x} + \frac{1}{2}\ln(1+x)\right](x)$

$y_p =$

General Sol is

$y = y_c + y_p$

$= c_1\frac{1}{x} + c_2 x - \frac{1}{2x}\ln(1+x) + x\left(-\frac{1}{2}\ln x - \frac{1}{2x} + \frac{1}{2}\ln(1+x)\right)$

10.6-9

(12) ~~Find~~ the general sol of $x(x-2)\frac{d^2y}{dx^2} - \ldots \frac{dy}{dx} + (\ldots)y = \ldots$

$$x(x-2)\frac{d^2y}{dx^2} - (x^2-2)\frac{dy}{dx} + 2(x-1)y = 3x^2(x-2)^2 e^x \quad \text{---} (1)$$

given that $y_1 = x^2$ is a sol of associated Homo Eq.

Sol from (1) $\frac{d^2y}{dx^2} - \frac{(x^2-2)}{x(x-2)}\frac{dy}{dx} + \frac{2(x-1)}{x(x-2)}y = \frac{3x^2(x-2)^2 e^x}{x(x-2)}$

$$y'' - \frac{(x^2-2)}{x(x-2)}y' + \frac{2(x-1)y}{x(x-2)} = 3x(x-2)e^x$$

Compare with $y'' + Py' + Qy = F(x)$

$$P = -\frac{(x^2-2)}{x(x-2)}, \quad Q = \frac{2(x-1)}{x(x-2)}, \quad F(x) = 3x(x-2)e^x$$

$$\because P + Qx = -\frac{(x^2-2)}{x(x-2)} + \frac{2(x-1)}{x(x-2)}x \neq 0$$

Now check

$$1 + P + Q = 1 + \left[-\frac{(x^2-2)}{x(x-2)}\right] + \frac{2(x-1)}{x(x-2)}$$

$$= \frac{x(x-2) - x^2 + \cancel{2} + 2x - \cancel{2}}{x(x-2)} = \frac{\cancel{x^2} - \cancel{2x} - \cancel{x^2} + \cancel{2x}}{x(x-2)} = 0$$

thus $y_2 = e^x$ is another sol of associated Homogeneous eq.

So $y_c = c_1 y_1 + c_2 y_2$

$y_c = c_1 x^2 + c_2 e^x$

$y_p = u_1 x^2 + u_2 e^x$ (supposed)

$W = y_1 y_2' - y_1' y_2$

$= x^2 e^x - 2x e^x$

$W = x e^x (x-2)$

$\because u_1 = \int -\frac{y_2 F(x)}{W} dx$

$= -\int \frac{\cancel{e^x}\, 3\cancel{x}\cancel{(x-2)}e^x}{\cancel{x}\cancel{e^x}\cancel{(x-2)}} dx$

$= -3\int e^x dx$

$u_1 = -3e^x$

$u_2 = \int \frac{y_1 F(x)}{W} dx$

$= \int \frac{x^2 . 3\cancel{x}\cancel{(x-2)}\cancel{e^x}}{\cancel{x}\cancel{e^x}\cancel{(x-2)}} dx$

$= 3\int x^2 dx$

$u_2 = 3\frac{x^3}{3} = x^3$

Put u_1, u_2 in y_p

$y_p = (-3e^x)x^2 + (x^3)e^x$

G.Sol is $y = y_c + y_p$

$y = c_1 x^2 + c_2 e^x + x^3 e^x - 3e^x x^2$ Ans

$\boxed{10\cdot 6-10}$

(13) Find sol of $(\sin^2 x)\frac{d^2y}{dx^2} - (\sin 2x)\frac{dy}{dx} + (1+\cos^2 x)y = \sin^3 x$ given that $y_1 = \sin x$ & $y_2 = x\sin x$ are linearly independent sol of associated homogeneous eq.

Sol $\because y_1 = \sin x$ & $y_2 = x\sin x$ (Given)

Note from (1) $\frac{d^2y}{dx^2} - \frac{\sin 2x}{\sin^2 x}\frac{dy}{dx} + \frac{(1+\cos^2 x)y}{\sin^2 x} = \frac{\sin^3 x}{\sin^2 x}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sin 2x}{\sin^2 x}\frac{dy}{dx} + \frac{1+\cos^2 x}{\sin^2 x}y = \sin$ (1)

$\therefore F(x) = \sin x$

Hence $y_c = c_1 y_1 + c_2 y_2$

$= c_1 \sin x + c_2 x\sin x$

$y_p = u_1 \sin x + u_2 x\sin x$

$W = y_1 y_2' - y_1' y_2$

$= \sin x(\sin x + x\cos x) - \cos x\, x\sin x$

$= \sin^2 x + \cancel{x\sin x\cos x} - \cancel{x\cos x\sin x}$

$W = \sin^2 x$

$u_1 = \int \frac{-y_2 F(x)}{W}$

$= -\int \frac{x\cancel{\sin x}\,\cancel{\sin x}}{\cancel{\sin^2 x}}dx$

$= -\int x\,dx = \boxed{-\frac{x^2}{2}}$

$u_2 = \int \frac{y_1 F(x)}{W}$

$= \int \frac{\sin x \sin x}{\sin^2 x}dx$

$= \int dx = \boxed{x}$

$y_p = -\frac{x^2}{2}\sin x + x^2 \sin x$

G.Sol $y = y_c + y_p$

$= c_1 \sin x + c_2 x\sin x - \frac{x^2}{2}\sin x + x^2 \sin x$

$= \left(c_1 + c_2 x - \frac{x^2}{2} + x^2\right)\sin x$

$= \left(c_1 + c_2 x + \frac{x^2}{2}\right)\sin x$ Ans.

(14) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = \dfrac{2e^x}{x^2}$ —— (1)

The characteristic eq is $D^3 - 3D^2 + 3D - 1 = 0$

$(D-1)^3 = 0$ therefore $D = 1, 1, 1$

C.F of (1) is $\quad y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

Let $\quad y_p = u_1 e^x + u_2 x e^x + u_3 x^2 e^x$

Here $y_1 = e^x \qquad (\because y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$ assumed)

$y_2 = x e^x$

$y_3 = x^2 e^x \quad , \quad F(x) = \dfrac{2e^x}{x^2}$

Substituting values in $u_1' y_1 + u_2' y_2 + u_3' y_3 = 0 \Rightarrow u_1' e^x + u_2' x e^x + u_3' x^2 e^x = 0$

Substituting values in $u_1' y_1' + u_2' y_2' + u_3' y_3' = 0 \Rightarrow u_1' e^x + u_2'(e^x + x e^x) + u_3'(2x e^x + x^2 e^x) = 0$

Substituting values in $u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = F(x)$

$\Rightarrow u_1' e^x + u_2'(2e^x + x e^x) + u_3'(2e^x + 4x e^x + x^2 e^x) = \dfrac{2e^x}{x^2}$

Solving these eqs for u_1', u_2', u_3' by Cramer's Rule

$$u_1' = \frac{\begin{vmatrix} 0 & x e^x & x^2 e^x \\ 0 & e^x + x e^x & 2x e^x + x^2 e^x \\ \frac{2e^x}{x^2} & 2e^x + x e^x & 2e^x + 4x e^x + x^2 e^x \end{vmatrix}}{\begin{vmatrix} e^x & x e^x & x^2 e^x \\ e^x & e^x + x e^x & 2x e^x + x^2 e^x \\ e^x & 2e^x + x e^x & 2e^x + 4x e^x + x^2 e^x \end{vmatrix}} = \frac{\frac{2e^x}{x^2}\begin{vmatrix} x e^x & x^2 e^x \\ e^x + x e^x & 2x e^x + x^2 e^x \end{vmatrix}}{\begin{vmatrix} e^x & x e^x & x^2 e^x \\ e^x & e^x + x e^x & 2x e^x + x^2 e^x \\ e^x & 2e^x + x e^x & 2e^x + 4x e^x + x^2 e^x \end{vmatrix}}$$

$$u_1' = \frac{\frac{2e^x}{x^2}\left[x e^x (2x e^x + x^2 e^x) - x^2 e^x (e^x + x e^x)\right]}{\begin{vmatrix} e^x & x e^x & x^2 e^x \\ 0 & e^x & 2x e^x \\ 0 & 2e^x & 2e^x + 4x e^x \end{vmatrix}} \qquad \begin{matrix} -R_1 + R_2 \\ -R_1 + R_3 \end{matrix}$$

$\boxed{10.6-12}$

$$= \frac{\frac{2e^x}{x^2}\left(2x^2e^{2x} + \cancel{x^3e^{2x}} - x^2e^{2x} - \cancel{x^3e^{2x}}\right)}{e^x\left[e^x(2e^x+4xe^x) - 2e^x(2xe^x)\right]}$$

$$= \frac{\frac{2e^x}{x^2}(x^2e^{2x})}{e^x\left[2e^{2x} + \cancel{4xe^{2x}} - \cancel{4xe^{2x}}\right]} = \frac{2e^{3x}}{2e^{3x}} = 1$$

$$u_2' = \frac{\begin{vmatrix} e^x & 0 & x^2e^x \\ e^x & 0 & 2xe^x + x^2e^x \\ e^x & \frac{2e^x}{x^2} & 2e^x + 4xe^x + x^2e^x \end{vmatrix}}{2e^{3x}} \text{ (as solved above)} = \frac{-\frac{2e^x}{x^2}\begin{vmatrix} e^x & x^2e^x \\ e^x & 2xe^x + x^2e^x \end{vmatrix}}{2e^{3x}}$$

$$= \frac{-\frac{2e^x}{x^2}\left[e^x(2xe^x + x^2e^x) - e^x(x^2e^x)\right]}{2e^{3x}} = \frac{-\frac{2e^{2x}}{x^2}\left(2xe^x + \cancel{x^2e^x} - \cancel{x^2e^x}\right)}{2e^{3x}}$$

$$= \frac{-4xe^{3x}}{x^2} \cdot \frac{1}{2e^{3x}} = -\frac{2}{x}$$

$$u_3' = \frac{\begin{vmatrix} e^x & xe^x & 0 \\ e^x & e^x + xe^x & 0 \\ e^x & 2e^x + xe^x & \frac{2e^x}{x^2} \end{vmatrix}}{2e^{3x}} = \frac{\frac{2e^x}{x^2}\begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}}{2e^{3x}}$$

$$= \frac{\frac{2e^x}{x^2}\left[e^x(e^x + xe^x) - e^x(xe^x)\right]}{2e^{3x}} = \frac{\frac{2e^x}{x^2}\left(e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}}\right)}{2e^{3x}}$$

$$= \frac{2e^{3x}}{x^2\, 2e^{3x}} = \frac{1}{x^2}$$

$\therefore\ u_1 = \int dx = x$

$u_2 = \int -\frac{2}{x}\,dx = -2\ln|x|$

$u_3 = \int \frac{dx}{x^2} = -\frac{1}{x}$

$y_p = u_1y_1 + u_2y_2 + u_3y_3$

$= xe^x + xe^x(-2\ln|x|) + x^2e^x(-\frac{1}{x})$

$= \cancel{xe^x} - 2xe^x\ln|x| - \cancel{xe^x}$

$y_p = -2xe^x\ln|x|$

$y = c_1e^x + c_2xe^x + c_3x^2e^x - 2xe^x\ln|x|$

$\boxed{10.6}$

(15) $\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} - 4y = e^{-x}\tan x$ $\boxed{10.6.13}$

Sol $D^3 - 2D - 4 = 0$

$(D-2)(D^2+2D+2) = 0$

$D = 2, -1 \pm i$

By synthetic division

$$\begin{array}{c|cccc} 2 & 1 & 0 & -2 & -4 \\ & \downarrow & 2 & 4 & 4 \\ \hline & 1 & 2 & 2 & \llcorner 0 \end{array}$$

$\therefore 2$ is Root
$D = 2$.

$D^2 + 2D + 2 = 0$.

$D = \dfrac{-2 \pm \sqrt{4 - 4\cdot1\cdot2}}{2(1)} = \dfrac{-2 \pm \sqrt{4-8}}{2}$

$= \dfrac{-2 \pm \sqrt{-4}}{2} = \dfrac{\cancel{2}(-1 \pm i)}{\cancel{2}}$

$y_c = c_1 e^{2x} + (c_2 \cos x + c_3 \sin x)e^{-x}$

Let $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$

where $y_1 = e^{2x}$, $y_2 = e^{-x}\cos x$, $y_3 = e^{-x}\sin x$

$y_1' = 2e^{2x}$ $\qquad y_1'' = 4e^{2x}$

$y_2' = -e^{-x}\cos x - e^{-x}\sin x$

$y_2'' = e^{-x}\cancel{\cos x} + e^{-x}\sin x + e^{-x}\sin x - \cancel{e^{-x}\cos x}$

$y_2'' = 2e^{-x}\sin x$

$y_3' = -e^{-x}\sin x + e^{-x}\cos x$

$y_3'' = e^{-x}\cancel{\sin x} - e^{-x}\cos x - e^{-x}\cos x - \cancel{e^{-x}\sin x}$

$y_3'' = -2e^{-x}\cos x$

Putting values in $u_1' y_1 + u_2' y_2 + u_3' y_3 = 0$ $\qquad u_1' e^{2x} + u_2' e^{-x}\cos x + u_3' e^{-x}\sin x = 0$

Putting values in $u_1' y_1' + u_2' y_2' + u_3' y_3' = 0$ $\qquad u_1' 2e^{2x} + u_2'(-e^{-x}\cos x - e^{-x}\sin x) + u_3'(-e^{-x}\sin x + e^{-x}\cos x) = 0$

Putting values in $u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = f(x)$ $\qquad u_1' 4e^{2x} + u_2' 2e^{-x}\sin x + u_3'(-2e^{-x}\cos x) = e^{-x}\tan x$

Solving by Cramers Rule

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-x}\cos x & e^{-x}\sin x \\ 0 & -e^{-x}\cos x - e^{-x}\sin x & -e^{-x}\sin x + e^{-x}\cos x \\ e^{-x}\tan x & 2e^{-x}\sin x & -2e^{-x}\cos x \end{vmatrix}}{\begin{vmatrix} e^{2x} & e^{-x}\cos x & e^{-x}\sin x \\ 2e^{2x} & -e^{-x}\cos x - e^{-x}\sin x & -e^{-x}\sin x + e^{-x}\cos x \\ 4e^{2x} & 2e^{-x}\sin x & -2e^{-x}\cos x \end{vmatrix}}$$

$$= \frac{e^{-x}\tan x \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ -e^{-x}\cos x - e^{-x}\sin x & -e^{-x}\sin x + e^{-x}\cos x \end{vmatrix}}{e^{2x}\cdot e^{-x}\cdot e^{-x}\begin{vmatrix} 1 & \cos x & \sin x \\ 2 & -\cos x - \sin x & -\sin x + \cos x \\ \end{vmatrix}}$$

Taking e^{2x} Common from c_1
e^{-x} Common from c_2
e^{-x} Common from c_3

10.6-14

Taking e^{-x} common from C_1 & C_2

$$U_1' = \frac{-e^{-3x}\tan x \begin{vmatrix} \cos x & \sin x \\ -\cos x-\sin x & -\sin x+\cos x \end{vmatrix}}{1\left[2\cos^2 x+2\cancel{\sin x\cos x}+2\sin^2 x-2\cancel{\sin x\cos x}\right]-\cos x(-4\cos x+4\sin x-4\cos x)+\sin x(4\sin x+4\cos x+4\sin x)}$$

$$=\frac{-e^{-3x}\tan x\left[-\sin x\cancel{\cos x}+\cos^2 x+\sin x\cancel{\cos x}+\sin^2 x\right]}{2(\cos^2 x+\sin^2 x)+8\cos^2 x-4\sin x\cancel{\cos x}+8\sin^2 x+4\sin x\cancel{\cos x}}=\frac{e^{-3x}\tan x}{2+8(\sin^2 x+\cos^2 x)}$$

$$\boxed{U_1' = \frac{e^{-3x}\tan x}{10}}$$

$$U_2' = \frac{\begin{vmatrix} e^{2x} & 0 & e^{-x}\sin x \\ 2e^{2x} & 0 & -e^{-x}\sin x+e^{-x}\cos x \\ 4e^{2x} & e^{-x}\tan x & -2e^{-x}\cos x \end{vmatrix}}{10}$$

$$U_2' = -e^{-x}\tan x\frac{\begin{vmatrix} e^{2x} & e^{-x}\sin x \\ 2e^{2x} & -e^{-x}\sin x+e^{-x}\cos x \end{vmatrix}}{10} = \frac{-\cancel{e^{-x}}\tan x\,\cancel{e^{2x}}\,\cancel{e^{-x}}\begin{vmatrix} 1 & \sin x \\ 2 & -\sin x+\cos x \end{vmatrix}}{10}$$

$$U_2' = -\frac{\tan x(-\sin x+\cos x-2\sin x)}{10} = +\frac{\tan x}{10}(3\sin x-\cos x)$$

$$U_3' = \frac{\begin{vmatrix} e^{2x} & e^{-x}\cos x & 0 \\ 2e^{2x} & -e^{-x}\cos x-e^{-x}\sin x & 0 \\ 4e^{2x} & 2e^{-x}\sin x & e^{-x}\tan x \end{vmatrix}}{10} = e^{-x}\tan x\frac{\begin{vmatrix} e^{2x} & e^{-x}\cos x \\ 2e^{2x} & -e^{-x}\cos x-e^{-x}\sin x \end{vmatrix}}{10}$$

$$= e^{-x}\tan x\cdot e^{2x}\cdot e^{-x}\frac{\begin{vmatrix} 1 & \cos x \\ 2 & -\cos x-\sin x \end{vmatrix}}{10} = \frac{\tan x(-\cos x-\sin x-2\cos x)}{10}$$

$$U_3' = \frac{\tan x}{10}(-3\cos x-\sin x)$$

$$U_1 = \frac{1}{10}\int e^{-3x}\tan x\,dx$$

$$U_2 = \frac{1}{10}\int(3\tan x\sin x \mp \tan x\cos x)dx = \frac{1}{10}\int\left(3\frac{\sin^2 x}{\cos x} \mp \frac{\sin x}{\cos x}\cos x\right)dx$$

$$= \frac{1}{10}\int\left(\frac{3(1-\cos^2 x)}{\cos x} \mp \sin x\right)dx = \frac{1}{10}\int(3\sec x - 3\cos x \mp \sin x)dx$$

$$= \frac{1}{10}\left[3\ln|\sec x+\tan x| - 3\sin x + \cos x\right]$$

$U_3 =$

$\boxed{10.6\text{-}15}$

$$u_3 = \frac{1}{10}\int\left(-3\tan x\cos x - \tan x\sin x\right)dx$$

$$= \frac{1}{10}\int\left(-3\,\frac{\sin x}{\cancel{\cos x}}\cancel{\cos x} - \frac{\sin^2 x}{\cos x}\right)dx$$

$$= \frac{1}{10}\int\left\{-3\sin x - \frac{(1-\cos^2 x)}{\cos x}\right\}dx$$

$$= \frac{1}{10}\int\left(-3\sin x - \sec x + \cos x\right)dx$$

$$= \frac{1}{10}\left[3\cos x - \ln|\sec x+\tan x| + \sin x\right]$$

$$y_p = u_1y_1 + u_2y_2 + u_3y_3$$

$$= \frac{e^{2x}}{10}\int e^{-3x}\tan x\,dx + \frac{e^{-x}\cos x}{10}\left[3\ln|\sec x+\tan x| - 3\sin x+\cos x\right]$$
$$+ \frac{e^{-x}\sin x}{10}\left[3\cos x - \ln|\sec x+\tan x| + \sin x\right]$$

$$= \frac{e^{2x}}{10}\int e^{-3x}\tan x\,dx + \frac{e^{-x}}{10}\left[3\cos x\ln|\sec x+\tan x| - 3\sin x\cos x + \cos^2 x\right]$$
$$+ \frac{e^{-x}}{10}\left[3\cos x\sin x - \sin x\ln|\sec x+\tan x| + \sin^2 x\right]$$

$$= \frac{e^{2x}}{10}\int e^{-3x}\tan x\,dx + \frac{e^{-x}}{10}\left[(\ln|\sec x+\tan x|)\{3\cos x - \sin x\} + \cos^2 x + \sin^2 x \cancel{- 3\sin x\cos x + 3\sin x\cos x}\right]$$

$$= \frac{e^{2x}}{10}\int e^{-3x}\tan x\,dx + \frac{e^{-x}}{10}\left[(3\cos x - \sin x)\ln|\sec x+\tan x| + 1\right]$$

$$y = y_c + y_p$$

$$= c_1e^{2x} + (c_2\cos x + c_3\sin x)e^{-x} + \frac{e^{2x}}{10}\int e^{-3x}\tan x\,dx$$
$$+ \frac{e^{-x}}{10}\left[(3\cos x - \sin x)\ln|\sec x+\tan x| + 1\right]$$

10.7 – 1

1

~~One Variable absent: Given a diff eq of the~~ form

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots\ldots, \frac{dy}{dx}, x\right) = 0 \quad \text{———} (1)$$

i.e., an eq. in which the dependent variable y is missing.

In order to solve eq. (1)

we put $\boxed{\frac{dy}{dx} = p}$

then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$

$\frac{d^3y}{dx^3} = \frac{d^2p}{dx^2}$

& so on.

then eq. (1) becomes

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \frac{d^{n-2}p}{dx^{n-2}}, \ldots\ldots, p, x\right)$$

So that order of eq. (1) has been lowered by one.

Similarly Consider the eq.

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots\ldots, \frac{dy}{dx}, y\right).$$

Here x is absent.

In order to solve it, put $\boxed{\frac{dy}{dx} = p}$

then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = \boxed{p\frac{dp}{dy}}$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(p\frac{dp}{dy}\right) = \frac{d}{dy}\left(p\frac{dp}{dy}\right)\cdot\frac{dy}{dx}$$

$$= \left[p\cdot\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2\right]p = \boxed{p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2}$$

Then eq. (2) will be transformed into an eq. in p & y of order $n-1$.

Exercise No. 10.7

Solve :

③ $x\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^3 - \dfrac{dy}{dx} = 0$

Sol. Given

$$x\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0 \quad \text{——— ①}$$

Here y is absent

Put $\dfrac{dy}{dx} = p$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}$$

Put values in ①

$$x\frac{dp}{dx} - p^3 - p = 0$$ order has been lowered to '1' from '2'

or $$x\frac{dp}{dx} = p(p^2+1)$$

Separating variables

$$\int \frac{1}{p(p^2+1)}dp = \int \frac{dx}{x} \quad \text{——— Ⓐ}$$

Now $$\frac{1}{p(p^2+1)} = \frac{A}{p} + \frac{Bp+C}{p^2+1}$$

$$\Rightarrow 1 = A(p^2+1) + (Bp+c)p \quad \text{Ⓘ}$$

Put $p = 0$

So $\boxed{A = 1}$

Comparing Coff. of p^2, p & Constt

$$\left.\begin{aligned} A+B &= 0 \\ C &= 0 \\ A &= 1 \end{aligned}\right\} \begin{aligned} &\Rightarrow \boxed{B = -1} \\ &\Rightarrow \boxed{C = 0} \\ &\Rightarrow \boxed{A = 1} \end{aligned}$$

So from Ⓐ, we have

2nd Method

$$x\frac{dp}{dx} - p^3 - p = 0$$

$$\frac{dp}{dx} - \frac{p^3}{x} - \frac{p}{x} = 0$$

$$\frac{dp}{dx} - p\cdot\frac{1}{x} = \frac{p^3}{x} \quad \text{Bernoulli's}$$

$$\frac{1}{p^3}\frac{dp}{dx} - \frac{p}{p^3}\frac{1}{x} = \frac{1}{x}$$

$$p^{-3}\frac{dp}{dx} - p^{-2}\left(\frac{1}{x}\right) = \frac{1}{x}$$

Let $p^{-2} = z$

So $-2p^{-3}\dfrac{dp}{dx} = \dfrac{dz}{dx}$

$p^{-3}\dfrac{dp}{dx} = \dfrac{1}{-2}\dfrac{dz}{dx}$

$$\therefore \frac{-1}{2}\frac{dz}{dx} - z\left(\frac{1}{x}\right) = \frac{1}{x}$$

$$\frac{dz}{dx} + \frac{2}{x}z = -\frac{2}{x} \quad (\text{Linear in } z)$$

$$I.F = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = \boxed{x^2}$$

$$\therefore (z)x^2 = \int\left(\frac{-2}{x}\right)x^2\,dx$$

$$= -2\int x\,dx$$

$$= -2\left(\frac{x^2}{2}\right) + c_1$$

$$z = \frac{c_1 - x^2}{x^2}$$

$$\frac{1}{p^2} = \frac{c_1 - x^2}{x^2}$$

$$p^2 = \frac{x^2}{c_1 - x^2}$$

P.T.O

10.7 – 3

③

$$\int\left(\frac{1}{p}-\frac{p}{(p^2+1)}\right)dp=\int\frac{dx}{x}$$

$$\ln p-\frac{1}{2}\ln(p^2+1)=\ln x+\ln c_1$$

$$\ln p-\ln(p^2+1)^{+1/2}=\ln c_1x$$

$$\ln\left(\frac{p}{\sqrt{p^2+1}}\right)=\ln c_1x$$

$$\Rightarrow\quad\frac{p}{\sqrt{p^2+1}}=c_1x$$

squaring or $\dfrac{p^2}{p^2+1}=c_1^2x^2$

or $p^2=p^2c_1^2x^2+c_1^2x^2$

$$p^2(1-c_1^2x^2)=c_1^2x^2$$

or $p^2=\dfrac{c_1^2x^2}{1-c_1^2x^2}$

or $p=\pm\dfrac{c_1x}{\sqrt{1-c_1^2x^2}}$

$$\frac{dy}{dx}=\pm\frac{c_1x}{c_1\sqrt{\frac{1}{c_1^2}-x^2}}$$

$$dy=\pm\frac{x}{\sqrt{\left(\frac{1}{c_1}\right)^2-x^2}}dx$$

Integrating $\displaystyle\int dy=\pm\int\frac{x}{\sqrt{\left(\frac{1}{c_1}\right)^2-x^2}}dx$

$$p=\pm\frac{x}{\sqrt{c_1-x^2}}$$

$$\frac{dy}{dx}=\pm\frac{x}{\sqrt{c_1-x^2}}$$

$$dy=\pm(c_1-x^2)^{-1/2}x\,dx$$

$$\int dy=\pm\int(c_1-x^2)^{-1/2}\frac{2x}{2}dx$$

$$y=\pm\frac{\sqrt{(c_1-x^2)}}{\cancel{2}\cdot\frac{1}{\cancel{2}}}+c_2$$

$$y=\pm\sqrt{c_1-x^2}+c_2$$

Ans

10.7 - 4

(4) 1057

$$\int dy = \pm\int \left[\left(\frac{1}{c_1}\right)^2 - x^2\right]^{-\frac{1}{2}} x\,dx$$

$$y = \mp\frac{1}{2}\int\left[\left(\frac{1}{c_1}\right)^2 - x^2\right]^{-\frac{1}{2}}(-2x)dx$$

$$= \mp\frac{1}{2}\frac{\left(\left(\frac{1}{c_1^2}\right) - x^2\right)^{\frac{1}{2}}}{1/2} + c_2$$

$$y = \mp\left(\frac{1}{c_1^2} - x^2\right)^{\frac{1}{2}} + c_2$$

$$y - c_2 = \mp\left(\frac{1}{c_1^2} - x^2\right)^{\frac{1}{2}}$$

$$(y - c_2)^2 = \frac{1}{c_1^2} - x^2$$

$$x^2 + (y - c_2)^2 = \frac{1}{c_1^2}$$

Ext ~~Q.No2~~

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2 \quad \text{--- (1)}$$

y-absent.

$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 3x^2$ Cauchy Euler Eq

Put $\frac{dy}{dx} = p$ $\quad \frac{d^2y}{dx^2} = \frac{dp}{dx}$

(1) becomes $x\frac{dp}{dx} - p = 3x^2$

$$\frac{dp}{dx} - \frac{p}{x} = 3x \quad \text{--- (2)}$$

PTO

10.7-5

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or $\frac{dp}{dx} - \frac{1}{x} p = 3x$ ——— (2)

It is a linear diff. eq. in p

$$I.F. = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

Multiply both sides of eq. (2) by I.F. $\frac{1}{x}$

$$\int d\left(p \cdot \frac{1}{x}\right) = \int 3\, dx$$

$$\frac{p}{x} = 3x + c_1$$

$$\therefore \quad p = 3x^2 + c_1 x$$

$$\frac{dy}{dx} = 3x^2 + c_1 x$$

$$\Rightarrow \quad y = \int (3x^2 + c_1 x)\, dx$$

$$\therefore \quad y = x^3 + \frac{c_1 x^2}{2} + c_2$$

$$\therefore \quad y = x^3 + c_1 x^2 + c_2$$

(8) $2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 4 = 0$

Sol. Given that

$$2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 4 = 0 \quad \text{——— (1)}$$

It is independent of y

Put $\frac{dy}{dx} = p$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx}$$

Put in (1)

$$2\frac{dp}{dx} - p^2 + 4 = 0$$

or

$$\frac{dp}{dx} = \frac{p^2 - 4}{2}$$

11.0.7-6

⑥

or $\int \frac{2}{p^2-4}\,dp = \int dx$

$$2\cdot\frac{1}{2\cdot 2}\ln\left(\frac{p-2}{p+2}\right) = x+\ln c \qquad \because \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a}$$

$$\frac{1}{2}\ln\left(\frac{p-2}{p+2}\right) = x+\ln c$$

$$\ln\sqrt{\frac{p-2}{p+2}} = x+\ln c$$

$$\ln\sqrt{\frac{p-2}{p+2}} = \ln e^x+\ln c = \ln ce^x$$

$$\sqrt{\frac{p-2}{p+2}} = ce^x$$

$$\Rightarrow \frac{p-2}{p+2} = c^2e^{2x}$$

$$\text{or } \frac{p+2}{p-2} = \frac{1}{c^2e^{2x}}$$

By Comp & Dividendo

$$\frac{(p+2)+(p-2)}{(p+2)-(p-2)} = \frac{1+c^2e^{2x}}{1-c^2e^{2x}}$$

$$\frac{2p}{4} = \frac{1+c^2e^{2x}}{1-c^2e^{2x}}$$

$$\text{or } \frac{p}{2} = \frac{1+c^2e^{2x}}{1-c^2e^{2x}}$$

$+, -, c^2e^{2x}$

$$\frac{dy}{dx} = 2\left[\frac{(1-c^2e^{2x})+2c^2e^{2x}}{(1-c^2e^{2x})}\right]$$

$$= 2\left[\cancel{\frac{(1-c^2e^{2x})}{(1-c^2e^{2x})}}\right]+\left[\frac{2c^2e^{2x}}{1-c^2e^{2x}}\right]$$

10.7-7

$$\cancel{\int dy = \int 2 + \frac{4c^2e^{2x}}{(1-c^2e^{2x})}dx}$$

$$\int dy = \int 2dx - 2\int \frac{-2c^2e^{2x}}{1-c^2e^{2x}}dx$$

or $y = 2x - 2\ln(1-c^2e^{2x}) + C_2$

$y = 2x - 2\ln(1-c_1e^{2x}) + C_2$

---

④ $x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 12x^3$; $y(1)=0$, $y'(1)=1$, $y''(1)=0$

Sol. Given

$$x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 12x^3 \quad \text{——— (1)}$$

Here y is absent

($\because \frac{d^2y}{dx^2}$ is lowest order in eq(1)) Put $\frac{d^2y}{dx^2} = p$

then $\frac{d^3y}{dx^3} = \frac{dp}{dx}$

Put in (1)

$$x\frac{dp}{dx} - 2p = 12x^3$$

or

$$\frac{dp}{dx} - \frac{2}{x}p = 12x^2 \quad \text{——— (2)}$$

It is a linear diff eq in p

I.F. $= e^{-\int 2/x\, dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$

Multiply both sides of (2) by I.F. $\frac{1}{x^2}$

$$\int d\left(p.\frac{1}{x^2}\right) = \int 12\,dx$$

$$\frac{p}{x^2} = 12x + C$$

or $p = 12x^3 + Cx^2$

$$\frac{d^2y}{dx^2} = 12x^3 + Cx^2$$

~~But $y''(1) = 0$~~

$\Rightarrow \quad 0 = 12 + C \Rightarrow \boxed{C = -12}$

So

$$\frac{d^2y}{dx^2} = 12x^3 - 12x^2$$

$$\Rightarrow \int \frac{d^2y}{dx^2}\,dx = \int (12x^3 - 12x^2)\,dx$$

or $$\frac{dy}{dx} = 3x^4 - 4x^3 + C_2$$

But $y'(1) = 1$

So $1 = 3 - 4 + C_2 \Rightarrow \boxed{C_2 = 2}$

So

$$\frac{dy}{dx} = 3x^4 - 4x^3 + 2$$

$$\Rightarrow \int \frac{dy}{dx}\,dx = \int (3x^4 - 4x^3 + 2)\,dx$$

$$y = \frac{3}{5}x^5 - x^4 + 2x + C_3$$

But $y(1) = 0$

So $0 = \frac{3}{5} - 1 + 2 + C_3$

$0 = \frac{8}{5} + C_3$

$\Rightarrow \boxed{C_3 = -8/5}$

So req. soln. is

$$y = \frac{3}{5}x^5 - x^4 + 2x - \frac{8}{5}$$

$$\therefore y = \frac{1}{5}(3x^5 - 5x^4 + 10x - 8)$$

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10.7

10.7-9

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Ext 5 $\frac{dy}{dx}\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^2=2\left(\frac{d^2y}{dx^2}\right)^2$

Sol. Given

$$\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^2=2\left(\frac{d^2y}{dx^2}\right)^2 \quad (1)$$

Put $\frac{dy}{dx}=p$

then

$$\frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}=p\frac{dp}{dy}$$

&

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(p\frac{dp}{dy}\right)=\frac{d}{dy}\left(p\frac{dp}{dy}\right)\cdot\frac{dy}{dx}$$

$$=\left[p\frac{d^2p}{dy^2}+\left(\frac{dp}{dy}\right)^2\right]p$$

Put in (1)

$$p^2\left[\left(\frac{dp}{dy}\right)^2+p\frac{d^2p}{dy^2}\right]+p^2=2p^2\left(\frac{dp}{dy}\right)^2$$

or

$$\left(\frac{dp}{dy}\right)^2+p\frac{d^2p}{dy^2}+1=2\left(\frac{dp}{dy}\right)^2$$

or $$p\frac{d^2p}{dy^2}=\left(\frac{dp}{dy}\right)^2-1 \quad (2)$$

Again

put $\frac{dp}{dy}=q$

$$\Rightarrow \frac{d^2p}{dy^2}=\frac{dq}{dy}=\frac{dq}{dp}\cdot\frac{dp}{dy}=q\frac{dq}{dp}$$

Put in (2)

$$pq\frac{dq}{dp}=q^2-1$$

$$\int\frac{q}{q^2-1}\,dq=\int\frac{dp}{p}$$

10.7-10

$$\frac{1}{2}\ln(q^2-1) = \ln p + \ln c_1$$

$$\ln\sqrt{q^2-1} = \ln pc_1$$

$$\Rightarrow \sqrt{q^2-1} = pc_1$$

or $$q^2-1 = p^2c_1^2$$

$$q^2 = 1+p^2c_1^2$$

$$q = \sqrt{1+p^2c_1^2}$$

$$\frac{dp}{dy} = \sqrt{1+p^2c_1^2}$$

$$\int\frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dp}{\sqrt{1+p^2c_1^2}} = dy$$

$$\frac{1}{c_1}\int\frac{dp}{\sqrt{\left(\frac{1}{c_1}\right)^2+p^2}} = \int dy$$

$$\frac{1}{c_1}\sinh^{-1}(c_1p) = y + c_2$$

or

$$\sinh^{-1}(c_1p) = c_1y + c_2c_1$$

$$c_1p = \sinh(c_1y+c_1c_2)$$

$$p = \frac{1}{c_1}\sinh(c_1y+c_1c_2)$$

$$\frac{dy}{dx} = \frac{1}{c_1}\sinh(c_1y+c_1c_2)$$

$$\int\frac{dy}{\sinh(c_1y+c_1c_2)} = \frac{1}{c_1}\int dx$$

$$\int \mathrm{Cosech}(c_1y+c_1c_2)\,dy = \frac{1}{c_1}\int dx$$

$$\ln\left[\mathrm{Cosh}(c_1y+c_1c_2) + \tanh(c_1y+c_1c_2)\right] = \frac{1}{c_1}x + c_3$$

10-7-11

Type 2

(5) $2y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 1$

Sol. Given

$$2y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 1 \quad \text{--- (1)}$$

Here x is absent

Put $\frac{dy}{dx} = p$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$$

Put in (1)

$$2yp\frac{dp}{dy} - p^2 = 1$$

$$2yp\frac{dp}{dy} = p^2+1$$

Separating variables

$$\int \frac{2p}{p^2+1}\,dp = \int \frac{dy}{y}$$

$$\ln(p^2+1) = \ln y + \ln c_1$$

$$\ln(p^2+1) = \ln c_1 y$$

$$\Rightarrow \quad p^2+1 = c_1 y$$

or $\quad p^2 = c_1 y - 1$

or

$$p = \pm\sqrt{c_1 y - 1}$$

$$\frac{dy}{dx} = \pm\sqrt{c_1 y - 1}$$

$$\int \frac{dy}{\sqrt{c_1 y - 1}} = \pm\int dx$$

$$\frac{1}{c_1}\int (c_1 y - 1)^{-1/2}(c_1)\,dy = \pm x + c_2$$

$$\frac{1}{c_1}\frac{(c_1y-1)^{1/2}}{1/2} = \pm x + c_2$$

$$\frac{1}{c_1} 2\sqrt{c_1y-1} = \pm x + c_2$$

$$2\sqrt{c_1y-1} = \pm c_1x + c_1c_2$$

$$2\sqrt{c_1y-1} = \pm c_1x + c_2$$

(b) $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 4y^2 \ln y$ ——— $y(1) = e$, $y'(1) = 2e$.

Sol. Given

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 4y^2 \ln y \quad \text{——— (1)}$$

Here x is absent.

Put $\frac{dy}{dx} = p$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$$

Put in (1)

$$yp\frac{dp}{dy} - p^2 = 4y^2\ln y$$

$$yp\,dp - (p^2 + 4y^2\ln y)\,dy = 0 \quad \text{——— (2)}$$

It is non exact diff. eq.

$M = yp$	$N = -p^2 - 4y^2\ln y$
$M_y = p$	$N_p = -2p$

Now

$$\frac{N_p - M_y}{M} = \frac{-2p-p}{yp} = -\frac{3}{y} \quad (\text{a fn. of } y)$$

So

$$\text{I.F.} = e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = e^{\ln y^{-3}} = \frac{1}{y^3}$$

Multiplying eq. (2) by I.F. $= \frac{1}{y^3}$

10.7-13

(11)

$\frac{p}{y^2}dp - \left(\frac{p^2+4y^2\ln y}{y^3}\right)dy = 0$

It is an exact diff. eq.

So $\int M\,dp + \int (\text{terms of } N \text{ free from } p)\,dy = c_1$

$$\int \frac{p}{y^2}dp + \int -4\frac{\ln y}{y}dy = c_1$$

$$\frac{p^2}{2y^2} - 4\left[\frac{(\ln y)^2}{2}\right] = c_1$$

or $\frac{p^2}{2y^2} - 2(\ln y)^2 = c_1$

$$p^2 - 4y^2(\ln y)^2 = 2c_1y^2$$

But $y'(1) = 2e$ & $y(1) = e$

So when $x = 1$, $y = e$, $y' = p = 2e$

So $4e^2 - 4e^2(\ln e)^2 = 2c_1e^2$

$\Rightarrow \boxed{c_1 = 0}$

Hence $p^2 - 4y^2(\ln y)^2 = 0$

$$p^2 = 4y^2(\ln y)^2$$

$$p = 2y\ln y$$

$$\frac{dy}{dx} = 2y\ln y$$

$$\int \frac{1}{y\ln y}dy = 2\int dx$$

$$\ln\ln y = 2x + c_2$$

But $y(1) = e$

So $\ln \ln e = 2 + c_2$

$0 = 2 + c_2$

$\Rightarrow \boxed{c_2 = -2}$

Hence

$\ln \ln y = 2x - 2$

Ext 1

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + 1 \quad \text{'y' absent}$$

Sol. Given

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + 1 \quad \text{——— (1)}$$

Put $\frac{dy}{dx} = p$

$\frac{d^2y}{dx^2} = \frac{dp}{dx}$

Put in (1)

$$\frac{dp}{dx} = p^2 + 1$$

separating

$$\int \frac{1}{p^2+1}\,dp = \int dx$$

$$\tan^{-1} p = x + c_1$$

$$p = \tan(x + c_1)$$

$$\int \frac{dy}{dx}\,dx = \int \tan(x+c_1)\,dx$$

$$y = \ln \sec(x+c_1) + \ln c_2$$

$$y = \ln c_2 \sec(x+c_1)$$

$$\Rightarrow e^y = \frac{c_2}{\cos(x+c_1)}$$

10.7-15

(B)

7. $(1+y^2)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx} = 0$

Sol. Given

$$(1+y^2)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx} = 0 \quad \text{--- (1)}$$

Here x is absent.

Put $\frac{dy}{dx} = p$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

Put in (1)

$$(1+y^2)p\frac{dp}{dy} + p^3 + p = 0$$

$$p\left[(1+y^2)\frac{dp}{dy} + (p^2+1)\right] = 0$$

or $(1+y^2)\frac{dp}{dy} + (p^2+1) = 0$

$$(1+y^2)\frac{dp}{dy} = -(p^2+1)$$

Separating variables $\int \frac{dp}{p^2+1} = -\int \frac{dy}{y^2+1}$

$$\tan^{-1}p = -\tan^{-1}y + c_1$$

$$\tan^{-1}p + \tan^{-1}y = c_1$$

$$\tan^{-1}\left(\frac{p+y}{1-py}\right) = c_1$$

$$\frac{p+y}{1-py} = \tan c_1 = c$$

$$p+y = c - cpy$$

$$p(1+cy) = c-y$$

$$p = \frac{c-y}{1+cy}$$

$$\frac{dy}{dx} = \frac{c-y}{cy+1}$$

$$\frac{dy}{dx} = -\frac{y-c}{cy+1}$$

10.7-16

$$\int \frac{cy+1}{y-c}\,dy = -\int dx$$

$$y-c\,\overline{)\,cy+1}\quad c$$
$$\underline{-cy\quad \mp c^2}$$
$$1+c^2$$

$$\int c + \frac{1+c^2}{y-c}\,dy = -\int dx$$

$$cy + (1+c^2)\ln(y-c) = -x + c_2$$

$$\therefore x = c_2 - cy - (1+c^2)\ln|y-c|$$

$$x = c_1 + c_1 y - (1+c_1^2)\ln|y+c_1| \qquad \text{when } c_1 = -c$$

⑧ $y\dfrac{d^2y}{dx^2} + 4y^2 - \dfrac{1}{2}\left(\dfrac{dy}{dx}\right)^2 = 0$; $y(0) = 1$, $y'(0) = \sqrt{8}$

Sol. Given that

$$y\frac{d^2y}{dx^2} + 4y^2 - \frac{1}{2}\left(\frac{dy}{dx}\right)^2 = 0 \quad \text{——— ①}$$

Here x is absent

Put $\dfrac{dy}{dx} = p$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$$

Put in ①

$$yp\frac{dp}{dy} + 4y^2 - \frac{1}{2}p^2 = 0$$

$$yp\,dp + \left(4y^2 - \frac{1}{2}p^2\right)dy = 0 \quad \text{——— ②}$$

Here

$M = yp$	$N = 4y^2 - \frac{1}{2}p^2$
$M_y = p$	$N_p = -p$

So eq. ② is non exact

Now $\dfrac{N_p - M_y}{M} = \dfrac{-p-p}{yp} = -\dfrac{2}{y}$ (fn. of y).

10.7-17

(15)

So I.F. $= e^{-2\int \frac{1}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = \frac{1}{y^2}$

Multiply both sides of (2) by I.F. $\frac{1}{y^2}$

$$\left(\frac{p}{y}\right)dp + \left(4 - \frac{p^2}{2y^2}\right)dy = 0$$

It is an exact diff. eq.

$$\int M\,dP + \int (\text{terms of N free from p})\,dy = c$$

$$\int \frac{p}{y}\,dp + \int 4\,dy = c$$

$$\frac{p^2}{2y} + 4y = c$$

But $y(0) = 1$ & $y'(0) = \sqrt{8}$

So $\frac{8}{2} + 4 = c$

$\Rightarrow \boxed{c = 8}$

Hence

$$\frac{p^2}{2y} + 4y = 8$$

$$p^2 + 8y^2 = 16y$$

$$p^2 = 16y - 8y^2$$

$$p^2 = 8(2y - y^2)$$

$$p = \sqrt{8}\cdot\sqrt{2y - y^2}$$

$$\frac{dy}{dx} = \sqrt{8}\,\sqrt{2y - y^2}$$

$$\int \frac{dy}{\sqrt{-(y^2 - 2y)}} = \sqrt{8}\int dx$$

$$\int \frac{dy}{\sqrt{-(y^2 - 2y + 1) + 1}} = \sqrt{8}\int dx$$

$$\int \frac{dy}{\sqrt{1 - (y-1)^2}} = \sqrt{8}\int dx$$

10.7-18

$\cancel{\sin^{-1}(y-1) = \sqrt{8}x + C}$

But $y(0) = 1$

So $\sin^{-1}(1-1) = C$

$\Rightarrow \boxed{C=0}$

So $\sin^{-1}(y-1) = \sqrt{8}\,x$

$\Rightarrow \sin^{-1}(y-1) = \sqrt{8}\,x$

$y-1 = \sin(\sqrt{8}\,x)$

or $y = 1 + \sin(\sqrt{8}\,x)$

Example 22

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ax = 0$$

Sol. Given that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ax = 0 \quad \text{——— (1)}$$

Here y is absent

Put $\frac{dy}{dx} = p$

$\frac{d^2y}{dx^2} = \frac{dp}{dx}$

Put in (1)

$$(1+x^2)\frac{dp}{dx} + xp + ax = 0$$

$$(1+x^2)\frac{dp}{dx} = -x(a+p)$$

$$\int\frac{dp}{a+p} = -\int\frac{x}{1+x^2}dx$$

$$\ln(a+p) = -\frac{1}{2}\ln(1+x^2) + \ln c_1$$

$$\ln(a+p) = \ln c_1(1+x^2)^{-1/2}$$

$$\Rightarrow a+p = \frac{c_1}{\sqrt{1+x^2}}$$

10.7-19

~~or $a + \frac{dy}{dx} = \frac{c_1}{\sqrt{1+x^2}}$~~

$$\frac{dy}{dx} = \frac{c_1}{\sqrt{1+x^2}} - a$$

$$\int dy = \int \left(\frac{c_1}{\sqrt{1+x^2}} - a\right) dx$$

$\therefore$ $y = c_1 \sinh^{-1} x - ax + c_2$ is the general soln.

Example 23 $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$

Sol:- Given that

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx} \quad \text{—— (1)}$$

Here x is absent

Put $\frac{dy}{dx} = p$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$$

Put in (1)

$$yp\frac{dp}{dy} + p^2 = p$$

or

$$y\frac{dp}{dy} + p = 1$$

or

$$\frac{dp}{dy} + \frac{1}{y}p = \frac{1}{y} \quad \text{—— (2)}$$

It is a linear diff. eq.

~~As~~ I.F. $= e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

Multiply both sides of (2) by I.F. y

$$\int d(p.y) = \int 1\, dy$$

$py = y + c_1$

or $y\dfrac{dy}{dx} = y + c_1$

$$\int \frac{y}{y+c_1}\,dy = \int dx$$

$$\int \frac{(y+c_1)-c_1}{y+c_1}\,dy = \int dx$$

$$\int 1 - \frac{c_1}{y+c_1}\,dy = \int dx$$

$$y - c_1 \ln(y+c_1) = x + c_2$$

or $y = x + c_1 \ln(y+c_1) + c_2$

Exact linear equations: If we differentiate the linear eq.

$$R_0 y^{(n-1)} + R_1 y^{(n-2)} + \cdots + R_{n-2}\, y' + R_{n-1}\, y = S(x) + c \quad \text{—} (1)$$

where

$R_0, R_1, R_2, \ldots, R_{n-2}, R_{n-1}$ are functions of x,

we obtain another linear eq.

$$P_0 y^{n} + P_1 y^{n-1} + \cdots + P_{n-2}\, y'' + P_{n-1}\, y' + P_n y = Q(x) \quad \text{—} (2)$$

where

$P_0, P_1, P_2, \ldots, P_{n-2}, P_{n-1}, P_n$ are functions of x s.t.

$$\left.\begin{aligned}
Q(x) &= S'(x) \\
P_0 &= R_0 \\
P_1 &= R_0' + R_1 \\
P_2 &= R_1' + R_2 \\
P_3 &= R_2' + R_3 \\
&\ \ \vdots \\
P_{n-1} &= R_{n-2}' + R_{n-1} \\
P_n &= R_{n-1}'
\end{aligned}\right\} \longrightarrow (3)$$

From (3), we obtain

$$S(x) = \int Q(x)\,dx + c$$

$$\boxed{R_0} = P_0$$

$$\boxed{R_1} = P_1 - R_0' = \boxed{P_1 - P_0'}$$

$$\boxed{R_2} = P_2 - R_1' = \boxed{P_2 - P_1' + P_0''}$$

$$\boxed{R_3} = P_3 - R_2' = \boxed{P_3 - P_2' + P_1'' - P_0'''}$$

~~$R_{n-2} = P_{n-2} - R'_{n-3}$~~

$$= P_{n-2} - P'_{n-3} + P''_{n-4} - \cdots + (-1)^{n-2} P_0^{(n-2)}$$

$$R_{n-1} = P_{n-1} - R'_{n-2}$$

$$= P_{n-1} - P'_{n-2} + P''_{n-3} - \cdots + (-1)^{n-1} P_0^{(n-1)}$$

Under this conditions, the linear eq ② of order n is said to be exact & ① is called its first integral.

From above, we obtain

$$\boxed{P_n - R'_{n-1} = P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \cdots + (-1)^n P_0^{(n)} = 0}$$

which is a necessary & sufficient condition for eq ② to be exact.

Linear D.Eq. of Ist order

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$I.F = e^{\int P\,dx}$$

10.7.23

1076 (21)

(4) ~~$(2x^2+3x)\frac{d^2y}{dx^2}+(6x+3)\frac{dy}{dx}+2y=(x+1)e^x$~~

Sol. Given that

$$(2x^2+3x)\frac{d^2y}{dx^2}+(6x+3)\frac{dy}{dx}+2y=(x+1)e^x \quad \text{— (1)}$$

Here

$\boxed{n=2}$, $P_0 = 2x^2+3x$, $P_1 = 6x+3$, $P_2 = 2$, $Q(x) = (x+1)e^x$

Also $\boxed{P_2 - P_1' + P_0''} = 2-6+4 = \boxed{0}$

Hence (1) is exact & its first Integral is $\boxed{R_0\frac{dy}{dx}+R_1y = S(x)}$ — (2)

Now $\boxed{R_0} = P_0 = \boxed{2x^2+3x}$

$\boxed{R_1} = P_1 - P_0' = (6x+3)-(4x+3) = \boxed{2x}$

$$R_0\frac{dy}{dx}+R_1y = S(x)$$

$S(x) = \int Q(x)dx$

$S(x) = \int \underset{I}{(x+1)}\underset{II}{e^x}\,dx$ IBP

from (2) $(2x^2+3x)\frac{dy}{dx}+2xy = (x+1)e^x - \int e^x.1\,dx$

$= (x+1)e^x - e^x + c_1$

$\therefore \quad (2x^2+3x)\frac{dy}{dx}+2xy = xe^x + c_1$

or

$$\frac{dy}{dx}+\frac{2xy}{2x^2+3x} = \frac{xe^x}{2x^2+3x}+\frac{c_1}{2x^2+3x}$$

or $$\frac{dy}{dx}+\frac{2\cancel{x}}{\cancel{x}(2x+3)}y = \frac{\cancel{x}e^x}{\cancel{x}(2x+3)}+\frac{c_1}{x(2x+3)} \quad \text{— (3)}$$ L.D.Eq

L.D.Eq.

$\therefore$ I.F. $= e^{\int\frac{2}{2x+3}dx} = e^{\ln(2x+3)} = 2x+3$

Multiplying both sides of (3) by I.F. $2x+3$

$\therefore \int d(y(2x+3)) = \int e^x\,dx + \int\frac{c_1}{x}\,dx$

$y(2x+3) = e^x + c_1\ln x + c_2$ Ans.

10.7-24 (22)

(19) $\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y\sin x = 0$

Sol. Given that

$$\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y\sin x = 0 \quad \text{--- (1)}$$

Here $\boxed{n=2}$, $P_0 = \sin x$, $P_1 = -\cos x$, $P_2 = 2\sin x$, $Q(x) = 0$

Also

$\boxed{P_2 - P_1' + P_0''} = 2\sin x - \sin x - \sin x = \boxed{0}$

$\therefore$ eq (1) is exact & its first integral is

$$\boxed{R_0 \frac{dy}{dx} + R_1 y = S(x)} \quad \text{--- (2)} \qquad S(x) = \int Q(x) = \int 0\, dx = c_1$$

where

$\boxed{R_0} = P_0 = \boxed{\sin x}$

$\boxed{R_1} = P_1 - P_0' = -\cos x - \cos x = \boxed{-2\cos x}$

from (2) $\sin x \frac{dy}{dx} - 2\cos x \cdot y = c_1$

or

$$\frac{dy}{dx} - 2\cot x \cdot y = \frac{c_1}{\sin x} \quad \text{--- (3)} \quad \text{L.D.Eq}$$

$\therefore$ I.F $= e^{-2\int \cot x} = e^{-2\ln \sin x} = e^{\ln \sin^{-2} x} = \frac{1}{\sin^2 x}$

Multiplying both sides of eq (3) by I.F. $\frac{1}{\sin^2 x}$

$$\int d\left(\frac{y}{\sin^2 x}\right) = c_1 \int \csc^3 x\, dx$$

Now $\int \csc^3 x\, dx = \int \csc x \cdot \csc^2 x\, dx = -\csc x \cot x - \int (-\cot x)(-\csc x \cot x)\, dx$

$= -\csc x \cot x - \int \csc x(\csc^2 x - 1)\, dx$

$\int \csc^3 x\, dx = -\csc x \cot x - \int \csc^3 x\, dx + \int \csc x\, dx$

or $\int \csc^3 x\, dx = \frac{1}{2}\left[-\csc x \cot x + \ln(\csc x - \cot x)\right]$

10-7-25

1078 (23)

$$\text{So } \frac{y}{\sin^2 x} = \cancel{-\frac{\operatorname{Cosec}x\operatorname{Cot}x}{2} + \frac{1}{2}\ln(\operatorname{Cosec}x - \operatorname{Cot}x) + C_2}$$

$$\text{or } y = -\frac{\operatorname{Cot}x \sin x}{2} + \frac{1}{2}\sin^2 x \ln(\operatorname{Cosec}x - \operatorname{Cot}x) + C_2 \sin x$$

(11) $(x+\sin x)\frac{d^3y}{dx^3} + 3(1+\cos x)\frac{d^2y}{dx^2} - 3\sin x\frac{dy}{dx} - y\cos x = -\sin x$

Sol. Given that

$$(x+\sin x)\frac{d^3y}{dx^3} + 3(1+\cos x)\frac{d^2y}{dx^2} - 3\sin x\frac{dy}{dx} - y\cos x = -\sin x \quad \text{--- (1)}$$

Here $\boxed{n=3}$

$P_0 = x+\sin x$, $P_1 = 3(1+\cos x)$, $P_2 = -3\sin x$, $P_3 = -\cos x$

and $Q(x) = -\sin x$

Also

$$\boxed{P_3 - P_2' + P_1'' - P_0'''} = -\cos x + 3\cos x - 3\cos x + \cos x = 0$$

Hence given eq. (1) is exact & its first integral is

$$\boxed{R_0\frac{d^2y}{dx^2} + R_1\frac{dy}{dx} + R_2 y = S(x)} \quad \text{--- (2)} \qquad S(x) = \int Q(x) = \int -\sin x\,dx$$

where

$\boxed{R_0} = P_0 = \boxed{x+\sin x}$

$\boxed{R_1} = P_1 - P_0' = 3+3\cos x - 1 - \cos x = \boxed{2(1+\cos x)}$

$\boxed{R_2} = P_2 - P_1' + P_0'' = -\cancel{3\sin x} + \cancel{3\sin x} - \sin x = \boxed{-\sin x}$

from (2) $(x+\sin x)\frac{d^2y}{dx^2} + 2(1+\cos x)\frac{dy}{dx} - \sin x \cdot y = \cos x + C_1$ --- (3)

Here in (3)

$\boxed{n=2}$, $P_0 = x+\sin x$, $P_1 = 2(1+\cos x)$, $P_2 = -\sin x$ & $Q(x) = \cos x + C_1$

Also $\boxed{P_2 - P_1' + P_0''} = -\sin x + 2\sin x - \sin x = \boxed{0}$ Hence exact

and integral is

$$\boxed{R_0\frac{dy}{dx} + R_1 y = S(x)} \quad \text{--- (4)} \qquad S(x) = \int Q(x)dx = \int(\cos x + C_1)dx$$

~~where $\boxed{R_0} = P_0 = \boxed{x+\sin x}$~~

$\boxed{R_1} = P_1 - P_0 = 2(1+\cos x) - (1+\cos x) = 1+\cos x$

from (4) $(x+\sin x)\dfrac{dy}{dx} + (1+\cos x)y = \sin x + c_1 x + c_2$

$\div$ by $(x+\sin x)$

$$\frac{dy}{dx} + \frac{(1+\cos x)}{(x+\sin x)}y = \frac{\sin x}{x+\sin x} + \frac{c_1 x}{x+\sin x} + \frac{c_2}{x+\sin x} \quad (5)$$

L.D.Eq

(5) is L.D.E

$$\therefore \text{I.F.} = e^{\int \frac{1+\cos x}{x+\sin x}dx} = e^{\ln(x+\sin x)} = (x+\sin x)$$

Multiply both sides of above eq. by I.F. $(x+\sin x)$

$$\int d\left(y(x+\sin x)\right) = \int \sin x + c_1 x + c_2 \, dx$$

$$y(x+\sin x) = -\cos x + c_1\frac{x^2}{2} + c_2 x + c_3$$

10.7-27

(23)

Ext ~~$\sin x \frac{d^3y}{dx^3} + (2\cos x+1)\frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} = \cos x$~~

Sol:- Given that

$$\sin x \frac{d^3y}{dx^3} + (2\cos x+1)\frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} = \cos x \quad \text{—— (1)}$$

Here $\boxed{n=3}$

$P_0 = \sin x,\quad P_1 = 2\cos x+1,\quad P_2 = -\sin x,\quad P_3 = 0,\quad Q(x)=\cos x$

Also $\boxed{P_3 - P_2' + P_1'' - P_0'''} = 0 + \cos x - 2\cos x + \cos x = \boxed{0}$

Hence given eq (1) is exact & its integral is

$$\boxed{R_0 \frac{d^2y}{dx^2} + R_1 \frac{dy}{dx} + R_2 y} = S(x) \quad \text{—— (2)}$$

$S(x) = \int Q(x)\,dx = \int \cos x\,dx$

where

$\boxed{R_0} = P_0 = \sin x$

$\boxed{R_1} = P_1 - P_0' = 2\cos x + 1 - \cos x = 1+\cos x$

$\boxed{R_2} = P_2 - P_1' + P_0'' = -\sin x + 2\sin x - \sin x = 0$

(2) becomes

$$\sin x \frac{d^2y}{dx^2} + (1+\cos x)\frac{dy}{dx} = \sin x + C_1 \quad \text{—— (3)}$$

Here

$n=2,\quad P_0 = \sin x,\quad P_1 = 1+\cos x,\quad P_2 = 0 \quad Q(x) = \sin x + C_1$

Also $\boxed{P_2 - P_1' + P_0''} = 0 + \sin x - \sin x = \boxed{0}$ Hence exact & its integral

Hence eq (3) is exact & its integral is

is $\boxed{R_0 \frac{dy}{dx} + R_1 y = S(x)}$ —— (4)

$S(x) = \int Q(x)\,dx = \int (\sin x + C_1)\,dx = -\cos x + C_1x + C_2$

where

$R_0 = P_0 = \sin x$ & $R_1 = P_1 - P_0' = 1 + \cancel{\cos x} - \cancel{\cos x} = 1$

from (4) $\sin x \frac{dy}{dx} + y = -\cos x + C_1 x + C_2$

$\div$ by $\sin x$

$$\frac{dy}{dx} + \frac{1}{\sin x} y = \frac{-\cos x}{\sin x} + \frac{C_1 x}{\sin x} + \frac{C_2}{\sin x} \quad \text{L.D.Eq (5)}$$

10.7-28 (26)

It is a linear diff. eq.

$$I.F. = e^{\int \operatorname{cosec} x\, dx} = e^{\ln \tan x/2} = \tan x/2$$

Multiply both sides of above eq. ⑤ by I.F. $\tan \frac{x}{2}$

$$\therefore \int d\left(y \tan \frac{x}{2}\right) = \int \tan \frac{x}{2}\left(-\cot x + c_1 x \operatorname{cosec} x + c_2 \operatorname{cosec} x\right) dx + c_3$$

Imp

⑲ $(e^{x}+2x)\frac{d^4y}{dx^4} + (4e^{x}+8)\frac{d^3y}{dx^3} + 6e^{x}\frac{d^2y}{dx^2} + 4e^{x}\frac{dy}{dx} + e^{x}y = \frac{1}{x^5}$

Sol. Given that

$$(e^{x}+2x)\frac{d^4y}{dx^4} + (4e^{x}+8)\frac{d^3y}{dx^3} + 6e^{x}\frac{d^2y}{dx^2} + 4e^{x}\frac{dy}{dx} + e^{x}y = \frac{1}{x^5} \quad (1)$$

Here $\boxed{n = 4}$

$P_0 = e^{x}+2x$, $P_1 = (4e^{x}+8)$, $P_2 = 6e^{x}$; $P_3 = 4e^{x}$, $P_4 = e^{x}$; $Q(x) = \frac{1}{x^5}$

Also

$$\boxed{P_4 - P_3' + P_2'' - P_1''' + P_0''''} = e^{x} - 4e^{x} + 6e^{x} - 4e^{x} + e^{x} = \boxed{0} \text{ Hence exact}$$

and 1st integral is

$$\boxed{R_0 \frac{d^3y}{dx^3} + R_1 \frac{d^2y}{dx^2} + R_2 \frac{dy}{dx} + R_3 y = S(x)} \quad (2) \qquad S(x) = \int Q(x) = \int x^{-5} dx$$

where

$R_0 = P_0 = \boxed{e^{x}+2x}$

$R_1 = P_1 - P_0' = (4e^{x}+8) - (e^{x}+2) = \boxed{3e^{x}+6}$

$R_2 = P_2 - P_1' + P_0'' = 6e^{x} - 4e^{x} + e^{x} = \boxed{3e^{x}}$

$R_3 = P_3 - P_2' + P_1'' - P_0''' = 4e^{x} - 6e^{x} + 4e^{x} - e^{x} = \boxed{e^{x}}$

from ②
$$(e^{x}+2x)\frac{d^3y}{dx^3} + (3e^{x}+6)\frac{d^2y}{dx^2} + 3e^{x}\frac{dy}{dx} + e^{x}y = \frac{x^{-4}}{-4} + c_1 \quad (3)$$

Here $n = 3$, $P_0 = e^{x}+2x$; $P_1 = 3e^{x}+6$, $P_2 = 3e^{x}$, $P_3 = e^{x}$

Also. $\boxed{P_3 - P_2' + P_1'' - P_0'''} = e^{x} - 3e^{x} + 3e^{x} - e^{x} = 0$ $\quad Q(x) = \frac{x^{-4}}{-4} + c_1$

Hence eq. ③ is also exact & its integral is

10.7-29

1062 (27)

$$\boxed{R_0\frac{d^2y}{dx^2}+R_1\frac{dy}{dx}+R_2y=S(x)} \quad (4) \qquad S(x)=\int Q(x)\,dx=\int\left(\frac{x^{-4}}{-4}+c_1\right)dx$$

$$S(x)=\frac{x^{-3}}{12}+c_1x+c_2$$

$\boxed{R_0}=P_0=e^x+2x$

$\boxed{R_1}=P_1-P_0'=(3e^x+6)-(e^x+2)=2e^x+4$

$\boxed{R_2}=P_2-P_1'+P_0''=3e^x-3e^x+e^x=e^x$

from (4) $(e^x+2x)\frac{d^2y}{dx^2}+(2e^x+4)\frac{dy}{dx}+e^xy=\frac{x^{-3}}{12}+c_1x+c_2$ (5)

Again

Here $\boxed{n=2}$, $P_0=e^x+2x$, $P_1=2e^x+4$, $P_2=e^x$, $Q(x)=\frac{x^{-3}}{12}+c_1x+c_2$

Also $\boxed{P_2-P_1'+P_0''}=e^x-2e^x+e^x=\boxed{0}$ Hence exact & its integral is

$$\boxed{R_0\frac{dy}{dx}+R_1y=S(x)} \quad (6) \qquad S(x)=\int Q(x)=\left(\frac{x^{-3}}{12}+c_1x+c_2\right)dx$$

$$=\frac{x^{-2}}{-24}+c_1\frac{x^2}{2}+c_2x+c_3$$

where

$\boxed{R_0}=P_0=e^x+2x$

$\boxed{R_1}=P_1-P_0'=(2e^x+4)-(e^x+2)=e^x+2$

from (6) $(e^x+2x)\frac{dy}{dx}+(e^x+2)y=\frac{x^{-2}}{-24}+c_1\frac{x^2}{2}+c_2x+c_3$

$\div$ by (e^x+2x)

$$\frac{dy}{dx}+\left(\frac{e^x+2}{e^x+2x}\right)y=\frac{\left(\frac{-1}{24x^2}+c_1\frac{x^2}{2}+c_2x+c_3\right)}{(e^x+2x)} \quad (7)$$

L.D.Eq

L.D.Eq

$$\therefore \text{I.F}=e^{\int\frac{e^x+2}{e^x+2x}dx}=e^{\ln(e^x+2x)}=(e^x+2x)$$

Multiply both sides of above eq (7) by I.F (e^x+2x)

$$\int d\left(y(e^x+2x)\right)=\int\frac{-1}{24x^2}+c_1\frac{x^2}{2}+c_2x+c_3\,dx+c_4$$

$$y(e^x+2x)=\frac{1}{24x}+\frac{c_1}{6}x^3+c_2\frac{x^2}{2}+c_3x+c_4$$

Example 24
$\cancel{(x^2+1)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 2\cos x - 2x}$

Sol. Given that

$$(x^2+1)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 2\cos x - 2x \quad \text{—— (1)}$$

Here $n = 2$, $P_0 = x^2+1$, $P_1 = 4x$, $P_2 = 2$

Also $P_2 - P_1' + P_0'' = 2 - 4 + 2 = 0$

Hence given eq (1) is exact & its integral is

$$R_0\frac{dy}{dx} + R_1 y = \int(2\cos x - 2x)\,dx + C_1$$

where

$R_0 = P_0 = x^2+1$

$R_1 = P_1 - P_0' = 4x - 2x = 2x$

So above eq becomes

$$(x^2+1)\frac{dy}{dx} + 2xy = 2\sin x - x^2 + C_1 \quad \text{—— (2)}$$

Also it is a linear diff eq.

Again here $n = 1$, $P_0 = x^2+1$, $P_1 = 2x$

Also $P_1 - P_0' = 2x - 2x = 0$

Hence eq (2) is exact & its integral is

$$R_0 y = \int(2\sin x - x^2 + C_1)\,dx + C_2$$

where

$R_0 = P_0 = x^2+1$

So above eq becomes

$$(x^2+1)y = -2\cos x - \frac{x^3}{3} + C_1 x + C_2$$

10·7–31

(29)

Exmpl 25 ~~$(2x-1)\frac{d^3y}{dx^3}+(4+x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$~~

Soln. Given that

$$(2x-1)\frac{d^3y}{dx^3}+(4+x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0 \quad \text{--- (1)}$$

Here $n = 3$

$P_0 = 2x-1, \quad P_1 = 4+x, \quad P_2 = 2, \quad P_3 = 0 \quad Q(x)=0.$

Also

$\boxed{P_3 - P_2' + P_1'' - P_0'''} = 0 - 0 + 0 - 0 = \boxed{0}$

Hence given eq. (1) is exact & its integral is

$$\boxed{R_0\frac{d^2y}{dx^2}+R_1\frac{dy}{dx}+R_2 y = S(x)} \quad \text{--- (2)}$$

$S(x)=\int Q(x)dx$
$=\int 0\, dx$
$= c_1$

where

$R_0 = P_0 = 2x-1$

$R_1 = P_1 - P_0' = 1+x-2 = 2+x$

$R_2 = P_2 - P_1' + P_0'' = 2-1+0 = 1$

So above eq. becomes

from (2) $$(2x-1)\frac{d^2y}{dx^2}+(2+x)\frac{dy}{dx}+y = c_1 \quad \text{--- (3)}$$

Here

$n = 2, \quad P_0 = 2x-1, \quad P_1 = (2+x) \ \& \ P_2 = 1, \quad Q(x) = c_1$

Also $\boxed{P_2 - P_1' + P_0''} = 1-1+0 = \boxed{0}$

Hence eq. (3) is exact & its integral is

$$\boxed{R_0\frac{dy}{dx}+R_1 y = S(x)} \quad \text{--- (4)}$$

$S(x)=\int Q(x)dx$
$=\int c_1\, dx$
$= c_1 x$

where

$R_0 = P_0 = (2x-1)$

$R_1 = P_1 - P_0' = 2+x-2 = x$

So above eq. (4) becomes

10.7-32 (30)

$(2x-1)\frac{dy}{dx} + xy = C_1x + C_2$ (5)

Here $n=1$, $P_0 = 2x-1$, $P_1 = x$

Also $P_1 - P_0' = x - 2 \neq 0$

Hence given eq. is not exact

Now from eq (5)

$$\frac{dy}{dx} + \left(\frac{x}{2x-1}\right)y = \frac{C_1x + C_2}{2x-1} \quad (6) \quad \text{L.D.Eq}$$

$$\text{I.F.} = e^{\int \frac{x}{2x-1}dx} = e^{\frac{1}{2}\int \frac{(2x-1)+1}{2x-1}dx} = e^{\frac{1}{2}\int 1 + \frac{1}{2x-1}dx} = e^{\frac{1}{2}[x + \frac{1}{2}\ln(2x-1)]}$$

$$= e^{x/2} e^{\frac{1}{4}\ln(2x-1)} = e^{x/2} e^{\ln(2x-1)^{1/4}} = e^{x/2}(2x-1)^{1/4}$$

Multiply both sides of above eq (6) by I.F. $e^{x/2}(2x-1)^{1/4}$

$$d\left(y e^{x/2}(2x-1)^{1/4}\right) = \frac{e^{x/2}(C_1x + C_2)}{(2x-1)^{3/4}}$$

$$\int d\left[y e^{x}(2x-1)^{1/4}\right] = \int \frac{e^{x/2}(C_1x + C_2)}{(2x-1)^{3/4}}dx + C_3$$

or $$y e^{x}(2x-1)^{1/4} = \int \frac{e^{x/2}(C_1x + C_2)}{(2x-1)^{3/4}}dx + C_3$$

Ex 10.7

(2) $$2x\frac{d^3y}{dx^3}\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dx^2}\right)^2 - a^2 \quad \text{--- (1)}$$

Putting $\frac{d^2y}{dx^2} = p$ in (1) & $\frac{d^3y}{dx^3} = \frac{dp}{dx}$

$$2xp\left(\frac{dp}{dx}\right) = p^2 - a^2$$

$$\frac{2p\,dp}{p^2-a^2} = \frac{dx}{x}$$

Integrating $\ln(p^2-a^2) = \ln x + \ln c_1$

$$p^2 - a^2 = c_1 x \quad \Rightarrow p^2 = c_1 x + a^2$$

$$p = \sqrt{c_1 x + a^2}$$

$$\frac{d^2y}{dx^2} = \sqrt{c_1 x + a^2} \quad \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = (c_1 x + a^2)^{\frac{1}{2}}$$

Integrating $\frac{dy}{dx} = \int (c_1 x + a^2)^{\frac{1}{2}} dx + c_2$

$$= \frac{1}{c_1}\int (c_1 x + a^2)^{\frac{1}{2}} c_1\,dx + c_2$$

$$\frac{dy}{dx} = \frac{2}{3c_1}(c_1 x + a^2)^{3/2} + c_2$$

Again Integrating

$$y = \frac{2}{3c_1}\int \frac{1}{c_1}(c_1 x + a^2)^{3/2} c_1\,dx + \int c_2\,dx$$

$$= \frac{2}{3c_1^2}\frac{2}{5}(c_1 x + a^2)^{5/2} + c_2 x + c_3$$

$$y = \frac{4}{15c_1^2}(c_1 x + a^2)^{5/2} + c_2 x + c_3$$

---

(13) $$x^5\frac{d^2y}{dx^2} + 3x^3\frac{dy}{dx} + (3-6x)x^2y = x^4 + 2x - 5 \quad \text{--- (1)}$$

$P_0 = x^5$

$P_1 = 3x^3$

$P_2 = 3x^2 - 6x^3$

$$P_2 - P_1' + P_0'' = 3x^2 - 6x^3 - 9x^2 + 20x^3$$
$$= 14x^3 - 6x^2 \neq 0$$

$\therefore$ (1) is not exact

We Multiply (1) by x^m and choose 'm' so as to make it exact.

$$x^{m+5}\frac{d^2y}{dx^2} + 3x^{m+3}\frac{dy}{dx} + (3-6x)x^{m+2}y = x^m(x^4 + 2x + 5) \quad \text{--- (2)}$$

Now $P_0 = x^{m+5}$

$P_1 = 3x^{m+3}$

$P_2 = (3-6x)x^{m+2}$

$$P_2 - P_1' + P_0'' = 3x^{m+2} - 6x^{m+3} - 3(m+3)x^{m+2} + (m+5)(m+4)x^{m+3}$$
$$= 3x^{m+2} - 6x^{m+3} - 9x^{m+2} - 3mx^{m+2} + m^2x^{m+3} + 9mx^{m+3} + 20x^{m+3}$$

$\boxed{10.7}$ (32) $\boxed{10.7\text{-}34}$

$= m^2 x^{m+3} + 9m x^{m+3} + 14 x^{m+3} - 6x^{m+2} - 3m x^{m+2}$

To make (2) exact put $= 0$.

$x^{m+3}(m^2+9m+14) - x^{m+2}(6x+3m) = 0$

$x^{m+3}(m^2+7m+2m+14) - x^{m+2}\,3(2x+m) = 0$

$x^{m+3}\left[m(m+7)+2(m+7)\right] - x^{m+2}\,3(2+m) = 0$

$x^{m+3}(m+2)(m+7) - x^{m+2}\,3(2+m) = 0$

$(m+2)\left[x^{m+3}(m+7) - 3x^{m+2}\right] = 0$

$m = -2$

Put $m=-2$ in (2)

$$x^3\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (3-6x)y = x^{-2}(x^4+2x-5)$$

$$x^3\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (3-6x)y = x^2 + \frac{2}{x} - \frac{5}{x^2}$$

This is an exact. $\because \begin{cases} P_0 = x^3 \\ P_1 = 3x \\ P_2 = 3-6x \end{cases}$ $\quad P_2 - P_1' + P_0'' = 3-6x-3+6x = 0$

Now first integral is $R_0\frac{dy}{dx} + R_1 y = \int S(x)\,dx$ $\quad S(x) = \int x^2 + \frac{2}{x} - \frac{5}{x^2}$

$R_0 = x^3$

$R_1 = P_1 - P_0'$

$R_1 = 3x - 3x^2$

$$x^3\frac{dy}{dx} + (3x-3x^2)y = \int x^2 + \frac{2}{x} - \frac{5}{x^2}\,dx$$

$$x^3\frac{dy}{dx} + 3x(1-x)y = \frac{x^3}{3} + 2\ln x + \frac{5}{x} + c_1$$

$$\frac{dy}{dx} + \frac{3}{x^2}(1-x)y = \frac{1}{3} + \frac{2\ln x}{x^3} + \frac{5}{x^4} + \frac{c_1}{x^3} \qquad \text{L.D.E}$$

$$I.F = e^{\int\left(\frac{3}{x^2} - \frac{3}{x}\right)dx} = e^{\int 3x^{-2}dx - 3\int\frac{dx}{x}} = e^{-\frac{3}{x} - 3\ln x}$$

$$I.F = e^{\ln\left(\frac{e^{-3/x}}{x^3}\right)} = \frac{1}{x^3}e^{-3/x}$$

$-\frac{3}{x} - 3\ln x$

$= -\frac{3}{x}\ln e - \ln x^3$

$= \ln e^{-\frac{3}{x}} - \ln x^3$

$= \ln\left(\frac{e^{-3/x}}{x^3}\right)$

$$\therefore \int d\left(y\cdot\frac{e^{-3/x}}{x^3}\right) = \int\frac{e^{-3/x}}{x^3}\left(\frac{1}{3} + \frac{2\ln x}{x^3} + \frac{5}{x^4} + \frac{c_1}{x^3}\right)dx$$

$$\frac{y}{x^3 e^{3/x}} = \int\frac{1}{x^3 e^{3/x}}\left(\frac{1}{3} + \frac{2\ln x}{x^3} + \frac{5}{x^4} + \frac{c_1}{x^3}\right)dx$$

$$\Rightarrow y = x^3 e^{\frac{3}{x}}\int\Big(\qquad\qquad \Big)\,dx$$

is the required solution

10.7 (33) 10.7-35

(14) (i) $\dfrac{d^3y}{dx^3} = \ln x$ (Integrate thrice ∵ order 3)

Integrating $\displaystyle\int \frac{d^3y}{dx^3} = \int \ln x\, dx$

I.B.P $\dfrac{d^2y}{dx^2} = \ln x \cdot x - \displaystyle\int \frac{1}{x}\cdot x\, dx$

$\dfrac{d^2y}{dx^2} = x\ln x - x + c_1$

Integrating $\displaystyle\int \frac{d^2y}{dx^2} = \int \underset{\mathrm{II}}{x}\underset{\mathrm{I}}{(\ln x - 1)}\,dx + \int c_1\,dx$

$\dfrac{dy}{dx} = (\ln x - 1)\dfrac{x^2}{2} - \displaystyle\int \frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}}{2}\,dx + c_1 x$

$\dfrac{dy}{dx} = \dfrac{x^2}{2}(\ln x - 1) - \dfrac{x^2}{4} + c_1 x + c_2$

Integrating

$\displaystyle\int \frac{dy}{dx} = \frac{1}{2}\int \underset{\mathrm{II}}{x^2}\underset{\mathrm{I}}{(\ln x - 1)}\,dx - \int \frac{x^2}{4}\,dx + c_1\int x\,dx + \int c_2\,dx$

$y = \dfrac{1}{2}\left(\dfrac{x^3}{3}(\ln x - 1) - \displaystyle\int \frac{1}{x}\,\frac{x^3}{3}\,dx\right) - \dfrac{x^3}{12} + c_1\dfrac{x^2}{2} + c_2 x + c_3$

$y = \dfrac{x^3}{6}(\ln x - 1) - \dfrac{x^3}{18} - \dfrac{x^3}{12} + c_1\dfrac{x^2}{2} + c_2 x + c_3$

$36y = 6x^3(\ln x - 1) - 2x^3 - 3x^3 + 18c_1x^2 + 36c_2x + 36c_3$

$36y = 6x^3\ln x - 11x^3 + c_1' x^2 + c_2' x + c_3'$

(14) (ii) $\dfrac{d^2y}{dx^2} = \underset{\mathrm{I}}{x^2}\,\underset{\mathrm{II}}{\sin x}$ (Integrate twice ∵ order 2)

Integrating $\dfrac{dy}{dx} = x^2(-\cos x) - \displaystyle\int \underset{\mathrm{I}}{2x}\cdot\underset{\mathrm{II}}{(-\cos x)}\,dx + c_1$

$\dfrac{dy}{dx} = -x^2\cos x + 2\left(x\sin x - \displaystyle\int 1\cdot\sin x\,dx\right) + c_1$

$\dfrac{dy}{dx} = -x^2\cos x + 2x\sin x + 2\cos x + c_1$

Integrating

$\displaystyle\int \frac{dy}{dx} = -\int \underset{\mathrm{I}}{x^2}\underset{\mathrm{II}}{\cos x}\,dx + 2\int \underset{\mathrm{I}}{x}\underset{\mathrm{II}}{\sin x}\,dx + 2\int \cos x\,dx + c_1$

$y = -\left(x^2\sin x - \displaystyle\int 2x(+\sin x)\,dx\right) + 2\left(x(-\cos x) - \int -\cos x\,dx\right) + 2\sin x + c_1 x$

$y = -x^2\sin x + 2\displaystyle\int \underset{\mathrm{I}}{x}\underset{\mathrm{II}}{\sin x}\,dx - 2x\cos x + 2\int \cos x\,dx + 2\sin x + c_1 x$

$y = -x^2\sin x + 2\left(x(-\cos x) - \displaystyle\int -\cos x\,dx\right) - 2x\cos x + 2\sin x + 2\sin x + c_1 x$

$y = -x^2\sin x - 4x\cos x + 2\sin x + 4\sin x + c_1 x + c_2$

$y = -x^2\sin x - 4x\cos x + 6\sin x + c_1 x + c_2$

(15) (i) $\dfrac{d^2y}{dx^2} = -\cot y\,\mathrm{cosec}^2 y$ —(1) $y'(0) = 1$ $y(0) = \dfrac{\pi}{2}$ (Integrate twice ∵ order 2)

× (1) by $\dfrac{dy}{dx}$ $\quad \dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right) = +\mathrm{cosec}\,y\,(-\mathrm{cosec}\,y\cot y)\dfrac{dy}{dx}$ $\quad \dfrac{d}{dx}(\mathrm{cosec}\,y) = -\mathrm{cosec}\,y\cot y\,\dfrac{dy}{dx}$

Integrating $\dfrac{1}{2}\left(\dfrac{dy}{dx}\right)^2 = \dfrac{\mathrm{cosec}^2 y}{2} + c_1$ $\quad y'(0) = 1,\ y(0) = \dfrac{\pi}{2}$

$\left(\dfrac{dy}{dx}\right)^2 = +\mathrm{cosec}^2 y + c_1$ $\quad \Rightarrow 1 = 1 + c_1$ or $\boxed{c_1 = 0}$

$\dfrac{dy}{dx} = \mathrm{cosec}\,y \Rightarrow \dfrac{dy}{\mathrm{cosec}\,y} = dx$ Integrating $\Rightarrow \displaystyle\int \sin y\,dy = \int dx$

$\Rightarrow -\cos y = x + c_2$

$y(0) = \dfrac{\pi}{2} \Rightarrow -\cos\dfrac{\pi}{2} = 0 + c_2$

$\boxed{0 = c_2}$

∴ $-\cos y = x$

$\cos y = -x$ is required sol.

K 120 R.S

(ii) $\frac{d^2y}{dx^2} = \frac{-a^2}{y^2}$

Multiply by $\frac{dy}{dx}$ $\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = -\frac{a^2}{y^2}\cdot\frac{dy}{dx} = -a^2(y)^{-2}\frac{dy}{dx}$

Integrating $\frac{\left(\frac{dy}{dx}\right)^2}{2} = -\frac{y^{-1}}{-1}a^2 + c \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{2a^2}{y} + c$

$= \frac{2a^2+yc}{y}$

$= \frac{2a^2\left(1+\frac{c}{2a^2}y\right)}{y}$

$\left(\frac{dy}{dx}\right)^2 = \frac{2a^2(1+c_1y)}{y}$

$\left(\frac{dy}{dx}\right) = \sqrt{2}\,a\frac{\sqrt{1+c_1y}}{\sqrt{y}}$

Let $\sqrt{1+c_1y} = t \Rightarrow 1+c_1y = t^2$

$c_1dy = 2t\,dt$

$dy = \frac{2}{c_1}t\,dt$

$1+c_1y = t^2$

$c_1y = t^2-1$

$y = \frac{t^2-1}{c_1}$

$\int\frac{\sqrt{y}}{\sqrt{1+c_1y}}dy = \sqrt{2}\,a\int dx$

$\int\frac{\sqrt{t^2-1}}{\sqrt{c_1}\,\cancel{t}}\frac{2}{c_1}\cancel{t}\,dt = \sqrt{2}\,ax + c_2$

$\frac{2}{c_1^{3/2}}\int\sqrt{t^2-1}\,dt = \sqrt{2}\,ax + c_2$

$\frac{2}{c_1^{3/2}}\left[\frac{t\sqrt{t^2-1}}{2} - \frac{1}{2}\ln\left(t+\sqrt{t^2-1}\right)\right] = \sqrt{2}\,ax + c_2$

Replace t by $\sqrt{1+c_1y}$

$\frac{2}{c_1}\left[\frac{\sqrt{1+c_1y}\sqrt{c_1y}}{2\sqrt{c_1}} - \frac{1}{2\sqrt{c_1}}\ln\left(\sqrt{1+c_1y}+\sqrt{c_1y}\right)\right] = \sqrt{2}\,ax + c_2$

$\frac{1}{c_1}\left[\frac{\sqrt{c_1y+c_1y^2}\,\cancel{\sqrt{c_1}}}{\cancel{\sqrt{c_1}}} - \frac{1}{\sqrt{c_1}}\ln\left(\sqrt{1+c_1y}+\sqrt{c_1y}\right)\right] = \sqrt{2}\,ax + c_2$

$\sqrt{y+c_1y^2} - \frac{1}{\sqrt{c_1}}\ln\left(\sqrt{1+c_1y}+\sqrt{c_1y}\right) = \sqrt{2}\,axc_1 + c_2'$

is general sol.

THE LAPLACE TRANSFORM

Chapter 11

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Merging Man and maths

Introduction :

Piecewise Continuous :- A real valued function f defined on an interval $[a, b]$ is said to be piecewise continuous in $[a, b]$ if there exists a partition

$$P = \{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$$

of $[a, b]$ such that f is continuous in the interior of each subinterval $[x_i, x_{i+1}]$ and has finite one-sided limits $f(x_i + 0)$ and $f(x_{i+1} - 0)$ at the end points of each subinterval $(i = 0, 1, 2, \ldots, (n-1))$.

The Laplace Transform of f :-

Let f be a real-valued piecewise continuous function defined on $[0, \infty[$. The Laplace transform of f, denoted by $\mathcal{L}(f) = F$, is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad \text{——— (1)}$$

provided the improper integral in (1) converges. The domain of F is the set of all real numbers s for which the above integral converges.

Note that the operation transforms the given function f of the variable t into a new function F of the variable s and is written

Symbolically,

$$F(s) = \mathcal{L}\{f(t)\}.$$

Inverse Laplace transform of F :-

If $F = \mathcal{L}\{f\}$ then the original function f is called the inverse Laplace transform of F and is denoted by $f = \mathcal{L}^{-1}\{F\}$. Clearly,

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\{\mathcal{L}(f)\} = f.$$

Thus if $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Example :-

Let $f(t) = 1$ on $[0, \infty[$. Then

$$\mathcal{L}\{f\} = \mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt$$

$$= \lim_{h\to\infty} \int_0^h e^{-st}\,dt$$

$$= \lim_{h\to\infty} \left[-\frac{e^{-st}}{s}\right]_0^h$$

$$= \lim_{h\to\infty} \left[-\frac{e^{-hs}}{s} + \frac{1}{s}\right]$$

$$= \frac{1}{s} = F(s), \quad \text{provided } s > 0.$$

Example :- Let $f(t) = t^n$, n being a positive integer. Evaluate $\mathcal{L}\{f(t)\}$.

Solution :- Here $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\, t^n\,dt$

Integrating by parts, taking t^n as first function

$$\mathcal{L}\{t^n\} = \left[t^n \cdot \frac{e^{-st}}{-s}\right]_0^\infty + \int_0^\infty n\,t^{n-1} \cdot \frac{e^{-st}}{s}\,dt$$

$$= -\left[\frac{t^n}{s e^{st}}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt$$

$$= \frac{n}{s}\,\mathcal{L}\{t^{n-1}\},$$

Hence $\mathcal{L}\{t^n\} = \frac{n}{s}\,\mathcal{L}\{t^{n-1}\}$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \mathcal{L}\{t^{n-2}\}$$

$$\vdots$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdots \frac{1}{s}\,\mathcal{L}\{1\}$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s}, \quad \left(\text{as } \mathcal{L}\{1\} = \frac{1}{s}\right)$$

$$= \frac{n!}{s^{n+1}} = F(s).$$

Example: Compute $\mathcal{L}\{e^{at}\}$, where a is constant and $s > a$.

Solution: Here $f(t) = e^{at}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\}$$

$$= \int_0^\infty e^{-st} \cdot e^{at}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt \qquad = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty$$

$$= \lim_{h\to\infty}\left[\frac{e^{(a-s)t}}{a-s}\right]_0^h \qquad = \frac{1}{a-s}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$= \lim_{h\to\infty}\left[\frac{e^{(a-s)h}}{a-s} - \frac{1}{a-s}\right]$$

$$= \frac{1}{a-s}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{1}{a-s}\left(e^{-\infty} - e^{0}\right) = \frac{1}{a-s}(0-1) = \frac{-1}{a-s} = \frac{1}{s-a}$$

$$= \begin{cases} \dfrac{1}{s-a} & \text{if } s > a \\ \infty & \text{if } s \le a. \end{cases}$$

Here $e^{-(s-a)h} \longrightarrow 0$ as $h \to \infty$ and $s > a$, while $e^{-(s-a)h} \longrightarrow \infty$ as $h \to \infty$ and $s < a$.

$a - s < 0$
$s - a > 0$
$s > a$

when $s = a$, $f(t) = e^{st}$ and

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot e^{st}\, dt = \int_0^\infty dt$$

$$= \{t\}_0^\infty = \infty.$$

Therefore, $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$.

<u>Example</u>: Find Laplace transforms of (i) $\sin at$, (ii) $\cos at$.

<u>Solution</u>: By definition,

$$\left.\begin{aligned} \mathcal{L}\{\cos at\} &= \int_0^\infty e^{-st} \cos at\, dt \\ \text{and } \mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st} \sin at\, dt. \end{aligned}\right\}$$

Therefore, $\mathcal{L}\{\cos at\} + i\,\mathcal{L}\{\sin at\}$

$$= \int_0^\infty e^{-st} \cos at\, dt + i \int_0^\infty e^{-st} \sin at\, dt$$

$L\{e^{at} \cos bt\}$

$$= \lim_{h\to\infty} \int_0^h e^{-st}\, e^{iat}\, dt = \lim_{h\to\infty} \int_0^h e^{(ia-s)t}\, dt$$

$= \int_0^\infty e^{-st} \cos bt\, dt$

$$= \lim_{h\to\infty} \left[\frac{e^{(ia-s)t}}{ia-s} \right]_0^h = \lim_{h\to\infty} \left[\frac{e^{(ia-s)h}}{ia-s} - \frac{1}{ia-s} \right]$$

$$= \begin{cases} \dfrac{1}{s-ia} & \text{if } s > 0 \\ \text{undefined} & \text{if } s < 0 \end{cases}$$

$$= \frac{s+ia}{s^2+a^2} \quad \text{if } s > 0.$$

Here $\lim\limits_{h\to\infty} \frac{e^{(ia-s)h}}{ia-s} = 0$ for $s>0$ and is undefined for $s<a$.

Equating real and imaginary parts, we get

(i) $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $s>0$

(ii) $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $s>0$

Example: Consider the function f defined by $f(t) = \frac{1}{t}$. For the Laplace transform of $\frac{1}{t}$, we first check the convergence of $\int_0^\infty \frac{e^{-st}}{t} dt$.

$$\int_0^\infty \frac{e^{-st}}{t} dt = \int_0^1 \frac{e^{-st}}{t} dt + \int_1^\infty \frac{e^{-st}}{t} dt$$

For $0 \le t \le 1$, we have $e^{-st} \geq e^{-s}$ if $s>0$.

Therefore, $\int_0^\infty \frac{e^{-st}}{t} dt \geq \int_0^1 \frac{e^{-s}}{t} dt + \int_1^\infty \frac{e^{-st}}{t} dt$

But $\int_0^1 \frac{e^{-s}}{t} dt = e^{-s} \lim\limits_{h\to 0} \left[\ln t\right]_h^1$

$$= \frac{1}{e^s} \lim_{h\to 0} (\ln 1 - \ln h)$$

$$= \infty$$

Hence $\int_0^1 \frac{e^{-st}}{t} dt$ also diverges to ∞. Consequently, $\int_0^\infty \frac{e^{-st}}{t} dt$ diverges and so by definition, $\mathcal{L}\left\{\frac{1}{t}\right\}$ does not exist.

Exponential Order a:

A function f defined on $[0,\infty[$ is said to be of exponential order a as $t \to \infty$. If there exists real constants a, $M>0$ and $T>0$ such that

$$|f(t)| \leq Me^{at} \quad \text{for} \quad t \geq T.$$

Theorem: Let f be a piecewise continuous function defined on $[0,\infty[$. If f is of exponential order a as $t \to \infty$ then $\mathcal{L}\{f(t)\}$ exists for all $s > a$.

Proof: Since f is of exponential order a, there exists positive real numbers M and T such that

$$|f(t)| \leq Me^{at} \quad , t \geq T \quad \text{——— (1)}$$

The theorem will be proved if we show that $\int_0^\infty e^{-st} f(t)\,dt$ converges.

$$\text{Now} \quad \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt \quad \text{(2)}$$

Since f is piecewise continuous, the first integral on the right of (2) exists. Thus the convergence of $\int_0^\infty e^{-st} f(t)\,dt$ depends on the convergence of $\int_T^\infty e^{-st} f(t)\,dt$. But

$$\left| \int_T^\infty e^{-st} f(t)\,dt \right| \leq \int_T^\infty |e^{-st} f(t)|\,dt$$

$$\leq \int_T^\infty e^{-st} Me^{at}\,dt , \quad \text{by (1)}$$

$$= M \int_T^{\infty} e^{(a-s)t}\, dt$$

$$= M \lim_{h\to\infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_T^h$$

$$= \lim_{h\to\infty} M \left[\frac{e^{(a-s)h}}{a-s} - \frac{e^{(a-s)T}}{a-s} \right]$$

$$= \begin{cases} M\left(0 - \dfrac{e^{(a-s)T}}{a-s}\right) & \text{if } a-s<0 \\ \infty & \text{if } a-s \geq 0. \end{cases}$$

Here $\lim\limits_{h\to\infty} \dfrac{e^{(a-s)h}}{a-s} = 0$ if $a-s<0$, and

$\lim\limits_{h\to\infty} \dfrac{e^{(a-s)h}}{a-s} = \infty$ if $a-s \geq 0$.

Thus $\mathcal{L}\{f(t)\}$ exists for all $s>a$.

Note: The Conditions stated above are sufficient but not neccessary. There exists functions which do not satisfy the hypothesis of above theorem.

Example: Consider the function f, defined by
$$f(t) = t^{-1/2}$$
Clearly f is not defined at $t=0$, but it will be shown that $\mathcal{L}\{t^{-1/2}\}$ exists. By definition, we have
$$\mathcal{L}\{t^{-1/2}\} = \int_0^{\infty} e^{-st}\, t^{-1/2}\, dt \quad \text{—— (i)}$$

Let $st = x$. Then $s\,dt = dx$.

or $dt = \frac{1}{s}dx$. So
$t^{-1/2} = \left(\frac{x}{s}\right)^{-1/2} = \sqrt{\frac{s}{x}}$.

Substituting these values of 't' and dt into (1), we have

$$\mathcal{L}\{t^{-1/2}\} = \frac{1}{\sqrt{s}}\int_0^\infty e^{-x}x^{-1/2}dx$$

$$= \frac{1}{\sqrt{s}}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{s}}$$

Thus $\mathcal{L}\{t^{-1/2}\}$ exists.

Properties Of The Laplace Transform:

Theorem: (The linearity property).
Let $f(t) = a\,g(t) + b\,h(t)$, where a, b are constants and $\mathcal{L}\{g(t)\}$ and $\mathcal{L}\{h(t)\}$ exists. Then $\mathcal{L}\{f(t)\}$ exists and

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{a g(t) + b h(t)\}$$

$$= a\mathcal{L}\{g(t)\} + b\mathcal{L}\{h(t)\}$$

Proof: By definition,

$$\mathcal{L}\{a\,g(t) + b\,h(t)\} = \int_0^\infty e^{-st}\{a g(t) + b h(t)\}dt$$

$$= a\int_0^\infty e^{-st}g(t)\,dt + b\int_0^\infty e^{-st}h(t)\,dt$$

$$= a\mathcal{L}\{g(t)\} + b\mathcal{L}\{h(t)\}$$

Theorem: (The Differentiation Formula). Let f be continuous on $[0,\infty[$ and of exponential order a. Let f' be piecewise continuous on every finite closed interval $0 \le t \le b$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof: By previous theorem, $\mathcal{L}\{f'(t)\}$ exists.

Let $F(s) = \mathcal{L}\{f(t)\}$. Then by definition,

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}\{f(t)\}$$

$$= s\,\mathcal{L}\{f(t)\} - f(0)$$

$$= sF(s) - f(0).$$

Corrollary: If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Proof: $\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0),$

$$= s\left[s\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0).$$

<u>Corollary</u>: $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots\cdots - f^{(n-1)}(0).$

<u>Proof</u>: $\mathcal{L}\{f^{(n)}(t)\} = s\,\mathcal{L}\{f^{(n-1)}(t)\} - f^{(n-1)}(0)$

$= s\left[s\{f^{(n-2)}(t)\} - f^{(n-2)}(0)\right] - f^{(n-1)}(0)$

$= s^2\mathcal{L}\{f^{(n-2)}(t)\} - sf^{(n-2)}(0) - f^{(n-1)}(0)$

$= s^2\left[s\,\mathcal{L}\{f^{(n-3)}(t)\} - f^{(n-3)}(0)\right] - sf^{(n-2)}(0) - f^{(n-1)}(0)$

$= s^3\mathcal{L}\{f^{(n-3)}(t)\} - s^2f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0)$

Continuing in this way we get the required result.

<u>Theorem</u>: (First Shifting property)(or Translation property).

Let $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > b$. For any constant a,

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \text{ for } s > a+b.$$

<u>Proof</u>: By definition,

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt$$

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$$

$$= F(s-a) \text{ if } s-a > b$$

or if $s > a+b$.

Example: Compute $\mathcal{L}\{\sinh at\}$ and $\mathcal{L}(\cosh at)$.

Solution: Here $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

we have

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\},$$

$$= \frac{1}{2}\cdot\frac{1}{s-a} - \frac{1}{2}\cdot\frac{1}{s+a}$$

$$= \frac{a}{s^2-a^2}$$

Similarly, $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$

Example: Compute $\mathcal{L}\{\cos^2 at\}$

Solution: Let $f(t) = \cos^2 at$

$$f'(t) = -2a\cos at \sin at$$

$$= -a\sin 2at$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

or $$s\mathcal{L}\{f(t)\} = \mathcal{L}\{f'(t)\} + f(0)$$

i.e., $$s\mathcal{L}\{\cos^2 at\} = -a\mathcal{L}\{\sin 2at\} + f(0)$$

$$= -a\,\frac{2a}{s^2+4a^2} + 1$$

$$= \frac{s^2+2a^2}{s^2+4a^2}$$

Therefore, $$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos^2 at\}$$

$$= \frac{s^2+2a^2}{s(s^2+4a^2)}$$

Example: Evaluate $\mathcal{L}\{e^{3t}(t^3+\sin 2t)\}$.

Solution:

$$\mathcal{L}\{e^{3t}(t^3+\sin 2t)\} = \mathcal{L}\{e^{3t}t^3 + e^{3t}\sin 2t\}$$
$$= \mathcal{L}\{e^{3t}t^3\} + \mathcal{L}\{e^{3t}\sin 2t\}$$

Now $\mathcal{L}\{t^3\} = \dfrac{3!}{s^4}$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

Now $\mathcal{L}\{e^{3t}t^3\} = \dfrac{3!}{(s-3)^4}$

and $\mathcal{L}\{e^{3t}\sin 2t\} = \dfrac{2}{(s-3)^2+4}$

Therefore, $\mathcal{L}\{e^{3t}(t^3+\sin 2t)\} = \dfrac{3!}{(s-3)^4} + \dfrac{2}{(s-3)^2+4}$

Example: Compute $\mathcal{L}\{te^{at}\cos bt\}$.

Solution: Consider $te^{at}e^{ibt} = te^{(a+ib)t}$

Let $f(t) = t$, then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \dfrac{1}{s^2}$.

Now,

$$\mathcal{L}\{te^{(a+ib)t}\} = \frac{1}{[s-(a+ib)]^2} \quad (\because \mathcal{L}(e^{at}f(t)) = F(s-a))$$

$$= \frac{1}{[(s-a)-ib]^2}$$

$$= \frac{[(s-a)+ib]^2}{[\{(s-a)-ib\}\{(s-a)+ib\}]^2}$$

$$= \frac{(s-a)^2 - b^2 + 2ib(s-a)}{[(s-a)^2+b^2]^2}$$

Equating real parts, we have

$$\mathcal{L}\{t e^{at}\cos bt\} = \frac{(s-a)^2 - b^2}{[(s-a)^2 + b^2]^2}$$

Theorem: Suppose $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$. Then $\mathcal{L}\{t f(t)\} = -F'(s)$.

Proof: Consider

$$\frac{d}{ds}\mathcal{L}\{f(t)\} = \frac{d}{ds}F(s)$$

$$= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{d}{ds} e^{-st} f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\mathcal{L}\{t f(t)\}$$

$$F'(s) = -\mathcal{L}\{t f(t)\}$$

or $\mathcal{L}\{t f(t)\} = -F'(s)$ as desired.

Using the above result repeatedly, we have

$$\mathcal{L}\{t^{n} f(t)\} = -\frac{d}{ds}\mathcal{L}\{t^{n-1} f(t)\}$$

$$= (-1)^2 \frac{d^2}{ds^2}\mathcal{L}\{t^{n-2} f(t)\}$$

$$\vdots$$

$$= (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}.$$

Example: Compute $\mathcal{L}\{t^3 e^{-t}\}$.

Solution: Let $f(t) = e^{-t}$. Then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}.$$

Now, $\mathcal{L}\{t^3 e^{-t}\} = (-1)^3 \cdot \frac{d^3}{ds^3} \mathcal{L}\{e^{-t}\}$

$$= -\frac{d^3}{ds^3}\left(\frac{1}{s+1}\right)$$

$$= -\frac{(-1)^3\, 3!}{(s+1)^4}$$

$$= \frac{6}{(s+1)^4}$$

<u>Theorem</u>: If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$ provided $\lim_{t\to\infty} \frac{f(t)}{t}$ exists

exists.

<u>Proof</u>: Let $\frac{f(t)}{t} = g(t)$. Then $f(t) = t\,g(t)$.

Now $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\,g(t)\}$

$$= -\frac{d}{ds}\mathcal{L}\{g(t)\},$$

Integrating, we have

$$\mathcal{L}\{g(t)\} = -\int_\infty^s F(u)\,du$$

$$= \int_s^\infty F(u)\,du.$$

<u>Theorem</u>: If f is piecewise continuous and is of exponential order a, then

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}.$$

<u>Proof</u>: The integral $g(t) = \int_0^t f(u)\,du$

is a continuous function of t. Since $f(t)$ is of exponential order a, $|f(t)| \le Me^{at}$. Therefore,

$$|g(t)| = \left|\int_0^t f(u)\,du\right| \le M\int_0^t e^{au}\,du$$

$$\le \frac{M}{a}\left\{e^{at}-1\right\}.$$

By the fundamental theorem of integral calculus, $g'(t) = f(t)$ except at points where f is discontinuous. Hence $g'(t)$ is piecewise continuous.

$\therefore$ We have $\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\}$

$$= s\mathcal{L}\{g(t)\} - g(0), \quad s > a$$

$$= s\mathcal{L}\{g(t)\}, \text{ since } g(0) = 0.$$

Thus $\mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\}$.

Example: Compute $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$.

Solution: Let $f(t) = \sin t$ so that

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = F(s).$$

Set $g(t) = \frac{\sin t}{t}$, so we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \frac{1}{1+u^2}\,du = \left[\arctan u\right]_s^\infty$$

$$= \frac{\pi}{2} - \text{arc tan } s.$$

Example: Evaluate $\mathcal{L}\left\{\int_0^t \frac{1-\cosh au}{u}\,du\right\}$

Solution: Let $f(t) = 1 - \cosh at$ so that

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{s}{s^2-a^2} = F(s)$$

Set $g(t) = \dfrac{1-\cosh at}{t}$ Then, we get

$$\mathcal{L}\left\{\frac{1-\cosh at}{t}\right\} = \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2-a^2}\right) du$$

$$= \left[\ln u - \frac{1}{2}\ln(u^2-a^2)\right]_s^\infty$$

$$= \lim_{u\to\infty}\left[\frac{1}{2}\ln u^2 - \frac{1}{2}\ln(u^2-a^2)\right]$$

$$+ \frac{1}{2}\ln(s^2-a^2) - \ln s$$

$$= \lim_{u\to\infty} \ln\left(\frac{u^2}{u^2-a^2}\right) + \frac{1}{2}\ln\left(\frac{s^2-a^2}{s^2}\right)$$

$$= \ln \lim_{u\to\infty}\left(\frac{u^2}{u^2-a^2}\right) + \frac{1}{2}\ln\left(\frac{s^2-a^2}{s^2}\right)$$

$$= \ln 1 + \frac{1}{2}\ln\left(\frac{s^2-a^2}{s^2}\right) = \frac{1}{2}\ln\left(\frac{s^2-a^2}{s^2}\right)$$

So $\mathcal{L}\left\{\int_0^t \frac{1-\cosh au}{u}\,du\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{1-\cosh at}{t}\right\}$

$$= \frac{1}{2s}\ln\left(\frac{s^2-a^2}{s^2}\right).$$

Unit Step Function: Let $a \geq 0$. The function u_a defined on $]0,\infty[$ by

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

is called the unit step function. If $a = 0$, then

$$u_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0. \end{cases}$$

Theorem: Let u_a be the unit step function. Then $\mathcal{L}\{u_a(t)\} = \dfrac{e^{-as}}{s}$.

Proof: By definition,

$$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-ts}\, u_a(t)\, dt$$

$$= \int_0^a 0\, e^{-st}\, dt + \int_a^\infty e^{-ts}\, dt = \int_a^\infty e^{-st}\, dt$$

$$= \lim_{h\to\infty} \left[\frac{e^{-ts}}{-s}\right]_a^h$$

$$= \lim_{h\to\infty} -\frac{e^{-hs}}{s} + \frac{e^{-as}}{s}$$

$$= \frac{e^{-as}}{s}, \quad s > 0, \text{ because } \lim_{h\to\infty} \frac{e^{-hs}}{s} = 0.$$

Theorem: Let f be a function of exponential order 'a' and $\mathcal{L}\{f(t)\} = F(s)$. For the function

$$u_a(t)\, f(t-a) = \begin{cases} 0 & \text{if } 0 < t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

$$\mathcal{L}\{u_a(t)\, f(t-a)\} = e^{-as} F(s).$$

Proof: $\mathcal{L}\{u_a(t)\, f(t-a)\} = \int_0^\infty e^{-st}\, u_a(t)\, f(t-a)\, dt$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt \qquad \text{—— (1)}$$

putting $t - a = z$ in (1), we get

$\Rightarrow dt = dz$

$$\mathcal{L}\{u_a(t)\, f(t-a)\} = e^{-as} \int_0^\infty e^{-sz} f(z)\, dz$$

$$= e^{-as} F(s).$$

This is known as the second translation property.

Example: Find the Laplace transform of
1

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{2} \\ \cos t & \text{if } t > \frac{\pi}{2} \end{cases}$$

Solution: By definition,

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{\pi/2} e^{-st} f(t)\, dt + \int_{\pi/2}^\infty e^{-st} f(t)\, dt$$

$$= 0 + \int_{\pi/2}^{\infty} e^{-st}\cos t\,dt$$

$$= \left[\frac{e^{-st}}{-s}\cos t\right]_{\pi/2}^{\infty} + \frac{1}{s}\int_{\pi/2}^{\infty} e^{-st}(-\sin t)\,dt$$

$$= -\frac{1}{s}\int_{\pi/2}^{\infty} e^{-st}\sin t\,dt$$

$$= \frac{1}{s^2}\left[e^{-st}\sin t\right]_{\pi/2}^{\infty} - \frac{1}{s^2}\int_{\pi/2}^{\infty} e^{-st}\cos t\,dt$$

$$= \frac{e^{-\pi/2\, s}}{s^2} - \frac{1}{s^2}F(s)$$

Therefore, $\left(1+\frac{1}{s^2}\right)F(s) = -\frac{e^{-\pi/2\, s}}{s^2}$

or $F(s) = -\frac{e^{-\frac{\pi}{2}s}}{s^2+1}$

Table Of Some Laplace Transforms:

	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
①	1	$\frac{1}{s}$, $s>0$
②	t	$\frac{1}{s^2}$
③	t^n	$\frac{n!}{s^{n+1}}$, $s>0$
④	t^a, $a>-1$	$\frac{\Gamma(\alpha+1)}{s^{a+1}}$
⑤	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$

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$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
(4) e^{at}	$\frac{1}{s-a}$, $s > a$
(5) te^{at}	$\frac{1}{(s-a)^2}$
(6) $\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
(9) $\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
(10) $\sinh at$	$\frac{a}{s^2-a^2}$, $s > \lvert a\rvert$
(11) $\cosh at$	$\frac{s}{s^2-a^2}$, $s > \lvert a\rvert$
(12) $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
(13) $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
(14) $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
(15) $t\sin at$	$\frac{2as}{(s^2+a^2)^2}$, $s > 0$
(16) $t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s > 0$
(17) $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$, $c > 0$
(18) $\int_0^t f(u)\,du$	$\frac{1}{s} F(s)$

(19)	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} F(s)$
(20)	$\frac{f(t)}{t}$ If limits exists, If Laplace exists	$\int_s^{\infty} F(u)\,du$
(21)	$u_a(t)$	$\frac{e^{-as}}{s}$
(22)	$u_a(t) f(t-a)$	$e^{-as} F(s)$
(23)	$\sin at - at\cos at$	$\frac{2a^3}{(s^2+a^2)^2}$
(24)	$f'(t)$	$sF(s) - f(0)$ or $s\mathcal{L}\{f(t)\} - f(0)$
(25)	$1-\cos at$	$\frac{a^2}{s(s^2+a^2)}$
(26)	$at - \sin at$	$\frac{a^3}{s^2(s^2+a^2)}$
(27)	$\sinh at - \sin at$	$\frac{2a^3}{s^4-a^4}$
(28)	$\cosh at - \cos at$	$\frac{2a^2 s}{s^4-a^4}$

(Exercise No. 11.1)

Compute Laplace transform of each of the following (1-28):

Q1 $t^2+6t-17$

Sol. Let $f(t) = t^2+6t-17$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2+6t-17\}$

$$= \mathcal{L}\{t^2\} + \mathcal{L}\{6t\} - \mathcal{L}\{17\}$$

$$= \mathcal{L}\{t^2\} + 6\mathcal{L}\{t\} - 17\mathcal{L}\{1\}$$

$$= \frac{2!}{s^3} + 6\cdot\frac{1!}{s^2} - 17\cdot\frac{1}{s}$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{6}{s^2} - \frac{17}{s} \qquad \text{where } s>0$$

Q2 e^{3t+5}

Sol. Let $f(t) = e^{3t+5}$

$$= e^{3t}\cdot e^{5}$$

$$f(t) = e^{5}e^{3t}$$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{5}\cdot e^{3t}\}$

$$= e^{5}\mathcal{L}\{e^{3t}\}$$

$$= e^{5}\cdot\frac{1}{s-3} \qquad s>3$$

$$= \frac{e^{5}}{s-3}$$

Q3 $\sin(7t+4)$

Sol. Let $f(t) = \sin(7t+4)$

$\Rightarrow f(t) = \sin 7t \cdot \cos 4 + \cos 7t \cdot \sin 4$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 7t \cos 4 + \cos 7t \sin 4\}$

$= \cos 4\, \mathcal{L}\{\sin 7t\} + \sin 4\, \mathcal{L}\{\cos 7t\}$

$= \cos 4 \cdot \dfrac{7}{(s)^2+(7)^2} + \sin 4 \cdot \dfrac{s}{(s)^2+(7)^2} \qquad s>0$

$= \dfrac{7\cos 4}{s^2+49} + \dfrac{s\sin 4}{s^2+49} \qquad s>0$

Q4 $\cos(at+b)$

Sol. Let $f(t) = \cos(at+b)$

$f(t) = \cos at \cos b - \sin at \sin b$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos at \cos b - \sin at \sin b\}$

$= \cos b\, \mathcal{L}\{\cos at\} - \sin b\, \mathcal{L}\{\sin at\}$

$= \cos b \cdot \dfrac{s}{s^2+a^2} - \sin b \cdot \dfrac{a}{s^2+a^2}$

$= \dfrac{s\cos b}{s^2+a^2} - \dfrac{a\sin b}{s^2+a^2}$

Q5 $\cosh(5t-3)$

Sol. Let $f(t) = \cosh(5t-3)$

or $f(t) = \cosh 5t \cosh 3 - \sinh 5t \sinh 3$

then
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cosh 5t \cosh 3 - \sinh 5t \sinh 3\}$$
$$= \cosh 3\,\mathcal{L}\{\cosh 5t\} - \sinh 3\,\mathcal{L}\{\sinh 5t\}$$
$$= \cosh 3 \cdot \frac{s}{s^2-(5)^2} - \sinh 3 \cdot \frac{5}{s^2-(5)^2}$$
$$= \frac{s\cosh 3}{s^2-25} - \frac{5\sinh 3}{s^2-25}$$

Q6 $(t^3-1)e^{-2t}$

Sol: let $f(t) = (t^3-1)e^{-2t}$

or $f(t) = t^3e^{-2t} - e^{-2t}$

then
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^3e^{-2t} - e^{-2t}\}$$
$$= \mathcal{L}\{t^3e^{-2t}\} - \mathcal{L}\{e^{-2t}\}$$
$$= \frac{3!}{(s-(-2))^4} - \frac{1}{s-(-2)} \qquad s > -2$$
$$= \frac{3!}{(s+2)^4} - \frac{1}{(s+2)}$$

Q7 $e^{-t}\sin 2t$

Sol: let $f(t) = e^{-t}\sin 2t$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\sin 2t\}$

$$= \frac{2}{(s-(-1))^2+(2)^2} \qquad s > -1$$

$$\mathcal{L}\{f(t)\} = \frac{2}{(s+1)^2+4}$$

Q8 $e^{3t}\cosh 4t$

Sol. Let $f(t) = e^{3t}\cosh 4t$

then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{3t}\cosh 4t\}$$

$$= \frac{s-3}{(s-3)^2-(4)^2} \qquad \left(\because \mathcal{L}\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}\right)$$

$$= \frac{s-3}{(s-3)^2-16}$$

Q9. $\cos t \cos 2t$

Sol. Let $f(t) = \cos t \cos 2t$

or $f(t) = \frac{1}{2}(2\cos 2t \cos t)$

$$= \frac{1}{2}\left[\cos(2t+t)+\cos(2t-t)\right]$$

$$= \frac{1}{2}\left[\cos 3t + \cos t\right]$$

$$f(t) = \frac{1}{2}\cos 3t + \frac{1}{2}\cos t$$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2}\cos 3t + \frac{1}{2}\cos t\right\}$

$$= \frac{1}{2}\mathcal{L}\{\cos 3t\} + \frac{1}{2}\mathcal{L}\{\cos t\}$$

$$= \frac{1}{2} \cdot \frac{s}{s^2+(3)^2} + \frac{1}{2} \cdot \frac{s}{s^2+(1)^2}$$

$$= \frac{1}{2} \cdot \frac{s}{s^2+9} + \frac{1}{2} \cdot \frac{s}{s^2+1}$$

$$\mathcal{L}\{f(t)\} = \frac{s}{2(s^2+9)} + \frac{s}{2(s^2+1)}$$

Q10. $\sin^3 t$

Sol. Let $f(t) = \sin^3 t$

or $f(t) = \frac{1}{4}(3\sin t - \sin 3t)$

$f(t) = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\}$

$$= \frac{3}{4}\mathcal{L}\{\sin t\} - \frac{1}{4}\mathcal{L}\{\sin 3t\}$$

$$= \frac{3}{4} \frac{1}{s^2+(1)^2} - \frac{1}{4} \cdot \frac{3}{s^2+(3)^2}$$

$$= \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$$

Q11 $te^{-3t}\sin at$

Sol. Let $f(t) = e^{-3t}\sin at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-3t} \cdot \sin at\}$

$$= \frac{a}{(s-(-3))^2+(a)^2}$$

$\mathcal{L}\{f(t)\} = \dfrac{a}{(s+3)^2+a^2} = F(s)$

Now

$$\mathcal{L}\{tf(t)\} = \mathcal{L}\{t e^{-3t}\sin at\}$$

$$= -\frac{d}{ds}\left[\frac{a}{(s+3)^2+(a)^2}\right]$$

$$= -a \cdot \frac{-1}{\left[(s+3)^2+a^2\right]^2} \cdot 2(s+3)$$

$$= \frac{2a(s+3)}{\left[(s+3)^2+a^2\right]^2}$$

Q12 $\sinh^2 at$

Sol. Let $f(t) = \sinh^2 at$

$$= \frac{\cosh 2at - 1}{2} \qquad (\because \cosh 2x = 1 + 2\sinh^2 x)$$

$$f(t) = \frac{1}{2}\cosh 2at - \frac{1}{2}$$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2}\cosh 2at - \frac{1}{2}\right\}$

$$= \frac{1}{2}\mathcal{L}\{\cosh 2at\} - \frac{1}{2}\mathcal{L}\{1\}$$

$$= \frac{1}{2}\cdot\frac{s}{s^2-(2a)^2} - \frac{1}{2}\cdot\frac{1}{s}$$

$$= \frac{1}{2}\left[\frac{s}{s^2-4a^2} - \frac{1}{s}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-(s^2-4a^2)}{s(s^2-4a^2)}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s^2}-\cancel{s^2}+4a^2}{s(s^2-4a^2)}\right]$$

$$\mathcal{L}\{f(t)\} = \frac{2a^2}{s(s^2-4a^2)}$$

<u>Q13.</u> Coshat.Sinat

<u>Soln</u>

Let $f(t) = \text{Cosh}\,at \cdot \text{Sin}\,at$

$$= \left(\frac{e^{at}+e^{-at}}{2}\right)\cdot \text{Sin}\,at$$

$$f(t) = \frac{1}{2}\left(e^{at}\,\text{Sin}\,at + e^{-at}\,\text{Sin}\,at\right)$$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2}\left(e^{at}\,\text{Sin}\,at + e^{-at}\,\text{Sin}\,at\right)\right\}$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\,\text{Sin}\,at\} + \frac{1}{2}\{e^{-at}\,\text{Sin}\,at\}$$

$$= \frac{1}{2}\cdot\frac{a}{(s-a)^2+a^2} + \frac{1}{2}\frac{a}{(s+a)^2+a^2}$$

$$= \frac{a}{2}\left[\frac{1}{(s-a)^2+a^2} + \frac{1}{(s+a)^2+a^2}\right]$$

$$= \frac{a}{2}\left[\frac{(s+a)^2+a^2+(s-a)^2+a^2}{[(s-a)^2+a^2][(s+a)^2+a^2]}\right]$$

$$= \frac{a}{2}\left[\frac{s^2+2\cancel{a}s+a^2+a^2+s^2-2\cancel{a}s+a^2+a^2}{(s-a)^2.(s+a)^2+a^2(s-a)^2+a^2(s+a)^2+a^4}\right]$$

$$= \frac{a}{2}\left[\frac{2s^2+4a^2}{\left[(s-a)(s+a)\right]^2+a^2\left[(s-a)^2+(s+a)^2\right]+a^4}\right]$$

$$= \frac{a}{2}\left[\frac{2s^2+4a^2}{(s^2-a^2)^2+a^2\left(2(s^2+a^2)\right)+a^4}\right]$$

$$= a\left[\frac{s^2+2a^2}{s^4-2\cancel{s^2}a^2+a^4+2\cancel{a^2}s^2+2a^4+a^4}\right]$$

$$= a\left[\frac{s^2+2a^2}{s^4+4a^4}\right]$$

$$\mathcal{L}\{f(t)\} = \frac{a(s^2+2a^2)}{s^4+4a^4}$$

Q14 Sinhat Cosat

Sol. Let $f(t) = \text{Sinh}\,at.\,\text{Cos}\,at$

$$= \left(\frac{e^{at}-e^{-at}}{2}\right)\text{Cos}\,at$$

$$f(t) = \frac{1}{2}\left(e^{at}\text{Cos}\,at - e^{-at}\text{Cos}\,at\right)$$

$$f(t) = \frac{1}{2}e^{at}\text{Cos}\,at - \frac{1}{2}e^{-at}\text{Cos}\,at$$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2}e^{at}\text{Cos}\,at - \frac{1}{2}e^{-at}\text{Cos}\,at\right\}$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\text{Cos}\,at\} - \frac{1}{2}\mathcal{L}\{e^{-at}\text{Cos}\,at\}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\cdot\frac{s-a}{(s-a)^2+a^2} - \frac{1}{2}\cdot\frac{(s+a)}{(s+a)^2+a^2}$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)\left[(s+a)^2+a^2\right] - (s+a)\left[(s-a)^2+a^2\right]}{\left[(s-a)^2+a^2\right]\left[(s+a)^2+a^2\right]}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)\left[s^2+2as+2a^2\right] - (s+a)\left[s^2-2as+2a^2\right]}{(s-a)^2(s+a)^2+a^2(s-a)^2+a^2(s+a)^2+a^4}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s^3}+\cancel{2as^2}+\cancel{2a^2s}-\cancel{as^2}-\cancel{2a^2s}-2a^3-\cancel{s^3}+2as^2-\cancel{2a^2s}-\cancel{as^2}+\cancel{2a^2s}-2a^3}{\left[(s-a)(s+a)\right]^2+a^2\left[(s-a)^2+(s+a)^2\right]+a^4}\right]$$

$$= \frac{1}{2}\left[\frac{2as^2-4a^3}{(s^2-a^2)^2+a^2\left[2(s^2+a^2)\right]+a^4}\right]$$

$$= \frac{as^2-2a^3}{s^4-\cancel{2a^2s^2}+a^4+\cancel{2a^2s^2}+2a^4+a^4}$$

$$\mathcal{L}\{f(t)\} = \frac{a(s^2-2a^2)}{s^4+4a^4}$$

Q15 Coshat . Cosbt

Sol:- Let $f(t) = \cosh at \cdot \cos bt$

$$= \left(\frac{e^{at}+e^{-at}}{2}\right)\cos bt$$

$$f(t) = \frac{1}{2}\left(e^{at}\cos bt + e^{-at}\cos bt\right)$$

then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\tfrac{1}{2}\left(e^{at}\cos bt + e^{-at}\cos bt\right)\right\}$$

$$= \tfrac{1}{2}\left(\mathcal{L}\{e^{at}\cos bt\} + \mathcal{L}\{e^{-at}\cos bt\}\right)$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+b^2} + \frac{s+a}{(s+a)^2+b^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)\left[(s+a)^2+b^2\right] + (s+a)\left[(s-a)^2+b^2\right]}{\left[(s-a)^2+b^2\right]\left[(s+a)^2+b^2\right]}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2+2as+a^2+b^2) + (s+a)(s^2-2as+a^2+b^2)}{(s^2-2as+a^2+b^2)(s^2+2as+a^2+b^2)}\right]$$

$$= \frac{1}{2}\left[\frac{s^3+\cancel{2as^2}+\cancel{a^2s}+sb^2-\cancel{as^2}-2a^2s-\cancel{a^3}-\cancel{ab^2}+s^3-\cancel{2as^2}+\cancel{a^2s}+b^2s+\cancel{as^2}-2a^2s+\cancel{a^3}+\cancel{ab^2}}{\left[(s^2+a^2+b^2)-2as\right]\left[(s^2+a^2+b^2)+2as\right]}\right]$$

$$= \frac{1}{2}\left[\frac{2s^3-2a^2s+2b^2s}{(s^2+a^2+b^2)^2-(2as)^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(s^3-a^2s+b^2s)}{(s^2+a^2+b^2)^2-4a^2s^2}\right]$$

$$\mathcal{L}\{f(t)\} = \frac{s(s^2-a^2+b^2)}{(s^2+a^2+b^2)^2-4a^2s^2}$$

Q16 $\quad t e^{at}\cos bt$

Sol. let $f(t) = e^{at}\cos bt$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\cos bt\}$

$$\mathcal{L}\{f(t)\} = \frac{s-a}{(s-a)^2+b^2} = F(s)$$

Now

$$\mathcal{L}\{t f(t)\} = \mathcal{L}\{t\, e^{at} \cos bt\}$$

$$= -\frac{d}{ds}\left(\frac{s-a}{(s-a)^2+b^2}\right)$$

$$= -\left[\frac{[(s-a)^2+b^2]\cdot 1-(s-a)\cdot 2(s-a)}{[(s-a)^2+b^2]^2}\right]$$

$$= -\left[\frac{(s-a)^2+b^2-2(s-a)^2}{[(s-a)^2+b^2]^2}\right]$$

$$= -\left[\frac{-(s-a)^2+b^2}{[(s-a)^2+b^2]^2}\right]$$

$$= \frac{(s-a)^2-b^2}{[(s-a)^2+b^2]^2}$$

Q17 t^{α}, $\alpha > -1$. Hence find $\mathcal{L}\{t^{5/2}\}$

Sol

Let $f(t) = t^{\alpha}$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^{\alpha}\}$

or $\mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st}\, t^{\alpha}\, dt$

Let $st = u$

$\Rightarrow t = \frac{u}{s}$

$dt = \frac{1}{s} du$

So

So $\mathcal{L}\{t^{\alpha}\} = \int_0^{\infty} e^{-u} \cdot \left(\frac{u}{s}\right)^{\alpha} \cdot \frac{1}{s} du$

$= \frac{1}{s}\int_0^{\infty} e^{-u} \cdot \frac{u^{\alpha}}{s^{\alpha}} du$

$= \frac{1}{s^{\alpha+1}}\int_0^{\infty} e^{-u} \cdot u^{\alpha} du$

$= \frac{1}{s^{\alpha+1}}\int_0^{\infty} u^{(\alpha+1)-1} \cdot e^{-u} du$

$\mathcal{L}\{t^{\alpha}\} = \frac{1}{s^{\alpha+1}} \Gamma(\alpha+1)$ —— Ans. $\left(\because \Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx\right)$

Now $\mathcal{L}\{t^{5/2}\} = \frac{1}{s^{5/2+1}} \Gamma(5/2+1)$

$= \frac{1}{s^{7/2}} \Gamma(7/2)$

$= \frac{1}{s^{7/2}} \cdot \frac{5}{2} \cdot \Gamma(5/2)$ $\quad (\because \Gamma(n) = (n-1)\Gamma(n-1))$

$= \frac{1}{s^{7/2}} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \Gamma(3/2)$

$= \frac{1}{s^{7/2}} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(1/2)$

$= \frac{1}{s^{7/2}} \cdot \frac{15}{8} \cdot \sqrt{\pi}$ $\quad (\because \Gamma(1/2) = \sqrt{\pi})$

$= \frac{15\sqrt{\pi}}{8 s^{7/2}}$

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Q18 $t^2 \sin at$

Sol:- Let $f(t) = t^2 \sin at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 \sin at\}$

$$= (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\sin at\} \qquad \left(\because \mathcal{L}\{t^n \sin at\} = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{\sin at\}\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) \qquad a^2(s^2+a^2)^{-1}$$

$$= \frac{d}{ds}\left[\frac{-a}{(s^2+a^2)^2} \cdot 2s\right] \qquad a(-1)(s^2+a^2)^{-1-1}(2s+$$

$$= \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right] \qquad \frac{-2as}{(s^2+a^2)^2}$$

$$= -2a \frac{d}{ds}\left[\frac{s}{(s^2+a^2)^2}\right], \qquad \frac{v.u' - u.v'}{v^2}$$

$$= -2a\left[\frac{(s^2+a^2)^2 \cdot 1 - s \cdot 2(s^2+a^2)^{2-1} \cdot 2s}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{(s^2+a^2)(s^2+a^2-4s^2)}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{a^2-3s^2}{(s^2+a^2)^3}\right]$$

$$= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

Q19 $t^2 \cos at$

Sol:- Let $f(t) = t^2 \cos at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 \cos at\}$ $\qquad ((s^2+a^2)^2)^2$

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$$\mathcal{L}\{f(t)\} = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\cos at\}$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2.(-2s) - (a^2-s^2).2(s^2+a^2).2s}{(s^2+a^2)^4}$$

$$= \frac{(s^2+a^2)^2(-2s) - 4s(a^2-s^2)(s^2+a^2)}{(s^2+a^2)^4}$$

$$= \frac{(s^2+a^2)(-2s) - 4s(a^2-s^2)}{(s^2+a^2)^3}$$

$$= \frac{-2s^3 - 2a^2s - 4a^2s + 4s^3}{(s^2+a^2)^3}$$

$$= \frac{2s^3 - 6a^2s}{(s^2+a^2)^3}$$

$$= \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

M **Q20** $t\sin^2 at$

Sol: Let $f(t) = t\sin^2 at$

then $f'(t) = 1.\sin^2 at + t.2\sin at.\cos at.a$

$f'(t) = \sin^2 at + 2at.\sin at.\cos at$

$$f'(t) = 2a\sin at\cos at + 2a\left\{1\cdot\sin at\cos at + t\cdot a\cos at\cos at + t\sin at\cdot(-a\sin at)\right\}$$

$$= a(2\sin at\cos at) + a(2\sin at\cos at) + 2a^2 t\cos^2 at - 2a^2 t\sin^2 at$$

$$= a\sin 2at + a\sin 2at + 2a^2t(\cos^2 at - \sin^2 at)$$

$$= 2a\sin 2at + 2a^2t(1 - 2\sin^2 at)$$

$$f''(t) = 2a\sin 2at + 2a^2t - 4a^2t\sin^2 at$$

Using formula

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{2a\sin 2at + 2a^2t - 4a^2t\sin^2 at\} = s^2\mathcal{L}\{f(t)\} - s(0) - 0$$

$$2a\mathcal{L}\{\sin 2at\} + 2a^2\mathcal{L}\{t\} - 4a^2\mathcal{L}\{f(t)\} = s^2\mathcal{L}\{f(t)\}$$

$$2a\cdot\frac{2a}{s^2+(2a)^2} + 2a^2\cdot\frac{1}{s^2} = (4a^2+s^2)\mathcal{L}\{f(t)\}$$

$$\frac{4a^2}{s^2+4a^2} + \frac{2a^2}{s^2} = (4a^2+s^2)\mathcal{L}\{f(t)\}$$

$$(4a^2+s^2)\cdot\mathcal{L}\{f(t)\} = \frac{4a^2}{s^2+4a^2} + \frac{2a^2}{s^2}$$

$$= \frac{4a^2s^2 + 2a^2(s^2+4a^2)}{s^2(s^2+4a^2)}$$

$$(s^2+4a^2)\mathcal{L}\{f(t)\} = \frac{4a^2s^2 + 2a^2s^2 + 8a^4}{s^2(s^2+4a^2)}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{6a^2s^2 + 8a^4}{s^2(s^2+4a^2)^2}$$

$$= \frac{2a^2(3s^2+4a^2)}{s^2(s^2+4a^2)^2}$$

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Q21 $t^2\cos^2 2t$

Sol. Let $f(t) = t^2\cos^2 2t$

$= t^2\left[\frac{1+\cos 4t}{2}\right]$

$f(t) = \frac{1}{2}t^2 + \frac{1}{2}t^2\cos 4t$

$\Rightarrow f'(t) = \frac{1}{2}(2t) + \frac{1}{2}\left[t^2 \cdot -4\sin 4t + 2t\cos 4t\right]$

$f'(t) = t - 2t^2\sin 4t + t\cos 4t$

$f''(t) = 1 - 2\left(t^2 \cdot 4\cos 4t + 2t\sin 4t\right) + t\left(-4\sin 4t\right) + \cos 4t \cdot 1$

$= 1 - 8t^2\cos 4t - 4t\sin 4t - 4t\sin 4t + \cos 4t$

$= 1 + \cos 4t - 8t\sin 4t - 8t^2\left(2\cos^2 2t - 1\right)$

$= 1 + \cos 4t - 8t\sin 4t - 16t^2\cos^2 2t + 8t^2$

$(-1)^n \frac{d}{ds} \mathcal{L}\{\sin 4t$

$f''(t) = 1 + 8t^2 + \cos 4t - 8t\sin 4t - 16t^2\cos^2 2t$

$\mathcal{L}\{f''(t)\} = \mathcal{L}\{1\} + 8\mathcal{L}\{t^2\} + \mathcal{L}\{\cos 4t\} - 8(-1)\frac{d}{ds}\mathcal{L}\{\sin 4t\} - 16\mathcal{L}\{f(t)\}$

$= \frac{1}{s} + 8\cdot\frac{2}{s^3} + \frac{s}{s^2+(4)^2} + 8\frac{d}{ds}\left(\frac{4}{s^2+16}\right) - 16\mathcal{L}\{f(t)\}$

$= \frac{1}{s} + \frac{16}{s^3} + \frac{s}{s^2+16} + 8\cdot\frac{-4\cdot 2s}{(s^2+16)^2} - 16\mathcal{L}\{f(t)\}$

$= \frac{s^2+16}{s^3} + \frac{s}{s^2+16} - \frac{64s}{(s^2+16)^2} - 16\mathcal{L}\{f(t)\}$

$\mathcal{L}\{f''(t)\} = \frac{(s^2+16)^3 + s^4(s^2+16) - 64s^4}{s^3(s^2+16)^2} - 16\mathcal{L}\{f(t)\}$

Using formula

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

Q23 $\dfrac{1-\cos at}{t}$

Sol: Let $f(t) = 1-\cos at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{1-\cos at\}$

$= \mathcal{L}\{1\} - \mathcal{L}\{\cos at\}$

So $\mathcal{L}\{f(t)\} = \dfrac{1}{s} - \dfrac{s}{s^2+a^2} = F(s)$

Now $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\} = \mathcal{L}\left\{\dfrac{1-\cos at}{t}\right\}$

$$= \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2+a^2}\right) du$$

$$= \int_s^\infty \left(\frac{1}{u} - \frac{2u}{2(u^2+a^2)}\right) du$$

$$= \left| \ln u - \frac{1}{2}\ln(u^2+a^2) \right|_s^\infty$$

$$= \left| \ln u - \ln\sqrt{u^2+a^2} \right|_s^\infty$$

$$= \left| \frac{1}{2}\ln u^2 - \frac{1}{2}\ln(u^2+a^2) \right|_s^\infty$$

$$= \frac{1}{2}\left| \ln u^2 - \ln(u^2+a^2) \right|_s^\infty$$

$$= \frac{1}{2}\left| \ln\left(\frac{u^2}{u^2+a^2}\right) \right|_s^\infty$$

$$= \lim_{u\to\infty}\left[\frac{1}{2}\ln\left(\frac{u^2}{u^2+a^2}\right) - \frac{1}{2}\ln\left(\frac{s^2}{s^2+a^2}\right)\right]$$

$$\frac{(s^2+16)^3+s^4(s^2+16)-64s^4}{s^3(s^2+16)^2} - 16\,\mathcal{L}\{f(t)\} = s^2\mathcal{L}\{f(t)\} - s(0) - 0$$

$$\frac{(s^2+16)^3+s^4(s^2+16)-64s^4}{s^3(s^2+16)^2} = (16+s^2)\,\mathcal{L}\{f(t)\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{(s^2+16)^3+s^4(s^2+16)-64s^4}{s^3(s^2+16)^3}$$

Q22 $\dfrac{\sin at}{t}$.

Sol:- Let $f(t) = \sin at$

then $\mathcal{L}\{f(t)\} = \dfrac{a}{s^2+a^2} = F(s)$

then $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\} = \mathcal{L}\left\{\dfrac{\sin at}{t}\right\}$

$$= \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \frac{a}{u^2+a^2}\,du$$

$$= a\int_s^\infty \frac{1}{a^2+u^2}\,du$$

$$= a\left|\frac{1}{a}\tan^{-1}\frac{u}{a}\right|_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \tan^{-1}\left(\frac{a}{s}\right)$$

$$= \lim_{u\to\infty}\left[\frac{1}{2}\ln\left(\frac{1}{1+a^2/u^2}\right) - \frac{1}{2}\ln\left(\frac{s^2}{s^2+a^2}\right)\right]$$

$$= \frac{1}{2}\ln\left(\frac{1}{1+0}\right) - \frac{1}{2}\ln\left(\frac{s^2}{s^2+a^2}\right)$$

$$= 0 - \frac{1}{2}\ln\left(\frac{s^2+a^2}{s^2}\right)^{-1}$$

$$= (-1)\left(-\frac{1}{2}\right)\ln\left(\frac{s^2+a^2}{s^2}\right)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{1}{2}\ln\left(\frac{s^2+a^2}{s^2}\right)$$

<u>Q24</u> $\int_0^t \frac{\sin au}{u}\,du$

<u>Sol</u> Consider $\int_0^t \frac{\sin au}{u}\,du$

First we find $\mathcal{L}\left\{\frac{\sin at}{t}\right\}$

let $f(t) = \sin at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\}$

$$= \frac{a}{s^2+a^2} = F(s)$$

then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \mathcal{L}\left\{\frac{\sin at}{t}\right\}$

$$= \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \frac{a}{u^2+a^2}\,du$$

$$= \left|\tan^{-1}\frac{u}{a}\right|_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}(s/a)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right) \quad \text{———} \quad (1)$$

We know that

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

So $$\mathcal{L}\left\{\int_0^t \frac{\sin au}{u}\,du\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{\sin at}{t}\right\}$$

$$= \frac{1}{s}\tan^{-1}\left(\frac{a}{s}\right) \qquad \text{using } (1)$$

Q25 $\displaystyle\int_0^t \frac{1-\cos au}{u}\,du$

Sol. Consider $\displaystyle\int_0^t \frac{1-\cos au}{u}\,du$

First we will find $\mathcal{L}\left\{\frac{1-\cos at}{t}\right\}$

Let $f(t) = 1-\cos at$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\{1-\cos at\}$

$$= \mathcal{L}\{1\} - \mathcal{L}\{\cos at\}$$

$$= \frac{1}{s} - \frac{s}{s^2+a^2} = F(s)$$

Now

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \mathcal{L}\left\{\frac{1-\cos at}{t}\right\}$$

$$= \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2+a^2}\right)du$$

$$= \int_s^\infty \left(\frac{1}{u} - \frac{2u}{2(u^2+a^2)} \right) du$$

$$= \left| \ln u - \frac{1}{2} \ln(u^2+a^2) \right|_s^\infty$$

$$= \left| \frac{1}{2} \ln u^2 - \frac{1}{2} \ln(u^2+a^2) \right|_s^\infty$$

$$= \frac{1}{2} \left| \ln\left(\frac{u^2}{u^2+a^2}\right) \right|_s^\infty$$

$$= \lim_{u \to \infty} \left[\frac{1}{2} \ln\left(\frac{u^2}{u^2+a^2}\right) - \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2}\right) \right]$$

$$= \lim_{u \to \infty} \left[\frac{1}{2} \ln\left(\frac{1}{1+\frac{a^2}{u^2}}\right) - \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2}\right) \right]$$

$$= \frac{1}{2} \ln\left(\frac{1}{1+0}\right) - \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2}\right)$$

$$= \frac{1}{2}(0) - \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2}\right)$$

$$= -\frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2}\right)$$

$$= \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2}\right)^{-1}$$

$$\mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2} \ln\left(\frac{s^2+a^2}{s^2}\right)$$

We know that

$$\mathcal{L}\left\{ \int_0^t f(u)\,du \right\} = \frac{1}{s} \mathcal{L}\{f(t)\}$$

So

$$\mathcal{L}\left\{ \int_0^t \frac{1-\cos au}{u}\,du \right\} = \frac{1}{s} \cdot \frac{1}{2} \ln\left(\frac{s^2+a^2}{s^2}\right)$$

$$= \frac{1}{2s} \ln\left(\frac{s^2+a^2}{s^2}\right)$$

Q26 : $\frac{\sinh at}{t}$

Sol Let $f(t) = \sinh at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sinh at\}$

$$= \frac{a}{s^2-a^2} = F(s)$$

Now $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \mathcal{L}\left\{\frac{\sinh at}{t}\right\}$

$$= \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \frac{a}{u^2-a^2}\,du$$

$$= a\int_s^\infty \frac{1}{u^2-a^2}\,du$$

$$= a\left|\frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right)\right|_s^\infty$$

$$= \frac{1}{2}\left|\ln\left(\frac{u-a}{u+a}\right)\right|_s^\infty$$

$$= \frac{1}{2}\lim_{u\to\infty}\left[\ln\left(\frac{u-a}{u+a}\right) - \ln\left(\frac{s-a}{s+a}\right)\right]$$

$$= \frac{1}{2}\lim_{u\to\infty}\left[\ln\left(\frac{1-a/u}{1+a/u}\right) - \ln\left(\frac{s-a}{s+a}\right)\right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{1-0}{1+0}\right) - \ln\left(\frac{s-a}{s+a}\right)\right]$$

$$= \frac{1}{2}\left[0 - \ln\left(\frac{s-a}{s+a}\right)\right]$$

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$$= -\frac{1}{2}\ln\left(\frac{s+a}{s-a}\right)^{-1}$$

$$= \frac{1}{2}\ln\left(\frac{s+a}{s-a}\right)$$

Q. $\ln t$

Sol: let $f(t) = \ln t$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\ln t\}$

$$= \int_0^\infty e^{-st}\cdot \ln t \, dt$$

$$= \int_0^\infty e^{-u}\cdot \ln\left(\frac{u}{s}\right)\cdot \frac{1}{s}du$$

Put $st = u$

$t = \frac{u}{s}$

$dt = \frac{1}{s}du$

$dt = \frac{1}{s}du$

$$= \frac{1}{s}\int_0^\infty e^{-u}(\ln u - \ln s)\, du$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\, du - \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln s\, du$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\, du - \frac{\ln s}{s}\int_0^\infty e^{-u} du$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\, du - \frac{\ln s}{s}\left| -e^{-u}\right|_0^\infty$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\, du - \frac{\ln s}{s}\left(-e^{-\infty} + e^{0}\right)$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\, du - \frac{\ln s}{s}(0+1)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\, du - \frac{\ln s}{s} \quad \text{——— (1)}$$

we know that

$$\Gamma(x+1) = \int_0^\infty u^x \cdot e^{-u}\,du$$

Diff. w.r.t. x

$$\Gamma'(x+1) = \frac{d}{dx}\int_0^\infty u^x \cdot e^{-u}\,du$$

$$= \int_0^\infty \frac{d}{dx}(u^x \cdot e^{-u})\,du$$

$$\Gamma'(x+1) = \int_0^\infty (u^x \cdot \ln u) \cdot e^{-u}\,du$$

Put $x = 0$

$$\Gamma'(1) = \int_0^\infty (u^0 \cdot \ln u)\, e^{-u}\,du$$

or $$\Gamma'(1) = \int_0^\infty e^{-u} \cdot \ln u\,du$$

Put in ①

$$\mathcal{L}\{f(t)\} = \frac{1}{s}\Gamma'(1) - \frac{\ln s}{s}$$

Q28 $f(t) = \begin{cases} 0 & \text{if } t<3 \\ (t-3)^3 & \text{if } t>3 \end{cases}$

Sol: Given that

$$f(t) = \begin{cases} 0 & \text{if } t<3 \\ (t-3)^3 & \text{if } t>3 \end{cases}$$

then $f(t) = u_3(t)(t-3)^3$

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then $\mathcal{L}\{f(t)\} = \mathcal{L}\{u_3(t)(t-3)^3\}$

$= e^{-3s}\mathcal{L}\{t^3\}$ $\quad \because \mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

$= e^{-3s}\cdot\frac{3!}{s^4}$

$= \frac{6e^{-3s}}{s^4}$

V.Imp

Q29 If $\mathcal{L}\{f(t)\} = F(s)$ for $s > a$ then show that

$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right)$, $c > 0$ & $s > ca$

Soln

Given that $\mathcal{L}\{f(t)\} = F(s)$

Now $\mathcal{L}\{f(ct)\} = \int_0^\infty e^{-st} f(ct)\,dt$

$= \int_0^\infty e^{-s\left(\frac{T}{c}\right)} f(T)\cdot\frac{1}{c}\,dT$ $\quad$ Put $ct = T$ $\Rightarrow t = \frac{T}{c}$ & $dt = \frac{1}{c}dT$

$= \frac{1}{c}\int_0^\infty e^{\left(-\frac{s}{c}\right)T} f(T)\,dT$

$= \frac{1}{c}\cdot F\left(\frac{s}{c}\right)$ where $\frac{s}{c} > a$ or $s > ca$

Q30 Compute $\mathcal{L}\{\sin\sqrt{t}\}$. Deduce $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Soln Let $f(t) = \sin\sqrt{t}$

$= \sin t^{1/2}$

$$= t^{1/2} - \frac{(t^{1/2})^3}{3!} + \frac{(t^{1/2})^5}{5!} - \cdots$$

$$f(t) = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots$$

Then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{ t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots \right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^{1/2}\} - \frac{1}{3!}\mathcal{L}\{t^{3/2}\} + \frac{1}{5!}\mathcal{L}\{t^{5/2}\} - \cdots$$

We know $\mathcal{L}\{t^{\alpha}\} = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

So
$$\mathcal{L}\{f(t)\} = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \cdots$$

$$= \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}{s^{5/2}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}{s^{7/2}} - \cdots$$

$$= \frac{\Gamma(1/2)}{2s^{3/2}} - \frac{3\Gamma(1/2)}{24s^{5/2}} + \frac{15\Gamma(1/2)}{(120)(8)s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{8s^{5/2}} + \frac{\sqrt{\pi}}{64s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{32s^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \left(\frac{1}{4s}\right) + \frac{\left(-\frac{1}{4s}\right)^2}{2!} - \cdots\right]$$

$$\mathcal{L}\{f(t)\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$$

or
$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}\, e^{-\frac{1}{4s}}}{2s^{3/2}} \quad \text{——— Ans.}$$

Deduction

Here $f(t) = \sin\sqrt{t}$

$$f'(t) = \cos\sqrt{t} \cdot \frac{1}{2\sqrt{t}} = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

Using the formula

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,\mathcal{L}\{\sin\sqrt{t}\} - 0$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,\frac{\sqrt{\pi}\,e^{-\frac{1}{4s}}}{2\,s^{3/2}}$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{2\sqrt{\pi}}{2\sqrt{s}}\,e^{-\frac{1}{4s}}$$

$$= \sqrt{\frac{\pi}{s}}\,e^{-\frac{1}{4s}}$$

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Ex 11·2

Inverse Laplace Transform:

If $F(s)$ is Laplace transform of a function $f(t)$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is denoted by $\mathcal{L}^{-1}\{F(s)\} = f(t)$

Theorem: (Linearity property). If $\mathcal{L}^{-1}\{F_1(s)\} = f_1(t)$ and $\mathcal{L}^{-1}\{F_2(s)\} = f_2(t)$, then

$$\mathcal{L}^{-1}\{aF_1(s) + bF_2(s)\} = a\mathcal{L}^{-1}\{F_1(s)\} + b\mathcal{L}^{-1}\{F_2(s)\}$$

$$= af_1(t) + bf_2(t).$$

Example: Compute $\mathcal{L}^{-1}\left\{\frac{5s}{s^2+5}\right\}$

Solution: Here $\frac{5s}{s^2+5} = 5 \cdot \frac{s}{s^2+(\sqrt{5})^2}$

So from the table of Laplace transforms, we have

$$\mathcal{L}^{-1}\left\{\frac{5s}{s^2+5}\right\} = 5\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{5})^2}\right\}$$

$$= 5\cos\sqrt{5}t.$$

Example: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-15}\right\}$.

Solution: Here $\frac{1}{s^2+2s-15} = \frac{1}{(s+5)(s-3)}$

$$= \frac{A}{(s+5)} + \frac{B}{s-3}$$

$$= \frac{1}{8}\left[\frac{1}{s-3} - \frac{1}{s+5}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-15}\right\} = \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= \frac{1}{8}e^{3t} - \frac{1}{8}e^{-5t}.$$

Example: Compute $\mathcal{L}^{-1}\left\{\frac{3s+17}{s^2+8s+25}\right\}$

Sol: Here $\frac{3s+17}{s^2+8s+25} = \frac{3(s+4)+5}{(s+4)^2+3^2} = 3\frac{s+4}{(s+4)^2+3^2} + 5\frac{1}{(s+4)^2+3^2}$

so $$\mathcal{L}^{-1}\left\{\frac{3s+17}{s^2+8s+25}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+4}{(s+4)^2+3^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2+3^2}\right\}$$

$$= 3e^{-4t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{5}{3}e^{-4t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= 3e^{-4t}\cos 3t + \frac{5}{3}e^{-4t}\sin 3t.$$

Example: Evaluate $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$

Solution: Here $\frac{s^2}{(s^2+a^2)^2} = \frac{1}{s^2+a^2} - \frac{a^2}{(s^2+a^2)^2}$

so $$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} - a^2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$$

$$= \frac{1}{a}\sin at - a^2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} \quad \text{--- (1)}$$

Now $$\mathcal{L}\{t\sin at\} = -\frac{d}{ds}\mathcal{L}\{\sin at\}$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{2as}{(s^2+a^2)^2}.$$

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So $\mathcal{L}^{-1}\left(\frac{2as}{(s^2+a^2)^2}\right) = t\sin at$

or $\quad 2a\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = t\sin at$

Now $\quad 2a\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\cdot\frac{1}{s}\right\}$

$$= \int_0^t u\sin au\,du \quad \left(\because \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du\right)$$

$$= -\frac{t\cos at}{a} + \frac{1}{a^2}\sin at$$

Substituting into (1), we have

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{1}{a}\sin at + \frac{t\cos at}{2} - \frac{\sin at}{2a}$$

$$= \frac{1}{2a}(\sin at + at\cos at).$$

Example: Compute $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$

Solution: Here $\mathcal{L}\{u_2(t)\} = \frac{e^{-2s}}{s}$.

Therefore $\quad \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = \int_0^t u_2(\tau)\,d\tau$

$$= (t-2)\,u_2(t).$$

Again, $\quad \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \int_0^t (\tau-2)\,u_2(\tau)\,d\tau$

$$= \frac{(t-2)^2}{2}\,u_2(t).$$

Alternatively:

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = u_2(t)f(t-2),$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2} = u_2(t)\cdot\frac{(t-2)^2}{2}$.

Example: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}$

Solution: We have $\mathcal{L}^{-1}\left\{\frac{1}{(s^2-a^2)}\right\} = \frac{1}{a}\sinh at$.

Now
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-a^2)}\right\} = \frac{1}{a}\int_0^t \sinh au\,du$$
$$= \frac{1}{a^2}\left[\cosh au\right]_0^t$$
$$= \frac{1}{a^2}\left[\cosh at - 1\right]$$

Applying the same property again, we obtain
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \frac{1}{a^2}\int_0^t (\cosh au - 1)\,du$$
$$= \frac{1}{a^3}\sinh at - \frac{1}{a^2}t.$$

Alternative method:
$$\frac{1}{s^2(s^2-a^2)} = \frac{1}{a^2}\left(\frac{1}{s^2-a^2} - \frac{1}{s^2}\right)$$

Therefore,
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \frac{1}{a^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2-a^2)}\right\} - \frac{1}{a^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$
$$= \frac{1}{a^3}\sinh at - \frac{1}{a^2}t$$

Example: compute $\mathcal{L}^{-1}\left\{\frac{2a^2}{(s^2+a^2)^2}\right\}$.

Solution: We have $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$.

By the Convolution property, we get

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$$\mathcal{L}^{-1}\left\{\frac{2a^2}{(s^2+a^2)^2}\right\} = 2(\sin at * \sin at)$$

$$= 2\int_0^t \sin a(t-u)\sin au\,du$$

$$= 2\int_0^t \{\sin at\cos au - \cos at\sin au\}\sin au\,du$$

$$= \sin at\int_0^t 2\cos au\sin au\,du - 2\cos at\int_0^t \sin^2 au\,du$$

$$= \sin at\int_0^t \sin 2au\,du - \cos at\int_0^t 2\sin^2 au\,du$$

$$= -\frac{\sin at(\cos 2at - 1)}{2a} - \cos at\left[t - \frac{\sin 2at}{2a}\right]$$

$$= \frac{1}{2a}(\sin 2at\cos at - \cos 2at\sin at) - t\cos at + \frac{1}{2a}\sin at$$

$$= \frac{1}{2a}\sin(2at - at) + \frac{1}{2a}\sin at - t\cos at$$

$$= \frac{1}{a}\sin at - t\cos at.$$

Ex: 11·3

Solution Of Initial Value Problems By the Laplace Transform :

In this section we will use the Laplace transform to solve constant co-efficients linear initial value problems. By the methods of Laplace transform, a differential equation can be converted into an algebraic equation. The independent variable will be t instead of x. If $y(t)$ is a solution of a differential equation then the Laplace transform of $y(t)$ will be denoted by $\mathcal{L}\{y(t)\} = Y(s)$.

The following procedure will be adopted:

(I) Given an initial value problem, take Laplace transform of both sides. Use initial conditions to convert the differential equation into an algebraic equation in $Y(s)$.

(II) Solve the algebraic equation for $Y(s)$.

(III) The inverse transform $\mathcal{L}^{-1}\{Y(s)\} = y(t)$ is the required solution of the given problem.

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(Exercise 11.2)

Compute the inverse Laplace transform of each of the following (Problems 1—20):

Q1 $\dfrac{s-2}{s^2-2}$

Sol. Let $F(s) = \dfrac{s-2}{s^2-2}$

$$F(s) = \frac{s}{s^2-2} - \frac{2}{s^2-2}$$

Then $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{ \dfrac{s}{s^2-2} - \dfrac{2}{s^2-2} \right\}$

$$= \mathcal{L}^{-1}\left\{ \frac{s}{s^2-(\sqrt{2})^2} \right\} - \mathcal{L}^{-1}\left\{ \frac{\sqrt{2}\cdot\sqrt{2}}{s^2-(\sqrt{2})^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s}{s^2-(\sqrt{2})^2} \right\} - \sqrt{2}\,\mathcal{L}^{-1}\left\{ \frac{\sqrt{2}}{s^2-(\sqrt{2})^2} \right\}$$

$$= \cosh\sqrt{2}t - \sqrt{2}\sinh\sqrt{2}t$$

Q2 $\dfrac{3s+1}{s^2-6s+18}$

Sol. Let $F(s) = \dfrac{3s+1}{s^2-6s+18}$

$$= \frac{3s+1}{s^2-6s+9+9}$$

$$= \frac{3(s-3)+9+1}{(s-3)^2+9}$$

$$= \frac{3(s-3)+10}{(s-3)^2+(3)^2}$$

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$$= \frac{3(s-3)}{(s-3)^2+(3)^2} + \frac{10}{(s-3)^2+(3)^2}$$

$$F(s) = 3\,\frac{s-3}{(s-3)^2+(3)^2} + \frac{10}{3}\,\frac{3}{(s-3)^2+(3)^2}$$

then $$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{3\,\frac{s-3}{(s-3)^2+(3)^2} + \frac{10}{3}\,\frac{3}{(s-3)^2+(3)^2}\right\}$$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+(3)^2}\right\} + \frac{10}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{(s-3)^2+(3)^2}\right\}$$

$$= 3e^{3t}\cos 3t + \frac{10}{3}e^{3t}\sin 3t$$

Q3 $$\frac{9s-67}{s^2-16s+39}$$

Sol. let $$F(s) = \frac{9s-67}{s^2-16s+64+39-64}$$

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39
25

$$= \frac{9s-67}{(s-8)^2-25}$$

$$= \frac{9(s-8)+72-67}{(s-8)^2-(5)^2}$$

$$= \frac{9(s-8)+5}{(s-8)^2-(5)^2}$$

$$= \frac{9(s-8)}{(s-8)^2-(5)^2} + \frac{5}{(s-8)^2-(5)^2}$$

$$F(s) = 9\,\frac{s-8}{(s-8)^2-(5)^2} + \frac{5}{(s-8)^2-(5)^2}$$

then $$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{9\,\frac{s-8}{(s-8)^2-(5)^2} + \frac{5}{(s-2)^2-(5)^2}\right\}$$

$$= 5\,\mathcal{L}^{-1}\left\{\frac{s-8}{(s-8)^2-(5)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{(s-8)^2-(5)^2}\right\}$$

$$\mathcal{L}^{-1}\{F(s)\} = 5e^{8t}\cosh 5t + e^{8t}\sinh 5t$$

Q4 $\dfrac{as+b}{s^2+2cs+d}$, $d > c^2 > 0$

Sol.

Let $F(s) = \dfrac{as+b}{s^2+2cs+d}$

$$= \frac{as+b}{s^2+2cs+c^2+d-c^2}$$

$$= \frac{as+b}{(s+c)^2+(d-c^2)} = \frac{as+ac+b-ac}{(s+c)^2+(d-c^2)}$$

$$= \frac{a(s+c)+b-ac}{(s+c)^2+(\sqrt{d-c^2})^2}$$

$$= \frac{a(s+c)}{(s+c)^2+(\sqrt{d-c^2})^2} + \frac{b-ac}{(s+c)^2+(\sqrt{d-c^2})^2}$$

$$F(s) = a\frac{s+c}{(s+c)^2+(\sqrt{d-c^2})^2} + \frac{b-ac}{\sqrt{d-c^2}}\cdot\frac{\sqrt{d-c^2}}{(s+c)^2+(\sqrt{d-c^2})^2}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = a\mathcal{L}^{-1}\left\{\frac{s-(-c)}{(s-(-c))^2+(\sqrt{d-c^2})^2}\right\} + \frac{b-ac}{\sqrt{d-c^2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{d-c^2}}{(s-(-c))^2+(\sqrt{d-c^2})^2}\right\}$$

$$= a e^{-ct}\cos\sqrt{d-c^2}\,t + \frac{b-ac}{\sqrt{d-c^2}}e^{-ct}\sin\sqrt{d-c^2}\,t$$

Q5 $\dfrac{s}{(s+a)^2+b^2}$

Sol. Let $F(s) = \dfrac{s}{(s+a)^2+b^2}$

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$$F(s) = \frac{(s+a)-a}{(s+a)^2+b^2}$$

$$F(s) = \frac{s+a}{(s+a)^2+b^2} - \frac{a}{(s+a)^2+b^2}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2} - \frac{a}{b}\frac{b}{(s+a)^2+b^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s-(-a)}{(s-(-a))^2+b^2}\right\} - \frac{a}{b}\mathcal{L}^{-1}\left\{\frac{b}{(s-(-a))^2+b^2}\right\}$$

$$= e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt$$

Q6 $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

$$1 = (As+B)(s^2+a^2) + (Cs+D)(s^2+b^2)$$

$$= As^3 + Aa^2 + Bs^2 + Ba^2 + Cs^3 + Ca^2 + Ds^2 + Da^2$$

Comparing coefficient s^3, s^2, s

$$0 = A + C$$

$$0 = B + D$$

Sol. let $F(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)}$

$$= \frac{1}{(a^2-b^2)}\left[\frac{a^2-b^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{(s^2+a^2)-s^2-b^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{(s^2+a^2)-(s^2+b^2)}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{1}{s^2+b^2} - \frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{b^2-a^2}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$F(s) = \frac{1}{b^2-a^2}\left[\frac{1}{a}\cdot\frac{a}{s^2+a^2} - \frac{1}{b}\cdot\frac{b}{s^2+b^2}\right]$$

$$\text{then } \mathcal{L}^{-1}\{F(s)\} = \frac{1}{b^2-a^2}\left[\frac{1}{a}\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} - \frac{1}{b}\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\}\right]$$

$$= \frac{1}{b^2-a^2}\left\{\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right\}$$

Q7 $\dfrac{1}{(s-1)(s^2+4)}$

Sol. let $F(s) = \dfrac{1}{(s-1)(s^2+4)}$ —— Ⓐ

Consider $\dfrac{1}{(s-1)(s^2+4)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+4}$ —— I

or $1 = A(s^2+4)+(Bs+C)(s-1)$

To find A put $s=1$ in I

$1 = A(1+4)$

$1 = 5A$

$\Rightarrow \boxed{A = \frac{1}{5}}$

from I

$1 = A(s^2+4) + Bs^2 - Bs + Cs - C$

$1 = (A+B)s^2 + (C-B)s + (4A-C)$

Comparing coffs. of s^2 & s

$A+B = 0$ —— ①

$C-B = 0$ —— ②

① $\Rightarrow \frac{1}{5}+B = 0 \Rightarrow \boxed{B = -\frac{1}{5}}$

② $\Rightarrow C+\frac{1}{5} = 0 \Rightarrow \boxed{C = -\frac{1}{5}}$

S.

1402

$$\frac{1}{(s-1)(s^2+4)} = \frac{\frac{1}{5}}{s-1} + \frac{-\frac{1}{5}s-\frac{1}{5}}{s^2+4}$$

or $$\frac{1}{(s-1)(s^2+4)} = \frac{1}{5(s-1)} - \frac{s+1}{5(s^2+4)}$$

So from eq. (A)

$$F(s) = \frac{1}{5(s-1)} - \frac{s+1}{5(s^2+4)}$$

Then

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{s^2+4}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(2)^2}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+(2)^2}\right\}$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t$$

Q8 $$\frac{7s+5}{(3s-8)^2}$$

Sol. Let $$F(s) = \frac{7s+5}{(3s-8)^2}$$

$$= \frac{7s+5}{9\left(s-\frac{8}{3}\right)^2} = \frac{7\left[s-\frac{56}{3}\right]+\frac{56}{3}+5}{9\left(s-\frac{8}{3}\right)^2}$$

$$= \frac{7\left(s-\frac{8}{3}\right)+\frac{56}{3}+5}{9\left(s-\frac{8}{3}\right)^2}$$

$$= \frac{7\left(s-\frac{8}{3}\right)+\frac{71}{3}}{9\left(s-\frac{8}{3}\right)^2}$$

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$$F(S) = \frac{7}{9(S-\frac{8}{3})} + \frac{71}{27(S-\frac{8}{3})^2}$$

Then $\mathcal{L}^{-1}\{F(S)\} = \frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{S-\frac{8}{3}}\right\} + \frac{71}{27}\mathcal{L}^{-1}\left\{\frac{1}{(S-\frac{8}{3})^2}\right\}$

$$= \frac{7}{9}e^{\frac{8}{3}t} + \frac{71}{27}te^{\frac{8}{3}t}$$

Q9 $\frac{5S+3}{(S+7)^5}$

Sol. Let $F(S) = \frac{5S+3}{(S+7)^5} = \frac{5S+35+3-35}{(S+7)^5}$

$$= \frac{5(S+7)+3-35}{(S+7)^5}$$

$$= \frac{5(S+7)-32}{(S+7)^5}$$

$$F(S) = \frac{5}{(S+7)^4} - \frac{32}{(S+7)^5}$$

then $\mathcal{L}^{-1}\{F(S)\} = 5\mathcal{L}^{-1}\left\{\frac{1}{(S+7)^4}\right\} - 32\mathcal{L}^{-1}\left\{\frac{1}{(S+7)^5}\right\}$

$$= \frac{5}{6}\mathcal{L}^{-1}\left\{\frac{3!}{(S-(-7))^4}\right\} - \frac{32}{24}\mathcal{L}^{-1}\left\{\frac{4!}{(S-(-7))^5}\right\}$$

$$= \frac{5}{6}t^3e^{-7t} - \frac{4}{3}t^4e^{-7t}$$

$$= t^3e^{-7t}\left(\frac{5}{6} - \frac{4}{3}t\right)$$

Q10 $\frac{2S-3}{2S^3+3S^2-2S-3}$

Sol. let $F(S) = \frac{2S-3}{2S^3+3S^2-2S-3}$ —— (A)

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Consider $\dfrac{2s-3}{2s^3+3s^2-2s-3}$

$$= \frac{2s-3}{2s(s^2-1)+3(s^2-1)}$$

$$= \frac{2s-3}{(s^2-1)(2s+3)}$$

$$= \frac{2s-3}{(s-1)(s+1)(2s+3)}$$

Now

$$\frac{2s-3}{(s-1)(s+1)(2s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{2s+3}$$

or $2s-3 = A(s+1)(2s+3) + B(s-1)(2s+3) + C(s-1)(s+1)$ —— I

For A, put $s=1$ in I

$2-3 = A(2)(5) \Rightarrow \boxed{A = -\frac{1}{10}}$

For B, put $s=-1$ in I

$-2-3 = B(-2)(1) \Rightarrow \boxed{B = 5/2}$

For C, put $s = -3/2$ in I

$2(-3/2)-3 = C(-3/2-1)(-3/2+1)$

$-6 = C(-5/2)(-1/2)$

$-6 = C\left(\frac{5}{4}\right) \Rightarrow \boxed{C = -\frac{24}{5}}$

So $\dfrac{2s-3}{(s-1)(s+1)(2s+3)} = \dfrac{-1}{10(s-1)} + \dfrac{5}{2(s+1)} - \dfrac{24}{5(2s+3)}$

or $\dfrac{2s-3}{2s^3+3s^2-2s-3} = \dfrac{-1}{10(s-1)} + \dfrac{5}{2(s+1)} - \dfrac{24}{5(2s+3)}$

Put in Ⓐ

$$F(s) = \frac{-1}{10(s-1)} + \frac{5}{2(s+1)} - \frac{24}{5(2s+3)}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = \frac{-1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} - \frac{24}{5}\mathcal{L}^{-1}\left\{\frac{1}{2(s+3/2)}\right\}$$

$$= \frac{-1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} - \frac{12}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-(-3/2)}\right\}$$

$$= \frac{-1}{10}e^{t} + \frac{5}{2}e^{-t} - \frac{12}{5}e^{-3/2 t}$$

Q11 $\dfrac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)}$

Sol.

let $F(s) = \dfrac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)}$ ——— Ⓐ

Consider

$$\frac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)} = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+4s+13}$$

$$\Rightarrow 2s^3+6s^2+21s+52 = A(s+2)(s^2+4s+13) + Bs(s^2+4s+13) + (Cs+D)(s^2+2s)$$

For A put $s = 0$

$$52 = A(2)(13) \Rightarrow \boxed{A = 2}$$

For B, put $s = -2$

$$2(-8)+6(4)-42+52 = B(-2)(4-8+13)$$

$$-16+24+10 = -2B(9)$$

$$18 = -18B \Rightarrow \boxed{B = -1}$$

From above

$$2s^3+6s^2+21s+52 = A(s^3+4s^2+13s+2s^2+8s+26) + B(s^3+4s^2+13s)$$
$$+ Cs^3+2Cs^2+Ds^2+2Ds$$

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Equating Coff of s^3 + s^2

$A+B+C = 2$ ———— ①

$6A+4B+2C+D = 6$ ———— ②

① $\Rightarrow$ $2-1+C = 2$

$1+C = 2$

$\boxed{C=1}$

② $\Rightarrow$ $6(2)+4(-1)+2(1)+D = 6$

$10+D = 6$

$\boxed{D=-4}$

So

$$\frac{2s^3+6s^2+21s+82}{s(s+2)(s^2+4s+13)} = \frac{2}{s} - \frac{1}{s+2} + \frac{s-4}{s^2+4s+13}$$

Put in Ⓐ

$$F(s) = \frac{2}{s} - \frac{1}{s+2} + \frac{s-4}{s^2+4s+13}$$

then

$$\bar{\mathcal{L}}^{1}\{F(s)\} = 2\bar{\mathcal{L}}^{1}\left\{\frac{1}{s}\right\} - \bar{\mathcal{L}}^{1}\left\{\frac{1}{s+2}\right\} + \bar{\mathcal{L}}^{1}\left\{\frac{s-4}{s^2+4s+4+9}\right\}$$

$$= 2\bar{\mathcal{L}}^{1}\left\{\frac{1}{s}\right\} - \bar{\mathcal{L}}^{1}\left\{\frac{1}{s+2}\right\} + \bar{\mathcal{L}}^{1}\left\{\frac{s-4}{(s+2)^2+(3)^2}\right\}$$

$$= 2\bar{\mathcal{L}}^{1}\left\{\frac{1}{s}\right\} - \bar{\mathcal{L}}^{1}\left\{\frac{1}{s-(-2)}\right\} + \bar{\mathcal{L}}^{1}\left\{\frac{(s+2)-6}{(s+2)^2+(3)^2}\right\}$$

$$= 2\bar{\mathcal{L}}^{1}\left\{\frac{1}{s}\right\} - \bar{\mathcal{L}}^{1}\left\{\frac{1}{s-(-2)}\right\} + \bar{\mathcal{L}}^{1}\left\{\frac{s-(-2)}{(s-(-2))^2+(3)^2}\right\} - 2\bar{\mathcal{L}}^{1}\left\{\frac{3}{(s-(-2))^2+(3)^2}\right\}$$

$$= 2(1) - e^{-2t} + e^{-2t}\cos 3t - 2e^{-2t}\sin 3t$$

Q12 $\dfrac{1}{(s^2+4)(s^2+6s-5)}$

Sol.

Let $F(s) = \dfrac{1}{(s^2+4)(s^2+6s-5)}$ ——— (A)

Consider

$$\frac{1}{(s^2+4)(s^2+6s-5)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+6s-5}$$

$$\Rightarrow \quad 1 = (As+B)(s^2+6s-5) + (Cs+D)(s^2+4)$$

$$1 = As^3+6As^2-5As+Bs^2+6Bs-5B+Cs^3+4Cs+Ds^2+4D$$

$$1 = (A+C)s^3+(6A+B+D)s^2+(-5A+6B+4C)s+(-5B+4D)$$

Comparing coffs. on both sides

$$A+C = 0 \quad \text{——— (1)}$$
$$6A+B+D = 0 \quad \text{——— (2)}$$
$$-5A+6B+4C = 0 \quad \text{——— (3)}$$
$$-5B+4D = 1 \quad \text{——— (4)}$$

$4(1)-(3) \Rightarrow 4A+\cancel{4C}+5A-6B-\cancel{4C} = 0$

$$9A-6B = 0$$

or $\quad 3A-2B = 0$ ——— (5)

$4(2)-(4) \Rightarrow 24A+4B+\cancel{4D}+5B-\cancel{4D} = 0-1$

$$24A+9B = -1 \quad \text{——— (6)}$$

$8(5)-(6) \Rightarrow \cancel{24A}-16B-\cancel{24A}-9B = 0+1$

$$-25B = 1 \quad \Rightarrow \quad \boxed{B = \frac{-1}{25}}$$

$(5) \Rightarrow \quad 3A-2\left(\frac{-1}{25}\right) = 0$

$$3A+\frac{2}{25} = 0$$

$$3A = -\frac{2}{25} \quad \Rightarrow \quad \boxed{A = -\frac{2}{75}}$$

① $\Rightarrow \quad -\frac{2}{75} + C = 0 \quad$ or $\boxed{C = \frac{2}{75}}$

④ $\Rightarrow \quad -5\left(\frac{-1}{25}\right) + 4D = 1$

$$\frac{1}{5} + 4D = 1$$

$$4D = 1 - \frac{1}{5}$$

$$4D = \frac{4}{5} \quad \Rightarrow \quad \boxed{D = \frac{1}{5}}$$

So

$$\frac{1}{(s^2+4)(s^2+6s-5)} = \frac{-\frac{2}{75}s - \frac{1}{25}}{s^2+4} + \frac{\frac{2}{75}s + \frac{1}{5}}{s^2+6s-5}$$

$$= \frac{-2s-3}{75(s^2+4)} + \frac{2s+15}{75(s^2+6s-5)}$$

$$= \frac{-2s}{75(s^2+4)} - \frac{3}{75(s^2+4)} + \frac{2s}{75(s^2+6s-5)} + \frac{15}{75(s^2+6s-5)}$$

$$= -\frac{2}{75}\cdot\frac{s}{s^2+4} - \frac{1}{25}\cdot\frac{1}{s^2+4} + \frac{2}{75}\frac{s}{(s^2+6s-5)} + \frac{1}{5}\cdot\frac{1}{s^2+6s-5}$$

$$= -\frac{2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{25}\cdot\frac{1}{s^2+(2)^2} + \frac{2}{75}\frac{s}{s^2+6s+9-14} + \frac{1}{5}\cdot\frac{1}{s^2+6s+9-14}$$

$$= -\frac{2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{25}\cdot\frac{1}{s^2+(2)^2} + \frac{2}{75}\boxed{\frac{(s+3)-3}{(s+3)^2-(\sqrt{14})^2}} + \frac{1}{5}\cdot\frac{1}{(s+3)^2-(\sqrt{14})^2}$$

Put in Ⓐ

$$F(s) = -\frac{2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{25}\cdot\frac{1}{s^2+(2)^2} + \frac{2}{75}\frac{s+3}{(s+3)^2-(\sqrt{14})^2} + \left(-\frac{6}{75}+\frac{1}{5}\right)\frac{1}{(s+3)^2-(\sqrt{14})^2}$$

$$F(s) = -\frac{2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{50}\cdot\frac{2}{s^2+(2)^2} + \frac{2}{75}\cdot\frac{s+3}{(s+3)^2-(\sqrt{14})^2} + \frac{3}{25}\cdot\frac{1}{(s+3)^2-(\sqrt{14})^2}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{2}{75}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(2)^2}\right\} - \frac{1}{50}\mathcal{L}^{-1}\left\{\frac{2}{s^2+(2)^2}\right\} + \frac{2}{75}\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2-(\sqrt{14})^2}\right\} + \frac{3}{25\sqrt{14}}\mathcal{L}^{-1}\left\{\frac{\sqrt{14}}{(s+3)^2-(\sqrt{14})^2}\right\}$$

11·2-19

1409

$$L^{-1}\{F(s)\} = -\frac{2}{75}\cos 2t - \frac{1}{50}\sin 2t + \frac{2}{75}e^{-3t}\cosh(\sqrt{14}\,t) + \frac{3}{25\sqrt{14}}e^{-3t}\sinh\sqrt{14}\,t$$

Q12 $\dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$

Sol. let $F(s) = \dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$ ———— ①

Consider

$$\frac{s^3+3s^2-s-3}{(s^2+2s+5)^2} = \frac{As+B}{s^2+2s+5} + \frac{Cs+D}{(s^2+2s+5)^2}$$

or $s^3+3s^2-s-3 = (As+B)(s^2+2s+5) + (Cs+D)$

$= As^3+2As^2+5As+Bs^2+2Bs+5B+Cs+D$

or $s^3+3s^2-s-3 = As^3+(2A+B)s^2+(5A+2B+C)s+(5B+D)$

Comparing coeffs. on both sides

$A = 1$ ———— I

$2A+B = 3$ ———— II

$5A+2B+C = -1$ ———— III

$5B+D = -3$ ———— IV

I $\Rightarrow$ $\boxed{A=1}$

II $\Rightarrow$ $2+B=3$ $\Rightarrow$ $\boxed{B=1}$

III $\Rightarrow$ $5+2+C=-1$

or $C = -1-7$ or $\boxed{C=-8}$

IV $\Rightarrow$ $5+D=-3$ or $\boxed{D=-8}$

So $\dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2} = \dfrac{s+1}{s^2+2s+5} + \dfrac{-8s-8}{(s^2+2s+5)^2}$

$= \dfrac{s+1}{s^2+2s+5} - \dfrac{8(s+1)}{(s^2+2s+5)^2}$

Put in ①

$$F(s) = \frac{s+1}{s^2+2s+5} - 8\frac{s+1}{(s^2+2s+5)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} - 8L^{-1}\left\{\frac{s+1}{(s^2+2s+5)^2}\right\} \quad \text{——— (A)}$$

now

$$L^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{s+1}{s^2+2s+1+4}\right\}$$

$$= L^{-1}\left\{\frac{s+1}{(s+1)^2+(2)^2}\right\}$$

$$= e^{-t}\cos 2t$$

Consider

$$L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{1}{s^2+2s+1+4}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)^2+(2)^2}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{2}{(s+1)^2+(2)^2}\right\}$$

$$L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \frac{1}{2}e^{-t}\sin 2t$$

using formula

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}(F(s))\right\}$$

$$L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+2s+5}\right)\right\}$$

$$\frac{1}{2}e^{-t}\sin 2t = -\frac{1}{t}L^{-1}\left\{\frac{-1}{(s^2+2s+5)^2}\cdot(2s+2)\right\}$$

$$\frac{1}{2}t\,e^{-t}\sin 2t = L^{-1}\left\{\frac{2s+2}{(s^2+2s+5)^2}\right\}$$

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$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+2s+5)^2}\right\} = \frac{1}{4}te^{-t}\sin 2t$$

Putting values in (A)

$$\ldots(s)\} = e^{-t}\cos 2t - 8\cdot\frac{1}{4}te^{-t}\sin 2t$$

$$= e^{-t}\cos 2t - 2t\,e^{-t}\sin 2t$$

Q14. $\tan^{-1}\left(\frac{a}{s}\right)$

Sol. Let $F(s) = \tan^{-1}\left(\frac{a}{s}\right)$

using formula

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}(F(s))\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\tan^{-1}\left(\frac{a}{s}\right)\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{1+\frac{a^2}{s^2}}\cdot\frac{-a}{s^2}\right\}$$

$$= \cancel{-}\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{\cancel{s^2}}{s^2+a^2}\cdot\frac{\cancel{-}a}{\cancel{s^2}}\right\}$$

$$= \frac{1}{t}\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}$$

$$= \frac{1}{t}\cdot\sin at$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{\sin at}{t}$$

$$\mathcal{L}\{t f(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$$

$$= -\frac{d}{ds}F(s)$$

$$t f(t) = -\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$\frac{a}{s} = a s^{-1}$$

$$= a(-1)s^{-1-1}$$

$$= -\frac{a}{s^2}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$$

14112

7·

Q15 $\ln \frac{s^2+1}{(s-1)^2}$

Sol. Let $F(s) = \ln \frac{s^2+1}{(s-1)^2}$

using formula

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\ln\frac{s^2+1}{(s-1)^2}\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\ln(s^2+1) - 2\ln(s-1)\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s-1}\right\}$$

$$= -\frac{1}{t}\left[2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}\right]$$

$$= -\frac{1}{t}\left[2\cos t - 2e^{t}\right]$$

$$= -\frac{2}{t}\cos t + \frac{2}{t}e^{t}$$

Q16 $\ln \frac{s^2+a^2}{s^2+b^2}$

Sol. Let $F(s) = \ln \frac{s^2+a^2}{s^2+b^2}$

using formula

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}(F(s))\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\ln\frac{s^2+a^2}{s^2+b^2}\right)\right\}$$

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$$\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left[\ln(s^2+a^2)-\ln(s^2+b^2)\right]\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+a^2}-\frac{2s}{s^2+b^2}\right\}$$

$$= -\frac{1}{t}\left[2\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}-2\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\}\right]$$

$$= -\frac{1}{t}\left[2\cos at-2\cos bt\right]$$

$$= -\frac{2}{t}\cos at+\frac{2}{t}\cos bt$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{2}{t}(\cos at-\cos bt)$$

Qn 4m. $\dfrac{e^{-3s}}{s^2(s^2+9)}$ $\qquad \mathcal{L}^{-1}\{e^{-as}F(s)\} = U_a(t)f(t-a)$

Sol. let $F(s) = \dfrac{e^{-3s}}{s^2(s^2+9)}$

$$= e^{-3s}\left[\frac{1}{s^2(s^2+9)}\right]$$

$$= \frac{e^{-3s}}{9}\left[\frac{1}{s^2}-\frac{1}{s^2+9}\right]$$

or $F(s) = \dfrac{1}{9}\cdot\dfrac{e^{-3s}}{s^2}-\dfrac{1}{9}\dfrac{e^{-3s}}{s^2+9}$

Then $\mathcal{L}^{-1}\{F(s)\} = \dfrac{1}{9}\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2}\right\}-\dfrac{1}{27}\mathcal{L}^{-1}\left\{e^{-3s}\cdot\dfrac{3}{s^2+(3)^2}\right\}$ $\qquad \frac{1}{9}U_3(t)\mathcal{L}^{-1}\{\frac{1}{s^2}\}_{t\to t-3} - \frac{1}{9}U_3(t)\mathcal{L}^{-1}\{\frac{1}{s^2+9}\}_{t\to}$

$$= \frac{1}{9}U_3(t)\cdot(t-3)-\frac{1}{27}U_3(t)\cdot\sin 3(t-3)$$

$$= \frac{1}{9}U_3(t)\cdot(t-3)-\frac{1}{27}U_3(t)\cdot\sin(3t-9)$$

$$= \frac{1}{9}U_3(t)\,t\big|_{t\to t-3}-\frac{1}{27}U_3(t)\sin 3t\big|_{t\to t-3}$$

$$= \frac{1}{9}U_3(t)(t-3)-\frac{1}{27}U_3(t)\sin(3t-9)$$

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Q19 $e^{-\pi s}\dfrac{s}{s^2-4s+5}$

Sol. Let $F(s) = e^{-\pi s}\dfrac{s}{s^2-4s+5}$

$$= e^{-\pi s}\left[\frac{s}{s^2-4s+4+1}\right]$$

$$= e^{-\pi s}\left[\frac{(s-2)+2}{(s-2)^2+(1)^2}\right]$$

or $F(s) = e^{-\pi s}\left[\dfrac{s-2}{(s-2)^2+(1)^2} + \dfrac{2}{(s-2)^2+(1)^2}\right]$

$$F(s) = e^{-\pi s}\left(\frac{s-2}{(s-2)^2+(1)^2}\right) + 2e^{-\pi s}\left(\frac{1}{(s-2)^2+(1)^2}\right)$$

Then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s-2}{(s-2)^2+(1)^2}\right\} + 2\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{(s-2)^2+(1)^2}\right\}$$

$$= U_\pi(t)\cdot e^{2(t-\pi)}\cdot \cos(t-\pi) + 2U_\pi(t)\cdot e^{2(t-\pi)}\cdot \sin(t-\pi)$$

$$= U_\pi(t)\cdot e^{2(t-\pi)}\cdot \cos(\pi-t) - 2U_\pi(t)\cdot e^{2(t-\pi)}\sin(\pi-t)$$

$$= -U_\pi(t)\cdot e^{2(t-\pi)}\cdot \cos t - 2U_\pi(t)\cdot e^{2(t-\pi)}\sin t$$

$$= -U_\pi(t)\cdot e^{2(t-\pi)}(\cos t + 2\sin t)$$

Q19 $e^{-2s}\dfrac{s+6}{s^3-5s^2+6s}$

Sol. Let $F(s) = e^{-2s}\dfrac{s+6}{s^3-5s^2+6s}$

$$= e^{-2s}\left[\frac{s+6}{s(s^2-5s+6)}\right]$$

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$$F(s) = e^{-2s}\left[\frac{s+6}{s(s-2)(s-3)}\right] \text{ ——— } (A)$$

Consider

$$\frac{s+6}{s(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3}$$

or $s+6 = A(s-2)(s-3) + Bs(s-3) + Cs(s-2)$

For A, put $s = 0$

$$6 = A(-2)(-3)$$

$$6 = 6A \implies \boxed{A = 1}$$

For B, put $s = 2$

$$2+6 = B(2)(2-3)$$

$$8 = -2B \implies \boxed{B = -4}$$

For C, put $s = 3$

$$3+6 = C(3)(3-2)$$

$$9 = 3C \implies \boxed{C = 3}$$

So

$$\frac{s+6}{s(s-2)(s-3)} = \frac{1}{s} - \frac{4}{s-2} + \frac{3}{s-3}$$

Put in eq. (A)

$$F(s) = e^{-2s}\left[\frac{1}{s} - \frac{4}{s-2} + \frac{3}{s-3}\right]$$

$$F(s) = e^{-2s}\cdot\frac{1}{s} - 4e^{-2s}\cdot\frac{1}{s-2} + 3e^{-2s}\cdot\frac{1}{s-3}$$

Then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s-2}\right\} + 3\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s-3}\right\}$$

$$= U_2(t) - 4U_2(t)\cdot e^{2(t-2)} + 3U_2(t)\cdot e^{3(t-2)}$$

$$= U_2(t)\left(1 - 4e^{2t-4} + 3e^{6t-6}\right)$$

1416

Q20. $e^{-3s} \dfrac{3s-7}{s^2-10s+26}$ 74

Sol. Let $F(s) = e^{-3s} \dfrac{3s-7}{s^2-10s+26}$

$$\frac{3s-7}{s^2-10s+25-25+26}$$

$$= e^{-3s}\left[\frac{3s-7}{s^2-10s+25+1}\right]$$

$$\frac{3s-7}{(s-5)^2+1}$$

$$= e^{-3s}\left[\frac{3(s-5)+15-7}{(s-5)^2+(1)^2}\right]$$

$$= e^{-3s}\left[\frac{3(s-5)+8}{(s-5)^2+(1)^2}\right]$$

$$= e^{-3s}\left[\frac{3(s-5)}{(s-5)^2+(1)^2} + \frac{8}{(s-5)^2+(1)^2}\right]$$

$$F(s) = 3e^{-3s}\frac{s-5}{(s-5)^2+(1)^2} + 8e^{-3s}\frac{1}{(s-5)^2+(1)^2}$$

then

$$\mathcal{L}^{-1}\{F(s)\} = 3\mathcal{L}^{-1}\left\{e^{-3s}\frac{s-5}{(s-5)^2+(1)^2}\right\} + 8\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{(s-5)^2+(1)^2}\right\}$$

$$= 3U_3(t)\, e^{5(t-3)} \cos(t-3) + 8U_3(t)\, e^{5(t-3)} \sin(t-3)$$

In each of problems 21—23, use the Convolution property to evaluate the inverse Laplace transform:

Imp Q21 $\dfrac{1}{s^2(s+5)}$

Sol. Let $F(s) = \dfrac{1}{s^2}$ & $G(s) = \dfrac{1}{s+5}$

then $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$

& $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\frac{1}{s+5}\} = e^{-5t}$

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By: Convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = f(t) * g(t)$$

$$= \int_0^t u\, e^{-5(t-u)}\, du$$

$$= \int_0^t u \cdot e^{-5t+5u}\, du$$

$$= \int_0^t e^{-5t} \cdot u \cdot e^{5u}\, du$$

$$= e^{-5t}\int_0^t u \cdot e^{5u}\, du$$

$$= e^{-5t}\left[\left| u \cdot \frac{e^{5u}}{5}\right|_0^t - \int_0^t \frac{e^{5u}}{5} \cdot 1\, du\right]$$

$$= e^{-5t}\left[t \cdot \frac{e^{5t}}{5} - \frac{1}{5}\left|\frac{e^{5u}}{5}\right|_0^t \right]$$

$$= e^{-5t}\left[\frac{t e^{5t}}{5} - \frac{1}{25}\left(e^{5t} - e^{0}\right)\right]$$

$$= e^{-5t}\left[\frac{t \cdot e^{5t}}{5} - \frac{1}{25}\left(e^{5t} - 1\right)\right]$$

$$= e^{-5t}\left[\frac{t e^{5t}}{5} - \frac{1}{25}e^{5t} + \frac{1}{25}\right]$$

$$= \frac{t}{5} - \frac{1}{25} + \frac{1}{25}e^{-5t}$$

$$= \frac{5t - 1 + e^{-5t}}{25}$$

$$= \frac{1}{25}\left(e^{-5t} + 5t - 1\right)$$

2nd

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t e^{-5u}(t-u)\, du$$

$$= \int_0^t \cancel{e^{-5u}(t) - e^{-5u}u}\, du$$

$$= \int_0^t \cancel{e^{-5u} t}$$

1418

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Imp Q12 $\dfrac{s}{(s+1)(s^2+4)}$

Sol.

Let $F(s) = \dfrac{1}{s+1}$ & $G(s) = \dfrac{s}{s^2+4}$

then $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\} = e^{-t}$

& $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+(2)^2}\right\} = \cos 2t$

By Convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\} = f(t) * g(t)$$

$$= g(t) * f(t)$$

$$= \int_0^t \cos 2u \cdot e^{-(t-u)}\, du$$

$$= \int_0^t \cos 2u \cdot e^{-t+u}\, du$$

$$= e^{-t}\int_0^t \cos 2u \, e^{u}\, du \quad \text{——— (1)}$$

Consider $e^{-t}\int_0^t \cos 2u \cdot e^{u}\, du$

Integ. by parts.

$$= e^{-t}\left[\left|\cos 2u \cdot e^{u}\right|_0^t - \int_0^t e^{u} \cdot -2\sin 2u\, du\right]$$

$$= e^{-t}\left[(\cos 2t \cdot e^{t} - 1) + 2\int_0^t \sin 2u \cdot e^{u}\, du\right]$$

$$= e^{-t}(\cos 2t \cdot e^{t} - 1) + 2e^{-t}\int_0^t \sin 2u \cdot e^{u}\, du$$

$$\text{S.}\; e^{-t}\int_0^t \cos 2u \cdot e^{u}\, du = e^{-t}(\cos 2t \cdot e^{t} - 1) + 2e^{-t}\left[\left|\sin 2u \, e^{u}\right|_0^t - \int_0^t e^{u} \cdot 2\cos 2u\, du\right]$$

$$e^{-t}\int_0^t \cos 2u \cdot e^{u}\,du = e^{-t}\left(e^{t}\cos 2t - 1\right) + e^{-t}\left(\sin 2t \cdot e^{t}\right) - 4e^{-t}\int_0^t \cos 2u \cdot e^{u}\,du$$

$$5e^{-t}\int_0^t \cos 2u \cdot e^{u}\,du = \cos 2t - e^{-t} + 2\sin 2t$$

$$e^{-t}\int_0^t \cos 2u \cdot e^{u}\,du = \frac{1}{5}\left[\cos 2t + 2\sin 2t - e^{-t}\right]$$

Put in ①

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\} = \frac{1}{5}\left[\cos 2t + 2\sin 2t - e^{-t}\right]$$

Imp

Q23 $\dfrac{1}{(s^2+1)(s^2+4s+5)}$

Sol.

let $F(s) = \dfrac{1}{s^2+1}$

& $G(s) = \dfrac{1}{s^2+4s+5}$

$= \dfrac{1}{s^2+4s+4+1}$

or $G(s) = \dfrac{1}{(s+2)^2+(1)^2}$

then $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+(1)^2}\right\} = \sin t$

& $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^2+(1)^2}\right\} = e^{-2t}\cdot \sin t$

By the Convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = f(t) * g(t)$$

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$$= \int_0^t \sin u \cdot e^{-2(t-u)} \cdot \sin(t-u)\, du$$

$$= \int_0^t \sin u \cdot e^{-2t+2u} \cdot \sin(t-u)\, du$$

$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

$$= \int_0^t e^{-2t} \cdot e^{2u} \cdot (\sin u \sin(t-u))\, du$$

$$= -\frac{e^{-2t}}{2}\int_0^t e^{2u}(-2\sin u \cdot \sin(t-u))\, du$$

$$= -\frac{e^{-2t}}{2}\int_0^t e^{2u}(\cos(u+t-u) - \cos(u-t+u))\, du$$

$$= -\frac{e^{-2t}}{2}\int_0^t e^{2u}(\cos t - \cos(2u-t))\, du$$

$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+8)}\right\} = -\frac{e^{-2t}}{2}\int_0^t e^{2u}\cos t\, du + \frac{e^{-2t}}{2}\int_0^t e^{2u}\cos(2u-t)\, dt$$

$$= -\frac{e^{-2t}\cos t}{2}\int_0^t e^{2u}\, du - \frac{e^{-2t}}{2}\int_0^t e^{2u}\cdot\cos(2u-t)\, dt$$

$$= -\frac{e^{-2t}\cos t}{2}\left|\frac{e^{2u}}{2}\right|_0^t - \frac{e^{-2t}}{2}\left|\frac{e^{2u}}{4+4}(2\cos(2u-t) + 2\sin(2u-t))\right|_0^t$$

$\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx+c) + b\sin(bx+c))$

$$= -\frac{e^{-2t}\cos t}{4}(e^{2t} - e^{0}) - \frac{e^{-2t}}{16}\left[e^{2t}(2\cos t + 2\sin t) - e^{0}(2\cos(-t) + 2\sin(-t))\right]$$

$$= -\frac{e^{-2t}\cos t}{4}(e^{2t}-1) - \frac{e^{-2t}}{16}\left[2e^{2t}(\cos t + \sin t) - (2\cos t + 2\sin t)\right]$$

$$= -\frac{\cos t}{4} + \frac{e^{-2t}\cos t}{4} - \frac{1}{8}(\cos t + \sin t) + \frac{e^{-2t}}{8}(\cos t + \sin t)$$ — Ans.

Imp

Qu. Show that

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \cosh at \cdot \cos at$$

Sol. Consider

$$\mathcal{L}\{\cosh at \cdot \cos at\} = \mathcal{L}\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\cos at\right\}$$

$$= \mathcal{L}\left\{\frac{1}{2}(e^{at}\cos at) + \frac{1}{2}(e^{-at}\cos at)\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\cos at\} + \frac{1}{2}\mathcal{L}\{e^{-at}\cos at\}$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2}\right] + \frac{1}{2}\left[\frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)[(s+a)^2+a^2] + (s+a)[(s-a)^2+a^2]}{[(s-a)^2+a^2][(s+a)^2+a^2]}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)[s^2+2as+2a^2] + (s+a)[s^2-2as+2a^2]}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{1}{2}\left[\frac{s^3+\cancel{2as^2}+\cancel{2a^2s}-\cancel{as^2}-\cancel{2a^2s}-\cancel{2a^3}+s^3-\cancel{2as^2}+\cancel{2a^2s}+\cancel{as^2}-\cancel{2a^2s}+\cancel{2a^3}}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$= \frac{1}{2}\left[\frac{2s^3}{(s^2+2a^2)^2-4a^2s^2}\right]$$

$$= \frac{s^3}{s^4+\cancel{4a^2s^2}+4a^4-\cancel{4a^2s^2}}$$

$$\mathcal{L}\{\cosh at \cos at\} = \frac{s^3}{s^4+4a^4}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \cosh at \cos at$$

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11.2-32

Q25 Show that $\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{2a^2}\sinh at \sin at$

Sol. Consider

$$\mathcal{L}\{\sinh at \cdot \sin at\} = \mathcal{L}\left\{\left(\frac{e^{at}-e^{-at}}{2}\right)\sin at\right\}$$

$$= \mathcal{L}\left\{\frac{1}{2}e^{at}\sin at - \frac{1}{2}e^{-at}\sin at\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\sin at\} - \frac{1}{2}\mathcal{L}\{e^{-at}\sin at\}$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}\right] - \frac{1}{2}\left[\frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right] \qquad \frac{a}{s^2+a^2}$$

$$= \frac{1}{2}a\left[\frac{[(s+a)^2+a^2]-[(s-a)^2+a^2]}{[(s-a)^2+a^2][(s+a)^2+a^2]}\right]$$

$$= \frac{1}{2}a\left[\frac{(s^2+2as+2a^2)-(s^2-2as+2a^2)}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{1}{2}a\left[\frac{\cancel{s^2}+2as+\cancel{2a^2}-\cancel{s^2}+2as-\cancel{2a^2}}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$= \frac{1}{2}a\left[\frac{4as}{(s^2+2a^2)^2-(2as)^2}\right]$$

$$= \frac{2a^2 s}{s^4+\cancel{4a^2s^2}+4a^4-\cancel{4a^2s^2}}$$

$$\mathcal{L}\{\sinh at \sin at\} = \frac{2a^2 s}{s^4+4a^4}$$

$$\Rightarrow \sinh at \sin at = \mathcal{L}^{-1}\left\{\frac{2a^2 s}{s^4+4a^4}\right\}$$

or $$\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{2a^2}\sinh at \cdot \sin at$$

11.3

1423

(1) Take Laplace Transform and Apply
$\mathcal{L}\{f(t)\} = sF(s) - f(0)$
and it calculated
(2) Apply initial condition
(3) above eq. will be convert into algebraic Eq in $Y(s)$
(4) apply inverse Laplace Trans for

(Exercise 11.3)

Use the Laplace transform method to solve the following initial value problems:

Q1 $\frac{dy}{dt} - ky = ce^{kt}$; $y(0) = 0$

Sol. Given eq. is

$$\frac{dy}{dt} - ky = ce^{kt} \quad \text{---} (1)$$

taking Laplace transform of both sides of (1) ~~1st step~~

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} - k\mathcal{L}\{y(t)\} = c\mathcal{L}\{e^{kt}\}$$

$$sY(s) - y(0) - kY(s) = c \cdot \frac{1}{s-k}$$

$$(s-k)Y(s) = \frac{c}{s-k} \qquad \because y(0) = 0 \quad \text{~~2nd step~~}$$

$$Y(s) = \frac{c}{(s-k)^2} \quad \text{3rd}$$

then $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{c}{(s-k)^2}\right\}$ 4th step

$$y(t) = c\mathcal{L}^{-1}\left\{\frac{1}{(s-k)^2}\right\}$$

$\boxed{y(t) = cte^{kt}}$ is the req. soln.

Q2 $\frac{dy}{dt} + 4y = 2e^{t} - 4e^{-t}$; $y(0) = 0$

Sol. Given eq. is

$$\frac{dy}{dt} + 4y = 2e^{t} - 4e^{-t} \quad \text{---} (1)$$

taking Laplace transform of both sides of (1).

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$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\,\mathcal{L}\{y(t)\} = 2\,\mathcal{L}\{e^{t}\} - 4\,\mathcal{L}\{e^{-t}\}$$

$$sY(s) - y(0) + 4Y(s) = 2\cdot\frac{1}{s-1} - 4\cdot\frac{1}{s+1}$$

$$(s+4)Y(s) = \frac{2}{s-1} - \frac{4}{s+1} \qquad \because y(0)=0$$

$$= \frac{2s+2-4s+4}{(s-1)(s+1)}$$

$$(s+4)Y(s) = \frac{-2s+6}{(s-1)(s+1)}$$

$$\Rightarrow \quad Y(s) = \frac{-2s+6}{(s-1)(s+1)(s+4)} \quad \text{———} \quad (A)$$

Consider

$$\frac{-2s+6}{(s-1)(s+1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+4}$$

or $-2s+6 = A(s+1)(s+4) + B(s-1)(s+4) + C(s-1)(s+1)$ —— I

For A, put $s=1$ in I

$$-2+6 = A(1+1)(1+4)$$

$$4 = A(2)(5) \Rightarrow \boxed{A = \frac{2}{5}}$$

For B, put $s=-1$ in I

$$2+6 = B(-2)(3)$$

$$8 = B(-6) \Rightarrow \boxed{B = -\frac{4}{3}}$$

For C, put $s=-4$ in I

$$8+6 = C(-5)(-3)$$

$$14 = 15C \Rightarrow \boxed{C = \frac{14}{15}}$$

1425

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S.

$$\frac{-2S+6}{(S-1)(S+1)(S+4)} = \frac{2}{5(S-1)} - \frac{4}{3(S+1)} + \frac{14}{15(S+4)}$$

Put in (A)

$$Y(S) = \frac{2}{5(S-1)} - \frac{4}{3(S+1)} + \frac{14}{15(S+4)}$$

Now

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{S+1}\right\} + \frac{14}{15}\mathcal{L}^{-1}\left\{\frac{1}{S+4}\right\}$$

$$y(t) = \frac{2}{5}e^{t} - \frac{4}{3}e^{-t} + \frac{14}{15}e^{-4t}$$ is req. soln.

Q3 $\frac{dy}{dt} + y = f(t)$ where $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 5 & \text{if } t \ge 1 \end{cases}$; $y(0)=0$

Soln Given eq. is

$$\frac{dy}{dt} + y = f(t) \quad \text{——— } ①$$

taking Laplace transform of both sides of ①

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}$$

$$SY(s) - y(0) + Y(S) = \mathcal{L}\{5u_1(t)\}$$

$$(S+1)Y(S) = 5\frac{e^{-s}}{s}$$

$$Y(S) = 5\frac{e^{-s}}{s(s+1)}$$

$$= 5e^{-s}\left[\frac{1}{s(s+1)}\right]$$

$$= 5e^{-s}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$$

$\because y(0)=0$

$$= \int_0^1 e^{-st}(0)\,dt + \int_1^\infty e^{-st}(5)\,dt$$

$$= 0 + 5\int_1^\infty e^{-st}\,dt$$

$$= 5\left.\frac{e^{-st}}{-s}\right|_1^\infty$$

$$= -\frac{5}{s}\left[0 - e^{-s}\right]$$

$$= \frac{5}{s}e^{-s}$$

1426

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$$Y(s) = 5\frac{e^{-s}}{s} - 5\frac{e^{-s}}{s+1}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = 5\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} - 5\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+1}\right\}$$

$$y(t) = 5u_1(t) - 5u_1(t)\,e^{-(t-1)}$$

$$= 5u_1(t)\left(1 - e^{-(t-1)}\right)$$ is req. soln.

Q4. $\frac{dy}{dt} + 2y = f(t)$ where $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$ $y(0) = 0$

Sol. Given eq. is

$$\frac{dy}{dt} + 2y = f(t) \quad \text{——— (1)}$$

Here $f(t) = t - t\,u_1(t)$

$$= t - (t-1+1)u_1(t)$$

$$= t - u_1(t)(t-1) - u_1(t)$$

So from (1)

$$\frac{dy}{dt} + 2y = t - u_1(t)(t-1) - u_1(t)$$

taking Laplace transform of both sides

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{t - u_1(t)(t-1) - u_1(t)\}$$

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

or $$(s+2)Y(s) = \frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

$$(s+2)Y(s) = \frac{1}{s^2} - e^{-s}\left[\frac{1+s}{s^2}\right]$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)\,dt \int_1^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\,t\,dt + 0$$

$$= t\,\frac{e^{-st}}{-s}\Big|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\,dt$$

$$= \frac{e^{-s}}{-s} - 0 + \frac{1}{s}\int_0^1 e^{-st}\,dt$$

$$\frac{e^{-s}}{-s} + \frac{1}{s}\,\frac{e^{-st}}{-s}\Big|_0^1$$

$$\frac{e^{-s}}{-s} - \frac{1}{s^2}(e^{-s} - 1)$$

$$\frac{-e^{-s}}{s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}$$

$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{e^{-s}}{s}$$

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$$Y(s) = \frac{1}{s^2(s+2)} - e^{-s}\left(\frac{s+1}{s^2(s+2)}\right) \quad \text{——— (A)}$$

Consider

$$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$

or $1 = As(s+2) + B(s+2) + Cs^2$

For B put $s = 0$

$1 = B(2) \Rightarrow \boxed{B = \frac{1}{2}}$

For C, put $s = -2$

$1 = C(4) \Rightarrow \boxed{C = \frac{1}{4}}$

For A, equating coff. of s^2

$A + C = 0$

$A + \frac{1}{4} = 0 \Rightarrow \boxed{A = -\frac{1}{4}}$

So $$\frac{1}{s^2(s+2)} = \frac{-1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)}$$

Now Consider

$$\frac{s+1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$

or $s+1 = As(s+2) + B(s+2) + Cs^2$

For B, put $s = 0$

$1 = 2B \Rightarrow \boxed{B = \frac{1}{2}}$

For C put $s = -2$

$-1 = C(4) \Rightarrow \boxed{C = -\frac{1}{4}}$

For A, equating coff. of s^2

$A + C = 0 \Rightarrow A - 1/4 = 0$ or $\boxed{A = 1/4}$

So

$$\frac{s+1}{s^2(s+2)} = \frac{1}{4s} + \frac{1}{2s^2} - \frac{1}{4(s+2)}$$

Putting in (A)

$$Y(s) = \frac{-1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)} - e^{-s}\left(\frac{1}{4s} + \frac{1}{2s^2} - \frac{1}{4(s+2)}\right)$$

$$Y(s) = \frac{-1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)} - \frac{e^{-s}}{4s} - \frac{e^{-s}}{2s^2} + \frac{e^{-s}}{4(s+2)}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{-1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s^2}\right\}$$
$$+ \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s+2}\right\}$$

$$y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4}u_1(t) - \frac{1}{2}\underline{u_1(t)(t-1)} + \frac{1}{4}u_1(t)\cdot e^{-2(t-1)}$$

$$= -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}u_1(t)\left(1-2t + e^{-2(t-1)}\right) \text{ is req. soln.}$$

Q5 $\frac{dy}{dt} = \cos t + \int_0^t y(u)\cos(t-u)\,du \qquad y(0) = 1$

Soln. Given eq. is

$$\frac{dy}{dt} = \cos t + \int_0^t y(u)\cdot\cos(t-u)\,du \quad \text{——— (1)}$$

Let $f(t) = y(t)$ & $g(t) = \cos t$

then $f * g = \int_0^t y(u)\cdot\cos(t-u)\,du$

then

$$\mathcal{L}\{f*g\} = \mathcal{L}\left\{\int_0^t y(u)\cdot\cos(t-u)\,du\right\}$$

$$= Y(s)\cdot G(s)$$

where $Y(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{y(t)\}$ & $G(s) = \mathcal{L}\{g(t)\} = \frac{s}{s^2+1}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t)f(t-a)$

1429

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So
$$\mathcal{L}\left\{\int_0^t y(u)\cdot \cos(t-u)\,du\right\} = Y(s)\cdot\frac{s}{s^2+1}$$

taking Laplace transform of both sides of ①

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\{\cos t\} + \mathcal{L}\left\{\int_0^t y(u)\cdot\cos(t-u)\,du\right\}$$

$$sY(s) - y(0) = \frac{s}{s^2+1} + Y(s)\cdot\frac{s}{s^2+1}$$

$$sY(s) - 1 = \frac{s}{s^2+1} + Y(s)\cdot\frac{s}{s^2+1}$$

$$sY(s) - Y(s)\cdot\frac{s}{s^2+1} = \frac{s}{s^2+1} + 1$$

$$Y(s)\left(s - \frac{s}{s^2+1}\right) = \frac{s+s^2+1}{s^2+1}$$

$$Y(s)\left(\frac{s^3+\cancel{s}-\cancel{s}}{s^2+1}\right) = \frac{s^2+s+1}{s^2+1}$$

$$Y(s)(s^3) = s^2+s+1$$

$$Y(s) = \frac{s^2+s+1}{s^3}$$

$$Y(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}$$

then $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$

$y(t) = 1 + t + \frac{1}{2}t^2$ is req. sol.

Q6) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = e^t$ $\quad y(0) = 1,\ y'(0) = 0$

Sol. Given $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = e^t$ ——— ①

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9.

Q7

$$\frac{d^2y}{dt^2} + y = \cos t \qquad y(0) = 0, \; y'(0) = -1$$

Sol. Given eq. is

$$\frac{d^2y}{dt^2} + y = \cos t \quad \text{——— } (1)$$

taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{\cos t\}$$

$$s^2Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{s}{s^2+1}$$

$$s^2Y(s) - s(0) + 1 + Y(s) = \frac{s}{s^2+1}$$

$$(s^2+1)Y(s) = \frac{s}{s^2+1} - 1$$

$$Y(s) = \frac{s}{(s^2+1)^2} - \frac{1}{s^2+1}$$

$$Y(s) = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right) - \frac{1}{s^2+1}$$

Then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{(-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$\Rightarrow$ $y(t) = \frac{1}{2}t\sin t - \sin t$ is req. soln.

Q8 $\quad \frac{d^2y}{dt^2} + y = 4t\sin t \qquad y(0) = 0 = y'(0)$

Sol. Given

$$\frac{d^2y}{dt^2} + y = 4t\sin t \quad \text{——— } (1)$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{4t\sin t\}$$

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$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4(-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$s^2Y(s) - 0 - 0 + Y(s) = -4 \cdot \frac{-1}{(s^2+1)^2} \cdot 2s$$

$$(s^2+1)Y(s) = \frac{8s}{(s^2+1)^2}$$

$$Y(s) = \frac{8s}{(s^2+1)^2} \cdot \frac{1}{s^2+1} \quad \text{or} \quad Y(s) = \frac{8s}{(s^2+1)^3}$$

Let $F(s) = \frac{8s}{(s^2+1)^2}$ & $G(s) = \frac{1}{s^2+1}$ $\qquad \mathcal{L}\{t f(t)\} = (-1)^1 \frac{d}{ds} \mathcal{L}\{f(t)\}$

then $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{8s}{(s^2+1)^2}\right\} = 4\mathcal{L}^{-1}\left\{(-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} = 4t \sin t = f(t)$

& $\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = g(t)$

then

$$\mathcal{L}^{-1}\{Y(s)\} = f(t) * g(t)$$

or
$$y(t) = \int_0^t 4u \sin u \sin(t-u)\, du$$

$$= -2\int_0^t -2u \sin u \cdot \sin(t-u) \cdot du$$

$$= -2\int_0^t u[\cos(u+t-u) - \cos(u-t+u)]\,du$$

$$= -2\int_0^t u[\cos t - \cos(2u-t)]\,du$$

$$= 2\int_0^t u[\cos(2u-t) - \cos t]\,du$$

$$= 2\int_0^t \underbrace{u \cdot \cos(2u-t)\, du}_{\text{integ. by parts}} - 2\int_0^t u \cos t\, du$$

$$= 2\left[\left| u \cdot \frac{\sin(2u-t)}{2} \right|_0^t - \int_0^t \frac{\sin(2u-t)}{2} \cdot 1\,du \right] - 2\cos t \int_0^t u\,du$$ 92

$$= \left| u\sin(2u-t) \right|_0^t - \int_0^t \sin(2u-t)\,du \; - 2\cos t \left| \frac{u^2}{2} \right|_0^t$$

$$= t\sin(2t-t) - 0 - \left| \frac{\cos(2u-t)}{2} \right|_0^t - 2\cos t\left(\frac{t^2}{2}\right)$$

$$= t\sin t - \frac{1}{2}\left[\cos(2t-t) - \cos(-t)\right] - t^2\cos t$$

$$= t\sin t - \frac{1}{2}\left(\cancel{\cos} t - \cancel{\cos} t\right) - t^2\cos t$$

$y(t) = t\sin t - t^2\cos t$ is req. soln.

Q9 $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} = 20\,e^{-t}\cos t \qquad y(0) = 0 = y'(0)$

Soln. Given eq. is

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} = 20\,e^{-t}\cos t \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 2\mathcal{L}\left\{\frac{dy}{dt}\right\} = 20\,\mathcal{L}\{e^{-t}\cos t\}$$

$$s^2Y(s) - s\,y(0) - y'(0) - 2\left(sY(s) - y(0)\right) = 20\,\frac{s+1}{(s+1)^2+(1)^2}$$

$$s^2Y(s) - 0 - 0 - 2\left(sY(s) - 0\right) = 20\,\frac{s+1}{(s+1)^2+(1)^2}$$

$$(s^2-2s)Y(s) = 20\,\frac{s+1}{s^2+2s+2}$$

$$Y(s) = 20\,\frac{s+1}{(s^2-2s)(s^2+2s+2)}$$

$$Y(s) = \frac{20s+20}{s(s-2)(s^2+2s+2)} \quad \text{——— (A)}$$

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Consider

$$\frac{20S+20}{S(S-2)(S^2+2S+2)} = \frac{A}{S} + \frac{B}{S-2} + \frac{CS+D}{S^2+2S+2}$$

$\Rightarrow \; 20S+20 = A(S-2)(S^2+2S+2) + BS(S^2+2S+2) + (CS+D)(S^2-2S)$

For A, put $S=0$

$20 = A(-2)(2) \Rightarrow \boxed{A=-5}$

For B, put $S=2$

$40+20 = B(2)(4+4+2)$

$60 = B(20) \Rightarrow \boxed{B=3}$

From above eq.

$20S+20 = A(S^3+\cancel{2S^2}+2S-\cancel{2S^2}-4S-4) + B(S^3+2S^2+2S) + (CS^3-2CS^2+DS^2-2DS)$

$20S+20 = A(S^3-2S-4) + B(S^3+2S^2+2S) + (CS^3-2CS^2+DS^2-2DS)$

Comparing coff. of S^3 & S^2

$A+B+C = 0$ ———— I

$2B-2C+D = 0$ ———— II

I $\Rightarrow -5+3+C=0 \Rightarrow \boxed{C=2}$

II $\Rightarrow 2(3)-2(2)+D=0$

$6-4+D=0$

$2+D=0 \Rightarrow \boxed{D=-2}$

So

$$\frac{20S+20}{S(S-2)(S^2+2S+2)} = -\frac{5}{S} + \frac{3}{S-2} + \frac{2S-2}{S^2+2S+2}$$

Put in (A)

$$Y(S) = -\frac{5}{S} + \frac{3}{S-2} + \frac{2(S-1)}{S^2+2S+2}$$

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$$Y(s) = -\frac{5}{s} + \frac{3}{s-2} + \frac{2(s+1)-4}{(s+1)^2+(1)^2}$$

Then

$$\mathcal{L}^{-1}\{Y(s)\} = -5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{2(s+1)-4}{(s+1)^2+(1)^2}\right\}$$

$$y(t) = -5(1) + 3e^{2t} + 2\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+(1)^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

or $y(t) = -5 + 3e^{2t} + 2e^{-t}\cos t - 4e^{-t}\sin t$ is req. soln.

Q.10 $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 12e^{-3t}\sin 2t$ $\quad y(0) = 1,\ y'(0) = 0$

Soln Given eq. is

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 12e^{-3t}\sin 2t \quad \text{——— (1)}$$

taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dt}\right\} - 4\mathcal{L}\{y(t)\} = 12\mathcal{L}\{e^{-3t}\sin 2t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) - 4Y(s) = 12\frac{2}{(s+3)^2+(2)^2}$$

$$s^2Y(s) - s - 0 - 3(sY(s) - 1) - 4Y(s) = \frac{24}{s^2+6s+13}$$

$$s^2Y(s) - s - 3sY(s) + 3 - 4Y(s) = \frac{24}{s^2+6s+13}$$

$$(s^2-3s-4)Y(s) - s + 3 = \frac{24}{s^2+6s+13}$$

$$\Rightarrow (s^2-3s-4)Y(s) = s - 3 + \frac{24}{(s^2+6s+13)}$$

$$\Rightarrow Y(s) = \frac{s-3}{(s^2-3s-4)} + \frac{24}{(s^2-3s-4)(s^2+6s+13)}$$

From above eq.

$$24 = A(s^3+6s^2+13s+s^2+6s+13)+B(s^3+6s^2+13s-4s^2-24s-52)$$
$$+(Cs+D)(s^2-3s-4)$$

$$24 = A(s^3+7s^2+19s+13)+B(s^3+2s^2-11s-52)+C(s^3-3s^2-4s)$$
$$+D(s^2-3s-4)$$

Comparing Coffs: of s^3 & s^2

$$A+B+C=0 \text{ ——— I}$$
$$7A+2B-3C+D=0 \text{ ——— II}$$

$$\text{I} \Rightarrow \frac{24}{265}-\frac{3}{5}+C=0$$

$$C = \frac{3}{5}-\frac{24}{265}$$
$$= \frac{159-24}{265}$$
$$C = \frac{135}{265} \quad \text{or} \quad \boxed{C = \frac{27}{53}}$$

$$\text{II} \Rightarrow 7\left(\frac{24}{265}\right)+2\left(-\frac{3}{5}\right)-3\left(\frac{27}{53}\right)+D=0$$

$$\frac{168}{265}-\frac{6}{5}-\frac{81}{53}+D=0$$

$$D = \frac{81}{53}+\frac{6}{5}-\frac{168}{265}$$
$$= \frac{405+318-168}{265}$$
$$D = \frac{555}{265} \quad \text{or} \quad \boxed{D = \frac{111}{53}}$$

So

$$\frac{24}{(s-4)(s+1)(s^2+6s+13)} = \frac{\frac{24}{265}}{s-4}+\frac{-\frac{3}{5}}{s+1}+\frac{\frac{27}{53}s+\frac{111}{53}}{s^2+6s+13}$$

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$$\text{or } Y(s) = \frac{s-3}{(s-4)(s+1)} + \frac{24}{(s-4)(s+1)(s^2+6s+13)} \quad \text{—— (A)}$$

Consider

$$\frac{s-3}{(s-4)(s+1)} = \frac{A}{s-4} + \frac{B}{s+1}$$

$$\text{or} \quad s-3 = A(s+1) + B(s-4)$$

For A, put $s = 4$

$$4-3 = A(\cancel{4+1}) \Rightarrow \boxed{A = \frac{1}{5}}$$

For B, put $s = -1$

$$-1-3 = B(-1-4)$$

$$-4 = -5B \Rightarrow \boxed{B = \frac{4}{5}}$$

So

$$\frac{s-3}{(s-4)(s+1)} = \frac{1}{5(s-4)} + \frac{4}{5(s+1)}$$

Now Consider

$$\frac{24}{(s-4)(s+1)(s^2+6s+13)} = \frac{A}{s-4} + \frac{B}{s+1} + \frac{Cs+D}{s^2+6s+13}$$

$$\Rightarrow 24 = A(s+1)(s^2+6s+13) + B(s-4)(s^2+6s+13) + (Cs+D)(s-4)(s+1)$$

For A, put $s = 4$

$$24 = A(5)(16+24+13)$$

$$24 = A(5)(53)$$

$$\boxed{A = \frac{24}{265}}$$

For B, Put $s = -1$

$$24 = B(-5)(1-6+13)$$

$$24 = B(-5)(8) \Rightarrow B = -\frac{24}{40}$$

$$\text{or} \quad \boxed{B = -\frac{3}{5}}$$

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$$\frac{24}{(s-4)(s+1)(s^2+6s+13)} = \frac{24}{265(s-4)} - \frac{3}{5(s+1)} + \frac{27s+111}{53(s^2+6s+13)}$$

Put in (A)

$$Y(s) = \frac{1}{5(s-4)} + \frac{4}{5(s+1)} + \frac{24}{265(s-4)} - \frac{3}{5(s+1)} + \frac{27s+111}{53(s^2+6s+13)}$$

$$= \left(\frac{1}{5} + \frac{24}{265}\right)\cdot\frac{1}{s-4} + \frac{1}{5(s+1)} + \frac{1}{53}\cdot\frac{27s+111}{(s+3)^2+(2)^2}$$

$$= \left(\frac{53+24}{265}\right)\cdot\frac{1}{s-4} + \frac{1}{5(s+1)} + \frac{1}{53}\cdot\frac{27(s+3)+111-81}{(s+3)^2+(2)^2}$$

$$= \frac{77}{265}\cdot\frac{1}{s-4} + \frac{1}{5(s+1)} + \frac{1}{53}\cdot\frac{27(s+3)+30}{(s+3)^2+(2)^2}$$

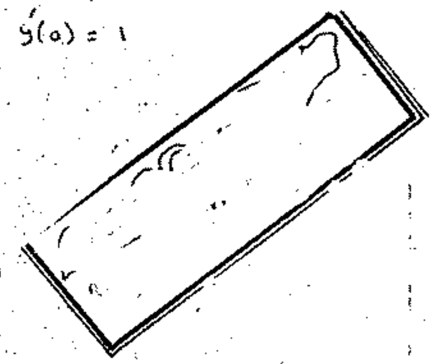
$$Y(s) = \frac{77}{265}\cdot\frac{1}{s-4} + \frac{1}{5(s+1)} + \frac{27}{53}\cdot\frac{s+3}{(s+3)^2+(2)^2} + \frac{30}{53}\cdot\frac{1}{(s+3)^2+(2)^2}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{77}{265}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{27}{53}\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+(2)^2}\right\} + \frac{15}{53}\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+(2)^2}\right\}$$

$$y(t) = \frac{77}{265}e^{4t} + \frac{1}{5}e^{-t} + \frac{27}{53}e^{-3t}\cos 2t + \frac{15}{53}e^{-3t}\sin 2t$$

Q11) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = u_3(t)$ $y(0)=0,\ y'(0)=1$



Sol. Given eq. is

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = u_3(t) \quad \text{———} (1)$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 4\mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{u_3(t)\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = \frac{e^{-3s}}{s}$$

$$s^2Y(s) - 0 - 1 - 4sY(s) + 4Y(s) = \frac{e^{-3s}}{s}$$

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$$(s^2-4s+4)Y(s) = 1+\frac{e^{-3s}}{s}$$

$$(s-2)^2 Y(s) = 1+\frac{e^{-3s}}{s}$$

then $Y(s) = \frac{1}{(s-2)^2} + \frac{e^{-3s}}{s(s-2)^2}$

$$\Rightarrow Y(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) + e^{-3s}\left[\frac{1}{s(s-2)^2}\right] \text{ ——— (A)}$$

Consider

$$\frac{1}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$\Rightarrow 1 = A(s-2)^2 + Bs(s-2) + Cs$$

For A; put $s=0$

$1 = A(4) \Rightarrow \boxed{A=\frac{1}{4}}$

For C, put $s=2$

$1 = C(2) \Rightarrow \boxed{C=\frac{1}{2}}$

For B, Comparing Coff. of s^2

$A+B = 0$

$\frac{1}{4}+B = 0 \Rightarrow \boxed{B=-\frac{1}{4}}$

So $$\frac{1}{s(s-2)^2} = \frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2(s-2)^2}$$

Put in (A)

$$Y(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) + e^{-3s}\left[\frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2(s-2)^2}\right]$$

$$Y(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) + \frac{1}{4}\cdot\frac{e^{-3s}}{s} - \frac{1}{4}\cdot e^{-3s}\cdot\frac{1}{s-2} + \frac{1}{2}e^{-3s}\cdot\frac{1}{(s-2)^2}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{(-1)^1\cdot\frac{d}{ds}\left(\frac{1}{s-2}\right)\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{s-2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{(s-2)^2}\right\}$$

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$$y(t) = te^{2t} + \frac{1}{4}U_3(t) - \frac{1}{4}U_3(t).e^{2(t-3)} + \frac{1}{2}U_3(t).(t-3).e^{2(t-3)}$$

$$y(t) = te^{2t} + \frac{1}{4}U_3(t)\left[1 - e^{2(t-3)} + 2(t-3).e^{2(t-3)}\right]$$ is req. soln.

Q12 $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = f(t)$ where $f(t) = \begin{cases} 0 & \text{if } 0<t<2 \\ 3 & \text{if } 2<t<5 \\ 0 & \text{if } t>5 \end{cases}$

$y(0) = 0 = y'(0)$

Sol. Given eq. is

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = f(t) \quad \text{——— (1)}$$

Now f(t) satisfying the above conditions can be re-written as

$$f(t) = 3(U_2(t) - U_5(t))$$

or $$f(t) = 3U_2(t) - 3U_5(t)$$

Put value in (1)

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 3U_2(t) - 3U_5(t)$$

Taking Laplace transform of both sides

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{y(t)\} = 3\mathcal{L}\{U_2(t)\} - 3\mathcal{L}\{U_5(t)\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = 3\frac{e^{-2s}}{s} - 3\frac{e^{-5s}}{s}$$

$$s^2Y(s) - 3sY(s) + 2Y(s) = 3\frac{e^{-2s}}{s} - 3\frac{e^{-5s}}{s}$$

$$(s^2 - 3s + 2)Y(s) = \frac{3e^{-2s}}{s} - \frac{3e^{-5s}}{s}$$

$$\Rightarrow Y(s) = \frac{3e^{-2s}}{s(s^2-3s+2)} - \frac{3e^{-5s}}{s(s^2-3s+2)}$$

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$$Y(s) = 3e^{-2s}\left[\frac{1}{s(s-1)(s-2)}\right] - 3e^{-5s}\left[\frac{1}{s(s-1)(s-2)}\right] \quad \text{——— (A)}$$

Consider

$$\frac{1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\Rightarrow \quad 1 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

For A, put $s = 0$

$$1 = A(-1)(-2) \quad \Rightarrow \boxed{A = \tfrac{1}{2}}$$

For B; put $s = 1$

$$1 = B(1)(-1) \quad \Rightarrow \boxed{B = -1}$$

For C, put $s = 2$

$$1 = C(2)(2-1)$$

$$1 = 2C \quad \Rightarrow \boxed{C = \tfrac{1}{2}}$$

So

$$\frac{1}{s(s-1)(s-2)} = \frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}$$

Put in (A)

$$Y(s) = 3e^{-2s}\left[\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right] - 3e^{-5s}\left[\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right]$$

$$Y(s) = \frac{3}{2}\cdot\frac{e^{-2s}}{s} - \frac{3e^{-2s}}{s-1} + \frac{3}{2}\cdot\frac{e^{-2s}}{s-2} - \frac{3}{2}\cdot\frac{e^{-5s}}{s} + \frac{3e^{-5s}}{s-1} - \frac{3}{2}\cdot\frac{e^{-5s}}{s-2}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-2}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s-1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s-2}\right\}$$

$$y(t) = \frac{3}{2}U_2(t) - 3U_2(t)e^{t-2} + \frac{3}{2}U_2(t)e^{2(t-2)} - \frac{3}{2}U_5(t) + 3U_5(t)e^{t-5} - \frac{3}{2}U_5(t)e^{2(t-5)}$$

$$y(t) = \frac{3}{2}U_2(t)\left[1 - 2e^{t-2} + e^{2(t-2)}\right] - \frac{3}{2}U_5(t)\left[1 - 2e^{t-5} + e^{2(t-5)}\right]$$

is req. soln.

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Q13 $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=2(t-3)U_3(t)$ $y(0)=2$, $y'(0)=1$

Sol. Given eq. is

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=2(t-3)U_3(t) \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\}+2\mathcal{L}\left\{\frac{dy}{dt}\right\}+\mathcal{L}\{y(t)\}=2\mathcal{L}\{U_3(t).(t-3)\}$$

$$s^2Y(s)-sy(0)-y'(0)+2\left(sY(s)-y(0)\right)+Y(s)=2.\frac{e^{-3s}}{s^2}$$

$$s^2Y(s)-2s-1+2\left(sY(s)-2\right)+Y(s)=2\frac{e^{-3s}}{s^2}$$

$$s^2Y(s)-2s-1+2sY(s)-4+Y(s)=2\frac{e^{-3s}}{s^2}$$

$$(s^2+2s+1)Y(s)-2s-5=2\frac{e^{-3s}}{s^2}$$

$$(s+1)^2Y(s)=2s+5+2\frac{e^{-3s}}{s^2}$$

$$\Rightarrow Y(s)=\frac{2s+5}{(s+1)^2}+\frac{2e^{-3s}}{s^2(s+1)^2}$$

$$\text{or } Y(s)=\frac{2s+5}{(s+1)^2}+e^{-3s}\left[\frac{2}{s^2(s+1)^2}\right] \quad \text{——— (A)}$$

Consider

$$\frac{2s+5}{(s+1)^2}=\frac{A}{s+1}+\frac{B}{(s+1)^2}$$

$$\Rightarrow 2s+5=A(s+1)+B$$

For B, put $s=-1$

$$-2+5=B \quad \Rightarrow \boxed{B=3}$$

Comparing coff. of s

$$\boxed{A=2}$$

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So $\dfrac{2S+5}{(S+1)^2} = \dfrac{2}{S+1} + \dfrac{3}{(S+1)^2}$

Now Consider

$$\frac{2}{S^2(S+1)^2} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S+1} + \frac{D}{(S+1)^2}$$

$$\Rightarrow 2 = AS(S+1)^2 + B(S+1)^2 + CS^2(S+1) + DS^2 \quad \text{———} (\alpha)$$

For B put $S=0$

$$2 = B(1) \quad \Rightarrow \boxed{B=2}$$

For D, put $S=-1$

$$2 = D(1) \quad \Rightarrow \boxed{D=2}$$

Comparing Coff. of S^3 & S^2

$$A + C = 0 \quad \text{———} \text{I}$$

$$2A + B + C + D = 0 \quad \text{———} \text{II}$$

$\text{I} \Rightarrow A = -C$

Put in II

$$-2C + B + C + D = 0$$

$$B - C + D = 0$$

$$2 - C + 2 = 0 \quad \Rightarrow \boxed{C=4}$$

Put in I

$$A + 4 = 0 \quad \Rightarrow \boxed{A=-4}$$

So $\dfrac{2}{S^2(S+1)^2} = -\dfrac{4}{S} + \dfrac{2}{S^2} + \dfrac{4}{S+1} + \dfrac{2}{(S+1)^2}$

Put in (A)

$$Y(S) = \frac{2}{S+1} + \frac{3}{(S+1)^2} + e^{-3S}\left[-\frac{4}{S} + \frac{2}{S^2} + \frac{4}{S+1} + \frac{2}{(S+1)^2}\right]$$

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$$\text{or } Y(s) = \frac{2}{s+1} + \frac{3}{(s+1)^2} - 4\frac{e^{-3s}}{s} + 2\frac{e^{-3s}}{s^2} + 4\frac{e^{-3s}}{(s+1)} + 2\frac{e^{-3s}}{(s+1)^2}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)}\right\} + 2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\}$$

$$y(t) = 2e^{-t} + 3te^{-t} - 4U_3(t) + 2t\,U_3(t) + 4U_3(t)\,e^{-(t-3)} + 2U_3(t)(t-3).e^{-(t-3)}$$

$$y(t) = 2e^{-t} + 3te^{-t} + U_3(t)\left[-4 + 2t + 4e^{-(t-3)} + 2(t-3).e^{-(t-3)}\right]$$

is req. soln.

Q14 $\frac{d^2y}{dt^2} + y = \begin{cases} \cos t & \text{if } 0 \le t < \pi/2 \\ 0 & \text{if } \frac{\pi}{2} < t < \infty \end{cases}$ $\quad y(0) = 3,\ y'(0) = -1$

Sol: Given eq. is

$$\frac{d^2y}{dt^2} + y = f(t) \quad \text{——— (1)}$$

where $f(t) = \begin{cases} \cos t & \text{if } 0 \le t < \pi/2 \\ 0 & \text{if } \frac{\pi}{2} < t < \infty \end{cases}$

Now $f(t)$ can be written as

$$f(t) = \cos t + U_{\pi/2}(t)\sin(t - \pi/2)$$

Put in (1)

$$\frac{d^2y}{dt^2} + y = \cos t + U_{\pi/2}(t).\sin(t - \pi/2)$$

taking Laplace transform of both sides

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{\cos t\} + \mathcal{L}\{U_{\pi/2}(t).\sin(t-\pi/2)\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{s}{s^2+1} + \frac{e^{-\pi/2\,s}}{s^2+1}$$

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$$s^2Y(s) - 3s + 1 + Y(s) = \frac{s}{s^2+1} + \frac{e^{-\frac{\pi}{2}s}}{s^2+1}$$

$$(s^2+1)Y(s) - 3s + 1 = \frac{s}{s^2+1} + \frac{e^{-\pi/2\, s}}{s^2+1}$$

$$(s^2+1)Y(s) = \frac{s}{s^2+1} + \frac{e^{-\pi/2\, s}}{s^2+1} + 3s - 1$$

$$\Rightarrow\quad Y(s) = \frac{s}{(s^2+1)^2} + \frac{e^{-\pi/2\, s}}{(s^2+1)^2} + \frac{3s-1}{(s^2+1)}$$

$$= \frac{3s-1}{s^2+1} + \frac{s}{(s^2+1)^2} + \frac{e^{-\frac{\pi}{2}s}}{(s^2+1)^2}$$

$$Y(s) = \frac{3s}{s^2+1} - \frac{1}{s^2+1} + \frac{1}{2}(-1)^1\frac{d}{ds}\left(\frac{1}{s^2+1}\right) + \frac{e^{-\frac{\pi}{2}s}}{(s^2+1)^2}$$

Then

$$\mathcal{L}^{-1}\{Y(s)\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{(-1)^1\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-\pi/2\, s}}{(s^2+1)^2}\right\}$$

$$y(t) = 3\cos t - \sin t + \frac{1}{2}t\sin t + U_{\pi/2}(t)\, f(t-\pi/2)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

$$f(t) = \frac{1}{2}(\sin t - t\cos t) \qquad \because \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$$

$$\Rightarrow f(t-\pi/2) = \frac{1}{2}\left[\sin(t-\pi/2) - (t-\pi/2)\cos(t-\pi/2)\right]$$

Put in above eq.

$$y(t) = 3\cos t - \sin t + \frac{1}{2}t\sin t + U_{\pi/2}(t)\cdot\frac{1}{2}\left[\sin(t-\pi/2) - (t-\tfrac{\pi}{2})\cos(t-\pi/2)\right]$$

$$= 3\cos t - \sin t + \frac{1}{2}t\sin t - \frac{1}{2}U_{\pi/2}(t)\left[\sin(\pi/2 - t) + (t-\pi/2)\cos(\pi/2 - t)\right]$$

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or
$$y(t) = 3\cos t - \sin t + \frac{1}{2}t\sin t - \frac{1}{2}U_{\pi/2}(t)\left(\cos t + (t-\pi/2)\sin t\right)$$
is req. soln.

Q15 $\frac{d^2y}{dt^2} + 4y = \sin t - U_{2\pi}(t)\cdot\sin t$ $\qquad y(0) = 0 = y'(0)$

Soln. Given eq. is

$$\frac{d^2y}{dt^2} + 4y = \sin t - U_{2\pi}(t)\sin t \quad \text{———} ①$$

Taking Laplace transform of both sides of ①

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{U_{2\pi}(t)\sin t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

$$s^2Y(s) + 4sY(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

$$(s^2+4s)Y(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{1}{(s^2+1)(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}$$

$$\text{or } Y(s) = \frac{1}{(s^2+1)(s^2+4)} - e^{-2\pi s}\frac{1}{(s^2+1)(s^2+4)} \quad \text{———} Ⓐ$$

Consider

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow 1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$1 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$$

$$1 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

Comparing Coffs. of both sides

$$A + C = 0 \quad \text{———} \mathrm{I}$$

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$B + D = 0$ ———— II

$4A + C = 0$ ———— III

$4B + D = 1$ ———— IV

I $\Rightarrow A = -C$

Put in III

$-4C + C = 0$

$-3C = 0 \Rightarrow \boxed{C = 0}$

Put in I

$A + 0 = 0 \Rightarrow \boxed{A = 0}$

II $\Rightarrow B = -D$

Put in IV

$4(-D) + D = 1$

$-3D = 1 \Rightarrow \boxed{D = -\frac{1}{3}}$

Put in II

$B - \frac{1}{3} = 0 \Rightarrow \boxed{B = \frac{1}{3}}$

So

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{0.s + \frac{1}{3}}{s^2+1} + \frac{0.s - \frac{1}{3}}{s^2+4}$$

$$= \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}$$

Put in (A)

$$Y(s) = \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)} - e^{-2\pi s}\left(\frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}\right)$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\}$$

$$y(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1^2}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{2}{s^2+2^2}\right\}$$

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$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{1}{3}U_{2\pi}(t)\sin(t-2\pi) + \frac{1}{6}U_{2\pi}(t)\sin 2(t-2\pi)$$

$$= \frac{1}{6}(2\sin t - \sin 2t) - \frac{1}{6}U_{2\pi}(t)\left[2\sin t - \sin 2t\right]$$

$$= \frac{1}{6}(2\sin t - \sin 2t)(1 - U_{2\pi}(t))$$ is req. soln.

Q16 $\dfrac{d^3y}{dt^3} - 4\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 6y = te^t$ $\quad y(0) = 0 = y'(0)$ & $y''(0) = 1$

Soln. Given eq. is

$$\frac{d^3y}{dt^3} - 4\frac{d^2y}{dt^2} + \frac{dy}{dt} + 6y = te^t \qquad (1)$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} - 4\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} + 6\mathcal{L}\{y(t)\} = \mathcal{L}\{te^t\}$$

$$\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] - 4\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) + y(0)\right] + 6Y(s) = -\frac{d}{ds}\left(\frac{1}{s-1}\right)$$

$$\left[s^3Y(s) - 0 - 0 - 1\right] - 4\left[s^2Y(s) - 0 - 0\right] + sY(s) + 0 + 6Y(s) = -\frac{-1}{(s-1)^2}$$

$$s^3Y(s) - 1 - 4s^2Y(s) + sY(s) + 6Y(s) = \frac{1}{(s-1)^2}$$

$$(s^3 - 4s^2 + s + 6)Y(s) = 1 + \frac{1}{(s-1)^2}$$

Solve by S.D.

-1	1	-4	1	6
	0	-1	5	-6
	1	-5	6	0

So $s = -1$

Hence $(s+1)$ is one factor

So above eq. becomes

$$(s+1)(s^2 - 5s + 6)Y(s) = \frac{(s-1)^2 + 1}{(s-1)^2}$$

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$$(s+1)(s-2)(s-3)Y(s) = \frac{s^2-2s+2}{(s-1)^2}$$ 1.8

or $$Y(s) = \frac{s^2-2s+2}{(s-1)^2(s+1)(s-2)(s-3)} \quad \text{——— (2)}$$

We resolve it into partial fractions.

Consider.

$$\frac{s^2-2s+2}{(s-1)^2(s+1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1} + \frac{D}{(s-2)} + \frac{E}{(s-3)}$$

Multiplying both sides by $(s-1)^2(s+1)(s-2)(s-3)$

$$s^2-2s+2 = A(s-1)(s+1)(s-2)(s-3) + B(s+1)(s-2)(s-3) + C(s-1)^2(s-2)(s-3) + D(s-1)^2(s+1)(s-3) + E(s-1)^2(s+1)(s-2)$$

For B, put $s = 1$

$$1-2+2 = B(2)(-1)(-2)$$

$$1 = 4B \Rightarrow \boxed{B = \tfrac{1}{4}}$$

For C put $s = -1$

$$1+2+2 = C(4)(-3)(-4)$$

$$5 = 48C \Rightarrow \boxed{C = \tfrac{5}{48}}$$

For D, put $s = 2$

$$4-4+2 = D(1)(3)(-1)$$

$$2 = -3D \Rightarrow \boxed{D = -\tfrac{2}{3}}$$

For E, put $s = 3$

$$9-6+2 = E(4)(4)(1) \quad \text{or} \quad 5 = 16E$$

$$\Rightarrow \boxed{E = 5/16}$$

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For A, Comparing Coff. of s^4

$$A+C+D+E = 0$$

or $A + \frac{5}{48} - \frac{2}{3} + \frac{5}{16} = 0$

$$A = \frac{2}{3} - \frac{5}{48} - \frac{5}{16}$$

$$= \frac{32-5-15}{48}$$

$$A = \frac{12}{48}$$

$$\Rightarrow \boxed{A = \frac{1}{4}}$$

So from B

$$Y(s) = \frac{1}{4(s-1)} + \frac{1}{4(s-1)^2} + \frac{5}{48(s+1)} - \frac{2}{3(s-2)} + \frac{5}{16(s-3)}$$

then

$$L^{-1}\{Y(s)\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \frac{5}{48}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{2}{3}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{16}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$y(t) = \frac{1}{4}e^{t} + \frac{1}{4}te^{t} + \frac{5}{48}e^{-t} - \frac{2}{3}e^{2t} + \frac{5}{16}e^{3t}$ is req. soln.

Q17 $\frac{d^3y}{dt^3} - 5\frac{d^2y}{dt^2} + 7\frac{dy}{dt} - 3y = 20\sin t$ where

$y(0) = 0 = y'(0)$, $y''(0) = -2$

Sol. Given eq. is

$$\frac{d^3y}{dt^3} - 5\frac{d^2y}{dt^2} + 7\frac{dy}{dt} - 3y = 20\sin t \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

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$$\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} - 5\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 7\mathcal{L}\left\{\frac{dy}{dt}\right\} - 3\mathcal{L}\{y(t)\} = \mathcal{L}\{20\sin t\}$$

$$\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] - 5\left[s^2Y(s) - sy(0) - y'(0)\right] + 7\left[sY(s) - y(0)\right] - 3Y(s)$$

$$= 20\,\mathcal{L}\{\sin t\}$$

$$\left[s^3Y(s) - 0 - 0 + 2\right] - 5\left[s^2Y(s) - 0 - 0\right] + 7\left[sY(s) - 0\right] - 3Y(s) = 20\cdot\frac{1}{s^2+1}$$

$$s^3Y(s) + 2 - 5s^2Y(s) + 7sY(s) - 3Y(s) = \frac{20}{s^2+1}$$

$$(s^3 - 5s^2 + 7s - 3)Y(s) = -2 + \frac{20}{s^2+1}$$

Solve by S.D.

Hence we have from above

$$\begin{array}{c|cccc}
1 & 1 & -5 & 7 & -3 \\
 & 0 & 1 & -4 & 3 \\
\hline
 & 1 & -4 & 3 & \vert\, 0 \\
1 & 0 & 1 & -3 & \\
\hline
 & 1 & -3 & \vert\, 0 &
\end{array}$$

$$(s-1)(s-1)(s-3)Y(s) = -2 + \frac{20}{s^2+1}$$

$$Y(s) = \frac{-2(s^2+1)+20}{(s-1)^2(s-3)(s^2+1)}$$

$$= \frac{-2(s^2+1-10)}{(s-1)^2(s-3)(s^2+1)}$$

$$= \frac{-2(s^2-9)}{(s-1)^2(s-3)(s^2+1)}$$

$$= \frac{-2\cancel{(s-3)}(s+3)}{(s-1)^2\cancel{(s-3)}(s^2+1)}$$

$$Y(s) = \frac{-2s-6}{(s-1)^2(s^2+1)} \quad \text{——— (2)}$$

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Consider

$$\frac{-2s-6}{(s-1)^2(s^2+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1}$$

Multiplying both sides by $(s-1)^2(s^2+1)$

$$-2s-6 = A(s-1)(s^2+1) + B(s^2+1) + (Cs+D)(s-1)^2 \quad \text{——— (3)}$$

For B, put $s=1$

$$-8 = B(2) \Rightarrow \boxed{B=-4}$$

Now (3) $\Rightarrow$

$$-2s-6 = A(s^3+s-s^2-1) + B(s^2+1) + (Cs+D)(s^2-2s+1)$$

$$-2s-6 = A(s^3-s^2+s-1) + B(s^2+1) + Cs^3-2Cs^2+Cs+Ds^2-2Ds+D$$

Comparing Coeff. of s^3, s^2, s, Const

$$A+C = 0 \quad \text{——— I}$$

$$-A+B-2C+D = 0 \quad \text{——— II}$$

$$A+C-2D = -2 \quad \text{——— III}$$

$$-A+B+D = -6 \quad \text{——— IV}$$

From I $A+C=0$, Put in III

$$0-2D = -2 \Rightarrow \boxed{D=1}$$

Put in IV

$$-A-4+1 = -6$$

$$-A-3 = -6$$

$$-A = -6+3$$

$$-A = -3$$

$$\boxed{A=3}$$

Put in I $3+C=0 \Rightarrow \boxed{C=-3}$

Hence from (2)

$$Y(s) = \frac{3}{(s-1)} - \frac{4}{(s-1)^2} + \frac{-3s+1}{(s^2+1)}$$

Now

$$\mathcal{L}^{-1}\{Y(s)\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-3s+1}{s^2+1}\right\}$$

$$y(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-3s}{s^2+1} + \frac{1}{s^2+1}\right\}$$

$$y(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$y(t) = 3e^{t} - 4te^{t} - 3\cos t + \sin t$ is req. soln

Q18 $\left(\frac{d^2}{dt^2} + 6\frac{d}{dt} + 7\right)^2 y = 0$, $y(0) = 0 = y'(0) = y''(0)$, $y'''(0) = 4\sqrt{2}$

Soln. Given eq. is

$$\left(\frac{d^2}{dt^2} + 6\frac{d}{dt} + 7\right)^2 y = 0$$

or $$\left[\left(\frac{d^2}{dt^2}\right)^2 + \left(6\frac{d}{dt}\right)^2 + (7)^2 + 2\left(\frac{d^2}{dt^2}\right)\left(6\frac{d}{dt}\right) + 2\left(6\frac{d}{dt}\right)(7) + 2\left(\frac{d^2}{dt^2}\right)(7)\right] y = 0$$

$$\left[\frac{d^4}{dt^4} + 36\frac{d^2}{dt^2} + 49 + 12\frac{d^3}{dt^3} + 84\frac{d}{dt} + 14\frac{d^2}{dt^2}\right] y = 0$$

$$\frac{d^4y}{dt^4} + 36\frac{d^2y}{dt^2} + 49y + 12\frac{d^3y}{dt^3} + 84\frac{dy}{dt} + 14\frac{d^2y}{dt^2} = 0$$

$$\frac{d^4y}{dt^4} + 12\frac{d^3y}{dt^3} + 50\frac{d^2y}{dt^2} + 84\frac{dy}{dt} + 49y = 0 \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} + 12\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} + 50\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 84\mathcal{L}\left\{\frac{dy}{dt}\right\} + 49\mathcal{L}\{y(t)\} = 0$$

$$[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] + 12[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)]$$
$$+ 50\left(s^2Y(s) - sy(0) - y'(0)\right) + 84[sY(s) - y(0)] + 49Y(s) = 0$$

$$\left[s^4 Y(s) - 0 - 0 - 0 - 4\sqrt{2}\right] + 12\left[s^3 Y(s) - 0 - 0 - 0\right] + 50\left[s^2 Y(s) - 0 - 0\right]$$ 113

$$+ 84\left[sY(s) - 0\right] + 49Y(s) = 0$$

$$s^4 Y(s) - 4\sqrt{2} + 12s^3 Y(s) + 50s^2 Y(s) + 84sY(s) + 49Y(s) = 0$$

$$(s^4 + 12s^3 + 50s^2 + 84s + 49)\,Y(s) = 4\sqrt{2}$$

$$(s^2 + 6s + 7)^2\,Y(s) = 4\sqrt{2}$$

$$\Rightarrow \quad Y(s) = \frac{4\sqrt{2}}{(s^2+6s+7)^2} \qquad \text{———} (\alpha)$$

To find $\mathcal{L}^{-1}\{Y(s)\}$, we use Convolution theorem.

Take $F(s) = \dfrac{1}{s^2+6s+7}$

& $G(s) = \dfrac{1}{s^2+6s+7}$

then $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+6s+7}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+9-2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2-(\sqrt{2})^2}\right\}$$

$$= \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+3)^2-(\sqrt{2})^2}\right\}$$

$$= \frac{1}{\sqrt{2}} e^{-3t} \sinh\sqrt{2}t$$

So $\mathcal{L}^{-1}\{F(s)\} = f(t)$

Also $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+6s+7}\right\}$

$$= \mathcal{L}^{-1}\{G(s)\} = g(t).$$

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Thm: $\mathcal{L}^{-1}\left\{F(s)\cdot G(s)\right\} = f(t) * g(t)$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+6s+7)^2}\right\} = \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t \frac{1}{\sqrt{2}}\cdot e^{-3u}\sinh\sqrt{2}u\cdot\frac{1}{\sqrt{2}}e^{-3(t-u)}\cdot\sinh\sqrt{2}(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-3t}\cdot\sinh\sqrt{2}u\cdot\sinh\sqrt{2}(t-u)\,du$$

$$= \frac{1}{4}\int_0^t e^{-3t}\left[2\sinh\sqrt{2}u\cdot\sinh\sqrt{2}(t-u)\right]du$$

$$= \frac{1}{4}\int_0^t e^{-3t}\left[\cosh\left(\sqrt{2}u+\sqrt{2}(t-u)\right)-\cosh\left(\sqrt{2}u-\sqrt{2}(t-u)\right)\right]du$$

$$= \frac{1}{4}\int_0^t e^{-3t}\left[\cosh\sqrt{2}t-\cosh(2\sqrt{2}u-\sqrt{2}t)\right]du$$

$$= \frac{e^{-3t}}{4}\int_0^t\left[\cosh\sqrt{2}t-\cosh(2\sqrt{2}u-\sqrt{2}t)\right]du$$

$$= \frac{e^{-3t}}{4}\int_0^t\cosh\sqrt{2}t\,du - \frac{e^{-3t}}{4}\int_0^t\cosh(2\sqrt{2}u-\sqrt{2}t)\,du$$

$$= \frac{e^{-3t}}{4}\cosh\sqrt{2}t\,\Big|u\Big|_0^t - \frac{e^{-3t}}{4}\left|\frac{\sinh(2\sqrt{2}u-\sqrt{2}t)}{2\sqrt{2}}\right|_0^t$$

$$= \frac{e^{-3t}}{4}\cosh\sqrt{2}t\,(t-0) - \frac{e^{-3t}}{8\sqrt{2}}\left[\sinh(2\sqrt{2}t-\sqrt{2}t)-\sinh(-\sqrt{2}t)\right]$$

$$= \frac{e^{-3t}}{4}\left[t\cosh\sqrt{2}t-\frac{1}{2\sqrt{2}}\left(\sinh\sqrt{2}t+\sinh\sqrt{2}t\right)\right]$$

$$= \frac{e^{-3t}}{4}\left[t\cosh\sqrt{2}t-\frac{1}{2\sqrt{2}}\sinh\sqrt{2}t-\frac{1}{2\sqrt{2}}\sinh\sqrt{2}t\right]$$

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$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+6s+7)^2}\right\} = \frac{e^{-3t}}{4}\left[t\cosh\sqrt{2}\,t - \frac{2}{2\sqrt{2}}\sinh\sqrt{2}\,t\right]$$

$$\mathcal{L}^{-1}\left\{\frac{4\sqrt{2}}{(s^2+6s+7)^2}\right\} = 4\sqrt{2}\cdot\frac{e^{-3t}}{4}\left[t\cosh\sqrt{2}\,t - \frac{1}{\sqrt{2}}\sinh\sqrt{2}\,t\right]$$

$$\mathcal{L}^{-1}\{Y(s)\} = \sqrt{2}\,e^{-3t}\left[t\cosh\sqrt{2}\,t - \frac{1}{\sqrt{2}}\sinh\sqrt{2}\,t\right]$$

$$y(t) = e^{-3t}\left[\sqrt{2}\,t\cosh\sqrt{2}\,t - \sinh\sqrt{2}\,t\right]$$ is req. soln.

Q19 $\frac{d^4y}{dt^4} + 5\frac{d^2y}{dt^2} + 4y = 1 - U_\pi(t)$, $y(0) = 0 = y'(0) = y''(0) = y'''(0)$

Sol. Given eq. is

$$\frac{d^4y}{dt^4} + 5\frac{d^2y}{dt^2} + 4y = 1 - U_\pi(t) \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} + 5\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{1 - U_\pi(t)\}$$

$$\left[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right] + 5\left[s^2Y(s) - sy(0) - y'(0)\right] + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$(s^4Y(s) - 0 - 0 - 0 - 0) + 5(s^2Y(s) - 0 - 0) + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$s^4Y(s) + 5s^2Y(s) + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$(s^4 + 5s^2 + 4)Y(s) = \frac{1 - e^{-\pi s}}{s}$$

$$(s^2+1)(s^2+4)Y(s) = \frac{1 - e^{-\pi s}}{s}$$

$$Y(s) = (1-e^{-\pi s})\left(\frac{1}{s(s^2+1)(s^2+4)}\right) \quad \text{——— (1)}$$

Consider

$$\frac{1}{s(s^2+1)(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{s^2+4}$$

Multiplying both sides by $s(s^2+1)(s^2+4)$

$$1 = A(s^2+1)(s^2+4) + (Bs+C)s(s^2+4) + (Ds+E)(s)(s^2+1) \quad \text{——— (A)}$$

For A, put $s = 0$ in (A)

$$1 = A(1)(4) \Rightarrow \boxed{A = \frac{1}{4}}$$

From (A)

$$1 = A(s^4+5s^2+4) + (Bs+C)(s^3+4s) + (Ds+E)(s^3+s)$$

Equating Coffs. of like powers of s

$$A + B + D = 0 \quad \text{——— I}$$

$$C + E = 0 \quad \text{——— II}$$

$$5A + 4B + D = 0 \quad \text{——— III}$$

$$4C + E = 0 \quad \text{——— IV}$$

$\text{II} \Rightarrow C = -E$

Put in IV

$$-4E + E = 0$$

$$-3E = 0 \Rightarrow \boxed{E = 0}$$

Hence $\boxed{C = 0}$

Subt. I from III

$$4A + 3B = 0$$

$$4(1/4) + 3B = 0$$

$$1 + 3B = 0$$

$$\boxed{B = -1/3}$$

Put in I

$\frac{1}{4} - \frac{1}{3} + D = 0$

$-\frac{1}{12} + D = 0$

$\boxed{D = \frac{1}{12}}$

So

$$Y(s) = (1 - e^{-\pi s})\left[\frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)}\right]$$

$$Y(s) = \left(\frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)}\right) - e^{-\pi s}\left(\frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)}\right)$$

Then

$$L^{-1}\{Y(s)\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}L^{-1}\left\{\frac{s}{s^2+(1)^2}\right\} + \frac{1}{12}L^{-1}\left\{\frac{s}{(s)^2+(2)^2}\right\} - \frac{1}{4}L^{-1}\left\{e^{-\pi s}\cdot\frac{1}{s}\right\}$$

$$+ \frac{1}{3}L^{-1}\left\{e^{-\pi s}\cdot\frac{s}{s^2+(1)^2}\right\} - \frac{1}{12}L^{-1}\left\{\frac{s}{(s)^2+(2)^2}\right\}$$

$$y(t) = \left(\frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t\right) - \frac{1}{4}U_\pi(t) + \frac{1}{3}U_\pi(t)\cos(t-\pi) - \frac{1}{12}U_\pi(t)\cos 2(t-\pi)$$

$$= \left(\frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t\right) - U_{\pi(t)}\left(\frac{1}{4} - \frac{1}{3}\cos(t-\pi) + \frac{1}{12}\cos 2(t-\pi)\right)$$

$y(t) = f(t) - U_\pi(t)f(t-\pi)$ where $f(t) = \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t$

is req. soln.

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220 $\quad t\frac{d^2y}{dt^2} + (t-1)\frac{dy}{dt} - y = 0$

$y(0) = 5, \quad y(\infty) = 0$

Sol. Given eq. is

$t\frac{d^2y}{dt^2} + (t-1)\frac{dy}{dt} - y = 0$ ——— ①

Taking Laplace transform of both sides of ①

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$$\mathcal{L}\left\{t\frac{d^2y}{dt^2}\right\} + \mathcal{L}\left\{(t-1)\frac{dy}{dt}\right\} - \mathcal{L}\{y(t)\} = 0$$ 118

$$\mathcal{L}\left\{t\frac{d^2y}{dt^2}\right\} + \mathcal{L}\left\{t\frac{dy}{dt}\right\} - \mathcal{L}\left\{\frac{dy}{dt}\right\} - \mathcal{L}\{y(t)\} = 0$$

$$-\frac{d}{ds}\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - \frac{d}{ds}\mathcal{L}\left\{\frac{dy}{dt}\right\} - (sY(s)-y(0)) - Y(s) = 0$$

$$\because \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$$

$$-\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right) - \frac{d}{ds}\left(sY(s) - y(0)\right) - sY(s) + y(0) - Y(s) = 0$$

$$-\frac{d}{ds}\left(s^2Y(s) - 5s - y'(0)\right) - \frac{d}{ds}\left(sY(s) - 5\right) - sY(s) + 5 - Y(s) = 0$$

$$-\left[s^2Y'(s) + 2sY(s) - 5\right] - \left[sY'(s) + Y(s)\right] - sY(s) + 5 - Y(s) = 0$$

$$-s^2Y'(s) - 2sY(s) + 5 - sY'(s) - Y(s) - sY(s) + 5 - Y(s) = 0$$

$$-(s^2+s)Y'(s) - 3sY(s) - 2Y(s) + 10 = 0$$

$$-(s^2+s)Y'(s) - (3s+2)Y(s) + 10 = 0$$

$$(s^2+s)Y'(s) + (3s+2)Y(s) = 10$$

$$Y'(s) + \frac{3s+2}{s^2+s}Y(s) = \frac{10}{s^2+s}$$

$$Y'(s) + \frac{3s+2}{s(s+1)}Y(s) = \frac{10}{s(s+1)} \quad \text{------} \quad (2)$$

It is a linear diff. eq. in $Y(s)$

$$\text{I.F.} = e^{\int \frac{3s+2}{s(s+1)} ds}$$

Consider $\frac{3s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$

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Multiplying both sides by $S(S+1)$

$$3S+2 = A(S+1) + BS$$

For A, put $S=0$

$$2 = A(1) \Rightarrow \boxed{A=2}$$

For B, put $S=-1$

$$-3+2 = B(-1)$$

$$-1 = -B \Rightarrow \boxed{B=1}$$

So $$\frac{3S+2}{S(S+1)} = \frac{2}{S} + \frac{1}{S+1}$$

So
$$\text{I.F.} = e^{\int\left(\frac{2}{S}+\frac{1}{S+1}\right)dS} = e^{2\ln S+\ln(S+1)} = e^{\ln S^2+\ln(S+1)} = e^{\ln S^2(S+1)} = S^2(S+1)$$

Multiplying both sides of ② by I.F. $S^2(S+1)$

$$d\left(Y(S).S^2(S+1)\right) = \frac{10}{S(S+1)}.S^2(S+1)\,dS$$

$$\int d\left(Y(S).S^2(S+1)\right) = \int 10\,S\,dS$$

$$Y(S).S^2(S+1) = 10\frac{S^2}{2} + C$$

or $$Y(S).S^2(S+1) = 5S^2 + C$$

$$Y(S) = \frac{5S^2+C}{S^2(S+1)}$$

$$Y(S) = \frac{5}{S+1} + \frac{C}{S^2(S+1)}$$

or $$Y(S) = \frac{5}{S+1} + C\left(\frac{1}{S^2(S+1)}\right) \quad \text{③}$$

Consider

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$$\frac{1}{S^2(S+1)} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S+1}$$

$\Rightarrow 1 = AS(S+1) + B(S+1) + CS^2$

For B, put $S=0$.

$1 = B(1) \Rightarrow \boxed{B=1}$

For C, put $S=-1$

$1 = C(-1)^2 \Rightarrow \boxed{C=1}$

For A, Comparing Coff. of S^2

$A+C = 0$

$\Rightarrow A+1 = 0 \Rightarrow \boxed{A=-1}$

So $\frac{1}{S^2(S+1)} = \frac{-1}{S} + \frac{1}{S^2} + \frac{1}{S+1}$

Hence from (3)

$$Y(S) = \frac{5}{S+1} + c\left(\frac{-1}{S} + \frac{1}{S^2} + \frac{1}{S+1}\right)$$

$$= \frac{5}{S+1} - \frac{c}{S} + \frac{c}{S^2} + \frac{c}{S+1}$$

$$Y(S) = \frac{c+5}{S+1} - c\left(\frac{1}{S} - \frac{1}{S^2}\right)$$

Now $\mathcal{L}^{-1}\{Y(S)\} = (c+5)\mathcal{L}^{-1}\left\{\frac{1}{S+1}\right\} - c\,\mathcal{L}^{-1}\left\{\frac{1}{S} - \frac{1}{S^2}\right\}$

$y(t) = (c+5)e^{-t} - c(1-t)$

Now $y(\infty) = 0 \Rightarrow y \to 0$ as $t \to \infty$

So $0 = 0 - c(1-t) \Rightarrow c(1-t) = 0 \Rightarrow c = 0$

Hence $y(t) = 5e^{-t}$ is req. soln.

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Q21 $\frac{dx}{dt} - x - 3y = 0$

$\frac{dy}{dt} - 5x - 3y = 0$ $\qquad x(0) = 2,\ y(0) = 1$

Sol: Given eqs. are

$$\left.\begin{aligned} \frac{dx}{dt} - x - 3y &= 0 \\ \frac{dy}{dt} - 5x - 3y &= 0 \end{aligned}\right]$$

Taking Laplace Transform of above eqs.

$$\left.\begin{aligned} \mathcal{L}\left\{\frac{dx}{dt}\right\} - \mathcal{L}\{x(t)\} - 3\mathcal{L}\{y(t)\} &= 0 \\ \mathcal{L}\left\{\frac{dy}{dt}\right\} - 5\mathcal{L}\{x(t)\} - 3\mathcal{L}\{y(t)\} &= 0 \end{aligned}\right]$$

or
$$\left.\begin{aligned} sX(s) - x(0) - X(s) - 3Y(s) &= 0 \\ sY(s) - y(0) - 5X(s) - 3Y(s) &= 0 \end{aligned}\right]$$

$$\left.\begin{aligned} sX(s) - 2 - X(s) - 3Y(s) &= 0 \\ sY(s) - 1 - 5X(s) - 3Y(s) &= 0 \end{aligned}\right]$$

or
$$\left.\begin{aligned} (s-1)X(s) - 3Y(s) - 2 &= 0 \\ -5X(s) + (s-3)Y(s) - 1 &= 0 \end{aligned}\right]$$

Solve by Cramer's rule

$$\frac{X(s)}{\begin{vmatrix} -3 & -2 \\ s-3 & -1 \end{vmatrix}} = \frac{-Y(s)}{\begin{vmatrix} s-1 & -2 \\ -5 & -1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} s-1 & -3 \\ -5 & s-3 \end{vmatrix}}$$

$$\frac{X(s)}{3 + 2s - 6} = \frac{-Y(s)}{-s + 1 - 10} = \frac{1}{s^2 - 4s + 3 - 15}$$

$$\frac{X(s)}{2s-3} = \frac{-Y(s)}{-s-9} = \frac{1}{s^2-4s-12}$$

$$\frac{X(s)}{2s-3} = \frac{Y(s)}{s+9} = \frac{1}{s^2-4s-12}$$

$$\Rightarrow X(s) = \frac{2s-3}{s^2-4s-12}$$

$$\& \quad Y(s) = \frac{s+9}{s^2-4s-12}$$

or $\quad X(s) = \dfrac{2s-3}{(s+2)(s-6)}$ ——— (A)

$\quad Y(s) = \dfrac{s+9}{(s+2)(s-6)}$ ——— (B)

Consider $\dfrac{2s-3}{(s+2)(s-6)} = \dfrac{A}{s+2} + \dfrac{B}{s-6}$

$$\Rightarrow 2s-3 = A(s-6) + B(s+2)$$

For A, put $s = -2$

$$-4-3 = A(-8) \Rightarrow \boxed{A = \frac{7}{8}}$$

For B, put $s = 6$

$$12-3 = B(6+2) \Rightarrow \boxed{B = \frac{9}{8}}$$

So $\dfrac{2s-3}{(s+2)(s-6)} = \dfrac{7}{8(s+2)} + \dfrac{9}{8(s-6)}$

Now Consider

$$\frac{s+9}{(s+2)(s-6)} = \frac{A}{s+2} + \frac{B}{s-6}$$

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$$\Rightarrow S+9 = A(S-6) + B(S+2)$$

For A, put $S = -2$

$$-2+9 = A(-8) \Rightarrow \boxed{A = -\frac{7}{8}}$$

For B, put $S = 6$

$$6+9 = B(6+2) \Rightarrow \boxed{B = \frac{15}{8}}$$

So

$$\frac{S+9}{(S+2)(S-6)} = -\frac{7}{8(S+2)} + \frac{15}{8(S-6)}$$

Put values in eqs. (A) & (B)

$$\left.\begin{aligned} X(S) &= \frac{7}{8(S+2)} + \frac{9}{8(S-6)} \\ Y(S) &= -\frac{7}{8(S+2)} + \frac{15}{8(S-6)} \end{aligned}\right]$$

Taking inverse Laplace transform

$$\mathcal{L}^{-1}\{X(S)\} = \frac{7}{8}\mathcal{L}^{-1}\left\{\frac{1}{S+2}\right\} + \frac{9}{8}\mathcal{L}^{-1}\left\{\frac{1}{S-6}\right\}$$

$$\mathcal{L}^{-1}\{Y(S)\} = -\frac{7}{8}\mathcal{L}^{-1}\left\{\frac{1}{S+2}\right\} + \frac{15}{8}\mathcal{L}^{-1}\left\{\frac{1}{S-6}\right\}$$

or

$$\left.\begin{aligned} x(t) &= \frac{7}{8}e^{-2t} + \frac{9}{8}e^{6t} \\ y(t) &= -\frac{7}{8}e^{-2t} + \frac{15}{8}e^{6t} \end{aligned}\right] \text{ is req. soln.}$$

Q22 $\frac{dx}{dt} - 4x - 5y = e^{-4t}$ $\qquad x(0) = 0$

$\frac{dy}{dt} + 4x + 4y = e^{4t}$ $\qquad y(0) = 0$

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Sol. Given eqs. are

$$\left.\begin{aligned} \frac{dx}{dt} - 4x - 5y &= e^{-4t} \\ \frac{dy}{dt} + 4x + 4y &= e^{4t} \end{aligned}\right]$$

Taking Laplace transform of both sides of above eqs.

$$\left.\begin{aligned} \mathcal{L}\left\{\frac{dx}{dt}\right\} - 4\mathcal{L}\{x(t)\} - 5\mathcal{L}\{y(t)\} &= \mathcal{L}\{e^{-4t}\} \\ \mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\mathcal{L}\{x(t)\} + 4\mathcal{L}\{y(t)\} &= \mathcal{L}\{e^{4t}\} \end{aligned}\right]$$

or $$\left.\begin{aligned} SX(S) - x(0) - 4X(S) - 5Y(S) &= \frac{1}{S+4} \\ SY(S) - y(0) + 4X(S) + 4Y(S) &= \frac{1}{S-4} \end{aligned}\right]$$

$$\left.\begin{aligned} (S-4)X(S) - 5Y(S) &= \frac{1}{S+4} \\ 4X(S) + (S+4)Y(S) &= \frac{1}{S-4} \end{aligned}\right]$$

$$\left.\begin{aligned} (S-4)X(S) - 5Y(S) - \frac{1}{S+4} &= 0 \\ 4X(S) + (S+4)Y(S) - \frac{1}{S-4} &= 0 \end{aligned}\right]$$

Solve by cramer's rule

$$\frac{X(S)}{\begin{vmatrix} -5 & -\frac{1}{S+4} \\ S+4 & \frac{-1}{S-4} \end{vmatrix}} = \frac{-Y(S)}{\begin{vmatrix} S-4 & \frac{-1}{S+4} \\ 4 & \frac{-1}{S-4} \end{vmatrix}} = \frac{1}{\begin{vmatrix} S-4 & -5 \\ 4 & S+4 \end{vmatrix}}$$

$$\frac{X(S)}{\frac{5}{S-4} + 1} = \frac{-Y(S)}{-1 + \frac{4}{S+4}} = \frac{1}{S^2 - 16 + 20}$$

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$$\frac{X(s)}{\frac{s+s-4}{s+4}} = \frac{-Y(s)}{\frac{-s-4+4}{s+4}} = \frac{1}{s^2+4}$$

$$\frac{X(s)}{\frac{s+1}{s-4}} = \frac{Y(s)}{\frac{s}{s+4}} = \frac{1}{s^2+4}$$

$$\Rightarrow X(s) = \frac{s+1}{(s-4)(s^2+4)} \quad \text{——— (A)}$$

$$\& \quad Y(s) = \frac{s}{(s+4)(s^2+4)} \quad \text{——— (B)}$$

Consider $\dfrac{s+1}{(s-4)(s^2+4)} = \dfrac{A}{s-4} + \dfrac{Bs+C}{s^2+4}$

$$\Rightarrow s+1 = A(s^2+4) + (Bs+C)(s-4)$$

For A, put $s = 4$

$$4+1 = A(16+4)$$

$$5 = 20A \Rightarrow \boxed{A = \tfrac{1}{4}}$$

Equating coffs. of s^2 & s

$$A + B = 0 \quad \text{——— I}$$

$$-4B + C = 1 \quad \text{——— II}$$

$$\text{I} \Rightarrow \tfrac{1}{4} + B = 0 \Rightarrow \boxed{B = -\tfrac{1}{4}}$$

$$\text{II} \Rightarrow -4\left(-\tfrac{1}{4}\right) + C = 1$$

$$1 + C = 1 \Rightarrow \boxed{C = 0}$$

So. $\dfrac{s+4}{(s-4)(s^2+4)} = \dfrac{1}{4(s-4)} + \dfrac{-\frac{1}{4}s}{s^2+4}$

$$= \frac{1}{4(s-4)} - \frac{s}{4(s^2+4)}$$

Consider

$$\frac{s}{(s+4)(s^2+4)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow \quad s = A(s^2+4) + (Bs+C)(s+4)$$

For A, put $s = -4$

$$-4 = A(16+4)$$

$$-4 = 20A \quad \Rightarrow \boxed{A = \frac{-1}{5}}$$

Equating coffs. of s^2 & s

$$A+B = 0 \quad \text{——— I}$$

$$4B + C = 1 \quad \text{——— II}$$

$$\text{I} \Rightarrow \quad -\frac{1}{5} + B = 0 \quad \Rightarrow \boxed{B = \frac{1}{5}}$$

$$\text{II} \Rightarrow \quad 4\left(\frac{1}{5}\right) + C = 1$$

$$C = 1 - \frac{4}{5}$$

$$\boxed{C = \frac{1}{5}}$$

So

$$\frac{s}{(s+4)(s^2+4)} = \frac{-1}{5(s+4)} + \frac{\frac{1}{5}s + \frac{1}{5}}{s^2+4}$$

$$= \frac{-1}{5(s+4)} + \frac{s+1}{5(s^2+4)}$$

Put values in (A) + (B)

$$X(s) = \frac{1}{4(s-4)} - \frac{s}{4(s^2+4)}$$

$$Y(s) = \frac{-1}{5(s+4)} + \frac{s+1}{5(s^2+4)}$$

Taking inverse Laplace transform of above eqs.

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$$\mathcal{L}^{-1}\{X(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{-1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{s^2+4}\right\}$$

or $$\mathcal{L}^{-1}\{X(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{(s)^2+(2)^2}\right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$\left.\begin{aligned} x(t) &= \frac{1}{4}e^{4t} - \frac{1}{4}\cos 2t \\ y(t) &= -\frac{1}{5}e^{-4t} + \frac{1}{5}\cos 2t + \frac{1}{10}\sin 2t \end{aligned}\right\} \text{ is req. soln.}$$

Q23 $$2\frac{dx}{dt} + \frac{dy}{dt} - x - y = 1 \qquad x(0) = 0$$

$$\frac{dx}{dt} + \frac{dy}{dt} + 2x - y = t \qquad y(0) = 0$$

Sol. Given eqs. are

$$2\frac{dx}{dt} + \frac{dy}{dt} - x - y = 1$$

$$\frac{dx}{dt} + \frac{dy}{dt} + 2x - y = t$$

Taking Laplace transform of both sides of above eqs.

$$2\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} - \mathcal{L}\{x(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{1\}$$

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{x(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$$

$$2(sX(s) - x(0)) + sY(s) - y(0) - X(s) - Y(s) = \frac{1}{s}$$

$$sX(s) - x(0) + sY(s) - y(0) + 2X(s) - Y(s) = \frac{1}{s^2}$$

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$$\left.\begin{aligned} 2SX(S) + SY(S) - X(S) - Y(S) &= \frac{1}{S} \\ SX(S) + SY(S) + 2X(S) - Y(S) &= \frac{1}{S^2} \end{aligned}\right]$$

$$(2S-1)X(S) + (S-1)Y(S) = \frac{1}{S} \quad \text{——— (A)}$$

$$(S+2)X(S) + (S-1)Y(S) = \frac{1}{S^2} \quad \text{——— (B)}$$

Subt. (B) from (A)

$$(2S-1-S-2)X(S) = \frac{1}{S} - \frac{1}{S^2}$$

$$(S-3)X(S) = \frac{S-1}{S^2}$$

$$\boxed{X(S) = \frac{S-1}{S^2(S-3)}}$$

Put in (B)

$$(S+2)\frac{(S-1)}{S^2(S-3)} + (S-1)Y(S) = \frac{1}{S^2}$$

$$\frac{S+2}{S^2(S-3)} + Y(S) = \frac{1}{S^2(S-1)}$$

$$Y(S) = \frac{1}{S^2(S-1)} - \frac{S+2}{S^2(S-3)}$$

$$= \frac{1}{S^2}\left[\frac{1}{S-1} - \frac{S+2}{S-3}\right]$$

$$= \frac{1}{S^2}\left[\frac{S-3-(S-1)(S+2)}{(S-1)(S-3)}\right]$$

$$= \frac{1}{S^2}\left[\frac{\cancel{S}-3-S^2-\cancel{S}+2}{(S-1)(S-3)}\right]$$

$$= \frac{-S^2-1}{S^2(S-1)(S-3)}$$

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$$Y(s) = -\frac{s^2+1}{s^2(s-1)(s-3)} \qquad \text{——— ②}$$

1-9

Consider

$$\frac{s-1}{s^2(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3}$$

$$\Rightarrow s-1 = As(s-3) + B(s-3) + Cs^2$$

For B, put $s=0$

$$-1 = -3B \Rightarrow \boxed{B = \frac{1}{3}}$$

For C, put $s=3$

$$2 = 9C \Rightarrow \boxed{C = \frac{2}{9}}$$

Equating coff. of s^2

$$A + C = 0$$

$$A + \frac{2}{9} = 0$$

$$\boxed{A = -\frac{2}{9}}$$

$$\frac{s-1}{s^2(s-3)} = -\frac{2}{9s} + \frac{1}{3s^2} + \frac{2}{9(s-3)}$$

Now Consider

$$\frac{-s^2-1}{s^2(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-3}$$

$$\Rightarrow -s^2-1 = As(s-1)(s-3) + B(s-1)(s-3) + Cs^2(s-3) + Ds^2(s-1)$$

For B, put $s=0$

$$-1 = B(-1)(-3) \Rightarrow \boxed{B = -\frac{1}{3}}$$

For C, put $s=1$

$$-1-1 = C(-2) \Rightarrow \boxed{C = 1}$$

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For D, put $S = 3$

$$-9-1 = D(9)(2)$$

$$-10 = 18D \Rightarrow \boxed{D = -\frac{5}{9}}$$

Equating Coff. of S^3

$$A + C + D = 0$$

$$\Rightarrow A + 1 - \frac{5}{9} = 0$$

$$A + \frac{4}{9} = 0 \Rightarrow \boxed{A = -\frac{4}{9}}$$

So

$$\frac{-S^2-1}{S^2(S-1)(S-3)} = -\frac{4}{9S} - \frac{1}{3S^2} + \frac{1}{S-1} - \frac{5}{9(S-3)}$$

Hence

$$X(S) = -\frac{2}{9S} + \frac{1}{3S^2} + \frac{2}{9(S-3)}$$

$$\& \quad Y(S) = -\frac{4}{9S} - \frac{1}{3S^2} + \frac{1}{S-1} - \frac{5}{9(S-3)}$$

Taking inverse Laplace transform of above eqs.

$$\mathcal{L}^{-1}\{X(S)\} = -\frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{S}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{S^2}\right\} + \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{S-3}\right\}$$

$$\mathcal{L}^{-1}\{Y(S)\} = -\frac{4}{9}\mathcal{L}^{-1}\left\{\frac{1}{S}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{S^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\} - \frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{S-3}\right\}$$

$$x(t) = -\frac{2}{9} + \frac{1}{3}t + \frac{2}{9}e^{3t}$$

$$y(t) = -\frac{4}{9} - \frac{1}{3}t + e^{t} - \frac{5}{9}e^{3t}$$

is req. soln.

Q24 $\quad \frac{dx}{dt} + \frac{dy}{dt} = t$

$$\frac{d^2x}{dt^2} - y = e^{-t}$$

$x(0) = 3, \quad x'(0) = -2, \quad y(0) = 0$

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Sol. Given eqs. are

$$\frac{dx}{dt} + \frac{dy}{dt} = t$$
$$\frac{d^2x}{dt^2} - y = e^{-t}$$

Taking Laplace transform of both sides of above eqs.

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\{t\}$$
$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{e^{-t}\}$$

$$sX(s) - x(0) + sY(s) - y(0) = \frac{1}{s^2}$$
$$s^2X(s) - sx(0) - x'(0) - Y(s) = \frac{1}{s+1}$$

$$sX(s) - 3 + sY(s) - 0 = \frac{1}{s^2}$$
$$s^2X(s) - 3s + 2 - Y(s) = \frac{1}{s+1}$$

or $$sX(s) + sY(s) = \frac{1}{s^2} + 3$$
$$s^2X(s) - Y(s) = \frac{1}{s+1} + 3s - 2$$

$$sX(s) + sY(s) = \frac{1+3s^2}{s^2}$$
$$s^2X(s) - Y(s) = \frac{1+(3s-2)(s+1)}{s+1}$$

$$sX(s) + sY(s) = \frac{1+3s^2}{s^2} \quad \text{——— (A)}$$
$$s^2X(s) - Y(s) = \frac{3s^2+s-1}{s+1} \quad \text{——— (B)}$$

Multiplying eq. (A) by s^2 & (B) by s

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$$S\,X(s) + S^3 Y(s) = 1+3S^2 \quad \text{——— (A)}$$

$$S^3 X(s) - S\,Y(s) = \frac{3S^3+S^2-S}{S+1} \quad \text{——— (B)}$$

Sub. (B) from (A)

$$S^3 Y(s) + S\,Y(s) = 3S^2+1 - \frac{3S^3+S^2-S}{S+1}$$

$$(S^3+S)Y(s) = \frac{(3S^2+1)(S+1)-(3S^3+S^2-S)}{(S+1)}$$

$$S(S^2+1)Y(s) = \frac{\cancel{3S^3}+3S^2+S+1-\cancel{3S^3}-S^2+S}{S+1}$$

$$S(S^2+1)Y(s) = \frac{2S^2+2S+1}{S+1}$$

$$\Rightarrow Y(s) = \frac{2S^2+2S+1}{S(S+1)(S^2+1)}$$

Now

Consider $\dfrac{2S^2+2S+1}{S(S+1)(S^2+1)} = \dfrac{A}{S} + \dfrac{B}{S+1} + \dfrac{CS+D}{S^2+1}$

$$\Rightarrow 2S^2+2S+1 = A(S+1)(S^2+1) + BS(S^2+1) + (CS+D)(S^2+S)$$

For A, put $S=0$

$$1 = A(1) \Rightarrow \boxed{A=1}$$

For B, put $S=-1$

$$2-2+1 = B(-1)(1+1)$$

$$1 = -2B \Rightarrow \boxed{B=-\tfrac{1}{2}}$$

Comparing Coeffs. of S^3 & S^2

$$A+B+C=0 \quad \text{——— I}$$

$$A+C+D=2 \quad \text{——— II}$$

$$\text{I} \Rightarrow 1-\tfrac{1}{2}+C=0$$

$\frac{1}{2} + C = 0 \quad \Rightarrow \boxed{C = -\frac{1}{2}}$ 133

$\text{II} \Rightarrow \quad 1 - \frac{1}{2} + D = 2$

$\frac{1}{2} + D = 2$

$D = 2 - \frac{1}{2} \quad \Rightarrow \boxed{D = \frac{3}{2}}$

So $Y(s) = \frac{1}{s} - \frac{1}{2(s+1)} + \frac{-\frac{1}{2}s + \frac{3}{2}}{s^2+1}$

$Y(s) = \frac{1}{s} - \frac{1}{2(s+1)} - \frac{s-3}{2(s^2+1)}$

Applying $\mathcal{L}^{-1}$ on both sides

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{s}{s^2+1} - \frac{3}{s^2+1}\}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{s}{s^2+1^2}\} + \frac{3}{2}\mathcal{L}^{-1}\{\frac{1}{s^2+1^2}\}$

$\boxed{y(t) = 1 - \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{3}{2}\sin t}$

Diff. w.r.t. t

$\frac{dy}{dt} = \frac{1}{2}e^{-t} + \frac{1}{2}\sin t + \frac{3}{2}\cos t$

Put in given eq

$\frac{dx}{dt} + \frac{1}{2}e^{-t} + \frac{1}{2}\sin t + \frac{3}{2}\cos t = t$

$\frac{dx}{dt} = t - \frac{1}{2}e^{-t} - \frac{1}{2}\sin t - \frac{3}{2}\cos t$

Integ. w.r.t. t

$x(t) = \frac{t^2}{2} + \frac{1}{2}e^{-t} + \frac{1}{2}\cos t - \frac{3}{2}\sin t + C$

But $x(0) = 3$

So $3 = 0 + \frac{1}{2} + \frac{1}{2} - 0 + C$

or $3 = 1 + c \Rightarrow \boxed{c = 2}$

So req. soln. is

$$\left.\begin{aligned} x(t) &= \frac{t^2}{2} + \frac{1}{2}e^{-t} \quad \cos t - \frac{3}{2}\sin t + 2 \\ y(t) &= 1 - \frac{1}{2}e^{-t} - \text{[illegible]} + \frac{3}{2}\sin t \end{aligned}\right]$$

<u>Q25</u>

$$\frac{dx}{dt} + 2\frac{d^2y}{dt^2} = e^{-t}$$

$$\frac{dx}{dt} + 2x - y = 1 \qquad x(0) = 0 = y(0) = y'(0)$$

<u>Sol.</u> Given eqs. are

$$\left.\begin{aligned} \frac{dx}{dt} + 2\frac{d^2y}{dt^2} &= e^{-t} \\ \frac{dx}{dt} + 2x - y &= 1 \end{aligned}\right]$$

Taking Laplace transform of both sides of above eqs.

$$\left.\begin{aligned} \mathcal{L}\left\{\frac{dx}{dt}\right\} + 2\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= \mathcal{L}\{e^{-t}\} \\ \mathcal{L}\left\{\frac{dx}{dt}\right\} + 2\mathcal{L}\{x(t)\} - \mathcal{L}\{y(t)\} &= \mathcal{L}\{1\} \end{aligned}\right]$$

$$\left.\begin{aligned} sX(s) - x(0) + 2\left(s^2Y(s) - sy(0) - y'(0)\right) &= \frac{1}{s+1} \\ sX(s) - x(0) + 2X(s) - Y(s) &= \frac{1}{s} \end{aligned}\right]$$

or
$$\left.\begin{aligned} sX(s) - 0 + 2s^2Y(s) &= \frac{1}{s+1} \\ sX(s) - 0 + 2X(s) - Y(s) &= \frac{1}{s} \end{aligned}\right]$$

$$sX(s) + 2s^2Y(s) = \frac{1}{s+1} \quad \text{——— (A)}$$

$$(s+2)X(s) - Y(s) = \frac{1}{s} \quad \text{——— (B)}$$

Multiplying eq. ⓑ by s^2

$$s\,X(s) + 2s^2 Y(s) = \frac{1}{s+1} \quad \text{------} \; Ⓐ$$

$$2s^2(s+2)X(s) - 2s^2 Y(s) = 2s \quad \text{------} \; Ⓑ$$

Adding Ⓐ + Ⓑ

$$\left[s + 2s^2(s+2)\right] X(s) = 2s + \frac{1}{s+1}$$

$$s\left(1 + 2s(s+2)\right) X(s) = \frac{2s^2+2s+1}{s+1}$$

$$s\left(1 + 2s^2 + 4s\right) X(s) = \frac{2s^2+2s+1}{s+1}$$

$$X(s) = \frac{2s^2+2s+1}{s(s+1)(2s^2+4s+1)}$$

Now Consider

$$\frac{2s^2+2s+1}{s(s+1)(2s^2+4s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{2s^2+4s+1}$$

$$\Rightarrow 2s^2+2s+1 = A(s+1)(2s^2+4s+1) + Bs(2s^2+4s+1) + (Cs+D)(s^2+s)$$

For A, put $s = 0$

$$1 = A(1) \Rightarrow \boxed{A = 1}$$

For B, put $s = -1$

$$2 - 2 + 1 = B(-1)(2-4+1)$$

$$1 = B(-1)(-1) \Rightarrow \boxed{B = 1}$$

Comparing Coff of s^3 & s^2

$$2A + 2B + C = 0 \quad \text{------} \; \mathrm{I}$$

$$5A + B + D = 2 \quad \text{------} \; \mathrm{II}$$

$$\mathrm{I} \Rightarrow 2(1) + 2(1) + C = 0 \Rightarrow \boxed{C = -4}$$

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$$\text{II} \Rightarrow \quad 5 + 1 + D = 2$$

$$6 + D = 2 \quad \Rightarrow \boxed{D = -4}$$

So

$$\frac{2s^2+2s+1}{s(s+1)(2s^2+4s+1)} = \frac{1}{s} + \frac{1}{s+1} + \frac{-4s-4}{2s^2+4s+1}$$

$$= \frac{1}{s} + \frac{1}{s+1} - \frac{4(s+1)}{(2s^2+4s+1)}$$

So

$$X(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{2(s+1)}{s^2+2s+\frac{1}{2}}$$

$$= \frac{1}{s} + \frac{1}{s+1} - \frac{2(s+1)}{s^2+2s+1+\frac{1}{2}-1}$$

$$= \frac{1}{s} + \frac{1}{s+1} - \frac{2(s+1)}{(s+1)^2-\frac{1}{2}}$$

$$X(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{2(s+1)}{(s+1)^2-\left(\frac{1}{\sqrt{2}}\right)^2}$$

Applying $\mathcal{L}^{-1}$ on both sides of eq.

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2-\left(\frac{1}{\sqrt{2}}\right)^2}\right\}$$

$$\boxed{x(t) = 1 + e^{-t} - 2e^{-t}\cosh\left(\frac{1}{\sqrt{2}}t\right)}$$

Diff. w.r.t. t, we get

$$\frac{dx}{dt} = -e^{-t} + 2e^{-t}\cosh\left(\frac{1}{\sqrt{2}}t\right) - 2\,e^{-t}\sinh\left(\frac{1}{\sqrt{2}}t\right)\cdot\frac{1}{\sqrt{2}}$$

Put in given derived eq