Geometry

Elijah Sheridan

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Contents

Chapter 1

Introduction

What follows endeavors to encapsulate the author's knowledge of geometry (and the notions upon which it depends, which decidedly extend well beyond the boundaries of geometry itself) as he studies theoretical particle physics and string theory.

Chapter 2

Manifolds

2.1 Introduction

Let M be a extcolor{magenta}{second-countable}[1], extcolor{magenta}{Hausdorff}[2], extcolor{magenta}{locally Euclidean topological space} of dimension n. We define an extcolor{magenta}{equivalence relation} on the set of homeomorphisms between extcolor{magenta}{open} subsets of M and $\mathbb{R}^n$ given by $\phi \sim \psi$ when $\psi \circ \phi^{-1}$ is extcolor{magenta}{smooth}. We then choose a $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ (i.e., $\phi_\alpha : U_\alpha \to \mathbb{R}^n$) such that the $\{U_\alpha\}$ cover M and the $\{\phi_\alpha\}$ are an equivalence class: this is denoted a extcolor{blue}{maximal atlas}[3]. We then say that M is an n-dimensional extcolor{blue}{smooth manifold}[4] (or manifold). Let $(\phi, U) \in \mathcal{U}$: ϕ is a extcolor{blue}{coordinate chart} (or chart) and the components of ϕ, x^i

[1]Arguably, the truly important property here is extcolor{magenta}{paracompactness}, which is slightly stronger and enables partitions of unity (enabling local-to-global promotions). However, it is a result that Hausdorff, second countable, extcolor{magenta}{locally compact} space is paracompact (and we get local compactness follows from locally Euclidean). Second countability also contributes to the feasibility of Euclidean embeddings and other nice, preferable behavior.

References: Second countability and manifolds

[2]Hausdorff topological spaces feature points which are sufficiently disjoint: in particular, calculus depends upon limits, and Hausdorff $\implies$ unique limits as desired (note, though, that the converse isn't true).

[3]Definitions vary here (indeed, it is more conventional to merely require "maximal" atlases) but the general motivation is as follows: given a chart ϕ on a manifold M, there are likely uncountably many collections of charts covering M containing ϕ, but there is a *unique* (i.e., canonical) choice of equivalence class of charts containing ϕ.

References: Axiom of choice and maximal atlases

[4]Our consideration of differential topology/geometry is motivated by physics, which interests itself in the dynamics (or change) of our universe. extcolor{magenta}{Calculus}, in a word, is the mathematics of change: hence, we are interested in studying the *least structured* space that permits the calculus. This is not Euclidean space itself but rather a smooth manifold, a space that need only resemble Euclidean space *locally*.

(i.e., $\phi_\alpha(m) = (x^1(m), \dots, x^n(m))$), are extcolor{blue}{coordinates}. We say real-valued maps are extcolor{blue}{functions} (e.g., the x^i are functions).

2.2 Smooth Maps

Given another manifold N, we say $f : V \to N$ is a extcolor{blue}{smooth map} (or smooth) for an open set $V \subseteq M$ when for all $m \in U$, there exist charts ϕ and ψ defined around m and $f(m)$ such that $\psi \circ f \circ \phi^{-1}$ is smooth. For arbitrary U, we say the same when there exists $F : W \to N$ for an open set $V \subset W \subseteq M$ such that $F_{|V} = f$ and F is smooth. We call smooth maps with smooth inverse extcolor{blue}{diffeomorphisms}. We use $C^\infty(M)$, $\mathrm{Diff}(M, N)$, and $\mathrm{Diff}(M)$ to denote the spaces of smooth functions on M, diffeomorphisms $M \to N$, and diffeomorphisms $M \to M$, respectively. From this point forward, all maps are smooth unless otherwise specified.

2.3 Tangent Spaces

Let T_mM denote the extcolor{magenta}{vector space} of extcolor{magenta}{linear derivations} on the (vector) space of extcolor{magenta}{germs} of functions defined around m, F_m. Alternatively, let T_mM be the extcolor{magenta}{quotient ring} $(F_m/F_m^2)^*$, where $*$ denotes the extcolor{magenta}{dual space}. T_mM has dimension n, and we call it the extcolor{blue}{tangent space} to M at m and elements of T_mM extcolor{blue}{vectors}. There is a natural map $f \mapsto f_*$ from the set of smooth functions $M \to N$, denoted $C^\infty(M, N)$, to the set of extcolor{magenta}{endomorphisms} $T_mM \to T_{f(m)}N$ given by $f_*X(g) \mapsto X(g \circ f)$ (where $X \in T_mM$ and $g \in C^\infty(M)$, the extcolor{magenta}{ring} of smooth functions on M). We call f_* the extcolor{blue}{pushfoward} of f. We define T_m^*M to be the extcolor{blue}{cotangent space} to M at m; there is a natural map $d : C^\infty(M) \to T_m^*M$ given by $f \mapsto df(m) = v \mapsto v(f)$, which we call the extcolor{blue}{differential}. We also have the dual map $f \mapsto f^*$, the extcolor{blue}{pullback}, acting as $T_{f(m)}^*N \to T_m^*M$ by $f^*A(X) = A(f_*X)$. Given a chart ϕ around M, a basis for T_mM is given by $\frac{\partial}{\partial x^i}$ or ∂_i, given by

$$\partial_i f = \frac{\partial(f \circ \phi^{-1})}{\partial r^i}\Big|_m \tag{2.1}$$

where r^i is the ith Euclidean coordinate. A basis is also given for T_m^*M by the dx^i. Finally, we define the extcolor{blue}{tangent bundle} $TM = \cup_{m\in M} T_mM$ and the extcolor{blue}{cotangent bundle} $T^*M = \cup_{m\in M} T_m^*M$; both are $2n$-dimensional smooth manifolds equipped with natural projection maps onto M.

Chapter 3

Fibre Bundles

Chapter 4

Lie Theory

Example (short) footnote[1].

Example (long) footnote[2]

[1] blah blah blah
[2] blaher blaher blaher

Chapter 5

Applications

Some *significant* applications are demonstrated in this chapter.

5.1 Example one

5.2 Example two

Chapter 6

Complex Manifolds

We have finished a nice book.