Module - 5

Numerical solutions of Ordinary Differential Equations

In this module we will study.

1. Taylor's series method.
2. Picards method
3. Euler's method
4. Modified Euler's method
5. Runge-Kutta method.
6. Predictor - corrector method.

consider the ODE is.

$$\frac{dy}{dx} = f(x,y) \quad \text{with } y(x_0) = y_0.$$

$y(x)$

Taylor's series method:

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0.$$

Expand $y(x)$ by using Taylor's series about point (x_0, y_0).

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots$$

$$\cdots \frac{(x-x_0)^n}{n!} y^n(x_0) + \cdots$$

① Solve $\frac{dy}{dx} = x - y^2$ with $y(0) = 1$, find $y(0.1)$

using Taylor's series method.

Solution: Taylor's series Expansion is given by.

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0)$$

$$+ \frac{(x-x_0)^4}{4!} y^{iv}(x) + \cdots \rightarrow ①$$

$$\frac{dy}{dx} = x - y^2 \quad y(0) = 1$$

$$x_0 = 0 \quad y_0 = 1$$

$$y' = \frac{dy}{dx} = x - y^2 \Rightarrow y'_0 = x_0 - y_0^2$$

$$= 0 - 1^2 = -1$$

$$y'_0 = -1.$$

$$y'' = 1 - 2yy' \rightarrow y''_0 = 1 - 2 \times 1 \times -1 = 1 + 2 = 3.$$

$$y''' = -2[yy'' + y'^2] \Rightarrow y'''_0 = -2[y_0 y''_0 + y_0'^2]$$

$$= -2[1 \times 3 + (-1)^2]$$

$$= -2[4] = -8$$

$$y'''_0 = -8.$$

$$y^{iv} = -2[yy''' + y'y'' + 2y'y'']$$

$$= -2[3y'y'' + yy''']$$

$$y_0^{iv} = -2[3y'_0 y''_0 + y_0 y'''_0]$$

$$= -2[3 \times (-1) \times 3 + 1 \times (-8)]$$

$$= -2[-9 - 8] = 34$$

$$y_0^{iv} = 34.$$

$$y(x) = y(0) + \frac{x}{1!}(-1) + \frac{x^2}{2!}(3) + \frac{x^3}{3!}(-8) + \frac{x^4}{4!}(34).$$

$x = 0.1.$

$$y(0.1) = 1 + \frac{(0.1)}{1!}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24} \times 34$$

$y(0.1) = 1 + 0.1 + 0.015 - 0.001333 + 0.00014166$

$y(0.1) = 0.91380866.$

$y(0.1) = 0.9138.$

This required solution.

Solve $\frac{dy}{dx} = 2y + 3e^x$ with $y(0) = 0$ at $x = 0.2$

using Taylor's series method. and

compare the numerical solution obtained with exact solution.

Solution:

Taylor's series expansion is given by.

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y_0^{iv} + \dots \rightarrow (1)$$

$y(0) = 0$

$x_0 = 0, \quad y_0 = 0.$

$$y(x) = y_0 + \frac{x}{1!} y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{4!} y_0^{(iv)} \quad \text{——} (2)$$

Now we find $y_0', y_0'', y_0''', y_0^{(iv)}$.

$$\frac{dy}{dx} = y' = 2y + 3e^x \Rightarrow y_0' = 2y_0 + 3e^{x_0}$$

$$= 2 \times 0 + 3e^0$$

$$y_0' = 0 + 3 = 3.$$

$$y'' = 2y' + 3e^x \Rightarrow y_0'' = 2y_0' + 3e^0$$

$$= 6 + 3 = 9$$

$$y_0'' = 9.$$

$y''' = 2y'' + 3e^x \Rightarrow y_0''' = 2y_0'' + 3e^0$

$= 2 \times 9 + 3$

$y_0''' = 21$

$y^{iv} = 2y''' + 3e^x \Rightarrow y_0^{iv} = 2y_0''' + 3e^0$

$= 42 + 3$

$= 45.$

$x = 0.2.$

$$y(0.2) = 0 + \frac{(0.2)}{1!} 3 + \frac{(0.2)^2}{2} 9 + \frac{(0.2)^3}{6} 21 + \frac{(0.2)^4}{24} 45.$$

$$= 0 + 0.6 + 0.18 + 0.028 + 0.003$$

$y(0.2) = 0.811.$

which is numerical solution.

$y(0.2) = 0.811.$

Analytical solution.

$$\frac{dy}{dx} = 2y + 3e^{x}.$$

$$\frac{dy}{dx} - 2y = 3e^{x}.$$

$$\frac{dy}{dx} + P(x)y = Q(x).$$

$P(x) = -2$, $Q(x) = 3e^{x}$.

$P(x)$ & $Q(x)$.

$$I.F = e^{\int P(x)\,dx}.$$

$$I.F = e^{\int -2\,dx} = e^{-2x}.$$

$$\boxed{y.I.F = \int Q(x).I.F\,dx + C}$$

$$y.I.F = \int Q(x)\,I.F.\,dx + c.$$

$$y.e^{-2x} = \int 3e^{x}.e^{-2x}\,dx + c.$$

$$y\,e^{-2x} = 3\int e^{-x}\,dx + c.$$

$$y\,e^{-2x} = -3e^{-x} + c \rightarrow (1)$$

$y(0) = 0$. $x_0 = 0$, $y = 0$.

$$y = -3e^{x} + ce^{2x} \rightarrow (1)$$

$$y(0) = -3e^{0} + ce^{2\times 0}.$$

$$0 = -3 + c$$

$$c = 3.$$

$$y = -3e^{x} + 3e^{2x}$$

$$\boxed{y(x) = -3e^{x} + 3e^{2x}}$$

$x = 0.2$

$$y(0.2) = -3e^{0.2} + 3e^{0.4}$$

$$= -3(1.2214) + 3(1.4918$$

$$= 0.81127.$$

$$y(0.2) = 0.81127.$$

Exact solution is $y(0.2) = 0.81127$

Picards method of successive approximations:

Consider the D.E $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$.

$$dy = f(x,y)\,dx.$$

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\,dx.$$

$$(y)_{x_0}^{x} = \int_{x_0}^{x} f(x,y)\,dx.$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx.$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx \rightarrow (1)$$

put $y = y_0$ in (1) we get

first approximation $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$.

second approximation $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$

Third ,, $y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\,dx$

$$y^{(n+1)} = y_0 + \int_{x_0}^{x} f(x, y^{(n)})\,dx. \quad n=0,1,2\ldots$$

This is general Iterative formula.

1. Employ picards method to obtain, correct to four places of decimal, solution of differential equation $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ given that $y = 0$ when $x = 0$.

Solution:

$$\frac{dy}{dx} = x^2 + y^2 \qquad x_0 = 0, \quad y_0 = 0.$$

By picards method

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx.$$

First approximation.

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

$$y_1 = 0 + \int_{0}^{x} (x^2 + 0^2)\, dx.$$

$$= \left(\frac{x^3}{3}\right)_0^x$$

$$y_1 = \frac{x^3}{3}$$

$x = 0.4$

$$y_1 = \frac{(0.4)^3}{3} = 0.021333$$

First approximation $y_1 = 0.0213$

Second approximation

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx.$$

$$= 0 + \int_0^x \cancel{f(x,}\,(x^2 + y_1^2)\, dx.$$

$$= \int_0^x \left(x^2 + \left(\frac{x^3}{3}\right)^2\right) dx$$

$$= \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx.$$

$$= \left(\frac{x^3}{3} + \frac{x^7}{63}\right)_0^x$$

$$y_2 = \frac{x^3}{3} + \frac{x^7}{63}.$$

at $x = 0.4$.

$$y_2 = \frac{(0.4)^3}{3} + \frac{(0.4)^3}{63}.$$

$$= 0.021333 + 0.000026$$

$$y_2 = 0.021356$$

$$y_2 = 0.0213.$$

$y_1 = 0.0213,$

$y_2 = 0.0213.$

$x = 0.4$ $\boxed{y = 0.0213}$