Runge-Kutta methods:

Consider the Differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0.$$

1. First order Runge-Kutta method:

$$y_1 = y_0 + h f(x_0, y_0).$$

Euler's method is the R-K method of the first order.

2. Second order R-K Method:

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f(x_0 + h, y_0 + k_1).$$

3. Third order R-K. Method.

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = h f(x_0 + h, y_0 + 2k_2 - k_1).$$

4. Fourth order R-K method:

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3).$$

Solve $\frac{dy}{dx} = x+y^2$, $y(0)=1$. Find $y(0.2)$ by Runge-Kutta method of order 4.

Solution:

$f(x,y) = x+y^2 \qquad x_0 = 0, \quad y_0 = 1. \quad h = 0.2.$

R-K method:

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

$$k_1 = hf(x_0, y_0).$$

$$= 0.2(0+1^2) = 0.2.$$

$$k_1 = 0.2$$

$$k_2 = hf(x_0+\tfrac{h}{2}, y_0+\tfrac{k_1}{2}).$$

$$= 0.2f(0.1 \;\; 1.1) = 0.2(0.1+(1.1)^2).$$

$$k_2 = 0.262$$

$$k_3 = hf(x_0+\tfrac{h}{2}, y_0+\tfrac{k_2}{2})$$

$$= 0.2\,[f(0.1 \;\; 1.131)] = 0.2(0.1+(1.131)^2)$$

$$k_3 = 0.2758$$

$$k_4 = hf(x_0+h, y_0+k_3)$$

$$= 0.2[f(0.2, 1.2758)] = 0.2(0.2+(1.2758)^2).$$

$$= 0.3655$$

$$y(0.2) = y_1 = 1 + \frac{1}{6}(0.2 + 2(0.262) + 2(0.2758) + 0.3655)$$

$$y_1 = 1.2735$$

Milne's Predictor - corrector formulae

consider the differential equation is

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 .$$

$$y^{(P)}_{n+1} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right).$$

i.e $$y^{(P)}_{n+1} = y_{n-3} + \frac{4h}{3}\left(2f_{n-2} - f_{n-1} + 2f_n\right)$$

This is called milne's predicter formula.

$$y^{(C)}_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'^{(P)}_{n+1}\right]$$

$$y^{(C)}_{n+1} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f^{(P)}_{n+1}\right]$$

This is called milne's corrector formula.

Using Taylor's series method, find y for $x = 0.1, 0.2, 0.3$. given that $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$ continue the solution at $x = 0.4$ using Milne's method.

Solution: $\frac{dy}{dx} = xy + y^2$ $\quad x_0 = 0, \; y_0 = 1, \; h = 0.1$.

Taylor's series method

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$$

$$y_0' = xy + y^2 = \cancel{0+1^2}$$

$$y_0' = 0 + 1^2 = 1$$

$$y'' = xy' + y + 2yy'$$

$$y_0'' = 0 + 1 + 2 \times 1 \times 1 = 3$$

$$y''' = xy'' + y' + y' + 2yy'' + 2y'^2$$

$$y_0''' = 0 + 2 + 6 + 2 = 10$$

$$y^{iv} = xy''' + y'' + 2y'' + 2yy''' + 2y'y'' + 4y'y''$$

$$y_0^{iv} = 0 + 3 + 6 + 20 + 6 + 12 = 47.$$

$$y_1 = y(0.1) = 1 + \frac{(0.1)}{1} 1 + \frac{(0.1)^2}{2} \times 3 + \frac{(0.1)^3}{6} 10 + \frac{(0.1)^4}{24} 47.$$

$$= 1 + 0.1 + 0.015 + 0.00166 + 0.000195 = 1.1168$$

$y(0.1) = 1.116$

$y(0.2) = 1.276$

$y(0.3) = 1.502$.

find above values ~~by~~ using Taylor's series method.

Milne's Predictor formula is

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$y_1' = x_1 y_1 + y_1^2$$

$$= (0.1)(1.116) + (1.116)^2$$

$$y_1' = 1.357$$

$$y_2' = x_2 y_2 + y_2^2$$

$$= (0.2)(1.276) + (1.276)^2$$

$$= 1.8833$$

$$y_3' = x_3 y_3 + y_3^2$$

$$= (0.3)(1.502) + (1.502)^2$$

$$= 2.7066$$

$$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(1.357) - 1.8833 + 2(2.7066)\right]$$

$$y_4^{(P)} = 1.8323.$$

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Milne's corrector formula.

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[y_2' + \cancel{4y_3}\, 4y_3' + y_4'^{(p)}\right]$$

$$y_4'^{(p)} = x_4 y_4^{(p)} + y_4^{p2}$$

$$= (0.4)(1.8323) + (1.8323)^2$$

$$= 4.0902$$

$$y_4^{(c)} = \overset{1.276}{y_2} + \frac{(0.1)}{3}\left[1.8833 + 4(2.7066) + 4.0902\right]$$

$$y_4^{(c)} = 1.836$$

$$\boxed{y(0.4) = 1.836}$$

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