Homework 2

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① a) $f(t)=1 \quad F(s)=\mathcal{L}\{1\}$

$$\mathcal{L}\{1\}=\int_0^\infty 1 e^{-st}\,dt=-\frac{1}{s}\int_0^\infty e^{-st}\,d(-st)=$$

$$=-\frac{1}{s}e^{-st}\Big|_0^\infty=-\frac{1}{s}e^{\infty}+\frac{1}{s}e^{0}=\frac{1}{s}$$

b) $f(t)=at^4+bt^2+c,\ F(s)=\mathcal{L}\{f(t)\}-?$

$$\mathcal{L}\{at^4+bt^2+c\}=\mathcal{L}(at^4)+\mathcal{L}\{bt^2\}+\mathcal{L}\{c\}$$

$$\mathcal{L}(at^4)=\int_0^\infty at^4\cdot e^{-st}\,dt=a\int_0^\infty t^4 e^{-st}\,dt=$$

$$=\left|\begin{matrix} u=t^4 & dv=e^{-st}dt \\ du=4t^3dt & v=-\frac{1}{s}e^{-st}\end{matrix}\right|=$$

$$=at^4\cdot\left(-\frac{e^{-st}}{s}\right)\Big|_0^\infty+a\int_0^\infty\frac{e^{-st}}{s}\cdot 4t^3dt=$$

$$=-\frac{a}{s}t^4\cdot e^{-st}\Big|_0^\infty+\frac{4a}{s}\overbrace{\int_0^\infty t^3e^{-st}\,dt}^{*}=$$

$$*:\left|\begin{matrix} u=t^3 & dv=e^{-st}dt \\ du=3t^2dt & v=-\frac{1}{s}e^{-st}\end{matrix}\right|=$$

$$=-\frac{a}{s}t^4\cdot e^{-st}\Big|_0^\infty+\frac{4a}{s}t^3\cdot\left(-\frac{1}{s}\right)e^{-st}\Big|_0^\infty-$$

$$-\left(-\frac{3}{s}\right)\frac{4a}{s}\underbrace{\int_0^\infty t^2e^{-st}\,dt}_{(**)}=\underbrace{-\frac{a}{s}t^4e^{-st}\Big|_0^\infty}_{(1*)}\ \ominus$$

$$(**):\left|\begin{matrix} u=t^2 & dv=e^{-st}dt \\ du=2tdt & v=-\frac{1}{s}e^{-st}\end{matrix}\right|$$

$$\ominus\ \underbrace{\frac{4a}{s^2}t^3e^{-st}\Big|_0^\infty}_{2*}+\underbrace{\frac{12a}{s^2}\cdot\left(-\frac{1}{s}\right)t^2e^{-st}\Big|_0^\infty}_{(3*)}\ \oplus$$

$$(+)\frac{12a}{s^3}\int_0^\infty t e^{-st}\,dt = \left| \begin{matrix} u=t & v=e^{-st}dt \\ d=dt & dv=\left(-\frac{1}{s}\right)e^{-st} \end{matrix} \right| (=)$$

$$(1*): -\frac{a}{s}t^4 e^{-st}\Big|_0^\infty = -\frac{a}{s}\infty^4 \cancel{e^{-\infty}}^{0} + \frac{a}{s}\cancel{t^4}^{0}\cancel{e^{0}}^{1} = 0$$

$$(2*): -\frac{4a}{s^3}t^3 e^{-st}\Big|_0^\infty = 0$$

$$(3*): 0.$$

$$= \frac{24a}{s^3}\cdot\left(-\frac{1}{s}\right)t\cdot e^{-st}\Big|_0^\infty \overset{=0}{} - \left(-\frac{1}{s}\right)\frac{24a}{s^3}\int_0^\infty e^{-st}\cdot dt =$$

$$= \frac{24a}{s^4}\cdot\left(-\frac{1}{s}\right)e^{-st}\Big|_0^\infty = -\frac{24a}{s^5}e^{-\infty} - \left(-\frac{24a}{s^5}\right)e^0 =$$

$$= \frac{24}{s^5} \Rightarrow \mathcal{L}(at^4) = \frac{24}{s^5}.$$

$$\mathcal{L}(bt^2) = \int_0^\infty bt^2 e^{-st}\,dt = \left| \begin{matrix} u=bt^2 & dv=e^{-st}dt \\ du=2bt\,dt & v=-\frac{1}{s}e^{-st} \end{matrix} \right.$$

$$= \left(-\frac{1}{s}\right)e^{-st}\cdot bt^2\Big|_0^\infty \overset{=0}{} - \left(-\frac{1}{s}\right)\int_0^\infty 2bt e^{-st}\,dt =$$

$$= \frac{2b}{s}\int_0^\infty t e^{-st}\,dt = \left| \begin{matrix} u=t & dv=e^{-st}dt \\ du=dt & v=\left(-\frac{1}{s}\right)e^{-st} \end{matrix} \right| =$$

$$= \frac{2b}{s}\cdot\left(-\frac{1}{s}\right)\cdot t\cdot e^{-st} - \left(-\frac{2b}{s^2}\right)\int_0^\infty e^{-st}\,dt =$$

$$= -\frac{2b}{s^3}e^{-st}\Big|_0^\infty = \left(-\frac{2b}{s^3}\right)\cancel{e^{-\infty}}^{0} - \left(-\frac{2b}{s^3}\right)\cdot e^0 =$$

$$= \frac{2b}{s^3} \Rightarrow \mathcal{L}(bt^2) = \frac{2b}{s^3}$$

$$\mathcal{L}\{c\} = \int_0^\infty c e^{-st}\,dt = -\frac{c}{s}e^{-st}\Big|_0^\infty = \frac{c}{s}$$

$$\mathcal{L}\{at^4+bt^2+c\} = \frac{24}{s^5} + \frac{2b}{s^3} + \frac{c}{s}$$

c) $f(t) = e^t \sin t$, $\mathcal{L}\{f(t)\} - ?$

$$\mathcal{L}\{e^t \sin t\} = \mathcal{L}\{e^t \cdot \frac{1}{2i}(e^{it} - e^{-it})\} =$$

where $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$

$$\mathcal{L}\{e^t \sin t\} = \int_0^\infty \sin t \cdot e^{-st+t}\, dt =$$

$$= \int_0^\infty \frac{e^{it} - e^{-it}}{2i} \cdot e^{-st}\, dt = \frac{1}{2i} \int_0^\infty e^{(i-s+1)}\, dt -$$

$$- \frac{1}{2i} \int_0^\infty e^{(-i+s+1)}\, dt = \frac{1}{2i} \cdot \frac{1}{s-i-1} - -$$

$$- \frac{1}{2i} \cdot \frac{1}{s+i-1} = \frac{\cancel{s}+i-\cancel{1}-\cancel{s}+i+\cancel{1}}{\cancel{2i}(s-i-1)(s+i-1)} = \frac{1}{(s-i-1)(s+i-1)} =$$

$$(s-i-1)(s+i-1) = s^2 + \cancel{si} - s - \cancel{is} - (-1) - s - \cancel{i} + \cancel{i} + 1$$

$$= s^2 - 2s + 2 = s^2 - 2s + 1 + 1 = (s-1)^2 + 1$$

$$= \frac{1}{(s-1)^2 + 1}$$

Task 2

$$f(t) = \frac{1}{(s-b)(s^2-a^2)}$$

Assume:

$$\frac{1}{(s-b)(s^2-a^2)} = \frac{A}{s-b} + \frac{B}{s^2-a^2}$$

$$A(s^2 - a^2) + B(s-b) = 1 \quad \emptyset$$

$$\frac{1}{(s-b)(s^2-a^2)} = \frac{A}{s-b} + \frac{B}{s-a} + \frac{C}{s+a}$$

$$A(s-a)(s+a) + B(s-b)(s+a) + C(s-b)(s-a) = 1$$

Assum $s = a$: $B(a-b)2a = 1$

$$B = \frac{1}{(a-b)(2a)}$$

$s=a$: $C(-2a)(-a-b)=1$

$C=\dfrac{1}{2a(a+b)}$

$s=b$: $A(b-a)(b+a)=1$

$A=\dfrac{1}{(b-a)(b+a)}$

$\mathcal{L}^{-1}\left(\dfrac{1}{(s-b)(s^2-a^2)}\right)=\mathcal{L}^{-1}\left\{\dfrac{1}{\cancel{2a(a-b)}}\;\dfrac{1}{(b-a)(b+a)}\;\dfrac{1}{s-b}\right\}+$

$+\mathcal{L}^{-1}\left\{\dfrac{1}{2a(a-b)}\;\dfrac{1}{(s-a)}\right\}+\mathcal{L}^{-1}\left\{\dfrac{1}{2a(a+b)}\;\dfrac{1}{s+a}\right\}$

$=\dfrac{1}{(b-a)(b+a)}\cdot e^{bt}+\dfrac{1}{2a(a-b)}e^{at}+\dfrac{1}{2a(a+b)}e^{-at}$

Task 3

a) $y''-5y'+6y=0,\ y(0)=2,\ y'(0)=0.$

$\mathcal{L}\{y''-5y'+6y\}=\mathcal{L}\{y''\}-5\mathcal{L}\{y'\}+$

$+6\mathcal{L}(y)=s^2Y-sy(0)-y'(0)-5sY+5y(0)$

$+6Y=s^2Y-2s-5sY+6Y+10=$

$=Y(s^2-5s+6)-2s+10$

$Y=\dfrac{2s-10}{s^2-5s+6}=\dfrac{2s-10}{(s-3)(s-2)}=\dfrac{A}{s-3}+\dfrac{B}{s-2}$

$D=25-4\cdot1\cdot6=1$

$s_1=\dfrac{5+1}{2}=3$

$s_2=\dfrac{5-1}{2}=2$

$A(s-2)+B(s-3)=2s-10$

$As-2A+Bs-3B=2s-10$

$(A+B)s-2A-3B=2s-10$

$\begin{cases}A+B=2 \Rightarrow B=2-A\\ -2A-3B=-10 \Rightarrow -2A-6+3A=-10 \Rightarrow \underline{A=-4}\Rightarrow\end{cases}$

$\Rightarrow B=2-(-4)=6 \Rightarrow \underline{B=6}$

Task 3 (Cont).

$$Y = -\frac{4}{s-3} + \frac{6}{s-2}$$

$$y = \mathcal{L}^{-1}(Y) = \underline{4e^{3t} + 6e^{2t}}$$

Task 4

a) $f(x) = e^{-a|x|}, \; a>0$

$$F(\xi) = \mathcal{F}\{f(x)\}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x}\, dx \Rightarrow$$

$$\mathcal{F}\{e^{-a|x|}\}(\xi) = \int_{-\infty}^{\infty} e^{-a|x|}\, e^{-i2\pi\xi x}\, dx =$$

There is $a>0$:

$$= \int_{-\infty}^{\infty} e^{-a|x| - i2\pi\xi x}\, dx = \int_{-\infty}^{0} e^{(+a-i2\pi\xi)x}\, dx +$$

$$+ \int_{0}^{\infty} e^{(-a-i2\pi\xi)x}\, dx = \frac{1}{a-i2\pi\xi}\, e^{(a-i2\pi\xi)x}\Big|_{-\infty}^{0} +$$

$$+ \frac{1}{-a-i2\pi\xi}\, e^{(-a-i2\pi\xi)x}\Big|_{0}^{\infty} = \frac{1}{a-i2\pi\xi} +$$

$$+ \frac{1}{+a+i2\pi\xi} + \frac{1}{a-i2\pi\xi}\, e^{\xrightarrow{}0}\, e^{-i\pi\xi} = \frac{a-i2\pi\xi + a + i2\pi\xi}{(a-i2\pi\xi)(+a+i2\pi\xi)} =$$

$$= \underline{\frac{2a}{+a^2 + 4\pi\xi^2}}$$

Task 5.

$$\mathcal{F}\{e^{iax}\cdot f(x)\} = F\left(\xi - \frac{a}{2\pi}\right)$$

$$\mathcal{F}\{e^{iax}\cdot f(x)\} = \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{i2\pi\xi x}\, dx =$$

$$= \int_{-\infty}^{\infty} e^{iax - i2\pi\xi x} f(x)\, dx = \int_{-\infty}^{\infty} e^{-i2\pi x\left(-\frac{a}{2\pi} + \xi\right)} f(x)\, dx$$

$$\Rightarrow \cancel{\mathcal{F}}\{ \quad F\{\xi - \frac{a}{2\pi}\}$$

Task 6 $\frac{\partial P(x,t)}{\partial t} = u\frac{\partial P(x,t)}{\partial x} + d\,\frac{\partial^2 P(x,t)}{\partial x^2}$

$P(x,t)$ – particle density function

u – constant drift

d – constant diffusion

Bound: $x \geq 0$, $P(x=0,t)=0$, $P(x=\infty,t)=0$, $P(x,t=0)=\delta(x-x_0)$

1) $\mathcal{F}\{\frac{\partial p}{\partial t}\} = \int_{-\infty}^{\infty} \frac{\partial P(x,t)}{\partial t} e^{-i2\pi\xi x} dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} P(x,t) e^{-i2\pi\xi x} dx =$

$\frac{\partial}{\partial t} P(\xi,t)$

2) $\mathcal{F}\{u\frac{\partial p}{\partial x} + d\frac{\partial^2 p}{\partial x^2}\} = u\,\mathcal{F}\{\frac{\partial p}{\partial x}\} + d\mathcal{F}\{\frac{\partial^2 p}{\partial x^2}\} =$

$i2\pi\xi u P\{\xi,t\} + (i2\pi\xi)^2 d\, P\{\xi,t\}$

$\frac{\partial}{\partial t} P(\xi;t) = 2\pi\xi\, P(\xi,t)(iu - d\cdot 2\pi\xi)$

$\frac{\partial P(\xi,t)}{P(\xi,t)} = 2\pi\xi(iu - d2\pi\xi)\cdot\partial t$

$\ln P(\xi,t) - \ln \overbrace{P\{\xi,t=0\}}^{\delta(x-x_0)} = 2\pi\xi(iu - d2\pi\xi)t$

$P(\xi,t) = \delta(x-x_0)\cdot e^{2\pi\xi(iu - d2\pi\xi)t}$

$\mathcal{F}\{P^{-1}\{\xi,t\}\}$