Euler's Method:

Consider the Differential equation

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0.$$

what is $y(x_n) = y_n$? when x_n given.

$$y_n = y(x_n) = y_{n+1} + h\, f(x_{n-1}, y_{n-1}). \rightarrow (1)$$

where $h = x_n - x_{n-1}$

$n=1,\quad y_1 = y_0 + h f(x_0, y_0)$

$n=2 \quad y_2 = y_1 + h f(x_1, y_1)$

$n=3 \quad y(x_3) = y_3 = y_2 + h f(x_2, y_2).$

$\vdots$

(1) Solve $\frac{dy}{dx} = x+y \quad y(0) = 1$ find $y(0.3)$ taking step size $h = 0.1$ Using Euler's Method.

By Euler's method

$f(x, y) = x+y. \quad x_0 = 0, \; y_0 = 1, \; h = 0.1$

$$y(0.1) = y_1 = y_0 + h f(x_0, y_0)$$

$$= 1 + (0+1)0.1$$

$$= 1 + 0.1$$

$$y(0.1) = 1.1$$

$$y(0.2) = y_2 = y_1 + h\, f(x_1, y_1)$$

$$= 1.1 + 0.1\,(0.1 + 1.1)$$

$$= 1.1 + 0.1\,(1.2)$$

$$= 1.22.$$

$$y(0.3) = y_3 = y_2 + h\, f(x_2, y_2).$$

$$= 1.22 + (0.1)\,(0.2 + 1.22).$$

$$= 1.22 + (0.1)\,(1.42)$$

$$= 1.362.$$

$$\boxed{y(0.3) = 1.362}$$

Modified Euler's method:

Consider the differential equation is

$$\frac{dy}{dx} = f(x,y) \quad \text{with } y(x_0) = y_0.$$

To find $y(x_1) = y_1$ at $x_1 = x_0 + h$.

Initial approximation

$$y_1^{(0)} = y_0 + h f(x_0, y_0).$$

First approximation.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

.

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]$$

Continue this procedure till two approximations are equal.

Now we have $\frac{dy}{dx} = f(x,y)$ $y(x_1) = y_1$

$y(x_2) = y_2$ at $x_2 = x_1 + h$.

We use above procedure again.

Find $y(0.1)$ and $y(0.2)$ using Euler's modified formula given that $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$.

Solution: $\frac{dy}{dx} = x^2 - y$ $\quad x_0 = 0, \quad y_0 = 1 \quad h = 0.1$

To find $y(0.1)$.

By Euler's modified formula

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0).$$

$$= 1 + (0.1)(0-1)$$

$$y_1^{(0)} = 0.9$$

First approximation.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[(0-1) + \left((0.1)^2 - 0.9\right)\right]$$

$$y_1^{(1)} = 0.9055$$

Second approximation.

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.1}{2}\left[(0-1) + \left[(0.1)^2 - 0.9055\right]\right]$$

$$y_1^{(2)} = 0.905225$$

Third approximation.

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + \frac{(0.1)}{2}\left[(0-1) + \left[(0.1)^2 - 0.905225\right)\right]\Big]$$

$$= 0.90523875$$

upto 4 decimal places.

$$y_1^{(2)} = y_1^{(3)} = 0.9052.$$

$$\boxed{y(0.1) = 0.9052}$$

To find $y(0.2)$.

$$y_2^{(0)} = y_1 + h f(x_1, y_1)$$

$$= 0.9052 + (0.1)\left((0.1)^2 - 0.9052\right)\Big)$$

$$= 0.81568$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2\ y_2^{(0)})\right]$$

$$= 0.9052 + \frac{(0.1)}{2}\left[(0.1)^2 - 0.9052\right) + \left((0.2)^2 - 0.81568\right]$$

$$= 0.9052 + \frac{(0.1)}{2}\left(-0.8952 + (0.04 - 0.81568)\right]\Big)$$

$$= 0.821656$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2\ y_2^{(1)})\right)$$

$$= 0.9052 + \frac{(0.1)}{2}\left[-0.8952 + (0.04 - 0.821656)\right]$$

$$= 0.8213572$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$y_2^{(3)} = 0.9052 + \frac{(0.1)}{2}\left[-0.8952 + (0.04 - 0.8213572)\right]$$

$$y_2^{(3)} = 0.82137214.$$

y_2 upto 4 decimal places.

$$y_2^{(2)} = y_2^{(3)} = 0.8213.$$

$$\boxed{y(0.2) = 0.8213}$$