Geometry

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2022-09-24

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Introduction

What follows endeavors to encapsulate the author's knowledge of geometry (and the notions upon which it depends, which decidedly extend well beyond the boundaries of geometry itself) as he studies theoretical particle physics and string theory.

Chapter 1

Manifolds

1.1 Construction

Let M be a extcolor{magenta}{second-countable}[1], extcolor{magenta}{Hausdorff}[2], extcolor{magenta}{locally Euclidean topological space} of dimension n. We define an extcolor{magenta}{equivalence relation} on the set of extcolor{magenta}{homeomorphisms} between extcolor{magenta}{open} subsets of M and $\mathbb{R}^n$ given by $\phi \sim \psi$ when $\psi \circ \phi^{-1}$ is extcolor{magenta}{smooth}. Let $\{\phi_\alpha\}$ then be an equivalence class such that the domains $\{U_\alpha\}$ of the maps ϕ_α cover M: the set $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ (where $\phi_\alpha : U_\alpha \to \mathbb{R}^n$) is then a extcolor{blue}{maximal atlas}[3]. When endowed with $\mathcal{U}$, we say M is an n-dimensional extcolor{blue}{smooth manifold}[4] (or manifol d). Let $(\phi, U) \in \mathcal{U}$: ϕ is a extcolor{blue}{coordinate chart} (or chart) and the components of ϕ, x^i

[1]Arguably, the truly important property here is extcolor{magenta}{paracompactness}, which is slightly stronger and enables partitions of unity (enabling local-to-global promotions). However, it is a result that Hausdorff, second countable, extcolor{magenta}{locally compact} space is paracompact (and we get local compactness follows from locally Euclidean). Second countability also contributes to the feasibility of Euclidean embeddings and other nice, preferable behavior.

References: Second countability and manifolds

[2]Hausdorff topological spaces feature points which are sufficiently disjoint: in particular, calculus depends upon limits, and Hausdorff $\implies$ unique limits as desired (note, though, that the converse isn't true).

[3]Definitions vary here (indeed, it is more conventional to merely require "maximal" atlases) but the general motivation is as follows: given a chart ϕ on a manifold M, there are likely uncountably many collections of charts covering M containing ϕ, but there is a *unique* (i.e., canonical) choice of equivalence class of charts containing ϕ.

References: Axiom of choice and maximal atlases

[4]Our consideration of differential topology/geometry is motivated by physics, which interests itself in the dynamics (or change) of our universe. extcolor{magenta}{Calculus}, in a word, is the mathematics of change: hence, we are interested in studying the *least structured* space that permits the calculus. This is not Euclidean space itself but rather a smooth manifold, a space that need only resemble Euclidean space *locally*.

(i.e., $\phi_\alpha(m) = (x^1(m), \dots, x^n(m))$), are extcolor{blue}{coordinates}. We say real-valued maps are extcolor{blue}{functions} (e.g., the x^i are function s).

1.2 Smooth Maps

Given another manifold N, we say $f : V \to N$ is a extcolor{blue}{smooth map} (or smooth) for an open set $V \subseteq M$ when for all $m \in U$, there exist charts ϕ and ψ defined around m and $f(m)$ such that $\psi \circ f \circ \phi^{-1}$ is smooth. Given $f : U \to N$ for arbitrary $U \subset M$, we say the same when f is the restriction of a smooth map on some open $W \supseteq V$. We call smooth maps with smooth inverse extcolor{blue}{diffeomorphisms}. We use $C^\infty(M)$, $\mathrm{Diff}(M,N)$, and $\mathrm{Diff}(M)$ to denote the spaces of smooth functions on M, diffeomorphisms $M \to N$, and diffeomorphisms $M \to M$, respectively. From this point forward, all maps are smooth unless otherwise specified.

1.3 Tangent Spaces

Let T_mM denote the extcolor{magenta}{vector space} of extcolor{magenta}{linear derivations} on the (vector) space of extcolor{magenta}{germs} of functions defined around m, F_m. Equivalently, let T_mM be the extcolor{magenta}{quotient ring} $(F_m/F_m^2)^*$, where $*$ denotes the extcolor{magenta}{dual space}[5]. T_mM has dimension n, and we call it the extcolor{blue}{tangent space} to M at m and elements of T_mM extcolor{blue}{vectors}. There is a natural map $f \mapsto f_*$ from the set of smooth functions $M \to N$, denoted $C^\infty(M,N)$, to the set of extcolor{magenta}{endomorphisms} $T_mM \to T_{f(m)}N$ given by $f_*X(g) \mapsto X(g \circ f)$ (where $X \in T_mM$ and $g \in C^\infty(M)$, the extcolor{magenta}{ring} of smooth functions on M). We call f_* the extcolor{blue}{pushfoward} of f. We define T_m^*M to be the extcolor{blue}{cotangent space} to M at m, and we have the dual map $f \mapsto f^*$, the extcolor{blue}{pullback}, acting as $T_{f(m)}^*N \to T_m^*M$ by $f^*A(X) = A(f_*X)$. There is additionally a natural map $d : C^\infty(M) \to T_m^*M$ given by $f \mapsto df(m) = v \mapsto v(f)$, which we call the extcolor{blue}{differential}. Given a chart ϕ around M, a basis for T_mM is given by $\frac{\partial}{\partial x^i}$ or ∂_i, given by

$$\partial_i f = \frac{\partial(f \circ \phi^{-1})}{\partial r^i}\Big|_m \tag{1.1}$$

where r^i is the ith Euclidean coordinate. A basis is also given for T_m^*M by the dx^i. Finally, we define the extcolor{blue}{tangent bundle} $TM = \cup_{m \in M} T_mM$ and the extcolor{blue}{cotangent bundle} $T^*M = \cup_{m \in M} T_m^*M$; both are $2n$-dimensional smooth manifolds equipped with natural projection maps onto M.

[5]TODO: prove equivalence of definitions.

Chapter 2

Fibre Bundles

2.1 Construction

Let M, F be manifolds and E be a manifold endowed with a surjective projection $\pi : E \to M$ such that M admits an open covering $\{U_\alpha\}$ for which each U_α has a diffeomorphism $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ acting by $p \to (\pi(p), \xi_\alpha(p))$ for $p \in P$ for $\xi_\alpha : U_\alpha \to \mathrm{Diff}(F)$. We then say (E, M, F, π) (or just E, or $E \to M$) is extcolor{blue}{fibre bundle} with extcolor{blue}{base} M and extcolor{blue}{fibre} F. Analogously to manifolds, we require that $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ be maximal and refer to it as a extcolor{blue}{local trivialization}. E is locally a product of M and F. Let $\phi : \pi^{-1}(U) \to U \times F$ as described we say and, in particular, letting $m \in M$, $\pi^{-1}(m)$ is diffeomorphic to F. Let $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta), (U_\delta, \phi_\delta) \in \mathcal{U}$: On $U_\alpha \cap U_\beta$ we have functions $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ given by $(m, x) \mapsto (m, \xi_{\alpha\beta}(m)(x))$ for some $\xi_{\alpha\beta}(m) \in \mathrm{Diff}(F)$ called extcolor{blue}{transition functions}. Sometimes we require $\xi_{\alpha\beta}(m) \in G$, a (topological) extcolor{blue}{group} acting on F on the left by diffeomorphisms (i.e., a subgroup of $\mathrm{Diff}(F)$). If a fibre bundle's local trivializations satisfy this maximally, we say E is a extcolor{blue}{G-bundle}, and that G is the extcolor{blue}{structure group}. We note that $\xi_{\alpha\alpha} = 1$, $\xi_{\alpha\beta} = \xi_{\beta\alpha}^{-1}$, and the extcolor{blue}{cocycle condition} $\xi_{\alpha\beta} \circ \xi_{\beta\delta} \circ \xi_{\delta\alpha} = 1$ holds on triple overlaps. Fibre bundles are uniquely determined by the base manifold and the transition functions. Given a manifold N and a map $g : N \to M$, the pullback bundle g^*E is the subset of $N \times E$ such that $g \circ \mathrm{proj}_1 = \pi \circ \mathrm{proj}_2$ with projection $\mathrm{proj}_1 : g^*E \to N$. A extcolor{blue}{vector bundle} is a fibre bundle whose fibres are vector spaces and whose local trivializations are fibre-wise linear isomorphisms.

2.2 Lie Groups

Let G be a manifold with group structure such that G that its binary operation is smooth: we say G is a extcolor{blue}{Lie group}[1]. Elements $g \in G$ induce $R_g, L_g, A_g \in \mathrm{Diff}(G)$ by $R_g : h \mapsto hg$, $L_g : h \mapsto gh$, and $A_g = L_g \circ R_{g^{-1}} : h \mapsto ghg^{-1}$.

2.3 Principal Bundles

Let P be a a G-bundle over M with fibre G and transition functions given by left multiplication: we say P is a extcolor{blue}{principle G-bundle} with base M. Equivalently, a principle G-bundle P is a fibre bundle with a extcolor{magenta}{regular} smooth right action by a Lie group G that preserves fibres and the ξ_α are G-equivariant[2]. Because left and right multiplication commute, we have an invariant right action of G on P[3].

2.4 Associated Bundles

Let N be a manifold with a left G action $\rho : G \to \mathrm{Diff}(N)$; this induces an action of G on $P \times N$ by $(p, n) \mapsto (pg, g^{-1}n)$ for $g \in G$. The quotient $E = (P \times N)/G$, which is characterized as a bundle by the base manifold M, fibre N, and transition functions given by the left action of the $\xi_{\alpha\beta}(m)$ on N, is denoted extcolor{blue}{associated bundle} to P by ρ with fibre N. For $p \in P$ such that $\pi(p) = m$, the local trivialization is explicitly $[(p, n)] \mapsto (m, \xi_\alpha(m)n)$[4]. In particular, the associated bundle E is also a G-bundle. Let $m = \dim(M)$; to M we can associated the extcolor{blue}{frame bundle} $F(M)$, the disjoint union of frames of each tangent space T_mM considered as a bundle over M: this is a principal $GL(m)$-bundle. The extcolor{blue}{tangent bundle} TM is the bundle associated to $F(M)$ via the fundamental representation on $\mathbb{R}^m$, (or equivalently, the disjoint union of the tangent spaces T_mM as a vector bundle over M). Continuing, the extcolor{blue}{cotangent bundle} T^*M is the bundle associated to $F(M)$ by the representation dual to the fundamental representation on $\mathbb{R}^n$ (disjoint union of cotangent spaces), and the extcolor{blue}{(k, ℓ) tensor bundle} is the bundle $T^k_\ell M$ associated to $F(M)$ by the tensor product of k copies of the fundamental representation and ℓ copies of its dual (disjoint union of k tensor products of tangent space and ℓ tensor products of the cotangent space). In particular, the bundle $\Lambda^k M$ is a subbundle of $T^0_k M$ given by only the totally antisymmetric tensors.

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[1]Some authors require the inverse map $g \mapsto g^{-1}$ to be smooth as well, but this follows from the smoothness of the binary operation.

[2]The second definition admits an exchange between fibre-preservation and $P/G = M$

[3]If $p \in \pi^{-1}(U_\alpha)$ and $\phi_\alpha(p) = (m, h)$, then $pg = \phi_\alpha^{-1}(m, hg)$.

[4]This is well-defined because under a local trivialization ϕ_α, letting $\xi_\alpha(\pi(p)) = g$, we have that $(p, n) \mapsto (\pi(p), gn)$ and $(ph, h^{-1}n) \mapsto (\pi(ph), (gh)h^{-1}n) = (\pi(p), gn)$.

References: Discussion of local trivialization on associated bundles

2.5 Sections

Let $\sigma : M \to E$ such that $\pi \circ \sigma = 1$, or a family of maps $\sigma_\alpha : U_\alpha \to F$ such that $\sigma_\alpha(m) = \xi_{\alpha\beta}(m)\sigma_\beta(m)$[5]; we say σ is a extcolor{blue}{section} of E[6]. We denote the space of sections of a fibre bundle E by $\Gamma(E)$. In particular, we define the spaces of extcolor{blue}{vector fields} $\Gamma(M) = \Gamma(TM)$, extcolor{blue}{(r,s) tensor fields} $\mathcal{T}^k_\ell(M) = \Gamma(T^k_\ell(M))$, and extcolor{blue}{differential k-forms} $\Omega^k(M) = \Gamma(\Lambda^k(M))$.

2.5.1 Vector Fields

The basis ∂_i for tangent spaces on M induced by a chart are naturally understood as vector fields on subsets of M and form a basis for vector fields on that subset. Given $\psi \in \mathrm{Diff}(M)$, we can define $\psi_* X$ by $(\psi_* X)_m = \psi_* X_{\psi^{-1}(m)}$ [^com:vector_field_pushforward]. Let $\Phi = I \times M \to M$ ($I \ni 0$ an interval in $\mathbb{R}$) satisfy $\phi_t \circ \phi_s = \phi_{t+s}$ for $\phi_t : M \to M$ given by $m \mapsto \Phi(t,m)$: we say Φ is a extcolor{blue}{one-parameter group of transformations} or extcolor{blue}{flow} on M[7]; Φ induces a $Y \in \mathcal{T}^1_0(M)$ by

$$Y(f)(m) = \lim_{t\to 0} \frac{(f \circ \phi_t)(m) - f(m)}{t}. \tag{2.1}$$

This correspondence has a partial inverse. A extcolor{blue}{curve} is a map $\gamma : (a,b) \subset \mathbb{R} \to M$. We say the tangent vector to γ at $\gamma(t) \in M$ is $\gamma_*(1)$ for $1 \in T_t\mathbb{R}$: this defines a smooth vector field $\dot\gamma$ on $\gamma((a,b))$. We say γ is an extcolor{blue}{integral curve} to X if $\dot\gamma = X_{|\gamma((a,b))}$. From the theory of extcolor{magenta}{ODEs}, we are assured that X induces integral curves γ_m at all m such that $\phi_t : m \mapsto \gamma_m(t)$ is smooth. Then $\Phi = \{\phi_t : M \to M\}_{t\in(-\varepsilon,\varepsilon)}$ is a (local) flow (for some $\varepsilon > 0$) induced by X. In particular, the vector field $\psi_* X$ and the flow $\psi\circ\phi_t\circ\psi^{-1}$ induce each other[8]. A extcolor{magenta}{Lie algebra} is a set endowed with an associative anticommuting binary operator satisfying the extcolor{magenta}{Jacobi identity}. The extcolor{blue}{Lie bracket} of vector

[5] TODO: show equivalence

[6] The idea here is that a section (smoothly) picks out a particular element of each fibre over the base manifold. For instance, a vector field in $\mathbb{R}^3$ is an example of a section of the trivial bundle $\mathbb{R}^6 \to \mathbb{R}^3$.

[7] "Flow" is sometimes used to refer specifically to the action of a one-parameter group of transformations on its manifold: we adopt the term more generally for its brevity and convenience.

[8]

$$\begin{aligned}
(\psi_* X)(f)(m) &= X(f \circ \psi)(\psi^{-1}(m)) \\
&= \lim_{t\to 0} \frac{(f \circ \psi \circ \phi_t)(\psi^{-1}(m)) - (f \circ \psi)(\psi^{-1}(m))}{t} \\
&= \lim_{t\to 0} \frac{(f \circ \psi \circ \phi_t \circ \psi^{-1})(m) - (f \circ \psi \circ \psi^{-1})(m)}{t} \\
&= \lim_{t\to 0} \frac{(f \circ [\psi \circ \phi_t \circ \psi^{-1}])(m) - f(m)}{t}
\end{aligned}$$

fields, $[X,Y] \in \mathcal{T}^1_0(M)$, defined by $[X,Y](f) = (X \circ Y)(f) - (Y \circ X)(f)$, endows $\mathcal{T}^1_0(M)$ with a Lie algebraic structure. The extcolor{blue}{Lie derivative}'[9] of a tensor field $T \in \mathcal{T}^k_\ell$ in the direction X, $\mathcal{L}_X T \in \mathcal{T}^k_\ell$, is given in terms of the one parameter group ϕ_t induced by X as

$$\mathcal{L}_X T = \lim_{t \to 0} \frac{T - (\phi_t)_* T}{t} \tag{2.2}$$

Alternatively, the Lie derivative is the unique operator $\Gamma(M) \times \mathcal{T}^k_\ell \to \mathcal{T}^k_\ell$ which obeys the Leibniz rule on tensor products and contractions, acts on functions $(k, \ell = 0)$ by merely applying the vector field argument, and commuting with the exterior derivative (to be defined shortly). In the special case that our argument is a vector field Y, we have the simpler form $\mathcal{L}_X Y = [X,Y]$.

2.5.2 Tensor Fields

Note that $\Gamma(T^\ell_k M)$ is identifiable with the $C^\infty(M)$-linear maps $\Gamma(T^k_\ell M) \to C^\infty(M)$, as the (k,ℓ) tensors in a section can act pointwise on the (ℓ,k) tensors in another section, thereby smoothly assigning real numbers to points on M, which constitutes an element of $C^\infty(M)$. This idea suggests an alternative—and preferable—definition. In particular,

$$\begin{aligned}
\mathcal{T}^k_\ell(M) &= \left(\bigotimes_{i=1}^{k} \mathcal{T}^0_1(M)\right) \otimes \left(\bigotimes_{j=1}^{\ell} \mathcal{T}^0_1(M)\right) \\
&= \left(\bigotimes_{i=1}^{k} \Gamma(T^0_1 M)\right) \otimes \left(\bigotimes_{j=1}^{\ell} \Gamma(T^1_0 M)\right)
\end{aligned}$$

except we construct these tensor products not from via free $\mathbb{R}$-vector spaces but from free $C^\infty(M)$- extcolor{magenta}{modules}.

2.5.3 Differential Forms

Upon differential forms there is a natural differentiation operation, the extcolor{blue}{exterior derivative} $d : \Omega^k(M) \to \Omega^{k+1}(M)$. Locally, the exterior derivative acts as

$$d(f\, dx^I) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\, dx^i \wedge dx^I \tag{2.3}$$

[9]We want to take derivatives of tensor fields, but we cannot compare tensors defined on distinct tangent spaces (i.e., in a limit) which certainly cannot be identified (i.e., $T_m M$ and $T_{m'} M$ for $m \neq m'$). This is a recurring theme in differential topology/geometry, and one with many different solutions. This is the first solution we encounter: the specification of a vector field provides sufficient directional and "connective' ' information (via the pushforward) to enable this kind of comparison, and thus, a derivative operator. Note that, in particular, because our definition utilizes a limit, the Lie derivative evaluated at a point m depends upon the local behavior of X, T, or the behavior in a neighborhood around m (i.e., not merely at the point m). This manifests itself algebraically in the fact that the Lie derivative is not $C^\infty(M)$-linear

for dx^I some k-fold wedge product of the canonical basis elements dx^i (i.e., I is a extcolor{blue}{multiindex}) and extends linearly. Equivalently, axiomatically the exterior derivative is the unique degree 1 extcolor{magenta}{antiderivation} $d : \Omega^k(M) \to \Omega^{k+1}(M)$ which agrees with the differential on $\Omega^0(M) = C^\infty(M)$ and squares to 0. We also have the following formula.

$$\begin{aligned} d\beta(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i v_i(\beta(X_0, \dots, \hat{X}_i, \dots, X_k)) \\ &+ \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} (-1)^{i+j} \beta([X_i, X_j], v_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \end{aligned} \tag{2.4}$$

Because $d^2 = 0$, we have a cohomological situation: elements of the kernel of d are extcolor{blue}{close d forms}, elements in the image are extcolor{blue}{exact forms}, and the extcolor{blue}{kth de Rham cohomology group} $H^k_{\mathrm{dR}}(M)$ is the quotient group of closed forms modulo exact forms. The wedge product endows these groups with ring structure, and the map $H^k_{\mathrm{dR}}(M) \times H^k(M) \to \mathbb{R}$ given by $[\omega], [c] \mapsto \int_c \omega$ establishes an isomorphism between de Rham and singular cohomology (de Rham's theorem), which depends essentially on the identity $\int_C d\omega = \int_{\partial C} \omega$ (Stokes' theorem).

TODO: bundle-valued differential forms

2.6 Integration on Manifolds

TODO

Chapter 3

Lie Theory

3.1 Lie Algebras

Let G be a Lie group. A vector field X on G satisfying $(L_g)_* X = X \circ L_g$ is called extcolor{blue}{left invariant}. The extcolor{blue}{Lie algebra} to G, $\mathfrak{g}$, is the set of all left-invariant vector fields on G; the name is natural as the Lie bracket induces a Lie algebraic structure on this set. Note that $\mathfrak{g}$ is naturally isomorphic to $T_e G$, where e is the identity element, as $Y \in T_e G$ induces a vector field X given by $X_g = (L_g)_* Y$; in particular, this means $\dim \mathfrak{g} = \dim G$. Given a basis $X_1, \dots, X_n$ of $\mathfrak{g}$, we have that $[X_i, X_j] = c_{ij}^h X_h$ and we say the c_{ij}^h are the extcolor{blue}{structure constants} associated with the basis. Identifying $\mathfrak{g} \cong T_e G$[1] there is a natural $\mathfrak{g}$-valued one-form θ on G defined by $v \mapsto (L_{g^{-1}})_*(v)$ for $v \in T_g G$. We call this the extcolor{blue}{Maurer-Cartan one-form} and it satisfies $d\theta + \frac{1}{2}[\theta, \theta]$. Noting that, if G is a matrix Lie group, $(L_g)_*$ coincides with the natural matrix multiplication action of the matrix g on TG, we have that $\theta(v) = g^{-1} v$ where the right hand side is matrix multiplication[2].

3.2 Exponential Map

A extcolor{blue}{one-parameter subgroup} on G is a continuous group homomorphism $\mathbb{R} \to G$. For $X \in \mathfrak{g}$, let $\phi_{X,t} : G \to G$ be the associated flow, and let $g_X(t) = \phi_{X,t}(e) \in G$; then g_X is a one parameter subgroup on G (i.e., $g_X(t) g_X(s) = g_S(t+s)$). Moreover, we have the extcolor{blue}{exponential map} $\exp : \mathfrak{g} \to G$ given by $X \mapsto g_X(1)$. From this it follows that $g_X(t) = \exp(tX)$; indeed, this is the most general form for a one parameter group.

[1] TODO: clarify this identification

[2] If we let G be embedded in a matrix Lie group by a map ϕ, then the Maurer-Carten form on $\phi(G)$ is $\theta = \phi(g^{-1})\phi_*$.

3.3 Adjoint Representation

Chapter 4

Connections

4.1 Construction

To each $A \in \mathfrak{g}$ we can naturally associate a extcolor{blue}{fundamental vector field} A^* on P given by $(A^{\#})(p) = (\sigma_p)_*(A)$, where $\sigma_p : G \to P$ is the map $g \mapsto pg$. Equivalently, $(A^{\#})(p)$ is the tangent vector to the curve $(\sigma_p \circ \exp)(At)$ at $t = 0$. Define $V_p = T_p \pi^{-1}(p) = \ker(\pi_*) \cap T_p P$, the extcolor{blue}{vertical subspace} of $T_p P$; $A \mapsto (A^*)_p$ is an isomorphism $\mathfrak{g} \mapsto V_p$. Moreover, $A \mapsto A^{\#}$ is equivariant with respect to the adjoint and principal actions of G and preserves the respective Lie brackets. The extcolor{blue}{vertical bundle} $VP = TP$ is the vector subbundle of these vertical subspaces.

To a principal bundle we can associate its extcolor{blue}{Atiyah sequence}.

where ι is the natural inclusion and $\overline{\pi}$ is the map $X \mapsto (\pi_{TP}(X), (\pi_P)_*(X))$. A extcolor{blue}{connection} on P is a choice of equivariant split for this sequence (direct sum, left, and right splits are equivalent). Explicitly, the G actions are the diagonal action (identifying $VP \cong P \times \mathfrak{g}$), the canonical action, and the induced action from $P \times TM$ leaving TM invariant, respectively.

In particular, a direct sum split φ is frequently identified with the vector subbundle $HP = \varphi^{-1}(0 \oplus \pi^* TM)$, which is complementary to VP and invariant under $(R_g)_*$ for $g \in G$ and referred to as an extcolor{blue}{Ehresmann connection}. Left splits can be understood as G-equivariant $\mathfrak{g}$-valued one-forms on P, commonly referred to as extcolor{blue}{connection one-forms}. Right splits are interpreted as maps sending a vector in $T_m M$ to one in any $T_p P$ for $p \in \pi^{-1}(m)$ and are known as extcolor{blue}{horizontal lifts}.

4.2 Properties

TODO

4.3 Exterior Covariant Derivative

TODO

4.4 Curvature

TODO

4.5 Bundle Automorphisms

TODO

4.6 On Associated Vector Bundles

TODO

Chapter 5

Complex Manifolds

5.1 Construction

A extcolor{blue}{complex manifold} is a manifold but with two definitional substitutions: $\mathbb{R}^n \leftrightarrow \mathbb{C}^n$ and holomorphic $\leftrightarrow$ smooth. extcolor{blue}{Holomorphic maps} between complex manifolds are defined analogously to smooth maps, and invertible maps which are both ways holomorphic are extcolor{blue}{biholomorphims}.

Let M be a complex manifold of dimension n: evidently this is a manifold of dimension $2n$, so in particular we can define the $2n$ dimensional vector spaces T_pM. We let $C^\infty(M)^{\mathbb{C}}$ denote the extcolor{magenta}{complexification} of $C^\infty(M)$: the complexification $T_pM^{\mathbb{C}}$ is a space of complex dimension $2n$ acts on $C^\infty(M)^{\mathbb{C}}$ through linear extension. T_p^*M complexifies in the same way, thereby enabling the complexification of all tensor bundles and their sections (also denoted with superscript $\mathbb{C}$).

If (U, ϕ) is a chart on M with ϕ acting by $m \mapsto (x^1(m) + iy^1(m), \dots, x^n(m) + iy^n(m))$, T_mM and T_m^*M are spanned by the dual bases $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}$ and dx^i, dy^j; we define new dual bases

$$\begin{aligned} &\frac{\partial}{\partial z^i} = \frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}, \quad \frac{\partial}{\partial \overline{z}^i} = \frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i} \\ &dz^i = dx^i + idy^i, \quad d\overline{z}^i = dx^i - idy^i \end{aligned} \tag{5.1}$$

Multiplying a vector in $T_pM^{\mathbb{C}}$ by i amounts to the substitutions $\frac{\partial}{\partial x^i} \to \frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial y^i} \to -\frac{\partial}{\partial x^i}$: this map defines a basis-independent $(1,1)$ tensor J_m on T_mM which squares to -1 and globally forms a section $J \in \mathcal{T}^1_1(M)$ known as the extcolor{blue}{almost complex structure} on M. This extends linearly to $J \in \mathcal{T}^1_1(M)^{\mathbb{C}}$: in particular, J is diagonal in the $\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^i}$ basis with eigenvalue i for the former and $-i$ for the latter. In particular, we have projection operators

$\mathcal{P}^{\pm}$ onto the $\pm i$ eigenspaces of J in $\Gamma(M)^{\mathbb{C}}$: elements of the image of $\mathcal{P}^{+}$ and $\mathcal{P}^{-}$ are extcolor{blue}{holomorphic vectors} and extcolor{blue}{antiholomorphic vectors}, respectively, with analogous terminology for vector fields.

Similarly, elements of $\Omega^k(M)^{\mathbb{C}}$ which decompose into tensor products of r terms in $\{dz^i\}$ and s terms in $\{d\overline{z}^i\}$ are said to have extcolor{blue}{bidegree} $(r, s$, the set of which is denoted $\Omega^{r,s}(M)$. It follows that $\Omega^k(M)^{\mathbb{C}} = \oplus_{r+s=k}\Omega^{r,s}(M)$. Moreover, linearly extending the exterior derivative gives a map $\Omega^{r,s}(M) \to \Omega^{r+1,s}(M) + \Omega^{r,s+1}(M)$, enabling the definition of the extcolor{blue}{Dolbeault operators} $\partial, \overline{\partial}$ which are the $\Omega^{r+1,s}(M)$ and $\Omega^{r,s+1}(M)$ parts of d, respectively (i.e., $d = \partial + \overline{\partial}$. It follows that $\partial^2 = \partial\overline{\partial} + \overline{\partial\partial} = \overline{\partial}^2 = 0$. Elements of the kernel of $\overline{\partial}$ in $\Omega^{k,0}(M)$ are called extcolor{blue}{holomorphic k-forms}; a k-form is holomorphic if and only if the $C^\infty(M)^{\mathbb{C}}$ coefficients of the decomposition into the dz^i basis are each holomorphic functions.

5.2 Hermitian and Kahler Geometry

Let M also be a Riemannian $2n$-dimensional manifold with metric g, which can be extended linearly to act on $\Gamma(M)^{\mathbb{C}}$. Moreover, let M have a metric g that is invariant under precomposition of both arguments with J: we say M is a extcolor{blue}{Hermitian manifold} and g a extcolor{blue}{Hermitian metric}. We use normal and overlined greek indices to denote the evaluation of the metric's arguments on $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \overline{z}^i}$ terms, respectively: in the Hermitian case, only mixed terms $g_{\mu\overline{\nu}}$ are non-zero.

Define the extcolor{blue}{K"ahler form} $\Omega \in \Omega^2(M)$ by $(X, Y) \mapsto g(J(X), Y)$; this is also invariant under precomposition by J, and extends linearly to an element of $\Omega^{1,1}(M)$. In particular, $\Omega = i g_{\mu\overline{\nu}} dz^\mu \wedge d\overline{z}^\nu$.

A extcolor{blue}{Hermitian connection} is a connection ∇ on $TM^{\mathbb{C}}$ such that, in the $z, \overline{z}$ basis, we have m$\nabla_\mu g = 0 = \nabla_{\overline{\mu}} g$ (metric compatibility) and the connection coefficients vanish when regular and overlined indices are mixed. Curvature and torsion tensors are formed as usual. Moreover, we have the extcolor{blue}{Ricci form} $\mathcal{R} = i\partial\overline{\partial}\log(\det(g_{\mu\overline{\nu}}))$. The hermitian covariant derivative of the almost complex structure vanishes.

A extcolor{blue}{K"ahler manifold} is a Hermitian manifold with closed K"ahler form, in which case the metric is called a extcolor{blue}{K"ahler metric}. This only happens if the Levi-Civita covariant derivative of the almost complex structure vanishes: or, in other words, the Hermitian and Levi-Civita connections are compatible. Indeed, the Ricci form of a K"ahler metric coincides with the Ricci curvature of the Levi-Civita connection (additionally, the Hermitian connection is torsion-free).

5.3 Dolbeault Cohomology and Hodge Theory

Replacing d in de Rham cohomology with $\overline{\partial}$ yields extcolor{blue}{Dolbeault cohomology} with extcolor{blue}{(r,s)-cocycles}, extcolor{blue}{(r,s)-coboundaries}, and extcolor{blue}{(r,s)th cohomology groups} being defined as expected. Recalling the Hodge star $*$, we have an inner product on $\Omega^{r,s}(M)$ by

$$(\alpha,\beta) = \int_M \alpha \wedge \overline{*}\beta \tag{5.2}$$

where $\overline{*}\beta$ is defined to be $\overline{*\beta} = *\overline{\beta}$.

From here we define $\partial^\dagger, \overline{\partial}^\dagger$ to be the extcolor{magenta}{adjoints} of the Dolbeault operators $\partial, \overline{\partial}$, respectively. They decrement the degree of the forms they act on, square to zero, and obey the following formulas.

$$\partial^\dagger = - * \overline{\partial} *, \ \ \overline{\partial}^\dagger = - * \partial * \tag{5.3}$$

We have Laplacians as follows.

$$\Delta = \{d, d^\dagger\}, \Delta_\partial = \{\partial, \partial^\dagger\}, \Delta_{\overline{\partial}} = \{\overline{\partial}, \overline{\partial}^\dagger\}, \tag{5.4}$$

where $\{\,\}$ here is the anticommutator. A form is extcolor{blue}{harmonic} with respect to $d, \partial, \overline{\partial}$ if it lies in the kernel of $\Delta, \Delta_\partial, \Delta_{\overline{\partial}}$, respectively. Hodge's exhibits shows that

$$\Omega^{r,s}(M) = \overline{\partial}\Omega^{r,s-1}(M) + \overline{\partial}^\dagger\Omega^{r,s+1}(M) + \mathrm{Harm}^{r,s}_{\overline{\partial}}(M) \tag{5.5}$$

with this sum being orthogonal with respect to the aforementioned inner product and with $\mathrm{Harm}^{r,s}_{\overline{\partial}}(M)$ the set of $\overline{\partial}$-harmonic (r,s) forms on M. Note that Dolbeault cohomology classes contain a unique $\overline{\partial}$-harmonic form. In the special case that M is K"ahler, $\Delta = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$.

Chapter 6

Characteristic Classes

6.1 Chern-Weil Theory

Consider $S_k(\mathbb{M}_n(\mathbb{C}))$ and $S(\mathbb{M}_n(\mathbb{C})) = \oplus_{i\in\mathbb{N}} S_k(\mathbb{M}_n(\mathbb{C}))$ (formal sums of tensors): in particular, we can define a product $S_p(\mathbb{M}_n(\mathbb{C})) \times S_q(\mathbb{M}_n(\mathbb{C})) \to S_{p+q}(\mathbb{M}_n(\mathbb{C}))$ by symmetrizing the tensor product of elements in these respective algebras.

Let ρ be an n-dimensional representation of a group G: then $\rho(G) \subset \mathbb{M}_n(\mathbb{C})$. Let $\overline{Q} \in S_k(\mathbb{M}_n(\mathbb{C}))$ be unchanged under precomposition by the n-fold product of the map $\rho(g) : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ for each $g \in G$. We say $\overline{Q}$ is extcolor{blue}{invariant}, and the space of invariant $(0, r)$ tensors is denoted $I_r(G, \rho)$, and $I(G, \rho)$ is defined analogously.

$\overline{Q}$ induces a map $Q : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$ by precomposing with the appropriate diagonal map: we say P is an extcolor{blue}{invariant polynomial}. Conversely, invariant polynomials Q' induce invariant tensors by taking the mapping from $A_1 \otimes \cdots \otimes A_k$ to the coefficient of $t^1 \dots t^k$ in the expansion of $Q'(A_i t^i)$ (where the t^i are merely variables).

Let $P \to M$ be a principal G-bundle. $\overline{Q} \in I_k(G, \mathrm{Ad})$ induces a map $\otimes_{i=1}^k \Omega^k(M, \mathfrak{g}) \to \Omega^k(M)$ by $(A_1 \otimes \eta_1) \otimes \cdots \otimes (A_k \otimes \eta_k) \mapsto \eta_1 \wedge \cdots \wedge \eta_k\, Q(A_1, \dots, A_k)$ for $A_i \in \rho(\mathrm{Ad}) = \mathfrak{g} \subset \mathbb{M}_n(\mathbb{C})$ and $\eta_i \in \Omega^k(M)$. Now let $\overline{Q}$ be invariant: an analogous action for the invariant polynomial Q is induced.

Consider the action of Q on a curvature Ω on P: the Chern-Weil theorem states that the resulting differential form is closed and that its (de Rham) cohomology class is independent of the particular curvature. This curvature class $\chi_P(\Omega)$ is the extcolor{blue}{characteristic class} of P given by Q. Moreover, $\chi_P(\Omega) \in \Omega^k(P)$ descends uniquely to an element of $\Omega^k(M)$, the map $\chi_P : I(G, \rho) \to H^*_{\mathrm{dR}}(M)$ is a morphism of algebras and is natural in the sense that $\chi_{f^*P} = f^*\chi_P$.

Chapter 7

Spin Geometry

Chapter 8

Algebraic Geometry

8.1 Construction

An extcolor{blue}{complex affine algebraic variety} or just a variety is the extcolor{magenta}{zero locus} zero locus of a finite set of polynomials. extcolor{blue}{Affine n-space} $\mathbb{A}^n$ is $\mathbb{C}^n$ endowed with the extcolor{blue}{Zariski topoplogy} wherein open sets are complements of varieties. We use the notation $\mathbb{V}(F_i)$ to denote the variety defined by the zero locus of the polynomials $F_i : \mathbb{A}^n \to \mathbb{C}$. Morphisms of varieties are restrictions of polynomial maps between affine spaces, and are continuous in the Zariski topology. An extcolor{blue}{irreducible variety} is a variety which cannot be expressed non-trivially as the union of two varieties: this enables the notion of the dimension of a variety, namely the longest chain of proper inclusions of its subvarieties (the number of inclusions, not the number of sets).

8.2 Commutative Ring Theory Interlude

For our purposes, a extcolor{blue}{commutative ring}, or just a ring, is a set endowed with the structure of an Abelian group (with operation known as "addition") and a monoid (with operation known as "multiplication") such that the latter operation distributes over the former. extcolor{blue}{Ring homomorphisms} preserve both structures and the identity element.

An extcolor{blue}{ideal} is a subset of a ring which is closed under addition and invariant under multiplication. The ideal generated by some elements is the intersection of all ideals containing those elements. The extcolor{blue}{radical} $\sqrt{I}$ of an ideal I is the set of elements which possess a power belonging to the ideal. An ideal is extcolor{blue}{maximal} if the only ideal properly containing it is the whole ring, extcolor{blue}{prime} if a product only belongs to it when both elements of the product belong to it, extcolor{blue}{radical} if it coincides with its radical, and extcolor{blue}{reduced} if it contains no nilpotent elements.

Rings can be quotiented by ideals, and the canonical surjection maps ideals containing the quotienting ideal to ideals in the quotient (preserving maximality, primeness and radicalness). A quotient is a extcolor{blue}{field} (multiplication has inverses) if and only if the ideal is maximal and a extcolor{blue}{domain} (there exist no extcolor{blue}{zero-divisors}, or non-zero elements mapping other non-zero elements to zero via multiplication) if and only if the ideal is prime. In particular, maximal ideals are prime. Moreover, prime ideals are radical and quotients are reduced if and only if the ideal is radical.

A extcolor{blue}{$\mathbb{C}$-algebra} is a ring containing $\mathbb{C}$, and suitable notions of homomorphism, etc. follow naturally. A extcolor{blue}{Noetherian ring} is a finitely generated ring, and Hilbert's basis theorem states that the rings of polynomials $\mathbb{C}[x_1, \dots, x_n]$ are Noetherian.

8.3 Algebra-Geometry Correspondence

Let $\mathbb{I}(V)$ denote the ideal of polynomials vanishing on the variety V. Immediately, we see that $\mathbb{V}(\mathbb{I}(V)) = V$, so $\mathbb{V}$ is a left inverse $\mathbb{I}$ to. Hilbert's Nullstellensatz reveals that $\mathbb{V}$ is also a partial right inverse: in particular, let I by a polynomial ideal, then $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$. In particular, $\mathbb{V}$ is a right inverse on radical ideals; said differently, the Nullstellensatz induces a bijection between varieties and radical ideals in $\mathbb{C}[x_1, \dots, x_n]$[1].

This has many corollaries. For instance, varieties are finite-dimensional; varieties are the zero locus of finitely many polynomials (in particular, the intersection of finitely many hypersurfaces); $\mathbb{I}(V)$ is radical; and prime ideals are in bijection with irreducible varieties.

Consider the extcolor{blue}{coordinate ring} $\mathbb{C}[V]$ of polynomials on V. This is $\mathbb{C}[x_1, \dots, x_n]/\mathbb{I}(V)$, hence it is a reduced, finitely generated $\mathbb{C}$-algebra. Variety morphisms induce (direction-reversing) morphisms between coordinate rings. The connection between varieties and coordinate rings is a bijection: namely, there is an equivalence (up to arrow-reversing) between the category of varieties and the category of finitely generated reduced $\mathbb{C}$-algebras.

[1] Alternatively, we can interpret this by defining an equivalence relationship on the ideals of $\mathbb{C}[x_1, \dots, x_n]$ by defining similarity between ideals sharing a radical and observing that the cosets of the quotient given by this equivalence are placed in bijection with varieties. This is equivalent to the given interpretation, which merely chooses that radical itself to be a candidate for the coset.

References

Appendix A

Tensors

A.1 Construction

Let V, W be n-dimensional vector spaces over $\mathbb{R}$ and consider the extcolor{magenta}{free} vector space $F(V \times W)$. We define $R(V \times W)$ as follows.

$$R(V \times W) = \Big\langle (av, bw) - ab(v, w), (v + w, v' + w') - (v, v') - (v, w') \\ - (w, v') - (w, w') \Big| a, b \in \mathbb{R}; v, v' \in V; w, w' \in W \Big\rangle \subset F(V \times W)$$

We define the extcolor{blue}{tensor product} of V and W by $V \otimes W = F(V \times W)/R(V \times W)$ and denote elements by $v \otimes w$ (i.e., (v, w) belongs to the extcolor{magenta}{coset} $v \otimes w$). Let $T^k_\ell(V) = (\bigotimes_{i=1}^k V) \otimes (\bigotimes_{j=1}^\ell V^*)$, the space of extcolor{blue}{(k, ℓ) tensors on V}. This space has dimension $n^{k+\ell}$, and we can identify $T^\ell_k(V)$ with $T^k_\ell(V)^*$'[1]. Moreover, observe that elements of $T^k_\ell(V)$ can be understood as a linear map $T^{k-r}_{\ell-s}(V) \to T^s_r(V)$ through partial evaluation: indeed, $T^\ell_k(V)$ is naturally isomorphic to the extcolor{blue}{hom set} $Hom(T^{k-r}_{\ell-s}(V), T^r_s(V))$. This also entails the natural conclusion $T^0_0(V) = \mathbb{R}$. Equivalently, $w \in T^{k-r}_{\ell-s}(V)$ is a linear map $T^k_\ell(V) \to T^r_s(V)$[2]. Finally, we note in passing that because $V^\ell \times (V^*)^k \subset F((V)^\ell \times (V^*)^k)$, we also have a multilinear action of $T^k_\ell(V)$ on $V^\ell \times (V^*)^k$ (indeed, one can define $T^r_s(V)$ as the set of multilinear maps on this space).

[1] To see this, let $v = v_1 \otimes \dots \otimes v_{k+\ell} \in T^\ell_k(V)$.

$$v \leftrightarrow (w_1 \otimes \dots \otimes w_{k+\ell} \mapsto v_1(w_1) \cdot \ldots \cdot v_{k+\ell}(w_{k+\ell})) \tag{A.1}$$

Linearity extends this to arbitrary elements of $T^\ell_k(V)$. Here we are exploiting $V \cong V^{**}$, enabling useful idea that vectors are linear functions on dual vectors and vice versa.

[2] Here we see the functional and argumentative behavior of tensors are indistinguishable, much like we have the symmetry $x(y) = y(x)$ for $x \in V, y \in V^*$.

A.2 Notation

Let e_a, σ^b constitute dual bases for V, V^*: then $v \in T^k_\ell(V)$ can be written in components as follows.

$$v = \sum_{\mu_1=1}^{n} \dots \sum_{\nu_k=1}^{n} v^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_\ell} e_{\mu_1} \otimes \dots \otimes e_{\mu_k} \otimes \sigma^{\nu_1} \otimes \dots \otimes \sigma^{\nu_\ell} \qquad (v^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_\ell} \in \mathbb{R}) \quad \text{(A.2)}$$

For brevity we adopt extcolor{blue}{Einstein summation notation} wherein a doubly-appearing index (once a superscript, once a subscript) is always assumed to be summed over, rending each Σ obsolete; we also suppress the $\otimes$ symbol. Thus, we rewrite v as follows[3].

$$v = v^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_\ell} e_{\mu_1} \dots e_{\mu_k} \sigma^{\nu_1} \dots \sigma^{\nu_\ell} \quad \text{(A.3)}$$

Finally, we adopt Penrose's extcolor{blue}{abstract index notation} wherein we append indices to v labelling its arguments (or, by duality, the arguments of other tensors that v can satisfy). Subscripts denote vector arguments, superscripts denote dual vector arguments, and summation over an abstract index is evaluation. This results in v being expressed as follows.

$$v = v^{a_1 \dots a_k}_{b_1 \dots b_\ell} \quad \text{(A.4)}$$

To avoid confusion, henceforth unsummed roman indices are abstract while unsummed greek indices denote particular components (e.g., of a tensor, given a basis).

A.3 Change of Basis

Note that $T^1_1(V)$ coincides with the space of extcolor{magenta}{endomorphisms} on V, so the extcolor{magenta}{automorphisms} are a subset. Suppose then that $\Lambda^b_a \in T^1_1(V)$ is a change of basis: i.e., we let $e_\mu \mapsto e'_\mu = \Lambda(e_\mu, \cdot) = \Lambda^a_b (e_\mu)^b$. The dual basis σ'^ν must satisfy $\delta^\nu_\mu = (e'_\mu)^a (\sigma'^\nu)_a = \Lambda^a_b (e_\mu)^b (\sigma'^\nu)_a = (e_\mu)^a (\sigma^\nu)_a$, so we conclude $\sigma'^\nu = \Lambda^{-1}(\cdot, \sigma^\nu) = (\Lambda^{-1})^b_a (\sigma^\nu)_b$, where Λ^{-1} is the inverse to Λ

[3] Our original tensor notation and the Einstein summation notation are related as follows.

$$\begin{aligned} v(\sigma^{\mu_1}, \dots, \sigma^{\mu_\ell}, e_{\nu_1}, \dots, e_{\nu_k}) &= v^{a_1 \dots a_k}_{b_1 \dots b_\ell} (\sigma^{\mu_1})_{a_1} \dots (\sigma^{\mu_k})_{a_k} (e_{\nu_1})^{b_1} \dots (e_{\nu_k})^{b_\ell} \\ &= v^{\kappa_1 \dots \kappa_k}_{\lambda_1 \dots \lambda_\ell} e_{\kappa_1}(\sigma^{\mu_1}) \dots e_{\kappa_k}(\sigma^{\mu_k}) \sigma^{\lambda_1}(e_{\nu_1}) \dots \sigma^{\lambda_\ell}(e_{\nu_k}) \\ &= v^{\kappa_1 \dots \kappa_k}_{\lambda_1 \dots \lambda_\ell} \delta^{\mu_1}_{\kappa_1} \dots \delta^{\mu_k}_{\kappa_k} \delta^{\lambda_1}_{\nu_1} \dots \delta^{\lambda_\ell}_{\nu_k} = v^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_k} \in \mathbb{R} \end{aligned}$$

(i.e., $(\Lambda^{-1})^a_b \Lambda^b_c = \delta^a_c$). The coefficients of v then transform as follows[4].

$$v'^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_\ell} = (\Lambda^{-1})^{\mu_1}_{\kappa_1} \ldots (\Lambda^{-1})^{\mu_k}_{\kappa_k} v^{\kappa_1 \ldots \kappa_k}_{\lambda_1 \ldots \lambda_\ell} \Lambda^{\lambda_1}_{\nu_1} \ldots \Lambda^{\lambda_\ell}_{\nu_\ell} \tag{A.5}$$

This exhibits two distinct transformation rules, one for upper indices and one for lower: we say the upper indices transform extcolor{blue}{contravariantly} and the lower indices transform extcolor{blue}{covariantly}. A third definition of a tensor is an assignment of a multidimensional array of numbers to each pair of bases of V and V^* such that the various assignments are related by the above transformation rule. The tensor v must remain invariant under a change in coordinates, so because $v = v^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_\ell} e_{\mu_1} \ldots e_{\mu_k} \sigma^{\nu_1} \ldots \sigma^{\nu_\ell}$ we are assured that $e_{\mu_1} \ldots e_{\mu_k} \sigma^{\nu_1} \ldots \sigma^{\nu_\ell}$ transforms inversely to $v^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_\ell}$, implying the following transformation law.

$$e'_{\mu_1} \ldots e'_{\mu_k} \sigma'^{\nu_1} \ldots \sigma'^{\nu_\ell} = \Lambda^{\kappa_1}_{\mu_1} \ldots \Lambda^{\kappa_k}_{\mu_k} e_{\kappa_1} \ldots e_{\kappa_k} \sigma^{\lambda_1} \ldots \sigma^{\lambda_\ell} (\Lambda^{-1})^{\nu_1}_{\lambda_1} \ldots (\Lambda^{-1})^{\nu_\ell}_{\lambda_\ell} \tag{A.6}$$

This (finally) motivates our choice of subscripts for vector bases and superscripts for dual vector bases: we are seeing that components of vectors and dual vector bases transform covariantly, while components of dual vectors and vector bases transform contravariantly (albeit by Λ and Λ^{-1}, respectively, in each case). By functional-argumentative duality, any element of $T^\ell_k(V)$ can yield an element of $T^{\ell-1}_{k-1}(V)$ by evaluating one vector (dual vector) argument on a dual vector (vector) argument: we refer to this as extcolor{blue}{contraction} and denote it by repeating an abstract index appropriately (e.g., the trace of a linear transformation is a contraction of its two arguments).

A.4 Tensor Algebra

We define the extcolor{blue}{tensor algebra} of V, $T(V) = \oplus^\infty_{k,\ell=0} T^\ell_k(V)$, which is a extcolor{magenta}{graded algebra} with multiplication operation $\otimes$. From we can also define the algebra ideal $I(V) = \langle v \otimes v \mid v \in V \rangle$ and consider

[4] We observe the following.

$$\begin{aligned}
v'^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_\ell} &= v^{a_1 \ldots a_k}_{b_1 \ldots b_\ell} (\sigma'^{\mu_1})_{a_1} \ldots (\sigma'^{\mu_k})_{a_k} (e'_{\nu_1})^{b_1} \ldots (e'_{\nu_\ell})^{b_\ell} \\
&= v^{a_1 \ldots a_k}_{b_1 \ldots b_\ell} (\Lambda^{-1})^{c_1}_{a_1} (\sigma^{\mu_1})_{c_1} \ldots (\Lambda^{-1})^{c_k}_{a_k} (\sigma^{\mu_k})_{c_k} \Lambda^{b_1}_{d_1} (e_{\nu_1})^{d_1} \ldots \Lambda^{b_\ell}_{d_\ell} (e_{\nu_\ell})^{d_\ell} \\
&= v^{\kappa_1 \ldots \kappa_k}_{\lambda_1 \ldots \lambda_\ell} e_{\kappa_1} ((\Lambda^{-1})^{c_1}_{a_1} (\sigma^{\mu_1})_{c_1}) \ldots e_{\kappa_k} ((\Lambda^{-1})^{c_k}_{a_k} (\sigma^{\mu_k})_{c_k}) \\
&\qquad \times \sigma^{\lambda_1} (\Lambda^{b_1}_{d_1} (e_{\nu_1})^{d_1}) \ldots \sigma^{\lambda_\ell} (\Lambda^{b_\ell}_{d_\ell} (e_{\nu_\ell})^{d_\ell}) \\
&= v^{\kappa_1 \ldots \kappa_k}_{\lambda_1 \ldots \lambda_\ell} (e_{\kappa_1})^{a_1} (\Lambda^{-1})^{c_1}_{a_1} (\sigma^{\mu_1})_{c_1} \ldots (e_{\kappa_k})^{a_k} (\Lambda^{-1})^{c_k}_{a_k} (\sigma^{\mu_k})_{c_k} \\
&\qquad \times (\sigma^{\lambda_1})_{b_1} \Lambda^{b_1}_{d_1} (e_{\nu_1})^{d_1} \ldots (\sigma^{\lambda_\ell})_{b_\ell} \Lambda^{b_\ell}_{d_\ell} (e_{\nu_\ell})^{d_\ell} \\
&= v^{\kappa_1 \ldots \kappa_k}_{\lambda_1 \ldots \lambda_\ell} (\Lambda^{-1})(\sigma^{\mu_1}, e_{\kappa_1}) \ldots (\Lambda^{-1})(\sigma^{\mu_k}, e_{\kappa_k}) \Lambda(\sigma^{\lambda_1}, e_{\nu_1}) \ldots \Lambda(\sigma^{\lambda_\ell}, e_{\nu_\ell}) \\
&= v^{\kappa_1 \ldots \kappa_k}_{\lambda_1 \ldots \lambda_\ell} (\Lambda^{-1})^{\mu_1}_{\kappa_1} \ldots (\Lambda^{-1})^{\mu_k}_{\kappa_k} \Lambda^{\lambda_1}_{\nu_1} \ldots \Lambda^{\lambda_\ell}_{\nu_\ell} \\
&= (\Lambda^{-1})^{\mu_1}_{\kappa_1} \ldots (\Lambda^{-1})^{\mu_k}_{\kappa_k} v^{\kappa_1 \ldots \kappa_k}_{\lambda_1 \ldots \lambda_\ell} \Lambda^{\lambda_1}_{\nu_1} \ldots \Lambda^{\lambda_\ell}_{\nu_\ell}
\end{aligned}$$

the quotient algebra $\Lambda(V) = T(V)/I(V)$, denoted the extcolor{blue}{exterior algebra}. Defining $I_k(V) = I(V) \cap T^0_k(V)$, we have the decomposition $\Lambda(V) = \oplus_{k=0}^{\infty} T^0_k(V)/I_k(V) = \oplus_{k=0}^{\infty} \Lambda_k(V)$; we refer to elements of $\Lambda_k(V)$ as extcolor{blue}{k-forms} on V, and $\dim \Lambda_k(V) = \binom{n}{k}$. Multiplication in $\Lambda(V)$ is denoted by $\wedge$ and referred to as the extcolor{blue}{wedge product}. The coset in $\Lambda(V)$ containing $v_1 \otimes \cdots \otimes v_m \in T(V)$ is exactly $v_1 \wedge \cdots \wedge v_{k+\ell}$. We can identify $\Lambda_k(V)$ with the set of extcolor{magenta}{alternating} $(k, 0)$ tensors; in abstract index notation, these satisfy $v_{a_1 \dots a_i \dots a_j \dots a_k} = -v_{a_1 \dots a_j \dots a_i \dots a_k}$. The tensor product coincides with the wedge product for such elements.

TODO: (anti)symmetric matrices

Appendix B

Algebraic Topology

B.1 Simplicial Homology

An extcolor{blue}{n-simplex} $p = \langle p_1 \ldots p_n \rangle$ is an ordered list of points representing its (oriented, or signed) convex hull of the points p_i. A extcolor{blue}{q-face} is the convex hull specified by $q+1$ of the p_i. A extcolor{blue}{simplicial complex} K is the finite union of n-simplexes (for possibly varying n) which is closed under taking faces and non-trivial intersections of two elements is a common face. The extcolor{blue}{r-chain group} $C_r(K)$ is the free Abelian group generated by the r-simplexes of K: elements are extcolor{blue}{r-chains}. The extcolor{blue}{boundary operators} $\partial_r : C_r(K) \to C_{r-1}(K)$ are defined by the linear extensions of

$$\langle p_0 \ldots p_r \rangle \mapsto \sum_i (-1)^i (p_0 \ldots \hat{p}_i \ldots p_r). \tag{B.1}$$

The extcolor{blue}{chain complex} for K is the induced long exact sequence

$$0 \to C_n(K) \to \ldots \to C_1(K) \to C_0(K) \to 0 \tag{B.2}$$

The extcolor{blue}{r-cycle group} $Z_r(K)$ (containing extcolor{blue}{r-cycles}) and the extcolor{blue}{r-boundary group} $B_r(K)$ (containing extcolor{blue}{r-boundaries}) are the kernel of ∂_r and the image of ∂_{r+1}, respectively. In particular, $\partial_r \circ \partial_{r+1} = 0$ so $B_r(K) \subset Z_r(K)$ and the quotient $Z_r(K)/B_r(K)$ is the extcolor{blue}{rth homology group} associated to K, whose elements are known as extcolor{blue}{homnology classes}

Appendix C

Clifford Algebra

TODO

Appendix D

Category Theory

A extcolor{blue}{category} C is a class $\mathrm{ob}(C)$ of extcolor{blue}{objects}; a class $\hom(C)$ extcolor{blue}{morphisms} (or maps, or arrows), each of which has a source object $a \in \mathrm{ob}(C)$ and a target object $b \in \mathrm{ob}(C)$ (written $a \to b$): in particular, $\hom(a, b)$ is the extcolor{blue}{hom-class} of morphisms $a \to b$ for $a, b \in \mathrm{ob}(C)$; and an associative binary operation with identity on $\hom(C)$ acting by $\hom(a, b) \times \hom(b, c) \to \hom(a, c)$. Given another category D, a extcolor{blue}{covariant functor} (or just functor) is a map $F : C \to D$ (i.e., $\mathrm{ob}(C) \to \mathrm{ob}(D)$ and $\hom(a, b) \to \hom(F(a), F(b))$) such that the binary operations and identity morphisms are preserved. A extcolor{blue}{covariant functor} is the same but sending $\hom(a, b) \to \hom(F(b), F(a))$ ("flipping the arrows").