HW3 Daniil Sherki

① Find the SVD of matrix.

$$A = UDV^T = U\Sigma V^T$$

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 3 & 1 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}$$

Singular values σ_i by findind the eigenvalues of AA^T.

$$AA^T = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1+1+9 & 3-1+3 \\ 3-13 & 9+1+1 \end{pmatrix} = \begin{pmatrix} 11 & 5 \\ 11 & 5 \end{pmatrix}$$

$$\det(AA^T - \lambda I) = \begin{vmatrix} 11-\lambda & 5 \\ 5 & 11-\lambda \end{vmatrix} =$$

$$= (11-\lambda)^2 - 25 = 121 - 22\lambda + \lambda^2 - 25 =$$

$$= \lambda^2 - 22\lambda + 96.$$

$$D = 484 - 4 \cdot 1 \cdot 96 = 100$$

$$\lambda_1 = \frac{22+10}{2} = 16 \quad \lambda_2 = \frac{22-10}{2} = 6$$

$\lambda_1 = 16 \quad \sigma_1 = \sqrt{16} = 4$

$\lambda_2 = 6 \quad \sigma_2 = \sqrt{6}.$

find singular vectors (the columns V) by finding ortho~~go~~normal set of eigenvectors of A^TA.

$$A^TA = \begin{pmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{pmatrix}$$

$$A^TA - 16I = \begin{pmatrix} -6 & 2 & 6 \\ 2 & -14 & -2 \\ 6 & -2 & -6 \end{pmatrix} R_3 + R_2 =$$

$$= \begin{pmatrix} -6 & 2 & 6 \\ 2 & -14 & -2 \\ 0 & 0 & 0 \end{pmatrix} 3R_2 + R_1 = \begin{pmatrix} 0 & -40 & 0 \\ 2 & -14 & -2 \end{pmatrix}$$

$$\begin{cases} -40x_2 = 0 \\ 2x_1 - 14x_2 - 2x_3 = 0 \end{cases} \begin{cases} x_2 = 0 \\ x_1 = -x_3 \end{cases}$$

$$V_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A^TA - 6I = \begin{pmatrix} 4 & 2 & 6 \\ 2 & +8 & -2 \\ 6 & -2 & 4 \end{pmatrix} R_3 + R_1 =$$

$$= \begin{pmatrix} 4 & 2 & 6 \\ 2 & +8 & -2 \\ 10 & 0 & 10 \end{pmatrix}$$

$$\begin{cases} 4x_1 + 2x_2 + 6x_3 = 0 \\ 10x_1 + 10x_3 = 0 \end{cases} \Rightarrow \begin{cases} 4x_1 + 2x_2 - 6x_3 = 0 \\ x_1 = -x_3 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 = +x_2 \\ x_1 = -x_3 \end{cases} \quad v_2 \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$

for last eigen vector

we could compute the kernel of A^TA or find a unit vector $\perp v_1, v_2$

$v_3 \perp v_1$ and $v_3 \perp v_2$

Assume that $v_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{cases} \frac{a}{\sqrt{2}} - \frac{c}{\sqrt{2}} = 0 \\ \frac{a}{\sqrt{3}} + \frac{b}{\sqrt{3}} + \frac{c}{\sqrt{3}} = 0 \\ a = c \end{cases}$$

$$v_3 = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} \quad \sqrt{a^2 + 0^2 + a^2} = 1$$

$$a = c = \frac{1}{\sqrt{2}}$$

Then

$$A = U \begin{pmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\sigma u_i = A v_i$

$u_i = \frac{1}{\sigma} \cdot A v_i$

$$u_1 = \frac{1}{4}\begin{pmatrix} 1 & -3 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

SRV:

$$A = \begin{pmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

② QR decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad A = QR$$

Q - column orthogonal (unitary)
R - upper ~~orthogonal~~ triangular.

$Q = n \times m$
$R = m \times m$ if $n \geq m$

$Rx = Q \cdot b$ $\quad a_1 = (1,1,0)$
$a_2 = (1,0,1)$
$a_3 = (0,1,1)$

$$q_1 = \frac{a_1}{\|a_1\|} =$$

$$q_2 = \frac{a_2}{\|a_2\|}$$

$$q_3 = \frac{a_3}{\|a_3\|}$$

$$\text{proj}_u a = \frac{\langle u, a \rangle}{\langle u, u \rangle} u$$

$u_1 = a_1$

$u_2 = a_2 - \text{proj}_{u_1} a_2$

$u_3 = a_3 - \text{proj}_{u_1} a_3 - \text{proj}_{u_2} a_3$

$e_1 = \frac{u_1}{\|u_1\|}$ $e_2 = \frac{u_2}{\|u_2\|}$ $e_3 = \frac{u_3}{\|u_3\|}$

$a_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $a_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$u_1 = a_1 = (1, 1, 0)$

$u_2 = a_2 - (a_2 \cdot e_1) e_1 = (1, 0, 1) - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$u_3 = a_3 - (a_3 \cdot e_1) e_1 - (a_3 \cdot e_2) a_2 =$

$0, 0, 1, - \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}, \cancel{\frac{1}{\sqrt{2}}}\, 0\right) - \frac{1}{\sqrt{6}} \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ $e_3 = \frac{u_3}{\|u_3\|^2} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$Q = [e_1, \dots, e_n]$

$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ $R = \begin{pmatrix} \langle e_1 a_1 \rangle & \langle e_1 a_2 \rangle & \langle e_1 a_3 \rangle \\ 0 & \langle e_2 a_2 & e_2 a_3 \\ 0 & 0 & e_3 a_3 \end{pmatrix}$

$R = \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$

③ Find eigenvalue decomposition of matrix A.

$$A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & 2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 0 & -3 \\ -9 & 2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{vmatrix} =$$

$$= \left((7-\lambda)(2-\lambda)(-8-\lambda) + 0 + 0\right) - \left(-54(2-\lambda)\right) =$$

$$= (2-\lambda)\left((7-\lambda)(-8-\lambda) + 54\right) = 0$$

$$-56 - 7\lambda + 8\lambda + \lambda^2 + 54 = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$D = 1 - 4 \cdot 1 \cdot (-2) = 9$$

$$\lambda_1 = \frac{-1+3}{2} = 1$$

$$\lambda_2 = \frac{-1-3}{2} = -2$$

$$\lambda_3 = 2.$$

$\lambda_1 = 1:$ $\begin{cases} 6x_1 - 3x_3 = 0 \\ -9x_1 + x_2 + 3x_3 \\ 18x_1 - 9 = 0 \end{cases}$

$\lambda_2 = -2$ $\begin{cases} 9x_1 - 3x_3 \\ -9x_1 + 4x_2 + 3x_3 = 0 \\ 3x_1 = x_3 \end{cases}$

$\lambda_3 = 2:$ $\begin{cases} 5x_1 - 3x_3 = 0 \\ -9x_1 + 3x_3 = 0 \\ 18x_1 = 10x_3 \end{cases}$

$$u_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix} \quad X^{-1} = \frac{1}{\det X} \cdot X^{T}$$

$$\det X = 1 \cdot (0 \cdot 0 - 3 \cdot 1) - 3 \cdot (1 \cdot 0 - 3 \cdot 0) +$$

$$+ 2(1 \cdot 1 - 0 \cdot 0) = -1$$

$$X^{T} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix} \quad X^{-1} = \frac{1}{\det x} \cdot X^{T} =$$

$$= (-1)^{1+1} \cdot \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} + (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix}$$

.... and other

$$A^{-1} = \begin{pmatrix} 3 & 0 & -1 \\ -2 & 0 & 1 \\ -9 & 1 & 3 \end{pmatrix}$$

and..

$$A = X \operatorname{diag}(\lambda)\, X^{-1}$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ -2 & 0 & 1 \\ -9 & 1 & 3 \end{pmatrix}$$

④ Find LU decomposition of matrix

$$A = \begin{pmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}$$

$A = LU$

L - lower triangular

U - upper triangular

1) $R_2 + R_1$

$$\begin{pmatrix} 0 & -2 & -1 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}$$

2) $2R_2 + R_3$

$$\begin{pmatrix} 0 & -2 & -1 & 2 \\ 0 & 6 & 2 & -5 \\ 6 & -4 & 6 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}$$

3) $9R_3 + 6R_4$

$$\begin{pmatrix} 0 & -2 & -1 & 2 \\ 0 & 6 & +2 & -5 \\ 0 & -6 & -30 & 27 \\ -9 & 5 & -5 & 12 \end{pmatrix}$$

4) $3R_1 + R_2$

$$\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 6 & 2 & -5 \\ 0 & -6 & -30 & 27 \\ -9 & 5 & -5 & 12 \end{pmatrix}$$

5) $R_3 + R_2$

$$\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -28 & 22 \\ 0 & -6 & -30 & 27 \\ -9 & 5 & -5 & 12 \end{pmatrix}$$

6) $28 R_1 - R_2$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & -28 & 22 \\ 0 & -6 & -30 & 27 \\ -9 & 5 & -5 & 12 \end{pmatrix}}_{L'}$$

— U:

$$\begin{array}{c} [4] \\ [3] \\ [1] \\ [2] \end{array} \begin{pmatrix} -9 & 5 & -5 & 12 \\ 6 & -4 & 0 & -5 \\ 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \end{pmatrix}$$

1) $R_3 + R_4$

$$\begin{pmatrix} -9 & 5 & -5 & 12 \\ 6 & -4 & 0 & -5 \\ 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \end{pmatrix}$$

2) $R_3 + (-0.5) R_2$

$$\begin{pmatrix} -9 & 5 & -5 & 12 \\ 6 & -4 & 0 & -5 \\ 0 & -5 & -2 & 4.5 \\ 0 & -2 & -1 & 2 \end{pmatrix}$$

3) $R_2 + \frac{2}{3} R_1$

$$\begin{pmatrix} -9 & 5 & -5 & 12 \\ 0 & -\frac{2}{3} & -\frac{10}{3} & 3 \\ 0 & -5 & -2 & 4.5 \\ 0 & -2 & -1 & 2 \end{pmatrix}$$

$$\begin{array}{c} [4] \\ [1] \\ [2] \\ [3] \end{array} 4) \begin{pmatrix} -9 & 5 & -5 & 12 \\ 0 & -5 & -2 & 4.5 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -3 & \frac{7}{3} \end{pmatrix}$$

5) $R_4 + \left(-\frac{1}{3}\right) \cdot R_3$

$$\begin{pmatrix} -9 & 5 & -5 & 12 \\ 0 & -5 & -2 & 4.5 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -3 & \frac{7}{3} \end{pmatrix}$$

6) $R_3 - \frac{2}{5} R_2$

$$\begin{pmatrix} -9 & 5 & -5 & 12 \\ 0 & -5 & -2 & 4.5 \\ 0 & 0 & -0.2 & 0.2 \\ 0 & 0 & -3 & \frac{7}{3} \end{pmatrix}$$

7) $R_4 - 15 R_1$

$$\underbrace{\begin{pmatrix} -9 & 5 & -5 & 12 \\ 0 & -5 & -2 & 4.5 \\ 0 & 0 & -0.2 & 0.2 \\ 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}}_{U}$$

Repeat all our workflow for

But.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$